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SEDIMENTATION OF NONCOLLOIDAL PARTICLES AT LOW REYNOLDS NUMBERS

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1. INTRODUCTION

Sedimentation, wherein particles fall under the action of gravity through a fluid in which they are suspended, is commonly used in the chemical and petroleum industries as a way of separating particles from fluid, as well as a way of separating particles with different settling speeds from each other. Examples of such separations include dewatering of coal slurries, clarification of waste water, and processing of drilling and mining fluids containing rock and mineral particles of various sizes. The separation of different particles by sedimentation is also the basis of some laboratory techniques for determining the distribution of particle sizes in a particulate dispersion.

Owing to the significance of the subject, there have been numerous experimental and theoretical investigations of the sedimentation of particles in a fluid. One of the earliest of these is Stokes' analysis of the translation of a single rigid sphere through an unbounded quiescent Newtonian fluid at zero Reynolds number, which led to his well-known law

$$u^{(0)} = \frac{2a^2(\rho_s - \rho)g}{9\mu}, \tag{1.1}$$

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where $u^{(0)}$ is the settling velocity of the sphere, a is its radius, ρ_s is its density, ρ is the fluid density, μ is the fluid viscosity, and g is the gravitational constant. Since then, research has focused on extending Stokes' law by considering nonspherical rigid particles, drops and bubbles, non-Newtonian fluids, nonzero Reynolds numbers, the presence of a wall near the particle, and interactions between particles.

In this paper, we do not attempt to present a complete review of the general subject of sedimentation, but rather we discuss recent developments in three areas:

1. *The sedimentation of monodisperse suspensions.* Here we focus on the theoretical and experimental determination of the average settling velocity of identical spherical particles in a suspension that is not infinitely dilute.
2. *The sedimentation of polydisperse suspensions.* For suspensions containing more than one particle species (i.e. different sizes and densities), we discuss recent studies that deal with the prediction of the local average settling velocity of a spherical particle in the midst of unlike spheres, and also with the macroscopic description of the sedimentation process as particles with different fall speeds separate from one another.
3. *Enhanced sedimentation in inclined channels.* The curious observation that the clarification rate of a suspension is increased merely by inclining a settling vessel from the vertical has recently been analyzed using the appropriate equations of continuum mechanics. We summarize the findings for the flow profiles and sedimentation rates as well as the results of a linear stability analysis used to predict the formation and growth of waves in inclined settling channels; these waves lead to a decrease in the efficiency of such processes.

Before proceeding with our review, we wish to call attention to several other articles that have appeared in this series in recent years that focus on properties of particulate suspensions other than sedimentation. Batchelor (1974) reviewed the effective transport properties of two-phase materials with random structure and included a description of the sedimentation of a dilute dispersion of equal spheres. Herczyński & Pieńkowska (1980) discussed the statistical theory of a suspension and, in particular, its rheological properties. Leal (1980) considered general particle motions in a viscous fluid in the presence of small inertia effects or small non-Newtonian effects, or when the particles (or bubbles) are slightly nonspherical. The Brownian motion of small colloidal particles suspended in liquids was reviewed by Russel (1981). In general, Brownian motion is only significant for particles that are so small that sedimentation due to gravity is negligibly slow, but circumstances exist where both effects are important. Finally, Schowalter (1984) discussed the stability and coagulation of particulate dispersions subjected to shear flows. Included in his review is a discussion of the interparticle forces, such as attractive van der Waals forces and repulsive electric double-layer forces, that characterize colloidal dispersions.

2. SEDIMENTATION OF MONODISPERSE SUSPENSIONS

We consider a suspension composed of rigid spherical particles of equal size and density sedimenting in a Newtonian fluid at very small particle Reynolds numbers. For typical hydrosol dispersions, the Reynolds number $\rho u^{(0)}a/\mu$ is small compared with unity when the spheres are less than 0.1 mm in diameter. Furthermore, let us suppose that the suspension is stable, i.e. that any Brownian motion and attractive van der Waals forces are too weak to cause the suspension to coagulate [see Spielman (1970) and Valoulis & List (1984) for detailed criteria for the onset of coagulation of suspended particles]. Then, an initially well-mixed suspension will separate into three regions when subjected to batch sedimentation in a vessel with vertical sidewalls. A layer of clarified fluid will form at the top of the settling vessel, whose depth increases with time as the particles settle. Below this layer lies the suspension region, sometimes referred to as the *clarification zone*. Since the spheres are identical, the interface between the clear fluid and the suspension will remain fairly sharp, though it may spread somewhat owing to particle diffusion. If the suspension is infinitely dilute, the particles settle with their Stokes velocity given by (1.1). On the other hand, for particle volume fractions as small as 1%, the average settling velocity of the spheres is noticeably lower than that given by Stokes' law. This phenomenon can be represented by a hindered settling function $f(c)$ such that the average fall velocity of a sphere in the suspension is given by $v = u^{(0)}f(c)$. It is generally assumed that $f(c)$ depends only on the solids volume fraction c and that it is a monotonically decreasing function with $f(0) = 1$. However, for colloidal dispersions, this correction to Stokes' law will also depend on the nature of any interparticle forces present. Strong double-layer repulsive forces will tend to keep the particles well separated, and hence will decrease the average fall speed—whereas strong attractive van der Waals forces will lead to an excess of spheres that are close partners to another sphere, and hence will increase the average settling velocity of the suspension. It is conceivable that if the attractive forces are sufficiently strong, $f(c)$ may exceed unity; however, it is expected that in this case the suspension would be unstable, and that the increased settling rate would then be accompanied by the formation of doublets, triplets, etc.

The third region within the settling vessel is a sediment layer that forms on the bottom of the vessel. Often it is assumed that c_s, the solids concentration in this layer, is a constant and is equal to the maximum random packing density of the spheres, which is about 0.6. In reality, the solids gradually compact in this zone as the liquid in the interstices between the spheres is slowly squeezed out. Thus, the sediment layer at the bottom of the vessel is sometimes called the *compression zone* or *thickening zone*. Mathematical models of the gravity thickening process have been reviewed by Dixon (1979). In general, the settling or compaction rate of the particles depends not only on the local solids concentration in the sediment but also on the concentration gradient, the depth of the sediment layer, and the interparticle forces between the closely packed spheres.

Discontinuities in the solids volume fraction c occur at the interfaces that separate the suspension from the clear fluid at the top and from the sediment layer at the bottom of the vessel. If the suspension is sufficiently concentrated, gradients or discontinuities in the solids volume fraction may also appear within the interior of the suspension. The well-known kinematic-wave theory describing this phenomenon was first given by Kynch (1952), and it is not repeated here. A cohesive account of the theory and detailed criteria for the presence of internal concentration gradients can be found in the book by Wallis (1969). Recent developments include the application of one-dimensional kinematic-wave theory to settling vessels with nonconstant cross-section (Baron & Wajc 1979, Schneider 1982) and also the modification of Kynch's theory of sedimentation in order to account for the layer of compacting sediment that forms at the vessel bottom (Fitch 1983).

Application of kinematic-wave theory to sedimentation, as well as the design of both batch and continuous settling vessels, requires knowledge of the hindered settling function $f(c)$. Thus, considerable effort has been expended in the past to determine this function either experimentally or theoretically.

