Stat 610 Lab 4: Newton's method

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The Cauchy location family with location family θ has probability density function

$$f(x; \theta) = \frac{1}{\pi}\left[\frac{1}{1 + (x - \theta)^2}\right]$$

Its log likelihood is therefore

$$\ell(\theta) = \sum_{i=1}^{n}(-\log(1 + (x_i - \theta)^2) - \log(\pi))$$

In this lab you'll look at ways of estimating θ.

You may recall from your other statistics classes that the Cauchy distribution is very poorly behaved and so many of the normal ways of estimating parameters do not work well.

We will be using Newton's method here, and so we will need to know the first and second derivatives of ℓ with respect to θ. You can verify for yourself that these are:

$$\ell'(\theta) = \sum_{i=1}^{n} \frac{2(x_i - \theta)}{1 + (x_i - \theta)^2}$$

$$\ell''(\theta) = \sum_{i=1}^{n} \frac{-2 + 2(x_i - \theta)^2}{(1 + (x_i - \theta)^2)^2}$$

Some R functions that compute these quantities are given below:

```
ell <- function(x, theta) {
    return(sum(log(pi^(-1) * 1 / (1 + (x - theta)^2))))
}
ellprime <- function(x, theta) {
    num = 2 * (x - theta)
    denom = 1 + (x - theta)^2
    return(sum(num / denom))
}
elldoubleprime <- function(x, theta) {
    num = 2 * (x - theta)^2 - 2
    denom = (1 + (x - theta)^2)^2
    return(sum(num / denom))
}
```

1. Suppose you are given the following ten data points: $-2.09, -2.68, -1.92, -1.76, -2.12, 2.21, 1.97, 1.61, 1.99, 2.18$. Compute the log likelihood of this dataset for a range of values of θ between -4 and 4. Make a plot of the log likelihood as a function of θ. Are there going to be any problems using Newton's method in this model?

2. Write a function that takes one Newton step. Your function should take a data vector and a value for θ.

3. Use the function that you wrote in the previous part to experiment with starting at different values of θ and taking just a couple (maybe around 10) Newton steps. You can either print out the updated value of $\hat{\theta}$ after each step, or you can add them to the plot you made in question 1 using `abline(v = thetahat)` (this only works if you used base R to plot in question 1).

4. When we implement Newton's method, we usually take Newton steps until either the difference between the current estimate and the previous estimate is very small (we have "converged") or until we have reached some maximum number of steps. Write such a function for estimating θ in the Cauchy location family.

   Try your function out on the values given in question 1. Does it give good results? Is there any extra information you could give a user that would tell them whether the value returned is a local maximum, a local minimum, or something else?