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1 Notation

I have attempted to introduce all non-standard¹ as and when it used, but to save ink here are some is some common notation. If any notation is unclear in this document please email me and I'll add it here.

- $\mathbb{N} = \{1, 2, 3, \dots\}$ denotes the positive integers
- $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ denotes the nonnegative integers.
- $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ denotes the integers
- $\mathbb{Q} = \{a/b : a, b \in \mathbb{Z}\}$ denotes the rational numbers
- \mathbb{R} denotes the real numbers, \mathbb{R}_+ the positive reals and $\mathbb{R}_{\geq 0}$ the nonnegative reals.
- \mathbb{C} denotes the complex numbers
- $\mathbb{Z}/n\mathbb{Z}$ denotes the integers modulo n .
- $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x \in \mathbb{R}$
- $\lceil x \rceil$ denotes the smallest integer greater than or equal to $x \in \mathbb{R}$.
- We write $a \mid b$ if $a \in \mathbb{Z}$ divides $b \in \mathbb{Z}$, i.e. if there exists some $k \in \mathbb{Z}$ with $b = ka$.
- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ denotes the binomial coefficient².
- We use iff on occasion as shorthand for if and only if, or equivalently \Leftrightarrow .
- \propto denotes proportionality, i.e. $a \propto b$ if there exists $c > 0$ with $a = bc$.
- We write $\binom{A}{k}$ to denote the set of subsets size k of A .
- We write $\mathcal{P}(A)$ for the set of all subsets of a set A .
- When clear from context, we use the combinatorial notation $[n] = \{1, \dots, n\}$.
- We denote the empty set $\{\}$ by \emptyset .
- When defining a to be b , we may write $a := b$ to make it clear this is a definition.

¹That is, non-standard to a student familiar with the A-Level mathematics syllabus.

²This is sometimes written as nCr and said as “ n choose k ”. At any reasonable level of mathematical maturity one tends to deviate from this nomenclature and as such, so will we.

2 BMO 1

2.1 1999 - 2000

2.1.1 Question 2

Problem. Prove that

$$2000 \mid 121^n - 25^n + 1900^n - (-4)^n$$

for any $n \in \mathbb{N}$

Idea: Factor 2000 and do some modular arithmetic.

Solution. Let

$$f : \mathbb{N} \rightarrow \mathbb{N}, n \mapsto 121^n - 25^n + 1900^n - (-4)^n$$

Factor $2000 = 2^4 \cdot 5^3$, then it suffices to prove

$$5^3 = 125 \mid f(n) \quad \text{and} \quad 2^4 = 16 \mid f(n)$$

for each $n \in \mathbb{N}$. These are easily seen working modulo 16 and 125. Observe

$$f(n) \equiv 9^n - 9^n + (-4)^n - (-4)^n \equiv 0 \pmod{16}$$

and

$$f(n) \equiv (-4)^n - 25^n + 25^n - (-4)^n \equiv 0 \pmod{125}$$

Hence, we deduce $2000 \mid f(n)$. □

Remark. A competitor not yet familiar with modular arithmetic may prove, for integer k and positive integer n ,

$$a \mid b^n \implies a \mid (b + ka)^n$$

by using the fact

$$a \mid b \implies a \mid b + ka$$

and expanding $(b + ka)^n$. All of this is elementary and leads to an analogous solution.

2.1.2 Question 3

Problem. Let ABC be a triangle with $\angle BAC = 90^\circ$. Find the point P on the perimeter of ABC such that the quantity

$$|AP| + |BP| + |CP|$$

is minimised.

Idea: Notice an invariant that reduces the problem to three cases. The non-trivial case is to be handled using the equality of areas in congruent triangles.

Solution. Suppose P is on BC . Then it is clear $|BC| = |BP| + |CP|$ and hence we have an invariant. Thus, it suffices to find P on BC such that $|AP|$ is minimised, which is when AP is orthogonal to BC . Identical arguments can be made for when P is on AB and (resp. AC) to see we have minimality when CP is orthogonal to AB (resp. BP orthogonal to AC).

Case 1: P on BC :

P is the foot of the altitude from A . Hence the triangle APB is congruent³ to APC . In particular they both have equal area, which allows us to deduce

$$\frac{|AP| \cdot |BP|}{2} = \frac{|AB| \cdot |AC|}{4} = \frac{|AP| \cdot |CP|}{2} \implies |AP| = \frac{|AB| \cdot |AC|}{|BP| + |CP|} = \frac{|AB| \cdot |AC|}{|BC|}$$

where the implication follows via simple algebra. All together

$$|AP| + |BP| + |CP| = \frac{|AB||AC|}{|BC|} + |BC|$$

Case 2: P on AB

AC is the altitude from C and has foot A . Hence, $P = A$ and

$$|AP| + |BP| + |CP| = |AB| + |AC|$$

Case 3: P on AC

By an identical argument see that, again, $P = A$ and

$$|AP| + |BP| + |CP| = |AB| + |AC|$$

Let $|AB| = a$, $|AC| = b$ and $|BC| = c$. Suppose $P = A$, which is the case iff

$$\frac{ab}{c} + c > a + b$$

³By SAS or AAA.

Use the Pythagorean theorem to express

$$\frac{ab}{c} + c = \sqrt{a^2 + b^2} + \frac{ab}{\sqrt{a^2 + b^2}}$$

It is equivalent to check⁴

$$\left(\sqrt{a^2 + b^2} + \frac{ab}{\sqrt{a^2 + b^2}} \right)^2 > (a + b)^2$$

which reduces to

$$\frac{a^2 b^2}{a^2 + b^2} > 0$$

and hence it is clear that the original inequality holds everywhere, meaning $P = A$. \square

Remark. If we didn't spot the invariant, we could've taken a purely coordinate theoretic approach to the problem. This looks ugly, but there could be a nice trick.