Empirical Formulae for the Hindered Settling Function

The results of many sedimentation and fluidization experiments reported in the literature have been summarized by Barnea & Mizrahi (1973) and Garside & Al-Dibouni (1977). The former authors proposed that the semiempirical formula

$$f(c) = \frac{(1-c)^2}{(1+c^{1/3})\exp(5c/3(1-c))} \tag{2.1}$$

provides the best fit to existing experimental data for very small particle Reynolds numbers. On the other hand, the most commonly used empirical

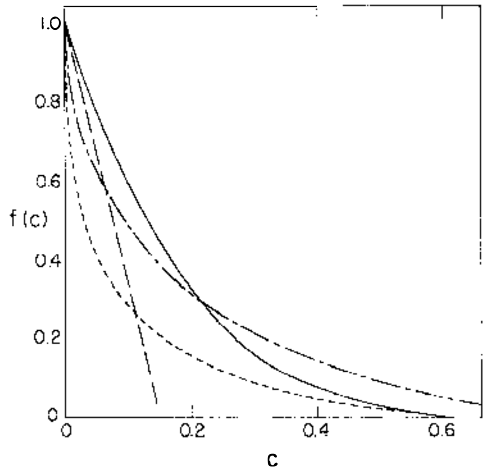

Figure 1 The hindered settling function $f(c)$; ——— the empirical correlation given by Equation (2.2); — — — the empirical correlation given by Equation (2.1); — — Batchelor's (1972) theory for dilute suspensions; ---- the exact theoretical results of Sangani & Acrivos (1982) for simple cubic arrays.

correlation is that attributed to Richardson & Zaki (1954),

$$f(c) = (1-c)^n, \tag{2.2}$$

where, according to Garside & Al-Dibouni (1977), a value of $n = 5.1$ most accurately represents their data for small Reynolds number. These two functions are plotted in Figure 1 for comparison, and it is evident that their behavior is quite different, especially for small c. In the dilute limit, Equation (2.1) has the form

$$f(c) \sim 1 - c^{1/3}, \tag{2.3}$$

whereas Equation (2.2), with $n = 5.1$, becomes,

$$f(c) \sim 1 - 5.1c. \tag{2.4}$$

The experimental data reported by Garside & Al-Dibouni (1977) generally fall between the predictions of (2.1) and (2.2).

Theoretical Formulae for the Hindered Settling Velocity

We now direct our attention to the theoretical models that have been proposed for computing the settling function $f(c)$. Among the earlier and more widely used are the so-called "cell models" or "self-consistent" schemes, which involve solving the creeping-flow equations within a fluid cell (typically assumed spherical) encasing a representative particle. The ratio of the particle volume to cell volume is set equal to the volume fraction of the particles in the suspension. One then obtains the formula $f(c) = 1 - \beta c^{1/3}$, but since the numerical value of the coefficient β depends on an ad hoc choice of the cell shape and boundary conditions on the surface of this cell, the reliability of such models is open to question.

From a fundamental point of view, it is therefore preferable to compute $f(c)$ via an exact analysis of the multiparticle flow problem. Unfortunately, owing to the complexity of the mathematical system, the analysis has been restricted so far to dilute systems and is therefore of limited usefulness, although it has shed light on the reasons behind the discrepancy between (2.3) and (2.4). To begin with, when $c \rightarrow 0$, it is natural to suppose that for the purpose of computing their influence on a representative or test particle, it suffices to replace all the other particles in the suspension by point singularities such as point forces, point dipoles, etc. The analysis is complicated, however, by the fact that in a system containing an infinite number of particles, the resulting sums or integrals are often divergent. This difficulty can be avoided in a number of ways; at present, the most popular is the renormalization technique first introduced by Batchelor (1972) in his analysis of the sedimentation of a statistically homogeneous, dilute suspension of monodisperse spheres. This technique has been elaborated on by Jeffrey (1974) and O'Brien (1979), and its salient features have also been summarized by Batchelor (1974). This method has been modified recently by Feuillebois (1984) in order to investigate the sedimentation of monodisperse spheres in a dilute suspension that is homogeneous in any horizontal plane but in which a vertical concentration profile is prescribed. Another approach is that of Hinch (1977), who constructs a hierarchy of equations describing the behavior of a two-phase macroscopically homogeneous material and shows how bulk parameters such as the sedimentation velocity can be computed in a systematic way. Another method is to use Fourier integrals and generalized functions (Saffman 1973). At any rate, it is generally accepted that the earlier difficulties involving nonabsolutely convergent sums or integrals can be overcome, and that, in principle at least, it should be possible to compute the settling function $f(c)$ from first principles.

Even in dilute systems, however, the departure of $f(c)$ from unity depends on the configuration of the particles. Saffman (1973), who appears to have been the first to discuss this point in detail, considers three distinct possibilities: (I) regular periodic arrays of particles, (II) random fixed arrays, and (III) random free arrays. In a regular array, the force and the velocity are the same for each particle and, as first shown by Hasimoto (1959) and in more detail by Sangani & Acrivos (1982), $f(c) \sim 1 - \beta c^{1/3} + O(c)$, where β is a constant whose value depends on the assumed lattice configuration, e.g. $\beta = 1.76$ for a simple cubic lattice. On the other hand, for case II, first studied by Brinkman (1947), the corresponding expression for f is, according to Hinch (1977),

$$f(c) \sim 1 - \frac{3}{\sqrt{2}} c^{1/2} - \frac{135}{64} c \ln c - 12.0c + \ldots .$$

The most relevant case for sedimentation is, however, that of random free arrays, but here the probability density function $P(\mathbf{r})$, which denotes the probability of finding a sphere with its center at $\mathbf{r}$ given that there is a sphere with its center at the origin, is not known a priori and can vary from system to system, depending on the conditions of the experiment. As a consequence, the functional dependence of the hindered settling function on c can assume different forms, since $f(c)$ is determined in part by $P(\mathbf{r})$. A simple explanation of this dependence of the settling velocity on the structure of the suspension is as follows. The first correction to the Stokes settling velocity from the presence of a second sphere is of order $u^{(0)}a/d$, where d is the distance between the two sphere centers. In a random dispersion, for which $P(\mathbf{r})$ is a constant for $|\mathbf{r}| > 2a$ and vanishes for $0 \leq |\mathbf{r}| \leq 2a$, there is an $O(c)$ number of spheres, on the average, within a distance $d = O(a)$. Since these near neighbors exert the greatest influence, we expect the correction to Stokes' law to be of the form $f(c) \sim 1 - O(c)$, and in fact Batchelor's (1972) exact analysis yields $f(c) \sim 1 - 6.55c + O(c^2)$. If, on the other hand, the nearest neighbors are at a distance approximately equal to the average separation, i.e. $d = O(ac^{-1/3})$, and there are an $O(1)$ number of such neighbors, then we should expect that $f(c) \sim 1 - O(c^{1/3})$. Thus, which of the two empirical formulas (2.3) and (2.4) is predicted by theory depends on the assumptions made regarding the statistical structure of the suspension.

The question that needs to be resolved now is whether a sedimenting suspension would be expected to have a random or an ordered microscale structure. Certainly, if Brownian motion is strong, the suspension will remain random. Indeed, in the experiments reported by Cheng & Schachman (1955), Kops-Werkhoven & Fijnaut (1981), Buscall et al. (1982), and Tackie et al. (1983) using micron- and submicron-sized uniform colloidal spheres, the sedimentation results at low concentrations were of the form $f(c) \sim 1 - \gamma c + O(c^2)$. The observed values of γ were generally in the range 4 to 6, and the linear form of the hindered settling function was accurate provided that c was less than about 0.08. Batchelor & Wen (1982) discuss these experiments more fully and conclude that the difference between the observed values of γ and Batchelor's (1972) predicted value of $\gamma = 6.55$ could be a consequence of van der Waals attractive forces causing an excess number of close pairs whose common speed of fall exceeds the fall speed of an isolated sphere. On the other hand, for the suspensions containing larger particles listed by Barnea & Mizrahi (1973), the correction of Stokes' settling law was more closely represented by (2.3), which, from our discussion of the theory, indicates that the particles were not randomly distributed. Although Batchelor (1972) has shown that pairwise interactions alone will not give rise to the prediction of order in the suspension, this may not be the case when interactions between many particles are involved.

