NE 523: Homework 1

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1 Two-species Boltzmann Equation

1. Begin with the generic Non-linear Boltzmann Equation.

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\nabla_r + \frac{\mathbf{F}}{m}\nabla_v\right) f(\mathbf{r}, \mathbf{v}, t) = \left(\frac{\partial f}{\partial t}\right)_{coll} \tag{1}$$

2. However, we have two particles with different masses and different external forces, which are given as:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_f\nabla_r + \frac{\mathbf{F}}{m}\nabla_v\right) f(\mathbf{r}, \mathbf{v}, t) = \left(\frac{\partial f}{\partial t}\right)_{coll} \tag{2a}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_g\nabla_r + \frac{\mathbf{Q}}{M}\nabla_v\right) g(\mathbf{r}, \mathbf{v}, t) = \left(\frac{\partial g}{\partial t}\right)_{coll} \tag{2b}$$

3. Let's define the following notation for convenience. Similar quantities are implicitly defined for $g(\mathbf{r}, \mathbf{v}, t)$.

$$f(\mathbf{r}, \mathbf{v}_x, t) = f_x \quad \text{and} \quad f(\mathbf{r}, \mathbf{v}'_x, t) = f'_x \tag{3}$$

4. We know every part of the previous equations except for the collision terms. We will take the collision terms to be the difference between the gain and loss terms. As there are two particle types with only binary collisions assumed, we will have four collision interactions (ff, fg, gg, gf). The loss term (R) is defined as follows:

$$R_{ff}\delta t d^3r d^3v_{f1} = \left[f_1 d^3r d^3v_{f1} \int d^3v_{f2} \int_{4\pi} d\hat{\Omega}\sigma_{ff}(\hat{\Omega}) f_2 \left|\mathbf{v}_{f1} - \mathbf{v}_{f2}\right|\right] \delta t \tag{4a}$$

$$R_{fg}\delta t d^3r d^3v_{f1} = \left[f_1 d^3r d^3v_{f1} \int d^3v_{g2} \int_{4\pi} d\hat{\Omega}\sigma_{fg}(\hat{\Omega}) g_2 \left|\mathbf{v}_{f1} - \mathbf{v}_{g2}\right|\right] \delta t \tag{4b}$$

$$R_{gg}\delta t d^3 r d^3 v_{g1} = \left[g_1 d^3 r d^3 v_{g1} \int d^3 v_{g2} \int_{4\pi} d\hat{\Omega}\sigma_{gg}(\hat{\Omega}) g_2 \left|\mathbf{v}_{g1} - \mathbf{v}_{g2}\right|\right] \delta t \tag{4c}$$

$$R_{gg}\delta t d^3 r d^3 v_{g1} = \left[g_1 d^3 r d^3 v_{g1} \int d^3 v_{f2} \int_{4\pi} d\hat{\Omega}\sigma_{gf}(\hat{\Omega}) f_2 \left|\mathbf{v}_{g1} - \mathbf{v}_{f2}\right|\right] \delta t \tag{4d}$$

5. Canceling like terms from each side of the loss expressions:

$$R_{ff} = \int d^3 v_{f2} \int_{4\pi} d\hat{\Omega}\sigma_{ff}(\hat{\Omega}) \left|\mathbf{v}_{f1} - \mathbf{v}_{f2}\right| f_1 f_2 \tag{5a}$$

$$R_{fg} = \int d^3 v_{g2} \int_{4\pi} d\hat{\Omega}\sigma_{fg}(\hat{\Omega}) \left|\mathbf{v}_{f1} - \mathbf{v}_{g2}\right| f_1 g_2 \tag{5b}$$

$$R_{gg} = \int d^3 v_{g2} \int_{4\pi} d\hat{\Omega}\sigma_{gg}(\hat{\Omega}) \left|\mathbf{v}_{g1} - \mathbf{v}_{g2}\right| g_1 g_2 \tag{5c}$$

$$R_{gg} = \int d^3 v_{f2} \int_{4\pi} d\hat{\Omega}\sigma_{gf}(\hat{\Omega}) \left|\mathbf{v}_{g1} - \mathbf{v}_{f2}\right| g_1 f_2 \tag{5d}$$

6. Next, we will define the gain terms ($\bar{R}$). Similar to the loss terms, there are four different loss terms. Where the prime terms represent the particle after the collision.

