Problem 0.0 (constructing K_n)

Let G be a graph with n vertices and m edges. Prove we can express the complete graph K_n as the union of $O(n^2 \log n/m)$ isomorphic copies of G.

Solution. Isomorphisms are just permutations of the vertices of G. We will work by randomly sampling, independently and uniformly, these permutations.

Let $\sigma \in S_n$ be a permutation of $[n]$, the vertices of G, denoting the resulting graph G_σ. Then, denoting the adjacency relation in an arbitrary graph H by $i \overset{H}{\sim} j$, we have $i \overset{G}{\sim} j$ iff $\sigma(i) \overset{G_\sigma}{\sim} \sigma(j)$. Thus, it suffices to find $\sigma_1, \ldots, \sigma_\ell$ that have

$$\forall v \neq w \in [n] \, \exists 1 \leq k \leq \ell \, \exists i \overset{G}{\sim} j \;:\; \sigma_k(i) = v, \sigma_k(w) = j$$

Let $A_{v,w}$ be the event that we have such a permutation for the pair (v, w) and fix $\ell \in \mathbb{N}$.

$\square$