Lynch & Herbolzheimer (1983) recently reported the first results of a numerical simulation designed to test this possibility. Although their results suggested that near-pairs of particles are less likely to persist in a sedimenting suspension, such a conclusion is only tentative at this stage. These authors also noted on theoretical grounds, however, that if multiparticle hydrodynamic interactions do cause the particles to become well spaced, then when the suspension is sheared the microscale arrangement should be altered so as to cause the sedimentation velocity to increase. This prediction was verified in their Couette device experiments in that, at asymptotically high shear rates, the settling function was found to decrease as $1-4c$, whereas when the shear was turned off, it returned to the form $1-c^{1/3}$. These results tend to further support the notion that there exists in sedimenting suspensions a microscale "structure" whose origin is at present unknown.

3. SEDIMENTATION OF POLYDISPERSE SUSPENSIONS

The majority of theoretical and experimental investigations of the sedimentation process have focused on monodisperse suspensions, whereas in most cases of practical importance, the suspensions contain particles having a range of settling velocities due to variations in their size, shape, and density. Again, we restrict our discussion to rigid spheres that have very small particle Reynolds numbers. We also assume that the suspension does not coagulate. First, let us note, however, that in contrast to monodisperse suspensions, the particles in a polydisperse suspension move relative to one another as a result of gravity. If this relative motion brings two particles sufficiently close to one another, then they will form a permanent doublet as a result of the attractive van der Waals force acting between them. The conditions governing the stability of a sedimenting polydisperse suspension and the rate of coagulation of an unstable suspension have been presented in the recent works of Melik & Fogler (1984), Davis (1984), and Wen & Batchelor (1984).

Preliminary investigations of stable, i.e. noncoagulating, suspensions containing a small number of distinct species indicate that the particle concentration does not remain uniform. Instead, as the settling develops, the faster-falling particles move away from the others, thereby creating different regions in the interior of the suspension. The lower region, i.e. that just above the sediment layer, contains all the particle species at their initial concentrations, whereas the region immediately above it is devoid of the fastest-settling species. Each successive region contains one fewer species than the region below, with the upper region of the suspension containing only particles of the slowest-settling species (see Figure 2). In most cases, these distinct regions are separated by a near-discontinuity, or "shock," in the species concentration distribution; these shocks can be quite steep, but diffusion of the particles ultimately limits the steepening so that there is a sharp but continuous transition across each shock separating two regions of uniform concentration. For continuous distributions of particle sizes and densities, no distinct shocks develop, but rather the species concentrations vary continuously as a function of height in the settling vessel.

In describing the above behavior, we have assumed that the spatial distribution of particles in any horizontal plane is uniform, and this is indeed true for dilute suspensions. However, in the settling of bidisperse suspensions at high enough solids concentrations, the remarkable observation has been made that vertical streams develop in the suspension. Whitmore (1955) was the first to describe this phenomenon after perform-

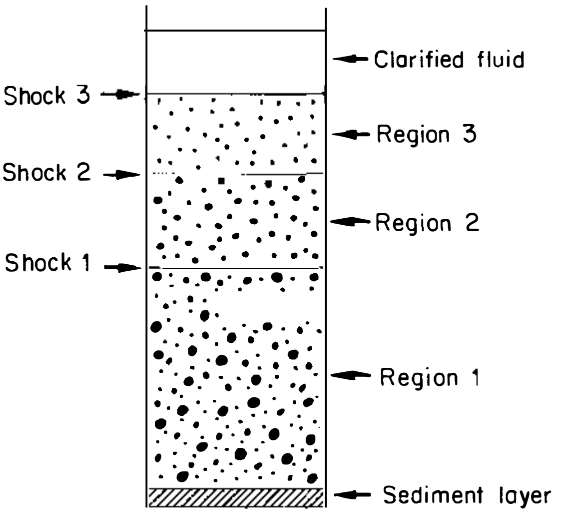

Figure 2 The regions that develop during the sedimentation of a mixture of three distinct species of particles. Region 1 contains all three species of particles, region 2 is devoid of the fastest-settling species, and region 3 contains only the slowest-settling species.

ing experiments with a suspension containing a mixture of heavy particles and neutrally buoyant particles. He reported that, within a few moments after the well-mixed suspension was allowed to stand, a lateral segregation of the particles set in that evolved into vertical fingers of a few millimeters in width. These fingers were composed of fluid and only the particle species that was present in lower concentration embedded within a continuum composed of fluid and only the other particle species. The fingers containing the neutrally buoyant particles were convected upward, whereas the streams containing the heavy particles descended toward the bottom of the vessel at a rate appreciably greater than that prevailing in the absence of the neutrally buoyant particles.

More recently, Whitmore's discovery of lateral segregation and subsequent convection in bidisperse suspensions has been explored in more detail by R. H. Weiland and coworkers. Upon adding positively buoyant particles to an otherwise uniformly settling suspension, Weiland & McPherson (1979) also observed a rapid lateral segregation of the two species of particles into vertically directed streams. This phenomenon is depicted in Figure 3. The suspending fluid is aqueous glycerol with density 1.12 g cm^{-3} and viscosity 4.95 cP, the heavy particles are polyvinylchloride spheres having a density of 1.41 g cm^{-3} and a mean diameter of 137 μm, and the buoyant particles are hollow ceramic spheres with a density of 0.595 g cm^{-3} and a mean diameter also of 137 μm. The heavy particles were dyed and appear as the light areas in the photograph, whereas fluid containing only buoyant particles is represented by the dark areas. Although the observed fingering structure develops most rapidly in systems containing buoyant and heavy particles, Weiland et al. (1984) report that fingering can also result when all the particles are more dense than the fluid but are of two distinct species, distinguishable by either density or size.

The phenomenon described above is of course fascinating from the point of view of fluid mechanics, but in fact (contrary to some of the statements that have been made in the literature) it is of relatively minor importance as a means of increasing the overall settling rate or, equivalently, of decreasing the total time required to settle the suspension. This is because, as a result of the vigorous agitation and convection brought about by the presence of the vertical streamers, a suspension containing two types of particles (say, heavy particles and neutrally buoyant ones) will quickly stratify into two layers: a bottom layer containing only the heavy particles but at a concentration c_H^* that is greater than the original concentration c_H, and a top layer with all the lighter particles. The heavy particles then continue to settle as in vertical sedimentation, and although the volume occupied by this more concentrated suspension is smaller than that of the initial suspension, the fact that $f(c_H^*) < f(c_H)$ implies that the heavy particles will settle slower than would have been the case before. Thus, the total settling time may not be greatly reduced, and under some conditions it may even increase. In fact, in a recent paper that presents both a model of the vertical streams and also quantitative results of two-species sedimentation experiments, Fessas & Weiland (1984) have reported that the settling rate of glass spheres in carbon tetrachloride can be accelerated up to sixfold upon the addition of PVC particles; however, the total settling time, which includes both the accelerated convection of the vertical streamers and the subsequent hindered settling of the concentrated heavy particles, has been observed to be reduced by no more than a factor of two (Weiland & McPherson 1979). Often the observed enhancement is considerably less than this. Finally, we note that if the total solids concentration is too low

