03: Linear Algebra - Review

Matrices - overview

- Rectangular array of numbers written between square brackets
  - 2D array
  - Named as capital letters (A,B,X,Y)
- Dimension of a matrix are [Rows x Columns]
  - Start at top left
  - To bottom left
  - To bottom right
  - $R^{[r \times c]}$ means a matrix which has r rows and c columns

$$A = \begin{bmatrix} 1402 & 191 \\ 1371 & 821 \\ 949 & 1437 \\ 147 & 1448 \end{bmatrix}$$

    - Is a [4 x 2] matrix

- Matrix elements
  - $A_{(i,j)}$ = entry in i^{th} row and jth column

$$A = \begin{bmatrix} 1402 & 191 \\ 1371 & 821 \\ 949 & 1437 \\ 147 & 1448 \end{bmatrix}$$

$A_{11} = 1402$

$A_{12} = 191$

$A_{32} = 1437$

$A_{41} = 147$

- Provides a way to organize, index and access a lot of data

Vectors - overview

- Is an n by 1 matrix
  - Usually referred to as a lower case letter
  - n rows
  - 1 column
  - e.g.

$$y = \begin{bmatrix} 460 \\ 232 \\ 315 \\ 178 \end{bmatrix}$$

- Is a 4 dimensional vector
  - Refer to this as a vector R4
- Vector elements
  - $v_i = i^{th}$ element of the vector
  - Vectors can be 0-indexed (C++) or 1-indexed (MATLAB)
  - In math 1-indexed is most common
    - But in machine learning 0-index is useful
  - Normally assume using 1-index vectors, but be aware sometimes these will (explicitly) be 0 index ones

Matrix manipulation

- ***Addition***
  - Add up elements one at a time
  - Can only add matrices of the *same dimensions*
    - Creates a new matrix of the same dimensions of the ones added

$$\begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 0.5 \\ 2 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0.5 \\ 4 & 10 \\ 3 & 2 \end{bmatrix}$$

- ***Multiplication by scalar***
  - Scalar = real number
  - Multiply each element by the scalar
  - Generates a matrix of the same size as the original matrix

$$3 \times \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 6 & 15 \\ 9 & 3 \end{bmatrix}$$

- ***Division by a scalar***
  - Same as multiplying a matrix by 1/4
  - Each element is divided by the scalar
- ***Combination of operands***
  - Evaluate multiplications first

