# Combinatorics

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**Abstract**

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# Preface

## Notation

- $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
- $[n] = \{1, \dots, n\}$
- if $a$ is defined to be $b$ we may say $a := b$ (but I will often forget)
- $A^k = \overbrace{A \times \cdots \times A}^{k \text{ times}}$ for a set $A$, $\times$ being the Cartesian product.

# Contents

# 1 Elementary Combinatorics

In combinatorics we are concerned with counting the number of objects from a collection. For example, we may count the positive integers divisible by 3 that are less than 1000. Here the objects are positive integers are the objects and the collection is those objects that are both divisible by 3 and less than 1000.

As a set is just is collection of objects, we will often phrase things set theoretically.

## 1.1 Fundamental Principles

To begin counting anything interesting, we'll need the following principles. The first two let us count things inductively (e.g. counting wine bottles by counting in threes or counting the number of wine & cheese pairings by multiplying the number of wine bottles and the number of cheeses). The final simply says it doesn't matter what order we count each element (e.g. wine bottle) in.

**Theorem 1.1 (addition principle)**

Let $n \in \mathbb{N}$. Let $A_1, \dots, A_n$ be finite sets. One has[a]

$$A_1, \dots, A_n \text{ disjoint} \implies |A_1 \cup \cdots \cup A_n| = |A_1| + \cdots + |A_n|$$

[a] $A_1, \dots, A_n$ are said to be disjoint if $\cap_{i=1}^n A_i = \emptyset$

To prove this, we simply count both sides one term at a time.

*Proof.* Let $x \in \cup_{i=1}^n A_i$. Then $x \in A_i$ for exactly one $1 \le i \le n$ and hence

$$|A_1| + \cdots + |A_n| = \sum_{x \in A_1} 1 + \cdots + \sum_{x \in A_n} 1 = \sum_{x \in \cup_{i=1}^n A_i} 1 = |A_1 \cup \cdots \cup A_n|$$

by counting both sides term by term. □

**Theorem 1.2 (multiplication principle)**

Let $n \in \mathbb{N}$. Let $A_1, \dots, A_n$ be finite sets. One has

$$|A_1 \times \cdots \times A_n| = |A_1| \times \cdots \times |A_n|$$

This time we count terms in each slot recursively.

*Proof.* Consider an arbitrary element $x = (x_1, \dots, x_n) \in A_1 \times \cdots \times A_n$ and let $1 \le i \le n$. Then fixing $x_j, j \neq i$ we have $|A_i|$ possible choices for $x_i$. Hence, we have

$$|A_1 \times \cdots \times A_n| = \sum_{i=1}^n \sum_{x \in A_i} |\cap_{j \neq i} A_j| = \sum_{i=1}^n |A_i| \sum_{x \in A_i} |\cap_{j \neq i} A_j| = \cdots = |A_1| \times \cdots \times |A_n|$$

by repeated application of this reasoning. □

**Theorem 1.3 (bijection principle)**

Let $A$ and $B$ be sets. If there is a bijection[a] $\pi : A \to B$ then $|A| = |B|$

[a] A bijection is a one-to-one mapping

*Proof.* This is a truism. □

The addition principle can be thought of as emptying two containers of $n$ and $m$ wine bottles and containing how many bottles you have in total $(n+m)$, whereas the multiplication principle can be thought of as counting the number of ways of pairing $b$ boys with $g$ girls. The bijection principle says if we take $s$ students and select them one by one at random, $t$ times, we'll always have selected $t$ students. We will often look for such interpretations in combinatorics.

With just these principles, we can count all the following quantities.

**Problem 1.1** **($k$-words with repetitions)**

How many words length $k$ can be formed from an alphabet with $n$ letters.

*Solution.* To count the number of words with repetitions allowed, we sample with replacement to get

$$\underbrace{n \times \cdots \times n}_{k \text{ times}} = n^k$$

$k$-words (or $k$-sequences) on $n$ symbols.

□

**Problem 1.2** **($k$-words without repetitions)**

How many words length $k$ with distinct letters can be formed from an alphabet with $n$ letters.