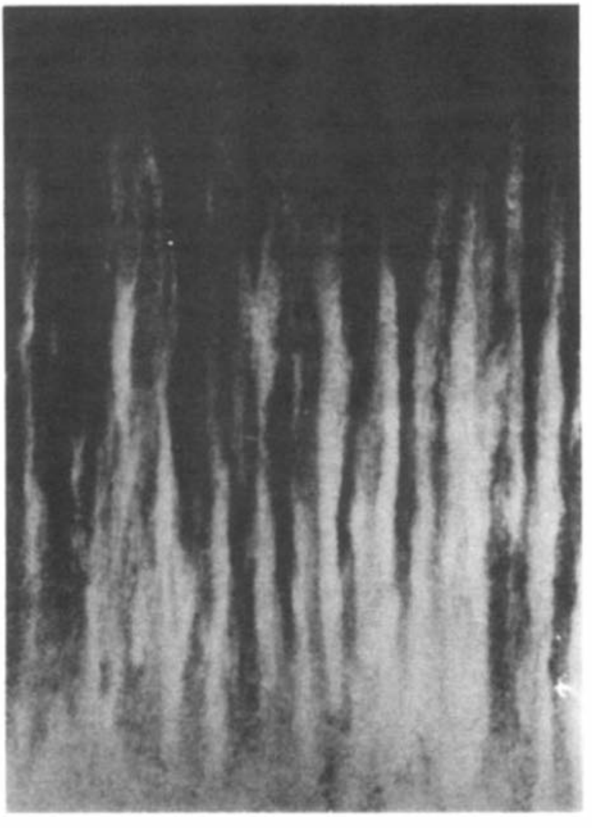

Figure 3 Vertical finger formation during sedimentation of a mixture of buoyant and heavy particles. The photograph was taken 60 seconds after mixing was completed, and the heavy particles are dyed. (From Weiland et al. 1984.)

(less than about 10% by volume), lateral segregation and fingering are not observed, and both solids components settle at reduced rates. Moreover, the cause of the lateral segregation of two species has heretofore not been addressed in the literature. This appears to be a question of hydrodynamic stability involving multiparticle interactions, certainly a fascinating area for future research.

We now return to our original description of the polydisperse sedimentation process and restrict our attention to suspensions where the spatial distribution of particles in any horizontal plane remains uniform. When a dispersion is extremely dilute, each individual particle falls with its Stokes velocity given by (1.1). Stokes' law provides a quantitative description of the sedimentation process for such dilute suspensions and forms the basis for the design of many particle-size analyzers. However, as stated earlier, even when the total volume fraction of particles exceeds only about 0.01, the behavior differs significantly from the prediction of Stokes' law as a consequence of interactions between particles of the various species. In particular, when a polydisperse suspension is not extremely dilute, the volume fraction of any one particle species differs in the various regions in the suspension. For a discrete distribution of N distinct particle species, Smith (1966) has derived from the requirement of particle flux continuity that

$$c_i^{(k+1)}(v_k^{(k)}-v_i^{(k+1)}) = c_i^{(k)}(v_k^{(k)}-v_i^{(k)}), \qquad i = k+1,\dots,\mathrm{N}, \tag{3.1}$$

where the superscript (k) refers to the conditions in region k (the regions are numbered sequentially starting at the bottom of the vessel—see Figure 2), and the subscripts refer to the particle species, which are numbered such that the fastest-settling species in region k is denoted as species k, i.e. $c_i^{(k)} = 0$ for $i < k$. A particle of species i has radius a_i, density ρ_i, volume fraction c_i, and a Stokes settling velocity $u_i^{(0)}$. Its average fall speed v_i depends on the local concentrations of all of the particles present, and we express this as

$$v_i = u_i^{(0)} f_i(\mathbf{c}), \tag{3.2}$$

where $\mathbf{c}$ is the vector of the volume fractions of all of the particle species. In general, f_i will differ for each particle species, and it will depend also on the effects of Brownian motion and interparticle forces, as discussed below.

In view of the fact that the average settling velocities are related to the volume fraction of all species present, Equation (3.1) represents a system of coupled, nonlinear algebraic equations for the unknowns $c_i^{(k+1)}$ that can be solved numerically using standard iterative techniques. Knowledge of the $c_i^{(k+1)}$ values, together with (3.2), suffices to give a complete description of the separation process. For continuous distributions of particles, Davis et al. (1982) and Greenspan & Ungarish (1982) have shown that the difference equations given by (3.1) are replaced by differential equations. However, these differential equations can be solved by finite-difference techniques that yield a set of coupled equations similar to (3.1) with N large.

In order to carry out the calculation described above, it is of course necessary to know the hindered settling function $f_i(\mathbf{c})$ for polydisperse suspensions. Smith (1965, 1966, 1967) has presented a theoretical cell model for the sedimentation of multisized particles that is an extension of the fluid cell model originally proposed by Happel (1958) for the settling of equisized particles. Smith's theory successfully predicts the trends of measured settling velocities with changing solids concentration, but it underestimates the velocities, just as does the cell theory for the settling of a single species. Lockett & Al-Habbooby (1973, 1974) carried out sedimentation and counter-current solid-liquid vertical-flow experiments with binary particle mixtures. They proposed that the monodisperse Richardson & Zaki (1954) correlation [Equation (2.2)] could be applied to particles of each of the two sizes in a binary suspension simply by using the total local solids volume fraction $c = \sum c_i$ without any reference to the individual particle species. Mirza & Richardson (1979) extended the Lockett & Al-Habbooby model to the sedimentation of multisized particle systems. The models of both of these sets of authors were found to predict sedimentation velocities that were larger than those found experimentally. In order to obtain a better representation of their experimental data, Mirza & Richardson (1979) applied an additional empirical correction factor of $(1-c)^{0.4}$ to the predicted sedimentation velocities. In a similar study, Selim et al. (1983) treated equidensity particles and used the total local volume fraction of solids in the Richardson & Zaki correlation for the slip velocity of the particles relative to the fluid; in addition, they proposed that the Stokes settling velocity of a particle of species i be modified by replacing the fluid density ρ in (1.1) with the average density of a suspension consisting of fluid and of particles of sizes smaller than that of species i. Clearly, this approach reprcsents only an ad hoc model for the effects of particle interactions between different species; nonetheless, using concentrated suspensions with $0.12 \leq c \leq 0.45$, Selim et al. (1983) found very good agreement between their model and experiments.

In an application of steady-state continuous sedimentation of polydisperse suspensions in an inclined settling channel (see the following section), Davis et al. (1982) measured the total solids concentration as a function of height in the channel for a feed suspension containing a known "bell-shaped" particle-size distribution. These authors compared their data with theoretical predictions based on the assumption of Lockett & Al-

Habbooby (1973, 1974) that the hindered settling effect is adequately represented by empirical correlations for monodisperse suspensions. For total solids volume fractions in the feed of 2 and 5%, the experimental data lay in-between the theoretical predictions using (2.1) and (2.2) for the hindered settling function.

In each of the theoretical models discussed above, the results for monodisperse suspensions were used to predict the effects of particle-particle interactions in polydisperse systems, or else an ad hoc modification of the monodisperse results was introduced. However, for general systems containing large variations in particle sizes or densities, we would not expect that polydisperse settling phenomena could be accurately described by this type of approach. On the other hand, for suspensions that are sufficiently dilute that pairwise particle interactions are dominant, the hindered settling effect can be calculated exactly. Such a calculation has been presented recently by Batchelor (1982). Following his earlier investigation of monodisperse suspensions, Batchelor showed that the mean velocity of a particle of species i in a stable dispersion of N distinct species is

$$v_i = u_i^{(0)}\left\{1+\sum_{j=1}^{N} S_{ij}c_j\right\}, \qquad i = 1,2,\ldots,N, \tag{3.3}$$

correct to order c. The dimensionless sedimentation coefficient S_{ij} is a function of (a) the size ratio $\lambda = a_j/a_i$, (b) the reduced density ratio $\gamma = (\rho_j-\rho)/(\rho_i-\rho)$, ($c$) the Peclet number of the relative motion of an i particle and a j particle, and (d) a dimensionless measure of the interparticle force potential. The Peclet number, whose inverse measures the effect of Brownian diffusion compared with the effect of relative motion due to gravity, is typically of order unity for micron-sized spheres in water, and it increases in proportion to $(a_i+a_j)^4$ for a given λ.

