Combinatorics

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Abstract

Suspendisse vitae elit. Aliquam arcu neque, ornare in, ullamcorper quis, commodo eu, libero. Fusce sagittis erat at erat tristique mollis. Maecenas sapien libero, molestie et, lobortis in, sodales eget, dui. Morbi ultrices rutrum lorem. Nam elementum ullamcorper leo. Morbi dui. Aliquam sagittis. Nunc placerat. Pellentesque tristique sodales est. Maecenas imperdiet lacinia velit. Cras non urna. Morbi eros pede, suscipit ac, varius vel, egestas non, eros. Praesent malesuada, diam id pretium elementum, eros sem dictum tortor, vel consectetuer odio sem sed wisi.

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Preface

Notation

- $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
- $[n] = \{1, \dots, n\}$
- if a is defined to be b we may say $a := b$ (but I will often forget)
- $A^k = \overbrace{A \times \cdots \times A}^{k \text{ times}}$ for a set A, $\times$ being the Cartesian product.

Contents

1 Elementary Combinatorics

In combinatorics we are concerned with counting the number of objects from a collection. For example, we may count the positive integers divisible by 3 that are less than 1000. Here the objects are positive integers are the objects and the collection is those objects that are both divisible by 3 and less than 1000.

As a set is just is collection of objects, we will often phrase things set theoretically.

1.1 Fundamental Principles

To begin counting anything interesting, we'll need the following principles. The first two let us count things inductively (e.g. counting wine bottles by counting in threes or counting the number of wine & cheese pairings by multiplying the number of wine bottles and the number of cheeses). The final simply says it doesn't matter what order we count each element (e.g. wine bottle) in.

Theorem 1.1 (addition principle)

Let $n \in \mathbb{N}$. Let $A_1, \dots, A_n$ be finite sets. One has[a]

$$A_1, \dots, A_n \text{ disjoint} \implies |A_1 \cup \cdots \cup A_n| = |A_1| + \cdots + |A_n|$$

[a] $A_1, \dots, A_n$ are said to be disjoint if $\cap_{i=1}^n A_i = \emptyset$

To prove this, we simply count both sides one term at a time.

Proof. Let $x \in \cup_{i=1}^n A_i$. Then $x \in A_i$ for exactly one $1 \leq i \leq n$ and hence

$$|A_1| + \cdots + |A_n| = \sum_{x \in A_1} 1 + \cdots + \sum_{x \in A_n} 1 = \sum_{x \in \cup_{i=1}^n A_i} 1 = |A_1 \cup \cdots \cup A_n|$$

by counting both sides term by term. $\square$

Theorem 1.2 (multiplication principle)

Let $n \in \mathbb{N}$. Let $A_1, \dots, A_n$ be finite sets. One has

$$|A_1 \times \cdots \times A_n| = |A_1| \times \cdots \times |A_n|$$

This time we count terms in each slot recursively.

Proof. Consider an arbitrary element $x = (x_1, \dots, x_n) \in A_1 \times \cdots \times A_n$ and let $1 \leq i \leq n$. Then fixing $x_j, j \neq i$ we have $|A_i|$ possible choices for x_i. Hence, we have

$$|A_1 \times \cdots \times A_n| = \sum_{i=1}^n \sum_{x \in A_i} |\cap_{j \neq i} A_j| = \sum_{i=1}^n |A_i| \sum_{x \in A_i} |\cap_{j \neq i} A_j| = \cdots = |A_1| \times \cdots \times |A_n|$$

by repeated application of this reasoning. $\square$

Theorem 1.3 (bijection principle)

Let A and B be sets. If there is a bijection[a] $\pi : A \to B$ then $|A| = |B|$

[a] A bijection is a one-to-one mapping

Proof. This is a truism. $\square$

The addition principle can be thought of as emptying two containers of n and m wine bottles and containing how many bottles you have in total $(n + m)$, whereas the multiplication principle can be thought of as counting the number of ways of pairing b boys with g girls. The bijection principle says if we take s students and select them one by one at random, t times, we'll always have selected t students. We will often look for such interpretations in combinatorics.