$$\bar{R}_{ff}\delta t d^3 r d^3 v_{f1} = \delta t d^3 r \int d^3 v'_{f2} \int_{4\pi} d\hat{\Omega}\sigma'_{ff}(\hat{\Omega}) \left|\mathbf{v}'_{f2} - \mathbf{v}'_{f1}\right| f'_2 f'_1 d^3 v'_{f1} \tag{6a}$$

$$\bar{R}_{fg}\delta t d^3 r d^3 v_{f1} = \delta t d^3 r \int d^3 v'_{g2} \int_{4\pi} d\hat{\Omega}\sigma'_{fg}(\hat{\Omega}) \left|\mathbf{v}'_{g2} - \mathbf{v}'_{f1}\right| g'_2 f'_1 d^3 v'_{f1} \tag{6b}$$

$$\bar{R}_{gg}\delta t d^3 r d^3 v_{g1} = \delta t d^3 r \int d^3 v'_{g2} \int_{4\pi} d\hat{\Omega}\sigma'_{gg}(\hat{\Omega}) \left|\mathbf{v}'_{g2} - \mathbf{v}'_{g1}\right| g'_2 g'_1 d^3 v'_{g1} \tag{6c}$$

$$\bar{R}_{gf}\delta t d^3 r d^3 v_{g1} = \delta t d^3 r \int d^3 v'_{f2} \int_{4\pi} d\hat{\Omega}\sigma'_{gf}(\hat{\Omega}) \left|\mathbf{v}'_{f2} - \mathbf{v}'_{g1}\right| f'_2 g'_1 d^3 v'_{g1} \tag{6d}$$

7. Next we will cancel the like terms from each side of the gain expressions. Note, we are also applying the fact that $d^3 v'_{xi} = d^3 v_{xi}$ to cancel the velocity element from the left sides. We also assume the forward and backwards cross sections are the same.

$$\bar{R}_{ff} = \int d^3 v_{f2} \int_{4\pi} d\hat{\Omega}\sigma_{ff}(\hat{\Omega}) \left|\mathbf{v}'_{f2} - \mathbf{v}'_{f1}\right| f'_2 f'_1 \tag{7a}$$

$$\bar{R}_{fg} = \int d^3 v_{g2} \int_{4\pi} d\hat{\Omega}\sigma_{fg}(\hat{\Omega}) \left|\mathbf{v}'_{g2} - \mathbf{v}'_{f1}\right| g'_2 f'_1 \tag{7b}$$

$$\bar{R}_{gg} = \int d^3 v_{g2} \int_{4\pi} d\hat{\Omega}\sigma_{gg}(\hat{\Omega}) \left|\mathbf{v}'_{g2} - \mathbf{v}'_{g1}\right| g'_2 g'_1 \tag{7c}$$

$$\bar{R}_{gf} = \int d^3 v_{f2} \int_{4\pi} d\hat{\Omega}\sigma_{gf}(\hat{\Omega}) \left|\mathbf{v}'_{f2} - \mathbf{v}'_{g1}\right| f'_2 g'_1 \tag{7d}$$

8. We would like to deal with the $\mathbf{v}'$ terms. For the monotypical collisions, we know the magnitude of the velocity does not change after the collision. However, this is not the case for the collisions involving both particle species. Therefore, we need to calculate the final velocities using known quantities.

   (a) We will start by defining the center of mass velocity as:

$$v_c = \frac{mv_f + Mv_g}{m + M} \tag{8}$$

   (b) Convert both velocities into the center of mass frame, where $v_{rel} = v_f - v_g$.

$$v_{f,c} = v_f - v_c = v_f - \frac{mv_f + Mv_g}{m+M} = \frac{v_f(m+M)}{m+M} - \frac{mv_f + Mv_g}{m+M} = \frac{M}{m+M} v_{rel} \tag{9a}$$

$$v_{g,c} = v_g - v_c = v_g - \frac{mv_f + Mv_g}{m+M} = \frac{v_g(m+M)}{m+M} - \frac{mv_f + Mv_g}{m+M} = -\frac{m}{m+M} v_{rel} \tag{9b}$$

   (c) We know, in the center of mass frame, the final center of mass velocity of a given particle is equal to the opposite of the initial center of mass velocity of that particle.

