GAUSS'S LAW

Learning outcomes:

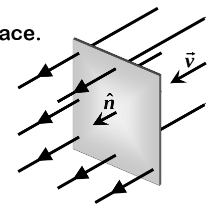
At the end of this chapter you should be able to…

- Calculate the electric flux through a surface.
- Use Gauss's Law to calculate the electric field due to symmetric charge distributions.
- Use Gauss's Law to determine the charge distribution on hollow conductors in electric fields.

FLUX

$\hat{n}$ is the unit vector normal to the surface.

$\vec{A} = A\hat{n}$ is the surface's area vector.



The amount of flow, Φ, through the surface depends on:

- the area of the surface, A;
- the angle between $\vec{v}$ and $\vec{A}$.
  (Φ is a maximum when $\theta = 0°$; a minimum for $\theta = 90°$.)

Hence: $\Phi = vA\cos\theta$ or just: $\Phi = \vec{v}\cdot\vec{A}$

Replacing the velocity vector with the electric field vector $\vec{E}$, we get: $\boxed{\Phi_e = \vec{E}\cdot\vec{A}}$

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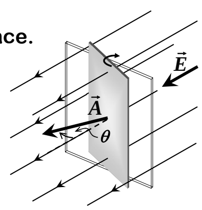

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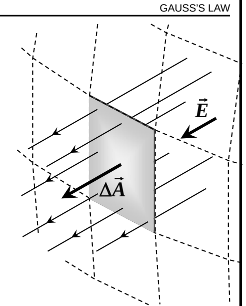
FLUX

For a surface made up of small elements, each ΔA, the total flux is

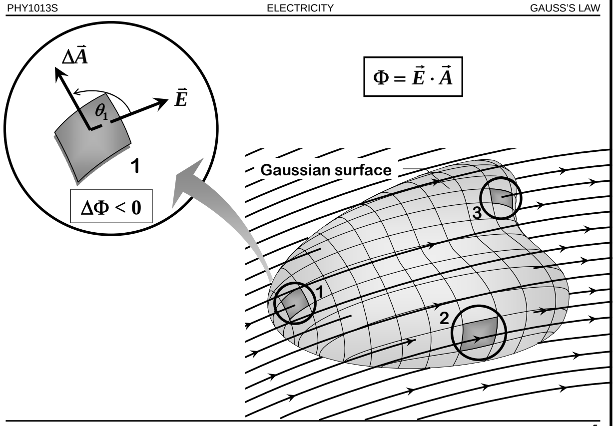
$$\Phi = \sum \vec{E}\cdot\Delta\vec{A}$$

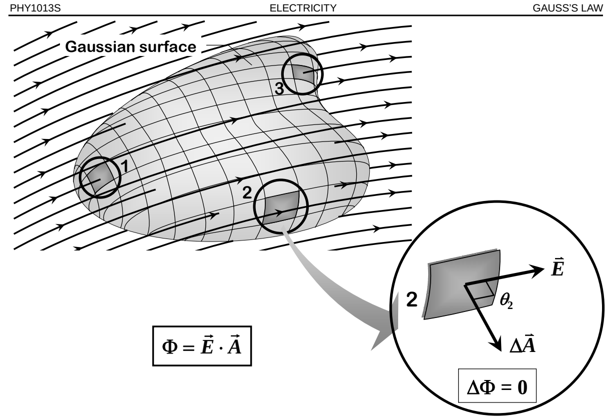


For a closed (Gaussian) surface, in the limit $\Delta A \to 0$,

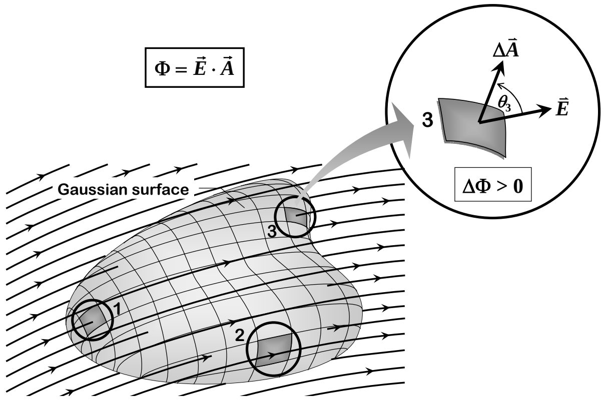
$$\boxed{\Phi = \oint \vec{E}\cdot d\vec{A}} \quad [\text{N m}^2/\text{C}]$$