Remark. There are many ways we can handle the resulting inequality after our geometry.

⁴As the map $x \mapsto x^2$ is monotone increasing on \mathbb{R}_+

2.1.3 Question 4

Problem. Define the sequence $(a_n)_{n \geq 0}$ for positive integers k by

$$a_0 = 1 \quad \text{and} \quad a_n = kn + (-1)^n a_{n-1}, \quad n \geq 1$$

Find all k such that 2000 is a term in (a_n) .

Idea: Look at the subsequences of even/odd terms of (a_n) . This will give us a closed form for (a_n) via some bash and it is then a number theory problem.

Solution. Run the recursion a second time to see, for $n \geq 2$,

$$a_n = kn + (-1)^n \left[k(n-1) + (-1)^{n-1} a_{n-2} \right] = k(n + (-1)^n(n-1)) - a_{n-2} \quad (\dagger)$$

If the $(-1)^n$ is constant we have easy recursions. Let (b_n) and (c_n) by $b_n = a_{2n}, n \geq 0$ and $c_n = a_{2n+1}, n \geq 0$ respectively. It is clear from (\dagger) that, for $n \geq 1$,

$$b_n = k(4n-1) - b_{n-1} \quad \text{and} \quad c_n = k - c_{n-1}$$

Hence we compute

$$\begin{aligned} b_n &= k(4n-1) - b_{n-1} \\ &= k(4n-1) - k(4(n-1)-1) + b_{n-2} \\ &= \dots \\ &= 4k \sum_{m=0}^{n-1} (-1)^m (n-m) - k \sum_{r=0}^{n-1} (-1)^r + (-1)^n \end{aligned}$$

using $b_0 = a_{2,0} = 1$. Similarly, we compute

$$c_n = k - c_{n-1} = k - k + c_{n-2} = \dots = k \mathbf{1}\{n \text{ odd}\} + (-1)^n (k-1)$$

using $c_0 = a_{2,0+1} = k-1$, where $\mathbf{1}\{\cdot\}$ is the indicator function⁵. Now observe, by reversing the order of summation (i.e. by substituting $m \mapsto n-1-m$),

$$\begin{aligned} \sum_{m=0}^{n-1} (-1)^m (n-m) &= \sum_{m=1}^n (-1)^{n-m} m = (-1)^n \sum_{m=1}^n (-1)^m m \\ &= (-1)^n \times \left(\underbrace{-1+2}_{=1} \underbrace{-3+4}_{=1} \underbrace{-5+6}_{=1} - \dots + (-1)^n n \right) = \begin{cases} n/2 & : 2 \mid n \\ (n+1)/2 & : 2 \nmid n \end{cases} \end{aligned}$$

⁵That is, for a subset $A \subseteq \mathbb{N}$ one has $\mathbf{1}\{n \in A\} = \begin{cases} 1 & : n \in A \\ 0 & : n \notin A \end{cases}$

All together, we get

$$b_n = \begin{cases} 1 + 2kn & : n \equiv 0 \pmod{2} \\ 2k(n+1) - k - 1 & : n \equiv 1 \pmod{2} \end{cases}$$

It is clear that

$$c_n = \begin{cases} k - 1 & : n \equiv 0 \pmod{2} \\ 1 & : n \equiv 1 \pmod{2} \end{cases}$$

It suffices to solve $b_n = 2000$ and $c_n = 2000$. For the latter, it is clear only $k = 2001$ gives a solution. For the former, if $n = 2m$, we seek the existence of pairs (m, k) with

$$1 + 4km = 2000$$

By parity, it is immediately clear there are no such pairs. For $n = 2m + 1$, solve

$$k(4m + 3) = 2001 = 3 \cdot 23 \cdot 29$$

Reducing modulo 4, any solution (m, k) must have $k \equiv 3 \pmod{4}$. Hence, writing $k = 3^a 23^b 29^c$ for $a, b, c \in \{0, 1\}$, we have $k \in \{3, 23, 3 \cdot 29, 23 \cdot 29\}$. Appending the $k = 2001$ solution, obtain

$$k \in \{3, 23, 87, 667, 2001\}$$

as our set of possible k . □

Remark. It is quicker to list terms, guess the general form of (a_n) and prove it by induction. I jumped straight into computations, hence the inelegance.

Remark. I wonder if there is a clever trick that allows us to solve without computing a closed-form expression for a_n ? It doesn't look likely admittedly.

2.1.4 Question 5

Problem. Suppose there are 7 people who wish to form 4 teams to compete in a quiz. In how many ways can they do this? Suppose another person joins, now how many ways can they form the teams?

Idea: Fix the team sizes and count each case.

Solution. We will split the count into the cases of possible team sizes. WLOG we fix a descending order on the team sizes.

Case 1: (4, 1, 1, 1)

Once we've chosen our four person team, there is nothing left to choose. Hence we get $\binom{7}{4} = 35$ combinations.

Case 2: (3, 2, 1, 1)

Once we've chosen our three person team, we have 4 remaining players to choose our two person team from. Hence we get $\binom{7}{3}\binom{4}{2} = 210$ combinations.

Case 3: (2, 2, 2, 1)

We choose our two-person teams one after the other as before, but note this time we will count some repetitions. Taking this approach, we count the $3!$ ways of arranging the three two-person teams in a line and hence we have $\binom{7}{2}\binom{5}{2}\binom{3}{2}\frac{1}{3!} = 105$ combinations.

Thus, summing over our disjoint possibilities, we obtain 350 combinations in total.

For the second part of this problem we take an essentially identical approach to see that there are 1701 combinations. This is left to the reader. \square

Remark. We may instead compute the number of combinations in case 3 as

$$\binom{7}{2, 2, 2} \frac{1}{3!} = \frac{7!}{2!2!2!3!} = 105$$

Remark. We could also recognise these as the Stirling numbers of the 2nd kind $S(7, 4)$ and $S(8, 4)$. These obey

$$S(n+1, k) = kS(n, k) + S(n, k-1)$$

which trivialises the second part.

