Fluid Dynamics of the Environment

J.M.F.T.

January 20, 2015

Contents

1 Internal gravity waves

2 Turbulence primer

Most environmental and geophysical flows occur at high Reynolds numbers. At high Reynolds numbers, laminar solutions to the Navier-Stokes equations are unstable, and the flows are turbulent.

2.1 Basics of turbulence

Throughout this and the next section, assume that the density ρ is constant and uniform.

2.1.1 What is turbulence?

Although there is no simple definition for turbulence, turbulent flows can be characterised by a few properties:

- Turbulent flows are dominated by inertia, since $\mathrm{Re} \gg 1$. Viscosity plays a small (but crucial) role.

- Turbulent flows are vortical, not irrotational. However, vortical flows need not be turbulent.

- Turbulent flows involve a range of length scales. Vortices exist at both large and small length scales: "Big whorls have little whorls, which feed on their velocity, // And little whorls have lesser whorls, and so on to viscosity." —Lewis Richardson.

Turbulent flows are unpredictable and simulating them numerically may be difficult as they may be numerically unstable. However, it may be possible to predict certain statistics of the flow, such as the behaviour of the mean flow or mean pressure (noting that fluctuations about these means may be great). Some features of turbulence can be shown to be universal.

Lewis Richardson's couplet describes the two processes at work in a turbulent flow. *Vortex stretching* is responsible for creating small vortices out of large ones, while *energy dissipation* causes small vortices to decay.

2.1.2 Vortex stretching

Taking the curl of the Navier-Stokes equation gives the *vorticity equation*:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \nu \boldsymbol{\nabla}^2 \boldsymbol{\omega} \tag{1}$$

The second term on the RHS of 1 means that vorticity is diffused with diffusion constant ν. This is a minor effect in flows with $\mathrm{Re} \gg 1$. More importantly, the first term means that when a parcel of fluid is stretched into a column, its vorticity increases and becomes aligned with the column. Thus stretching a vortex takes energy to a smaller scale. However, if the vortex column is compressed, not nearly as much energy is taken back to the large scale, since the compression in general de-aligns the vorticity vector and the column.

2.1.3 Energy dissipation: Kolmogorov's theory (1941)

We know that energy is dissipated in a fluid at a rate

$$\int_V \mu |\boldsymbol{\nabla} \boldsymbol{u}|^2 dV \tag{2}$$

(or expressed in terms of the strain rate tensor, using $|\boldsymbol{\nabla} \boldsymbol{u}|^2 = 2e_{ij}e_{ij}$). For a laminar flow with characteristic velocity U in a box with sides L, we might suppose that this scales as $\mu \frac{U^2}{L^2} L^3 = \mu U^2 L$. However, for a turbulent flow, the energy dissipation is much faster than this, because there is a transfer

of energy from large to small scales, and so $|\nabla u|^2$ is in fact very large: viscosity acts more strongly at small lengthscales.

This suggests that although the large-scale structure of the flow might have characteristic lengthscale L, the small-scale structure of the flow should be described by another characteristic lengthscale η, called the *Kolmogorov length scale* or the *Kolmogorov microscale*. This is the lengthscale at which the effects of vortex stretching and dissipation balance each other.

To summarise: The flow is externally driven (continually or initially) at the lengthscale L, energy cascades to smaller lengthscales; as the energy is transferred to smaller lengthscales, viscosity acts more and more strongly; until the energy reaches the lengthscale η and can proceed no further.

Let $U(k)$ be the Fourier transform of the velocity $u(x)$, using the normalisation

$$U(k) = \frac{1}{\sqrt{2\pi}} \int_V u(x) e^{-ik \cdot x} dx. \tag{3}$$

Let

$$E(k) = \frac{1}{2} \rho U(k) \cdot U(k)* \tag{4}$$

be the energy density spectrum. (In defining these Fourier transforms, we are assuming that ρ is constant and uniform.) The total kinetic energy is

$$KE = \int E(k) dk \tag{5}$$

with the integral taken over wavenumber space. For isotropic turbulence, U and E depend only on the magnitude $k = |k|$.

Kolmogorov made two hypotheses about isotropic turbulent flows. These hypotheses allow us to make progress by dimensional analysis, but they do not follow from first principles, and their predictions must be verified experimentally.

Since density is constant, we will find it convenient to rewrite the above expressions without ρ, so that the dimension M may be set to 1. We therefore work with the energy dissipation rate

$$\epsilon = \int_V \nu |\nabla u|^2 dV \tag{6}$$

which has dimensions $L^2 T^{-3}$, and with the energy spectrum

$$E(k) = \frac{1}{2} U(k) \cdot U(k)* \tag{7}$$

which has dimensions $L^3 T^{-2}$. We write

$$KE = \int E(k) dk \tag{8}$$

for the kinetic energy per unit mass, with dimensions $L^2 T^{-2}$.