The electric flux through a Gaussian surface is proportional to the *net* number of electric field lines passing through that surface.





$$\Phi = \vec{E} \cdot \vec{A}$$



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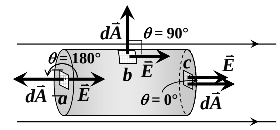
A Gaussian surface in the form of a cylinder of radius R is immersed in a uniform electric field $\vec{E}$, with the cylinder axis parallel to the field.



Determine the (net) flux of the field through this surface.

$$\Phi = \oint \vec{E} \cdot d\vec{A} = \int_a \vec{E} \cdot d\vec{A} + \int_b \vec{E} \cdot d\vec{A} + \int_c \vec{E} \cdot d\vec{A}$$

$$\therefore \Phi = \int_a E(\cos 180°)dA + \int_b E(\cos 90°)dA + \int_c E(\cos 0°)dA$$

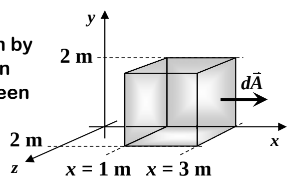
$$\therefore \Phi = -E\int_a dA + E\int_b (0)dA + E\int_c dA$$

$$\therefore \Phi = -EA + 0 + EA \qquad \therefore \underline{\Phi = 0}$$

HRW$_6$ 547 SP24-2

A non-uniform electric field given by $\vec{E} = 3.0x\hat{\mathrm{i}} + 4.0\hat{\mathrm{j}}$ pierces a Gaussian cube situated on the x-axis between x = 1.0 m and x = 3.0 m.

Determine the electric flux through the right face.



$$d\vec{A} = dA\hat{\mathrm{i}} \qquad \therefore \Phi = \int \vec{E} \cdot d\vec{A} = \int \left(3.0x\hat{\mathrm{i}} + 4.0\hat{\mathrm{j}}\right) \cdot \left(dA\hat{\mathrm{i}}\right)$$

$$\therefore \Phi = \int \left[(3.0x)(dA)\hat{\mathrm{i}} \cdot \hat{\mathrm{i}} + (4.0)(dA)\hat{\mathrm{j}} \cdot \hat{\mathrm{i}}\right]$$

$$\therefore \Phi = \int \left(3.0x\,dA + 0\right) = 3.0\int x\,dA$$

$$\therefore \Phi = 3.0\int (3.0)\,dA = 9.0\int dA = (9.0\,\mathrm{N/C})\left(4.0\,\mathrm{m}^2\right)$$

$$\therefore \underline{\Phi = 36\ \mathrm{N\,m^2/C}}$$

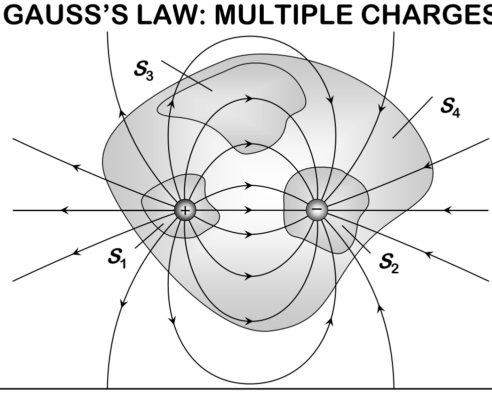
GAUSS' LAW

The net electric flux through a Gaussian surface is proportional to the net charge enclosed: $\varepsilon_0\Phi = Q_{in}$

Substituting for Φ, $\boxed{\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = Q_{in}}$

Notes:
- Q_{in} is the *net, enclosed* charge.
- $\vec{E}$ is the *total* field through the surface.
- These equations are valid only in vacuum (or air).
- Gauss's Law is both easier to use and more universal/fundamental than Coulomb's Law.

