Random Geometric Graphs

A collection of notes from my time working under the supervision of Mathew Penrose on random geometric graph theory and related topics.

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Abstract

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Keywords: Random geometric graphs, Poisson Point Processes, Combinatorial Geometry, Sphere Packing

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1 Introduction

1.1 What is a random geometric graph?

> **Definition 1.1 (geometric graph)**
>
> Let $\mathcal{X} \subset \mathbb{R}^d$ be a finite collection of points and $r > 0$. The *geometric graph* $G(\mathcal{X}, r)$ is the graph with vertex set $\mathcal{X}$ and edge set $\{\{x, y\} \subset \mathcal{X} \ : \ |x - y| \leq r\}$, where $|\cdot|$ denotes the Euclidean norm.

We can turn this into a random graph by letting the set $\mathcal{X} \subset \mathbb{R}^d$ be random. The resulting structure $G(\mathcal{X}, r)$ is said to be the *random geometric graph* (A.K.A the *Gilbert graph*). The first structure we'll consider is found by uniformly scattering n points in the d-dimensional hypercube $[0, 1]^d$.

> **Example 1.2 (fixed scale geometric graph)**
>
> Let $\xi_1, \xi_2, \ldots \overset{\text{i.i.d}}{\sim} \text{Unif}[0, 1]^d$ and fix a sequence $(r_n)_{n \geq 1}$ in $\mathbb{R}_{>0}$. Let $\mathcal{X}_n := \{\xi_1, \ldots, \xi_n\}$. Then $G(\mathcal{X}_n, r_n)$ is the *fixed scale geometric graph* at time $n \geq 1$.

We will also consider what happens when the number of vertices is Poisson.

> **Example 1.3 (Poisson scale geometric graph)**
>
> Let $\xi_1, \xi_2, \ldots \overset{\text{i.i.d}}{\sim} \text{Unif}[0, 1]^d$ and $N_n \sim \text{Poisson}(n)$ for $n \geq 1$ be independent of $(\xi_1, \xi_2, \ldots)$. Let $\mathcal{P}_n := \{\xi_1, \ldots, \xi_{N_n}\}$ and fix a sequence $(r_n)_{n \geq 1}$ in $\mathbb{R}_{>0}$. Then the random geometric graph $G(\mathcal{P}_n, r_n)$ is the d-dimensional *Poisson scale geometric graph* at time $n \geq 1$.

As one may expect, the vertices of this graph form a Poisson point process in $\mathbb{R}^d$. This was originally an exercise (1.1) in [2].

> **Proposition 1.4 ($\mathcal{P}_n(\cdot)$ is a Poisson point process)**
>
> Let $\mathcal{P}_n$ be the vertex set in the Poisson scale geometric graph. Then $\mathcal{P}_n(\cdot) : A \mapsto |\mathcal{P}_n \cap A|$, where $A \subset [0, 1]^d$ is measurable and $|\cdot|$ is the counting measure, is a *Poisson point process* with intensity $\lambda(A)$, $\lambda(\cdot)$ the Lebesgue measure on $\mathbb{R}^n$.

See Appendix A for the definition of the Poisson point process.

Proof. Write $\mathcal{P}_n(A_j) = \sum_{i=1}^{N_n} \mathbf{1}\{\xi_i \in A_j\}$. Then, conditioning on $\{N_n = m\}$, each $\mathcal{P}_n(A_j)$ is a $\text{Binom}(m, \lambda(A_j))$ random variable, making the joint distribution (also conditioned on $\{N_n = m\}$) a $\text{Multinom}(m, \lambda(A_1), \ldots, \lambda(A_k))$. Thus one has

$$\mathbb{P}(\mathcal{P}_n(A_1) = a_1, \ldots, \mathcal{P}_n(A_k) = a_k) = \underbrace{\left(\frac{e^{-n} n^m}{m!} \right)}_{\text{Poisson}(n;m)} \times \underbrace{\binom{m}{j_1, \ldots, j_k} \prod_{i=1}^{k} (\lambda(A_i))^{j_i}}_{\text{Multinom}(m, \lambda(A_1), \ldots, \lambda(A_k); j_1, \ldots, j_k)}$$

$$= \prod_{i=1}^{k} \frac{e^{-n\lambda(A_i)} (n\lambda(A_i))^{j_i}}{(j_i)!}$$

which is a product of the distributions $\text{Poisson}(n\lambda(A_i)), 1 \leq i \leq k$, giving us both of our defining properties of a Poisson point process. $\square$

The exercise (1.2) in [2] states the following result, which we'll prove in an analogous way.

> **Proposition 1.5 (expected degree of kth vertex in fixed scale RGG)**
>
> Consider the fixed scale random geometric graph $G(\mathcal{X}_n, r_n)$ with $r_n \to 0$ as $n \to \infty$. Let $D_{k,n}$ be the degree of the first vertex ξ_k, then $\mathbb{E}[D_{k,n}] \sim \theta n r_n^d$.

Proof. Write $D_{k,n} = \sum_{i \neq k} \mathbf{1}\{|\xi_k - \xi_i| \leq r\}$, then we have $D_{k,n} \sim \text{Binom}(n-1, p)$ with

$$p = \lambda(B_{r_n}(\xi_k) \cap [0,1]^d)$$

where $B_{r_n}(\xi_k)$ is the hypersphere radius r_n centred at ξ_k and λ the Lebesgue measure on $\mathbb{R}^d$. Now, conditioning on ξ_k, we have

$$\mathbb{E}[D_{k,n}|\xi_k] = (n-1)\lambda(B_{r_n}(\xi_k) \cap [0,1]^d)$$

and hence, by the total law of expectation,

$$\begin{aligned}
\mathbb{E}[D_{k,n}] &= \mathbb{E}[\mathbb{E}[D_{k,n}|\xi_k]] \\
&= (n-1)\int_{[0,1]^d} \lambda(B_{r_n}(\xi_k) \cap [0,1]^d)\mathrm{d}\xi_k \\
&\sim n\int_{[0,1]^d} \lambda(B_{r_n}(\xi_k))\mathrm{d}\xi_k \sim n(\theta r_n^d)
\end{aligned}$$

