Random Geometric Graphs

A collection of notes from my time working under the supervision of Mathew Penrose on random geometric graph theory and related topics.

Jacob Green
Department of Mathematical Sciences, The University of Bath

October 23, 2024

Abstract

Suspendisse vitae elit. Aliquam arcu neque, ornare in, ullamcorper quis, commodo eu, libero. Fusce sagittis erat at erat tristique mollis. Maecenas sapien libero, molestie et, lobortis in, sodales eget, dui. Morbi ultrices rutrum lorem. Nam elementum ullamcorper leo. Morbi dui. Aliquam sagittis. Nunc placerat. Pellentesque tristique sodales est. Maecenas imperdiet lacinia velit. Cras non urna. Morbi eros pede, suscipit ac, varius vel, egestas non, eros. Praesent malesuada, diam id pretium elementum, eros sem dictum tortor, vel consectetuer odio sem sed wisi.

Keywords: Random geometric graphs, Poisson Point Processes, Combinatorial Geometry, Sphere Packing

Contents

1 Introduction

1.1 What is a random geometric graph?

Definition 1.1 (geometric graph)

Let $\mathcal{X} \subset \mathbb{R}^d$ be a finite collection of points and $r > 0$. The *geometric graph* $G(\mathcal{X}, r)$ is the graph with vertex set $\mathcal{X}$ and edge set $\{\{x, y\} \subset \mathcal{X} \ : \ |x - y| \leq r\}$, where $|\cdot|$ denotes the Euclidean norm.

We can turn this into a random graph by letting the set $\mathcal{X} \subset \mathbb{R}^d$ be random. The resulting structure $G(\mathcal{X}, r)$ is said to be the *random geometric graph* (A.K.A the *Gilbert graph*). The first structure we'll consider is found by uniformly scattering n points in the d-dimensional hypercube $[0, 1]^d$.

Example 1.1 (fixed scale geometric graph)

Let $\xi_1, \xi_2, \ldots \overset{\text{i.i.d}}{\sim} \text{Unif}[0, 1]^d$ and fix a sequence $(r_n)_{n \geq 1}$ in $\mathbb{R}_{>0}$. Let $\mathcal{X}_n := \{\xi_1, \ldots, \xi_n\}$. Then $G(\mathcal{X}_n, r_n)$ is the *fixed scale geometric graph* at time $n \geq 1$.

We will also consider what happens when the number of vertices is Poisson.

Example 1.2 (Poisson scale geometric graph)

Let $\xi_1, \xi_2, \ldots \overset{\text{i.i.d}}{\sim} \text{Unif}[0, 1]^d$ and $N_n \sim \text{Poisson}(n)$ for $n \geq 1$ be independent of $(\xi_1, \xi_2, \ldots)$. Let $\mathcal{P}_n := \{\xi_1, \ldots, \xi_{N_n}\}$ and fix a sequence $(r_n)_{n \geq 1}$ in $\mathbb{R}_{>0}$. Then the random geometric graph $G(\mathcal{P}_n, r_n)$ is the d-dimensional *Poisson scale geometric graph* at time $n \geq 1$.

As one may expect, the vertices of this graph form a Poisson point process in $\mathbb{R}^d$. This was originally an exercise (1.1) in [2].

Proposition 1.1 ($\mathcal{P}_n(\cdot)$ is a Poisson point process)

Let $\mathcal{P}_n$ be the vertex set in the Poisson scale geometric graph. Then $\mathcal{P}_n(\cdot) : A \mapsto |\mathcal{P}_n \cap A|$, where $A \subset [0, 1]^d$ is measurable and $|\cdot|$ is the counting measure, is a *Poisson point process* with intensity $\lambda(A)$, $\lambda(\cdot)$ the Lebesgue measure on $\mathbb{R}^n$.

See Appendix A for the definition of the Poisson point process.

Proof. Write $\mathcal{P}_n(A_j) = \sum_{i=1}^{N_n} \mathbf{1}\{\xi_i \in A_j\}$. Then, conditioning on $\{N_n = m\}$, each $\mathcal{P}_n(A_j)$ is a $\text{Binom}(m, \lambda(A_j))$ random variable, making the joint distribution (also conditioned on $\{N_n = m\}$) a $\text{Multinom}(m, \lambda(A_1), \ldots, \lambda(A_k))$. Thus one has

$$\mathbb{P}(\mathcal{P}_n(A_1) = a_1, \ldots, \mathcal{P}_n(A_k) = a_k) = \underbrace{\left(\frac{e^{-n} n^m}{m!}\right)}_{\text{Poisson}(n;m)} \times \underbrace{\binom{m}{j_1, \ldots, j_k} \prod_{i=1}^{k} (\lambda(A_i))^{j_i}}_{\text{Multinom}(m, \lambda(A_1), \ldots, \lambda(A_k); j_1, \ldots, j_k)}$$
$$= \prod_{i=1}^{k} \frac{e^{-n\lambda(A_i)} (n\lambda(A_i))^{j_i}}{(j_i)!}$$

which is a product of the distributions $\text{Poisson}(n\lambda(A_i)), 1 \leq i \leq k$, giving us both of our defining properties of a Poisson point process. $\square$

The exercise (1.2) in [2] states the following result, which we'll prove in an analogous way.

Proposition 1.2 (expected degree of k^{th} vertex in fixed scale RGG)

Consider the fixed scale random geometric graph $G(\mathcal{X}_n, r_n)$ with $r_n \to 0$ as $n \to \infty$. Let $D_{k,n}$ be the degree of the first vertex ξ_k, then $\mathbb{E}[D_{k,n}] \sim \theta n r_n^d$.

Proof. Write $D_{k,n} = \sum_{i \neq k} \mathbf{1}\{|\xi_k - \xi_i| \leq r\}$, then we have $D_{k,n} \sim \text{Binom}(n-1, p)$ with

$$p = \lambda(B_{r_n}(\xi_k) \cap [0,1]^d)$$

where $B_{r_n}(\xi_k)$ is the hypersphere radius r_n centred at ξ_k and λ the Lebesgue measure on $\mathbb{R}^d$. Now, conditioning on ξ_k, we have

$$\mathbb{E}[D_{k,n}|\xi_k] = (n-1)\lambda(B_{r_n}(\xi_k) \cap [0,1]^d)$$

and hence, by the total law of expectation,

$$\begin{aligned}
\mathbb{E}[D_{k,n}] &= \mathbb{E}[\mathbb{E}[D_{k,n}|\xi_k]] \\
&= (n-1) \int_{[0,1]^d} \lambda(B_{r_n}(\xi_k) \cap [0,1]^d) \mathrm{d}\xi_k \\
&\sim n \int_{[0,1]^d} \lambda(B_{r_n}(\xi_k)) \mathrm{d}\xi_k \sim n(\theta r_n^d)
\end{aligned}$$

Where the first asymptotic equality follows via the dominated convergence theorem (to see this, observe $B_{r_n}(\xi_k) \cap [0,1]^d \to B_{r_n}(\xi_k)$ A.E. and $B_{r_n}(\xi_k)$ also dominating). Note θ is a constant depending on d, and is the coefficient of r_n^d in the formula for the volume of a hypersphere radius r_n. ☐

1.2 Counting Edges of $G(\mathcal{X}_n, r_n)$ and $G(\mathcal{P}_n, r_n)$

The number of edges $\mathcal{E}_n, \mathcal{E}'_n$ in $G(\mathcal{X}_n, r_n)$ and $G(\mathcal{P}_n, r_n)$ respectively is a Poisson random variable. To prove this, we'll use a so called *dependency graph*. We handle the two RGGs $G(\mathcal{X}_n, r_n)$ and $G(\mathcal{P}_n, r_n)$ seperately.