The computation of the sedimentation coefficient S_{ij} for a pair of unlike spheres is much more complicated than for equal spheres, because the pair-distribution function—defined as the probability of finding the center of a particle of species j in unit volume at a position $\mathbf{r}$ relative to the center of a particle of species i—is nonuniform and must be determined as part of the solution. This distribution function is governed (a) by a particle-pair conservation equation that in a dilute dispersion depends on the relative motion of two particles through the suspending fluid due both to gravity and to any interactive force that may act between the two particles; and (b) by Brownian diffusion of the two particles. For details of the calculation of the pair-distribution function and the sedimentation coefficients, the reader is referred to Batchelor (1982) and to a sequel paper by Batchelor & Wen (1982), in which numerical results for these quantities have been compiled.

4. ENHANCED SEDIMENTATION IN INCLINED CHANNELS

The sedimentation process is often very slow, especially when the particles are small, and hence there exists a need for constructing simple devices that could accomplish this solid-liquid separation more rapidly. One such class of devices, known commercially as "supersettlers" or "lamella settlers," consists of settling vessels having inclined walls and in which the retention times can be reduced by an order of magnitude or more below those in corresponding vertical settlers. These settlers may be composed either of a narrow tube or channel inclined from the vertical or of a large tank containing several closely spaced tilted plates.

The phenomenon of enhanced sedimentation in inclined channels has a long history, and one of the first references to appear in the literature is attributed to the physician Boycott (1920), who observed that "when . . . blood is put to stand in narrow tubes, the corpuscles sediment a good deal faster if the tube is inclined than when it is vertical." Subsequently, many investigators have studied the phenomenon for a variety of suspensions and have reported that a severalfold increase in the sedimentation rate could indeed be achieved; an excellent summary of the early papers on the subject is given by Hill (1974).

The enhancement in the sedimentation rate can be viewed as resulting from the fact that whereas particles can only settle onto the bottom in a channel with vertical walls, such particles can also settle onto the upward-facing wall in a tilted channel, as shown in Figure 4. These particles then

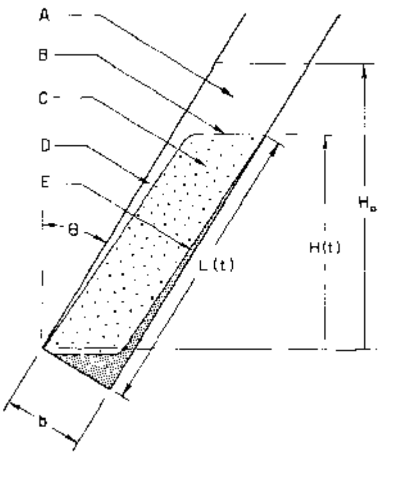

Figure 4 The different regions of the flow field during sedimentation in an inclined channel: (*A*) region of particle-free fluid above the suspension, (*B*) interface between the particle-free fluid and the suspension, (*C*) suspension, (*D*) thin particle-free fluid layer beneath the downward-facing surface, (*E*) concentrated sediment.

form a thin sediment layer that rapidly slides down toward the bottom of the vessel under the action of gravity. Thus, one can interpret the increase in the settling rate as due to an increase in the surface area available for settling. Of course, since the suspension is incompressible, coincident with the removal of particles from the suspension, there must also occur an accompanying production of clarified fluid. This takes place at the top of the suspension and along the downward-facing wall, where a thin clear-fluid layer is formed (cf. Figure 4). This particle-free layer is buoyant compared with the bulk suspension, and the fluid within it flows rapidly to the top of the vessel. Also, this layer reaches a steady-state thickness in a relatively short time, because the fluid that is being entrained into the layer exerts a drag force on the particles at the interface that counterbalances the normal component of the gravitational force on these particles.

Although this so-called Boycott effect of enhanced sedimentation has been known for many years, it has only recently been analyzed using first principles. Hill et al. (1977) were the first to study the phenomenon theoretically using the appropriate equations of continuum mechanics. These authors used a numerical solution to obtain the flow profiles and sedimentation rates in various upward-pointing cones. Subsequently, Probstein et al. (1981), Rubinstein (1980), and Leung & Probstein (1983) performed an analytical analysis of continuous steady-state settling in relatively long and narrow inclined channels and treated the problem as a two-dimensional viscous channel flow. Their analysis includes three stratified layers: a clarified layer, a suspension layer, and a sediment layer. Equations were developed for the velocity profiles and fluxes in each of these layers, as well as for the location of the interfaces separating them. Concurrently, Acrivos & Herbolzheimer (1979) and Herbolzheimer & Acrivos (1981) presented analytical derivations of the laminar-flow profiles in two-dimensional inclined channels for both batch and continuous settling. The former paper applies to low-aspect-ratio vessels in which the clarified-fluid layer that forms beneath the downward-facing wall of the channel is much smaller than the spacing between the walls (see Figure 4), whereas the latter paper considers very tall and narrow vessels in which the clarified-fluid layer occupies an appreciable portion of the channel.

A result common to each of the analyses discussed above is the verification of the expression for the clarification rate given via an elementary kinematic model by Ponder (1925) and independently by Nakamura & Kuroda (1937)—the so-called PNK theory. This theory states that the rate of production of clarified fluid is equal to the vertical settling velocity of the particles multiplied by the horizontal projection of the channel area available for settling. For the parallel-plate geometry of

Figure 4, the expression is

$$S^*(t) = v_0 b \left(\cos \vartheta + \frac{L}{b} \sin \vartheta \right), \tag{4.1}$$

where $S^*(t)$ is the volumetric rate of production of clarified fluid per unit depth in the third dimension of the vessel (hereafter referred to as the settling rate), v_0 is the vertical settling velocity [i.e. (1.1) multiplied by the hindered settling function $f(c)$ discussed earlier], ϑ is the angle inclination of the plates from the vertical, b is the spacing between the plates, and L is the length of the vessel. It is evident from (4.1) that the augmentation in the rate of settling should be $O(L/b)$, which, in principle, could be made arbitrarily large by decreasing the spacing between the inclined walls.

The enhancement in the settling rate predicted by (4.1) holds true only so long as the flow in the channel remains laminar. Under some conditions, however, waves have been observed to form at the interface between the suspension and the layer of particle-free fluid underneath the downward-facing wall (Pearce 1962, Probstein & Hicks 1978, Leung 1983, Herbolzheimer 1983, Davis et al. 1983). These waves grow as they travel up the vessel and often break before reaching the top of the suspension. Evidently, the occurrence of such an instability limits the efficiency of the inclined settling process, because when the waves break, fluid that has already been clarified is remixed with the suspension. A photograph depicting this phenomenon is shown in Figure 5. The particles are glass spheres having a density of 2.44 g cm^{-3} and an average diameter of 130 μm. The suspending fluid is a synthetic Newtonian lubricant having a density of 1.05 g cm^{-3} and a viscosity of 10 cP.

Recently, classical linear stability theory has been applied to the inclined settling process by Herbolzheimer (1983), Davis et al. (1983), and Leung (1983) in order to elucidate the conditions under which the flow becomes unstable and to determine the growth rate of the waves. In what follows, we present the findings of these stability analyses, preceded by a brief discussion of the laminar-flow profiles.