*Solution.* For $k$-words without repetition, we sample without replacement for each letter to get

$$n \times (n-1) \times \cdots \times (n-k+1) =: n^{\underline{k}}$$

$k$-words without repetition on $n$ symbols.[1] □

If we were to consider words of length $n$ on $n$ symbols without repetition, then we would simply be *permuting* (i.e. reordering) the $n$ symbols. Hence we have $n \times (n-1) \times \cdots \times 1 =: n!$ *permutations* on $n$ symbols. Using this notation we have $n^{\underline{k}} = n!/(n-k)!$. From this, we have the following result.

**Problem 1.3** **($k$-subsets)**

How many subsets size $k$ are there of a set size $n$?

*Solution.* Let $x$ be the desired quantity. For each subset size $k$, we may reorder it's elements in $k!$ ways. If we care about the order of the elements in the subset, then the number of such subsets is precisely the number of $k$-words without repetition. Hence

$$x \cdot k! = \frac{n!}{(n-k)!}$$

and we have counted the desired quantity. □

Under the hood there we used the bijection principle, as we implicitly assumed (quite reasonably) the existence of a bijection from the collection of subsets size $k$ where reordering matters and the $k$-words on $n$ symbols.

We call $\binom{n}{k} := \frac{n!}{k!(n-k)!}$ the *binomial coefficient*, and would say this as "$n$ choose $k$". We will dedicate an entire section (in fact, next section) to these.

**Problem 1.4** **($k$-multisubsets)**

How many multisubsets[a] of size $k$ are there in a set of $n$ objects?

[a] a multiset is a set that allows repetition, see https://en.wikipedia.org/wiki/Multiset

[1] We call $n^{\underline{k}}$ the falling factorial of $n$

*Solution.* Let $X = \{x_1, \ldots, x_n\}$. We call the number of times $x_i, 1 \leq i \leq n$ appears in a multisubset of $X$ the *multiplicity* of $x_i$. We know that the size of a multiset is the sum of its multiplicities, thus it suffices to count solutions $(\ell_1, \ldots, \ell_n)$ of nonnegative integers to

$$\ell_1 + \cdots + \ell_n = k$$

which by setting $\ell' := \ell + 1$ is the same as counting positive integer solutions to

$$\ell_1' + \cdots + \ell_n' = k + n$$

Now write this as

$$\underbrace{1 + \cdots + 1}_{\ell_1' \text{ times}} + \cdots + \underbrace{1 + \cdots + 1}_{\ell_n' \text{ times}} = k + n$$

We are tasked with dividing up the $k+n$ ones into $n$ groups, which can be done by insering $n-1$ stars in between two ones (not the edges as we require $n$ nonempty collections of ones) to signifying the beginning of a new group. This gives $\binom{n-k-1}{n-1} = \binom{n-k-1}{k}$ possibilities. □

**Problem 1.5** (*k*-partitions with a given sizing)

How many ways are there to divide a set partition a set $X$ of size $n$ into sets $X_1, \ldots, X_k$ with size $n_1, \ldots, n_k$ respectively (where $n_1 + \cdots + n_k = n$)?

*Solution.* There are

$$\binom{n - n_1 - \cdots - n_{i-1}}{n_i}$$

ways of choosing the $X_i$ once $X_1, \ldots, X_{i-1}$ have been chosen, and

$$\binom{n}{k}\binom{n-k}{m} = \frac{n!}{k!m!(n-k-m)!}$$

So all together one obtains

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-(n_1+\cdot+n_{k-1})}{n_k} = \frac{n!}{n_1!\cdots n_k!} =: \binom{n}{n_1, \ldots, n_k}$$

such partitions.[2] □

### Exercises

**Exercise 1.1**

In how many ways can $n$ people be seated at a circular table with $n$ seats, where we do not classify between seatings that can be obtained from one another by rotating the table?

**Exercise 1.2**

How many subsets are there of the set $\{1, 2, \ldots, n\} =: [n]$? How many are of even size?

**Exercise 1.3**

There are $2n$ people at a party. How many ways can the $2n$ people split into $n$ pairs?

**Exercise 1.4**

A robot is placed on the bottom left of an $n \times n$ chessboard. The robot has two moves, go vertically up one square or horizontally along (to the right) one square. How many sequences of moves can the robot make to get to the top right square?

