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LETTERS

A constitutive law for dense granular flows

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A continuum description of granular flows would be of considerable help in predicting natural geophysical hazards or in designing industrial processes. However, the constitutive equations for dry granular flows, which govern how the material moves under shear, are still a matter of debate[1–10]. One difficulty is that grains can behave[11] like a solid (in a sand pile), a liquid (when poured from a silo) or a gas (when strongly agitated). For the two extreme regimes, constitutive equations have been proposed based on kinetic theory for collisional rapid flows[12], and soil mechanics for slow plastic flows[13]. However, the intermediate dense regime, where the granular material flows like a liquid, still lacks a unified view and has motivated many studies over the past decade[14]. The main characteristics of granular liquids are: a yield criterion (a critical shear stress below which flow is not possible) and a complex dependence on shear rate when flowing. In this sense, granular matter shares similarities with classical visco-plastic fluids such as Bingham fluids. Here we propose a new constitutive relation for dense granular flows, inspired by this analogy and recent numerical[15,16] and experimental work[17–19]. We then test our three-dimensional (3D) model through experiments on granular flows on a pile between rough sidewalls, in which a complex 3D flow pattern develops. We show that, without any fitting parameter, the model gives quantitative predictions for the flow shape and velocity profiles. Our results support the idea that a simple visco-plastic approach can quantitatively capture granular flow properties, and could serve as a basic tool for modelling more complex flows in geophysical or industrial applications.

Advances in our understanding of dense granular flows have recently been made by comparing different flow configurations[14]. The simplest configuration from a rheological point of view is the one sketched in the inset to Fig. 1. A granular material confined under a normal stress P in between two rough planes is sheared at a given shear rate $\dot{\gamma}$ by applying a shear stress τ. In refs 15 and 16, for stiff particles the shear stress is shown, using dimensional arguments and numerical simulations, to be proportional to the normal stress, with a coefficient of proportionality that is a function of a single dimensionless number, called the inertial number I:

$$\tau = \mu(I)P \text{ with } I = \dot{\gamma} d/(P/\rho_s)^{0.5} \qquad (1)$$

where $\rho(I)$ is the friction coefficient, d is the particle diameter and μ_s is the particle density. They found that the volume fraction Φ of the sample is also a function of I but varies only slightly in the dense regime. The inertial number, which is the square root of the Savage number[20] or of the Coulomb number[21] introduced previously in the literature, can be interpreted as the ratio between two timescales, a macroscopic deformation timescale ($1/\dot{\gamma}$) and an inertial timescale $(d^2\rho_s/P)^{0.5}$. By confronting results from the simple shear test with experimental measurements of granular flows on rough inclined planes[17,22], it can be shown that the friction coefficient $\mu(I)$ has the shape given in Fig. 1. It starts from a critical value of μ_s at zero shear rate and converges to a limiting value of μ_2 at high I. The following friction law can then be proposed, compatible with the experiments[19]:

$$\mu(I) = \mu_s + (\mu_2 - \mu_s)/(I_0/I + 1) \qquad (2)$$

where I_0 is a constant. Very recently, this simple description of granular flows has been successful in predicting two-dimensional configurations, capturing velocity profiles on inclined planes[14,23] and important features of flows on a pile[19]. However, the simple scalar law (equation (1)) cannot be applied in more complex flows where shear in different

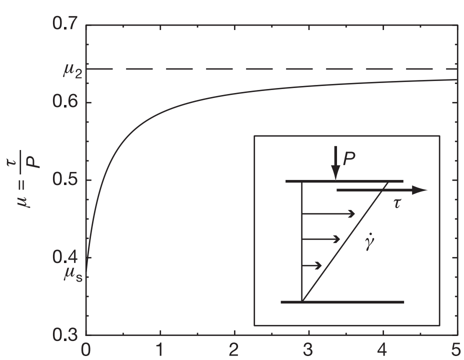

Figure 1 | Friction coefficient μ as a function of the dimensionless parameter I ($\mu_s = \tan(20.9)$, $\mu_2 = \tan(32.76)$ and $I_0 = 0.279$). Inset, definition of the pressure P, the shear stress τ, and the shear rate $\dot{\gamma}$ in the simple plane shear configuration.