$$v'_{f,c} = -v_{f,c} = -\frac{M}{m+M} v_{rel} \tag{10a}$$

$$v'_{g,c} = -v_{g,c} = \frac{m}{m+M} v_{rel} \tag{10b}$$

   (d) Then, convert back to the stationary frame to find the velocities before and after the collision by adding the center of mass velocity. When $m = M$, both equations reduce to the expected result of the particle velocities switching.

$$v'_f = v'_{f,c} + v_c = \frac{-Mv_f + Mv_g}{m+M} + \frac{mv_f + Mv_g}{m+M} = \frac{v_f(m-M) + 2Mv_g}{m+M} \tag{11a}$$

$$v'_g = v'_{g,c} + v_c = \frac{mv_f - mv_g}{m+M} + \frac{mv_f + Mv_g}{m+M} = \frac{v_g(M-m) + 2mv_f}{m+M} \tag{11b}$$

9. With the final particle velocities written in terms of known quantities, we can evaluate the following two differences:

$$\mathbf{v}'_f - \mathbf{v}'_g = \frac{-Mv_f + Mv_g}{m+M} - \frac{mv_f - mv_g}{m+M} = -\frac{v_f(m+M)}{m+M} + \frac{v_g(m+M)}{m+M} = \mathbf{v}_g - \mathbf{v}_f \tag{12a}$$

$$\mathbf{v}'_g - \mathbf{v}'_f = -(\mathbf{v}'_f - \mathbf{v}'_g) = \mathbf{v}_f - \mathbf{v}_g \tag{12b}$$

10. We can now write the following equalities, which allow us to combine the loss and gain terms when the particles are dissimilar:

$$\left|\mathbf{v}'_{g2} - \mathbf{v}'_{f1}\right| = \left|\mathbf{v}_{f1} - \mathbf{v}_{g2}\right| \tag{13a}$$

$$\left|\mathbf{v}'_{f2} - \mathbf{v}'_{g1}\right| = \left|\mathbf{v}_{g1} - \mathbf{v}_{f2}\right| \tag{13b}$$

11. Now, we can write the collision term for the particles in the f distribution.

$$\left(\frac{\partial f}{\partial t}\right)_{coll} = \left(\bar{R}_{ff} - R_{ff}\right) + \left(\bar{R}_{fg} - R_{fg}\right) = \tag{14}$$

$$\int d^3 v_{f2} \int_{4\pi} d\hat{\Omega}\sigma_{ff}(\hat{\Omega})\left|\mathbf{v}_{f1} - \mathbf{v}_{f2}\right|\left(f'_1 f'_2 - f_1 f_2\right) \tag{15}$$

$$+\int d^3 v_{g2} \int_{4\pi} d\hat{\Omega}\sigma_{fg}(\hat{\Omega})\left|\mathbf{v}_{f1} - \mathbf{v}_{g2}\right|\left(f'_1 g'_2 - f_1 g_2\right) \tag{16}$$

12. Similarly for the particles in the g distribution:

$$\left(\frac{\partial g}{\partial t}\right)_{coll} = \left(\bar{R}_{gg} - R_{gg}\right) + \left(\bar{R}_{gf} - R_{gf}\right) = \tag{17}$$

$$\int d^3 v_{g2} \int_{4\pi} d\hat{\Omega}\sigma_{gg}(\hat{\Omega})\left|\mathbf{v}_{g1} - \mathbf{v}_{g2}\right|\left(g'_1 g'_2 - g_1 g_2\right) \tag{18}$$

$$+\int d^3 v_{f2} \int_{4\pi} d\hat{\Omega}\sigma_{gf}(\hat{\Omega})\left|\mathbf{v}_{g1} - \mathbf{v}_{f2}\right|\left(g'_1 f'_2 - g_1 f_2\right) \tag{19}$$

We can write the Boltzmann equation for the particles of mass m.

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_f \nabla_r + \frac{\mathbf{F}}{m}\nabla_v\right) f(\mathbf{r}, \mathbf{v}, t) =$$
$$\int d^3 v_{f2} \int_{4\pi} d\hat{\Omega}\sigma_{ff}(\hat{\Omega})\left|\mathbf{v}_{f1} - \mathbf{v}_{f2}\right|\left(f'_1 f'_2 - f_1 f_2\right)$$
$$+\int d^3 v_{g2} \int_{4\pi} d\hat{\Omega}\sigma_{fg}(\hat{\Omega})\left|\mathbf{v}_{f1} - \mathbf{v}_{g2}\right|\left(f'_1 g'_2 - f_1 g_2\right)$$

We can also write the Boltzmann equation for the particles of mass M.