Where the first asymptotic equality follows via the dominated convergence theorem (to see this, observe $B_{r_n}(\xi_k) \cap [0,1]^d \to B_{r_n}(\xi_k)$ A.E. and $B_{r_n}(\xi_k)$ also dominating). Note θ is a constant depending on d, and is the coefficient of r_n^d in the formula for the volume of a hypersphere radius r_n. $\qquad\square$

1.2 Counting Edges of $G(\mathcal{X}_n, r_n)$ and $G(\mathcal{P}_n, r_n)$

The number of edges $\mathcal{E}_n, \mathcal{E}_n'$ in $G(\mathcal{X}_n, r_n)$ and $G(\mathcal{P}_n, r_n)$ respectively is a Poisson random variable. To prove this, we'll use a so called *dependency graph*. We handle the two RGGs $G(\mathcal{X}_n, r_n)$ and $G(\mathcal{P}_n, r_n)$ seperately.

Counting edges of $G(\mathcal{X}_n, r_n)$

An elementary question to ask is "what is the expected edge count at time n?". Via the previously derived vertex degree asymptotics, we have an easy asymptotic result.

Proposition 1.6 (expected edge count in fixed scale RGG)

Let $\mathcal{E}_n$ and $\mathcal{E}_n'$ be the number of edges in $G(\mathcal{X}_n, r_n)$ and $G(\mathcal{P}_n, r_n)$ respectively. Then,

$$\mathbb{E}[\mathcal{E}_n] \sim \theta n^2 r_n^d/2$$

Proof of lemma. By the handshaking lemma we have $2\mathbb{E}[\mathcal{E}_n] = \sum_{k=1}^n \mathbb{E}[D_{k,n}] \sim n^2 r_n^d$. $\qquad\square$

Now we introduce the *dependency graph*.

Definition 1.7 (dependency graph)

Let $(V, \sim)$ be a finite simple graph w/ edge relation $\sim$ and vertex set V. We say $(V, \sim)$ is a *dependency graph* for the random variables $(W_\alpha)_{\alpha \in V}$ if whenever $A, B \subset V$ are disjoint with no $\alpha \in A$, $\beta \in B$ such that $\alpha \sim \beta$ (i.e. A and B lie in unique connected components) then

$$(W_\alpha)_{\alpha \in A} \perp\!\!\!\perp (W_\beta)_{\beta \in B} \quad \text{i.e. are independent families}$$

When we take the random variable associated with each vertex to be 0-1, we obtain the following bound.

Lemma 1.8 (Poisson approximation lemma for Bernoulli sums)

Let $(\xi_i)_{i \in I}$ be a finite collection of Bernoulli random variables with dependency graph $(I, \sim)$. Set $p_i := \mathbb{P}(\xi_i = 1)$, $p_{ij} := \mathbb{P}(\xi_i = 1; \xi_j = 1)$ and suppose $\lambda := \sum_{i \in I} p_i$ is finite. Then, letting $W := \sum_{i \in I} \xi_i$, we have (LHS is just total variation distance)

$$\sum_{k \geq 0} \left|\mathbb{P}[W = k] - \mathbb{P}(\text{Po}(\lambda) = k)\right| \leq \min\{6, 2\lambda^{-1}\}\left(\sum_{i \in I}\sum_{\mathcal{N}(i)\setminus\{i\}} p_{ij} + \sum_{i \in I}\sum_{j \in \mathcal{N}(i)} p_i p_j\right)$$

Proof. Omitted, consult [4]. □

Theorem 1.9 (total variation between edge and Poisson distribution)

Let $\lambda_n := \mathbb{E}[\mathcal{E}_n]$. Then

$$\sum_{k \geq 0} |\mathbb{P}(\mathcal{E}_n = k) - \mathbb{P}(\text{Po}(\lambda_n) = k)| = O(nr^d)$$

The idea of this proof is to write $\mathcal{E}_n = \sum_{1 \leq i < j \leq n} \mathbf{1}\{|\xi_i - \xi_j| \leq r_n\}$, i.e. the number of edges as the sum of the indicators over all possible edges indicating whether this edge exists. This has a rather obvious dependency graph and hence we can apply the Poisson approximation lemma (1.7) with some work on the asymptotics.

Proof. Let $V\{\{i, j\} : 1 \leq i < j \leq\}$ and define $\sim$ by $\alpha \sim \beta$ if $\alpha \cap \beta \neq \emptyset$ and $\alpha \neq \beta$. Define the random variables $W_\alpha = \mathbf{1}\{|\xi_i - \xi_j| \leq r_n\}$ for $\alpha \in V$ and let $\lambda_n := \sum_{\alpha \in V} p_\alpha$.

> **Claim:** $G(V, \sim)$ is a dependency graph for $(W_\alpha)_{\alpha \in V}$.

Proof of claim. Follows immediately from the independence of the ξ_i. □

Let $p_\alpha := \mathbb{P}(W_\alpha = 1)$ and $p_{\alpha\beta} := \mathbb{P}(W_\alpha = 1; W_\beta = 1)$, as in the setup of lemma 1.7.