First hypothesis   The rate of dissipation of energy (per unit density), ϵ (dimensions $L^2 T^{-3}$), depends only on viscosity ν (dimensions $L^2 T^{-1}$) and on the microscale η. In particular, ϵ and η are not to depend on the large lengthscale L.

By dimensional analysis, this hypothesis allows us to construct η:

$$\eta = (\nu^3/\epsilon)^{1/4} \tag{9}$$

Note that in a system in equilibrium, the rate of dissipation of energy is equal to the rate of energy input. [1]

---

[1] We can have a dynamic equilibrium, but not a static one, since turbulent flow is not steady.

Second hypothesis   The energy density $E(k)$ (dimensions $L^3 T^{-2}$) depends only on the rate of dissipation ϵ and on k (dimensions L^{-1}).

By dimensional analysis,

$$E(k) = C\epsilon^{2/3}k^{-5/3} \tag{10}$$

for some constant C.

This second hypothesis is not completely valid. If the flow is in a bounded domain with characteristic lengthscale L, or if the flow is driven at the lengthscale L, then the hypothesis is not valid at wavenumbers $k \sim 1/L$ or below. And the hypothesis is also not valid when $k \sim 1/\eta$, where the details of dissipation start to be important.

However, for the range $1/L \ll k \ll 1/\eta$, known as the *inertial range*, the hypothesis is justified and is supported by experimental evidence: (10) is reproduced by experimental data.

2.1.4   Two-dimensional turbulence

A turbulent flow in two dimensions is different from one in three dimensions, since for a two-dimensional flow the vortex stretching term in (1) is always zero. Vorticity is conserved, and there is no cascade of energy to smaller lengthscales. Dotting (1) with $\boldsymbol{\omega}$ gives us the *enstrophy equation*:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\right)\left(\frac{1}{2}|\boldsymbol{\omega}|^2\right) = \nu\left(\boldsymbol{\nabla}^2 \frac{1}{2}|\boldsymbol{\omega}|^2 - |\boldsymbol{\nabla}\boldsymbol{\omega}|^2\right). \tag{11}$$

The scalar quantity $\frac{1}{2}|\boldsymbol{\omega}|^2$ is called the *enstrophy density*.

We obtain a very similar equation for the kinetic energy density by dotting the Navier-Stokes equation with $\boldsymbol{u}$:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\right)\left(\frac{1}{2}|\boldsymbol{u}|^2\right) = \nu\left(\boldsymbol{\nabla}^2 \frac{1}{2}|\boldsymbol{u}|^2 - |\boldsymbol{\nabla}\boldsymbol{u}|^2\right). \tag{12}$$

Equations **??** and 11 tell us that in a 2D flow, enstrophy and kinetic energy are redistributed by diffusion (first term on RHS) as well as dissipated (second term).

Since total enstrophy (enstrophy density integated across the volume) must decrease, there must be a net movement of energy to *larger* lengthscales (i.e. smaller k). Total enstrophy is proportional to $\int k^2 E(k)dk$, so if $E(k)$ increases at high k, this increase must be balanced by a greater increase in $E(k)$ at low k, so that $\int k^2 E(k)dk$ never increases.

Hence there is an *anticascade* of energy to larger scales, and a *cascade* of enstrophy to smaller scales.

2.2   Simplistic approaches to modelling turbulence

Instead of starting from first principles and trying to solve the Navier-Stokes equations for a turbulent system directly, it is often easier, and fruitful, to model the effects of turbulence instead.

2.2.1   Molecular diffusion

A direct approach is to model all scales down to the microscale $\eta = (\nu^3/\epsilon)^{1/4}$. The energy spectrum (10) predicts that the kinetic energy per unit volume will be

$$\text{KE density} = \int_{1/L}^{1/\eta} E(k)dk \propto (\epsilon L)^{2/3} \tag{13}$$

giving a characteristic velocity

$$u \propto \sqrt{\text{KE density}} \propto (\epsilon L)^{1/3} \tag{14}$$

and therefore a Reynolds number

$$Re = \frac{uL}{\nu} \propto \frac{\epsilon^{1/3} L^{4/3}}{\nu}. \tag{15}$$

Note that we calculate this Reynolds number using a characteristic velocity calculated only from the turbulent properties, not taking into account any large-scale velocities (such as one driving the flow).