Counting edges of $G(\mathcal{X}_n, r_n)$

An elementary question to ask is "what is the expected edge count at time n?". Via the previously derived vertex degree asymptotics, we have an easy asymptotic result.

Proposition 1.3 (expected edge count in fixed scale RGG)

Let $\mathcal{E}_n$ and $\mathcal{E}'_n$ be the number of edges in $G(\mathcal{X}_n, r_n)$ and $G(\mathcal{P}_n, r_n)$ respectively. Then,

$$\mathbb{E}[\mathcal{E}_n] \sim \theta n^2 r_n^d / 2$$

Proof of lemma. By the handshaking lemma we have $2\mathbb{E}[\mathcal{E}_n] = \sum_{k=1}^n \mathbb{E}[D_{k,n}] \sim n^2 r_n^d$. ☐

Now we introduce the *dependency graph.*

Definition 1.2 (dependency graph)

Let $(V, \sim)$ be a finite simple graph w/ edge relation $\sim$ and vertex set V. We say $(V, \sim)$ is a *dependency graph* for the random variables $(W_\alpha)_{\alpha \in V}$ if whenever $A, B \subset V$ are disjoint with no $\alpha \in A$, $\beta \in B$ such that $\alpha \sim \beta$ (i.e. A and B lie in unique connected components) then

$$(W_\alpha)_{\alpha \in A} \perp\!\!\!\perp (W_\beta)_{\beta \in B} \quad \text{i.e. are independent families}$$

When we take the random variable associated with each vertex to be 0-1, we obtain the following bound.

Lemma 1.1 (Poisson approximation lemma for Bernoulli sums)

Let $(\xi_i)_{i \in I}$ be a finite collection of Bernoulli random variables with dependency graph $(I, \sim)$. Set $p_i := \mathbb{P}(\xi_i = 1)$, $p_{ij} := \mathbb{P}(\xi_i = 1; \xi_j = 1)$ and suppose $\lambda := \sum_{i \in I} p_i$ is finite. Then, letting $W := \sum_{i \in I} \xi_i$, we have (LHS is just total variation distance)

$$\sum_{k \geq 0} \Big| \mathbb{P}[W = k] - \mathbb{P}(\text{Po}(\lambda) = k) \Big| \leq \min\{6, 2\lambda^{-1}\} \left(\sum_{i \in I} \sum_{\mathcal{N}(i) \setminus \{i\}} p_{ij} + \sum_{i \in I} \sum_{j \in \mathcal{N}(i)} p_i p_j \right)$$

Proof. Omitted, consult [4]. □

Theorem 1.1 (total variation between edge and Poisson distribution)

Let $\lambda_n := \mathbb{E}[\mathcal{E}_n]$. Then

$$\sum_{k\geq 0} |\mathbb{P}(\mathcal{E}_n = k) - \mathbb{P}(\text{Po}(\lambda_n) = k)| = O(nr^d)$$

The idea of this proof is to write $\mathcal{E}_n = \sum_{1\leq i<j\leq n} \mathbf{1}\{|\xi_i - \xi_j| \leq r_n\}$, i.e. the number of edges as the sum of the indicators over all possible edges indicating whether this edge exists. This has a rather obvious dependency graph and hence we can apply the Poisson approximation lemma (1.7) with some work on the asymptotics.

Proof. Let $V\{\{i,j\} : 1 \leq i < j \leq\}$ and define $\sim$ by $\alpha \sim \beta$ if $\alpha \cap \beta \neq \emptyset$ and $\alpha \neq \beta$. Define the random variables $W_\alpha = \mathbf{1}\{|\xi_i - \xi_j| \leq r_n\}$ for $\alpha \in V$ and let $\lambda_n := \sum_{\alpha\in V} p_\alpha$.

Claim: $G(V, \sim)$ is a dependency graph for $(W_\alpha)_{\alpha\in V}$.

Proof of claim. Follows immediately from the independence of the ξ_i. □

Let $p_\alpha := \mathbb{P}(W_\alpha = 1)$ and $p_{\alpha\beta} := \mathbb{P}(W_\alpha = 1; W_\beta = 1)$, as in the setup of lemma 1.7.

Claim: $p_\alpha \sim \theta r_n^d$

Proof of claim. Fix $\alpha = \{i, j\} \in V$. Then, letting $\lambda(\cdot)$ be the Lebesgue measure on $\mathbb{R}^d$,

$$\begin{aligned} p_\alpha = \mathbb{P}(|\xi_i - \xi_j| \leq r_n) &= \int_{v\in[0,1]^d} \mathbb{P}(|\xi_i - \xi_j| \leq r_n | \xi_i = v)\mathrm{d}v \\ &= \int_{v\in[0,1]^d} \lambda(B_{r_n}(v) \cap [0,1]^d)\mathrm{d}v \\ &\sim \int_{v\in[0,1]^d} \lambda(B_{r_n}(v))\mathrm{d}v \sim \theta r_n^d \end{aligned}$$

where the first asymptotic equality follows from the dominated convergence theorem. □

Claim: $p_{\alpha\beta} \sim (\theta r_n^d)^2$

Proof of claim. An analogous argument works. □

Now we have all our ingredients, lets cook. Observe, from the previously computed asymptotics,

$$\lambda_n \sim n^2\theta r_n^d/2 \quad \text{and} \quad \sum_{\alpha\in V} p_\alpha^2 \sim \theta\lambda_n r_n^d$$

and hence by counting $|\mathcal{N}(\alpha)| = 2(n-2)$, obtain

$$\sum_{\alpha\in V}\sum_{\beta\sim\alpha} (p_{\alpha\beta} + p_\alpha p_\beta) \sim \binom{n}{2} \times 2(n-2) \times 2(\theta r_n^d)^2 = O(n\lambda_n r_n^d)$$

which by the Poisson approximation lemma gives

$$\sum_{k\geq 0} |\mathbb{P}(\mathcal{E}_n = k) - e^{-\lambda_n}\lambda_n^k/k!| = O(nr_n^d)$$

as claimed. □

Corollary 1.1 (edge distribution of the fixed scale random geometric graph)

If the limit $\lambda_n \to \lambda \in (0, \infty)$ exists and $nr_n^d \to 0$, then $\mathcal{E}_n \xrightarrow{\mathcal{D}} \text{Poisson}(\lambda)$

Proof. By proposition 1.6, we have $\lambda \sim \lambda_n \sim \theta n^2 r_n^d/2 \Rightarrow n^2 r_n^d \sim 2\lambda/\theta \in (0, \infty)$ which forces $nr_n^d \to 0$ and hence lemma 1.8 applies giving $\mathcal{E}_n \xrightarrow{\mathcal{D}} \text{Poisson}(\lambda)$. $\square$

Counting edges of $G(\mathcal{P}_n, r_n)$

We now spend the rest of the (sub)section proving similar results for $G(\mathcal{P}_n, r_n)$. Before proceeding we have the following lemma from [2]. We write $\mathcal{P}_{<\infty}(A)$ for the family of finite subsets of A.