> **Claim:** $p_\alpha \sim \theta r_n^d$

Proof of claim. Fix $\alpha = \{i, j\} \in V$. Then, letting $\lambda(\cdot)$ be the Lebesgue measure on $\mathbb{R}^d$,

$$p_\alpha = \mathbb{P}(|\xi_i - \xi_j| \leq r_n) = \int_{v \in [0,1]^d} \mathbb{P}(|\xi_i - \xi_j| \leq r_n | \xi_i = v) \mathrm{d}v$$

$$= \int_{v \in [0,1]^d} \lambda(B_{r_n}(v) \cap [0,1]^d) \mathrm{d}v$$

$$\sim \int_{v \in [0,1]^d} \lambda(B_{r_n}(v)) \mathrm{d}v \sim \theta r_n^d$$

where the first asymptotic equality follows from the dominated convergence theorem. □

> **Claim:** $p_{\alpha\beta} \sim (\theta r_n^d)^2$

Proof of claim. An analogous argument works. □

Now we have all our ingredients, lets cook. Observe, from the previously computed asymptotics,

$$\lambda_n \sim n^2 \theta r_n^d / 2 \quad \text{and} \quad \sum_{\alpha \in V} p_\alpha^2 \sim \theta \lambda_n r_n^d$$

and hence by counting $|\mathcal{N}(\alpha)| = 2(n-2)$, obtain

$$\sum_{\alpha \in V} \sum_{\beta \sim \alpha} (p_{\alpha\beta} + p_\alpha p_\beta) \sim \binom{n}{2} \times 2(n-2) \times 2(\theta r_n^d)^2 = O(n\lambda_n r_n^d)$$

which by the Poisson approximation lemma gives

$$\sum_{k \geq 0} |\mathbb{P}(\mathcal{E}_n = k) - e^{-\lambda_n} \lambda_n^k / k!| = O(nr_n^d)$$

as claimed. □

> **Corollary 1.10 (edge distribution of the fixed scale random geometric graph)**
>
> If the limit $\lambda_n \to \lambda \in (0, \infty)$ exists and $nr_n^d \to 0$, then $\mathcal{E}_n \xrightarrow{\mathcal{D}} \text{Poisson}(\lambda)$

Proof. By proposition 1.6, we have $\lambda \sim \lambda_n \sim \theta n^2 r_n^d/2 \Rightarrow n^2 r_n^d \sim 2\lambda/\theta \in (0, \infty)$ which forces $nr_n^d \to 0$ and hence lemma 1.8 applies giving $\mathcal{E}_n \xrightarrow{\mathcal{D}} \text{Poisson}(\lambda)$. $\qquad\square$

Counting edges of $G(\mathcal{P}_n, r_n)$

We now spend the rest of the (sub)section proving similar results for $G(\mathcal{P}_n, r_n)$. Before proceeding we have the following lemma from [2]. We write $\mathcal{P}_{<\infty}(A)$ for the family of finite subsets of A.

> **Lemma 1.11 (Mecke formula)**
>
> Let $k \in \mathbb{N}$. For any measurable $f : (\mathbb{R}^d)^k \times \mathcal{P}_{<\infty}([0,1]^d) \to \mathbb{R}$, when the expectation exists,
>
> $$\mathbb{E} \sum_{X_1,\ldots,X_k \in \mathcal{P}_n}^{\neq} f(X_1, \ldots, X_k, \mathcal{P}_n \setminus \{X_1, \ldots, X_n\}) = n^k \int \cdots \int \mathbb{E} f(x_1, \ldots, x_k, \mathcal{P}_n) \mathrm{d}x_1 \ldots \mathrm{d}x_k$$
>
> where $\sum^{\neq}$ denotes the sum over the ordered k-tuples of distinct points (in $\mathcal{P}_n$).

Proof. The idea is to condition on the number of points in $\mathcal{P}_n$. Observe,

$$
\begin{aligned}
& \mathbb{E} \sum_{X_1,\ldots,X_k \in \mathcal{P}_n}^{\neq} f(X_1, \ldots, X_k, \mathcal{P}_n \setminus \{X_1, \ldots, X_n\}) \\
& = \mathbb{E}\left[\mathbb{E}\left[\sum_{X_1,\ldots,X_k \in \mathcal{P}_n}^{\neq} f(X_1, \ldots, X_k, \mathcal{P}_n \setminus \{X_1, \ldots, X_n\}) \middle| |\mathcal{P}_n| = m \right] \right] \\
& = \sum_{m \geq k} \left(\frac{e^{-n} n^m}{m!} \right) \sum_{X_1,\ldots,X_k \in \{\xi_1,\ldots,\xi_m\}}^{\neq} \mathbb{E} f(X_1, \ldots, X_k, \{X_{k+1}, \ldots, X_m\}) \\
(\dagger) \quad & = \sum_{m \geq k} \left(\frac{e^{-n} n^m}{m!} \right) \left(\prod_{i=1}^{k}(m-i+1) \right) \mathbb{E} f(X_1, \ldots, X_k, \{X_{k+1}, \ldots, X_m\}) \\
& = \sum_{m \geq k} \left(\frac{e^{-n} n^m}{m!} \right) \left(\prod_{i=1}^{k}(m-i+1) \right) \int_{[0,1]^d} \cdots \int_{[0,1]^d} f(x_1, \ldots, x_k, \{x_{k+1}, \ldots, x_m\}) \mathrm{d}x_1 \ldots \mathrm{d}x_m \\
& = n^k \sum_{m \geq k} \left(\frac{e^{-n} n^{m-k}}{(m-k)!} \right) \int_{[0,1]^d} \cdots \int_{[0,1]^d} f(x_1, \ldots, x_k, \{x_{k+1}, \ldots, x_m\}) \mathrm{d}x_1 \ldots \mathrm{d}x_m \\
& = n^k \int_{[0,1]^d} \cdots \int_{[0,1]^d} f(x_1, \ldots, x_k, \{x_{k+1}, \ldots, x_m\}) \mathrm{d}x_1 \ldots \mathrm{d}x_m \\
& = n^k \int_{[0,1]^d} \cdots \int_{[0,1]^d} \mathbb{E} f(x_1, \ldots, x_k, \mathcal{P}_n) \mathrm{d}x_1 \ldots \mathrm{d}x_k
\end{aligned}
$$

Where ($\dagger$) follows from the ξ_i being i.i.d, so all expectations are equal and it suffices to count distinct k-tuples in $\{\xi_1, \ldots, \xi_m\}$. [CHECK THIS PROOF W/ PENROSE] $\qquad\square$

We can now use this lemma to find the expected number of edges in $G(\mathcal{P}_n, r_n)$.