Since $\eta = (\nu^3/\epsilon)^{1/4}$, we have

$$Re = \left(\frac{L}{\eta}\right)^{4/3} \gg 1. \tag{16}$$

For moderately large Re, direct numerical simulation (DNS) may be possible, but usually Re is so large that DNS is impractical or impossible.

The problem would be even worse if the fluid were not of uniform and constant density. The mass diffusivity κ and the momentum diffusivity ν are compared by the *Schmidt number* $Sc = \nu/\kappa$. For salt water, $Sc \approx 1000$. In order to simulate the diffusion of mass, we would have to look at an even smaller scale, the *Batchelor scale* $\lambda_B = \eta/Sc^{1/2}$.

2.2.2 The closure problem

A turbulent flow $\boldsymbol{u}$, and the associated pressure field p, can be decomposed into its mean part $\overline{\boldsymbol{u}}$, and a fluctuating part $\boldsymbol{u}'$ that fluctuates about $\boldsymbol{0}$: $\overline{\boldsymbol{u}'} = \boldsymbol{0}$. We can model a turbulent flow by ignoring the details of these fluctuations, and looking only at how these fluctuations affect the mean part. [2]

Beginning with the Navier-Stokes equations:

$$\rho \left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\right) \boldsymbol{u} = -\boldsymbol{\nabla} p + \mu \boldsymbol{\nabla}^2 \boldsymbol{u} + \boldsymbol{f} \tag{17}$$

and averaging, we obtain

$$\rho \left(\frac{\partial \overline{\boldsymbol{u}}}{\partial t} + \overline{\boldsymbol{u}} \cdot \boldsymbol{\nabla} \overline{\boldsymbol{u}}\right) = -\boldsymbol{\nabla} \overline{p} + \mu \boldsymbol{\nabla}^2 \overline{\boldsymbol{u}} + \boldsymbol{f} - \rho \overline{\boldsymbol{u}' \cdot \boldsymbol{\nabla} \boldsymbol{u}'} \tag{18}$$

(assuming that the body force $\boldsymbol{f}$ is a constant). The fluctuations are still present in the quadratic term $-\rho \overline{\boldsymbol{u}' \cdot \boldsymbol{\nabla} \boldsymbol{u}'}$ on the RHS, because (13) is nonlinear. [3] We cannot remove such nonlinear terms by more averaging, as that simply introduces further nonlinear terms. This is the *closure problem*: we cannot close the system as we do not know how to deal with these terms. Many approximate closures have been proposed, but none are universally applicable.

Note that

$$-\rho \overline{\boldsymbol{u}' \cdot \boldsymbol{\nabla} \boldsymbol{u}'} = -\rho \boldsymbol{\nabla} \cdot (\overline{\boldsymbol{u}' \boldsymbol{u}'}) \tag{19}$$

and so the fluctuations affect the mean flow as though they were an additional stress force, with stress tensor

$$\boldsymbol{\sigma}^{(R)} = -\rho \overline{\boldsymbol{u}' \boldsymbol{u}'}. \tag{20}$$

This 'stress' is called the *Reynolds stress*.

2.2.3 The k-ϵ model

The most commonly used closure model is the k-ϵ model, where $k = \frac{1}{2} \boldsymbol{u}' \cdot \boldsymbol{u}'$ is the *turbulent kinetic energy* (per unit density), and ϵ is the energy dissipation. The model states that

$$\rho \left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\right) k = \boldsymbol{\nabla} \cdot \left(\frac{\mu_T}{\sigma_k} \boldsymbol{\nabla} k\right) + 2\mu_T \boldsymbol{S} : \boldsymbol{S} - \rho \epsilon \tag{21}$$

---

[2]The mean can be taken as a time average or as a local spatial average, but this detail is unimportant for what follows.
[3]If we had a non-constant ρ, then we would have further such quadratic terms, such as the turbulent buoyancy flux $\overline{\boldsymbol{u}' \cdot \boldsymbol{\nabla} \rho}$.

where

$$\mu_T = C_\mu \rho \frac{k^2}{\epsilon} \tag{22}$$

is the *turbulent viscosity*,

$$\boldsymbol{S} = \frac{1}{2}(\boldsymbol{\nabla}\overline{\boldsymbol{u}} + (\boldsymbol{\nabla}\overline{\boldsymbol{u}})^T) \tag{23}$$

is the strain rate tensor corresponding to the mean flow, and $C_\mu \approx 0.09$ and $\sigma_k \approx 1.00$ are constants that are determined empirically. Equations 14 and 17 together specify the problem for $\overline{\boldsymbol{u}}$ completely.