GAUSS'S LAW: MULTIPLE CHARGES



USING GAUSS'S LAW TO DETERMINE ELECTRIC FIELDS

1. Draw the situation.
2. Choose a Gaussian surface appropriate to the symmetry.
3. Apply Gauss's law: $\boxed{\varepsilon_0 \oint \vec{E} \cdot d\vec{A} = Q_{in}}$

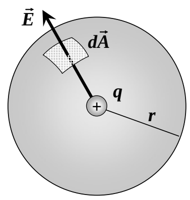
GAUSS ⇔ COULOMB

- Draw the situation…
- Choose a Gaussian surface appropriate to the symmetry… (For a point charge, choose a concentric sphere as a Gaussian surface.)
  The electric field is constant over the surface and directed radially outwards, so $\vec{E}\cdot d\vec{A} = E\,dA$
- Apply Gauss's law…

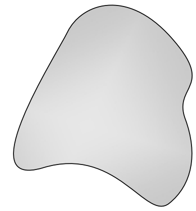
$$Q_{\text{in}} = \varepsilon_0 \oint \vec{E}\cdot d\vec{A} = \varepsilon_0 \oint E\,dA$$

$$\therefore Q_{\text{in}} = \varepsilon_0 E \oint dA$$

$$\therefore Q_{\text{in}} = \varepsilon_0 E 4\pi r^2 \qquad \boxed{E = \frac{q}{4\pi\varepsilon_0 r^2}}$$



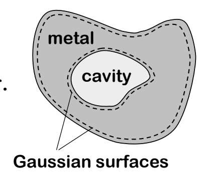
ZERO FIELD



If there is no electric field *at all*, quite obviously the net flux through any Gaussian surface is also zero.

CHARGE ON AN ISOLATED CONDUCTOR

Any excess charge added to an isolated conductor moves entirely to the external surface of that conductor.



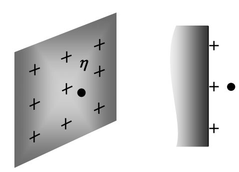
- The field inside the conductor must be zero (otherwise there would be currents inside the conductor)…
- Therefore there is no flux through the Gaussian surfaces…
- $\therefore$ there can be no charge *within* the Gaussian surfaces…
- $\therefore$ all the charge must lie *outside* the Gaussian surfaces…

i.e. …
All the charge lies on the external surface of the conductor.

EXTERNAL FIELD DUE TO A CHARGED CONDUCTOR

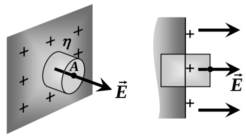
For a non-spherical conductor, the surface charge density, η, varies over the surface, and the field established around the conductor is very complex.

However, for a point just outside the surface, the adjacent section of surface is small enough to be considered as flat, and the charge density as uniform…



EXTERNAL FIELD DUE TO A CHARGED CONDUCTOR

A cylindrical Gaussian surface with an end cap area of A is embedded in the surface of the conductor as shown.



The flux through the outer cap is EA.

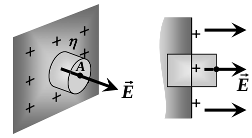
The total charge enclosed by the cylinder is given by ηA.

Therefore, according to Gauss's law: $\varepsilon_0 EA = \eta A$

and hence: $\boxed{E = \frac{\eta}{\varepsilon_0}}$

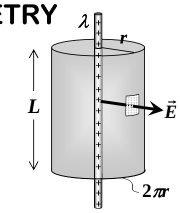
EXTERNAL FIELD DUE TO A CHARGED CONDUCTOR

Summary:



- The electric field is zero everywhere *inside* the conductor.
- Any excess charge is all on the *surface*.
- The external field lies perpendicular to the surface and is given by $E = \frac{\eta}{\varepsilon_0}$.
- On an irregularly shaped conductor the charge collects around sharp points, but $E = \frac{\eta}{\varepsilon_0}$ still holds true.

