Crossing Bridges

An investigation into a bridge crossing problem under funky constraints

Jacob Green
Department of Mathematical Sciences, The University of Bath

October 30, 2024

Abstract

Suspendisse vitae elit. Aliquam arcu neque, ornare in, ullamcorper quis, commodo eu, libero. Fusce sagittis erat at erat tristique mollis. Maecenas sapien libero, molestie et, lobortis in, sodales eget, dui. Morbi ultrices rutrum lorem. Nam elementum ullamcorper leo. Morbi dui. Aliquam sagittis. Nunc placerat. Pellentesque tristique sodales est. Maecenas imperdiet lacinia velit. Cras non urna. Morbi eros pede, suscipit ac, varius vel, egestas non, eros. Praesent malesuada, diam id pretium elementum, eros sem dictum tortor, vel consectetuer odio sem sed wisi.

Keywords: Logic, Discrete Optimisation

1 Problem Statement & Cursory Remarks

Problem **(N-crossing problem)**

Suppose N people must cross a bridge. They may cross at most in pairs and they require a flashlight to cross, of which there is only 1. If a pair is to cross, they must walk at the speed of the slowest member. If person $1 \leq i \leq N$ takes 2^i minutes to cross, what is the fastest time all N people can cross the bridge?

To get a feel for the problem, lets solve the $N = 3$ case.

Solution for 3-crossing problem. Let the start of the bridge be A and the end be B. Call a trip from $A \to B$ a *crossing* and a trip from $B \to A$ a *return*. Then clearly there must be at least two crossings and one return. Also, we know that in order to be optimal, crossings will always be done in pairs and returns will always be done alone. One can argue that this is true rather simply (exercise 1).

Assume for the moment the torch is always at A. Then the fastest way to get everyone across, noting we must always cross in pairs, is $2^3 + 2^2 = 12$ minutes. The fastest we can return is in $2^1 = 2$ minutes, giving us a lower bound of $12 + 2 = 14$ minutes.

We now seek a way to cross achieving this lower bound. If person 1 and 2 cross first, person 1 returns and person 1 and 3 cross this will take a total of $2^2 + 2^1 + 2^3 = 14$ minutes. Hence 14 minutes is the fastest achievable time. □

The structure here was rather intuitive. We select the fastest person to be the "torch bearer", i.e. the one who carries out the return trip, and then we are forced to be in optimal conditions (note 1 and 3 crossing first works out the exact same). Lets see if this logic prevails in the $N = 4$ case.

Solution to the 4-crossing problem. One may argue, as before, that we need at least 3 crossings and 2 returns. The fastest way to get everyone across, in exactly 3 crossings (noting we always cross in pairs), is $2^4 + 2^2 + 2^2 = 22$ minutes. The fastest way to do 2 returns is $2^1 + 2^1 = 4$ minutes. So we have a lower bound of 28 minutes.

Unlike last time, this lower bound cannot be achieved. To see this, note in order to have a total return time of 4 minutes, we would need person 1 to return twice. This forces our 3 crossings to be person 1 with person i, $2 \leq i \leq 4$, in some order. If this is the case, we take a total of $2^2 + 2^3 + 2^4 + 2^1 + 2^1 = 32$ minutes. Contradiction!

We know our total time must be even, as all individual crossings/returns are of even length. Thus the next possible best bound is 30. This can be acheived as follows:

1. Person 1 and 2 cross together.
2. Person 1 returns.
3. Person 3 and 4 cross together.
4. Person 2 returns.
5. Person 1 and 2 cross together.

This gives a total time of $2^2 + 2^1 + 2^4 + 2^2 + 2^2 = 30$ minutes, and hence this is our best possible time. □

The logic almost prevails. We have the fastest 2 as the torch bearers, but we don't let the fastest bear the torch twice.

One thing we observed was that, in order to minimise time spent crossing, people would always cross in pairs and return alone. This is due to the simple reasoning that crossing alone requires one person to go back with the torch, and thus the number of people at side B would remain constant but we'd increase our total time. Equivalent logic can be used to argue for solo returns. While a trivial observation, this will be key in solving the full problem.

This time, it was also optimal to pair up the slowest two walkers. This, as one can easily verify, is not true in the $N = 3$ case. With similar arguing one can find the minimum in the $N = 5$ case to be 56, which is also obtained by pairing up the slowest two walkers.

2 Formal Statement & Lemmas

Now let us formalise this problem. Let $\mathcal{P}_k^{=}(S)$ denote the subsets size k of a set S, and $\mathcal{P}_k^{\leq}(S)$ the subsets of size at most k. Let $[N] := \{1, \ldots, N\}$.

We know in order to be optimal, we will always be crossing in pairs and returning alone (except in the special case when a crossing gets everyone to B, then no one will return). Thus we may consider a strategy of getting everyone across as a sequence of *moves* $\omega = (\omega_1, \omega_2) \in \mathcal{P}_2^{=}([N]) \times \mathcal{P}_1^{\leq}([N])$, noting the order we choose the two people in our crossing pair doesn't affect the crossing time and we have the possibility of no one returning. We say ω_1 is the *crossing pair* and ω_2 the *returner* for a move ω.

Let us also track the number of people at B after a given sequence of moves. Let $f : [N] \to \{0, 1\}$ map $f(i\ ;\ \omega_1, \ldots \omega_k) = \mathbf{1}\{\text{person } i \text{ is at } B\}$ and set $F(\omega_1, \ldots, \omega_k) = \sum_{i=1}^{N} f(i\ ;\ \omega_1, \ldots, \omega_k)$. We will call a sequence of moves $\Omega = (\omega_1, \ldots, \omega_k)$ *complete* if $F(\Omega) = N$.

Lemma 2.0 (optimal number of moves)

If Ω is complete, then $|\Omega| = N - 1$.

Proof. Let Ω be a complete sequence of moves and let Ω_k denote the first k of these moves. Then

$$F(\Omega_k) < N - 2 \implies F(\Omega_{k+1}) = 1 + F(\Omega_k)$$

and

$$F(\Omega_k) = N - 2 \implies F(\Omega_{k+1}) = 2 + F(\Omega_k)$$

as each move will leave one new person at side B of the bridge, unless there are exactly two people remaining in which case we can send both people over and make no return trip. Thus, setting $\Omega_0 = \emptyset$ so that $F(\Omega_0) = 0$, we have $F(\Omega_{N-1}) = N$ and hence $|\Omega| = N - 1$. $\square$