

Combinatorics

Jacob Green
Department of Mathematical Sciences, The University of Bath

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Abstract

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Preface

Notation

- $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
- $[n] = \{1, \dots, n\}$
- if a is defined to be b we may say $a := b$ (but I will often forget)
- $A^k = \overbrace{A \times \dots \times A}^{k \text{ times}}$ for a set A , \times being the Cartesian product.

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1 Elementary Combinatorics

In combinatorics we are concerned with counting the number of objects from a collection. For example, we may count the positive integers divisible by 3 that are less than 1000. Here the objects are positive integers are the objects and the collection is those objects that are both divisible by 3 and less than 1000.

As a set is just is collection of objects, we will often phrase things set theoretically.

1.1 Fundamental Principles

To begin counting anything interesting, we'll need the following principles. The first two let us count things inductively, the final says it doesn't matter what order we count each element in. These are essentially common sense principles. Regardless, being mathematicians, we will prove these.

Theorem 0 (addition principle)

Let $n \in \mathbb{N}$. Let A_1, \dots, A_n be finite sets. One has^a

$$A_1, \dots, A_n \text{ disjoint} \implies |A_1 \cup \dots \cup A_n| = |A_1| + \dots + |A_n|$$

^a A_1, \dots, A_n are said to be disjoint if $\bigcap_{i=1}^n A_i = \emptyset$

To prove this, we simply count both sides one term at a time.

Proof. Let $x \in \bigcup_{i=1}^n A_i$. Then $x \in A_i$ for exactly one $1 \leq i \leq n$ and hence

$$|A_1| + \dots + |A_n| = \sum_{x \in A_1} 1 + \dots + \sum_{x \in A_n} 1 = \sum_{x \in \bigcup_{i=1}^n A_i} 1 = |\bigcup_{i=1}^n A_i|$$

by counting both sides term by term. □

Theorem 1 (multiplication principle)

Let $n \in \mathbb{N}$. Let A_1, \dots, A_n be finite sets. One has

$$|A_1 \times \dots \times A_n| = |A_1| \times \dots \times |A_n|$$

This time we count terms in each slot recursively.

Proof. Consider an arbitrary element $x = (x_1, \dots, x_n) \in A_1 \times \dots \times A_n$ and let $1 \leq i \leq n$. Then fixing $x_j, j \neq i$ we have $|A_i|$ possible choices for x_i . Hence, we have

$$|A_1 \times \dots \times A_n| = \sum_{i=1}^n \sum_{x \in A_i} |\bigcap_{j \neq i} A_j| = \sum_{i=1}^n |A_i| \sum_{x \in A_i} |\bigcap_{j \neq i} A_j| = \dots = |A_1| \times \dots \times |A_n|$$

by repeated application of this reasoning. □

Theorem 2 (bijection principle)

Let A and B be sets. If there is a bijection (i.e. one-to-one mapping) $\pi : A \rightarrow B$ then $|A| = |B|$

Proof. This is a truism. □

The addition principle can be thought of as emptying two containers of n and m wine bottles and containing how many bottles you have in total ($n + m$), whereas the multiplication principle can be thought of as counting the number of ways of pairing b boys with g girls. The bijection principle says if we take s students and select them one by one at random, t times, we'll always have selected t students. We will often look for such interpretations in combinatorics.

With just these principles, we can count all the following quantities.

Problem 3 (k -words with repetitions)

How many words length k can be formed from an alphabet with n letters.

Solution. To count the number of words with repetitions allowed, we sample with replacement to get

$$\underbrace{n \times \cdots \times n}_{k \text{ times}} = n^k$$

k -words (or k -sequences or k -tuples) on n symbols by the multiplication principle. □

Problem 4 (k -words without repetitions)

How many words length k with distinct letters can be formed from an alphabet with n letters.

Solution. For k -words without repetition, we sample without replacement for each letter to get

$$n \times (n-1) \times \cdots \times (n-k+1) =: n^{\underline{k}}$$

k -words without repetition on n symbols by the multiplication principle.¹ □

If we were to consider words of length n on n symbols without repetition, then we would be *permuting* (i.e. reordering/relabelling) the n symbols. Hence we have $n \times (n-1) \times \cdots \times 1 =: n!$ *permutations* on n symbols. Using this notation we have $n^{\underline{k}} = n!/(n-k)!$.

Suppose we wish to count the number of subsets length 2 on n symbols. We have $n(n-1)$ ways of choosing 2 unique elements (i.e. 2-words without repetition) and we have $2!$ ways of arranging the 2 unique elements. Hence, as each 2 element subset we count gives rise to $2! = 2$ 2-words, we may count these and divide by 2, obtaining $\frac{1}{2}n(n-1)$ unique 2-element subsets². This method is often referred to as overcounting and correcting, and gives us another way to count unknown objects with known objects.

Problem 5 (k -subsets)

How many subsets size k are there of a set size n ?

Solution. Let x be the desired quantity. For each subset size k , we may reorder it's elements in $k!$ ways and we may select k unique elements by counting k -words on n symbols without repetition, thus

$$x \cdot k! = \frac{n!}{(n-k)!}$$

and dividing by $k!$ gives $x = n!/k!(n-k)! =: \binom{n}{k}$ □

We call $\binom{n}{k}$ the *binomial coefficient*, and would say this as “ n choose k ” (as this counts the ways of choosing k things from n things). We will dedicate an entire section (in fact, next section) to these.