2.2 2000 - 2001

2.2.1 Question 1

Problem. Find all two digit $n \in \mathbb{N}$ for which the digit sum of $10^n - n$ is divisible by 170.

Idea: $10^n - n = 99 \cdots 9ab$, so count the 9s and hence compute the digit sum explicitly.

Solution. Let $d(m)$ be the digit sum of an integer m . Then,

$$d(10^n - n) = 9(n - 2) + d(100 - n) \quad (\dagger)$$

Write $n = 10x + y$ for some $1 \leq x \leq 9$ and $0 \leq y \leq 9$. We deduce from (\dagger)

$$d(10^n - n) = \begin{cases} 89x - 8 & : y = 0 \\ 89x + 8y + 1 & : \text{otherwise} \end{cases}$$

If $y = 0$ we check for divisibility by 10 and see that $x = 2$ is forced, which we can verify gives $n = 20$ as a solution. Now, for $y \neq 0$, we first observe

$$170 \mid 89x + 8y + 1 \implies 89x + 8y \geq 169 \implies x \geq 2$$

Further, by parity, we see that x must be odd. Now we check $x \in \{3, 5, 7, 9\}$ individually. If $n = 10x + y$ is a viable n then

$$x = 3 \implies 170 \mid 268 + 8y \implies y = 9 \quad (1)$$

$$x = 5 \implies 170 \mid 446 + 8y \implies y = 8 \quad (2)$$

$$x = 7 \implies 170 \mid 624 + 8y \implies y = 7 \quad (3)$$

$$x = 9 \implies 170 \mid 802 + 8y \implies y = 6 \quad (4)$$

Combining (1), (2), (3), (4) and recalling that when $y = 0$ we had $n = 20$ forced, our possible solutions can be listed as $n = 20, 39, 58, 77, 96$. \square

Remark. We could've just manually checked each $x \in \{0, 1, \dots, 9\}$, rather than ruling out most of these by parity / bounding, making this problem highly accessible.

2.2.2 Question 3

Problem. A *tetromino* is a shape composed of four unit squares, each adjacent to at least one other.

- (i) Prove there are exactly 7 tetrominos up to rotation.
- (ii) Can we tile⁶ the 7 tetrominos into a 4×7 rectangle?

Idea: (i) fix the maximum number of adjacent squares (ii) colour and spot an invariant

Solution to i. For each tetromino A , let $\omega(A)$ be the maximal number of adjacent squares in T . It is immediately clear $\omega(T) \in \{2, 3, 4\}$.

$$\omega(A) = 4$$

Clearly there is only one possibility, the I -tetromino.

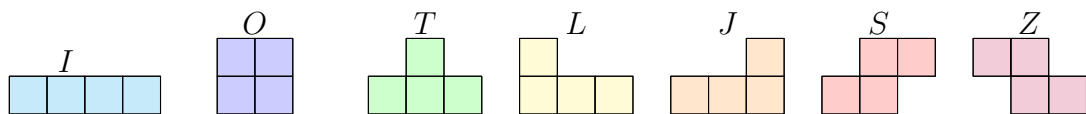
$$\omega(A) = 3$$

Here we observe three possibilities, the L -tetromino, J -tetromino or T -tetromino. Fix an orientation and place 3 squares in a straight line. There are $\binom{3}{1} = 3$ ways to place the two squares on top, and each resulting tetromino uniquely corresponds to a tetromino constructed by placing the remaining square underneath. Hence there are 3 such tetrominos.

$$\omega(A) = 2$$

Here we observe three possibilities, the L -tetromino, J -tetromino or T -tetromino. Repeat the $\omega(A) = 3$ argument.

Hence, we count $1 + 3 + 3 = 7$ tetrominos in total. Explicit constructions are given below.



□

Solution to ii. Colour the squares red and blue in an alternating pattern, so that there are an even number of red and an even number of blue squares. Notice the T -tetromino, no matter where we place it, will contain either 3 red or 3 blue squares. The remaining tetrominos all contain 2 red and 2 blue squares no matter where we place them. Deduce that any tiling of the tetrominos into the 4×7 board must contain an odd number of red and blue squares. Hence, we conclude no such tiling exists. □

⁶Equivalently, can we pack the 7 tetrominos into a 4×7 rectangle such that no two tetrominos overlap?

2.2.3 Question 4

Problem. Let $\{x\}$ denote the nearest integer⁷ to $x \in \mathbb{R}$. Define (a_n) by $a_n = n + \{\sqrt{n}\}$. What is the smallest $k \in \mathbb{N}$ such that $a_k, a_{k+1}, \dots, a_{k+2000}$ are consecutive integers?

Idea: Observe this is equivalent to finding the minimal k with $\{\sqrt{k}\} = \{\sqrt{k+2000}\}$ and do some bounding to solve this equation.

