NGS modes in laser tomography AO systems

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Abstract

This note reviews the NGS modes modelling for laser tomography AO systems. Two models are presented: *i)* tilt-tomography using a combination of tilt and high-altitude quadratic modes that produce pure tilt through cone projected ray-tracing through the wave-front profiles and *ii)* a spatio-angular MMSE tilt estimation anywhere in the field that is more general. I also discuss the (straightforward) generalisation to dynamic controllers using quasi-Markovian time-progression models from [Correia et al., 2015].

Contents

1 NGS modes in laser-tomography AO: LTAO and MCAO

The NGS modes in laser-tomography AO are defined as the null modes of the high-order LGS measurement space, i.e., modes that produce average slope $\neq 0$, but that due to the LGS tilt indetermination, cannot be measured by the latter. In other words, the null space can be thought as the combination of all the modes that have non-null projection onto the angle-of-arrival (= not just Zernike tip and tilt but also higher order Zernike modes).

1.1 Tilt tomography for 2-DM LGS-assisted MCAO

The classical approach to derive tilt anisoplanatism is to consider that field-dependent tilt is a linear combination of aperture-plane tilt and a combination of pupil and high-altitude quadratic modes of opposite signs whose resultant through ray-racing produces pure tilt (the quadratic terms vanish) [Flicker and Rigaut, 2002, Ellerbroek and Rigaut, 2001]. A similar approach can still be used in laser tomography provided the quadratic terms are scaled by the cone shrinking factor – this is the general case shown next. At least 3 independent measurements of tilt are required in the field (i.e. 6 measurements) from which 2 tip/tilt and 3 quadratic modes (differential focus and astigmatisms) are estimated. In real systems, since the LGS cannot measure focus neither due to Na-range fluctuations, one of the NGS WFSensors is actually a 2x2 WFS providing focus and astigmatism measurements. We thus end up with a total of 12 measurements to estimate 6 modes – which is somehow sub-optimal since we're not making use of the full information contained in the measurements – as was done more recently in [Gilles et al., 2011].

In the following we assume a split-tomography framework [Gilles and Ellerbroek, 2008] and follow on the footsteps and notation of [Correia et al., 2013].

For natural tomography AO (with tilt-removed high-order WFS measurements) is can be computed from what follows with ease.

Following the above considerations the resulting aperture-plane wave-front (WF) is conveniently expanded onto a truncated orthonormal Zernike polynomial's basis defined over 2 layers. For the NGS modes model, only modes $Z_{2\cdots6}$ are used that correspond to the TT and the quadratic modes of Eq. (1).

The total aperture-plane wave-front error induced by the TT/TA modes at time instant t, in direction $\boldsymbol{\theta}$ is given by

$$W(\boldsymbol{\rho},\boldsymbol{\theta},t)=\sum_{i=2}^{6}\alpha_i(t)Z_i\left(\frac{\boldsymbol{\rho}}{R_0}\right)+\sum_{i=2}^{6}\beta_i(t)Z_i\left(\frac{\boldsymbol{\rho}+\boldsymbol{\theta}h_{\text{DM}}}{R_h}\right) \tag{1}$$

where $\boldsymbol{\rho}$ is a two-dimensional space coordinate vector and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are vectors containing the Zernike coefficients (following the ordering of [Noll, 1976]) defined over the lower and upper DM-conjugate planes of radius $R_0 = D_0/2$ and $R_h = D_h/2$ Note $i \in \{2, \cdots, 6\}$ which effectively represents 5 modes; $i = 1$ refers to piston and is disregarded as this mode has no impact on AO performance. Global TT have the corresponding Zernike modes applied to the ground DM only, *i.e.* $\beta_2 = \beta_3 = 0$.

