High-Precision Semantics Extraction for Mathematics

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- MSc Student, Computer Science
- FAU (**F**riedrich-**A**lexander-**U**niversität Erlangen-Nürnberg) (in Southern Germany)
- KWARC research group
  - Led by Michael Kohlhase
  - Knowledge representation and reasoning techniques
  - Focus on mathematical content

- A **S***emantic,* **M***ultilingual* **Glo***ssary of* **M***athematics*
- Definitions of mathematical terms
- Semantic information about dependencies

1714 words defined, the language coverage is:

English	94.0%
German	69.8%
Chinese (simplified)	8.5%
Romanian	3.5%
...	
Afrikaans	0.0%

. . . let's use GF!

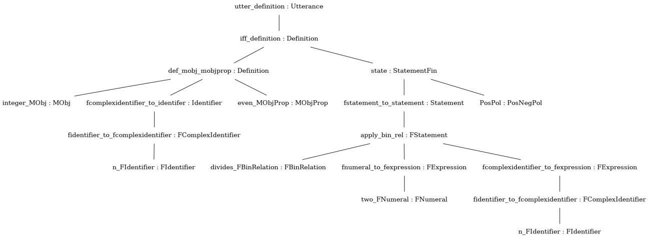
Example adapted from [GIJ+16]:

"A non-empty graph G is said to be **connected***, if any two of its nodes are linked by a path in G."*

"Ein nicht-leerer Graph G heißt **zusammenhängend***, wenn je zwei seiner Knoten durch einen Weg in G verbunden sind."*

"An integer n is called even iff $2|n$*."*

$\Downarrow$ **parse**

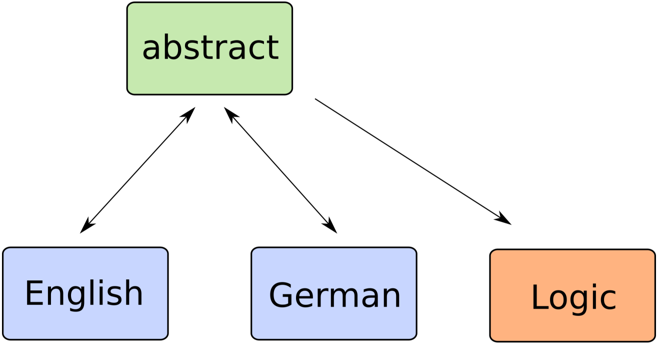


$\Downarrow$ **linearize**

"Eine ganze Zahl n heißt gerade genau dann, wenn $2|n$*."*

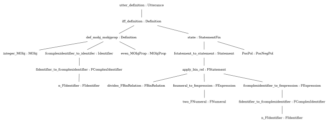
- Let's try to formalize sentences
- Example: $\forall n.\mathbf{int}(n) \Rightarrow (\mathbf{even}(n) \Leftrightarrow \mathbf{divides}(2, n))$
- I will present two different approaches

The formal representation is just another language:



