CS 2850 – Networks HW 3

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1. Let b_i be the bid made by the i-th bidder, and let b_j be the bid made by the j-th bidder where $i, j \in \{2, 3\}$ and $i \neq j$. Suppose that bidder i bids at least as much as bidder j (i.e. $b_i \geq b_j$). In order to win, we must bid $b_1 > b_i$ for the item. Since this is a second-price sealed-bid auction, we will pay b_i for the item if we win which yields a payoff of $v_1 - b_i$. If instead we lose the auction, then our payoff is zero. From this, we consider two cases:

   - **Case 1: $\mathbf{b_i > v_1}$** If $b_i > v_1$ is the second highest bet and we won the auction, then it must be the case that we bid more than v_1. In this case, we are guaranteed to have a payoff of $v_1 - b_i \leq 0$. Notice that winning when $b_i > v_1$ is at best as valuable as not playing at all. Thus, we should never bid more than v_1 when $b_i > v_1$.
   - **Case 2: $\mathbf{b_i \leq v_1}$** In this case, we know that both b_i and b_j are at most v_i. We will still pay b_i for the item, but we are guaranteed to have a payoff of at least zero. More specifically, our payoff is $v_1 - b_i \geq 0$. Notice that when $b_i \leq v_1$, a bid of $b_1 = v_i$ is guaranteed to yield the exact same payoff as a bid of $b_1 > v_i$.

   Thus, bidding above $v_1 = 30$ never increases the payoff and only adds additional risk. Therefore, our friend is incorrect and we should never bid more than 30.

2. Let $\psi_i\left(R, v_i, b_i, b_j\right)$ be the payoff for bidder i when placing a bid of b_i for an object valued at v_i while facing off in a two-buyer auction against some other bidder j who places a bid of b_j. In addition, the auction is a second-price auction with a reserve price R. Let us further define this payoff function as

$$\psi_i\left(R, v_i, b_i, b_j\right) = \begin{cases} 0 & \text{if } b_i \leq b_j \text{ or } b_i \leq R, \\ v_i - \max\{b_j, R\} & \text{if } b_i > b_i \text{ and } b_i > R. \end{cases}.$$

   (a) Let us refer to the opposing buyer as bidder j and ourselves as bidder i. From the question, we know that $R = 10$ and our value is $v_i = 15$. Our goal is to bid such that our expected payoff is maximized, or rather, pick a value b_i such that $\mathbb{E}\left[\psi_i\left(R, v_i, b_i, b_j\right)\right]_{b_j}$ is maximized, where the expected payoff is given by

$$\mathbb{E}\left[\psi_i\left(R, v_i, b_i, b_j\right)\right]_{b_j} = \sum_{b \in \{5,10,15\}} \mathbb{P}\left(b_j = b\right) \cdot \psi_i\left(R = 10, v_i = 15, b_i, b_j = b\right).$$

Computing each value of b_i gives us

$$\begin{aligned}
\mathbf{b_i = 5:} \quad & \mathbb{E}\left[\psi_i\left(R, v_i, b_i = 5, b_j\right)\right]_{b_j} &&= \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 0 \\
& &&= 0, \\
\mathbf{b_i = 10:} \quad & \mathbb{E}\left[\psi_i\left(R, v_i, b_i = 10, b_j\right)\right]_{b_j} &&= \frac{1}{3}(15 - 10) + \frac{1}{3}(15 - 10) + \frac{1}{3} \cdot 0 \\
& &&= \frac{5}{3}, \\
\mathbf{b_i = 15:} \quad & \mathbb{E}\left[\psi_i\left(R, v_i, b_i = 15, b_j\right)\right]_{b_j} &&= \frac{1}{3}(15 - 15) + \frac{1}{3}(15 - 10) + \frac{1}{3} \cdot 0 \\
& &&= \frac{5}{3}.
\end{aligned}$$

Since choosing $b_i = 15$ has a higher win probability than any other value and also yields an expected at least as good as the next choice, we should bid $b_i = 15$.

(b) Let m_0 be the expected revenue earned from the second-price auction with a reserve price of $R = 0$. We can compute m_0 by summing over all possible values of $\min\{b_i, b_j\}$ (note we take the min since this is a second-price auction) and multiplying by the probability of each value occurring. This gives us

$$m_0 = \sum_{b_i \in \{5,10,15\}} \sum_{b_j \in \{5,10,15\}} \frac{1}{9} \min\{b_i, b_j\} = \$7.78.$$

(c) If in this case $R = 10$ and the expected revenue is m_{10}, then we only include values of b_i, b_j if $b_i \geq 10$ or $b_j \geq 10$. This gives us

$$m_{10} = m_0 - \frac{1}{9} \min\{b_i = 5, b_j = 5\} = \$7.22.$$

(d) Let m_p be the expected revenue earned for the object when the seller posts it for price p. We can compute m_p by multiplying p by the probability that at least one of the buyers want to buy the object at a price of at least p–let E_p be this event. This gives us

$$\begin{aligned} m_p &= p \cdot \mathbb{P}\left(E_p\right), \\ m_5 &= 5 \cdot \mathbb{P}\left(E_5\right) = 5 \cdot \frac{9}{9} = \$5, \\ m_{10} &= 10 \cdot \mathbb{P}\left(E_{10}\right) = 10 \cdot \frac{8}{9} = \$8.89, \\ m_{15} &= 15 \cdot \mathbb{P}\left(E_{15}\right) = 15 \cdot \frac{5}{9} = \$8.33. \end{aligned}$$