The coefficients β_i can be worked out such that $W(\boldsymbol{\rho},\boldsymbol{\theta},t)$ only produces TT in the LGS WFSs (but not on the NGS WFSs). This constraint provides $\beta_i = -r_l^{-2}\alpha_i$, with r_l given by the ratio of the cone-intersected pupil and meta-pupil in the DM conjugate range is

$$r_l = r_n\underbrace{\left(1-\frac{h_{\text{DM}}}{h_{\text{Na}}}\right)}_{r_c}, \tag{2}$$

where $r_c \triangleq 1 - h_{\text{DM}}/h_{\text{Na}}$ is the shrinking factor of the cone-intersected meta-pupil diameter with respect to the cylinder-intersected meta-pupil diameter at h_{DM} km, translating the cone effect for a DM conjugated to range h_{DM} and an LGS at range h_{Na} and

$$r_n \triangleq \frac{D_0}{D_0 + FoV \times h_{\text{DM}} \times 1000}, \tag{3}$$

normalizes the upper modal coefficient to the particular choice of underlying meta-pupils over which the modes are defined, with FoV the field-of-view in radians, h_{DM} the conjugate altitude of the upper DM in km.

Setting $\beta_i = -r_l^{-2}\alpha_i$ in Eq. (1) and solving for the aperture-plane field-dependent TT

$$\mathsf{TT}(\boldsymbol{\rho},\boldsymbol{\theta},t)=\sum_{i=2}^{3}\xi_i(\theta,t)Z_i\left(\frac{\boldsymbol{\rho}}{R_0}\right) \tag{4}$$

with the field-dependent TT coefficients $\boldsymbol{\xi}(\boldsymbol{\theta},t)$ [Flicker et al., 2003]

$$\boldsymbol{\xi}(\boldsymbol{\theta},t)=\begin{pmatrix} 1 & 0 & -r_c^{-2}\frac{2\sqrt{3}h\theta_x}{R_0} & -r_c^{-2}\frac{\sqrt{6}h\theta_y}{R_0} & -r_c^{-2}\frac{\sqrt{6}h\theta_x}{R_0} & 0 \\ 0 & 1 & -r_c^{-2}\frac{2\sqrt{3}h\theta_y}{R_0} & -r_c^{-2}\frac{\sqrt{6}h\theta_x}{R_0} & r_c^{-2}\frac{\sqrt{6}h\theta_y}{R_0} & 0 \end{pmatrix}\boldsymbol{\chi}(t) \tag{5}$$

In Eq. (5) $\boldsymbol{\chi}(t) \in \Re^{6\times 1}$ is a vector with the coefficient of the NGS modes (numerically equal to $\boldsymbol{\alpha}(t)$ in Eq. (1) but not to be confounded with it) plus a sixth coefficient to model pure focus induced by N_a-layer that does not produce TT.

Expressing the Zernike polynomials using cartesian coordinates one gets

$$x_k^{\mathsf{tip}} = 2(x_0, 0); \tag{6a}$$

$$x_k^{\mathsf{tilt}} = 2(y_0, 0); \tag{6b}$$

$$x_k^{\Delta F} = \sqrt{3}([x_0^2 + y_0^2]; -[x_c^2 + y_c^2]/r_l^2); \tag{6c}$$

$$x_k^{\Delta A_0} = \sqrt{6}([x_0^2 - y_0^2]; -[x_c^2 - y_c^2]/r_l^2); \tag{6d}$$

$$x_k^{\Delta A_{45}} = \sqrt{6}([x_0 y_0]; -[x_c y_c]/r_l^2); \tag{6e}$$

where x_0, y_0 and x_c, y_c are the actuator coordinates on the ground and upper DMs, normalised by the meta-pupil radius at 0 and h_{DM}.

