Fluid Dynamics of the Environment: Example Sheet postmortems

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Contents

1 Question 1: Equipartition of energy

> Consider linear surface waves of amplitude η_0 in a fluid of depth H. Prove that there is an equipartition between kinetic and potential energy, averaged over a wave length. Derive also a relationship between energy flux, energy density (sum of kinetic and potential energy densities) and group velocity.

Begin with the linearised inviscid irrotational equations:

$$\rho \boldsymbol{u}_t = -\boldsymbol{\nabla} p + \rho \boldsymbol{g}. \tag{1}$$

These hold in $z \in (-H, 0)$. Write $\boldsymbol{u} = \boldsymbol{\nabla}\phi$; then $\boldsymbol{\nabla}^2\phi = 0$. And $w = \phi_z = 0$ on $z = -H$. Hence

$$\phi = \phi_0 \cos(kx - \omega t)\cosh(k(z + H)) \tag{2}$$

$$u = -\phi_0 k \sin(kx - \omega t)\cosh(k(z + H)) \tag{3}$$

$$w = \phi_0 k \cos(kx - \omega t)\sinh(k(z + H)). \tag{4}$$

The surface height η will be of the form

$$\eta = \Re(\eta_0 e^{i(kx - \omega t)}); \tag{5}$$

and the usual free-surface BCs on this surface give

$$\eta = \phi_0 \frac{k}{\omega} \sinh(kH)\sin(kx - \omega t) \tag{6}$$

and the dispersion relation

$$\omega^2 = gk \tanh kH. \tag{7}$$

Now dot (1) with $\boldsymbol{u}$ to get an energy equation:

$$\underbrace{\left(\frac{1}{2}\rho \boldsymbol{u}^2\right)_t}_{\text{KE density}_t} + \underbrace{\rho g w}_{\text{PE density}_t} + \boldsymbol{\nabla}\cdot(\underbrace{p\boldsymbol{u}}_{\substack{\text{energy}\\\text{flux}\\\text{density}}}) = 0 \tag{8}$$

Thus we compute $K = \frac{1}{2}\rho\boldsymbol{u}^2$ and integrate over $-H < z < 0$ to find the kinetic energy per unit area. For potential energy, we are interested in the *perturbation* to the PE rather than the total PE. We have

$$\Delta PE \text{ per unit area} = \int_0^\eta \rho g z dz = \frac{1}{2}\rho g \eta^2. \tag{9}$$

Then it can be shown that $\langle KE \text{ p.u.a.}\rangle = \langle \Delta PE \text{ p.u.a.}\rangle$

2 Question 2

3 Question 3

Figure 1: Incident, reflected and transmitted waves.

4 Question 4: Reflexion and transmission of internal gravity waves

> Consider a plane internal wave beam of frequency ω propagating from $z < 0$ where the fluid has a linear stratification given by buoyancy N_1 into the region $z > 0$ where the buoyancy frequency is $N_2 < N_1$. The density field is continuous across $z = 0$.

If $\omega < N_2$ then the incident wave gives rise to both a transmitted and a reflected wave. But if $N_2 < \omega < N_1$ then the 'transmitted wave' is evanescent, and we have total internal reflexion. Consider $\omega < N_2$:

The boundary conditions at $z = 0$ are that pressure perturbations p' and vertical velocities $\boldsymbol{u} \cdot \boldsymbol{n}$, or equivalently vertical displacements, must be continuous. Using the dispersion relation $\omega = N \cos\theta$ and the fact that ω is the same everywhere, we have

$$\omega = N_1 \cos\theta_I = N_2 \cos\theta_T \tag{10}$$

so the transmitted wave makes an angle

$$\cos\theta_T = \frac{N_1}{N_2} \cos\theta_I \tag{11}$$

with the vertical.

The wavevectors of the three rays are

$$\boldsymbol{k}_I = k_I(\cos\theta_I, -\sin\theta_I) \tag{12}$$
$$\boldsymbol{k}_R = k_R(\cos\theta_I, +\sin\theta_I) \tag{13}$$
$$\boldsymbol{k}_T = k_T(\cos\theta_T, -\sin\theta_T) \tag{14}$$

as shown in Figure 1.