CYLINDRICAL SYMMETRY



For a long, thin, cylindrical *insulator* with a uniform linear charge density of λ … we choose a cylindrical Gaussian surface with radius r and height L:

The area of the curved surface is $2\pi r L$.

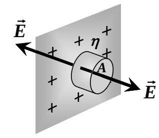
By symmetry, the total flux through the surface is $E\,2\pi r L$.

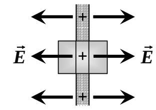
The total charge enclosed by the cylinder is λL.

Therefore, according to Gauss's law: $\varepsilon_0 E\,2\pi r L = \lambda L$

and hence: $\boxed{E = \dfrac{\lambda}{2\pi\varepsilon_0 r}}$

PLANAR SYMMETRY





For a large, flat, thin *insulating* sheet with a uniform surface charge density of η … we choose a cylindrical Gaussian surface with end cap area A, which pierces the sheet perpendicularly.

According to Gauss's law: $\varepsilon_0(E A + E A) = \eta A$

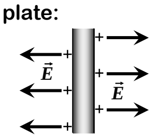
and hence: $\boxed{E = \dfrac{\eta}{2\varepsilon_0}}$

PLANAR SYMMETRY

For a charged large, flat, thin *conducting* plate:

surface charge density = η_1

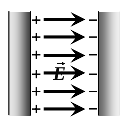
$\boxed{E = \dfrac{\eta_1}{\varepsilon_0}}$



When two oppositely charged plates are brought close together, all the charge moves to the inner faces:

surface charge density $\eta = 2\eta_1$

$\boxed{E = \dfrac{2\eta_1}{\varepsilon_0} = \dfrac{\eta}{\varepsilon_0}}$



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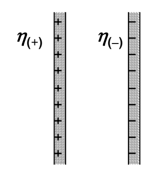
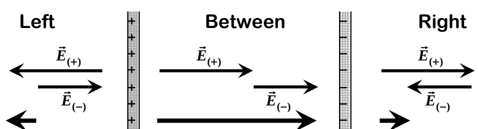
Two large, parallel, non-conducting sheets each carry a fixed uniform charge on one side: $\eta_{(+)}$ = 6.8 μC/m^2 and $\eta_{(-)}$ = 4.3 μC/m^2.

Determine the electric field (a) to the left, (b) between, and (c) to the right of the sheets.



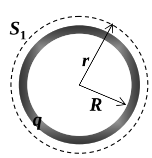
$E_{(+)} = \dfrac{\eta_{(+)}}{2\varepsilon_0} = 3.84\times10^5$ N/C $\qquad E_{(-)} = \dfrac{\eta_{(-)}}{2\varepsilon_0} = 2.43\times10^5$ N/C



$\vec{E}_L = 1.4\times10^5$ N/C, left $\qquad \vec{E}_B = 6.3\times10^5$ N/C, right $\qquad \vec{E}_L = 1.4\times10^5$ N/C, right

SPHERICAL SYMMETRY

A spherical shell of radius R and charge q is surrounded by a concentric spherical Gaussian surface (S_1) with radius r (where $r \geq R$).



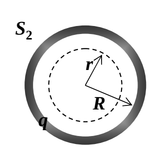
According to Gauss's law: $\varepsilon_0 E\,4\pi r^2 = q$

and hence, for S_1: $E = \dfrac{q}{4\pi\varepsilon_0 r^2}$

I.e. A uniform spherical shell of charge acts, on all charges outside it, as if all its charge were concentrated at its centre. [Shell theorem 1]

SPHERICAL SYMMETRY

S_2 is a concentric spherical Gaussian surface with radius r lying *within* the spherical shell of charge q (i.e. $r \leq R$).



According to Gauss, for S_2: $E = 0$

I.e. A uniform spherical shell of charge exerts no electrostatic force on a charged particle located inside it. [Shell theorem 2]