[2] The quantity $\binom{n}{n_1,\ldots,n_k}$ is called the multinomial coefficient

## 1.2 Binomial Coefficients

**Lemma 1.1 (addition formula)**

Let $n \in \mathbb{N}$ and $1 \leq k \leq n$. The identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

holds.

*Proof 1.* A simple algebraic verification works, I leave this to the reader. □

We can also prove this identity with the technique of "double counting". Intuitively, this is just saying that if we can count things in two ways they must be equal. For example, if we had an $n \times n$ grid full of numbers, we could count the sum of all the numbers by summing along the rows or the columns. For certain selections of numbers, this can yield interesting equalities.

*Proof 2.* We count the $k$-subsets of $[n]$ in two ways. First, all at once, getting $\binom{n}{k}$ such sets. Secondly, by considering seperately the cases when 1 is in our subset and isn't. This gives $\binom{n-1}{k-1}$ choices with 1 and $\binom{n-1}{k}$ choices without, giving us the desired equality. □

**Theorem 1.4 (binomial theorem)**

For all integers $n \geq 0$ and $x, y \in \mathbb{C}$, one has

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

*Proof 1.* We first use a straight forward induction argument. Observe, assuming the hypothesis for some $n \geq 0$ (and noting it trivially holds for $n = 0$),

$$\begin{aligned}
(x+y)^{n+1} &= (x+y)(x+y)^n \\
&= (x+y) \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k+1} \\
&= \sum_{k=0}^{n+1} \left( \binom{n}{k-1} + \binom{n}{k} \right) x^k y^{n-k+1} \qquad (\dagger) \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{(n+1)-k}
\end{aligned}$$

where $(\dagger)$ follows from an index shift $(k \to k-1)$ and defining $\binom{n}{k} := 0$ for $k > n$ or $k < 0$. □

As before, we can double count ourselves a proof of this theorem.

*Proof 2.* Consider the coefficient of $x^k y^{n-k}$ of

$$(x+y)^n = \overbrace{(x+y) \cdots (x+y)}^{n \text{ times}}$$

When computing this expansion, we are multiplying exactly one of $x$ or $y$ from each $(x+y)$ term on the RHS. Our entire expansion will be the sum of all such choices. Thus, to compute the coefficient of $x^k y^{n-k}$ in $(x+y)^n$, it suffices to count the number of ways to choose exactly $k$ $x$s out the $n$ possible choices. This is clearly $\binom{n}{k}$ and hence the result follows. □

As an immediate corollary we have the following.

**Corollary 1.1 (sum and alternating sum of binomial coefficients)**

$$\sum_{k=0}^{n}\binom{n}{k}=2^n \qquad \text{and} \qquad \sum_{k=0}^{n}(-1)^k\binom{n}{k}=0$$

*Proof 1.* Take $x=y=1$ and $x=1, y=-1$ in the Binomial Theorem. □

We may also take a more combinatorial approach to proving these identities.

*Proof 2.* There are $2^n$ subsets of $[n]$ by counting directly (see exercise 1.2) and by counting each of the subsets size $k$ in groups we get $\sum_{k=0}^{n}\binom{n}{k}$, giving the first equivalence. For the second, note there are an equal number of odd and even subsets of $[n]$ (also exercise 1.2) so we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}=\sum_{k=0}^{n}\sum_{k=0,2|k}^{n}\binom{n}{k}-\sum_{k=0,2|k}^{n}\binom{n}{k}=0$$

giving the latter equivalence. □

Let us recall the following lemma from (elementary) complex analysis, concerning sums of powers of roots of unity. If you are not yet familiar with these tools please skip the next lemma and problem.

**Lemma 1.2 (sums of powers of roots of unity)**

Let $1, \xi_1, \ldots, \xi_{n-1}$ be the $n$-roots of unity, that is the solutions to $z^n=1, z\in\mathbb{C}$. Then, one has

$$1+\sum_{j=1}^{n-1}\xi_j^{\ell}=\begin{cases} n: \ell\equiv 0 \mod n \\ 0: \ell\not\equiv 0 \mod n\end{cases}$$