Lemma 1.2 (Mecke formula)

Let $k \in \mathbb{N}$. For any measurable $f : (\mathbb{R}^d)^k \times \mathcal{P}_{<\infty}([0, 1]^d) \to \mathbb{R}$, when the expectation exists,

$$\mathbb{E} \sum_{X_1,\dots,X_k \in \mathcal{P}_n}^{\neq} f(X_1, \dots, X_k, \mathcal{P}_n \setminus \{X_1, \dots, X_n\}) = n^k \int \cdots \int \mathbb{E} f(x_1, \dots, x_k, \mathcal{P}_n) \mathrm{d}x_1 \dots \mathrm{d}x_k$$

where $\sum^{\neq}$ denotes the sum over the ordered k-tuples of distinct points (in $\mathcal{P}_n$).

Proof. The idea is to condition on the number of points in $\mathcal{P}_n$. Observe,

$$\begin{aligned}
&\mathbb{E} \sum_{X_1,\dots,X_k \in \mathcal{P}_n}^{\neq} f(X_1, \dots, X_k, \mathcal{P}_n \setminus \{X_1, \dots, X_n\}) \\
&= \mathbb{E}\left[\mathbb{E}\left[\sum_{X_1,\dots,X_k \in \mathcal{P}_n}^{\neq} f(X_1, \dots, X_k, \mathcal{P}_n \setminus \{X_1, \dots, X_n\}) \Big| |\mathcal{P}_n| = m\right]\right] \\
&= \sum_{m \geq k} \left(\frac{e^{-n} n^m}{m!}\right) \sum_{X_1,\dots,X_k \in \{\xi_1,\dots,\xi_m\}}^{\neq} \mathbb{E} f(X_1, \dots, X_k, \{X_{k+1}, \dots, X_m\}) \\
(\dagger) \qquad &= \sum_{m \geq k} \left(\frac{e^{-n} n^m}{m!}\right) \left(\prod_{i=1}^{k} (m - i + 1)\right) \mathbb{E} f(X_1, \dots, X_k, \{X_{k+1}, \dots, X_m\}) \\
&= \sum_{m \geq k} \left(\frac{e^{-n} n^m}{m!}\right) \left(\prod_{i=1}^{k} (m - i + 1)\right) \int_{[0,1]^d} \cdots \int_{[0,1]^d} f(x_1, \dots, x_k, \{x_{k+1}, \dots, x_m\}) \mathrm{d}x_1 \dots \mathrm{d}x_m \\
&= n^k \sum_{m \geq k} \left(\frac{e^{-n} n^{m-k}}{(m-k)!}\right) \int_{[0,1]^d} \cdots \int_{[0,1]^d} f(x_1, \dots, x_k, \{x_{k+1}, \dots, x_m\}) \mathrm{d}x_1 \dots \mathrm{d}x_m \\
&= n^k \int_{[0,1]^d} \cdots \int_{[0,1]^d} f(x_1, \dots, x_k, \{x_{k+1}, \dots, x_m\}) \mathrm{d}x_1 \dots \mathrm{d}x_m \\
&= n^k \int_{[0,1]^d} \cdots \int_{[0,1]^d} \mathbb{E} f(x_1, \dots, x_k, \mathcal{P}_n) \mathrm{d}x_1 \dots \mathrm{d}x_k
\end{aligned}$$

Where ($\dagger$) follows from the ξ_i being i.i.d, so all expectations are equal and it suffices to count distinct k-tuples in $\{\xi_1, \dots, \xi_m\}$. [CHECK THIS PROOF W/ PENROSE] $\square$

We can now use this lemma to find the expected number of edges in $G(\mathcal{P}_n, r_n)$.

Proposition 1.4 (expected edge count in Poisson scale RGG)

Let $G(\mathcal{P}_n, r_n)$ be the Poisson scale RGG and $\mathcal{E}_n$ be the number of edges at time n. Then,

$$\mathbb{E}[\mathcal{E}_n] \sim \theta n^2 r_n^d/2$$

Proof. Write $\mathcal{E}_n = \frac{1}{2}\sum^{\neq}_{X_1,X_2\in\mathcal{P}_n} \mathbf{1}\{\|X_1 - X_2\| \leq r_n\}$. Then we can use Mecke's formula as follows.

$$
\begin{aligned}
\mathbb{E}[\mathcal{E}_n] &= \frac{1}{2}\mathbb{E}\left[\sum^{\neq}_{X_1,X_2\in\mathcal{P}_n} \mathbf{1}\{\|X_1 - X_2\| \leq r_n\}\right] \\
\text{(Mecke)} \qquad &= \frac{n^2}{2}\int_{[0,1]^d}\int_{[0,1]^d} \mathbb{E}[\mathbf{1}\{\|x_1 - x_2\| \leq r_n\}]\mathrm{d}x_1\mathrm{d}x_2 \\
&= \frac{n^2}{2}\int_{[0,1]^d}\int_{[0,1]^d} \mathbf{1}\{\|x_1 - x_2\| \leq r_n\}\mathrm{d}x_1\mathrm{d}x_2 \\
&= \frac{n^2}{2}\int_{[0,1]^d} \lambda(B(x_1, r_n)\cap[0,1]^d)\mathrm{d}x_1 \sim \theta n^2 r_n^d/2
\end{aligned}
$$

where the final asymptotic equality follows from the dominated convergence theorem and volume of a d-dimensional hypersphere. Note we have already seen the asymptotics for the final integral. □

Rest follows on from Exercise (2.2), too small brained to figure it out right now will attempt again tomorrow :<

1.3 A central limit theorem for our edge counts

Lemma 1.3 (Falling factorial moment for the Poisson distribution)

Let $X \sim \text{Poisson}(\lambda)$ and fix $k \in \mathbb{N}$. Then $\mathbb{E}[X(X-1)\dots(X-k+1)] = \lambda^k$

Proof. This follows by the following simple calculation.

$$
\begin{aligned}
\mathbb{E}[X(X-1)\cdots(X-k+1)] &= \sum_{x\geq 0} x(x-1)\cdots(x-k+1)\frac{e^{-\lambda}\lambda^x}{x!} \\
= e^{-\lambda}\sum_{x\geq k} x(x-1)\cdots(x-k+1)\frac{\lambda^x}{x!} &= \lambda^k e^{-\lambda}\sum_{x\geq k}\frac{\lambda^{x-k}}{(x-k)!} = \lambda^k
\end{aligned}
$$

□

2 The sphere packing problem and geometric graphs

Here I take some notes on chapter 7 of Pach & Agarwals combinatorial geometry [WILEY GIVES BAD CITATION], hoping to find bridges between random geometric graphs and combinatorial geometry.

As the title suggests, we spend this section looking for the existence of high density packings in $\mathbb{R}^d$. We take a non construct approach, instead turning to the probabilistic method. For a convex body $C \subset \mathbb{R}^d$, denote the maximal packing (respectively, minimal covering) density by $\delta(C)$ (respectively, $\vartheta(C)$).

Let us first consider $C = B^d \subset \mathbb{R}^d$, the d-dimensional ball. Then one has the following easy observation.

Proposition 2.0 **(easy lower bound for $\delta(B^d)$)**

Let $B^d \subset \mathbb{R}$ be the d-dimensional unit ball. Then,

$$\delta(B^d) \geq 2^{-d}\vartheta(B^d) \geq 2^{-d}$$

Proof. The latter inequality is trivial ($\vartheta(C) \geq 1$). For the former, let us consider a *saturated packing* $\mathcal{B} = \{B^d + c_i | i \in \mathbb{N}\}$ (that is, a packing such that no body can be added without intersecting a pre-existing body) and the factor 2 enlargement $\mathcal{B}' = \{2B^d + c_i | i \in \mathbb{N}\}$.

Claim: $\mathcal{B}'$ is a covering of $\mathbb{R}^d$.