> **Proposition 1.12 (expected edge count in Poisson scale RGG)**
>
> Let $G(\mathcal{P}_n, r_n)$ be the Poisson scale RGG and $\mathcal{E}_n$ be the number of edges at time n. Then,
>
> $$\mathbb{E}[\mathcal{E}_n] \sim \theta n^2 r_n^d/2$$

Proof. Write $\mathcal{E}_n = \frac{1}{2} \sum_{X_1, X_2 \in \mathcal{P}_n}^{\neq} \mathbf{1}\{\|X_1 - X_2\| \leq r_n\}$. Then we can use Mecke's formula as follows.

$$
\begin{aligned}
\mathbb{E}[\mathcal{E}_n] &= \frac{1}{2} \mathbb{E}\left[\sum_{X_1, X_2 \in \mathcal{P}_n}^{\neq} \mathbf{1}\{\|X_1 - X_2\| \leq r_n\} \right] \\
(\text{Mecke}) \quad &= \frac{n^2}{2} \int_{[0,1]^d} \int_{[0,1]^d} \mathbb{E}[\mathbf{1}\{\|x_1 - x_2\| \leq r_n\}] \mathrm{d}x_1 \mathrm{d}x_2 \\
&= \frac{n^2}{2} \int_{[0,1]^d} \int_{[0,1]^d} \mathbf{1}\{\|x_1 - x_2\| \leq r_n\} \mathrm{d}x_1 \mathrm{d}x_2 \\
&= \frac{n^2}{2} \int_{[0,1]^d} \lambda(B(x_1, r_n) \cap [0,1]^d) \mathrm{d}x_1 \sim \theta n^2 r_n^d / 2
\end{aligned}
$$

where the final asymptotic equality follows from the dominated convergence theorem and volume of a d-dimensional hypersphere. Note we have already seen the asymptotics for the final integral. $\square$

Rest follows on from Exercise (2.2), too small brained to figure it out right now will attempt again tomorrow :<

1.3 A central limit theorem for our edge counts

Lemma 1.13 (Falling factorial moment for the Poisson distribution)

Let $X \sim \text{Poisson}(\lambda)$ and fix $k \in \mathbb{N}$. Then $\mathbb{E}[X(X-1)\dots(X-k+1)] = \lambda^k$

Proof. This follows by the following simple calculation.

$$
\mathbb{E}[X(X-1)\cdots(X-k+1)] = \sum_{x \geq 0} x(x-1)\cdots(x-k+1) \frac{e^{-\lambda}\lambda^x}{x!}
$$

$$
= e^{-\lambda} \sum_{x \geq k} x(x-1)\cdots(x-k+1) \frac{\lambda^x}{x!} = \lambda^k e^{-\lambda} \sum_{x \geq k} \frac{\lambda^{x-k}}{(x-k)!} = \lambda^k
$$

$\square$

2 Appendices

2.1 Appendix A: Poisson Point Processes

For a full treatment, consult [3]. Here I will simply give the relevent definitions and results from this text, leaving proofs (unless containing a particularly important idea) to [3].

Definition 2.14 (point process)

Let $(\mathbb{X}, \mathcal{X})$ be a measure space and let $\mathbf{N}(\mathbb{X}) \equiv \mathbf{N}$ be the family of measures that can be written as a countable sum of finite measures on $\mathbb{X}$ with image in $\mathbb{N}_0$. Let $\mathcal{N}(\mathbb{X}) \equiv \mathcal{N}$ be the σ-algebra generated by the collection of subsets $\{\mu \in \mathbf{N} : \mu(A) = k\}$ over $A \in \mathcal{X}, k \in \mathbb{N}_0$. A *point process* on $\mathbb{X}$ is a random element η of the measure space $(\mathbf{N}, \mathcal{N})$

Definition 2.15 (Poisson point process)

Let $\mathbb{X}$ be a space and λ an s-finite measure on $\mathbb{X}$. A *Poisson point process* with intensity measure λ is a point process η on $\mathbb{X}$ with

(i) $\eta(A) \sim \text{Poisson}(\lambda(A))$, that is $\eta(A)$ is Poisson with parameter $\lambda(A)$

(ii) If $A_1, \ldots, A_n \in \mathbb{X}$ are pairwise disjoint then $\eta(A_1), \ldots, \eta(A_n)$ are independent.

2.2 Appendix B: Existence Of High Density Packings

Here I take some notes on chapter 7 of Pach & Agarwals combinatorial geometry [WILEY GIVES BAD CITATION], hoping to find bridges between random geometric graphs and combinatorial geometry.

As the title suggests, we spend this section looking for the existence of high density packings in $\mathbb{R}^d$. We take a non construct approach, instead turning to the probabilistic method. For a convex body $C \subset \mathbb{R}^d$, denote the maximal packing (respectively, minimal covering) density by $\delta(C)$ (respectively, $\vartheta(C)$).

Let us first consider $C = B^d \subset \mathbb{R}^d$, the d-dimensional ball. Then one has the following easy observation.

Proposition 2.16 (easy lower bound for $\delta(B^d)$)

Let $B^d \subset \mathbb{R}$ be the d-dimensional unit ball. Then,

$$\delta(B^d) \geq 2^{-d}\vartheta(B^d) \geq 2^{-d}$$

Proof. The latter inequality is trivial ($\vartheta(C) \geq 1$). For the former, let us consider a *saturated packing* $\mathcal{B} = \{B^d + c_i | i \in \mathbb{N}\}$ (that is, a packing such that no body can be added without intersecting a pre-existing body) and the factor 2 enlargement $\mathcal{B}' = \{2B^d + c_i | i \in \mathbb{N}\}$.

Claim: $\mathcal{B}'$ is a covering of $\mathbb{R}^d$.