Essentially, the model describes how turbulent kinetic energy is governed. We can interpret (17) as follows:

- The first term on the RHS represents the diffusion of turbulent kinetic energy, at some (non-constant) rate μ_T/σ_k.

- The second term represents the generation of turbulent kinetic energy, by means of vortex stretching.

- The third term represents the dissipation of turbulent kinetic energy.

Equation 17 can be re-written as

$$\rho\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\epsilon = \boldsymbol{\nabla}\cdot\left(\frac{\mu_T}{\sigma_\epsilon}\boldsymbol{\nabla}\epsilon\right) + 2C_{1\epsilon}\frac{\epsilon}{k}\mu_T\boldsymbol{S}:\boldsymbol{S} - C_{2\epsilon}\rho\frac{\epsilon^2}{k} \tag{24}$$

where $\sigma_\epsilon \approx 1.30$, $C_{1\epsilon} \approx 1.44$ and $C_{2\epsilon} \approx 1.92$ are constants that are determined empirically.

Experiments and simulations find that these constants are *universal*.

2.2.4   Turbulent diffusion models

As discussed, the vortex stretching process is irreversible and takes energy to smaller lengthscales. There is little back-scatter: smaller lengthscales do not strongly affect the larger scales, except as a sink of energy. This is why we can consider the mean flow as in (14) and ignore the behaviour of the fluctuations, except as a sink of energy.

This suggests replacing the Reynolds stress with an additional viscosity term that mimics the diffusion of momentum and dissipation of energy without actually describing how this happens. That is, we take

$$-\overline{\boldsymbol{u}'\boldsymbol{u}'} = \nu_T\boldsymbol{S} \tag{25}$$

where ν_T is called the *(turbulent) eddy viscosity*. We also write $\mu_T = \rho\nu_T$. The force due to Reynolds stresses is therefore

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^{(R)} = \boldsymbol{\nabla}\cdot(-\rho\overline{\boldsymbol{u}'\boldsymbol{u}'}) \tag{26}$$
$$= \rho\boldsymbol{\nabla}\cdot(\nu_T\boldsymbol{S}) \tag{27}$$
$$= \rho(\boldsymbol{\nabla}\nu_T\cdot\boldsymbol{S} + \nu_T\boldsymbol{\nabla}\cdot\boldsymbol{S}) \tag{28}$$
$$= \rho(\boldsymbol{\nabla}\nu_T\cdot\boldsymbol{S} + \frac{1}{2}\nu_T\boldsymbol{\nabla}^2\overline{\boldsymbol{u}}). \tag{29}$$

Putting this into the averaged Navier-Stokes equations gives us

$$\left(\frac{\partial\overline{\boldsymbol{u}}}{\partial t} + \overline{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\overline{\boldsymbol{u}}\right) = -\frac{1}{\rho}\boldsymbol{\nabla}\overline{p} + \left(\nu + \frac{1}{2}\nu_T\right)\boldsymbol{\nabla}^2\overline{\boldsymbol{u}} + \frac{1}{\rho}\boldsymbol{f} + \boldsymbol{\nabla}\nu_T\cdot\boldsymbol{S}. \tag{30}$$

Closure models that do this are called *turbulent diffusion models*. Different models propose different expressions for ν_T: this is usually not constant, but may depend on the position or on $\overline{\boldsymbol{u}}$.

6

2.2.5 Prandtl's mixing length model

Prandtl's mixing length model is a turbulent diffusion model, in which

$$\nu_T = ku'l \tag{31}$$

where l is some *mixing length*, $u' = \sqrt{\overline{|\boldsymbol{u}'|^2}}$ is the *turbulence intensity*, and the *Prandtl ratio* $k \approx 0.4$ is another empirically-determined constant. The mixing length represents the lengthscale of turbulent eddies; it may vary with position, and depends on the geometry of the problem.

The law of the wall For example, in the domain $z > 0$ bounded by a wall at $z = 0$, the lengthscale l of turbulent eddies is assumed to scale with z. The turbulence intensity is assumed to be some constant q. Hence $\nu_T = kqz$. The horizontal shear stress is also assumed to be constant.

Assuming that the mean flow is $\overline{\boldsymbol{u}} = U(z)\boldsymbol{e}_x$, the x-component of Equation 25 gives us

$$0 = \frac{1}{\rho}G + \left(\nu + \frac{1}{2}\nu_T\right)\frac{d^2U}{dz^2} + \frac{1}{2}kq\frac{dU}{dz} \tag{32}$$

where $G = -\frac{\partial \overline{p}}{\partial x}$ is the horizontal pressure gradient, which is some constant.