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_g \nabla_r + \frac{\mathbf{Q}}{M}\nabla_v\right) g(\mathbf{r}, \mathbf{v}, t) =$$
$$\int d^3 v_{g2} \int_{4\pi} d\hat{\Omega}\sigma_{gg}(\hat{\Omega})\left|\mathbf{v}_{g1} - \mathbf{v}_{g2}\right|\left(g'_1 g'_2 - g_1 g_2\right)$$
$$+\int d^3 v_{f2} \int_{4\pi} d\hat{\Omega}\sigma_{gf}(\hat{\Omega})\left|\mathbf{v}_{g1} - \mathbf{v}_{f2}\right|\left(g'_1 f'_2 - g_1 f_2\right)$$

2 Invariance of Phase-space Volume Element

1. We start knowing the position and velocity vectors are given below. Additionally, we have an external force field that is a not a function of velocity, but can be a function of position $\mathbf{F}(\mathbf{r})$.

$$\mathbf{r} = \langle x, y, z \rangle \tag{20a}$$

$$\mathbf{v} = \langle v_x, v_y, v_z \rangle \tag{20b}$$

2. I could carry this derivation out in the full 3D expression for position and velocity, but the equations end up being identical for each coordinate direction. Additionally, each direction is independent of the other directions, so none of the directions are coupled. As such, I am going to work with a 1D expression for position and velocity. I could carry out the exact same steps for $\hat{y}$ and $\hat{z}$, but doing so would not add anything substantive. The expressions I will work with are given as follows:

$$\mathbf{r} = \langle x \rangle = r \tag{21a}$$

$$\mathbf{v} = \langle v_x \rangle = v \tag{21b}$$

3. Next, after a small time δt, the position and velocity will change.

$$r' = x + v\delta t \tag{22a}$$

$$v' = v + \frac{F(x)}{m}\delta t \tag{22b}$$

4. If we can show the determinant of the Jacobian equal one, we can show the two elements have the same area. First, we will calculate the Jacobian, which can be used to determine how the area maps between $drdv$ and $dr'dv'$.

$$J = \begin{pmatrix} \frac{\partial r'}{\partial r} & \frac{\partial r'}{\partial v} \\ \frac{\partial v'}{\partial r} & \frac{\partial v'}{\partial v} \end{pmatrix} \tag{23}$$

5. Let's find all the components of the Jacobian.

$$\frac{\partial r'}{\partial r} = \partial r\,(r + v\delta t) = 1 \tag{24a}$$

$$\frac{\partial r'}{\partial v} = \partial v\,(r + v\delta t) = \delta t \tag{24b}$$

$$\frac{\partial v'}{\partial r} = \partial r\left(v + \frac{F(r)}{m}\delta t\right) = \frac{\delta t}{m}F'(x) \tag{24c}$$

$$\frac{\partial v'}{\partial v} = \partial v\left(v + \frac{F(r)}{m}\delta t\right) = 1 \tag{24d}$$

6. Next, let's find the determinant of the Jacobian.

$$\det(J) = \det\begin{pmatrix} 1 & \delta t \\ \frac{\delta t}{m}F'(r) & 1 \end{pmatrix} = (1)(1) - (\delta t)\left(\frac{\delta t}{m}F'(r)\right) \tag{25}$$

7. However, we are ignoring the quadratic terms in δt, so the determinant simplifies to the following.

$$\det(J) = 1 - \cancelto{0}{(\delta t)\left(\frac{\delta t}{m}F'(r)\right)} = 1 \tag{26}$$

8. Because the determinant of the Jacobian equals one, when we map from $drdv \rightarrow dr'dv'$, the area gets scaled by one. As the area does not change between the two coordinate systems, we conclude $drdv = dr'dv'$ for the 1D case.

9. As previously mentioned, we could compute the Jacobian for the complete mapping $d^3\mathbf{r}d^3\mathbf{v} \rightarrow d^3\mathbf{r}'d^3\mathbf{v}'$. However, we would just be repeating the 1D case two more times.