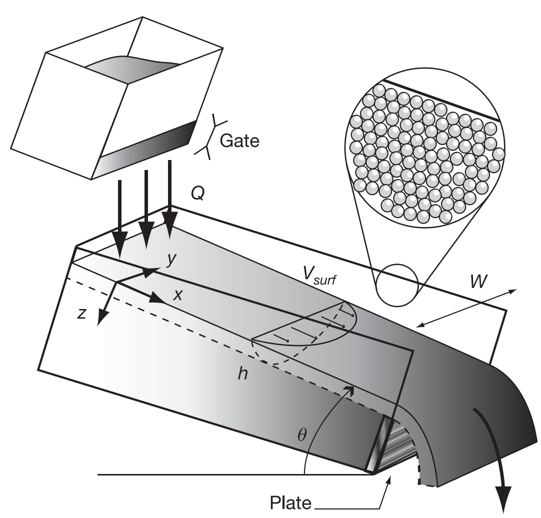

Figure 2 | Experimental set-up of granular flows on a pile between rough sidewalls. The channel is partially closed at the bottom end to create a static pile on top of which the grains flow[19]. The sidewalls are made rough by gluing one layer of beads on them. The channel is 120 cm long and the width W varies from 0.9 cm (16.5d) up to 28.9 cm (546d).

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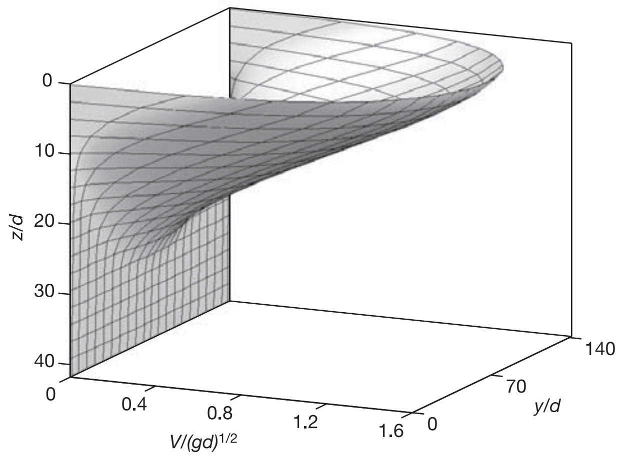

Figure 3 | Typical 3D velocity profile predicted by the rheology ($W = 142d$, $\theta = 22.6°$, $Q/d^{3/2}g^{1/2} = 15.2$). For clarity only one quarter of the lines of the 71 × 80 computational grid is plotted.

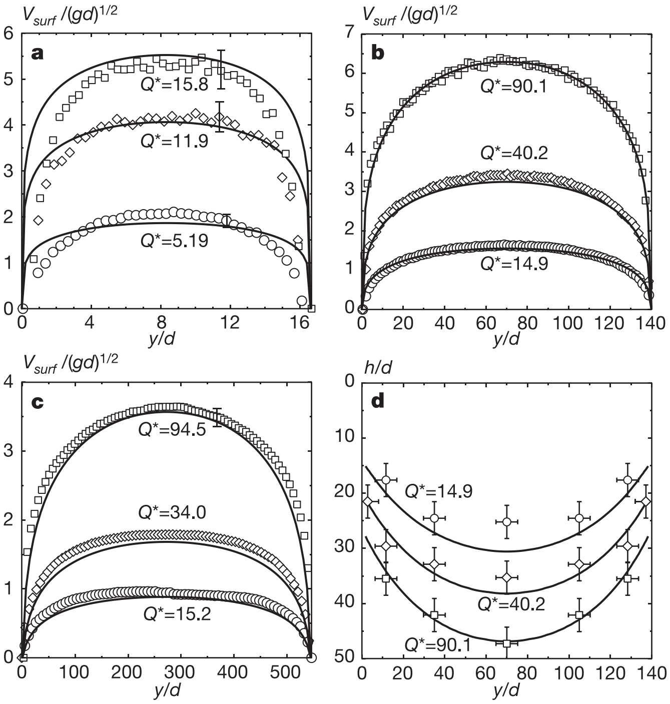

Figure 4 | Comparison of 3D simulations (lines) and experimental results (symbols) for different flow rates ($Q^* = Q/d^{3/2}g^{1/2}$). a, b, c, Free-surface velocity profiles for channel width $W = 16.5d$ (**a**), $W = 140d$ (**b**) and $W = 546d$ (**c**). **d**, Depths of the flowing layer across the channel for $W = 140d$. The experimental and computational flow rates are equal within 2.5%. The error bars represent the dispersion of the measurements for different experiments.

directions is present and where a full three-dimensional rheology is needed.