In the case $G = 0$, we can solve (**??**) exactly to give us

$$U(z) = \frac{A}{kq}\log(2\nu + kqz) \tag{33}$$

where A is some constant. The condition that shear stress is (approximately) constant allows us to write A in terms of the *bed shear stress*, the shear stress on $z = 0$.

For $z \ll \frac{\nu}{kq}$, (**??**) says that $U(z) \propto z$, as we would expect for a boundary layer. In this region, molecular viscosity ν is dominant. However, for larger z, $U(z)$ is logarithmic in z:

$$U(z) \sim \frac{u_*}{\kappa}\log\frac{z}{z_0} \tag{34}$$

This is the *law of the wall*. Here u_* is called the *slip velocity* and z_0 the *roughness height*; $\kappa \approx 0.41$ is a dimensionless constant called *von Karmen's constant*.

The law of the wall does not hold when z becomes too large, when other effects such as small pressure gradients take hold, or when the mixing length is no longer $l \sim z$, but it is a good approximation for boundary layers.

2.2.6 Entrainment: Diffusion of turbulence

Consider a turbulent patch of length scale l and turbulence intensity u', surrounded by irrotational fluid outside the patch. The vortical motion within the patch will draw in the external fluid, straining and diffusing vorticity into it. Thus the size of the patch will increase; its boundaries move out at a rate proportional to u'.

The diffusivity of turbulence is called the *turbulent diffusivity*, written κ_T, with $\kappa_T \sim u'l$. This is not to be confused with turbulent viscosity, though both are $\sim u'l$ with similar (but not necessarily equal) constants of proportionality.

This diffusivity of turbulence can explain the growth of the Rayleigh-Taylor instability in a tall tube.

2.3 Mixing

We now drop the assumption that ρ is constant and uniform.

2.3.1 What is mixing?

Mixing is the blending of fluid particles with different properties, such as density, salinity or temperature. (These properties are related to each other.) There are two processes responsible for mixing:

- *Stirring* is the intermingling of fluid particles of different properties. This produces large gradients in these properties.

- *Diffusion* drives a flux down a gradient that reduces these gradients between adjacent fluid particles.

Diffusion is an irreversible process: there is a 'preferred' direction for fluxes, down a gradient; but stirring (on its own) is reversible. Diffusion is a slow process that occurs at the molecular level, whereas stirring can happen quickly and over large lengthscales. Together, stirring and diffusion can cause irreversiible mixing.

In general, a scalar property S is governed by the advection-diffusion equation

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\right) S = \kappa \boldsymbol{\nabla}^2 S \tag{35}$$

where κ (dimensions $L^2 T^{-1}$) is the molecular diffusivity of S. This is often very small: the diffusivity of heat in air is $\approx 10^{-5} m^2 s^{-1}$ and the diffusivity of salt in water is $\approx 10^{-9} m^2 s^{-1}$. The timescale for diffusing across a length l is therefore $t \sim l^2/\kappa$, which is huge: heat takes a day to diffuse across a metre in air, and salinity takes thirty years to diffuse a metre.

However, stirring (represented in the advection term) can help to create large gradients in S, which increases the rate of diffusion (by reducing l).

2.3.2 The energy budget

Consider an incompressible Boussinesq fluid with a linear equation of state. [4] Assume that dissipative heating is unimportant, and ignore any heating effects from dilution, chempical potential energy, and so on.

Begin with the momentum equation for a variable-density fluid:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\right) \rho \boldsymbol{u} = -\boldsymbol{\nabla} p - \rho g \boldsymbol{e}_z + \rho \nu \boldsymbol{\nabla}^2 \boldsymbol{u} \tag{36}$$

and dot with $\boldsymbol{u}$. Since

$$\boldsymbol{u} \cdot \boldsymbol{\nabla} p = \boldsymbol{\nabla} \cdot (p\boldsymbol{u}) \tag{37}$$

for an incompressible fluid, and

$$\boldsymbol{u} \cdot \rho g \boldsymbol{e}_z = \left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\right)(\rho g z) - \boldsymbol{\nabla} \cdot (g z \kappa \boldsymbol{\nabla} \rho) + g \kappa \frac{\partial \rho}{\partial z} \tag{38}$$

