Report MK0.1

The Magnificent Me

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Definitions

Definition of the metric

$$ds^2 = (1+2\chi)dr^2 + \xi_{;i} dr dx^i + e^{2A(r)}\left[1 + 2(\psi\delta_{ij} - \partial_i\partial_j W)\right] dx^i dx^j \tag{1}$$

Define $a(r) = e^{A(r)}$
Gauge invariant variables

$$\begin{aligned} R &= \chi - (a^2(\xi + W'))' \\ \Psi &= \psi - aa'(\xi + W') \end{aligned} \tag{2}$$

Primes refer to derivatives with respect to r
Near boundary expansion: use a coordinate change such that $\frac{d\rho}{dr} = -\frac{2\rho}{L}$ where L is the AdS radius. Now the metric components and the scalar field have the following near boudary expansions

$$\begin{aligned} \psi &= h_{(0)} + h_{(2)}\rho + \rho^2(h_{(4)} + hh_{1(4)}\log\rho + hh_{2(4)}\log^2\rho) + ... \\ \chi &= \chi_{(0)} + \chi_{(2)}\rho + \rho^2(\chi_{(4)} + h\chi_{1(4)}\log\rho + h\chi_{2(4)}\log^2\rho) + ... \\ W &= H_{(0)} + H_{(2)}\rho + \rho^2(H_{(4)} + hH_{1(4)}\log\rho + hH_{2(4)}\log^2\rho) + ... \end{aligned} \tag{3}$$

We consider the active and inert scalar fields Φ_a and Φ_i with the following perturbation decomposition

$$\begin{aligned} \Phi_a &= \phi_a(r) + \widetilde{\phi}_a(r,x) \\ \Phi_i &= \phi_i + \widetilde{\phi}_i(r,x) \end{aligned} \tag{4}$$

Obviously enough, the background and perturbations near-boundary expansions are similar.

Einstein equations

The gauge invariant Einstein equations are as follow

$$\begin{aligned} 12a'a\Psi' - 3p^2\Psi &= a^2\kappa^2(-\widetilde{\phi}'_a\phi'_a + \widetilde{\phi}_a V_{\phi_a} + \widetilde{\phi}_i V_{\phi_i} + 2RV) \\ \left(-3aa'(R' - 4\Psi') - 6\left(aa'' + a'^2\right)(R - \Psi) + 3\Psi''a^2 + \nabla^2(R + 2\Psi)\right)\delta_{ij} &- [R + 2\Psi]_{;ij} \\ = a(r)^2\kappa^2\left(\widetilde{\phi}'_a\phi'_a + \widetilde{\phi}_a V_{\phi_a} + \widetilde{\phi}_i V_{\phi_i} - R(\phi'_a)^2 + \Psi(\phi'_a)^2 + 2\Psi V\right)&\delta_{ij} \\ \left[\frac{3a'(r)}{a(r)}R - 3\Psi'\right]_{;i} &= -\kappa^2\widetilde{\phi}_{a;i}\phi'_a \end{aligned} \tag{5}$$

From the second equation with $i \neq j$, we get $R = -2\Psi$ which makes the equation considerably easier. It is possible to combine these equations to get

$$R'' + \left(2A' - \frac{A'''}{A''}\right)R' + \left(4A'' - \frac{2A'A'''}{A''} - e^{-2A}p^2\right) = 0 \tag{6}$$

Where we used the background Einstein equations to replace the potential and background active scalar in (5) with

$$\begin{aligned} V &= \frac{3}{2\kappa^2}(4A'^2 + A'') \\ (\phi_a')^2 &= \frac{3}{\kappa^2}A'' \end{aligned} \tag{7}$$

It is convenient to do the change of variable to ρ to solve equation (6) and find the solution

$$R(p^2,\rho) = -2\frac{\rho}{p^2}F\left(\frac{3}{2} + \frac{\sqrt{1-p^2L^2}}{2}, \frac{3}{2} - \frac{\sqrt{1-p^2L^2}}{2}; 3; 1-\rho\right) \tag{8}$$