We therefore propose the following 3D generalization of the friction law for a granular material. The basic assumption consists in neglecting the small variation of the volume fraction observed in the dense regime. The granular material is then described as an incompressible fluid with the internal stress tensor given by the following relations:

$$\sigma_{ij} = -P\delta_{ij} + \tau_{ij} \text{ and } \tau_{ij} = \eta(|\dot\gamma|, P)\dot\gamma_{ij}$$
$$\text{with } \eta(|\dot\gamma|, P) = \mu(I)P/|\dot\gamma| \text{ and } I = |\dot\gamma|d/(P/\rho_s)^{0.5} \qquad (3)$$

where $\dot\gamma_{ij} = \partial u_i/\partial x_j + \partial u_j/\partial x_i$ is the strain rate tensor and $|\dot\gamma| = (0.5\dot\gamma_{ij}\dot\gamma_{ij})^{0.5}$ is the second invariant of $\dot\gamma_{ij}$. In this rheology, P represents an isotropic pressure, and $\eta(|\dot\gamma|, P)$ is an effective viscosity, which definition is related to the friction coefficient $\mu(I)$ (equation (2)). An important property of the proposed constitutive law is that the effective viscosity diverges to infinity when the shear rate goes to zero. This divergence ensures that a yield criterion exists. Looking at equation (3) in the limit of $|\dot\gamma|$ going to zero, we can show that the material flows only if the following condition is satisfied:

$$|\tau| > \mu_s P \text{ where } |\tau| = (0.5\tau_{ij}\tau_{ij})^{0.5} \qquad (4)$$

The yield criterion then takes the form of a Drucker–Prager-like criterion[24]. Below the threshold, the medium behaves locally as a rigid body. It is interesting to note that within this framework, the granular media can be viewed as a visco-plastic fluid[25]. The specificity compared to classical Bingham or Herschel–Bulkley fluids is that the effective viscosity depends both on the shear rate and on the local pressure. This property is linked to the frictional nature of stresses in granular media.

To test this rheology we performed experiments of granular flows on a heap as sketched in Fig. 2. This set-up is similar to our previous study[19] except that here sidewalls are made rough by gluing one layer of beads on them. This imposes a well-defined no-slip boundary condition at the walls. This configuration represents a severe test for the model, since it gathers in a single configuration several specificities of granular flows. First, when grains are released from the hopper, a steady regime is reached with a strongly sheared layer flowing on top of a static zone. The slope and the thickness of the flowing layer are selected by the system. Second, owing to the rough sidewalls used here, a significant shear exists also in the transverse direction, the flow pattern being thus fully three-dimensional. The experiments are carried out using glass beads 0.53 mm in diameter and the two control parameters are the width W of the channel and the flow rate per unit of width Q. The present study focuses on the steady and uniform regime characterized by a constant slope and a velocity aligned along the x direction and invariant along the flow (a tiny y component can be observed close to the wall, which remains 20 times smaller than the stream-wise velocity). We performed systematic measurements of the free-surface inclination θ, of the free-surface-velocity profile $V_{\mathrm{surf}}(y)$ using particle-imaging velocimetry, and we get estimates of the thickness of the flowing layer $h(y)$ using an erosion method[19].

To compare the experimental results with the predictions of the local rheology, we perform numerical simulations of a granular fluid described by the constitutive law equation (3) and flowing in an inclined U-shaped channel with a no-slip boundary condition at the three walls. The velocity $u(y,z)$ is assumed to be aligned with x and to depend only on y and z. To get the 3D steady velocity profile, we solve the incompressible Navier–Stockes equations with the internal stress being given by equation (3) using a finite difference scheme. For the rheological parameters μ_s, μ_2 and I_0 coming into play in equation (2), we choose the values given by the experimental data of flows on the inclined planes[18] where the same particles were used (see ref. 19 for how to compute these parameters): $\mu_s = \tan(20.9)$, $\mu_2 = \tan(32.76)$ and $I_0 = 0.279$. This choice means that no fitting parameter will exist when we compare results from the simulations to the experimental data. A typical velocity profile obtained by the model is shown in Fig. 3. We first observe that a static zone develops at the base of the channel. The limit of the static zone varies across the channel, the flowing layer being larger in the centre than close to the walls. The second observation is that the velocity profile is truly 3D and sheared in both y and z directions.

We then tried to quantitatively compare the velocity profiles predicted by the simulations with the ones measured experimentally. In the simulation we impose the inclination and compute the flow