Because each coordinate (x, y, z) yields the same result as the 1D case, we also conclude $d^3\mathbf{r}d^3\mathbf{v} \rightarrow d^3\mathbf{r}'d^3\mathbf{v}'$.

3 Moments of the Maxwell-Boltzmann Equilibrium Distribution

Before beginning the problem, I will describe the variables and assumptions used in multiple parts.

Convenient Variable Definitions

1. Begin with the Maxwell-Boltzmann distribution

$$f_0(\mathbf{v}) = n\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{m|\mathbf{v}-\mathbf{v}_0|^2}{2kT}\right) \tag{27}$$

2. We are told the distribution is in equilibrium. As such, we know the bulk velocity, $\mathbf{v}_0$, of the gas does not change. We define a relative particle velocity, u, and du as:

$$\mathbf{u} = \mathbf{v} - \mathbf{v}_0, \quad \mathbf{v} = \mathbf{u} + \mathbf{v}_0 \tag{28a}$$

$$\frac{d\mathbf{u}}{d\mathbf{v}} = 1 \Rightarrow d\mathbf{u} = d\mathbf{v} \tag{28b}$$

3. This leaves the distribution function as:

$$f_0(\mathbf{v}) = n\left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{m\mathbf{u}^2}{2kT}\right) \tag{29}$$

4. We can simplify our notation by using the following variables:

$$\alpha = \frac{m}{2\pi kT} \tag{30a}$$

$$\beta = \frac{m}{2kT} \tag{30b}$$

$$\alpha\pi = \beta \tag{30c}$$

5. Rewriting our distribution function using the aforedefined variables.

$$f_0(\mathbf{v}) = n\alpha^{3/2} \exp\left(-\beta\mathbf{u}^2\right) \tag{31}$$

Velocity Space Differential Volume Elements

We are expected to integrate over dv^3, but we changed the variable in our integrand to be a function of the relative velocity $\mathbf{u}$. Therefore, we will show the volume elements of $\mathbf{v}$ and $\mathbf{u}$ are equal.

1. We know $\mathbf{u} = \mathbf{v} - \mathbf{v}_0$, which can be written component-wise as:

$$(u_x, u_y, u_z) = (v_x, v_y, v_z) - (v_{0x}, v_{0y}, v_{0z}) \tag{32}$$

2. This gives the following equations:

$$u_x = v_x - v_{0x}, \quad u_y = v_y - v_{0y}, \quad u_z = v_z - v_{0z} \tag{33}$$

3. We can take the derivative of u_x to find:

$$\frac{du_x}{dv_x} = 1 \Rightarrow du_x = dv_x \tag{34}$$

4. Similarly to the equation above, $du_y = dv_y$ and d $u_z = dv_z$. Therefore, we can write:

$$dv_x dv_y dv_z = d^3v = d^3u = du_x du_y du_z \tag{35}$$

5. We see any d^3v can be rewritten as d^3u.

3.1 Part A: the Zeroth Moment

Before starting, we should think about the expected result. We are integrating a distribution function scaled by the particle density over all velocity space. Therefore, we should expect to obtain the total number of particles/cc.

1. First, we will start with Eq. 31 as $f_0(\mathbf{v})$ and substitute the expression for $g(\mathbf{v})$.

$$\int d^3v \, [1] \left[n\alpha^{3/2} \exp\left(-\beta \mathbf{u}^2\right) \right] \tag{36}$$

2. Factor out all variables not dependent on v:

$$n\alpha^{3/2} \int d^3v \exp\left(-\beta \mathbf{u}^2\right) \tag{37}$$

3. For now, omit the constants and focus on the integral. We can rewrite d^3v as d^3u (Eq. 35) and $\mathbf{u}^2$ as its components.

$$\int du_x du_y du_z \exp\left(-\beta\left(u_x^2 + u_y^2 + u_z^2\right)\right) \tag{38}$$

   (a) where $d^3u = du_x du_y du_z$,

   (b) $\mathbf{u}^2 = u_x^2 + u_y^2 + u_z^2$

4. We can separate the velocity in the exponential and factor like terms.

$$\int_{-\infty}^{\infty} du_x \exp\left(-\beta u_x^2\right) \int_{-\infty}^{\infty} du_y \exp\left(-\beta u_y^2\right) \int_{-\infty}^{\infty} du_z \exp\left(-\beta u_z^2\right) \tag{39}$$