The coefficients $\boldsymbol{\chi}(t)$ can be found from least-squares fitting the tilt and quadratic modes to the atmosphere (in passing, this is by no means different from the DM fitting step to an estimated tomographic phase – in the latter the DM influence functions are used instead of the quadratic modes defined previously).

$$\boldsymbol{\chi} = \arg\min_{\boldsymbol{\chi}} \left\langle |\mathbf{H}_\alpha \boldsymbol{\varphi} - \mathbf{H}_\alpha^\chi \mathbf{H}_{\mathsf{PS2Q}} \boldsymbol{\chi}|^2 \right\rangle_{\mathrm{FoV}} \tag{7}$$

where

$$\boldsymbol{\psi} = \mathbf{H}_\alpha \boldsymbol{\varphi} \tag{8}$$

is the pupil-plane integrated wave-front profile in direction $\boldsymbol{\alpha}$ and

$$\mathsf{TT}_{\boldsymbol{\alpha}} = \mathbf{H}_\alpha^\chi \mathbf{H}_{\mathsf{PS2Q}} \boldsymbol{\chi} \tag{9}$$

is the tip/tilt produced by the TT/TTA modes on those directions.

In addition,

$$\mathbf{H}_{\mathsf{PS2Q}} \triangleq \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/r_l^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/r_l^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/r_l^2 & 0 \end{pmatrix}, \tag{10}$$

providing the 2-layer Zernike modal decomposition of the WF from the NGS modal coefficients.

By setting $M \triangleq \mathbf{H}_\alpha^\chi \mathbf{H}_{\mathsf{PS2Q}}$ one then gets

$$\boldsymbol{\chi} = \left\langle M^\intercal M \right\rangle_{\mathrm{FoV}}^{-1} \left\langle M^\intercal \mathbf{H}_{\boldsymbol{\alpha}} \right\rangle_{\mathrm{FoV}} \tag{11}$$

In practice this is done over a few directions in the FoV with weights applied on them (e.g. Simpson weights).

Although the NGS modes produce only field-dependent TT not seen by the LGS WFS, for a NGS looking upwards through a cylinder and not a cone, focus and astigmatisms are indeed probed.

1.2 Tilt correction in NTAO and LTAO systems

In the NTAO[1]/LTAO case, the tilt compensation is greatly simplified since an explicit tomographic estimate of tilt is not mandatory. We thus fall in the spatio-angular case whereby the computation of field-dependent covariances is done off-line and applied directly on-line.

We now develop the *minimum mean square error* (MMSE) solution with a simplified measurement model involving the pupil-plane turbulence only

$$\mathbf{s}_{\boldsymbol{\alpha}}(t) = \mathbf{G}\boldsymbol{\psi}_{\boldsymbol{\alpha}}(t) + \boldsymbol{\eta}(t) \tag{12}$$

Assuming $\mathbf{s}$ and $\boldsymbol{\psi}$ are zero-mean and jointly Gaussian, direct application of the MMSE solution to estimate the aperture-plane phase in the N_β-science directions of interest yields [Anderson and Moore, 1995]

$$\mathcal{E}\{\boldsymbol{\psi}_{\boldsymbol{\beta}}|\mathbf{s}_{\boldsymbol{\alpha}}\} \triangleq \boldsymbol{\Sigma}_{(\boldsymbol{\psi}_{\boldsymbol{\beta}},\mathbf{s}_{\boldsymbol{\alpha}})}\boldsymbol{\Sigma}_{\mathbf{s}_{\boldsymbol{\alpha}}}^{-1}\mathbf{s}_{\boldsymbol{\alpha}} = \widehat{\boldsymbol{\psi}}_{\boldsymbol{\beta}} \tag{13}$$

where $\mathcal{E}\{X|Y\}$ stands for mathematical expectation of X conditioned to Y. Since in general $\boldsymbol{\beta} \neq \boldsymbol{\alpha}$, Eq. (13) follows from $\mathcal{E}\{\boldsymbol{\psi}_{\boldsymbol{\beta}}|\mathbf{s}_{\boldsymbol{\alpha}}\} = \mathcal{E}\{\boldsymbol{\psi}_{\boldsymbol{\beta}}|\mathcal{E}\{\boldsymbol{\psi}_{\boldsymbol{\alpha}}|\mathbf{s}_{\boldsymbol{\alpha}}\}\}$. Matrices $\boldsymbol{\Sigma}_{(\boldsymbol{\psi}_{\boldsymbol{\beta}},\mathbf{s}_{\boldsymbol{\alpha}})}$ and $\boldsymbol{\Sigma}_{\mathbf{s}_{\boldsymbol{\alpha}}}$ are spatio-angular covariance matrices that relate the tilt on direction θ to that on direction β. These matrices are computed using formulae in [Whiteley et al., 1998b] from the simulator OOMAO [Conan and Correia, 2014].