Proof of claim. Suppose $x \notin \cup_{B \in \mathcal{B}'} B$. Then we have $|x - c_i| > 2$ for som $i \geq 1$, which gives

$$(B^d + c_i) \cap (B^d + x) = \emptyset$$

Contradiction! □

Hence, letting $d(C, \mathbb{R}^d)$ denote the density of a body C in $\mathbb{R}^d$, we clearly have

$$\vartheta(B^d) \leq d(\mathcal{B}', \mathbb{R}^d) = 2^d d(\mathcal{B}, \mathbb{R}^d) \leq 2^d \delta(B^d)$$

and the result is proven. □

In high dimensional Euclidean space, this trivial bound was the best bound known, up to a constant, for a long time! In fact, no construction of saturated lattice packings of balls is known (again, in high dimensional space), worth checking if this has changed since time of writing. This is surprising, given the obvious drawbacks of this argument. If we saturate a packing by continuously adding balls, we can't control the structure of this packing.

Let $\delta_L(C)$ be the maximal lattice packing density of C into $\mathbb{R}^d$. Using techniques developed by Minkowski and Hlawka, we can prove $\delta_L(B^d) > 2^{-d}$. Namely, we can affirm the existence of a lattice packing strictly outperforming our previously derived lower bound of 2^{-d}.

Definition 2.0 **(star-shaped body)**

Let $C \subset \mathbb{R}^d$ be compact. We call C a *star-shaped body* if $\mathbf{0}$ is in the interior of C and has the relation $\mathbf{x} \in C \Rightarrow \lambda\mathbf{x} \in C$ for all $\lambda \in [0, 1]$.

Definition 2.1 **(admissible latice)**

Let $\Lambda = \Lambda(u_1, \ldots, u_d) = \{m_1 u_1 + \cdots + m_d u_d | m_1, \ldots, m_d \in \mathbb{Z}\}$ be a lattice in $\mathbb{R}^d$ and let C be a star-shaped body. Then Λ is said to be *admissible* for C if it contains no interior point of $C \setminus \{\mathbf{0}\}$.

Definition 2.2 (critical determinant)

Let C be a star-shaped body. The *critical determinant* of C is defined by

$$\Delta(C) := \inf\{\det \Lambda | \Lambda \text{ is admissible for } C\}$$

Lemma 2.0 (Mahler selection theorem - 1946)

Let $\Lambda_1, \Lambda_2, \ldots$ be an infinite sequence of lattices in $\mathbb{R}^d$ that have constants $\alpha, \beta > 0$ satisfying, for $i \geq 1$,

(i) Λ_i is admissible for αB^d

(ii) $\det \Lambda_i \leq \beta$

Then we can select a convergent subsequence $\Lambda_{i_1}, \Lambda_{i_2}, \cdots \to \Lambda$.

Proof. Given as exercise, solve at some point. □

This lemma allows us to prove that the infimum is obtained (verify this also), giving us the following reformulation of Minkowski's fundamental theorem (or as I prefer, Minkowski's pidgeonhole principle).

Theorem 2.0 (Minkowski's critical determinant lower bound)

For any centrally symmetric convex body $C \subset \mathbb{R}^d$,

$$\frac{\Delta(C)}{\text{vol}C} \geq \frac{1}{2^d}$$

Finding an upper bound for the ratio $\Delta(C)/\text{vol}C$ turns out the be much harder. Minkowski was able to give an upper bound, when $C = B^d$, of $\frac{1}{2}(\zeta(d))^{-1}$ where $\zeta(\cdot)$ is the Reimman zeta function. We prove, thanks to Hlawka, a generalisation of Minkowski's result.

Lemma 2.1 (Davenport & Rogers)

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous map that vanishes outside of some bounded region. For $\gamma \in \mathbb{R}$ set

$$V(\gamma) := \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1, \ldots, x_{d-1}, \gamma) \mathrm{d}x_1 \cdots \mathrm{d}x_{d-1}$$

Fix $\delta > 0$ and let Λ' be the integer lattice in the hyperplane $x_d = 0$. For a vector $y \in \mathbb{R}^d$ of the form $y = (y_1, \ldots, y_{d-1}, \delta)$ define Λ_y as the lattice generated by y and Λ'. Then,

$$\int_0^1 \cdots \int_0^1 \left(\sum_{x \in \Lambda_y, x_d \neq 0} f(x) \right) \mathrm{d}y_1 \cdots \mathrm{d}y_{d-1} = \sum_{i \in Z \setminus \{0\}} V(i\delta)$$

Proof. Looks like a routine computation but theres one line dependent on voodoo magic, can't wrap my head around that bloody jump, but I imagine its something dumb I'm missing :(□

Now for our generalisation. We work, as suggested, with the probabilistic method.

Theorem 2.1 (Hlawka's theorem)

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a bounded Reimann integrable function that vanishes outside of some bounded region and let $\epsilon > 0$. Then there exists a unit lattice Λ in $\mathbb{R}^d$ with

$$\sum_{\mathbf{0} \neq x \in \Lambda} f(x) < \int_{\mathbb{R}^d} f(x) \mathrm{d}x + \epsilon$$

Proof. Skill issue + lack of relevance. Moving onto non-lattice packings, if this becomes interesting again I'll revisit. □

3 Irregular sphere packings: A breakthrough in high dimensional sphere packing

We denote the maximal density unit-volume sphere packing in $\mathbb{R}^d$ by $\theta(d)$. Formally, we call a collection of points $\mathcal{P} \subset \mathbb{R}^d$ a *packing* if $\|x-y\|_2 \geq 2r_d$ for all $x, y \in \mathcal{P}$, where r_d is the radius of the unit-volume d-dimensional hypersphere. Letting $\mathcal{P}$ be the collection of all packings in $\mathbb{R}^d$, we may define

$$\theta(d) := \sup_{\mathcal{P} \in \mathcal{P}} \limsup_{R \to \infty} \frac{|\mathcal{P} \cap B_{\mathbf{0}}(R)|}{\mathrm{Vol}(B_{\mathbf{0}}(R))}$$

where $\mathrm{Vol}(\cdot)$ is the Lebesgue measure in $\mathbb{R}^d$ and $B_{\mathbf{0}}(R)$ is the ball radius R centred at $\mathbf{0} \in \mathbb{R}^d$.

In this section, we follow the work of Campos & Co to prove $\theta(d) \to (1-o(1))\frac{d \log d}{2^{d+1}}$ as $d \to \infty$. Campos & Co utilise a Poisson random geometric graph subject to some regularity conditions to improve Hayes' 1947 bound on the maximal density of a sphere packing in high dimensional $\mathbb{R}^d$ by a factor of $\log d$. It is worth noting that this is still an exponential factor behind the best upper bound, which currently is $2^{(-0.599\cdots+o(1))d}$. The explicit value of $\theta(d)$ is only known for $d \in \{1, 2, 3, 8, 24\}$.

Of particular interest is the structural irregularity of the packings used to prove this theorem. Previous work relied on lattice packings, whereas this work constructs packings by recursively adding random spheres to $\mathbb{R}^d$ that are centred on points of an appropriately modified Poisson process.

Here is The Theorem™

Theorem 3.1 **($\theta(d)$ bound / theorem 1.1 in [1])**

As $d \to \infty$, one has

$$\theta(d) \geq (1-o(1))\frac{d \log d}{2^{d+1}}$$

To prove this, we need a tight bound for the size of the maximal independent set in an "almost-regular" graph. Campos & Co introduce a new, sharp (up to some constants), bound for the size of this set. With this tool the proof of The Theorem™is remarkably easy, however proving this graph theoretic tool will take some work.