(where we have used the advection-diffusion equation for ρ, with mass diffusivity κ), we have:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\right)\left(\frac{1}{2}\rho|\boldsymbol{u}|^2 + \rho g z\right) + \boldsymbol{\nabla} \cdot \left(p\boldsymbol{u} - \frac{1}{2}\rho\nu\boldsymbol{\nabla}|\boldsymbol{u}|^2 - g z \kappa \boldsymbol{\nabla}\rho\right) = -g\kappa\frac{\partial\rho}{\partial z} - \rho\nu|\boldsymbol{\nabla}\boldsymbol{u}|^2. \tag{39}$$

This is an equation desciribing kinetic and potential energy density. The energy of a parcel of fluid changes due to diffusion, because:

- work is done on it by pressure from neighbouring parcels, causing an energy flux $p\boldsymbol{u}$;

- viscosity causes the diffusion of kinetic energy, with energy flux $-\frac{1}{2}\rho\nu\boldsymbol{\nabla}|\boldsymbol{u}|^2$;

- mass diffusion causes the diffusion of potential energy, with energy flux $-g z \kappa \boldsymbol{\nabla}\rho$.

---

[4]Incompressibility means that $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$, but this does not mean that the density ρ is constant; ρ is governed by the advection-diffusion equation.

The energy of a parcel also changes due to:

- mass diffusion raising the centre of mass, changing the potential energy at a rate $-g\kappa\frac{\partial\rho}{\partial z}$;

- dissipation due to viscosity, decreasing the kinetic energy at a rate $\rho\nu|\boldsymbol{\nabla}\boldsymbol{u}|^2$.

These processes, whose terms appear on the RHS of (31), are irreversible.

If there are no fluxes of mass or momentum from the boundaries of V, then integrating the energy equation gives us:

$$\frac{d}{dt}(KE + PE) = W - \epsilon \tag{40}$$

where $KE = \int_V \frac{1}{2}\rho|\boldsymbol{u}|^2 dV$ and $PE = \int_V \rho gz\, dV$ be the total kinetic and potential energy of the system, $W = -\int_S p\boldsymbol{u}\cdot\boldsymbol{n}\, dS$ is the rate of pressure working and $\epsilon = \int_V \rho\nu\boldsymbol{\nabla}|\boldsymbol{u}|^2 dV$ is the rate of dissipation.

Energy is dissipated into heat. Kinetic and potential energy can be converted into each other, but mixing and dissipation cause irreversible.

2.4   Stably stratified flows

2.5   Mixing efficiency

2.6   Internal mixing

2.6.1   Kelvin-Helmholtz instability

2.6.2   Stratified shear flow

2.6.3   Holmboe instability

2.6.4   Entrainment by internal mixing

3   Shallow water

4   Particle-laden and granular flow

We consider the flow of a current consisting of a fluid with a well-mixed suspension of particles, with the particles denser than the fluid. The difference in density means that *buoyancy* has a role to play. The flow is driven by a pressure gradient and affected by the effects of buoyancy. Viscosity is negligible, so buoyancy and inertia balance each other.

Particles can settle out of suspension (*sedimentation*) or be *entrained* into suspension. This means the concentration of the particles in the suspension changes. So the density difference between the suspension and the ambient fluid changes too.

4.1   Types of particulate flows

We model four types of particulate gravity currents. The models are ad-hoc and apply to different geophysical situations, where different assumptions may be made about the flows and the particles:

Grain suspension by fluid turbulence   Examples include pyroclastic flows , turbidity currents [5] and powder snow [6]. In such flows, the particles may be very concentrated and denser than the fluid, but they are kept in suspension by the turbulent flow of the fluid; any particles which settle quickly undergo *resuspension*.

Liquefied and fluidised flow   Examples include dense snow avalanches and *some* mudflows. In these cases, the flow is dominant and the particles move with the flow; in this regime, particles do not interact with each other (*cohesionlessness*), as they are separated from each other (*grain dispersion*)). Their effect is to increase the viscosity of the fluid. However, this limits the possibility of turbulence, and so particles can settle out at the base of the fluid.

Dry grain flow and interactions   Examples include sand dune avalanches and rock slides. Now the particles interact with each other, colliding fequently; the fluid acts as a lubricant.

Grain-supported matrix   Examples include debris flows and lahars, or large boulders suspended in a muddy matrix. Now cohesion has a large rôle to play. The flow of mud and slurries are non-Newtonian. Effects of cohesion, friction, pore pressure, etc. may be modelled using other stress-shear rate relationships (*rheological models*), including power laws, Bingham plastic models nad Herschel-Bulkley models. In the latter two models, the fluid has a *yield strength*: a minimum shear stress must be applied before any flow occurs.

---

[5]Wikipedia: A turbidity current is a current of rapidly moving, sediment-laden water moving down a slope through water, or another fluid. The current moves because it has a higher density than the fluid through which it flowsthe driving force of a turbidity current derives from its sediment, which renders the turbid water denser than the clear water above. The deposit of a turbidity current is called a turbidite.