Under the spatio-angular approach, it is also possible to compute $\boldsymbol{\psi}(\boldsymbol{\rho}, \boldsymbol{\beta}, t + \Delta)$, i.e., temporally predict the TT ahead in time by adjusting the angles over which the correlations are computed [Correia et al., 2015].

Given that the conditioning relates only to the very last available measurement (as opposed to present and previous measurements), these

[1]Assume tilt-removed measurements.

reconstructors are labelled as *static*. Developing terms in Eq. (13) using Eq. (12) the reconstructor becomes

$$\widehat{\boldsymbol{\psi}}_{\boldsymbol{\beta}} \triangleq \left\langle \boldsymbol{\psi}_{\boldsymbol{\beta}} \boldsymbol{\psi}_{\boldsymbol{\alpha}}^{\mathsf{T}} \right\rangle \mathbf{G}^{\mathsf{T}} \left(\mathbf{G} \left\langle \boldsymbol{\psi}_{\boldsymbol{\alpha}} \boldsymbol{\psi}_{\boldsymbol{\alpha}}^{\mathsf{T}} \right\rangle \mathbf{G}^{\mathsf{T}} + \left\langle \boldsymbol{\eta}\boldsymbol{\eta}^{\mathsf{T}} \right\rangle \right)^{-1} \mathbf{s}_{\boldsymbol{\alpha}} \tag{14}$$

This MMSE reconstructor is dubbed *spatio-angular* (SA) reconstructor on account of the nature of the covariance matrices involved in its definition. It can be seen as a generalization of the work of Whitely *et al* [Whiteley et al., 1998b] in seeking the optimal anisoplanatic reconstuctor in classical AO to the tomographic, multiple sensor case. It has several convenient features: is much faster to compute off-line than the *explicit tomography* reconstructor for large modal sets and circumvents the truncated expansion on a modal basis.

When phase is expanded onto the Zernike polynomials the spatio-angular cross-covariance functions can be analytically computed for the infinite outer-scale case of turbulence [Valley and Wandzura, 1979, Chassat, 1989]

$$\langle \phi_i(0)\phi_j(\xi)\rangle = 3.895 \left(\frac{D}{r_0}\right)^{\frac{5}{3}} \frac{\int_0^{h_{\max}} C_n^2(h) I_{ij}\left(\frac{\xi h}{R}\right) \mathrm{d}h}{\int_0^{\infty} C_n^2(h)\mathrm{d}h} \tag{15}$$

with $D = 2R$ the telescope diameter, r_0 the Fried parameter, h the altitude ξ the angle between the pupils over which the Zernike polynomials are defined, C_n^2 the atmospheric vertical profile and $I_{ij}(x)$ a term involving 1-D numerical integration. Equation (15) has been extended for the finite outer-scale case in [Takato and Yamaguchi, 1995] and later extensively used and generalized in [Whiteley et al., 1998a, Whiteley et al., 1998b]. The layered spatial covariance matrix $\left\langle \boldsymbol{\varphi}\boldsymbol{\varphi}^{\mathsf{T}} \right\rangle$ is block-diagonal (layers are independent) and can be found in [Noll, 1976] for the infinite outer scale case and in [Winker, 1991] for the finite case.