"An integer n is called even iff $2|n$*."*

$\Downarrow$ **parse**



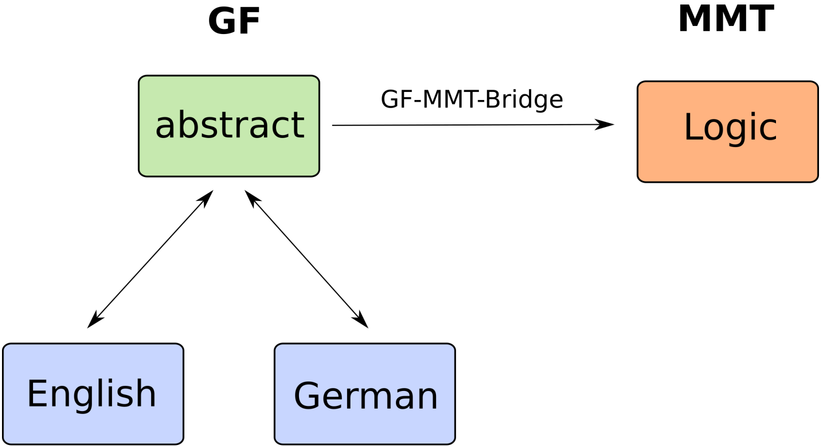
$\Downarrow$ **linearize**

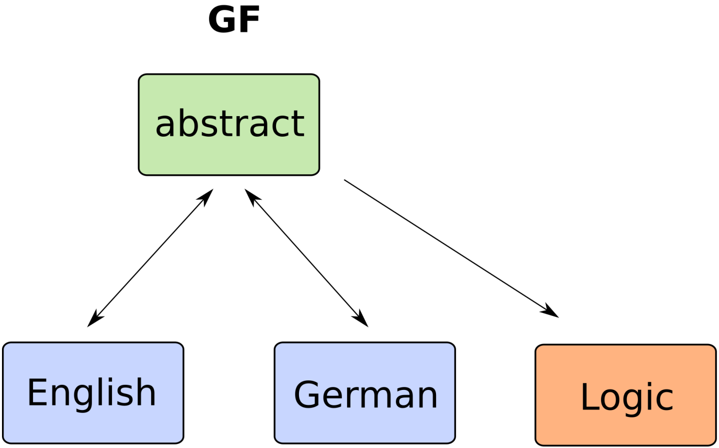
$$(\forall n.((\lambda x.\mathbf{int}(x))n) \Rightarrow ((\lambda x.\mathbf{even}(x))n \Leftrightarrow \mathbf{divides}(2, n)))$$

$\Downarrow$ external simplifier

$$\forall n.\mathbf{int}(n) \Rightarrow (\mathbf{even}(n) \Leftrightarrow \mathbf{divides}(2, n))$$

Use an external system (MMT) for the logic:




GF
abstract
English
German
Logic

There are formulas in the text:

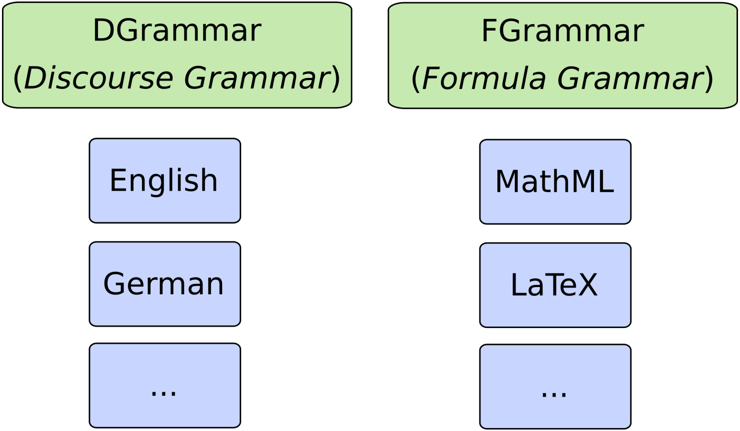
"A sequence $x = \{x_n\}_{n=1}^{\infty} \in l^{\infty}(V)$ *is called quasi-almost convergent to* $v \in V$ *if* $\forall L \in \Pi$, $L(x - \widetilde{v}) = 0$.*"* [5]

There are formulas in the text:

"A sequence $x = \{x_n\}_{n=1}^{\infty} \in l^{\infty}(V)$ *is called quasi-almost convergent to* $v \in V$ *if* $\forall L \in \Pi$, $L(x - \widetilde{v}) = 0$*."* [5]

There is text in the formulas:

"$H(P) = \{\alpha \in \mathbb{N}_0 \mid \textit{there exists a rational function } f \textit{ on } C \textit{ with } (f)_\infty = \alpha P\}$" [2]



Depends on language

Depends on representation

Example:

"$x^2 + 1$ is greater than or equal to $\sqrt{x^2 + 1}$."

