Convex Two-Region Segmentation

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目录

1 introduction

Let us start with a Mumford-Shah-like model with two regions (forground/background) and fixed color models:

$$\min_{\Omega_1} \int_{\Omega_1} f_1(x)\, dx + \int_{\Omega-\Omega_1} f_2(x)\, dx + \nu|\partial\Omega_1|,$$

with integrals over $\Omega_1 \subset \Omega$ and its complement $\Omega - \Omega_1$. The integrands may for example arise

- from a Gaussian color model for each region:

$$f_i(x) = \frac{(I(x) - \mu_i)^2}{2\sigma_i^2} + \log\sigma_i$$

- or from general color distribution p_i:

$$f_i(x) = -\log p_i(I(x)).$$

The term $|\partial\Omega_1|$ denotes the length of the boundary $\partial\Omega_1$.

In Chan, Esedoglu, Nikolova, Trans. on Image Proc. 2006, the authors propose to encode the two-region segementation by a binary indicator function

$$u : \Omega \to \{0, 1\}, \quad u(x) = \begin{cases} 1, & \text{if } x \in \Omega_1, \\ 0, & \text{else.} \end{cases}$$

In terms of u, the segmentation problem is

$$E(u) = \int_\Omega f_1(x)u(x)\, dx + \int_\Omega f_2(x)(1 - u(x))\, dx + \nu \int_\Omega |\nabla u(x)|\, dx.$$

It is related to the Chan-Vese model by associating $u \equiv H(\phi)$.

The above functional is convex in u because the first two terms are linear in u and the total variation of u is also convex. The overall optimization problem is not convex because the space of binary functions u is not a convex space: Convex combination of binary functions are typically no longer binary.

The two-region segmentation problem is defined over the space $BV(\Omega; \{0, 1\})$, the space of functions of bounded variation, i.e., functions u for wich the total variation $TV(u)$ is finite.

Relaxation denotes the technique of simply dropping certain constraints from the overall optimization problem. Convex Relaxation means that unpon relaxation the problem becomes convex.

Chan et al.(2006) convexify the two-region segmentation problem by simply dropping the constraint that u must be binary. They allow u to take on values in the entire interval $[0, 1]$, which is the convex hull of the original domain:

$$\min_{u\in BV(\Omega;[0,1])} E(u).$$

By construction, this is a convex optimization problem. The hard labeling of each pixel as 0 or 1 is replaced by a soft labeling of each pixel with some value between 0 and 1.

In general, the optimum of relaxed problem

$$u^* = \arg \min_{u\in BV(\Omega;[0,1])} E(u)$$

is not binary. A binary function is obtained by thresholding:

$$1_{u^*>\theta}(x) = \begin{cases} 1, & \text{if } u^*(x) > \theta, \\ 0, & \text{else.} \end{cases}$$

Such relaxation techniques can be applied to many optimization problems. In general, one loses optimality as the thresholded solution is typically not an optimum for the original binary labelling problem.

Surprisingly, this is not the case for the functional considered here. More specifically, one can show that the thresholded solution $1_{u^*>\theta}$ has the same energy as the relaxed solution u^*. As a consequence, it is indeeded a global optimum of the original binary labeling problem.

Let

$$u^* = \arg \min_{u \in BV(\Omega;[0,1])} \int_\Omega fu + |\nabla u| \, dx$$

be a global minimizer of the relaxed problem with an arbitrary function f. Then the function $1_{u^*>\theta}$ is a global minimizer of the corresponding binary optimization problem for any threshold value $\theta \in (0,1)$.

2 General definition of total variation

So far, we worked with a definition of total variation which is not differentiable and which only applies to differentiable functions.

A remedy is given by introducing a dual variable $\boldsymbol{\xi} \in \mathbb{R}^2$:

$$|\nabla u| = \sup_{|\boldsymbol{\xi}| \leq 1} \boldsymbol{\xi} \cdot \nabla u.$$

where the supremum is attained at $\boldsymbol{\xi} = \frac{\nabla u}{|\nabla u|}$ if $\nabla u \neq 0$.

It allows to generalize the total variation to a differentiable expression which which also applies to discontinuous functions u:

$$TV(u) := \sup_{\boldsymbol{\xi} \in \mathcal{K}} \int_\Omega u \, \mathrm{div} \boldsymbol{\xi} \, dx = \sup_{\boldsymbol{\xi} \in \mathcal{K}} \int_\Omega \boldsymbol{\xi} \cdot \nabla u \, dx = \int_\Omega |\nabla u| \, dx,$$

with the dual variable $\boldsymbol{\xi}$ being a differentiable vector field with compact support (i.e. $\boldsymbol{\xi} = 0$ at the boundary), constrined to the unit disc at every point $x \in \Omega$:

$$\mathcal{K} = \left\{ \boldsymbol{\xi} \in C_c^1(\Omega; \mathbb{R}^2) : |\boldsymbol{\xi}(x)| \leq 1 \quad \forall x \in \Omega \right\}.$$

3 Minimization with Primal-Dual Algorithm

The two-region segmentation with known color models can be solved by thresholding the solution of the relaxed (convex) problem which is of the form

$$\min_{u \in C} \int_\Omega fu \, dx + TV(u) = \min_{u \in C} \sup_{\boldsymbol{\xi} \in \mathcal{K}} \int_\Omega fu + u \, \mathrm{div} \boldsymbol{\xi} \, dx,$$

where $C = BV(\Omega; [0,1])$.

An efficient algorithm for minimizing this saddle point problem was proposed in Pock, Cremers, Chambolle, Bischof, ICCV 2009. It amounts to an alternating projected gradient descent/ascent with an extrapolation step:

$$\begin{cases} \boldsymbol{\xi}^{n+1} = \Pi_{\mathcal{K}}(\boldsymbol{\xi}^n - \sigma \nabla \bar{u}^n), \\ u^{n+1} = \Pi_C(u^n - \tau(\, \mathrm{div} \boldsymbol{\xi}^{n+1} + f), \\ \bar{u}^{n+1} = u^{n+1} + (u^{n+1} - u^n) = 2u^{n+1} - u^n, \end{cases}$$

where $\Pi_{\mathcal{K}}$ and Π_C denote the back-projections onto $\mathcal{K}$ and C. It provably converges for sufficient small step sizes σ and τ.

For the primal variable u the projection onto the set $C = BV(\Omega; [0,1])$ is done by clipping:

$$(\Pi_C u)(x) = \max\left\{1, \min\{0, u(x)\}\right\} = \begin{cases} u(x), & \text{if } u(x) \in [0,1], \\ 1, & \text{if } u(x) > 1, \\ 0, & \text{if } u(x) < 0. \end{cases}$$

For the dual variable $\boldsymbol{\xi}$ projection onto the unit disk $\mathcal{K}$ is done as follows:

$$(\Pi_{\mathcal{K}} \boldsymbol{\xi})(x) = \frac{\boldsymbol{\xi}(x)}{\max\{1, |\boldsymbol{\xi}(x)|\}}.$$