$$3 \times \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} / 3$$

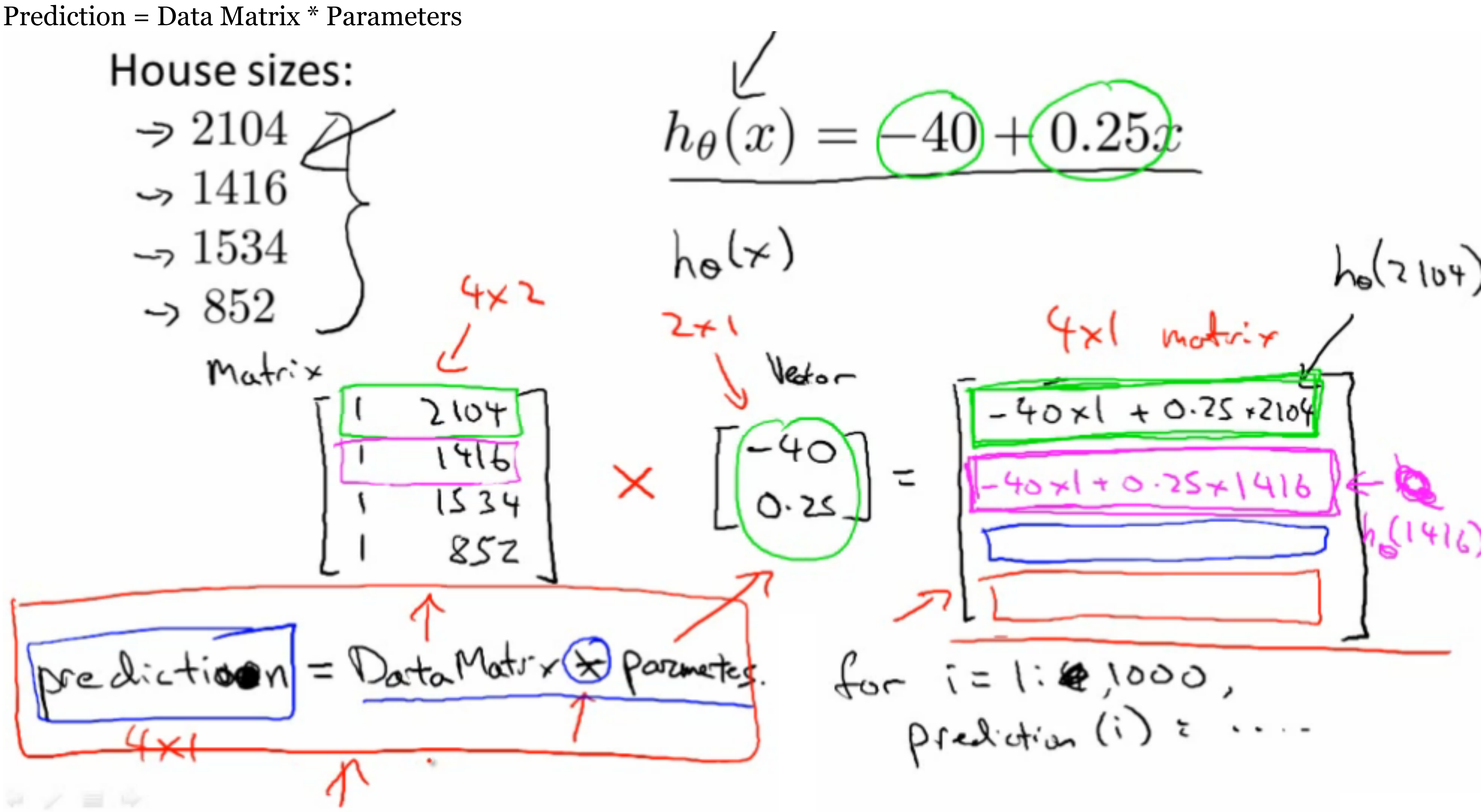
- **Matrix by vector multiplication**
  - [3 x 2] matrix * [2 x 1] vector
    - New matrix is [3 x 1]
      - More generally if [a x b] * [b x c]
        - Then new matrix is [a x c]
    - How do you do it?
      - Take the two vector numbers and multiply them with the first row of the matrix
        - Then add results together - this number is the first number in the new vector
      - The multiply second row by vector and add the results together
      - Then multiply final row by vector and add them together

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 4 & 0 \\ 2 & 1 \end{bmatrix}}_{3\times2} \underbrace{\begin{bmatrix} 1 \\ 5 \end{bmatrix}}_{2\times1} = \begin{bmatrix} 16 \\ 4 \\ 7 \end{bmatrix}$$

$$1\times1 + 3\times5 = 16$$
$$4\times1 + 0\times5 = 4$$
$$2\times1 + 1\times5 = 7$$

- Detailed explanation
  - A * x = y
    - A is m x n matrix
    - x is n x 1 matrix
    - n must match between vector and matrix
      - i.e. inner dimensions must match
    - Result is an m-dimensional vector
  - To get y_i - multiply A's i^{th} row with all the elements of vector x and add them up

- Neat trick
  - Say we have a data set with four values
  - Say we also have a hypothesis $h_\theta(x) = -40 + 0.25x$
    - Create your data as a matrix which can be multiplied by a vector
    - Have the parameters in a vector which your matrix can be multiplied by
  - Means we can do
    - Prediction = Data Matrix * Parameters