There is No Standard Formula Representation

Example:

"*$x^2 + 1$ is greater than or equal to $\sqrt{x^2 + 1}$.*"

First Idea:

```
x^2 + 1 is greater than or equal to SQRT(x^2 + 1).
```

Example:

"$x^2 + 1$ *is greater than or equal to* $\sqrt{x^2 + 1}$*."*

First Idea:

```
x^2 + 1 is greater than or equal to SQRT(x^2 + 1).
```

LaTeX:

```
$x^2 + 1$ is greater than or equal to $\sqrt{x^2 + 1}$.
```

Formula: $x^2 + 1$

Presentation MathML

```
<math>
    <mrow>
        <msup>
            <mi>x</mi>
            <mn>2</mn>
        </msup>
        <mo>+</mo>
        <mn>1</mn>
    </mrow>
</math>
```

Content MathML

```
<math>
  <apply>
    <plus />
    <apply>
      <power />
      <ci>x</ci>
      <cn>2</cn>
    </apply>
    <cn>1</cn>
  </apply>
</math>
```

Formula: $x^2 + 1$

Presentation MathML

```
<math>
    <mrow>
        <msup>
            <mi>x</mi>
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        </msup>
        <mo>+</mo>
        <mn>1</mn>
    </mrow>
</math>
```

Content MathML

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<math>
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    <apply>
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      <ci>x</ci>
      <cn>2</cn>
    </apply>
    <cn>1</cn>
  </apply>
</math>
```

$\LaTeX \xrightarrow{\text{LaTeXML}} \text{MathML}$ (see [Mil])

DGrammar
(*Discourse Grammar*)

English

German

...

Depends on language

FGrammar
(*Formula Grammar*)

MathML

LaTeX

...

Depends on representation

Definitions:

"An integer n is called **even** *iff $2|n$."*

Declarations:

"Let $\widetilde{H}$ be a numerical semigroup." [2]

Statements:

"For any graph G, $\alpha_1(G) \leq \frac{nb(G)}{4}$." [4]

Mathematical Objects:

"An integer n is called **even** *iff* $2|n$.*"*

"Let $\widetilde{H}$ *be a numerical semigroup."* [2]

"For any graph G, $\alpha_1(G) \leq \frac{nb(G)}{4}$.*"* [4]

Mathematical Properties:

"An integer n is called **even** *iff* $2|n$.*"*

"If $x = \{x_n\}_{n=1}^{\infty} \in l^{\infty}(V)$ *is strongly almost convergent in* V, *then [...]"* [5]

Remember: Grammar Architecture

DGrammar
(*Discourse Grammar*)

English

German

...

Depends on language

FGrammar
(*Formula Grammar*)

MathML

LaTeX

...

Depends on representation

Formula Statement:

"An integer n is called **even** *iff* $2|n$*."*

"Since $uv \notin A$*, this implies that* $uz \in A$ *and* $vz \in A$*"* [4]

Objects:

"$\bigcup_{i\in I} O_i$ *is open in* Y*."*

"The stabilizer in $G^{\mathbb{C}}$ *for* p *is an algebraic group."* [1]

(Restricted) Identifier:

"An integer n *is called* **even** *iff 2|n."*

"Let $r \geq 2$ *be an integer."*

Numerals: 2, 19, ...

Identifiers: n, φ, G, ...

Binary Operators: $+$, $\cap$, ...

Binary Relations: $\geq$, $\in$, ...

Example: $n^2 + 19 \geq n^2$

See also [Gin11]

Example: "$n > 1$"

```
-- abstract grammar:
greater_than : FExpression -> FExpression -> FStatement;

-- concrete grammar:
greater_than a b = a ++ ">" ++ b;
```

Parsing a Formula - Binary Relations

Example: "$n > 1$"

```
-- abstract grammar:
greater_than : FExpression -> FExpression -> FStatement;

-- concrete grammar:
greater_than a b = a ++ ">" ++ b;
```

Better way:

```
-- abstract grammar:
apply_bin_rel : FBinRelation -> FExpression ->
        FExpression -> FStatement;

-- concrete grammar could be:
apply_bin_rel rel a b = a ++ rel ++ b;
```

Parsing a Formula - Binary Relations

The first solution couldn't handle the following cases nicely:

- "$1 < m < n$"
- "$0 \leq r < 1$"
- "$0 = t_0 < t_1 < \ldots < t_n = 1$"

Parsing a Formula - Binary Relations

The first solution couldn't handle the following cases nicely:

- "$1 < m < n$"
- "$0 \leq r < 1$"
- "$0 = t_0 < t_1 < \ldots < t_n = 1$"

Cases like "$0 \leq r < 1$" are easy with better approach:

```
-- abstract grammar:
apply_tern_rel : FBinRelation -> FBinRelation->
        FExpression -> FExpression -> FExpresion ->
        FStatement;
```