Proof of claim. Suppose $x \notin \cup_{B \in \mathcal{B}'} B$. Then we have $|x - c_i| > 2$ for each $i \geq 1$, which gives

$$(B^d + c_i) \cap (B^d + x) = \emptyset, \quad i \geq 1$$

Contradiction! $\qquad\square$

Hence, letting $d(C, \mathbb{R}^d)$ denote the density of a body C in $\mathbb{R}^d$, we clearly have

$$\vartheta(B^d) \leq d(\mathcal{B}', \mathbb{R}^d) = 2^d d(\mathcal{B}, \mathbb{R}^d) \leq 2^d \delta(B^d)$$

and the result is proven. $\qquad\square$

In high dimensional Euclidean space, this trivial bound was the best bound known, up to a constant, for a long time! In fact, no construction of saturated lattice packings of balls is known (again, in high dimensional space), worth checking if this has changed since time of writing. This is surprising, given the obvious drawbacks of this argument. If we saturate a packing by continuously adding balls, we can't control the structure of this packing.

Let $\delta_L(C)$ be the maximal lattice packing density of C into $\mathbb{R}^d$. Using techniques developed by Minkowski and Hlawka, we can prove $\delta_L(B^d) > 2^{-d}$. Namely, we can affirm the existence of a lattice packing strictly outperforming our previously derived lower bound of 2^{-d}.

Definition 2.17 (star-shaped body)

Let $C \subset \mathbb{R}^d$ be compact. We call C a *star-shaped body* if $\mathbf{0}$ is in the interior of C and has the relation $\mathbf{x} \in C \Rightarrow \lambda\mathbf{x} \in C$ for all $\lambda \in [0, 1]$.

Definition 2.18 (admissible latice)

Let $\Lambda = \Lambda(u_1, \ldots, u_d) = \{m_1 u_1 + \cdots + m_d u_d | m_1, \ldots, m_d \in \mathbb{Z}\}$ be a lattice in $\mathbb{R}^d$ and let C be a star-shaped body. Then Λ is said to be *admissible* for C if it contains no interior point of $C \setminus \{\mathbf{0}\}$.

Definition 2.19 (critical determinant)

Let C be a star-shaped body. The *critical determinant* of C is defined by

$$\Delta(C) := \inf\{\det \Lambda | \Lambda \text{ is admissible for } C\}$$

Lemma 2.20 (Mahler selection theorem - 1946)

Let $\Lambda_1, \Lambda_2, \ldots$ be an infinite sequence of lattices in $\mathbb{R}^d$ that have constants $\alpha, \beta > 0$ satisfying, for $i \geq 1$,

(i) Λ_i is admissible for αB^d

(ii) $\det \Lambda_i \leq \beta$

Then we can select a convergent subsequence $\Lambda_{i_1}, \Lambda_{i_2}, \cdots \to \Lambda$.

Proof. Given as exercise, solve at some point. $\qquad\square$

This lemma allows us to prove that the infimum is obtained (verify this also), giving us the following reformulation of Minkowski's fundamental theorem (or as I prefer, Minkowski's pidgeonhole principle).

Theorem 2.21 (Minkowski's critical determinant lower bound)

For any centrally symmetric convex body $C \subset \mathbb{R}^d$,

$$\frac{\Delta(C)}{\text{vol} C} \geq \frac{1}{2^d}$$

Finding an upper bound for the ratio $\Delta(C)/\text{vol} C$ turns out the be much harder. Minkowski was able to give an upper bound, when $C = B^d$, of $\frac{1}{2}(\zeta(d))^{-1}$ where $\zeta(\cdot)$ is the Reimman zeta function. We prove, thanks to Hlawka, a generalisation of Minkowski's result.

Lemma 2.22 (Davenport & Rogers)

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous map that vanishes outside of some bounded region. For $\gamma \in \mathbb{R}$ set

$$V(\gamma) := \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1, \ldots, x_{d-1}, \gamma) \mathrm{d}x_1 \cdots \mathrm{d}x_{d-1}$$

Fix $\delta > 0$ and let Λ' be the integer lattice in the hyperplane $x_d = 0$. For a vector $y \in \mathbb{R}^d$ of the form $y = (y_1, \ldots, y_{d-1}, \delta)$ define Λ_y as the lattice generated by y and Λ'. Then,

$$\int_0^1 \cdots \int_0^1 \left(\sum_{x \in \Lambda_y, x_d \neq 0} f(x) \right) \mathrm{d}y_1 \cdots \mathrm{d}y_{d-1} = \sum_{i \in Z \setminus \{0\}} V(i\delta)$$

Proof. Looks like a routine computation but theres one line dependent on voodoo magic, can't wrap my head around it :($\qquad\square$

Now for our generalisation. We work, as suggested, with the probabilistic method.

Theorem 2.23 (Hlawka's theorem)

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a bounded Reimann integrable function that vanishes outside of some bounded region and let $\epsilon > 0$. Then there exists a unit lattice Λ in $\mathbb{R}^d$ with

$$\sum_{\mathbf{0} \neq x \in \Lambda} f(x) < \int_{\mathbb{R}^d} f(x) \mathrm{d}x + \epsilon$$

Proof. $\qquad\square$

2.3 Appendix C: A new bound for sphere packing in $\mathbb{R}^d$ as $d \to \infty$

Notes based off of [1]. Campos utilises a Poisson random geometric graph to improve Hayes' 1947 bound on the density of the sphere packing problem in high dimensional $\mathbb{R}^d$ by a factor of $\log d$.

We denote the maximal density unit-volume sphere packing in $\mathbb{R}^d$ by $\theta(d)$. In this note, we follow the work of Campos et al to prove $\theta(d) \to (1 - o(1))\frac{d \log d}{2^{d+1}}$ as $d \to \infty$. It is worth noting that this is still an exponential factor behind the best upper bound, which currently is $2^{(-0.599\cdots+o(1))d}$. An explicit value of $\theta(d)$ is only known for $d \in \{1, 2, 3, 8, 24\}$.

> **Theorem 2.24 ($\theta(d)$ bound / theorem 1.1 in [1])**
>
> As $d \to \infty$, one has
> $$\theta(d) \geq (1 - o(1))\frac{d \log d}{2^{d+1}}$$

Of particular importance is the structural irregularity of the packings used to prove this theorem. Previous work relied on lattice packings, whereas this work constructs packings by recursively adding random spheres to $\mathbb{R}^d$ that are centred on points of an appropriately modified Poisson process.