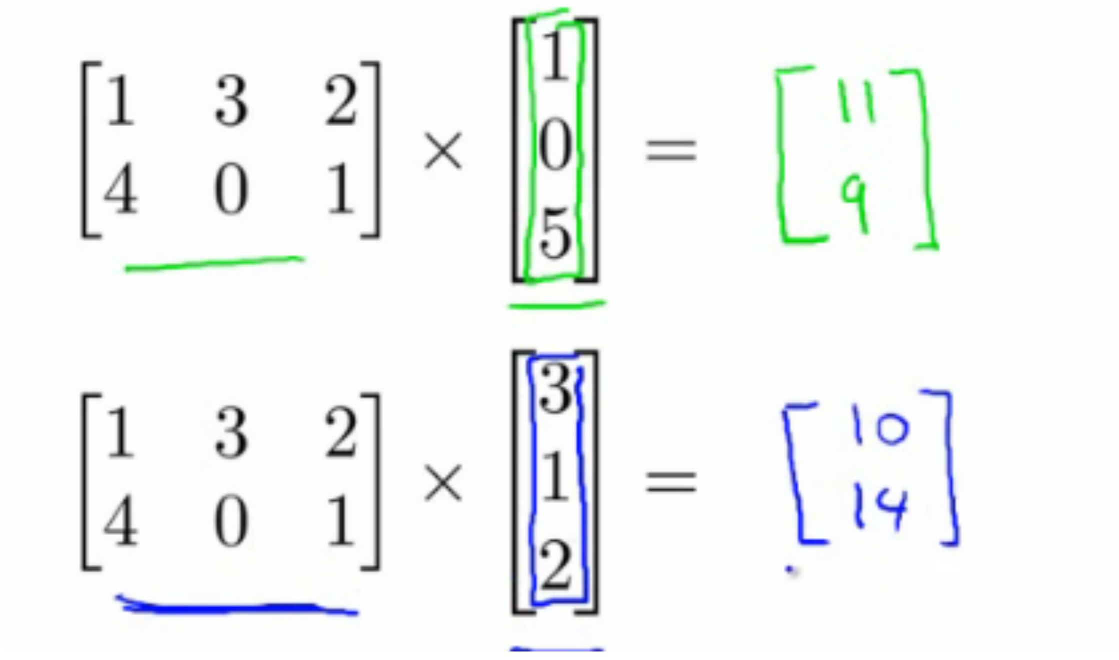
    - Here we add an extra column to the data with 1s - this means our θ_0 values can be calculated and expressed

- The diagram above shows how this works
  - This can be far more efficient computationally than lots of for loops
  - This is also easier and cleaner to code (assuming you have appropriate libraries to do matrix multiplication)

- ***Matrix-matrix multiplication***
  - General idea
    - Step through the second matrix one column at a time
    - Multiply each column vector from second matrix by the entire first matrix, each time generating a vector
    - The final product is these vectors combined (not added or summed, but literally just put together)
  - Details
    - A x B = C
      - A = [m x n]
      - B = [n x o]
      - C = [m x o]
        - With vector multiplications o = 1
    - Can only multiply matrix where columns in A match rows in B
  - Mechanism
    - Take column 1 of B, treat as a vector
    - Multiply A by that column - generates an [m x 1] vector
    - Repeat for each column in B
      - There are o columns in B, so we get o columns in C
  - Summary
    - *The i^{th} column of matrix C is obtained by multiplying A with the i^{th} column of B*
  - Start with an example
  - A x B

$$\begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 5 & 2 \end{bmatrix}$$