Let $\Delta(G)$ be the maximum vertex degree in a graph G and let $\Delta_{\mathrm{co}}(G)$ denote the maximum codegree (that is, the maximal size of $|\mathcal{N}(u) \cap \mathcal{N}(v)|$ over vertices $u, v \in V$).

Theorem 3.2 **($\alpha(G)$ bound / theorem 1.3 in [1])**

Let G be a graph on n vertices with $\Delta(G) \leq \Delta$ and $\Delta_{\mathrm{co}}(G) \leq C\Delta(\log \Delta)^{-c}$. Then,

$$\alpha(G) \geq (1-o(1))\frac{n \log \Delta}{\Delta}$$

where $o(1) \to 0$ as $\Delta \to \infty$.

In fact, one can take $C = 2^{-7}$ and $c = 7$.

This bound is sharp up to some constants c and C. To see this, ... [Not sure I fully get this, pls fix].

Campos & Co also managed to adapt their techniques to the problem of spherical codes. I will elect to leave that to their paper (see Theorem 1.2 & Appendix A. in [1]).

3.1 Proof of The Theorem™

Our plan of attack is as follows.

(i) Discretize space with a Poisson process at a carefully chosen intensity.

(ii) Impose additional uniformity properties on the discrete point set given in (i), giving $X \subset \mathbb{R}^d$.

(iii) Consider geometric graph $G = G(X, 2r_d)$, r_d the radius of the d-dimensional unit-volume ball.

(iv) Bound $\alpha(G)$, the largest independent set in G.

Assuming for the moment our $\alpha(G)$ bound is true, we can prove The Theorem™with ease. All we will need is the following lemma, which says we can take a large collection of points X that has the geometric graph $G(X, 2r_d)$ in the conditions of Theorem 1.2 (our $\alpha(G)$ bound).

Lemma 3.1 (large uniform geometric graphs / lemma 2.1 in [1])

Let $\Omega \subset \mathbb{R}^d$ be bounded and measurable. Then $\forall d \geq 1000\ \exists X \subset \Omega$ with

$$|X| \geq (1 - 1/d)\frac{\Delta}{2^d}\mathrm{Vol}(\Omega), \quad \text{where} \quad \Delta = \left(\frac{\sqrt{d}}{4\log d}\right)^d$$

and, setting $G = G(X, r_d)$ we have

$$\Delta(G) \leq \Delta(1 + \Delta^{-1/3}) \quad \text{and} \quad \Delta_{\mathrm{co}} \leq \Delta e^{-(\log d)^2/8}$$

To prove lemma 2.1 we'll consider a Poisson point process with intensity $\lambda = 2^{-d}\Delta = \left(\frac{\sqrt{d}}{8\log d}\right)^d$ and remove "bad" points $x \in X$ with

$$|X \cap B_x(2r_d)| \geq \Delta(1 + \Delta^{-1/3}) \quad \text{or} \quad |X \cap B_x(2_r) \cap B_y(2r_d)| \geq \Delta e^{-(\log d)^2/8} \tag{$\heartsuit$}$$

for some $y \in X$. These bad points are of course those that have too high a degree or codegree to let us be in the conditions of our $\alpha(G)$ bound for $G = G(X, 2r_d)$. To prove we don't delete too many points, we will make repeated use of the previously seen Mecke equation, stated in the following form.

Lemma 3.2 (univariate Mecke)

Let $\Omega \subset \mathbb{R}^d$ be bounded and measurable, and let $X \sim \mathrm{Po}_\lambda(\Omega)$ be a Poisson point process rate λ on Ω. Then, for events $(A_x)_{x\in\Omega}$

$$\mathbb{E}[|x \in X \cap \Omega : A_x \text{ holds on } X|] = \lambda \int_\Omega \mathbb{P}(A_x \text{ holds on } X \cup \{x\})dx$$

Proof. Just a special case of lemma 1.2. □

We will also make use of the following elementary tail bound for Poisson random variables

Lemma 3.3 (Poisson tail bound)

Let $Y \sim \mathrm{Po}(\lambda)$ be a Poisson random variable with rate λ. Then, for all $t > 0$,

$$\mathbb{P}(Y - \lambda \geq \lambda t) \leq \exp(-\lambda \min\{t, t^2\}/3)$$

Proof. Smells like a Chernoff type bound, some napkin mathematics (using chernoff and $1 + x \leq e^x$) gives me a candidate RHS of $e^{-\lambda t^2}$. I can't, off the top, of my head beat this though. □

Firstly, we'll prove the number of points with too high a degree is small.

Lemma 3.4 (number of high degree points is small / lemma 2.2 in [1])

Let $\Omega \subset \mathbb{R}^d, d \geq 4$ be a bounded & measurable subset and $\mathcal{X} \sim \mathrm{Po}_\lambda(\Omega)$. Then,

$$\mathbb{E}[|\{x \in X : |X \cap B_x(2r_d)| \geq \Delta(1 + \Delta^{-1/3})\}|] \leq \frac{1}{2d}\mathbb{E}[|X|], \quad \text{with } \lambda = 2^{-d}\Delta = \left(\frac{\sqrt{d}}{8\log d}\right)^d$$

Proof. Fix $s > 0$. Then Mecke's equation gives us

$$\mathbb{E}[|\{x \in \mathcal{X} : |\mathcal{X} \cap B_x(2r_d)| \geq s\}|] = \lambda \int_\Omega \mathbb{P}[|\mathcal{X} \cap B_x(2r_d)| \geq s - 1] \tag{$\dagger$}$$

Now, for fixed $x \in \Omega$, $|\mathcal{X} \cap B_x(2r_d)|$ is a Poisson random variable of rate at most $\lambda 2^d =: \Delta$, which follows from noting $\mathrm{Vol}(B_x(2r_d) \cap X) \leq 2^d$. Hence, via our Poisson tail bounds, one has

$$\mathbb{P}[|\mathcal{X} \cap B_x(2r_d)| \geq \Delta(t+1)] \leq \exp\left(-\min\{t, t^2\}\Delta/3\right) \quad \text{for } t > 0$$

Taking $t = \Delta^{-1/3} - \Delta^{-1}$ and substuting this into ($\dagger$) gives us the result. $\square$

Now we do the same with the codegree, which will take more work.

Lemma 3.5 (number of high codegree points is small)

Let $\Omega \subset \mathbb{R}^d$ be bounded and measurable, $\mathcal{X} \sim \mathrm{Po}_\lambda(\Omega)$ and set $\eta = e^{-(\log d)^2/8}$. Then,

$$\mathbb{E}\left[|\{x \in \mathcal{X} : \exists y \in \mathcal{X} \text{ with } |\mathcal{X} \cap B_x(2r_d) \cap B_y(2r_d)| \geq \eta\Delta\}|\right] \leq \frac{1}{2d}\mathbb{E}[|\mathcal{X}|]$$

Proof. Set $I_{x,y} = |\mathcal{X} \cap B_x(2r_d) \cap B_y(2r_d)|$. Then applying Meckes, obtain

$$\mathrm{LHS} = \mathbb{E}\,|\{x \in \mathcal{X} : \exists y \in \mathcal{X} \text{ with } I_{x,y} \geq \eta\Delta\}| = \lambda \int_\Omega \mathbb{P}(\exists y \in \mathcal{X} : I_{x,y} \geq \eta\Delta - 1)\mathrm{d}x$$

Thus it suffices to prove that $\mathbb{P}(\exists y \in \mathcal{X} : I_{x,y} \geq \eta\Delta - 1) \leq \frac{1}{2d}$.