Theory and Experiments for Laminar Flow

We consider here the suspension to be composed of a Newtonian fluid and identical solid spheres of vanishingly small particle Reynolds number. The sedimentation of polydisperse suspensions in inclined channels has been treated by Davis et al. (1982), to which the reader is referred for details. A key feature of our analysis is to treat the suspension as an effective fluid whose properties depend on the solids volume fraction. The particles are assumed to move with the bulk average velocity plus a slip velocity in the direction of gravity equal to the vertical settling velocity of the suspension. The latter is determined either from a vertical sedimentation experiment or from an empirical correlation. Aside from the vessel geometry and solids volume fraction, a complete description of the settling process is governed by two dimensionless parameters: R, a sedimentation Reynolds number; and Λ, the ratio of a sedimentation Grashof number to R, where $R \equiv \rho H v_0/\mu$ and $\Lambda \equiv H^2 g(\rho_s - \rho)c_0/v_0\mu$, with H being the vertical height of the suspension and c_0 the volume fraction of solids (which remains uniform everywhere within the interior of the suspension). Since in most cases of interest, Λ is $O(10^6)$–$O(10^8)$ while R is only $O(1)$–$O(10)$, an asymptotic analysis for determining the details of the motion was developed by Acrivos & Herbolzheimer (1979) and Herbolzheimer & Acrivos (1981) by taking $\Lambda \gg 1$ with $R\Lambda^{-1/3} \ll 1$. In this limit, the flow along the channel in the thin clarified-fluid layer to leading order is parallel and fully developed. In particular, if a coordinate system is defined with x directed along the downward-facing wall of the channel and y perpendicular to it, then

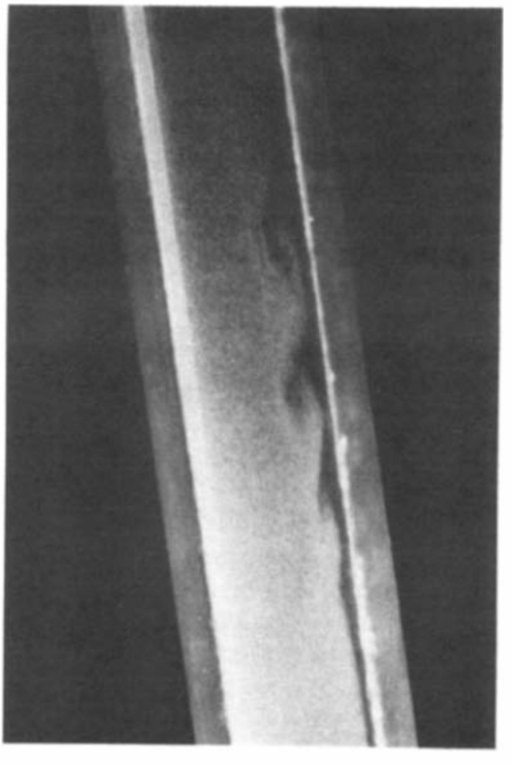

Figure 5 Wave formation and breakage during sedimentation in an inclined channel.

provided that the aspect ratio of the channel is not too large, the velocity in the x direction is given by

$$u = \Lambda^{1/3}(\tilde{y}\tilde{\delta} - \tfrac{1}{2}\tilde{y}^2) \cos \vartheta, \tag{4.2}$$

where the velocity u has been rendered dimensionless by v_0, $\tilde{\delta} \equiv \Lambda^{1/3}\delta$, $\tilde{y} \equiv \Lambda^{1/3}y$, and y is the distance from the wall made dimensionless by H. Also, after a short initial time period, the thickness of the clear-fluid layer reaches a steady-state thickness, which for the parallel-plate geometry is

$$\delta(x) = \Lambda^{-1/3}(3x \tan \vartheta)^{1/3}, \tag{4.3}$$

where both δ and x have been made dimensionless by H. An experimental determination of the clarified-fluid layer thickness provides a sensitive test of the theory, and in a series of experiments reported by Acrivos & Herbolzheimer (1979), measurements of $\delta(x)$ and of the sedimentation rate were in excellent agreement with the predictions. Since the thickness of the clear-fluid layer scales as $\Lambda^{-1/3}$, it will remain thin relative to the channel width so long as $b/H \gg \Lambda^{-1/3}$. Also, in view of the fact that the clarified-fluid layer has a very large velocity at its interface with the adjacent suspension, it drives a boundary-layer flow in the suspension region. The details of the boundary-layer analysis that describes this flow are given in Acrivos & Herbolzheimer (1979).

When the aspect ratio of the channel, H/b, is of order $\Lambda^{1/3}$ or greater, the clear-fluid layer occupies an appreciable fraction of the channel. Under these conditions, the flow along the channel to leading order is parallel and fully developed in both the clear-fluid layer and the adjacent suspension layer. In particular, the velocity in the up-channel direction is given by (Rubinstein 1980, Herbolzheimer & Acrivos 1981)

$$u = \Lambda^{1/3}\left\{\frac{6Q}{\tilde{b}}\, \bar{y}(1-\bar{y}) - \bar{y}[(\bar{\delta}+1/2)\bar{y} - \bar{\delta}]\,(1-\bar{\delta})^2\tilde{b}^2 \cos \vartheta\right\}, \tag{4.4a}$$

in the pure-fluid layer (i.e. for $0 \leq \bar{y} \leq \bar{\delta}$), and by

$$u_s = \Lambda^{1/3}\left\{\frac{6Q}{\tilde{b}}\bar{y}(1-\bar{y}) - (1-\bar{y})\,[(3/2-\bar{\delta})\bar{y} - 1/2]\bar{\delta}^2\tilde{b}^2 \cos \vartheta\right\}, \tag{4.4b}$$

in the suspension layer (i.e. for $\bar{\delta} \leq \bar{y} \leq 1$). The variables have been made dimensionless using H_0 and v_0 as the characteristic length and velocity, respectively. The variable Q is the net flow rate of material across any plane of constant x, $\tilde{b} \equiv \Lambda^{1/3}b/H_0$ is an $O(1)$ quantity by definition, $\bar{y}$ is the fractional distance across the channel (i.e. $\bar{y} \equiv \Lambda^{1/3}y/\tilde{b}$), and $\bar{y} = \bar{\delta}(x, t)$ is the equation for the interface separating the pure fluid and the suspension. For simplicity, the presence of the sediment layer on the upward-facing wall has been neglected, and the viscosity of the suspension has been set equal to that of the pure fluid. These approximations are accurate in the dilute limit, and removing them is relatively straightforward (Herbolzheimer 1980, Leung & Probstein 1983).

In order to complete the description of the laminar-flow field in these high-aspect-ratio vessels, it is necessary to determine the thickness of the clear-fluid layer $\bar{\delta}(x,$
batch settling, one surprising result is that the clear-fluid layer attains a steady shape only along the lower portion of the channel given by

$$\bar{\delta} = \tfrac{1}{2}\{1-[1-4(3x \tan \vartheta)^{1/3}/\tilde{b}]^{1/2}\}, \tag{4.5a}$$

while, in contrast, its thickness is independent of position and sweeps across the upper portion of the vessel linearly with time above some critical point

$$\bar{\delta} = \Lambda^{1/3} t \sin \vartheta/\tilde{b}. \tag{4.5b}$$

In other words, when viewed as a function of distance along the channel, the thickness of the clear-fluid layer becomes discontinuous. The equation for the point of discontinuity separating the two parts of the interface is given by Equation (3.7) of Herbolzheimer & Acrivos (1981), who also report experimental verification of this interesting behavior.