With $F(a,b;c;z)$ being a hypergeometric function which admits the following expansion

$$\begin{aligned} F(a,b;a+b+m;z)/\Gamma(a+b+m) = &\frac{\Gamma(m)}{\Gamma(a+m)\Gamma(b+m)}\sum_{n=0}^{m-1}\frac{(a)_n(b)_n}{(1-m)_n n!}(1-z)^n \\ &+ \frac{(1-z)^m(-1)^m}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a+m)_n(b+m)_n}{(n+m)!n!}\left[\alpha_n - \log(1-z)\right](1-z)^n \end{aligned} \tag{9}$$

With

$$\alpha_n = \psi(n+1) + \psi(n+m+1) - \psi(a+n+m) - \psi(b+n+m) \tag{10}$$

In the case of equation (8) we have $m=0$ and so the first sum in equation (9) is to be interpreted as null. It is easy to see that $a = \frac{3}{2} + \frac{\sqrt{1-p^2L^2}}{2}$ and $b = \frac{3}{2} - \frac{\sqrt{1-p^2L^2}}{2}$. When computing the Ricci scalar near the boundary, one can observe that in order to have a constant curvature it is necessary that $\xi = 0$ which will be imposed from here on out. We will study the results in the synchronous gauge, meaning that we set $\chi = 0$

Results for the GPPZ solution

In the GPPZ solution the background scalar field solution is

$$\phi_a = \frac{\sqrt{3}}{2}\log\left[\frac{1+\sqrt{\rho}}{1-\sqrt{\rho}}\right] \tag{11}$$

And $\phi_i = 0$. Both scalars have an expansion going like

$$\phi = \sqrt{\rho}(\phi_{(0)} + \rho(\phi_{(2)} + \psi_{(2)}\log\rho + ...) \tag{12}$$

The scale factor's solution is

$$a^2 = e^{2A(r)} = \frac{1-\rho}{\rho} \tag{13}$$

And the potential takes the form

$$V(\Phi_a,\Phi_i) = -\frac{3}{2L^2}\left[\frac{1}{4}\cosh^2\left(\frac{2\Phi_a}{\sqrt{3}}\right) + \cosh\left(\frac{2\Phi_a}{\sqrt{3}}\right)\cosh(2\Phi_i) - \frac{1}{4}\cosh^2(2\Phi_i) + 1\right] \tag{14}$$

Plugging these expansions in the Klein-Gordon equation of $\widetilde{\phi}_a$ (which will take on perturbation terms coming from the metric), one gets

$$h_{(2)} = \frac{L^2p^2}{8}\left[h_{(0)} + \frac{\sqrt{3}}{6}\phi_{a(0)}\right] + \frac{\sqrt{3}}{6}\left[\phi_{a(0)} + \frac{1}{2}\psi_{a(2)}\right] \tag{15}$$

This is then used in the solutions found using the Einstein equations to get

$$\phi_{a(2)} + \psi_{a(2)} = \frac{\sqrt{3}}{8}L^2p^2h_{(0)} + \frac{1}{16}L^2p^2\phi_{a(0)} + \frac{\phi_{a(0)}}{2} + \frac{\sqrt{3}}{\gamma}\left[\frac{11 - 20\alpha_0 + 8\alpha_1}{2L^2p^2} - \frac{9}{8} + \frac{\alpha_1}{2}\right] \tag{16}$$

Where we defined $\gamma = \Gamma\left(\frac{3}{2} - \frac{1}{2}\sqrt{1 - L^2p^2}\right)\Gamma\left(\frac{3}{2} + \frac{1}{2}\sqrt{1 - L^2p^2}\right)$ and the tilde has been droped but the $\phi_{(n)}$ are coeffecients of the asymptotic expansion of $\widetilde{\phi}_a$.

It is possible to follow a similar procedure for the inert scalar. Namely, we expand the action to second order in perturbation and apply the least action principle. Unlike for the active scalar, the KG equation of $\widetilde{\phi}_i$ will not introduce metric perturbations since ϕ_i is constant. Doing this, at first non-vanishing order in ρ, one will get

$$\psi_{i(2)} = \frac{L^2p^2}{4}\phi_{i(0)} - 2\phi_{i(0)} \tag{17}$$