```
-- concrete grammar:
apply_tern_rel rel1 rel2 a b c = a ++ rel1 ++ b ++
                                    rel2 ++ c;
```

Formula Statement:

"An integer n is called **even** *iff* $2|n$*."*

"Since $uv \notin A$*, this implies that* $uz \in A$ *and* $vz \in A$*"* [4]

Objects:

"$\bigcup_{i\in I} O_i$ *is open in* Y*."*

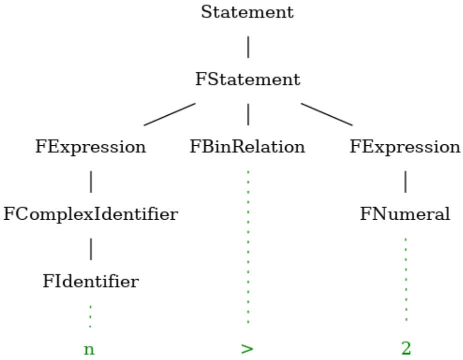
"The stabilizer in $G^{\mathbb{C}}$ *for* p *is an algebraic group."* [1]

(Restricted) Identifier:

"An integer n *is called* **even** *iff* $2|n$*."*

"Let $r \geq 2$ *be an integer."*

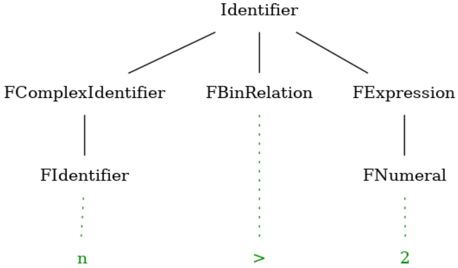
"we know that $n > 2$*"*



In logic: $\mathbf{greater}(n, 2)$

Example: "$n > 2$" as Identifier

"there is an integer $n > 2$ such that..."



$$(\exists n.(\lambda x.\mathbf{int}(x))n \wedge \mathbf{greater}(n, 2) \wedge (\ldots))$$

$\Downarrow$ external simplifier

$$\exists n.\mathbf{int}(n) \wedge \mathbf{greater}(n, 2) \wedge \ldots$$

We need to extract "n" from "$n > 2$":

```
-- Identifier record (simplified)
Identifier = {
    formula : Str;  -- the restriction, e.g. greater(n, 2)
    core :  Str;    -- the identifier, e.g. n
};
```

Example: “$n > 2$” as Identifier

We need to extract “n” from “$n > 2$”:

```
-- Identifier record (simplified)
Identifier = {
    formula : Str;   -- the restriction, e.g. greater(n, 2)
    core :  Str;     -- the identifier, e.g. n
};
```

```
-- example usage
fcid_fbinrel_fexpr_to_identifier a r1 b = {
    formula = r1 ++ "(" ++ a ++ "," ++ b ++ ")";
    core = a;
};
```

```
-- for statements like (∃n.(λx.int(x))n ∧ greater(n,2))
exists_statement obj id = inp("∃" ++ id.core ++ "."
        ++ lwrap(obj) ++ id.core ++ and_formula(id));
```

Examples:

- *"integer"*
- *"an even integer"*
- *"There is a bijective map from $(0,1)$ to $\mathbb{R}$"*
- *"There is a bijective map f from $(0,1)$ to $\mathbb{R}$"*
- *"Let C be a complete nonsingular irreducible curve over an algebraically closed field k of characteristic 0"* [2]

```
exists_suchthat :
    PosNegPol          -- is/isn't
    -> MObj            -- integer
    -> Identifier      -- n
    -> StatementFin    -- ...
    -> StatementFin;
```

"there is an integer n such that ..."

$\exists n.\mathbf{int}(n) \wedge \ldots$

```
exists_suchthat :
    PosNegPol               -- is/isn't
    -> MObj                 -- positive even integer
    -> Identifier           -- n
    -> StatementFin         -- ...
    -> StatementFin;
```

"there is a positive even integer n such that . . . "

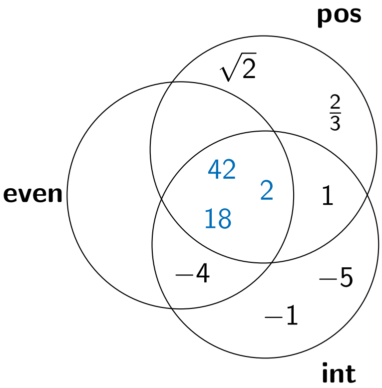
$\exists n.\mathbf{pos}(n) \wedge \mathbf{even}(n) \wedge \mathbf{int}(n) \wedge \ldots$