The horizontal displacements of fluid particles satisfy

$$\eta_x = \begin{cases} e^{-i\omega t} \sin\theta_I (\hat{\eta}_I e^{i\boldsymbol{k}_I \cdot \boldsymbol{x}} + \hat{\eta}_R e^{i\boldsymbol{k}_R \cdot \boldsymbol{x}}) & z < 0 \\ e^{-i\omega t} \sin\theta_T \hat{\eta}_T e^{i\boldsymbol{k}_R \cdot \boldsymbol{x}} & z > 0 \end{cases} \tag{15}$$

where the $\hat{\eta}_{I,R,T}$ are displacement amplitudes along the ray, and the $\sin\theta_{I,T}$ appear because the rays are travelling in different directions, and and we need to pick out the horizontal components of displacement.

Considering dependences on x, we see that

$$k_I \cos\theta_I = k_R \cos\theta_I = k_T \cos\theta_T \tag{16}$$

and so $k_I = k_R$ and $k_T = \frac{N_2}{N_1} k_I$. That is, wavenumbers in the x direction are all equal to each other.

Meanwhile, pressure perturbations are given by

$$\hat{p} = \frac{i\rho_0\omega^2\hat{\eta}\tan\theta}{|k|} \tag{17}$$

and these are continuous at $z = 0$. Hence:

$$\frac{\hat{\eta}_I \tan\theta_I}{k_I} + \frac{\hat{\eta}_R \tan\theta_I}{k_I} = \frac{\hat{\eta}_T \tan\theta_T}{k_T} \tag{18}$$

$$\tan\theta_I(\hat{\eta}_I + \hat{\eta}_R) = \tan\theta_T \frac{\cos\theta_T}{\cos\theta_I}\hat{\eta}_T \tag{19}$$

$$\sin\theta_I(\hat{\eta}_I + \hat{\eta}_R) = \sin\theta_T\hat{\eta}_T \tag{20}$$

Finally, vertical displacements are given by

$$\eta_y = \begin{cases} e^{-i\omega t}\cos\theta_I(\hat{\eta}_I e^{i\boldsymbol{k}_I\cdot\boldsymbol{x}} - \hat{\eta}_R e^{i\boldsymbol{k}_R\cdot\boldsymbol{x}}) & z < 0 \\ e^{-i\omega t}\cos\theta_T\hat{\eta}_T e^{i\boldsymbol{k}_R\cdot\boldsymbol{x}} & z > 0 \end{cases} \tag{21}$$

and are also continuous across the boundary. Hence

$$\cos\theta_I(\hat{\eta}_I - \hat{\eta}_R) = \cos\theta_T\hat{\eta}_T \tag{22}$$

$$\tag{23}$$

So, Equations 20 and 23 together give:

$$\begin{pmatrix} -1 & \frac{\sin\theta_T}{\sin\theta_I} \\ 1 & \frac{\cos\theta_T}{\cos\theta_I} \end{pmatrix} \begin{pmatrix} \hat{\eta}_R \\ \hat{\eta}_T \end{pmatrix} = \hat{\eta}_I \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{24}$$

This is solved to give

$$\hat{\eta}_R = \hat{\eta}_I \frac{\sin(\theta_T - \theta_I)}{\sin(\theta_T + \theta_I)} \tag{25}$$

$$\hat{\eta}_T = \hat{\eta}_I \frac{\sin(2\theta_I)}{\sin(\theta_T + \theta_I)} \tag{26}$$

Now we should check that η_x is continuous at $z = 0$, i.e. that

$$\sin\theta_I(\hat{\eta}_I + \hat{\eta}_R) = \sin\theta_T\hat{\eta}_T. \tag{27}$$

This is easily done.

What happens in the limit $\omega = N_2$?

If $\omega = N_2$, then $\cos\theta_T = 1$ and $\sin\theta_T = 0$, i.e. the transmitted wave points vertically upwards. The amplitude of the transmitted wave $\hat{\eta}_T$ does not vanish, but the group velocity of the transmitted wave is zero and so no energy is transmitted. The reflected wave has the same amplitude as the incident wave: $\hat{\eta}_R = -\hat{\eta}_I$.