Random Geometric Graphs

A collection of notes from my time working under the supervision of Mathew Penrose on random geometric graph theory and related topics.

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Abstract

Suspendisse vitae elit. Aliquam arcu neque, ornare in, ullamcorper quis, commodo eu, libero. Fusce sagittis erat at erat tristique mollis. Maecenas sapien libero, molestie et, lobortis in, sodales eget, dui. Morbi ultrices rutrum lorem. Nam elementum ullamcorper leo. Morbi dui. Aliquam sagittis. Nunc placerat. Pellentesque tristique sodales est. Maecenas imperdiet lacinia velit. Cras non urna. Morbi eros pede, suscipit ac, varius vel, egestas non, eros. Praesent malesuada, diam id pretium elementum, eros sem dictum tortor, vel consectetuer odio sem sed wisi.

Keywords: Random geometric graphs, Poisson Point Processes

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1 Introduction

1.1 What is a random geometric graph?

Definition 1.1 (geometric graph)

Let $\mathcal{X} \subset \mathbb{R}^d$ be a finite collection of points and $r > 0$. The *geometric graph* $G(\mathcal{X}, r)$ is the graph with vertex set $\mathcal{X}$ and edge set $\{\{x, y\} \subset \mathcal{X} \ : \ |x - y| \leq r\}$, where $|\cdot|$ denotes the Euclidean norm.

We can turn this into a random graph by letting the set $\mathcal{X} \subset \mathbb{R}^d$ be random. The resulting structure $G(\mathcal{X}, r)$ is said to be the *random geometric graph* (A.K.A the *Gilbert graph*). The first structure we'll consider is found by uniformly scattering n points in the d-dimensional hypercube $[0, 1]^d$.

Example 1.2 (fixed scale geometric graph)

Let $\xi_1, \xi_2, \ldots \overset{\text{i.i.d}}{\sim} \text{Unif}[0, 1]^d$ and fix a sequence $(r_n)_{n\geq 1}$ in $\mathbb{R}_{>0}$. Let $\mathcal{X}_n := \{\xi_1, \ldots, \xi_n\}$. Then $G(\mathcal{X}_n, r_n)$ is the *fixed scale geometric graph* at time $n \geq 1$.

We will also consider what happens when the number of vertices is Poisson.

Example 1.3 (Poisson scale geometric graph)

Let $\xi_1, \xi_2, \ldots \overset{\text{i.i.d}}{\sim} \text{Unif}[0, 1]^d$ and $N_n \sim \text{Poisson}(n)$ for $n \geq 1$ be independent of $(\xi_1, \xi_2, \ldots)$. Let $\mathcal{P}_n := \{\xi_1, \ldots, \xi_{N_n}\}$ and fix a sequence $(r_n)_{n\geq 1}$ in $\mathbb{R}_{>0}$. Then the random geometric graph $G(\mathcal{P}_n, r_n)$ is the d-dimensional *Poisson scale geometric graph* at time $n \geq 1$.

As one may expect, the vertices of this graph form a Poisson point process in $\mathbb{R}^d$. This was originally an exercise (1.1) in [1].

Proposition 1.4 ($\mathcal{P}_n(\cdot)$ is a Poisson point process)

Let $\mathcal{P}_n$ be the vertex set in the Poisson scale geometric graph. Then $\mathcal{P}_n(\cdot) : A \mapsto |\mathcal{P}_n \cap A|$, where $A \subset [0, 1]^d$ is measurable and $|\cdot|$ is the counting measure, is a *Poisson point process* with intensity $\lambda(A)$, $\lambda(\cdot)$ the Lebesgue measure on $\mathbb{R}^n$.

See Appendix A for the definition of the Poisson point process.

Proof. Write $\mathcal{P}_n(A_j) = \sum_{i=1}^{N_n} \mathbf{1}\{\xi_i \in A_j\}$. Then, conditioning on $\{N_n = m\}$, each $\mathcal{P}_n(A_j)$ is a $\text{Binom}(m, \lambda(A_j))$ random variable, making the joint distribution (also conditioned on $\{N_n = m\}$) a $\text{Multinom}(m, \lambda(A_1), \ldots, \lambda(A_k))$. Thus one has

$$\begin{aligned}
\mathbb{P}(\mathcal{P}_n(A_1) = a_1, \ldots, \mathcal{P}_n(A_k) = a_k) &= \underbrace{\left(\frac{e^{-n}n^m}{m!}\right)}_{\text{Poisson}(n;m)} \times \underbrace{\binom{m}{j_1, \ldots, j_k} \prod_{i=1}^{k} (\lambda(A_i))^{j_i}}_{\text{Multinom}(m,\lambda(A_1),\ldots,\lambda(A_k);j_1,\ldots,j_k)} \\
&= \prod_{i=1}^{k} \frac{e^{-n\lambda(A_i)}(n\lambda(A_i))^{j_i}}{(j_i)!}
\end{aligned}$$

which is a product of the distributions $\text{Poisson}(n\lambda(A_i)), 1 \leq i \leq k$, giving us both of our defining properties of a Poisson point process. $\square$

An exercise (1.2) in [1] states the following result, which we'll prove in a (somewhat?) analogous way.