```
exists_suchthat :
    PosNegPol               -- is/isn't
    -> MObj                 -- positive even integer
    -> Identifier           -- n
    -> StatementFin         -- ...
    -> StatementFin;
```

"there is a positive even integer n such that . . . "

$\exists n.\mathbf{pos}(n) \wedge \mathbf{even}(n) \wedge \mathbf{int}(n) \wedge \ldots$

$\exists n.(\lambda x.\mathbf{pos}(x) \wedge \mathbf{even}(x) \wedge \mathbf{int}(x))n \wedge \ldots$



- *"Positive even integers"* are the intersection of integers, even things and positive things
- This doesn't work for every adjective!

"An integer n is called even iff $2|n$."

Idea: $\mathbf{even}(n) \Leftrightarrow \mathbf{divides}(2, n)$

"An integer n is called even iff $2|n$*."*

Idea: $\mathbf{even}(n) \Leftrightarrow \mathbf{divides}(2, n)$

With quantifier: $\forall n.\mathbf{int}(n) \Rightarrow (\mathbf{even}(n) \Leftrightarrow \mathbf{divides}(2, n))$

What's generated:
$(\forall n.((\lambda x.\mathbf{int}(x))n) \Rightarrow ((\lambda x.\mathbf{even}(x))n \Leftrightarrow \mathbf{divides}(2, n)))$

"A positive integer n is called prime, iff there is no integer $1 < m < n$ such that $m|n$"

Translation to (from) German:

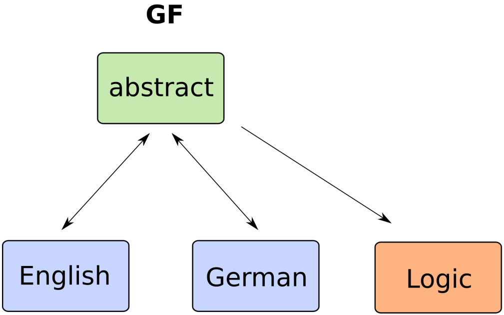
"Eine positive ganze Zahl n ist prim genau dann, wenn es keine ganze Zahl $1 < m < n$ gibt, sodass $m|n$"

Formalization:

$$(\forall n.((\lambda x.\mathbf{pos}(x) \wedge \mathbf{int}(x))n) \Rightarrow ((\lambda x.\mathbf{prime}(x))n \Leftrightarrow (\neg\exists m.(\lambda x.\mathbf{int}(x))m \wedge \mathbf{less}(1, m) \wedge \mathbf{less}(m, n) \wedge (\mathbf{divides}(m, n)))))$$

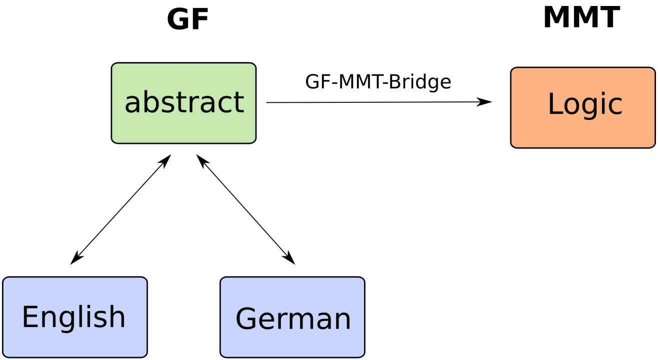
$\Downarrow$ external simplifier

$$\forall n.\mathbf{pos}(n) \wedge \mathbf{int}(n) \Rightarrow (\mathbf{prime}(n) \Leftrightarrow \neg\exists m.\mathbf{int}(m) \wedge \mathbf{divides}(m, n) \wedge \mathbf{less}(1, m) \wedge \mathbf{less}(m, n))$$



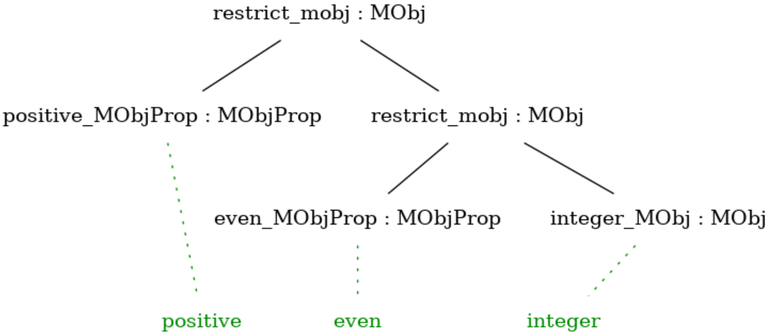