[6]Wikipedia: Freshly fallen, uncompacted snow. The density and moisture content of powder snow can vary widely; snowfall in coastal regions and areas with higher humidity is usually heavier than a similar depth of snowfall in an arid or continental region. Light, dry (low moisture content, typically 47% water content) powder snow is prized by skiers and snowboarders. It is often found in the Rocky Mountains of North America and in most regions in Japan.

4.2 Modelling a particulate gravity current

4.3 Physics within a particle gravity current

A particle gravity current can be modelled as having a main body and a base. In the main body, there is a well-mixed suspension of particles in a turbulent flow; in the base, the particles and fluid have low velocity and particles are settling out.

Let

- ρ_0 denote the density of the fluid;
- μ_0 denote the viscosity of the fluid;
- ρ_p denote the density of the particles;
- ϕ denote the concentration of particles.

Then the *bulk density* of the fluid-particle mixture is

$$\rho_l = \rho_0 + \phi(\rho_p - \rho_0) \tag{41}$$

and the *reduced gravity* is

$$g' = g\frac{\rho_l - \rho_0}{\rho_0} = \phi g\frac{\rho_p - \rho_0}{\rho_0} = \phi g'_0 \tag{42}$$

where

$$g'_0 = g\frac{\rho_p - \rho_0}{\rho_0}. \tag{43}$$

4.4 Sedimenting particle-laden flows

We now assume that particles are small compared to the scale of motion, so that the presence of an individual particle has a negligible effect on the bulk flow. However, the collective of particles will have an effect. We also assume that particles sediment slowly, over a timescale much larger than that of the flow; and that particles are numerous and well-mixed, so that we can talk of the 'concentration' of particles, rather than having to study individual particles.

Although we use the word 'particles', this theory could just as well apply to bubbles or to droplets; a bubble can be regarded as a particle with density negligible compared to that of the fluid; the reduced gravity is negative.

The suspension changes the bulk properties of the fluid. As discussed, the suspension has density

$$\rho_l = \rho_0 + \phi(\rho_p - \rho_0). \tag{44}$$

The viscosity of the suspension is given by a power law

$$\mu = \mu_0 \left(1 - \frac{\phi}{\phi_{max}}\right)^{-n\phi_{max}} \tag{45}$$

where ϕ_{max} is the maximum suspension concentration that can be supported. For spherical particles, $n = \frac{5}{2}$.

Particles settle towards the bottom at some settling velocity u_s which we will determine. Let D denote the particle diameter. We define the *particle Reynolds number*

$$Re_p = \frac{\rho_l D u_s}{\mu} \tag{46}$$

which governs many properties of the flow.

When particles sediment out of suspension, we are left with a *clarified* fluid, devoid of particles, and a concentrated suspension at the bottom where all the particles gather. Sedimenting flows are classified into four types:

- Type I: Free settling of individual particles

- Type II: Flocculant settling: Coaleescence of particles

- Type III: Hindered (zone) settling: Restricted, fluid motion

- Type IV: Compression settling: Mechanical support

We will discuss Types I and III in detail.

4.4.1   Type I: Free settling of individual particles

This occurs when particles are very well-separated from each other, and may be treated independently of each other.

The particle Reynolds number   When a sphere of diameter D moves at a speed u_s through otherwise unmoving fluid, its behaviour depends on the particle Reynolds number $Re_p = \frac{\rho_l D u_s}{\mu}$. If $Re_p \ll 1$ then the flow past the sphere is *laminar*; the sphere travels smoothly. But if $10^3 \ll Re_p \ll 2 \cdot 10^5$ then the flow is in the *inertial regime*; the sphere travels roughly, and the boundary layer is turbulent.

Various other regimes are possible for different values of Re_p; they are discussed in Middleton and Southard (1984).

The drag coefficient   By dimensional analysis, the drag F_D on a particle travelling at u_s through a fluid must be proportional to $\frac{1}{2}\rho_l u_s^2 A_p$:

$$F_D = C_D \frac{1}{2}\rho_l u_s^2 A_p \tag{47}$$

where A_p is the planar area of the particle. (See Prandtl and Tietjens, 1957 for details.) The dimensionless coefficient C_D is called the *drag coefficient*, and depends on the Reynolds number. The dependence may be found empirically:

- When $Re_p \ll 1$, then C_D is inversely proportional to Re_p:

$$C_D = \frac{24}{Re_p} \tag{48}$$

- When $0.2 < Re_p < 10^3$, there is a transitional region.