Of greater practical interest than batch settling is the continuous operation of inclined settlers. Even though (as mentioned earlier) the thickness of the clear-fluid layer remains transient in the upper portion of the vessel for batch systems, the analyses of Herbolzheimer & Acrivos (1981) and Leung & Probstein (1983) predict that high-aspect-ratio settlers can be operated continuously provided the feed and widthdrawal locations are chosen properly. A thorough discussion of the feed and withdrawal conditions that should lead to a steady operation of such settlers is found in Section 4 of Herbolzheimer & Acrivos (1981). We focus here on two such regimes that are of the most widespread interest. In the first regime (regime 1), the feed is introduced at the bottom of the vessel while clarified fluid is removed from the top, so that a net flow of fluid up the channel is maintained. In this case, $\bar{\delta}(x)$ is governed by

$$(3-2\bar{\delta})\bar{\delta}^2 Q + \frac{\tilde{b}^3}{3}\bar{\delta}^3(1-\bar{\delta})^3 \cos \vartheta = x \sin \vartheta, \tag{4.6}$$

where [from (4.1)] $Q = \tan \vartheta + \Lambda^{-1/3}\tilde{b} \sec \vartheta \cong \tan \vartheta$. Numerical solutions of (4.6) are possible for all values of $\tilde{b}$, and experimental verification of these solutions has been reported by Davis et al. (1983).

In regime 2 both the feed and withdrawal are located at the top of the vessel. Thus $Q = 0$ everywhere along the channel, and (4.6) has two steady solutions,

$$\bar{\delta}(x) = \tfrac{1}{2}[1 \pm (1-4(3x \tan \vartheta)^{1/3}/\tilde{b})^{1/2}]. \tag{4.7}$$

The existence of these two solutions was first predicted by Probstein et al. (1981). In the solution with the negative root, termed the "subcritical mode," the thickness of the clear-fluid layer vanishes at the bottom of the vessel and then grows with x—as is always observed in batch settling—whereas in the second solution, termed the "supercritical mode," the suspension layer vanishes at the bottom of the vessel and then grows with x. Which of these two modes is realized in practice cannot be predicted from the fully developed flow theory discussed thus far. However, Rubinstein (1980) has presented a two-dimensional model for the entry-flow problem that enables one to relate the realization of a particular mode to the feed and withdrawal conditions at the top of the channel. The supercritical mode of operation was first observed experimentally by Probstein & Hicks (1978), and later by Leung & Probstein (1983), who argued that this mode may be advantageous in suppressing the effects of any remixing of the particle-free fluid and the suspension; such remixing might occur if the interface between these regions became unstable. However, as can be seen from (4.7), real solutions for regime 2 exist only for $\tilde{b} > 4(3 \sec \vartheta \tan \vartheta)^{1/3}$; this restriction limits the maximum aspect ratio that can be used for the top-feeding modes. On the other hand, a vessel with the feed at the bottom can be operated, in principle, for arbitrarily large values of the aspect ratio. Hence, if the flow remains stable, the highest possible enhancement in the settling rate can be achieved using regime 1. The aim of the stability analysis that follows is to predict the conditions under which the flow will remain stable.

Theory and Experiments for Unstable Flow

It has been noted earlier that in certain cases the interface between the clear-fluid slit and the suspension becomes wavy. Evidently, the occurrence of such an instability limits the efficiency of the inclined settling process, especially when the waves break. This phenomenon of wave formation and growth is similar to the well-known problem of a falling liquid film on an inclined surface, which has also been observed to become unstable. In order to investigate the origins of this problem, a linear-stability analysis has been performed for the laminar-flow profiles presented in the previous section. The details of this analysis can be found in the recent papers by Herbolzheimer (1983) and Davis et al. (1983).

The theoretical development proceeds along the lines of the classical works by Yih (1963) and by Benjamin (1957), who studied the instability of the falling liquid film. The disturbances to the base-state flow are assumed to be two dimensional and infinitesimally small and are expressed as superpositions of normal modes of the general form

$$\Psi(x, y, t) = \psi(y) \exp i(\alpha x - \omega t), \tag{4.8a}$$

$$\Phi(x, y, t) = \varphi(y) \exp i(\alpha x - \omega t), \tag{4.8b}$$

where, since the waves are spatially (rather than temporally) growing, the wave number of the disturbance α is treated as complex, while the frequency ω is a real constant. Also, Ψ and Φ are the normal-mode disturbance stream functions in the clear-fluid layer and in the adjacent suspension layer, respectively. A similar expression applies for the displaced position of the interface separating these two regions. The analysis is a local (quasi-parallel and quasi-steady) one, because the base-state profiles vary parametrically with x and t but only slowly compared with the variations in the disturbances.

When (4.8a,b) are substituted into the linearized momentum equations for the disturbances, the well-known Orr-Sommerfeld equation results, together with the associated boundary conditions of no-slip at the walls and matching of velocity and stress at the interface. This system represents an eigenvalue problem with α as the complex eigenvalue. If the solution of this problem yields a positive value for α_I, the imaginary part of the eigenvalue, then the disturbances are damped as they travel up the vessel; whereas, if α_I is negative, then these same disturbances grow and the system is unstable.

The Orr-Sommerfeld equation was first solved asymptotically for small frequencies. [This corresponds to long wavelengths, since the wave speed, scaled with the average velocity in the clear-fluid layer, remained $O(1)$.] These results were then supplemented with a numerical analysis because the most highly amplified waves were found to occur for moderate values of the frequency and the Reynolds number. For sedimentation in low-aspect-ratio vessels, the asymptotic results of the linear theory show that small disturbances always grow, i.e. that the presumably laminar flow in the narrow slit is always unstable, provided that the local Reynolds number Rx is greater than $140\bar{\delta}/(57 \cos \vartheta)$, which is generally very small. An equivalent result was found by Leung (1983), who performed a long-wavelength analysis of the stability of the "supercritical" mode having a very thin suspension layer. However, the rate of growth of the disturbances in each case depends on R, Λ, and the geometrical and physical properties of the system, so that under many circumstances these disturbances do not grow to a visible size over the length of the vessel. Thus, in many of the experiments described earlier, the flow appeared to be stable. The local exponential amplification factor for the wave growth is $-\alpha_I$, which is seen plotted in Figure 6 for the most unstable mode as a function of ϑ and for various values of the local Reynolds number Rx. Clearly, according to the predictions of the theory, the waves have a maximum amplification that generally occurs in the range $\vartheta = 5$–$15°$.

Using the linear-stability analysis as a guide, experiments for which a visible observation of the instability was expected were performed by Herbolzheimer (1983), both on a batch and a continuous basis, under the

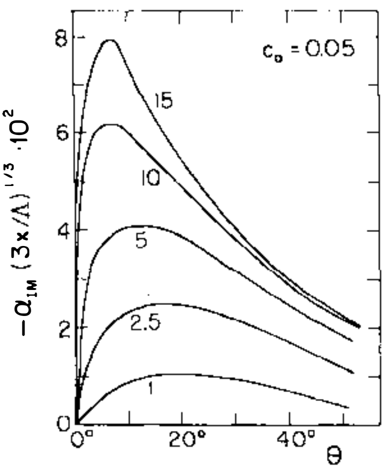

Figure 6 The parameter $-\alpha_{IM}\,(3x/\Lambda)^{1/3}$ for the most unstable mode versus the angle of inclination ϑ for a low-aspect-ratio vessel with $c_0 = 0.05$ and $Rx = 1$, 2.5, 5, 10, 15. (From Herbolzheimer 1983.)

following set of conditions: $8 \leq H_0/b \leq 15$, $5 \leq \vartheta \leq 50°$, $4 \leq R_0 \leq 25$, $7 \times 10^6 \leq \Lambda_0 \leq 2 \times 10^8$, $0.005 \leq c_0 \leq 0.15$ (where H_0 refers to the initial height of the suspension, and R_0 and Λ_0 are equivalent to R and Λ, respectively, only with L replacing H in their definitions). The experimental data were found to follow the general trends predicted by the linear theory. In particular, the distance along the channel at which waves first become visible, x_i, was measured because the inverse of this distance provides an experimental indication of the average growth rate of the waves (Herbolzheimer 1983, Davis et al. 1983). As an example of these measurements, $1/x_i$ is plotted in Figure 7 as a function of ϑ for $c_0 = 0.05$ and various values of R_0. From the figure, it is evident that the shapes of the curves are very similar to the corresponding theoretical predictions for the maximum amplification rates shown in Figure 6.