Proposition 1.5 (expected degree of first vertex in fixed scale RGG)

Consider the fixed scale random geometric graph $G(\mathcal{X}_n, r_n)$ with $r_n \to 0$ as $n \to \infty$. Let $D_{1,n}$ be the degree of the first vertex ξ_1. Prove that $\mathbb{E}[D_{1,n}] \sim \theta n r_n^d$

Proof. Write $D_{1,n} = \sum_{i=2}^{n} \mathbf{1}\{|\xi_1 - \xi_i| \leq r\}$, then by the ξ_i being i.i.d uniformly in $[0,1]^d$ we have $D_{1,n} \sim \text{Binom}(n-1, p)$ where p is the probability of a given $\xi_i, i \geq 2$ being within r_n of ξ_1. By the uniformity of the ξ_i this is computed as $p = \lambda(B_{r_n}(\xi_1) \cap [0,1]^d)$ where $B_{r_n}(\xi_1)$ is the hypersphere radius r_n centred at ξ_1 and λ the Lebesgue measure on $\mathbb{R}^d$. Now, conditioning on ξ_1, we have

$$\mathbb{E}[D_{1,n}|\xi_1 = v] = (n-1)\lambda(B_{r_n}(v) \cap [0,1]^d) \sim \theta n r_n^d$$

where the asymptotic equality follows from noting that n sufficiently large forces $B_{r_n}(v) \subset [0,1]^d$ by $r_n \to 0$. Note θ depends on d, and is the coefficient of the volume of the d-dimensional hypersphere radius r_n. Finally, by taking the expectation over ξ_1 and applying the total law of expectations we obtain $\mathbb{E}[D_{1,n}] \sim \theta n r_n^d$ by noting the independence of v on our asymptotic expression for $\mathbb{E}[D_{1,n}|\xi_1 = v]$. □

1.2 Counting Edges of $G(\mathcal{X}_n, r_n)$ and $G(\mathcal{P}_n, r_n)$

The number of edges $\mathcal{E}_n, \mathcal{E}'_n$ in $G(\mathcal{X}_n, r_n)$ and $G(\mathcal{P}_n, r_n)$ respectively is a Poisson random variable. To prove this, we'll use the so called *dependency graph*. For a more detailed treatment of thiese consult Appendix B.

Definition 1.6 (dependency graph)

Let $(V, \sim)$ be a finite simple graph w/ edge relation $\sim$ and vertex set V. We say $(V, \sim)$ is a *dependency graph* for the random variables $(W_\alpha)_{\alpha \in V}$ if whenever $A, B \subset V$ are disjoint with no $\alpha \in A$, $\beta \in B$ such that $\alpha \sim \beta$ (i.e. A and B lie in unique connected components) then

$$(W_\alpha)_{\alpha \in A} \perp\!\!\!\perp (W_\beta)_{\beta \in B} \quad \text{i.e. are independent}$$

From [3] (Theorem 2.1), we have the following result concerning dependency graphs of finite Bernoulli collections.

Lemma 1.7 (Poisson approximation lemma for Bernoulli sums)

Let $(\xi_i)_{i \in I}$ be a finite collection of Bernoulli random variables with dependency graph $(I, \sim)$. Set $p_i := \mathbb{P}(\xi_i = 1)$, $p_{ij} := \mathbb{P}(\xi_i = 1; \xi_j = 1)$ and suppose $\lambda := \sum_{i \in I} p_i$ is finite. Then, letting $W := \sum_{i \in I} \xi_i$, we have (LHS is just total variation distance)

$$\sup_{A \subseteq \mathbb{Z}} \left| \mathbb{P}[W \in A] - \mathbb{P}(\text{Po}(\lambda) \in A) \right| \leq \min\{3, \lambda^{-1}\} \left(\sum_{i \in I} \sum_{\mathcal{N}(i) \setminus \{i\}} p_{ij} + \sum_{i \in I} \sum_{j \in \mathcal{N}(i)} p_i p_j \right)$$

2 Appendices

2.1 Appendix A: Poisson Point Processes

For a full treatment, consult [2]. Here I will simply give the relevent definitions and results from this text, leaving proofs (unless containing a particularly important idea) to [2].

Definition 2.8 (point process)

Let $(\mathbb{X}, \mathcal{X})$ be a measure space and let $\mathbf{N}(\mathbb{X}) \equiv \mathbf{N}$ be the family of measures that can be written as a countable sum of finite measures on $\mathbb{X}$ with image in $\mathbb{N}_0$. Let $\mathcal{N}(\mathbb{X}) \equiv \mathcal{N}$ be the σ-algebra generated by the collection of subsets $\{\mu \in \mathbf{N} : \mu(A) = k\}$ over $A \in \mathcal{X}, k \in \mathbb{N}_0$. A *point process* on $\mathbb{X}$ is a random element η of the measure space $(\mathbf{N}, \mathcal{N})$

Definition 2.9 (Poisson point process)

Let $\mathbb{X}$ be a space and λ an s-finite measure on $\mathbb{X}$. A *Poisson point process* with intensity measure λ is a point process η on $\mathbb{X}$ with

(i) $\eta(A) \sim \text{Poisson}(\lambda(A))$, that is $\eta(A)$ is Poisson with parameter $\lambda(A)$

(ii) If $A_1, \ldots, A_n \in \mathbb{X}$ are pairwise disjoint then $\eta(A_1), \ldots, \eta(A_n)$ are independent.

2.2 Appendix B: Dependency graphs

References

[1] Michael Krivelevich, Konstantinos Panagiotou, Mathew Penrose, and Colin McDiarmid. *Random Graphs, Geometry and Asymptotic Structure*. London Mathematical Society Student Texts. Cambridge University Press, 2016.

[2] Günter Last and Mathew Penrose. *Lectures on the Poisson Process*. Institute of Mathematical Statistics Textbooks. Cambridge University Press, 2017.

[3] Mathew Penrose. *Random Geometric Graphs*. Oxford University Press, 05 2003.