Campos & Co also managed to adapt their techniques to the problem of spherical codes. I will elect to leave that to their paper (see thm1.2 [1]).

Proof of $\theta(d)$ bound

Here I describe the method.

(i) Discretize space with a Poisson process at a carefully chosen intensity.

(ii) Impose additional uniformity properties on the discrete point set given in (i), giving $X \subset \mathbb{R}^d$.

(iii) Consider geometric graph $G = G(X, 2r_d)$, r_d the radius of the d-dimensional unit-volume ball.

(iv) Bound $\alpha(G)$, the largest independent set in G.

In regards to step (4) we have the following new tool, certainly of independent interest. Let $\Delta(G)$ be the maximum vertex degree and let $\Delta_{\mathrm{co}}(G)$ denote the maximum codegree (that is, the maximal size of $|\mathcal{N}(u) \cap \mathcal{N}(v)|$ over vertices $u, v \in V$).

> **Theorem 2.25 ($\alpha(G)$ bound / theorem 1.3 in [1])**
>
> Let G be a graph on n vertices with $\Delta(G) \leq \Delta$ and $\Delta_{\mathrm{co}}(G) \leq C\Delta(\log \Delta)^{-c}$. Then,
> $$\alpha(G) \geq (1 - o(1))\frac{n \log \Delta}{\Delta}$$
> where $o(1) \to 0$ as $\Delta \to \infty$.

In fact, one can take $C = 2^{-7}$ and $c = 7$. In fact, this bound is sharp up to some constants c and C. To see this, ... [Not sure I fully get this].

With this tool, the proof of our $\theta(d)$ bound is remarkably easy. We just need the following lemma.

> **Lemma 2.26 (large uniform geometric graphs / lemma 2.1 in [1])**
>
> Let $\Omega \subset \mathbb{R}^d$ be bounded and measurable. Then $\forall d \geq 1000 \; \exists X \subset \Omega$ with
>
> $$|X| \geq (1 - 1/d)\frac{\Delta}{2^d}\text{Vol}(\Omega), \quad \text{where} \quad \Delta = \left(\frac{\sqrt{d}}{4\log d}\right)^d$$
>
> and, setting $G = G(X, r_d)$ we have
>
> $$\Delta(G) \leq \Delta(1 + \Delta^{-1/3}) \quad \text{and} \quad \Delta_{\text{co}} \leq \Delta e^{-(\log d)^2/8}$$

Proof of $\theta(d)$ bound / Theorem 1.1 in [1]. Fix $R > 0$ and set $\Omega = B_{\mathbf{0}}(R)$. It suffices to prove that we can place

$$(1 - o(1))\text{Vol}(\Omega)\frac{d\log d}{2^{d+1}}$$

points each at pairwise distance less than $2r^d$. The previous lemma gives us $X \subset \Omega$ with

$$|X| \geq (1 - o(1))\text{Vol}(\Omega)\Delta 2^{-d}$$

with $G(X, 2r_d)$ satisfying the conditions of our $\alpha(G)$ bound. Applying this gives an independent set I with

$$|I| \geq (1 - o(1))\frac{|X|\log\Delta}{\Delta} \geq (1 - o(1))\text{Vol}(\Omega)\frac{d\log d}{2^{d+1}}$$

and we're done. $\qquad\square$

To prove lemma 2.1 we'll consider a Poisson point process with intensity $\lambda = 2^{-d}\Delta = \left(\frac{\sqrt{d}}{8\log d}\right)^d$ and remove "bad" points $x \in X$ with

$$|X \cap B_x(2r_d)| \geq \Delta(1 + \Delta^{-1/3}) \quad \text{or} \quad |X \cap B_x(2_r) \cap B_y(2r_d)| \geq \Delta e^{-(\log d)^2/8} \qquad (\heartsuit)$$

for some $y \in X$. These bad points are of course those that have too high a degree or codegree to let us be in the conditions of our $\alpha(G)$ bound for $G = G(X, 2r_d)$. To prove we don't delete too many points, we will make repeated use of the previously seen Mecke equation, stated in the following form.

> **Lemma 2.27 (univariate Mecke)**
>
> Let $\Omega \subset \mathbb{R}^d$ be bounded and measurable, and let $X \sim \text{Po}_\lambda(\Omega)$ be a Poisson point process rate λ on Ω. Then, for events $(A_x)_{x \in \Omega}$
>
> $$\mathbb{E}[|x \in X \cap \Omega : A_x \text{ holds on } X|] = \lambda \int_\Omega \mathbb{P}(A_x \text{ holds on } X \cup \{x\})dx$$

Proof. Just a special case of lemma 1.11. $\qquad\square$

We will also make use of the following elementary tail bound for Poisson random variables

> **Lemma 2.28 (Poisson tail bound)**
>
> Let $Y \sim \text{Po}(\lambda)$ be a Poisson random variable with rate λ. Then, for all $t > 0$,
>
> $$\mathbb{P}(Y - \lambda \geq \lambda t) \leq \exp(-\lambda\min\{t, t^2\}/3)$$

Proof. Smells like a Chernoff type bound, some napkin mathematics (using chernoff and $1 + x \leq e^x$) gives me a candidate RHS of $e^{-\lambda t^2}$. I can't, off the top, of my head beat this though. $\qquad\square$

Firstly, we'll prove the number of points with too high a degree is small.