*Proof.* Write $\xi_j=\exp(2\pi ij/n)$. If $\ell=mn, m\in\mathbb{Z}$, then $\xi_j^{\ell}=\exp(2\pi im)=1$ and hence

$$1+\sum_{j=1}^{n-1}\xi_j^{\ell}=n$$

Conversely, if $\ell=mn+r, m\in\mathbb{Z}, 0<r<n$ then $\xi_j^{\ell}=\exp(2\pi i(mn+r)j/n)=\exp(2\pi ijr/n)$ and

$$1+\sum_{j=1}^{n-1}\xi_j^{\ell}=\sum_{j=0}^{n-1}(\exp(2\pi ir/n))^j=\frac{1-\exp(2\pi ir)}{1-\exp(2\pi ir/n)}=0$$

where we used the standard result for the sum of a geometric series in equality 2. □

With this, we can prove the following result. Generalising further the previous problem.

**Problem 1.6 (sums of binomial multiples)**

Compute the following quantity.

$$\sum_{k=0}^{n}\binom{mn}{mk}$$

*Solution.* Let $\xi_0, \ldots, \xi_{m-1}$ be the $m^{\text{th}}$ roots of unity, with $\xi_0=1$. Then, by our previous lemma and the Binomial theorem, we can deduce

$$\sum_{k=0}^{n}\binom{mn}{mk}=\sum_{k=0}^{mk}\binom{nm}{k}\left(\frac{1}{m}\sum_{\ell=0}^{m-1}\xi_\ell^k\right)=\frac{1}{m}\sum_{\ell=0}^{m-1}\sum_{k=0}^{mn}\binom{mn}{k}\xi_i^k=\frac{1}{m}\sum_{\ell=0}^{m-1}(1+\xi_\ell)^{mn}$$

The term of the RHS can be further simplified, I elect to leave it here. □

THINGS TO ADD:

- VANDERMONDE'S
- HOCKEYSTICK LEMMA

### Exercises

**Exercise 1.5**

How many words length $n$ can be formed from an alphabet of $\ell$ letters $\mathcal{A} = \{A_1, \ldots, A_\ell\}$ such that the first letter $A_1$ occurs an even number of times?

## 1.3 The Pigeonhole Principle

If I have 5 pigeons and 4 containers, each only able to fit one pigeon, can I fit all the pigeons into my containers? Of course not! It turns out this common-sense principle allows us to discover many, many, combinatorial facts. . .

**Theorem 1.5 (pigeonhole principle)**

Given a set $X$ of size $n$, any partition of $X$ into $m < n$ subsets $X_1, \ldots, X_m$ must have at least one $X_i$ with $|X_i| > 1$.

*Proof.* Suppose each $|X_i| \leq 1$, then $|X| = |\cup_{i=1}^m X_i| = \sum_{i=1}^m |X_i| \leq m < n$. Absurd! □

We can slightly better than this. What if my containers can fit 2 pigeons and I only have 2 this time. Then I still can't fit my pigeons into my containers. We make this rigorous as follows.

**Theorem 1.6 (full pigeonhole principle)**

Given a set $X$ of size $n = km + 1$, any partition of $X$ into $m$ subsets $X_1, \ldots, X_m$ must have at least one $X_i$ with $|X_i| > k$

*Proof.* Same idea. □

We also have a infinite pigeonhole principle. If I can only fit a finite number of pigeons into each container then, assuming I only have finitely many containers, I cannot fit an infinite number of pigeons into my containers.

**Theorem 1.7 (infinite pigeonhole principle)**

Given a set $X$ of infinite cardinality, any partition of $X$ into finitely many sets $X_1, \ldots, X_m$ must have some $X_i$ also of infinite cardinality.

*Proof.* Guess :) □

### Exercises

## 1.4 The Principle of Inclusion-Exclusion

It is often much easier to count how many objects have properties $A$ AND $B$ than $A$ OR $B$. Thankfully we have the following principle to relate the two.

## 1.5 Generating Functions

**Exercises**

## 1.6 Miscellaneous Topics

**Exercises**

## 1.7 Additional Problems

# 2 The Probabilistic Method

## 2.1 Introducing the Probabilistic Method with Ramsey Bounds

### Exercises

## 2.2 Linearity of Expectation

**Exercises**

## 2.3 Alteration

**Exercises**

## 2.4 Lovasź Local Lemma

**Exercises**

# 3 Solutions

Here I list the exercise solutions in a random order. I will store which exercises have a solution in the README of this repository.