1.3 Measurement model with time-averaged variables

Assume the following measurement model

$$\begin{aligned}
\boldsymbol{s}_k &= \int_{(k-1)T_s}^{kT_s} \mathbf{G}\left(\boldsymbol{\psi}(\tau)^{\mathsf{tur}} - \boldsymbol{\psi}^{\,\mathsf{cor}}(\tau)\right) \mathrm{d}\tau + \boldsymbol{\eta}_k && \text{(16a)}\\
&= \mathbf{G}\left(\left\langle \boldsymbol{\psi}^{\mathsf{tur}}\right\rangle_k - \left\langle \boldsymbol{\psi}^{\mathsf{cor}}\right\rangle_k\right) + \boldsymbol{\eta}_k && \text{(16b)}\\
&= \mathbf{G}\left\langle \boldsymbol{\psi}^{\mathsf{res}}\right\rangle_k + \boldsymbol{\eta}_k && \text{(16c)}
\end{aligned}$$

where $\boldsymbol{s}_k \in \Re^{12\times 1}$ are the T_s-averaged slopes over each OIWFS sub-aperture and over the integration time T_s, $\mathbf{G} \in \mathbb{R}^{12\times 9}$ is the wave-front-to-measurements matrix, 'dm' and 'res' stand respectively for correction and residual phase; $\boldsymbol{\eta}_k$ is a zero-mean Gaussian-distributed spectrally white noise vector with known covariance matrix Σ_η – $\boldsymbol{\eta} \sim \mathcal{N}(0, \Sigma_\eta)$.

Symbol $\langle\cdot\rangle$ represents time average, *i.e.*

$$\langle\boldsymbol{\psi}\rangle_k \triangleq \int_{(k-1)T_s}^{kT_s} \boldsymbol{\psi}(\tau)\mathrm{d}\tau \tag{17}$$

The modal matrix $\mathbf{G}$ translates modal coefficients of TT, TT and TTFA modes into average slopes over the illuminated sub-region of each sub-aperture

$$\begin{aligned}\mathbf{G} &\triangleq \begin{pmatrix} \mathbf{G}_{\mathsf{TT1}} \\ \mathbf{G}_{\mathsf{TT2}} \\ \mathbf{G}_{\mathsf{TTFA}} \end{pmatrix} \\ &= \begin{pmatrix} \gamma_\mathsf{T} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_\mathsf{T} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_\mathsf{T} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_\mathsf{T} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_\mathsf{T} & 0 & \gamma_\mathsf{F} & \gamma_\mathsf{A} & \gamma_\mathsf{A} \\ 0 & 0 & 0 & 0 & 0 & \gamma_\mathsf{T} & \gamma_\mathsf{F} & \gamma_\mathsf{A} & -\gamma_\mathsf{A} \\ 0 & 0 & 0 & 0 & \gamma_\mathsf{T} & 0 & -\gamma_\mathsf{F} & \gamma_\mathsf{A} & -\gamma_\mathsf{A} \\ 0 & 0 & 0 & 0 & 0 & \gamma_\mathsf{T} & \gamma_\mathsf{F} & -\gamma_\mathsf{A} & -\gamma_\mathsf{A} \\ 0 & 0 & 0 & 0 & \gamma_\mathsf{T} & 0 & -\gamma_\mathsf{F} & -\gamma_\mathsf{A} & -\gamma_\mathsf{A} \\ 0 & 0 & 0 & 0 & 0 & \gamma_\mathsf{T} & -\gamma_\mathsf{F} & -\gamma_\mathsf{A} & \gamma_\mathsf{A} \\ 0 & 0 & 0 & 0 & \gamma_\mathsf{T} & 0 & \gamma_\mathsf{F} & -\gamma_\mathsf{A} & \gamma_\mathsf{A} \\ 0 & 0 & 0 & 0 & 0 & \gamma_\mathsf{T} & -\gamma_\mathsf{F} & \gamma_\mathsf{A} & \gamma_\mathsf{A} \end{pmatrix},\end{aligned} \tag{18}$$

For the TTF OIWFS, the average slope produced by the TT on the quarter of the aperture $S_{\frac{1}{4}} = \frac{\pi}{4}$ (aperture units are normalised by the aperture radius) is given by

$$\begin{aligned}\gamma_\mathsf{T} &= \frac{1}{S_{\frac{1}{4}}} \int_0^1 \int_0^{\sqrt{1-y^2}} \frac{\partial}{\partial x} Z_{2,3}(x,y)\partial x \partial y \\ &= 2\end{aligned} \tag{19}$$