Solution. We begin by observing $a_{n+i} \geq a_n + i$ with equality if and only if $\{\sqrt{n}\} = \{\sqrt{n+i}\}$. Hence it suffices to find the smallest k such that $\{\sqrt{k}\} = \{\sqrt{k+2000}\}$. We now work by bounding. Firstly, see that $\{\sqrt{k+2000}\}$ will round up if $\sqrt{k+2000} \geq \sqrt{k} + 1$, hence we require⁸

$$\sqrt{k+2000} < \sqrt{k} + 1 \iff k + 2000 < k + 2\sqrt{k} + 1 \iff \sqrt{k} > \frac{1999}{2} \iff k > \frac{1999^2}{4}$$

Now we claim, for $x \in \mathbb{R}$, if $\{\sqrt{x}\} = \{\sqrt{x+2000}\}$ then $\{\sqrt{y}\} = \{\sqrt{y+2000}\}$ for all $y \geq x$. One way to see this is to notice $\sqrt{y+2000} - \sqrt{y}$ is a decreasing function⁹. Let us consider the critical value $x = \frac{1999^2}{4}$. It is clear $\sqrt{x} = 1999/2$ and hence $\{\sqrt{x}\} = 1000$. Further,

$$\{\sqrt{x+2000}\} = \left\{ \sqrt{\frac{1999^2}{4} + 2000} \right\} = \left\{ \sqrt{\left(\frac{1999}{2} + 1\right)^2} \right\} = 1001$$

Unfortunately, these quantities are thus not equal. However, our feasible gap between $\sqrt{x+2000} - \sqrt{x}$ will be strictly less than 1 on $(2001^2/2, 2001^2/2)$. Hence, we will have $x \geq 2001^2/2$. In the case of equality, we have $\{\sqrt{x}\} = 1001$ and

$$\{\sqrt{x+2000}\} = \left\{ \sqrt{\frac{2001^2}{4} + 2000} \right\} = \left\{ \sqrt{\left(\frac{2001}{2} + 1\right)^2 - 2} \right\} = 1001$$

Hence, any $k \geq \frac{2001^2}{4}$ works. Take

$$k = \left\lceil \frac{2001^2}{4} \right\rceil = \left\lceil \frac{2000^2}{4} + \frac{2 \cdot 2000}{4} + \frac{1}{4} \right\rceil = \left\lceil 1001000 + \frac{1}{4} \right\rceil = 1001001$$

minimally and the problem is solved. □

Remark. I'm not particularly pleased with this solution. We handwave the jump from $x \geq \frac{1999^2}{4}$ to $x \geq \frac{2001^2}{4}$. A rigorous justification is desired.

⁷We round up the case when $x = a + 0.5$, $a \in \mathbb{Z}$

⁸Note we may use $x < y \iff x^2 < y^2$ as both sides are positive and $x \mapsto x^2$ is strictly increasing on \mathbb{R}_+

⁹We can prove this with calculus, the Cauchy-Schwarz inequality or Jensen's inequality.

2.2.4 Question 5

Problem. Let ABC be a triangle with side lengths a, b, c and circumcircle Γ radius r . Prove ABC is right-angled if and only if $a^2 + b^2 + c^2 = 8r^2$.

Idea: Relate the angles at A, B and C and the radius R using the law of sines.

Solution. We begin with the following claim, which for some will be a well known fact.

Claim:

$$\frac{a}{\sin \angle BAC} = \frac{b}{\sin \angle ABC} = \frac{c}{\sin \angle ACB} = 2r$$

Proof of claim. Let O be the centre of Γ and draw the altitude from O to BC , letting D be the foot. By the inscribed angle theorem we have $\angle BOC = 2\angle BAC$ and hence, as the triangle OBC is isosceles, $\angle OBC = \angle OCB = 90^\circ - \angle BAC$. Thus,

$$r = \frac{|BD|}{\sin(90^\circ - \angle OCB)} = \frac{|BD|}{\sin \angle BAC} = \frac{a}{2 \sin \angle BAC}$$

Repeating for the altitudes O to AB and O to AC proves our claim. \square

Using our claim, we see that¹⁰

$$a^2 + b^2 + c^2 = 8r^2 \iff \sin^2 \angle BAC + \sin^2 \angle ABC + \sin^2 \angle ACB = 2$$

Now the problem reduces to trigonometric bash. Let $\angle BAC = A$, $\angle ABC = B$ and $\angle ACB = C$. Then

$$\begin{aligned} \cos^2 A + \cos^2 B + \cos^2 C &= \cos^2 A + \cos^2 B + \cos^2(180^\circ - (A + B)) \\ &= \cos^2 A + \cos^2 B + \cos^2(A + B) \\ &= \cos^2 A + \cos^2 B + (\cos A \cos B - \sin A \sin B)^2 \\ &= \cos^2 A + \cos^2 B + \cos^2 A \cos^2 B + \sin^2 A \sin^2 B - 2 \sin A \sin B \cos A \cos B \\ &= 1 + \cos^2 A \cos^2 B + (1 - \cos^2 A)(1 - \cos^2 B) - 2 \sin A \sin B \cos A \cos B \\ &= 1 + 2 \cos^2 A \cos^2 B - 2 \sin A \sin B \cos A \cos B \\ &= 1 - 2 \cos A \cos B \cos C \end{aligned}$$

Hence,

$$\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C$$

Clearly, this is equal to 2 iff one of A, B or C are equal to 90° so we're done. \square

Remark. The first claim we made, as with the trig identity we proved, are fairly well known¹¹. If one knows both of these facts the problem is trivial.

¹⁰Note we assume, as Γ is a circle, $r > 0$ to divide through by $4r^2$.

¹¹Fairly well known in olympiad circles that is, I didn't know them!

2.3 2001 - 2002

2.3.1 Question 1

Problem. Find all $(m, n) \in \mathbb{N} \times \mathbb{N}$ such that n is odd and

$$\frac{1}{m} + \frac{4}{n} = \frac{1}{12}$$

Idea: Rearrange and turn into a divisibility problem.

Solution. Observe,

$$\frac{1}{m} + \frac{4}{n} = \frac{1}{12} \implies 12n + 48m = mn \implies (m - 12)(n - 48) = 12 \cdot 48 = 2^6 3^2$$

Hence, $m - 12 = 2^a 3^b$ and $n - 48 = 2^{6-a} 3^{2-b}$ for integer a, b . Further, n odd $\implies a = 6$ so

$$(m, n) \in \{(588, 49), (204, 51), (76, 57)\}$$

are the only possible solutions. □

Here is a more obvious solution, taking the natural approach to solve $m = f(n)$.