5. Focus on the integral over du_x. We recognize the following integral as the error function, which we know the value of.

$$\int_{-\infty}^{\infty} du_x \exp\left(-\beta u_x^2\right) = \sqrt{\frac{\pi}{\beta}} \tag{40}$$

6. We notice there is no u_x dependence in the solution to the integral. Therefore, we realize the integrals in du_y and du_z are equal to the integral over du_x.

$$\int_{-\infty}^{\infty} du_x \exp\left(-\beta u_x^2\right) \int_{-\infty}^{\infty} du_y \exp\left(-\beta u_y^2\right) \int_{-\infty}^{\infty} du_z \exp\left(-\beta u_z^2\right) = \left(\frac{\pi}{\beta}\right)^{3/2} \tag{41}$$

7. Reintroduce the previously omitted constants.

$$n\alpha^{3/2}\left(\frac{\pi}{\beta}\right)^{3/2} \tag{42}$$

8. Use the relation from Eq. 30c and recognize all constants besides n cancel.

$$n\left(\frac{\alpha\pi}{\beta}\right)^{3/2} = n(1)^{3/2} = n \tag{43}$$

We arrive at the expected expression for the zeroth moment:

$$\int d^3v f_0(\mathbf{v}) = n \tag{44}$$

3.2 Part B: the First Moment

1. Begin with the expression for the first moment. We will substitute d^3v with d^3u according to Eq. 35 and $\mathbf{v}$ with $\mathbf{u}$ according to Eq. 28a.

$$\int d^3v\mathbf{v}f_0(\mathbf{v}) = \int d^3u\left(\mathbf{u}+\mathbf{v}_0\right)f_0(\mathbf{v}) \tag{45}$$

2. Split the integrals.

$$\int d^3u\mathbf{u}f_0(\mathbf{v}) + \int d^3u\mathbf{v}_0f_0(\mathbf{v}) \tag{46}$$

3. Start with the first integral. Use Eq. 31 to rewrite the distribution function.

$$\int d^3u\mathbf{u}f_0(\mathbf{v}) = \int d^3u\mathbf{u}\left(n\alpha^{3/2}\exp\left(-\beta\mathbf{u}^2\right)\right) \tag{47}$$

   (a) Because these are vector equations, we need to separate each component and integrate independently. Note: $\mathbf{u} = (u_x, u_y, u_z)$ and $\mathbf{u^2} = (u_x^2, u_y^2, u_z^2)$.

$$\hat{i} : n\alpha^{3/2}\int_{-\infty}^{\infty} du_xu_x\exp\left(-\beta u_x^2\right) \tag{48a}$$

$$\hat{j} : n\alpha^{3/2}\int_{-\infty}^{\infty} du_yu_y\exp\left(-\beta u_y^2\right) \tag{48b}$$

$$\hat{k} : n\alpha^{3/2}\int_{-\infty}^{\infty} du_zu_z\exp\left(-\beta u_z^2\right) \tag{48c}$$

   (b) Focusing on the $\hat{i}$ equation, we will neglect the constants for now.

$$\int_{-\infty}^{\infty} du_xu_x\exp\left(-\beta u_x^2\right) \tag{49}$$

   (c) Say some variable $q = u_x^2$. Thus, $dq/du_x = 2u_x$. Apply these substitutions keeping in mind q is non-negative as $u_x \in \mathbb{R}$, so $u_x^2 = y \geq 0$.

$$\int_{-\infty}^{\infty} \frac{dq}{2u_x}u_x\exp\left(-\beta|q|\right) = \frac{1}{2}\int_{-\infty}^{\infty} dq\exp\left(-\beta|q|\right) = \left[-\frac{1}{2\beta}|q|\right]_{-\infty}^{\infty} = 0 \tag{50}$$

   (d) Similar to the $\hat{i}$ integral, the $\hat{j}$ and $\hat{k}$ integrals are also equal to zero. Therefore, we find the first integral to be equal to zero.