> **Lemma 2.29 (number of high degree points is small / lemma 2.2 in [1])**
>
> Let $\Omega \subset \mathbb{R}^d, d \geq 4$ be a bounded & measurable subset and $\mathcal{X} \sim \text{Po}_\lambda(\Omega)$. Then,
>
> $$\mathbb{E}[|\{x \in X : |X \cap B_x(2r_d)| \geq \Delta(1 + \Delta^{-1/3})\}|] \leq \frac{1}{2d}\mathbb{E}[|X|], \quad \text{with } \lambda = 2^{-d}\Delta = \left(\frac{\sqrt{d}}{8\log d}\right)^d$$

Proof. Fix $s > 0$. Then Mecke's equation gives us

$$\mathbb{E}[|\{x \in \mathcal{X} : |\mathcal{X} \cap B_x(2r_d)| \geq s\}|] = \lambda \int_\Omega \mathbb{P}[|\mathcal{X} \cap B_x(2r_d)| \geq s - 1] \tag{†}$$

Now, for fixed $x \in \Omega$, $|\mathcal{X} \cap B_x(2r_d)|$ is a Poisson random variable of rate at most $\lambda 2^d =: \Delta$, which follows from noting $\text{Vol}(B_x(2r_d) \cap X) \leq 2^d$. Hence, via our Poisson tail bounds, one has

$$\mathbb{P}[|\mathcal{X} \cap B_x(2r_d)| \geq \Delta(t+1)] \leq \exp\left(-\min\{t, t^2\}\Delta/3\right) \quad \text{for } t > 0$$

Taking $t = \Delta^{-1/3} - \Delta^{-1}$ and substuting this into (†) gives us the result. $\square$

Now we do the same with the codegree, which will take more work.

> **Lemma 2.30 (number of high codegree points is small)**
>
> Let $\Omega \subset \mathbb{R}^d$ be bounded and measurable, $\mathcal{X} \sim \text{Po}_\lambda(\Omega)$ and set $\eta = e^{-(\log d)^2/8}$. Then,
>
> $$\mathbb{E}\left[|\{x \in \mathcal{X} : \exists y \in \mathcal{X} \text{ with } |\mathcal{X} \cap B_x(2r_d) \cap B_y(2r_d)| \geq \eta\Delta\}|\right] \leq \frac{1}{2d}\mathbb{E}[|\mathcal{X}|]$$

Proof. Set $I_{x,y} = |\mathcal{X} \cap B_x(2r_d) \cap B_y(2r_d)|$. Then applying Meckes, obtain

$$\text{LHS} = \mathbb{E}|\{x \in \mathcal{X} : \exists y \in \mathcal{X} \text{ with } I_{x,y} \geq \eta\Delta\}| = \lambda \int_\Omega \mathbb{P}(\exists y \in \mathcal{X} : I_{x,y} \geq \eta\Delta - 1)\mathrm{d}x$$

Thus it suffices to prove that $\mathbb{P}(\exists y \in \mathcal{X} : I_{x,y} \geq \eta\Delta - 1) \leq \frac{1}{2d}$.

> **Claim: (1)**
>
> $$\mathbb{P}(\exists y \in \mathcal{X} : I_{x,y} \geq \eta\Delta - 1) \leq \mathbb{E}|B_x(\log d) \cap \mathcal{X}| + \mathbb{E}|\{y \in \mathcal{X} \setminus B_x(\log d) : I_{x,y} \geq \eta\Delta - 1\}|$$

Proof of claim (1). Observe, by applying Markov's inequality twice,

$$\begin{aligned}
&\mathbb{P}(\exists y \in \mathcal{X} : I_{x,y} \geq \eta\Delta - 1) \\
&= \mathbb{P}(\exists y \in \mathcal{X} \cap B_x(\log d) : I_{x,y} \geq \eta\Delta - 1) + \mathbb{P}(\exists y \in \mathcal{X} \setminus B_x(\log d) : I_{x,y} \geq \eta\Delta - 1) \\
&= \mathbb{P}(|\{y \in \mathcal{X} \cap B_x(\log d) : I_{x,y} \geq \eta\Delta - 1\}| \geq 1) + \mathbb{P}(|\{y \in \mathcal{X} \setminus B_x(\log d) : I_{x,y} \geq \eta\Delta - 1\}| \geq 1) \\
&\leq \mathbb{E}|\{y \in \mathcal{X} \cap B_x(\log d) : I_{x,y} \geq \eta\Delta - 1\}| + \mathbb{E}|\{y \in \mathcal{X} \setminus B_x(\log d) : I_{x,y} \geq \eta\Delta - 1\}| \quad \text{(Markov)} \\
&\leq \mathbb{E}|\mathcal{X} \cap B_x(\log d)| + \mathbb{E}|\{y \in \mathcal{X} \setminus B_x(2r_d) : I_{x,y} \geq \eta\Delta - 1\}|
\end{aligned}$$

and we're done :) $\square$

Now we'll bound the two terms in our summation. For the leftmost term, simply observe

$$\mathbb{E}|\mathcal{X} \cap B_x(\log d)| = \lambda\text{Vol}(B_x(\log d)) = \left(\frac{\sqrt{d}}{8\log d}\right)^d \cdot \left(\frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}(\log d)^d\right) = \frac{\left(\frac{d\pi}{64}\right)^{d/2}}{\Gamma(\frac{d}{2}+1)}$$

Which with $\Gamma(d/2 + 1) \geq 2^{-d/2}e^{-d/2}d^{d/2}$ (Stirling's approximation) gives

$$\frac{\left(\frac{d\pi}{64}\right)^{d/2}}{\Gamma(\frac{d}{2}+1)} \leq \left(\frac{2\pi e}{32}\right)^{d/2}$$

Now Campos & Co jump from this to a bound above of $\frac{1}{4d}$. I make some comments about this below.

For the rightmost term we'll need the following result.