Alternative solution. Observe, by simple algebra,

$$\frac{1}{m} + \frac{4}{n} = \frac{1}{12} \implies m = \frac{12n}{n - 48}$$

Hence it suffices to solve the equation

$$n - 48 \mid 12n \implies n - 48 \mid 12n - 12(n - 48) = 576 = 2^6 \times 3^2$$

Thus, we have $n = 2^a 3^b$ for $a = 0, \dots, 6$ and $b = 0, 1, 2$ and n odd gives $a = 0$. We once again deduce

$$(m, n) \in \{(588, 49), (204, 51), (76, 57)\}$$

are the only possible solutions. □

Remark. This is one of the rare BMO problems where working iteratively, taking the obvious step each time, does actually work. The less obvious, i.e. non alternative, does not depend on the knowledge that, for all $k \in \mathbb{Z}$,

$$a \mid b \implies a \mid b - ka$$

making it accessible to even the most inexperienced candidates.

2.3.2 Question 3

Problem. Solve over $x \in \mathbb{R}_+$ the equation

$$x + \left\lfloor \frac{x}{6} \right\rfloor = \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{2x}{3} \right\rfloor$$

Idea: Reduce to an integer equation and check cases modulo 6.

Solution. Observe that the RHS is always in \mathbb{Z} , whereas the LHS is in \mathbb{Z} if and only if $x \in \mathbb{Z}$. Hence, it suffices to solve over $x = n \in \mathbb{N}$. To do this, we look at the cases on n modulo 6. Write $n = 6m + k, k = 0, \dots, 5$. Then it is clear our equation reduces to

$$k = \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{2k}{3} \right\rfloor$$

which is true if and only if $k \neq 1$. Hence our solution set is

$$x \in \mathbb{N} \setminus \{n \in \mathbb{N} : n \equiv 1 \pmod{6}\}$$

and the problem is solved. □

Remark. Boring and easy, I would expect this to be a question 1 on a modern paper.

2.3.3 Question 4

Problem. Suppose 12 people are seated around a circular table. In how many ways can 6 pairs shake hands so that no sets of arms cross?

Solution.

□

2.3.4 Question 5

Problem. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be strictly increasing with

$$f(n + f(m)) = f(n) + m + 1 \quad (\dagger)$$

for all $m, n \in \mathbb{N}$. What are the possible values of $f(2001)$?

Idea: Notice f has constant differences and hence is linear.

Solution. Substituting $n \mapsto n + 1$ into (\dagger) and taking the difference with $n \mapsto n$, see

$$f(n + f(m) + 1) - f(n + f(m)) = f(n + 1) - f(n) \quad (\ddagger)$$

for all $m, n \in \mathbb{N}$. Hence, a shift by $f(m)$ does not alter the difference $f(n + 1) - f(n)$. We naturally conjecture that f has constant differences.

Claim: $f(n + k + 1) - f(n + k) = f(n + 1) - f(n)$ for all $n, k \in \mathbb{N}$.

Proof of claim. It is clear $f : \mathbb{N} \rightarrow \mathbb{N}$ strictly increasing $\Rightarrow f$ unbounded. Hence, we may choose $m \in \mathbb{N}$ with $f(m) \leq k < f(m + 1)$. Again, using the monotonicity of f , see

$$f(n + f(m)) \leq f(n + k) \leq f(n + f(m + 1))$$

and

$$f(n + f(m) + 1) \leq f(n + k + 1) \leq f(n + f(m + 1) + 1)$$

Subtracting these inequalities and using (\ddagger) , we deduce our claim. \square

Our claim makes it clear¹² that f is linear, i.e. there exist constants $a, b \in \mathbb{Z}$ such that for all $n \in \mathbb{N}$ we have $f(n) = an + b$. Using (\dagger) with $n = m$, observe

$$a(n + an + b) + b = an + b + n + 1 \iff (a^2 - 1)n = 1 - ab$$

For the LHS to be constant, $a \in \{\pm 1\}$ is forced. f strictly increasing $\Rightarrow a = 1 = b$ so that $f(n) = n + 1$ for all $n \in \mathbb{N}$. We conclude $f(2001) = 2002$ is the only possibility. \square

Remark. The phrasing of this problem suggests it is possible to do this without solving the functional equation. I cannot see a feasible way.

¹²To see this, use $f(n + 1) - f(n) = c \in \mathbb{N}$ and backwards induct $f(n) = c(n - 1) + f(1)$.

2.4 2002 - 2003

2.4.1 Question 3

Problem. Let $x, y, z \in \mathbb{R}_+$. Given that $x^2 + y^2 + z^2 = 1$, prove

$$x^2yz + xy^2z + xyz^2 \leq \frac{1}{3}$$

Solution. Let \cdot denote the dot product in \mathbb{R}^3 . By the Cauchy-Schwarz inequality,

$$x^2yz + xy^2z + xyz^2 = (xyz, xyz, xyz) \cdot (x, y, z) \leq \|(xyz, xyz, xyz)\| \|(x, y, z)\| = \sqrt{3}xyz$$

By the AM-GM inequality,

$$\sqrt[3]{x^2y^2z^2} \leq \frac{x^2 + y^2 + z^2}{3} = \frac{1}{3} \implies xyz \leq \frac{1}{3\sqrt{3}}$$

The result immediately follows. □

Remark. There is an elementary solution using only the trivial inequality, but this is far from obvious. See [the mathematics stack exchange](#)

2.4.2 Question 4

Problem. Let $m, n \in \mathbb{N}$ and consider a $m \times n$ grid R of lattice points¹³ in the plane. What is the maximum number of points in R we may colour red so that no collection of 3 red points form a right-angled triangle with sides parallel to the axis?

