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CSC 440 Assignment 1
Coronavirus Problem
Due 2020-02-05

In this problem, we are given a 5×5 grid of students, some of them initially infected with coronavirus. After each passing day, any non-infected students adjacent to two infected students becomes infected. Two students are adjacent if they touch each other on one of their four sides; diagonal students are not adjacent. Infected students never become not infected.

1. *What is the maximum number of initially infected students such that at least one initially healthy student always remains healthy?*

   The answer here is 20 students, or more generally, for an $n \times n$ grid of students, the answer to this is $n^2 - n$. These 20 students would need to be arranged such that the 5 healthy students are all in a single row or column aligned *against* an edge of the grid (classroom). In the case where not all students eventually become infected, the fewest number of healthy students remaining is 5.

2. *What is the minimum number of initially infected students such that there is some arrangement of that many initially infected students will result in every student eventually becoming infected?*

   This minimum number is 5, and this can achieved by arranging the initially infected students along the main diagonal (or the opposite diagonal). This answer was found through trial and error, but it a provable solution that can be found with some more thought. The more general case will be described later.

3. *Can you arrange this minimum number of infected students in such a way that the infection never spreads to any healthy student?*

   Yes, there are many ways to achieve this in a 5×5 grid, but the simplest arrangement is to line them up in a single row or column. In this arrangement, no healthy block is adjacent to more than one infected block.

4. *How would your answers change if there were n^2 students in an $n \times n$ grid?*

   This problem and the solutions to the above questions can be generalized to n^2 students in an $n \times n$ grid.

   a. For question 1, the maximum number of initially infected students such that at least one initially healthy student always remains healthy is $n^2 - n$.

Before going further, let us describe a certain property of this problem. We can define the perimeter of the infected students as the total length of the boundaries surrounding the infected students. For example, a classroom with a single infected student has a perimeter of 4, and a classroom only containing two students adjacent to each other has a perimeter of 6. It should also be noted that a classroom where all students are infected has a perimeter of $4n$.

What we notice is that the perimeter of the infected students can never *increase* after subsequent days. This is because when a student is newly infected, the new block adds up to 2 to the perimeter, but also reduces the perimeter by at least 2. So, the addition of a newly-infected student can cause the perimeter to decrease up to 2, or stay the same.

So, for question 1, if we have $n^2 - n$ students arranged so that the n healthy students are all in a single row or column aligned against an edge of the grid, the perimeter is $4n - 2$. The whole grid can only become infected if the perimeter is initially at least $4n$. The addition of any more infected blocks would increase the perimeter to $4n$ and cause the rest of the students to become infected (this is not to imply that having a perimeter of $4n$ is the reason that the rest of the students will become infected).

b. For question 2, the fewest number of initially infected students required to eventually infect the whole classroom is n students arranged along the main diagonal (or the other diagonal). In this case, the perimeter is $4n$, and the perimeter doesn't change as new students are infected and the whole class eventually becomes infected. If there were any fewer than n initially-infected students, the perimeter would be less than $4n$ and thus could never increase to the $4n$ perimeter required by a fully-infected class.

c. The answer for question 3 doesn't change.

5. *Does it matter if n is even or odd?*

   No, it does not matter if n is odd. Nothing about the above reasoning relies on n being even or odd.

6. *What property about the set of infected students never changes as the days pass and the infection spreads?*

   Invariants:
   1. The perimeter of the infected region never increases as days pass
   2. A student that is infected never becomes healthy