The average slope produced by the focus mode is given by

$$\begin{aligned}\gamma_\mathsf{F} &= \frac{1}{S_{\frac{1}{4}}} \int_0^1 \int_0^{\sqrt{1-y^2}} \frac{\partial}{\partial x} Z_4(x,y)\partial x \partial y \\ &= \frac{16\sqrt{3}}{3\pi}\end{aligned} \tag{20}$$

whereas for the astigmatisms

$$\gamma_{\mathsf{A}} = \frac{1}{S_{\frac{1}{4}}} \int_0^1 \int_0^{\sqrt{1-y^2}} \frac{\partial}{\partial x} Z_{5\cdots 6}(x,y) \partial x \partial y \qquad (21)$$
$$= \frac{8\sqrt{6}}{3\pi}$$

The signal $\pm$ attached to γ_{F} and γ_{A} in Eq. (18) is a function of the exact quadrant where each sub-aperture is located.

1.4 Noise model

1.4.1 Diffraction-limited case

In the following, the noise model detailed in [Clare et al., 2006] is used. It is assumed that spots are diffraction limited. Therefore, these equations apply for a Nyquist-sampled spot, *i.e.* with 2×2 pixels, by other words a quadrant detector.

The noise added to each sub-aperture measurement is given by (in angle rms units)

$$\sigma_\eta = \frac{\theta_b}{\mathsf{SNR}}, \qquad (22)$$

where θ_b is the effective spot size of the sub-aperture, and SNR is the signal-to-noise ratio of a single sub-aperture. For a quadrant detector, the SNR is given by

$$\mathsf{SNR} = \frac{N_p}{\sqrt{N_p + 4N_b + 4\sigma_e^2}}, \qquad (23)$$

where N_p is the number of photo-detection events per sub-aperture, N_b is the number of background photo-detection events per sub-aperture, and σ_e is the rms detector read noise per pixel.

In the IR (H band), the NGS images are assumed to contain a diffraction-limited core, for which case the effective spot size is given by [Hardy Eq. 5.13]

$$\theta_b = \frac{3\pi\lambda\sqrt{N_{sa}}}{16D_0}, \qquad (24)$$

where N_{sa} is the total number of sub-apertures for the NGS WFS. The 2×2 NGS WFS is therefore noisier than any of the two single sub-aperture NGS WFS. Note θ_b is twice that of the latter, since N_{sa} is 4 instead of 1 and that the number of photo-detections per sub-aperture is also cut by a factor of 4 – providing a $\sim$ 2 times smaller SNR. One thus ends up with a factor $\sim$

4 noisier measurement. The same reasoning applied to a general SH WFS leads to standard estimations of the full-aperture gain.

To convert to mas rms, multiply $\sigma_\eta[mas] = 180/\pi \times 3600 \times 1000 \times \sigma_\eta[rad]$

1.4.2 Seeing-limited case

For the seeing-limited case, use instead for the photon noise

$$\sigma_\eta^2 = \frac{\pi^2}{2ln(2)}\frac{1}{n_{ph}}\left(\frac{N_T}{N_D}\right)^2 [rad^2], \tag{25}$$

where N_T and N_D are the FWHM of image and sub-aperture respectively. With $N_T = \lambda/r_0$ and $N_D = \lambda/D$ one finds

$$\sigma_\eta^2 = \frac{\pi^2}{2ln(2)}\frac{1}{n_{ph}}\left(\frac{D}{r_0}\right)^2 [rad^2], \tag{26}$$

For the read-out noise we use

$$\sigma_\eta^2 = \frac{\pi^2}{3}\frac{\sigma_e^2}{n_{ph}^2}\left(\frac{N_S^2}{N_D}\right)^2 [rad^2], \tag{27}$$

where σ_e is the rms number of photoelectron event per pixel and per frame, N_S^2 is the total number of pixels in the CoG computation.

1.5 Performance assessment

For the MMSE case, the standard performance assessment formulae apply.