Solution. Let $\chi_{m,n}$ denote the desired quantity. It is clear we have $\chi_{1,n} = n$ and $\chi_{m,1} = m$, as we may colour each point in the lines $1 \times n$ and $m \times 1$ without inducing a right-triangle (or in fact, any non-degenerate triangle). Now suppose $m, n \geq 2$ and denote the points in R by $(r_{i,j})_{(i,j) \in [m] \times [n]}$. For fixed j , it is clear if we colour $r_{i_1,j}$ and $r_{i_2,j}$ with $i_1 \neq i_2$ we cannot colour any other points in the rows $(r_{i_1,j})_{j \in [n]}, (r_{i_2,j})_{j \in [n]}$ without inducing a right-triangle with sides parallel to the axis. Suppose we have a colouring with two distinct points coloured in two distinct columns. Then it is clear we may only colour at most m of these without inducing a right-triangle with sides parallel to the axis. We can repeat our argument for the rows of R to see, if we have two rows with two distinct points coloured that we may only colour at most n points. We deduce $\chi_{n,m} = \max\{m, n\}$ if we colour at least 2 distinct points in at least 2 distinct rows or columns.

Instead, suppose there is only one column with at least 2 distinct coloured points. We can colour $m - 1$ of points in this “colourful” column and $n - 1$ points in the row that isn’t coloured in the “colourful” row. It is clear this is the best we can do with a colourful column. An analogous argument for a colourful row, giving $\chi_{m,n} = m + n - 2$ in this case. Hence, $\chi_{m,n} = \max\{\max\{m, n\}, m + n - 2\} = m + n - 2$. \square

Remark. This could probably be written in a better way, but the problem is boring when you spot the “colourful row” strategy so I’m leaving it here. The problem where we disclude any right-triangle seems more interesting. I can’t find an obvious approach like in this question. Perhaps the pigeonhole principle works. Exercise for the motivated reader!

¹³That is, points with integer coordinates.

2.4.3 Question 5

Problem. Solve over $a, b, c \in \mathbb{N}$ the equation

$$a!b! = a! + b! + c! \tag{(\dagger)}$$

Idea: Keep bounding things until it all comes together

Solution. WLOG $a \leq b$. Notice, by factoring the terms containing a or b ,

$$a!b! = a! + b! + c! \implies a! - 1 = \frac{c! + 1}{b! - 1}$$

Hence, if (a, b, c) forms a solution we must have

$$b! - 1 \mid c! + 1 \implies b < c$$

We also spot immediately that $a \geq 3$ is forced. Now, see that

$$a!b! = a! + b! + c! \implies b! = 1 + \frac{b!}{a!} + \frac{c!}{a!}$$

As $a \geq 3$, $b \geq 3$ and hence the LHS is even. For the RHS to be even, it is clear we require at least one of $b \leq a + 1$ or $c \leq a + 1$ to be true. As $b < c$, we deduce $a \leq b \leq a + 1$. Suppose $a = b$, then

$$(a!)^2 - 2a! - c! = 0$$

which has integral roots¹⁴ only if $\sqrt{1 + c!} \in \mathbb{Z}$. This can only happen if $c! = (m - 1)(m + 1)$ with $\mathbb{Z} \ni m < c$. We can bound $c! \leq c(c + 2)$ for all $c \geq 5$ easily, hence we need only check $c \in \{1, 2, 3, 4\}$. By hand we see only $c = 4$ works. In this case we're left with

$$(a!)^2 - 2a! - 4! = 0 \implies (a! - 6)(a! + 4) = 0 \implies a = 3$$

Hence, the only solution with $a = b$ is $(a, b, c) = (3, 3, 4)$. Now for $b = a + 1$, see

$$a!(a + 1)! = a! + (a + 1)! + c! \implies (a + 1)! = 1 + (a + 1) + c!$$

Now as $c > a + 1$ the RHS has remainder 1 when divided by $a + 1$, whereas the LHS clearly has remainder 0. Hence, the only solution to (\dagger) is $(a, b, c) = (3, 3, 4)$. \square

¹⁴Over the variable a that is.

3 BMO 2

3.1 1999 - 2000

3.1.1 Question 2

Problem. For $x, y, z \in \mathbb{R}_+$, minimise $x^2 + 4xy + 4y^2 + 2z^2$ subject to $xyz = 32$.

Idea: Apply AM-GM to reduce to one variable and apply AM-GM again to finish.

Solution. Let $f(x, y, z) = x^2 + 4xy + 4y^2 + 2z^2$. Notice

$$f(x, y, z) = (x + 2y)^2 + 2z^2$$

By AM-GM we have

$$x + 2y \geq 2\sqrt{2xy} \text{ with equality iff } x = 2y$$

and hence, fixing z , $f(x, y, z)$ is minimised¹⁵ at $x = 2y$. We can thus reduce to

$$g(y, z) = 16y^2 + 2z^2$$

Now $xyz = 32 \Rightarrow z = 16/y^2$ so we further reduce to

$$h(y) = 16y^2 + 2 \cdot \frac{16^2}{y^4}$$

Applying AM-GM again, we see

$$h(y) = 8y^2 + 8y^2 + 2 \cdot \frac{16^2}{y^4} \geq 3(8 \cdot 8 \cdot 2 \cdot 16^2)^{1/3} = 96$$

with equality iff $8y^2 = 8y^2 = 2 \cdot \frac{16^2}{y^4} \Rightarrow y = 2$. Hence 96 is our minimal value and this occurs at the point $(x, y, z) = (4, 2, 4)$. \square

Remark. When applying the AM-GM inequality we want the RHS to be constant, or we will not be able to deduce the minimal value is at equality. This is why we split into 3 terms on the second application.

Remark. We could've used calculus to minimise $h(y)$ but calculus + olympiad = bad.

¹⁵This is as $xyz = 32 \Rightarrow xy = 32/z$ and hence the product on the RHS is fixed. In fact, it is well known (proof also via AM-GM) that the sum of two positive variables $u + v$ with fixed product uv is minimised at $u = v$ so could've instead have just quoted this, though proving things yourself is always preferred!