4. Now, move to the second integral

$$\int d^3u\mathbf{v}_0 f_0(\mathbf{v}) \tag{51}$$

   (a) We know v_0 is constant, so factor v_0 out.

$$\mathbf{v}_0 \int d^3u f_0(\mathbf{v}) \tag{52}$$

   (b) The integral can be recognized as the zeroth moment. Therefore, the second integral is given as:

$$\mathbf{v}_0 \int d^3u f_0(\mathbf{v}) = \mathbf{v}_0 n \tag{53}$$

5. With both integrals evaluated, we find the final expression for the first moment to be:

$$\int d^3v\mathbf{v} f_0(\mathbf{v}) = nv_0 \tag{54}$$

Our result of nv_0 is expected as the integral yielded the average particle velocity times the number density.

3.3 Part C: the Second Moment

1. Begin with the expression for the first moment. We will substitute d^3v with d^3u according to Eq. 35 and $\mathbf{v}$ with $\mathbf{u}$ according to Eq. 28a.

$$\int d^3u \left(\frac{1}{2}mv^2\right) f_0(\mathbf{v}) \tag{55}$$

2. We know $\mathbf{v}$ can be written as $\mathbf{u} + \mathbf{v}_0$ (Eq. 28a).

$$\frac{1}{2}m \int d^3u \langle \mathbf{u} + \mathbf{v}_0 \rangle^2 f_0(\mathbf{v}) \tag{56}$$

3. Expand the distribution function according to Eq. 31.

$$\frac{1}{2}m \int d^3u \langle \mathbf{u} + \mathbf{v}_0 \rangle^2 \left[n\alpha^{3/2} \exp\left(-\beta u^2\right) \right] \tag{57}$$

4. Compute the vector product.

$$\langle \mathbf{u} + \mathbf{v}_0 \rangle^2 = \tag{58a}$$

$$\langle u_x + v_{0,x}, u_y + v_{0,y}, u_z + v_{0,z} \rangle^2 = \tag{58b}$$

$$\left(u_x^2 + 2u_x v_{0x} + v_{0x}^2\right) + \left(u_y^2 + 2u_y v_{0y} + v_{0y}^2\right) + \left(u_z^2 + 2u_z v_{0z} + v_{0z}^2\right) = \tag{58c}$$

$$u^2 + v_0^2 + 2\left(u_x v_{0x} + u_x v_{0x} + u_x v_{0x}\right) \tag{58d}$$

5. Substitute the vector product expression.

$$\frac{1}{2} mn\alpha^{3/2} \int d^3u \left[u^2 + v_0^2 + 2\left(u_x v_{0x} + u_x v_{0x} + u_x v_{0x}\right)\right] \left[\exp\left(-\beta u^2\right)\right] \tag{59}$$

6. Convert to spherical coordinates. Noting we are neglecting the constants for now.

$$\int_0^\pi d\phi \sin\phi \int_0^{2\pi} d\theta \int_0^\infty du u^2 \left[u^2 + v_0^2 + 2\left(u_x v_{0x} + u_x v_{0x} + u_x v_{0x}\right)\right] \exp\left(-\beta u^2\right) \tag{60}$$

7. Next, we will convert the vector product to spherical coordinates noting we are only converting the **u** components.

$$\langle \mathbf{u} + \mathbf{v}_0 \rangle^2 = \tag{61a}$$

$$u^2 + v_0^2 + 2\left(u_x v_{0x} + u_x v_{0x} + u_x v_{0x}\right) = \tag{61b}$$

$$u^2 + v_0^2 + 2\left(v_{0x} u \sin\phi\cos\theta + v_{0y} u \sin\phi \sin\theta + v_{0z} u \cos\phi\right) \tag{61c}$$

8. Split the single integral up into five separate integrals.

$$\int_0^\pi d\phi \sin\phi \int_0^{2\pi} d\theta \int_0^\infty du u^4 \exp\left(-\beta u^2\right) + \tag{62}$$

$$v_0^2 \int_0^\pi d\phi \sin\phi \int_0^{2\pi} d\theta \int_0^\infty du u^2 \exp\left(-\beta u^2\right) + \tag{63}$$

$$v_{0x} \int_0^\pi d\phi \sin^2\phi \int_0^{2\pi} d\theta \cos\theta \int_0^\infty du u^2 \exp\left(-\beta u^2\right) + \tag{64}$$

$$v_{0y} \int_0^\pi d\phi \sin^2\phi \int_0^{2\pi} d\theta \sin\theta \int_0^\infty du u^2 \exp\left(-\beta u^2\right) + \tag{65}$$

$$v_{0z} \int_0^\pi d\phi \sin\phi\cos\phi \int_0^{2\pi} d\theta \int_0^\infty du u^2 \exp\left(-\beta u^2\right) \tag{66}$$