- When $10^3 < Re_p < 2 \cdot 10^5$, C_D is approximately constant for cylinders, spheres and discs. For these shapes, $C_D \approx 0.44$.

- At $Re_p \approx 2 \cdot 10^5$, the drag coefficient drops very quickly before increasing much more slowly again. This is called the *drag crisis regime*, and arises because of a flow separation.

The settling velocity   So what is the settling velocity u_s? We consider the momentum balance on a particle in suspension. The forces acting on the particle are drag, buoyancy and gravity, and so:

$$m\frac{du}{dt} = F_g - F_b - F_D \tag{49}$$

$$F_D = C_D \frac{1}{2}\rho_l u^2 A_p \tag{50}$$

$$F_b = \rho_l g V \tag{51}$$

$$F_g = \rho_p g V \tag{52}$$

where V is the volume of the particle. The expression for F_b is given by Archimedes' law: the buoyancy force is g times the mass of fluid displaced. (Note that the fluid displaced is assumed to have density

ρ_l rather than ρ_0, even though we are assuming that particles are well-separated from each other and do not interact!) Setting $\frac{du}{dt} = 0$ and solving for u gives us the terminal velocity, which is the settling velocity.

For a sphere of diameter D, we have $V = \frac{1}{6}\pi D^3$ and $A_p = \frac{1}{4}\pi D^2$, and so

$$u_s = \sqrt{\frac{4g(\rho_p - \rho_l)D}{3C_D\rho_l}}. \tag{53}$$

Recall that in the Stokes regime $Re_p < 0.2$, we have $C_D = \frac{24}{Re_p}$. Hence:

$$F_D = 3\pi u_s \mu D \tag{54}$$

$$u_s = \frac{g(\rho_p - \rho_l)D^2}{18\mu} \tag{55}$$

with $u_s \propto D^2$ and dependent on μ.

But in the inertial, turbulent regime, with $10^3 < Re_p < 2 \cdot 10^5$, we have that $C_D = 0.44$ is constant, and

$$F_D = 0.055\pi\rho_l u_s^2 D^2 \tag{56}$$

$$u_s = 1.74\sqrt{\frac{g(\rho_p - \rho_l)D}{\rho_l}} \tag{57}$$

with $u_s \propto D^{1/2}$ and not dependent on μ.

These predictions are confirmed by observations: the Stokes regime holds for fine grains such as silt, whereas the inertial regime holds for very coarse sand, granules and pebbles (provided that particle-particle interactions may be neglected). However, for D between 0.1mm and 1mm (such as for medium or coarse sand), neither the Stokes nor the turbulent predictions hold well. Many equations have been proposed to describe the intermediate region, including the *Ferguson-Church equation*. [7]

Advection and diffusion of particles   The concentration of particles ϕ is governed by an advection-diffusion equation:

$$\frac{\partial\phi}{\partial t} + \mathbf{u} \cdot \nabla\phi = D\nabla^2\phi \tag{58}$$

where D now denotes a diffusion coefficient, *not* the particle diameter. The particle velocity $\mathbf{u}$ is equal to the velocity of the fluid, plus the settling velocity:

$$\mathbf{u} = (u, v, w) + (0, 0, -u_s). \tag{59}$$

In the particular case of no background fluid flow, $\mathbf{u} = (u, v, w)$ and the advection-diffusion equation reduces to

$$\frac{\partial\phi}{\partial t} - u_s\frac{\partial\phi}{\partial z} = D\nabla^2\phi \tag{60}$$

The diffusion coefficient D depends on Re_p and whether the flow is laminar or turbulent. In the laminar regime, diffusion of particles is due to molecular (Brownian) diffusion:

$$D = D_B \sim \frac{kT}{6\pi\mu r} \tag{61}$$

where r is the radius of particles, k is the Boltzmann constant and T is the temperature. Molecular diffusion is a very slow process unless r is very small, such as for aerosol particles.

However, in the turbulent regime, diffusion of particles is due to turbulent diffusivity:

$$D = D_t \sim u^\star h \tag{62}$$

where $u^\star = \sqrt{\frac{\tau_b}{\rho}}$ is the shear velocity (friction velocity), τ_b is the shear stress at the bottom, and h is some length scale, such as the distance from the bottom.

---

[7]See `http://hinderedsettling.com/2013/08/09/grain-settling-python/`.

4.4.2 Type III: Hindered (zone) settling

When there are many particles, then particles no longer settle at u_s but at some slower velocity, u_h, which depends on ϕ and on u_s.

5 Jets, plumes and thermals