3.1.2 Question 3

Problem. Find $a, b \in \mathbb{N}$ with

$$(a^{1/3} + b^{1/3} - 1)^2 = 49 + 20\sqrt[3]{6}$$

Idea: Write $\sqrt{49 + 20\sqrt[3]{6}} = x + y\sqrt[3]{6} + z\sqrt[3]{36}$ for some $x, y, z \in \mathbb{Q}$.

Solution. We begin by conjecturing that $\sqrt{49 + 20\sqrt[3]{6}} = x + y\sqrt[3]{6} + z\sqrt[3]{36}$ for some x, y, z . This amounts to finding a solution to the system

$$2xz + y^2 = 0, \quad 3z^2 + xy = 10, \quad x^2 + 12yz = 49$$

over $x, y, z \in \mathbb{Z}$. By eliminating variables, we find $(x, y, z) = (-1, 2, 2)$ works. Hence,

$$\sqrt{49 + 20\sqrt[3]{6}} = -1 + 2\sqrt[3]{6} + 2\sqrt[3]{36}$$

To find a solution (a, b) it suffices to solve

$$a^{1/3} + b^{1/3} - 1 = -1 + 2\sqrt[3]{6} + 2\sqrt[3]{36}$$

It is then clear we may take $a = 48$ and $b = 288$. □

Remark. Motivating our conjecture takes some thought. Assuming the existence of¹⁶ such a, b , taking square roots tells us $\sqrt[3]{a} + \sqrt[3]{b} - 1 = \sqrt{49 + 20\sqrt[3]{6}}$. If we let a or b be cubes, we see this is doomed. Further, if there does not exist $x \in \mathbb{N}$ with $a = x^3b$ we are also doomed. From here, our conjecture becomes a fairly natural guess.

¹⁶Note the question tells us this exists, so we can cheese our way into this idea. The question “Are there $a, b \in \mathbb{N}$ with ...? If so, find such a pair.” would make this idea require a touch more ingenuity.

3.1.3 Question 4

Problem. Find a subset $A \subset \mathbb{N}$ of cardinality 10 with the property that no 6 element subset of A has sum divisible by 6. Can we find $A \subset \mathbb{N}$ of cardinality 11 with this property?

Solution. It is sufficient to construct a 10-tuple X in $(\mathbb{Z}/6\mathbb{Z})^{10}$ with the property that no 6 elements sum to 0. Clearly, we cannot have all elements of X being equal¹⁷. We try 2 unique elements. It suffices to find residues $a, b \in \mathbb{Z}/6\mathbb{Z}$ and multiplicities of a and b respectively $x, y \in \{1, \dots, 10\}$, $x + y = 10$ with our desired property. We need $x = y = 5$, or we can sum the same element 6 times to get sum 0. We can choose any a and b provided $az + b(6 - z) \neq 0$ for each $z \in \{1, \dots, 5\}$. By enumeration, we see we can take $a = 1, b = 2$ so that $X = (1, 1, 1, 1, 1, 2, 2, 2, 2, 2)$. We can then take

$$A = \{1, 7, 13, 19, 25, 2, 8, 14, 20, 26\}$$

to exhibit such a set.

A clean proof for the second part is desired. □

Remark. Let $\omega(A)$ denote the number of unique remainders mod 6 in A . It looks like case bashing on $\omega(A) \in \{1, 2, \dots, 6\}$ (noticing we can reduce to $\omega(A) \in \{3, 4, 5\}$ easily) should work, but I can't find a clean proof. You can rule out any case where there are 3 occurrences of the same remainder for example by parity, and I imagine similar arguments rule out the others but I couldn't see them :(

Remark. This is a special case of a theorem of Erdős, Ginzburg and Ziv, which states any sequence a_1, \dots, a_{2m-1} has a subset $I \subset \{1, \dots, 2m-1\}$ of size m with $\sum_{i \in I} a_i \equiv 0 \pmod{m}$. Proofs of this are highly nontrivial, and mimicking these seems infeasible. The most elementary approach uses the combinatorial Nullstellensatz.

¹⁷Or we may sum any 6 of them to obtain 0.

3.2 2000-2001

3.2.1 Question 1

Problem. Aegyptus and Brutus initially have p and q marbles respectively, where $p > q$. They take it in turns, starting with Aegyptus, giving each other exactly the number of marbles the other already possesses (e.g. the first turn will be Aegyptus giving Brutus q of his marbles, leaving him with $p - q$ marbles). After $2n$ turns, Aegyptus has q marbles and Brutus has p marbles. Find the ratio $\frac{p}{q}$ as a function of n .

Idea: Set up a recurrence relation and solve it.

Solution. Let $a(n)$ denote the number of marbles Aegyptus has after n turns, where $a(0) := p$. Set $p + q = N$. As Aegyptus moves on turn $2m + 1$, we have

$$a(2m + 1) = a(2m) - (N - a(2m)) = 2a(2m) - N$$

and, as Brutus moves on turn $2m + 2$,

$$a(2m + 2) = a(2m + 1) + a(2m + 1) = 4a(2m) - 2N$$

A quick induction¹⁸ shows that, for all $m \geq 0$,

$$a(2m) = 2^{2m}p - \sum_{k=0}^{m-1} 2^{2k} \times (2N)$$

where $\sum_{\emptyset} \cdots := 0$. We compute, using the well known formula for a geometric series,

$$\sum_{k=0}^{m-1} 2^{2k} \times (2N) = 2N \frac{2^{2m} - 1}{3}$$

Hence,

$$a(2m) = 2^{2m}p - 2N \frac{2^{2m} - 1}{3}$$

For our n , we know $a(2n) = q$. Thus, letting $x = \frac{p}{q}$, we deduce

$$q = 2^{2n}p - 2(p + q) \frac{2^{2n} - 1}{3} \implies 1 = 2^{2n}x - 2(x + 1) \frac{2^{2n} - 1}{3}$$

This is a simple linear equation, which can be solved to give

$$x = \frac{2 \cdot 4^n + 1}{4^n + 2}$$

Recalling $x = p/q$ we're done. □

¹⁸Work out the first few terms, guess a formula and verify.