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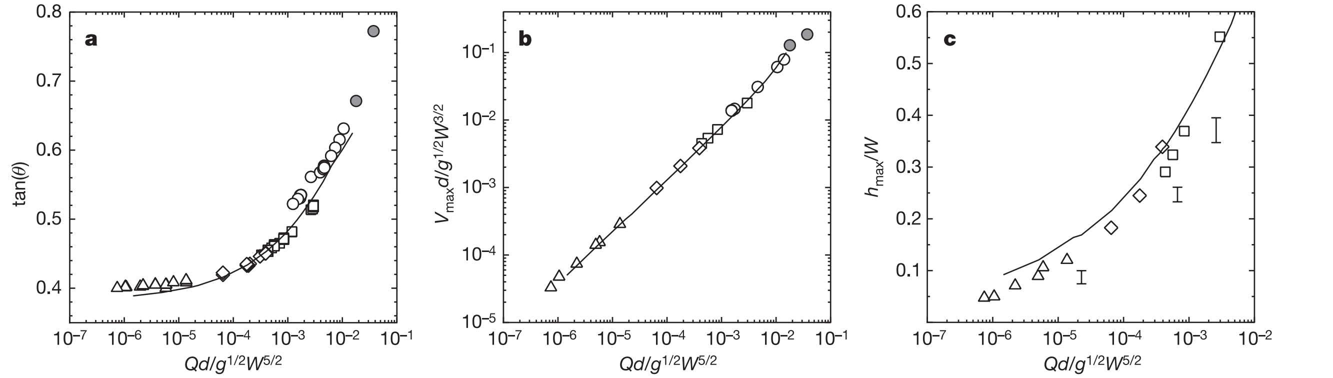

Figure 5 | Scaling laws for the experimental measurements (symbols) compared to the predictions of the model (lines). Free-surface inclination $\tan(\theta)$ (**a**), rescaled maximum free-surface velocity $V'_{\max}$ (**b**) and maximum flowing thickness $h_{\max}/W$ (**c**), as a function of the rescaled flow rate Q'. Data collapse onto single curves for all flow rates and all widths: $W = 16.5d$ (circles), $W = 55d$ (squares), $W = 140d$ (diamonds), $W = 546d$ (triangles). In **c**, the error bars refer to the dispersion of the measurements for each width (the dispersions are smaller than the size of the symbols in **a** and **b**). The grey circles (**a**, **b**) refer to measurements above the critical angle $\theta_2 = \arctan\mu_2$. In this case, no steady uniform flow is predicted by the model. Experimentally, we observe a transition to a collisional regime (see Supplementary Information and Supplementary Movies).

rate *a posteriori*, whereas in the experiments, the flow rate is controlled and the inclination is measured. Figures 4a-c show the free-surface velocity profiles obtained in the experiments and simulations for different widths and different flow rates. The experimental data are the symbols and the continuous lines are the prediction of the 3D rheology. The agreement is good and quantitative. A slight deviation between experiment and model is observed in the narrower channel, 16.5 particle-diameters wide. In Fig. 4d, we also compare the prediction of the theory for the thickness of the flowing layer. In both theory and experiments, the flowing layer is thicker in the centre than at the walls. A quantitative agreement is again observed, although the simulation systematically overestimates the flowing layer thickness. This could be due to the not-very-precise erosion method used for estimating the thickness. All these results show that the proposed rheology gives quantitative predictions for this complex 3D flow, a striking success for a model that has been entirely calibrated based on a different flow configuration.

We have systematically carried out experiments for a wide range of flow rates and channel widths, and within 15% a quantitative agreement is always observed. To compare experiments and simulations in a systematic way, it is interesting to notice that simple scalings can be predicted from the rheology proposed. It is easy to show analytically that one can get rid of the width of the channel in the equations of motion by using the following dimensionless variables: $z' = z/W$, $y' = y/W$, $V' = Vd/g^{0.5}W^{1.5}$ and $Q' = Qd/g^{0.5}W^{2.5}$ (see Supplementary Information). It follows that the inclination of the pile θ, the maximum velocity in the centreline of the channel $V'_{\max}$, and the maximum flowing thickness $h'_{\max}$ should all depend only on Q'. In Fig. 5, we show that the experimental measurements follow the predicted scaling and that the numerical simulations (continuous lines in Fig. 5) give quantitative predictions. One interesting result of this scaling analysis is that the thickness h scales with the width W, meaning that neither the thickness of the flowing layer nor the shear rate are intrinsic properties of the granular media but are controlled by the width of the channel and the flow rate.

We conclude that the simple visco-plastic constitutive law proposed seems to describe dense granular flows very well. Once calibrated on the inclined plane configuration, the model quantitatively captures the complex 3D sheared flow observed when grains flow in between two rough walls. Limits of the approach exist that mainly concern the yield criterion. Within the proposed constitutive law, the flow threshold is simply described by a Coulomb criterion. However, the transition between solid-like and liquid-like behaviour in granular matter seems much more complex, involving shear bands[26,27], intermittent flows[28] and hysteretic phenomena[29,30]. Such features should be included in a more comprehensive rheology. However, we believe that the simple visco-plastic rheology presented here represents a minimal model that quantitatively captures the basic features of granular flows important in many applications. We think this model could help to take into account more accurately the complex yield features specific to granular matter.

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Supplementary Information is linked to the online version of the paper at www.nature.com/nature.

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