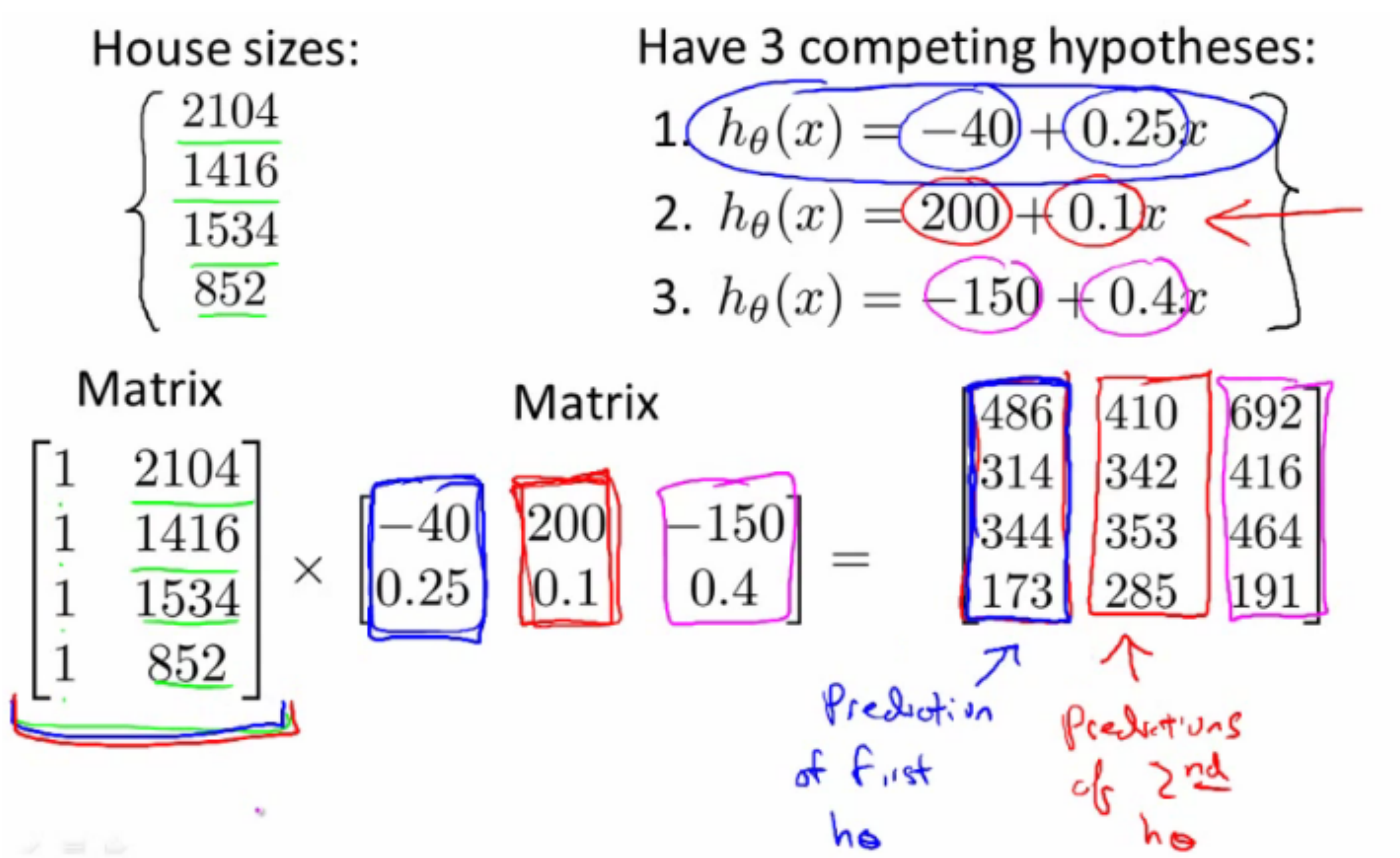
- Initially
  - Take matrix A and multiply by the first column vector from B
  - Take the matrix A and multiply by the second column vector from B



- 2 x 3 times 3 x 2 gives you a 2 x 2 matrix

Implementation/use

- House prices, but now we have three hypothesis and the same data set
- To apply all three hypothesis to all data we can do this efficiently using matrix-matrix multiplication
  - Have
    - Data matrix
    - Parameter matrix
  - Example
    - Four houses, where we want to predict the prize
    - Three competing hypotheses
    - Because our hypothesis are one variable, to make the matrices match up we make our data (houses sizes) vector into a 4x2 matrix by adding an extra column of 1s