With just these principles, we can count all the following quantities.

Problem 1.1 (**k-words with repetitions**)

How many words length k can be formed from an alphabet with n letters.

Solution. To count the number of words with repetitions allowed, we sample with replacement to get

$$\underbrace{n \times \cdots \times n}_{k \text{ times}} = n^k$$

k-words (or k-sequences) on n symbols.

□

Problem 1.2 (**k-words without repetitions**)

How many words length k with distinct letters can be formed from an alphabet with n letters.

Solution. For k-words without repetition, we sample without replacement for each letter to get

$$n \times (n-1) \times \cdots \times (n-k+1) =: n^{\underline{k}}$$

k-words without repetition on n symbols.[1] □

If we were to consider words of length n on n symbols without repetition, then we would simply be *permuting* (i.e. reordering) the n symbols. Hence we have $n \times (n-1) \times \cdots \times 1 =: n!$ *permutations* on n symbols. Using this notation we have $n^{\underline{k}} = n!/(n-k)!$. From this, we have the following result.

Problem 1.3 (**k-subsets**)

How many subsets size k are there of a set size n?

Solution. Let x be the desired quantity. For each subset size k, we may reorder it's elements in $k!$ ways. If we care about the order of the elements in the subset, then the number of such subsets is precisely the number of k-words without repetition. Hence

$$x \cdot k! = \frac{n!}{(n-k)!}$$

and we have counted the desired quantity. □

Under the hood there we used the bijection principle, as we implicitly assumed (quite reasonably) the existence of a bijection from the collection of subsets size k where reordering matters and the k-words on n symbols.

We call $\binom{n}{k} := \frac{n!}{k!(n-k)!}$ the *binomial coefficient*, and would say this as "n choose k". We will dedicate an entire section (in fact, next section) to these.

Problem 1.4 (**k-multisubsets**)

How many multisubsets[a] of size k are there in a set of n objects?

[a] a multiset is a set that allows repetition, see https://en.wikipedia.org/wiki/Multiset

[1] We call $n^{\underline{k}}$ the falling factorial of n

Solution. Let $X = \{x_1, \ldots, x_n\}$. We call the number of times $x_i, 1 \leq i \leq n$ appears in a multisubset of X the *multiplicity* of x_i. We know that the size of a multiset is the sum of its multiplicities, thus it suffices to count solutions $(\ell_1, \ldots, \ell_n)$ of nonnegative integers to

$$\ell_1 + \cdots + \ell_n = k$$

which by setting $\ell' := \ell + 1$ is the same as counting positive integer solutions to

$$\ell_1' + \cdots + \ell_n' = k + n$$

Now write this as

$$\underbrace{1 + \cdots + 1}_{\ell_1' \text{ times}} + \cdots + \underbrace{1 + \cdots + 1}_{\ell_n' \text{ times}} = k + n$$

We are tasked with dividing up the $k + n$ ones into n groups, which can be done by insering $n - 1$ stars in between two ones (not the edges as we require n nonempty collections of ones) to signifying the beginning of a new group. This gives $\binom{n-k-1}{n-1} = \binom{n-k-1}{k}$ possibilities. $\square$

Problem 1.5 **(k-partitions with a given sizing)**

How many ways are there to divide a set partition a set X of size n into sets $X_1, \ldots, X_k$ with size $n_1, \ldots, n_k$ respectively (where $n_1 + \cdots + n_k = n$)?