9. As the following integrals equal zero, the final three terms will cancel out.

$$\text{From the } v_{0x} \text{ integral:} \quad \int_0^{2\pi} d\theta \cos\theta = 0 \tag{67a}$$

$$\text{From the } v_{0y} \text{ integral:} \quad \int_0^{2\pi} d\theta \sin\theta = 0 \tag{67b}$$

$$\text{From the } v_{0z} \text{ integral:} \qquad \int_0^{\pi} d\phi \sin\phi \cos\phi = \left[-\frac{1}{2}\cos^2\phi\right]_0^{\pi} = 0 \tag{67c}$$

10. Then, the expression becomes:

$$\int_0^{\pi} d\phi \sin\phi \int_0^{2\pi} d\theta \int_0^{\infty} u^4 \exp\left(-\beta u^2\right) + v_0^2 \int_0^{\pi} d\phi \sin\phi \int_0^{2\pi} d\theta \int_0^{\infty} u^2 \exp\left(-\beta u^2\right) a = \tag{68}$$

$$4\pi \left[\int_0^{\infty} du u^4 \exp\left(-\beta u^2\right) + v_0^2 \int_0^{\infty} du u^2 \exp\left(-\beta u^2\right)\right] \tag{69}$$

11. Focusing on the first integral and neglecting constants for now:

$$\int_0^{\infty} du u^4 \exp\left(-\beta u^2\right) = \int_0^{\infty} du \partial_\beta^2 \left[\exp\left(-\beta u^2\right)\right] \tag{70}$$

   (a) Because integrals and derivatives are linear operators, we can "factor" out the partial derivatives.

$$\partial_\beta^2 \int_0^{\infty} du \exp\left(-\beta u^2\right) = \tag{71a}$$

$$\partial_\beta^2 \left[\frac{1}{2}\sqrt{\frac{\pi}{\beta}}\right] = \frac{\sqrt{\pi}}{2}\left(\frac{-1}{2}\frac{-3}{2}\beta^{-5/2}\right) = \frac{3\sqrt{\pi}}{8}\beta^{-5/2} \tag{71b}$$

12. Next, evaluate the second integral and neglect constants once again:

$$\int_0^{\infty} du u^2 \exp\left(-\beta u^2\right) = \int_0^{\infty} du (-1)\partial_\beta \left[\exp\left(-\beta u^2\right)\right] \tag{72a}$$

$$-\partial_\beta \int_0^{\infty} du \exp\left(-\beta u^2\right) = \tag{72b}$$

$$\partial_\beta \left[-\frac{1}{2}\sqrt{\frac{\pi}{\beta}}\right] = -\frac{\sqrt{\pi}}{2}\left(-\frac{1}{2}\beta^{-3/2}\right) = \frac{\sqrt{\pi}}{4}\beta^{-3/2} \tag{72c}$$

13. Plugging the evaluated integrals back into the expression.

$$\frac{1}{2} mn\alpha^{3/2}\left\{4\pi\left[\frac{3\sqrt{\pi}}{8}\beta^{-5/2} + \frac{\sqrt{\pi}}{4}v_0^2\beta^{-3/2}\right]\right\} = \tag{73a}$$

$$mn\left(\frac{\pi^{3/2}\alpha^{3/2}}{\beta^{3/2}}\right)\left[\frac{3}{4}\beta^{-1} + \frac{1}{2}v_0^2\right] = \tag{73b}$$

$$n\left[\frac{1}{2}mv_0^2 + \frac{3}{4}m\beta^{-1}\right] = \tag{73c}$$

$$n\left[\frac{1}{2}mv_0^2 + \frac{3}{4}m\frac{2kT}{m}\right] = \tag{73d}$$

$$n \left[\frac{1}{2} m v_0^2 + \frac{3}{2} kT \right] \tag{73e}$$

We arrive at the expected result for the second moment of the transport equation.

$$\int d^3v \left(\frac{1}{2} m v^2 \right) f_0(\mathbf{v}) = n \left[\frac{1}{2} m v_0^2 + \frac{3}{2} kT \right] \tag{74}$$

The first term represents the average motion of the bulk and the second term represents the energy from thermal motion of the particles, which are both multiplied by the particle density.