Claim: (1)

$$\mathbb{P}(\exists y \in \mathcal{X} : I_{x,y} \geq \eta\Delta - 1) \leq \mathbb{E}|B_x(\log d) \cap \mathcal{X}| + \mathbb{E}|\{y \in \mathcal{X} \setminus B_x(\log d) : I_{x,y} \geq \eta\Delta - 1\}|$$

Proof of claim (1). Observe, by applying Markov's inequality twice,

$$\begin{aligned}
&\mathbb{P}(\exists y \in \mathcal{X} : I_{x,y} \geq \eta\Delta - 1) \\
&= \mathbb{P}(\exists y \in \mathcal{X} \cap B_x(\log d) : I_{x,y} \geq \eta\Delta - 1) + \mathbb{P}(\exists y \in \mathcal{X} \setminus B_x(\log d) : I_{x,y} \geq \eta\Delta - 1) \\
&= \mathbb{P}(|\{y \in \mathcal{X} \cap B_x(\log d) : I_{x,y} \geq \eta\Delta - 1\}| \geq 1) + \mathbb{P}(|\{y \in \mathcal{X} \setminus B_x(\log d) : I_{x,y} \geq \eta\Delta - 1\}| \geq 1) \\
&\leq \mathbb{E}|\{y \in \mathcal{X} \cap B_x(\log d) : I_{x,y} \geq \eta\Delta - 1\}| + \mathbb{E}|\{y \in \mathcal{X} \setminus B_x(\log d) : I_{x,y} \geq \eta\Delta - 1\}| \qquad \text{(Markov)} \\
&\leq \mathbb{E}|\mathcal{X} \cap B_x(\log d)| + \mathbb{E}|\{y \in \mathcal{X} \setminus B_x(2r_d) : I_{x,y} \geq \eta\Delta - 1\}|
\end{aligned}$$

and we're done :) $\square$

Now we'll bound the two terms in our summation. For the leftmost term, simply observe

$$\mathbb{E}|\mathcal{X} \cap B_x(\log d)| = \lambda \mathrm{Vol}(B_x(\log d)) = \left(\frac{\sqrt{d}}{8\log d}\right)^d \cdot \left(\frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}(\log d)^d\right) = \frac{\left(\frac{d\pi}{64}\right)^{d/2}}{\Gamma(\frac{d}{2}+1)}$$

Which with $\Gamma(d/2+1) \geq 2^{-d/2}e^{-d/2}d^{d/2}$ (Stirling's approximation) gives

$$\frac{\left(\frac{d\pi}{64}\right)^{d/2}}{\Gamma(\frac{d}{2}+1)} \leq \left(\frac{2\pi e}{32}\right)^{d/2}$$

Now Campos & Co jump from this to a bound above of $\frac{1}{4d}$. I make some comments about this below.

For the rightmost term we'll need the following result.

Claim: (2) For $t \geq 0$, take $x, y \in \mathbb{R}^d$ with $|x - y| \geq t$. Then,

$$\mathrm{Vol}(B_x(2r_d) \cap B_y(2r_d)) \leq 2^d e^{-t^2/4}$$

Proof of claim 2. Easy calculations, just take a (carefully chosen) covering ball. See [1] fact 2.3. ☐

Now, using Mecke, observe (as $I_{x,y} = 0$ on $y \notin B_x(4r_d)$) setting $\Omega' := B_x(4r_d) \setminus B_x(\log d)$

$$\begin{aligned}
\mathbb{E}|\{y \in X \setminus B_x(\log d) : I_{x,y} \geq \eta\Delta - 1\}| &= \lambda \int_{\Omega'} \mathbb{P}(X_{x,y} \geq \eta\Delta - 2)\mathrm{d}y \\
&\leq \lambda \mathrm{Vol}(\Omega') \ \times \ \max_{y\in\Omega'} \mathbb{P}(X_{x,y} \geq \eta\Delta - 2) \\
&\leq \lambda \mathrm{Vol}(B_x(4r_d)) \ \times \ \max_{y\in\Omega'} \mathbb{P}(X_{x,y} \geq \eta\Delta - 2) \\
&= \mathbb{E}[X \cap B_x(4r_d)] \ \times \ \max_{y\in\Omega'} \mathbb{P}(X_{x,y} \geq \eta\Delta - 2)
\end{aligned}$$

Now some annoying bounding gives the same $\frac{1}{4d}$ upper bound. Gonna eat lunch then type this up.

NOTE: if this bound can be made exponentially decreasing in d, like it can with the leftmost term, we have some juice as overall as we'll get a larger number of points in lemma 2.1, maybe improving the bound? It'll still be $(1 - o(1)) \cdots$ but $o(1)$ will tend to 0 faster, giving us much better constants n shit. ☐

Remark 3.1 (on claim 1)

Could we get something tighter by initially summing over $y \in \mathcal{X}$ (disjoint probabilities) and then bounding the terms in the summand? Also the choice of our split set to be $B_x(\log d)$ is purely to cancel with the chosen $\log(d)$ factor in our intensity λ. Worth noting. Again, all this will do is give us a better $o(1)$, nothing ground breaking.

Remark 3.2 (on the bounding of $\mathbb{E}|\mathcal{X} \cap B_x(\log d)|$)

That final leap seems super lazy? We go from an exponential rate of decrease to linear. Surely this can be improved to achieve something non-trivial? Maybe not, worth investigating regardless. In the paper they get 8 as their final denominator, I get 32 so I'm over bounding by a factor of $\sqrt{2}$. ALL THIS ACHIEVES IS A BETTER RATE OF CONVERGENCE ON $o(1)$, PROVIDED I CAN MATCH IT FOR THE BOUNDING OF THE RIGHTMOST TERM.

Now we may finally prove our original lemma, lemma 2.1 (also lemma 2.1 in [1]).

Proof of lemma 2.1. Set S_1 be the collection of points with high degree and S_2 those with high codegree (i.e. the left/right conditions in (♡)) and $X = \mathcal{X} \setminus (S_1 \cup S_2)$ the space without these "bad" points. Then,

$$\mathbb{E}|X| \geq \mathbb{E}|\mathcal{X}| - \mathbb{E}|S_1| - \mathbb{E}|S_2| \geq \left(1 - \frac{1}{d}\right) \mathbb{E}|\mathcal{X}| = \left(1 - \frac{1}{d}\right) \mathrm{Vol}(\Omega)\Delta 2^{-d}$$

using our lemmas for the expected size of S_1 and S_2, so we have our $X \subset \Omega$ (by $\mathbb{P}(X \geq \mathbb{E}X) > 0$). ☐

Remark 3.3 (on the $d \geq 1000$ assumption)

Where was $d \geq 1000$ used? I can only spot a $d \geq 4$ in the use of Stirling's and also in the proof of claim (2), which requires $r_d \leq \sqrt{d/8}$ and a quick look at desmos affirms this is true for $d \geq 4$. I found a $d \geq 5$ using their bound of the $\mathbb{E}[\mathcal{X} \cap B_x(\log d)]$ in lemma 2.4, or with my (worse by a factor of $\sqrt{2}$) bound $d \geq 13$.

Now we're ready to prove The Theorem™. Super easy with our lemma!