Solution. There are

$$\binom{n - n_1 - \cdots - n_{i-1}}{n_i}$$

ways of choosing the X_i once $X_1, \ldots, X_{i-1}$ have been chosen, and

$$\binom{n}{k}\binom{n-k}{m} = \frac{n!}{k!m!(n-k-m)!}$$

So all together one obtains

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-(n_1+\cdot+n_{k-1})}{n_k} = \frac{n!}{n_1!\cdots n_k!} =: \binom{n}{n_1, \ldots, n_k}$$

such partitions.[2] $\square$

Exercises

Exercise 1.1

In how many ways can n people be seated at a circular table with n seats, where we do not classify between seatings that can be obtained from one another by rotating the table?

Exercise 1.2

How many subsets are there of the set $\{1, 2, \ldots, n\} =: [n]$? How many are of even size?

Exercise 1.3

There are $2n$ people at a party. How many ways can the $2n$ people split into n pairs?

Exercise 1.4

A robot is placed on the bottom left of an $n \times n$ chessboard. The robot has two moves, go vertically up one square or horizontally along (to the right) one square. How many sequences of moves can the robot make to get to the top right square?

[2]The quantity $\binom{n}{n_1,\ldots,n_k}$ is called the multinomial coefficient

1.2 Binomial Coefficients

Lemma 1.1 (addition formula)

Let $n \in \mathbb{N}$ and $1 \le k \le n$. The identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

holds.

Proof 1. A simple algebraic verification works, I leave this to the reader. □

We can also prove this identity with the technique of "double counting". Intuitively, this is just saying that if we can count things in two ways they must be equal. For example, if we had an $n \times n$ grid full of numbers, we could count the sum of all the numbers by summing along the rows or the columns. For certain selections of numbers, this can yield interesting equalities.

Proof 2. We count the k-subsets of $[n]$ in two ways. First, all at once, getting $\binom{n}{k}$ such sets. Secondly, by considering seperately the cases when 1 is in our subset and isn't. This gives $\binom{n-1}{k-1}$ choices with 1 and $\binom{n-1}{k}$ choices without, giving us the desired equality. □

Theorem 1.4 (binomial theorem)

For all integers $n \ge 0$ and $x, y \in \mathbb{C}$, one has

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Proof 1. We first use a straight forward induction argument. Observe, assuming the hypothesis for some $n \ge 0$ (and noting it trivially holds for $n = 0$),

$$\begin{aligned}
(x+y)^{n+1} &= (x+y)(x+y)^n \\
&= (x+y)\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k+1} \\
&= \sum_{k=0}^{n+1} \left(\binom{n}{k-1} + \binom{n}{k} \right) x^k y^{n-k+1} \qquad (\dagger) \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{(n+1)-k}
\end{aligned}$$

where $(\dagger)$ follows from an index shift $(k \to k-1)$ and defining $\binom{n}{k} := 0$ for $k > n$ or $k < 0$. □

As before, we can double count ourselves a proof of this theorem.

Proof 2. □

As an immediate corollary we have the following.

Corollary 1.1 (sum and alternating sum of binomial coefficients)

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \qquad \text{and} \qquad \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

Proof 1. Take $x = y = 1$ and $x = 1, y = -1$ in the Binomial Theorem. □

We may also take a more combinatorial approach to proving these identities.

Proof 2. There are 2^n subsets of $[n]$ by counting directly (see exercise 1.2) and by counting each of the subsets size k in groups we get $\sum_{k=0}^{n}\binom{n}{k}$, giving the first equivalence. For the second, note there are an equal number of odd and even subsets of $[n]$ (also exercise 1.2) so we have

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = \sum_{k=0}^{n}\sum_{k=0,2|k}^{n}\binom{n}{k} - \sum_{k=0,2|k}^{n}\binom{n}{k} = 0$$

giving the latter equivalence. □

Exercises

Exercise 1.5

How many words length n can be formed from an alphabet of ℓ letters $\mathcal{A} = \{A_1, \ldots, A_\ell\}$ such that the first letter A_1 occurs an even number of times?

1.3 Additional Problems

2 The Probabilistic Method

2.1

3 Solutions

Here I list the exercise solutions in a random order. I will store which exercises have a solution in the README of this repository.