- **+** Parsing and logic generation in one tool (GF)
- **–** Grammar engineering gets complicated
- **–** We need an external tool for simplification and reasoning

Second Approach: Semantics Modelling in MMT



What is MMT?

- *Meta Meta Tool*
- Foundation-independent
- A lot of features for mathematical knowledge management
- See e.g. [Rab16]
- We'll focus only on a small "slice" of MMT



$\lambda x.\mathbf{pos}(x) \wedge \mathbf{even}(x) \wedge \mathbf{int}(x)$

$\lambda x.\mathbf{pos}(x) \wedge \mathbf{even}(x) \wedge \mathbf{int}(x)$

Type declarations for atoms:

even_MObjProp : $\iota \longrightarrow o$ |
positive_MObjProp : $\iota \longrightarrow o$ |
integer_MObj : $\iota \longrightarrow o$ |

$\lambda x.\mathbf{pos}(x) \wedge \mathbf{even}(x) \wedge \mathbf{int}(x)$

Type declarations for atoms:

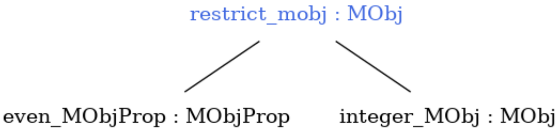
even_MObjProp : $\iota \longrightarrow o$
positive_MObjProp : $\iota \longrightarrow o$
integer_MObj : $\iota \longrightarrow o$

Better: Types for **`cat`**s in MMT:

MObj : type | = $\iota \longrightarrow o$
MObjProp : type | = $\iota \longrightarrow o$

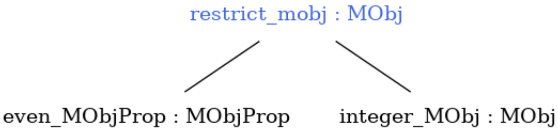
even_MObjProp : MObjProp
positive_MObjProp : MObjProp
integer_MObj : MObj

Grammar Nodes: **`restrict_mobj`**



Goal: **restrict even integer** $= \lambda x.$**even**$(x) \wedge$ **integer**(x)

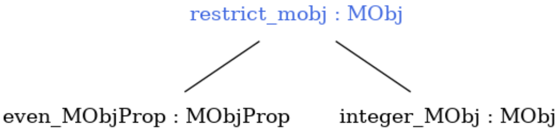
Grammar Nodes: `restrict_mobj`



Goal: **restrict even integer** $= \lambda x.$**even**$(x) \wedge$ **integer**(x)

restrict_MObj : MObjProp ⟶ MObj ⟶ MObj |
  = [mprop,mobj] [x] mprop(x) ∧ mobj(x) |

Grammar Nodes: `restrict_mobj`



Goal: **restrict even integer** $= \lambda x.$**even**$(x) \wedge$ **integer**(x)

restrict_MObj : MObjProp ⟶ MObj ⟶ MObj |
= [mprop,mobj] [x] mprop(x) ∧ mobj(x) |

We can map the GF tree to an MMT term:

`restrict_mobj even_MObjProp integer_MObj`

$\Downarrow$ GF-MMT-Bridge

restrict_MObj even_MObjProp integer_MObj