> **Claim:** (2) For $t \geq 0$, take $x, y \in \mathbb{R}^d$ with $|x - y| \geq t$. Then,
> $$\mathrm{Vol}(B_x(2r_d) \cap B_y(2r_d)) \leq 2^d e^{-t^2/4}$$

Proof of claim 2. Easy calculations, just take a (carefully chosen) covering ball. See [1] fact 2.3. $\qquad\square$

Now, using Mecke, observe (as $I_{x,y} = 0$ on $y \notin B_x(4r_d)$) setting $\Omega' := B_x(4r_d) \setminus B_x(\log d)$

$$\mathbb{E}|\{y \in X \setminus B_x(\log d) : I_{x,y} \geq \eta\Delta - 1\}| = \lambda \int_{\Omega'} \mathbb{P}(X_{x,y} \geq \eta\Delta - 2)\mathrm{d}y$$
$$\leq \lambda\mathrm{Vol}(\Omega') \times \max_{y \in \Omega'} \mathbb{P}(X_{x,y} \geq \eta\Delta - 2)$$
$$\leq \lambda\mathrm{Vol}(B_x(4r_d)) \times \max_{y \in \Omega'} \mathbb{P}(X_{x,y} \geq \eta\Delta - 2)$$
$$= \mathbb{E}[X \cap B_x(4r_d)] \times \max_{y \in \Omega'} \mathbb{P}(X_{x,y} \geq \eta\Delta - 2)$$

Now some annoying abounding gives the same $\frac{1}{4d}$ upper bound. Gonna eat lunch then type this up. $\quad\square$

> **Remark 2.31 (on claim 1)**
>
> Could we get something tighter by initially summing over $y \in \mathcal{X}$ (disjoint probabilities) and then bounding the terms in the summand? Also the choice of our split set to be $B_x(\log d)$ is purely to cancel with the chosen $\log(d)$ factor in our intensity λ. Worth noting.

> **Remark 2.32 (on the bounding of $\mathbb{E}|\mathcal{X} \cap B_x(\log d)|$)**
>
> That final leap seems super lazy? We go from an exponential rate of decrease to linear. Surely this can be improved to achieve something non-trivial? Maybe not, worth investigating regardless. In the paper they get 8 as their final denominator, I get 32? Did I improve this or make a mistake? Check it out!

Now we may finally prove our original lemma, lemma 2.26 / lemma 2.1 in [1].

Proof of lemma 2.26. Set S_1 be the collection of points with high degree and S_2 those with high codegree (i.e. the left/right conditions in $(\heartsuit)$) and $X = \mathcal{X} \setminus (S_1 \cup S_2)$ the space without these "bad" points. Then,

$$\mathbb{E}|X| \geq \mathbb{E}|\mathcal{X}| - \mathbb{E}|S_1| - \mathbb{E}|S_2| \geq \left(1 - \frac{1}{d}\right)\mathbb{E}|\mathcal{X}| = \left(1 - \frac{1}{d}\right)\mathrm{Vol}(\Omega)\Delta 2^{-d}$$

using our lemmas for the expected size of S_1 and S_2, so we have our $X \subset \Omega$ (by $\mathbb{P}(X \geq \mathbb{E}X) > 0$). $\quad\square$

> **Remark 2.33 (on the $d \geq 1000$ assumption)**
>
> Where was $d \geq 1000$ used? I can only spot a $d \geq 4$ in the use of Stirling's. Worth checking.

Controlling degrees and Codegrees

We now introduce the *Rödl nibble*, which will be the key to our proof of thm 1.3.
Throughout this (sub)section, we make the following assumptions.

$$d_G(v) \in \{\Delta - 1, \Delta\}, \ \Delta_{\mathrm{co}}(G) \leq \eta\Delta, \ \Delta \geq 1, \ \gamma \leq \frac{1}{2}, \ \eta \in [\Delta^{-1/2}, \gamma^2/8]$$

We spend this section showing that after each "nibble", the degrees and codegrees decrease appropriately. This result is summarised with the following proposition.

Let Δ, γ, η be as above, let $\alpha \in [2\gamma^2, \gamma]$ and let G be as above. Let $A \subseteq V(G)$ be a (γ/δ)-random set, and let $G' = G \setminus (A \cup \mathcal{N}_G(A))$. Then, $\forall u \neq v \in G$

$$\mathbb{P}(d_{G'}(v) \geq (1 - \delta + \alpha)d_G(v)|v \in V(G')) \leq \exp\left(\frac{-\alpha^2}{32\gamma\eta}\right)$$

$$\mathbb{P}(d_{G'}(u,v) \geq (1 - \delta + \alpha)\eta\Delta|v \in V(G')) \leq \exp\left(\frac{-\alpha^2}{32\gamma\eta}\right)$$

Before we're able to prove this we'll need some preliminary results concerning our nibble. First, we'll calculate the mean degree/codegree after each nibble.

Lemma 2.35 (mean degree/codegree post-nibble / lemma 3.2 in [1])

Let Δ, γ, η, G be as defined at the start of the (sub)section, let A and G' be as in prior proposition (lemma 3.1 in [1]). Then, for $u, v \in V(G)$,

$$\mathbb{E}[d_{G'}(v)|v \in V(G')] \leq (1 - \gamma + \gamma^2)d_G(v) \quad \text{and} \quad \mathbb{E}[d_{G'}(u,v)|u, v \in V(G')] \leq (1 - \gamma + \gamma^2)d_G(u,v)$$

We just prove the mean degree result, noting the proof for the codegree is almost identical.

Proof. Conditioned on the fact $v \in V(G')$, we know that A is a (γ/Δ)-random subset of $V(G) \setminus (\mathcal{N}_G(v) \cup \{v\})$ (as all the $\mathcal{N}_G(v) \cup \{v\}$ must survive). Let $w \in \mathcal{N}_G(v)$ and $d_w = |\mathcal{N}_G(w) \setminus (\mathcal{N}_G(v) \cup \{v\})|$. Then,

$$d_w = |\mathcal{N}_G(w)| - |\mathcal{N}_G(v) \cap \mathcal{N}_G(w)| - |\{v\}| = d_G(w) - d_G(w,v) - 1$$

(where the -1 comes from $w \in \mathcal{N}_G(v)$), so by our setup we may bound

$$d_w \geq (\Delta - 1) - \eta\Delta - 1 \geq (1 - \gamma^2/8)\Delta - 2$$

[SLIGHTLY OFF ON LAST BOUND THERE, CHECK PAPER AND SEE HOW TO GO FROM A -¿ B]. □

References

[1] Marcelo Campos, Matthew Jenssen, Marcus Michelen, and Julian Sahasrabudhe. A new lower bound for sphere packing, 2023.

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