Proof of $\theta(d)$ bound / Theorem 1.1 in [1]. Fix $R > 0$ and set $\Omega = B_{\mathbf{0}}(R)$. It suffices to prove that we can place

$$(1 - o(1))\mathrm{Vol}(\Omega)\frac{d \log d}{2^{d+1}}$$

points each at pairwise distance at least $2r^d$. Lemma 2.1 gives us $X \subset \Omega$ with

$$|X| \geq (1 - o(1))\mathrm{Vol}(\Omega)\Delta 2^{-d}$$

and the geometric graph $G(X, 2r_d)$ satisfies the conditions of Theorem 2.2. Applying said theorem, gives an independent set I with

$$|I| \geq (1 - o(1))\frac{|X| \log \Delta}{\Delta} \geq (1 - o(1))\text{Vol}(\Omega)\frac{d \log d}{2^{d+1}}$$

and we're done. $\square$

3.2 Nibbling on almost-uniform graphs

We now introduce the so called *Rödl nibble*, which will be the key to our proof of thm 1.3.

Throughout this (sub)section, we make the following assumptions.

$$d_G(v) \in \{\Delta - 1, \Delta\}, \; \Delta_{\text{co}}(G) \leq \eta\Delta, \; \Delta \geq 1, \; \gamma \leq \frac{1}{2}, \; \eta \in [\Delta^{-1/2}, \gamma^2/8]$$

We spend this (sub)section showing that after each "nibble", the degrees and codegrees decrease appropriately. This result is characterised with the following proposition.

Proposition 3.1 (nibbles dont kill degrees / lemma 3.1 in [1])

Let Δ, γ, η be as above, let $\alpha \in [2\gamma^2, \gamma]$ and let G be as above. Let $A \subseteq V(G)$ be a (γ/δ)-random set, and let $G' = G \setminus (A \cup \mathcal{N}_G(A))$. Then, $\forall u \neq v \in G$

$$\mathbb{P}(d_{G'}(v) \geq (1 - \delta + \alpha)d_G(v) | v \in V(G')) \leq \exp\left(\frac{-\alpha^2}{32\gamma\eta}\right)$$
$$\mathbb{P}(d_{G'}(u, v) \geq (1 - \delta + \alpha)\eta\Delta | v \in V(G')) \leq \exp\left(\frac{-\alpha^2}{32\gamma\eta}\right)$$

Before we're able to prove this we'll need some preliminary results concerning our nibble. First, we'll calculate the mean degree/codegree after each nibble.

Lemma 3.6 (mean degree/codegree post-nibble / lemma 3.2 in [1])

Let Δ, γ, η, G be as defined at the start of the (sub)section, let A and G' be as in prior proposition (lemma 3.1 in [1]). Then, for $u, v \in V(G)$,

$$\mathbb{E}[d_{G'}(v) | v \in V(G')] \leq (1 - \gamma + \gamma^2)d_G(v) \quad \text{and} \quad \mathbb{E}[d_{G'}(u, v) | u, v \in V(G')] \leq (1 - \gamma + \gamma^2)d_G(u, v)$$

We just prove the mean degree result, noting the proof for the codegree is almost identical.

The idea is to look individually at vertices in $\mathcal{N}_G(v)$, given v is not "eaten", and to bound the probability this vertex survives the "nibble".

Proof. Conditioned on the fact $v \in V(G')$, A is a (γ/Δ)-random subset of $V(G) \setminus (\mathcal{N}_G(v) \cup \{v\})$ (as all the $\mathcal{N}_G(v) \cup \{v\}$ must survive). Let $w \in \mathcal{N}_G(v)$ and $d_w = |\mathcal{N}_G(w) \setminus (\mathcal{N}_G(v) \cup \{v\})|$. Then,

$$d_w = |\mathcal{N}_G(w)| - |\mathcal{N}_G(v) \cap \mathcal{N}_G(w)| - |\{v\}| = d_G(w) - d_G(w, v) - 1$$

(where the -1 comes from $w \in \mathcal{N}_G(v)$), so by our setup we may bound

$$d_w \geq (\Delta - 1) - \eta\Delta - 1 \geq \; (1 - \gamma^2/8)\Delta - 2$$

[SLIGHTLY OFF ON LAST BOUND, SHOULD ARRIVE AT $(1 - \gamma^2/2)\Delta$].
For $w \in V(G')$ to be true (conditioned on $v \in V(G')$), we need none of the $w' \in \mathcal{N}_G(w) \setminus (\mathcal{N}_G(v) \cup \{v\})$ to be "eaten". Hence,

$$\mathbb{P}(w \in V(G') | v \in V(G')) = \left(1 - \frac{\gamma}{\Delta}\right)^{d_w}$$

which, using $d_w \in \mathbb{N}_0$ and the Binomial theorem, gives

$$\mathbb{P}(w \in V(G') | v \in V(G')) \leq 1 - \frac{\gamma}{\Delta}d_w + \frac{\gamma^2}{2\Delta^2}d_w \leq 1 - \gamma + \gamma^2$$

where we used $(1-\gamma^2/2)\Delta \leq d_w \leq \Delta$ in the final deduction. Now we may just sum over $w \in \mathcal{N}_G(v)$ to obtain the final result. □

We now state a one sided version of Freedman's inequality (a martingale generalisation of one of the Bernstein inequalities), courtesy of Chung & Lu.

Proposition 3.2 (Chung & Lu concentration inequality / Theorem 3.3 in [1])

Let $(S_i)_{i=0}^n$ be a martingale with respect to a filtration $(\mathcal{F}_i)_{i=0}^n$. Suppose (S_i) has increments $(\xi_j)_{j=1}^n$ with $\xi_j \leq R_j$ and $\mathbb{E}[|\xi_j|^2|\mathcal{F}_j] \leq \sigma_i^2$ almost surely. Then, for all $a \geq 0$,

$$\mathbb{P}(S_n - S_0 \geq a) \leq \exp\left(-\frac{a^2}{2\sum_{i=1}^n (R_i^2 + \sigma_i^2)}\right)$$

Lemma 3.7 (Lemma 3.4 in [1])

Let H be a bipartite graph with partition $X \cup Y$ satisfying $\forall x, x' \in X : d_H(x, x') \leq \ell$. Fix $p \in (0, 1/2]$ and let A be p-random in Y. Let $X' = X \setminus N(A)$ and $S = |X'|$, then $\forall a \geq 0$

$$\mathbb{P}(S - \mathbb{E}[S] \geq a) \leq \exp\left(-\frac{a^2}{4p(e_H(X, Y) + \ell|X|^2)}\right)$$

Proof. Enumerate the vertex set $Y = \{y_1, \ldots, y_n\}$ and let $\mathcal{F}_t$ be the σ-algebra generated by A_t where $A_t := A \cap \{y_1, \ldots, y_t\}$. Set $S_t = \mathbb{E}[S|\mathcal{F}_t]$

Claim: S_t is a martingale.

Proof of claim. Easy, just the accumulating information martingale. Write up at some point. □

Let our martingale $(S_t)_{t=0}^n$ have increments $(\xi_t)_{t=1}^n$. Then, by writing $S = \sum_{x \in X} \mathbf{1}\{x \in X'\}$, we have

$$\xi_t = \sum_{x \in X} (\mathbb{P}(x \in X'|\mathcal{F}_t) - \mathbb{P}(x \in X'|\mathcal{F}_{t-1}))$$

Now we bound this quantity, aiming to apply Chung & Lu's concentration inequality on (S_t).