3.2.2 Question 2

Problem. Find all $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ with

$$1 + x^2y = x^2 + 2xy + 2x + y$$

Solution. Observe

$$1 + x^2y = x^2 + 2xy + 2x + y \implies x \mid y - 1$$

by reducing mod x . Hence, if (x, y) is a solution there exists $m \in \mathbb{Z}$ with $y = mx + 1$. Hence, by substituting this in and cleaning things up, we see for $x \neq 0$ that

$$1 + x^2y = x^2 + 2xy + 2x + y \implies m(x^2 - 2x - 1) = 4$$

Hence, we have $x^2 - 2x - 1 \in \{\pm 1, \pm 2, \pm 4\}$. By checking discriminants we see only the x with $x^2 - 2x - 1 \in \{-1, \pm 2\}$ are viable, which yield possible $x \in \{-1, 1, 2, 3\}$. Now,

$$1 + x^2y = x^2 + 2xy + 2x + y \implies y = \frac{x^2 + 2x - 1}{x^2 - 2x - 1}$$

so we have viable pairs $(x, y) \in \{(-1, -1), (2, -7), (3, 7)\}$. Including the $x = 0$ possibility, we deduce

$$(x, y) \in \{(-1, -1), (0, 1), (2, -7), (3, 7)\}$$

is our solution set. □

3.2.3 Question 4

Problem. Suppose there are n dwarves of height $1, 2, \dots, n$ arranged in a circle. Let V denote the sum of the positive differences (i.e. total variations) between consecutive dwarves. Find the maximum and minimum, over all arrangements of the dwarves, of V .

Idea: For the minimum, consider the arcs with endpoints 1 and n for each permutation. For the maximum, employ a “centering” strategy to bound the sum.

Solution. We claim that the minimum value of V is $2n - 2$, which occurs when the dwarves are seated in consecutive order. We also claim the maximum value of V is $\lfloor n^2/2 \rfloor$, which occurs when the dwarves are placed in a “zig-zag” pattern.

Let $\sigma \in S_{n-1}$ be a permutation of $\{1, \dots, n-1\}$ and let $V(\sigma)$ be the total variation of the circle where the dwarf with height n is sat at the head of the table and the dwarf with height $\sigma(i)$ is sat $i \geq 1$ seats along from the head of the table. Let $\text{id}_{n-1} \in S_{n-1}$ denote the identity permutation, which corresponds to the consecutive order seating arrangement. It suffices to see $V(\sigma) \geq V(\text{id}_{n-1})$.

It is easy to see $V(\text{id}_{n-1}) = 2n - 2$. We have

$$V(\sigma) := \sum_{i=1}^n |\sigma(i+1) - \sigma(i)|$$

where $\sigma(n) := n$ is fixed and $\sigma(n+1) = \sigma(1)$ (i.e. arguments are taking mod n). Let us consider the arc $1 = \sigma(i), \sigma(i+1), \dots, \sigma(i+k) = n$. As each term in the sum is bounded below by 1, if $k = 1$ it is clear we have $V(\sigma) \geq V(\text{id}_{n-1})$. Otherwise, by the triangle inequality, we have

$$V(\sigma) = \sum_{j=1}^k |\sigma(i+j) - \sigma(i+j-1)| \geq |\sigma(i+k) - \sigma(i)| = n - 1$$

Repeating this argument for the remaining arc we deduce

$$V(\sigma) \geq 2(n-1) = V(\text{id}_{n-1})$$

Now for the maximum, let $f(x) := \left| x - \frac{n+1}{2} \right|$. By the triangle inequality, it is clear that

$$\sum_{i=1}^n |\sigma(i+1) - \sigma(i)| \leq \sum_{i=1}^n f(\sigma(i+1)) + f(\sigma(i)) = 2 \sum_{i=1}^n f(i) \quad ((\dagger))$$

We compute $\sum_{i=1}^n f(i)$ for the odd/even cases of n . Suppose $n = 2m$, then

$$\begin{aligned}
\sum_{i=1}^{2m} \left| i - \frac{2m+1}{2} \right| &= \sum_{i=1}^m \left| i - \frac{2m+1}{2} \right| + \sum_{i=m+1}^{2m} \left| i - \frac{2m+1}{2} \right| \\
&= \sum_{i=1}^m \left[\frac{2k+1}{2} - i \right] + \sum_{i=m+1}^{2m} \left[i - \frac{2k+1}{2} \right] \\
&= \sum_{i=1}^{2m} i - 2 \sum_{i=1}^m i \\
&= \frac{1}{2}(2m)(2m+1) - 2 \cdot \frac{1}{2}m(m+1) \\
&= m^2
\end{aligned}$$

The odd case is similar. For $n = 2m + 1$ we compute $\sum_{i=1}^{2m+1} f(i) = m(m+1)$. Hence, using (\dagger) ,

$$V(\sigma) \leq 2 \cdot \left\lfloor \frac{n^2}{4} \right\rfloor = \left\lfloor \frac{n^2}{2} \right\rfloor$$

We get equality when $|\sigma(i+1) - \sigma(i)| = f(\sigma(i+1)) - f(\sigma(i))$ for all $i \geq 1$, which can be obtained by taking $\sigma(1) = 2, \sigma(2) = n-1, \sigma(3) = 3, \sigma(4) = n-2, \dots, \sigma(n-1) = 1$ in an alternating matter about the midpoint. \square

Remark. Beautiful problem! I would be interested in seeing alternative approaches.