```
abstract Cats = {
  cat
    MObj;
    MObjProp;
}
```

```
theory Cats : ur:?LF =
   include ?FOL ❙

   MObj : type | = ι ⟶ o ❙
   MObjProp : type | = ι ⟶ o ❙
❚
```

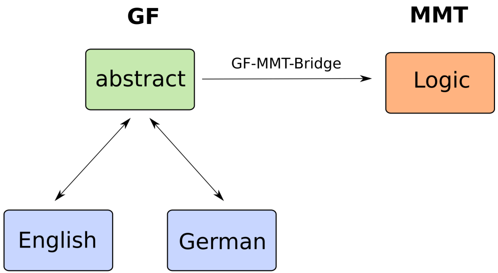
```
abstract Lexicon = Cats ** {
  fun
    even_MObjProp : MObjProp;
    positive_MObjProp :
                        MObjProp;
    integer_MObj : MObj;
}
```

```
theory Lexicon : ur:?LF =
   include ?Cats ❙

   even_MObjProp : MObjProp ❙
   positive_MObjProp : MObjProp ❙
   integer_MObj : MObj ❙
❚
```

```
abstract Grammar = Cats ** {
  fun
    restrict_MObj :
        MObjProp -> MObj ->
        MObj;
}
```

```
theory Grammar : ur:?LF =
   include ?Cats ❙
   restrict_MObj
      : MObjProp ⟶ MObj ⟶ MObj |
      = [mprop,mobj] [x]
                  (mprop x) ∧ (mobj x) ❙
❚
```



GF	(= *grammar development framework*)
+ MMT	(= *logic development framework*)
???	(= *semantics development framework*)

```
theory Cats : ur:?LF =
   include ?FOL ❙

   MObj : type | = ι ⟶ o ❙
   MObjProp : type | = ι ⟶ o ❙
❚
```

$\rightarrow$ Where does FOL come from?

```
theory Cats : ur:?LF =
  include ?FOL ❙

  MObj : type | = ι ⟶ o ❙
  MObjProp : type | = ι ⟶ o ❙
❚
```

→ Where does FOL come from?

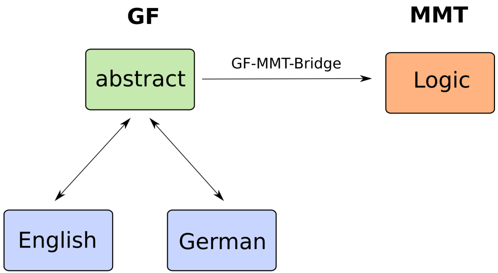
```
theory FOL : ur:?LF =
  include ?PL ❙
  // type for individuals ❙
  ind : type | # ι ❙

  // add quantifiers ❙
  forall : ( ι ⟶ o ) ⟶ o | # ∀ 1 ❙
  exists : ( ι ⟶ o ) ⟶ o | # ∃ 1 | = [p] ¬ ∀ [x] ¬ (p x) ❙
❚
```

```
theory PL : ur:?LF =
  // declare a type for propositions and introduce o as notation ❙
  prop : type ❘ # o ❙

  // declare basic logical operations ❙
  negation : o ⟶ o ❘ # ¬ 1 prec 30 ❙
  conjunction : o ⟶ o ⟶ o ❘ # 1 ∧ 2 prec 15 ❙