Claim: $-d_H(y_t) \leq \xi_t \leq p d_H(y_t) =: R_t$

Proof of claim. We have $x \in X'$ if and only if $x \notin \mathcal{N}_H(A)$ (i.e. if all the neighbours of x survive the nibble) so, setting $d_t(x) = \mathcal{N}_H(x) \cap \{y_{t+1}, \ldots, y_n\}$ to be our unseen neighbours, we have

$$\mathbb{P}(x \in X'|\mathcal{F}_t) = (1-p)^{d_t(x)} \mathbf{1}\{x \notin \mathcal{N}_H(A_t)\}$$

Hence, $\mathbb{P}(x \in X'|\mathcal{F}_t) = \mathbb{P}(x \in X'|\mathcal{F}_{t-1})$ whenever $x \notin \mathcal{N}_H(y_t)$, allowing us to write

$$\xi_t = \sum_{x \in \mathcal{N}_H(y_t)} \left((1-p)^{d_t(x)} \mathbf{1}\{x \notin \mathcal{N}_H(A_t)\} - (1-p)^{d_t(x)-1} \mathbf{1}\{x \notin \mathcal{N}_H(A_t)\}\right)$$

We know investigate the cases where y_t is "eaten"and when it isn't, along with some casework on our indicators, to obtain our bounds.

If $y_t \notin A$ (i.e. if y_t is not "eaten") then $A_t = A_{t-1}$ and hence (as the indicators are equal),

$$0 \leq \xi_t \leq \sum_{x \in \mathcal{N}_H(y_t)} (1-p)^{d_t(x)} - (1-p)^{d_t(x)+1} = \sum_{x \in \mathcal{N}_H(y_t)} p(1-p)^{d_t(x)} \leq p d_H(y_t)$$

by crudely bounding $1 - p \leq 1$.

If $y_t \in A$ (i.e. if y_t is "eaten") then $A_t \supseteq A_{t-1}$ and hence (as $\mathbf{1}\{x \notin \mathcal{N}_H(A_t)\} \leq \mathbf{1}\{x \notin \mathcal{N}_H(A_{t-1})\}$),

$$0 \geq \xi_t \geq -\sum_{x \in \mathcal{N}_H(y_t)} (1-p)^{d_t(x)} \geq -d_H(y_t)$$

by again, crudely bounding $1 - p \leq 1$. Hence the claim is proven. □

Claim: $\mathbb{E}[|\xi_t|^2 | \mathcal{F}_{t-1}] \leq p d_H(y_t)^2 + p^2 d_H(y_t)^2 =: \sigma_t^2$

Proof of claim. Again, we look at the cases when $y_t \in A$ and $y_t \notin A$ (i.e. y_t "eaten" or not "eaten"). Writing $\xi_t = \xi_t \mathbf{1}\{y_t \in A\} + \xi_t \mathbf{1}\{y_t \notin A\}$ we have

$$\begin{aligned}\mathbb{E}[|\xi_t||\mathcal{F}_{t-1}] &\leq (d_H(y_t))^2 \mathbb{E}[\mathbf{1}\{y_t \in A\}|\mathcal{F}_{t-1}] + p^2 (d_H(y_t))^2 \mathbb{E}[\mathbf{1}\{y_t \notin A\}|\mathcal{F}_{t-1}] \\ &= p(d_H(y_t))^2 + (1-p)p^2(d_H(y_t))^2 \\ &\leq p(d_H(y_t))^2 + p^2(d_H(y_t))^2 =: \sigma_t^2\end{aligned}$$

using the fact A is a p-random subset of Y. □

Now we're ready to apply Chung & Lu's inequality, first we bound the sum in the denominator.

Claim: $\sum_{t=1}^n (\sigma_t^2 + R_t^2) \leq 2p(e_H(X,Y) + \ell|X|^2)$

Proof of claim. Firstly, observe

$$\sum_{t=1}^n (\sigma_t^2 + R_t^2) = \sum_{t=1}^n (p(d_H(y_t))^2 + 2p^2(d_H(y_t))^2) \leq 2p \sum_{t=1}^n (d_H(y_t))^2$$

Now, by writing $d_H(y_t) = \sum_{x \in X} \mathbf{1}\{x \in \mathcal{N}_H(y_t)\}$, we can bound

$$\begin{aligned}\sum_{t=1}^n (d_H(y_t))^2 &= \sum_{t=1}^n \left(\sum_{x \in X} \mathbf{1}\{x \in \mathcal{N}_H(y_t)\}\right) = \sum_{t=1}^n \sum_{x,x' \in X} \mathbf{1}\{x, x' \in \mathcal{N}_H(y_t)\} \\ &= \sum_{x,x' \in X} \sum_{t=1}^n \mathbf{1}\{x, x' \in \mathcal{N}_H(y_t)\} = \sum_{x,x' \in X} d_H(x,x') \leq \ell |X|^2\end{aligned}$$

Campos & Co have an extra $e_H(X,Y)$ term in the bound? Why? Am I being stupid or is this just not needed... Anyway the result is proven. □

Now, upon applying Chung & Lu's inequality to $S = S_n$, noting $S_0 = \mathbb{E}[S]$, we have

$$\mathbb{P}(S - \mathbb{E}[S] \geq a) \leq \exp\left(-\frac{a^2}{4p\ell|X|^2}\right)$$

for any $a \geq 0$, and we're done. :) □

Remark 3.4 (magical denominator term)

Who's buggin, me or Campos? Where in God's name does that $e_H(X,Y)$ term come from?????

With this we can prove our goal lemma, concerning the preservation of regularity after a "nibble".

I'll do this another day, I've had enough graph theory for the moment.

3.3 Bounding the independence number

4 Appendices

4.1 Appendix A: Poisson Point Processes

For a full treatment, consult [3]. Here I will simply give the relevent definitions and results from this text, leaving proofs (unless containing a particularly important idea) to [3].

Definition 4.1 (point process)

Let $(\mathbb{X}, \mathcal{X})$ be a measure space and let $\mathbf{N}(\mathbb{X}) \equiv \mathbf{N}$ be the family of measures that can be written as a countable sum of finite measures on $\mathbb{X}$ with image in $\mathbb{N}_0$. Let $\mathcal{N}(\mathbb{X}) \equiv \mathcal{N}$ be the σ-algebra generated by the collection of subsets $\{\mu \in \mathbf{N} : \mu(A) = k\}$ over $A \in \mathcal{X}, k \in \mathbb{N}_0$. A *point process* on $\mathbb{X}$ is a random element η of the measure space $(\mathbf{N}, \mathcal{N})$

Definition 4.2 (Poisson point process)

Let $\mathbb{X}$ be a space and λ an s-finite measure on $\mathbb{X}$. A *Poisson point process* with intensity measure λ is a point process η on $\mathbb{X}$ with

(i) $\eta(A) \sim \text{Poisson}(\lambda(A))$, that is $\eta(A)$ is Poisson with parameter $\lambda(A)$

(ii) If $A_1, \ldots, A_n \in \mathbb{X}$ are pairwise disjoint then $\eta(A_1), \ldots, \eta(A_n)$ are independent.

References

[1] Marcelo Campos, Matthew Jenssen, Marcus Michelen, and Julian Sahasrabudhe. A new lower bound for sphere packing, 2023.

[2] Michael Krivelevich, Konstantinos Panagiotou, Mathew Penrose, and Colin McDiarmid. *Random Graphs, Geometry and Asymptotic Structure.* London Mathematical Society Student Texts. Cambridge University Press, 2016.

[3] Günter Last and Mathew Penrose. *Lectures on the Poisson Process.* Institute of Mathematical Statistics Textbooks. Cambridge University Press, 2017.

[4] Mathew Penrose. *Random Geometric Graphs.* Oxford University Press, 05 2003.