- What does this mean
  - Can quickly apply three hypotheses at once, making 12 predictions
  - Lots of good linear algebra libraries to do this kind of thing very efficiently

Matrix multiplication properties

- Can pack a lot into one operation
  - However, should be careful of how you use those operations
  - Some interesting properties
- **Commutativity**
  - When working with raw numbers/scalars multiplication is commutative
    - 3 * 5 == 5 * 3
  - This is not true for matrix
    - A x B != B x A
    - **Matrix multiplication is not commutative**
- **Associativity**
  - 3 x 5 x 2 == 3 x 10 = 15 x 2
    - Associative property
  - **Matrix multiplications is associative**
    - A x (B x C) == (A x B) x C
- **Identity matrix**
  - 1 is the identity for any scalar
    - i.e. 1 x z = z

            - for any real number
    - In matrices we have an identity matrix called I
        - Sometimes called $I_{\{n \times n\}}$

$$\underset{2 \times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \qquad \underset{3 \times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \qquad \underset{4 \times 4}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

- See some identity matrices above
    - Different identity matrix for each set of dimensions
    - Has
        - 1s along the diagonals
        - 0s everywhere else
    - 1x1 matrix is just "1"

- Has the property that any matrix A which can be multiplied by an identity matrix gives you matrix A back
    - So if A is [m x n] then
        - A * I
            - I = n x n
        - I * A
            - I = m x m
        - (To make inside dimensions match to allow multiplication)
- Identity matrix dimensions are implicit
- Remember that matrices are not commutative AB != BA
    - Except when B is the identity matrix
    - Then AB == BA

Inverse and transpose operations

- **Matrix inverse**
    - How does the concept of "the inverse" relate to real numbers?
        - 1 = "identity element" (as mentioned above)
            - Each number has an inverse
                - This is the number you multiply a number by to get the identify element
                - i.e. if you have x, x * 1/x = 1
        - e.g. given the number 3
            - $3 * 3^{-1} = 1$ (the identity number/matrix)
        - In the space of real numbers **not everything has an inverse**
            - e.g. 0 does not have an inverse
    - What is the inverse of a matrix
        - If A is an m x m matrix, then A inverse = A^{-1}
        - So $A*A^{-1} = I$
        - Only matrices which are m x m have inverses
            - Square matrices only!
    - Example
        - 2 x 2 matrix

$$\underbrace{\begin{bmatrix} 3 & 4 \\ 2 & 16 \end{bmatrix}}_{\substack{2\times2 \\ A}} \underbrace{\begin{bmatrix} 0.4 & -0.1 \\ -0.05 & 0.075 \end{bmatrix}}_{A^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A^{-1}A} = I_{2\times2}$$

        - How did you find the inverse
            - Turns out that you can sometimes do it by hand, although this is very hard
            - Numerical software for computing a matrices inverse
                - Lots of open source libraries
    - If A is all zeros then there is no inverse matrix
        - Some others don't, intuition should be matrices that don't have an inverse are a singular matrix or a degenerate matrix (i.e. when it's too close to 0)
        - So if all the values of a matrix reach zero, this can be described as reaching singularity

- **Matrix transpose**
    - Have matrix A (which is [n x m]) how do you change it to become [m x n] while keeping the same values
        - i.e. swap rows and columns!
    - How you do it;
        - Take first row of A - becomes 1st column of A^T
        - Second row of A - becomes 2nd column...
    - A is an m x n matrix
        - B is a transpose of A
        - Then B is an n x m matrix

- $A_{(i,j)} = B_{(j,i)}$

$$\underline{A} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 5 & 9 \end{bmatrix} \quad \underline{A^T} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 0 & 9 \end{bmatrix}$$