  // we can define other operations through ¬ and ∧ ❙
  disjunction : o ⟶ o ⟶ o ❘ = [a,b] ¬ (¬ a ∧ ¬ b) ❘ # 1 ∨ 2 prec 10 ❙
  implication : o ⟶ o ⟶ o ❘ = [a,b] ¬ a ∨ b ❘ # 1 ⇒ 2 prec 8 ❙
  equivalence : o ⟶ o ⟶ o ❘ = [a,b] (a ⇒ b) ∧ (b ⇒ a) ❘ # 1 ⇔ 2 prec 5 ❙
❚
```



GF	(= *grammar development framework*)
+ MMT	(= *logic development framework*)
???	(= *semantics development framework*)

Very hard!!

See also [Zin04] and [Wol13].

"From $A \subset B$ and $B \subset A$ it follows that $A = B$."

"$\subset$" might refer to "$\subseteq$" or to "$\subsetneq$"

$\rightarrow$ we get at least $2 \cdot 2 = 4$ parse trees

$\rightarrow$ we can discard some of them:

Example: Interpreting both "$\subset$" as "$\subsetneq$":

$(\subseteq (A, B) \land A \neq B) \land (\subseteq (B, A) \land A \neq B), \;\; A = B$

"an integer n is called even iff $2|n$*"*

even : int ⟶ o | = [n] div(2, n) |

- Math Linguistics/Translation
- Semantics Development Framework:

GF	*(= grammar development framework)*
+ MMT	*(= logic development framework)*
???	*(= semantics development framework)*

- Math Linguistics/Translation
- Semantics Development Framework:

GF	*(= grammar development framework)*
+ MMT	*(= logic development framework)*
???	*(= semantics development framework)*

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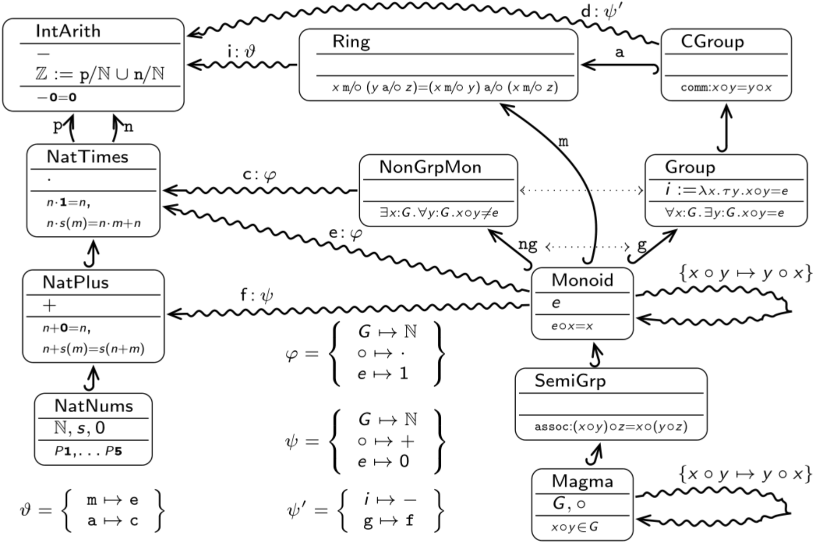
- *arXMLiv* corpus: 1,232,186 HTML5 scientific documents from the arXiv.org
- *GloVe* word embeddings for mathematics
- ...

```
theory PLSemantics : ur:?LF =
   include ?PLSyntax ❙

   ded : o ⟶ type | # ⊦ 1 ❙   // jder  not jvdash ❙

   trueIn : ⊦ ⊤ ❙
   andIn : {A,B} ⊦ A ⟶ ⊦ B ⟶ ⊦ A ∧ B | # andI 3 4 ❙
   negEl : {A} ⊦ A ⟶ ⊦ ¬ A ⟶ ⊦ ⊥ | # negE 2 3 ❙
   falseEl : {A} ⊦ ⊥ ⟶ ⊦ A | # falseE 2 ❙
   andElL : {A,B} ⊦ A ∧ B ⟶ ⊦ A | # andEL 3 ❙
```


IntArith
−
ℤ := p/ℕ ∪ n/ℕ
−0=0
Ring
x m/o (y a/o z)=(x m/o y) a/o (x m/o z)
CGroup
comm:x∘y=y∘x
d : ψ′
i : ϑ
a
p
n
NatTimes
·
n·1=n,
n·s(m)=n·m+n
c : φ
NonGrpMon
∃x:G.∀y:G.x∘y≠e
Group
i :=λx.ιy.x∘y=e
∀x:G.∃y:G.x∘y=e
m
e : φ
ng
g
NatPlus
+
n+0=n,
n+s(m)=s(n+m)
f : ψ
Monoid
e
e∘x=x
{x ∘ y ↦ y ∘ x}
φ = { G ↦ ℕ, ∘ ↦ ·, e ↦ 1 }
SemiGrp
assoc:(x∘y)∘z=x∘(y∘z)
NatNums
ℕ, s, 0
P1, . . . P5
ψ = { G ↦ ℕ, ∘ ↦ +, e ↦ 0 }
Magma
G, ∘
x∘y∈G
{x ∘ y ↦ y ∘ x}
ϑ = { m ↦ e, a ↦ c }
ψ′ = { i ↦ −, g ↦ f }

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