The linear stability analysis and experiments described above have been extended to determine the amplification rates of waves in tall, narrow channels with $H/b \geq O(\Lambda^{1/3})$. This case is of particular interest, since in light of (4.1), such vessels give very high enhancement in the clarification rate. It is important to know, therefore, under what conditions, if any, a stable operation of these vessels can be achieved. The analysis for these high-aspect-ratio vessels led to a critical Reynolds number R_c, below which the system can be expected to be stable to infinitesimal disturbances, unlike the case of low-aspect-ratio vessels, which were shown to be unstable for all but very small values of the Reynolds number. In general, it was shown by Davis et al. (1983) that R_c increased with increasing H/b and increasing ϑ. Thus, the same geometric factors that lead to the highest enhancement in the settling rate also give the most stable operating conditions—a

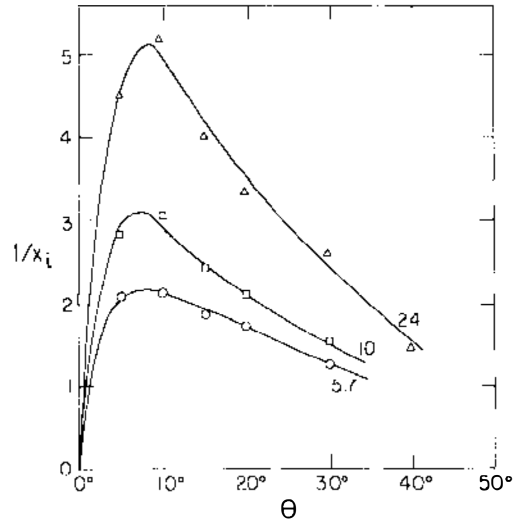

Figure 7 The inverse of the visually determined wave inception distance $1/x_i$ versus the angle of inclination ϑ for a low-aspect-ratio vessel, with $c_0 = 0.05$, $\Lambda_0 = 8.7 \times 10^7$, and $R_0 = 5.7, 10$, and 24. The solid lines are visual best-fit estimates through the data. (From Herbolzheimer 1983.)

surprising result that can be very useful in the future design of "supersettlers."

This exciting prediction was tested in the laboratory using a bottom-feeding continuous vessel with an aspect ratio of approximately 100, and the results were in very good agreement with the linear theory. This can be seen in Figures 8 and 9, where the amplification rates of the waves are plotted for theory and experiment, respectively, as functions of ϑ under the conditions $\Lambda_0 = 1.3 \times 10^7$, $\tilde{b}_0 = 2.4$, $c_0 = 0.01$, and for various values of the Reynolds number. The theoretical amplification rates are computed at the location midway up the channel. Compared with the low-aspect-ratio case (Figures 6 and 7), there is dramatic stabilization for $\vartheta > 20°$.

5. CONCLUSIONS

In this review, we have considered recent advances in the understanding of the sedimentation of monodisperse suspensions, the sedimentation of polydisperse suspensions, and enhanced sedimentation in inclined channels. Although sedimentation is a classical subject with a long history, it is our hope that the reader will be convinced that many opportunities for future research still very much exist. From our discussion, which has been limited to dispersions of rigid spherical particles having zero particle

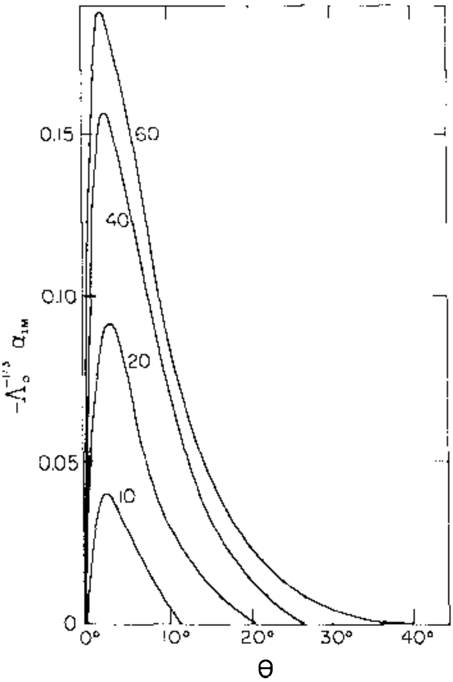

Figure 8 The scaled maximum amplification factor $-\Lambda_0^{-1/3}\alpha_{IM}$ versus the angle of inclination ϑ for regime 1, with $x_0 = 1/2$, $\tilde{b}_0 = 2.365$, and $R_0 = 10, 20, 40$, and 60. (From Davis et al. 1983.)

Reynolds numbers, it is evident that continued research is needed in many areas. Certainly, the problem of determining the effects of multiparticle interactions during the sedimentation of random, concentrated suspensions is both an important and a challenging one. Moreover, for dilute dispersions, the question still remains as to whether or not a settling suspension develops a structure having a length scale comparable to the mean particle spacing. Also, further experimental and theoretical research is needed in order to relate the macroscopic properties of a sedimenting polydisperse suspension to the local microphysical interactions between the different particles. The augmentation of the sedimentation rate in inclined channels holds great promise for the design of high-capacity settling vessels; continued work needed in this area includes studying the effects of inertia on the laminar-flow profiles in the channel and performing a nonlinear analysis in order to investigate the breaking of the waves at the liquid-suspension interface and to determine the extent of the subsequent remixing of the clarified fluid with the suspension.

Beyond this, there is considerable scope for further research on the

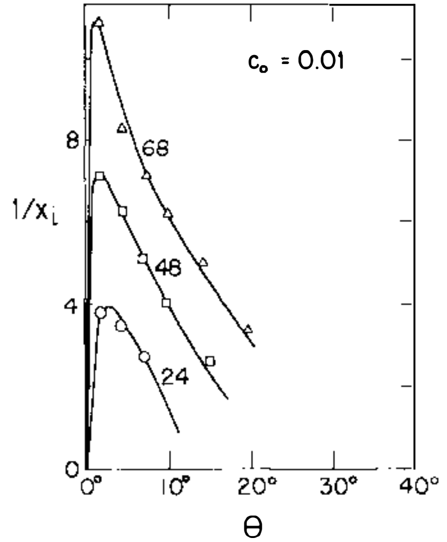

Figure 9 The inverse of the visually determined wave inception distance $1/x_i$ versus the angle of inclination ϑ for regime 1, with $c_0 = 0.01$, $\Lambda_0 = 1.3 \times 10^7$, $\bar{b}_0 = 2.4$, and $R_0 = 24$, 48, and 68. The solid lines are visual best-fit estimates through the data. (From Davis et al. 1983.)

sedimentation of nonspherical particles, of nonrigid particles such as drops and bubbles, and of particles having nonzero inertia. Also, as shown by the pioneering studies of Greenspan (1983) and Ungarish & Greenspan (1983, 1984), the effects of rotation and of the associated centrifugal forces on the separation of particles from a suspension give rise to a host of fascinating flow phenomena that are of interest from both the fundamental and practical points of view; these deserve a great deal more attention.

ACKNOWLEDGMENTS

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