3. (a) The solution can be obtained by solving the following system of equations:

$$\begin{aligned} x + y &= 100 \\ 10 + \frac{x}{10} &= \frac{y}{20} + 17 \end{aligned}$$

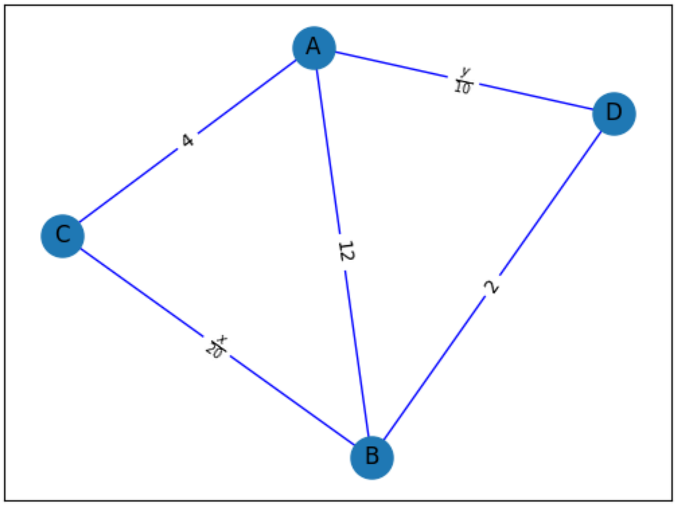
which yields a Nash equilibrium of $x = 80$ and $y = 20$.

(b) Let us refer to the new route (i.e. $A \to E \to B$) as Route III. Notice that $10 + \frac{t}{10} < 17 + \frac{t}{20}$ for all $0 \leq t \leq 100$. Thus, it will always be faster to take Route III over Route II. In other words, Route II is strictly dominated by Route III. It follows that $y = 0$ in the Nash equilibrium. We can determine the value of x, z by solving the following system of equations:

$$\begin{aligned} x + z &= 100 \\ 10 + \frac{x}{10} &= \frac{z}{10} + 10 \end{aligned}$$

which yields a Nash equilibrium of $x = 50$, $y = 0$, and $z = 50$.

4. (a) The following is a map of the game:



Notice that taking Route 3 is strictly dominated if there exists a Nash equilibrium such that

$$4 + \frac{x}{20} \leq 12, \tag{1}$$

$$2 + \frac{y}{10} \leq 12. \tag{2}$$

If we let $z = 0$ for now, then we can solve the following system of equations:

$$x + y = 200$$

$$4 + \frac{x}{20} = 2 + \frac{y}{10}$$

which yields $x = 120, y = 80$. Since these values satisfy equations (1) and (2), we have a Nash equilibrium of $x = 120, y = 80, z = 0$.

(b) **Project 1:** We can find a Nash equilibrium by solving the following system of equations:

$$x + y + z = 200$$

$$4 + \frac{x}{20} = 2 + \frac{y}{10}$$

$$2 + \frac{y}{10} = 5$$

which yields a Nash equilibrium of $x = 20, y = 30, z = 150$ and a total travel time of $t_1 = x\left(4 + \frac{x}{20}\right) + y\left(2 + \frac{y}{10}\right) + 5z = 1000$ hours.

Project 2: With the new road, travelers can simply go from $A \to C \to D \to B$ for a constant travel time of $4+0+2$ hours per traveler. This yields a Nash equilibrium of $x = 0, y = 0, z = 0$ and a total travel time of $t_2 = 200 * 6 = 1200$ hours.

Travelers could also start by taking the $A \to D$ path until $\frac{y}{10} = 4$–at which point it becomes more efficient to take the $A \to C$ path. Thus, $y = 40$ travelers will take the $A \to D$ path and 160 travelers will take the $A \to C$ path. From this point, all travelers have a choice to take the $C \to B$ path or the $D \to B$ path. Moreover, it is more efficient to take the $C \to B$ path until $\frac{x}{20} = 2$–at which point it becomes more efficient to take the $D \to B$ path. Thus, $x = 40$ travelers will take the $C \to B$ path and 160 travelers will take the $D \to B$ path. This yields a Nash equilibrium of $x = 40, y = 40, z = 0$ and a total travel time of $t_2 = 160 \cdot 4 + y \cdot \frac{y}{10} + 160 \cdot 2 + x \cdot \frac{x}{20} = 200 \cdot 4 + 200 \cdot 2 = 1200$ hours.

Conclusion: Project 1 should be chosen over Project 2 because it has a lower total travel time.