Starting from

$$\sigma^2(\boldsymbol{\beta}) = \left\langle \|\boldsymbol{\psi}(\boldsymbol{\beta}) - \widehat{\boldsymbol{\psi}}(\boldsymbol{\beta})\|^2 \right\rangle \tag{28}$$

with $\widehat{\boldsymbol{\psi}}(\boldsymbol{\beta}) = \mathbf{Rs}(\boldsymbol{\alpha})$ we get

$$\begin{aligned} \sigma^2(\boldsymbol{\beta}) &= \text{trace}\{\boldsymbol{\Sigma}_{\boldsymbol{\beta}}\} + \text{trace}\{\mathbf{RG}\boldsymbol{\Sigma}_{\boldsymbol{\alpha}}\mathbf{G}^\mathsf{T}\mathbf{R}^\mathsf{T}\} && (29)\\ &- \text{trace}\{\mathbf{RG}\boldsymbol{\Sigma}_{\boldsymbol{\beta},\boldsymbol{\alpha}}\} - \text{trace}\{\boldsymbol{\Sigma}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\mathbf{G}^\mathsf{T}\mathbf{R}^\mathsf{T}\} && (30)\\ &+ \text{trace}\{\mathbf{R}\boldsymbol{\Sigma}_{\boldsymbol{\eta}}\mathbf{R}^\mathsf{T}\} && (31) \end{aligned}$$

We can as well account for temporal errors by estimating instead the following

$$\sigma^2(\boldsymbol{\beta}, T_s) = \left\langle \|\boldsymbol{\psi}(\boldsymbol{\beta}, t+T_s) - \boldsymbol{\psi}(\boldsymbol{\beta}, t)\|^2 \right\rangle \tag{32}$$

leading to similar expressions as before but where a lag equal to T_s is considered when computing the integrals with a displacement between the meta-pupils that increases by vT_s with $\mathbf{v}$ the wind velocity vector ($v = |\mathbf{v}|$).

A more sound way to compute the residuals is by considering still Eq. (34) when the reconstructor has built-in prediction

$$\mathbf{R} = \boldsymbol{\Sigma}_{(\boldsymbol{\psi}_{\boldsymbol{\beta},k+1},\mathbf{s}_{\boldsymbol{\alpha},k})}\boldsymbol{\Sigma}^{-1}_{\mathbf{s}_{\boldsymbol{\alpha},k}} \tag{33}$$

and computing

$$\sigma^2(\boldsymbol{\beta}) = \text{trace}\{\boldsymbol{\Sigma}_{\boldsymbol{\beta}}\} + \text{trace}\{\mathbf{R}\mathbf{G}\boldsymbol{\Sigma}_{\boldsymbol{\alpha}}\mathbf{G}^\intercal\mathbf{R}^\intercal\} \tag{34}$$

$$- \text{trace}\{\mathbf{R}\mathbf{G}\boldsymbol{\Sigma}_{(\boldsymbol{\beta},k+1),(\boldsymbol{\alpha},k)}\} - \text{trace}\{\boldsymbol{\Sigma}_{(\boldsymbol{\alpha},k),(\boldsymbol{\beta},k+1)}\mathbf{G}^\intercal\mathbf{R}^\intercal\} \tag{35}$$

$$+ \text{trace}\{\mathbf{R}\boldsymbol{\Sigma}_{\boldsymbol{\eta}}\mathbf{R}^\intercal\} \tag{36}$$

where I omit the temporal dependence whenever possible due to stationarity.

2 Future work: Dynamic controller of NGS modes

We'll use the approach laid out in [Correia et al., 2015] in which the states are directly the pupil-integrated TT modes in the GS directions coupled to a state-transition model using a quasi-Markovian time-progression model. We can thus include information from a large number of layers but never explicitly estimate those layered modal contributions to the final NGS modes in the pupil in the GS directions.

The angular cross-covariance matrices give rise to order-1 Markovian models with non-null temporal cross-spectra (iff more that TT are considered). A comparison to decoupled high-order models (AR for instance) for the NGS-only modes must be conducted. A temporal TF analysis is most welcome – if this problem is tractable.

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