In this next problem we will first encounter the idea of “stars and bars” counting, which gives us a way of counting the ways to group n indistinguishable objects into k groups. This has many novel uses, like the one we’re about to encounter.

Problem 6 (k -multisubsets)

How many multisubsets^a of size k are there in a set of n objects?

^aa multiset is a set that allows repetition, see <https://en.wikipedia.org/wiki/Multiset>

¹We call $n^{\underline{k}}$ the falling factorial of n

²note this gives another proof that the product of 2 consecutive numbers is divisible by 2, a somewhat nontrivial bridge between combinatorics and number theory, and far from our last

Solution. Let $X = \{x_1, \dots, x_n\}$. We call the number of times $x_i, 1 \leq i \leq n$ appears in a multisubset of X the *multiplicity* of x_i . We know that the size of a multiset is the sum of its multiplicities, thus it suffices to count solutions (ℓ_1, \dots, ℓ_n) of nonnegative integers to

$$\ell_1 + \dots + \ell_n = k$$

which by setting $\ell' := \ell + 1$ is the same as counting positive integer solutions to

$$\ell'_1 + \dots + \ell'_n = k + n$$

Now write this as

$$\underbrace{1 + \dots + 1}_{\ell'_1 \text{ times}} + \dots + \underbrace{1 + \dots + 1}_{\ell'_n \text{ times}} = k + n$$

We are tasked with dividing up the $k + n$ ones into n groups, which can be done by inserting $n - 1$ stars in between two ones (not the edges as we require n nonempty collections of ones) to indicate the beginning of a new group. This gives $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$ possibilities. \square

Problem 7 (k -partitions with a given sizing)

How many ways are there to divide a set partition a set X of size n into sets X_1, \dots, X_k with size n_1, \dots, n_k respectively (where $n_1 + \dots + n_k = n$)?

Solution. There are

$$\binom{n - n_1 - \dots - n_{i-1}}{n_i}$$

ways of choosing the X_i once X_1, \dots, X_{i-1} have been chosen, and

$$\binom{n}{k} \binom{n-k}{m} = \frac{n!}{k!m!(n-k-m)!}$$

So all together one obtains

$$\binom{n}{n_1} \binom{n-n_1}{n_2} \dots \binom{n-(n_1+\dots+n_{k-1})}{n_k} = \frac{n!}{n_1! \dots n_k!} =: \binom{n}{n_1, \dots, n_k}$$

such partitions.³ \square

Exercises

Exercise 8

In how many ways can n people be seated at a circular table with n seats.

Exercise 9

How many subsets are there of the set $\{1, 2, \dots, n\} =: [n]$? How many are of even size?

Exercise 10

There are $2n$ people at a party. How many ways can the $2n$ people split into n pairs?

Exercise 11

A robot is placed on the bottom left of an $n \times n$ chessboard. The robot has two moves, go vertically up one square or horizontally along (to the right) one square. How many sequences of moves can the robot make to get to the top right square?

³The quantity $\binom{n}{n_1, \dots, n_k}$ is called the multinomial coefficient

1.2 Binomial Coefficients

We spend this section proving a myriad of identities concerning binomial coefficients, often providing both an algebraic proof and a combinatorial proof.

Lemma 1 (addition formula)

Let $n \in \mathbb{N}$ and $1 \leq k \leq n$. The identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

holds.

Proof 1. A simple algebraic verification works, I leave this to the reader. \square

We can also prove this identity with the technique of “double counting”. Intuitively, this is just saying that if we can count things in two ways they must be equal. For example, if we had an $n \times n$ grid full of numbers, we could count the sum of all the numbers by summing along the rows or the columns. For certain selections of numbers, this can yield interesting equalities.

Proof 2. We count the k -subsets of $[n]$ in two ways. First, all at once, getting $\binom{n}{k}$ such sets. Secondly, by considering separately the cases when 1 is in our subset and isn't. This gives $\binom{n-1}{k-1}$ choices with 1 and $\binom{n-1}{k}$ choices without, giving us the desired equality. \square

Theorem 2 (Binomial theorem)

For all integers $n \geq 0$ and $x, y \in \mathbb{C}$, one has

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

Proof 1. We first use a straight forward induction argument. Observe, assuming the hypothesis for some $n \geq 0$ (and noting it trivially holds for $n = 0$),

$$\begin{aligned} (x + y)^{n+1} &= (x + y)(x + y)^n \\ &= (x + y) \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} \\ &= \sum_{k=0}^n \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^n \binom{n}{k} x^k y^{n-k+1} \\ &= \sum_{k=0}^{n+1} \left(\binom{n}{k-1} + \binom{n}{k} \right) x^k y^{n-k+1} \quad (\dagger) \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{(n+1)-k} \end{aligned}$$

where (\dagger) follows from an index shift ($k \rightarrow k-1$) and defining $\binom{n}{k} := 0$ for $k > n$ or $k < 0$. \square

As before, we can double count ourselves a proof of this theorem.

Proof 2. Consider the coefficient of $x^k y^{n-k}$ of

$$(x + y)^n = \overbrace{(x + y) \cdots (x + y)}^{n \text{ times}}$$

When computing this expansion, we are multiplying exactly one of x or y from each $(x + y)$ term on the RHS. Our entire expansion will be the sum of all such choices. Thus, to compute the coefficient of $x^k y^{n-k}$ in $(x + y)^n$, it suffices to count the number of ways to choose exactly k x 's out the n possible choices. This is clearly $\binom{n}{k}$ and hence the result follows. \square

As an immediate corollary we have the following.

Corollary 3 (sum and alternating sum of binomial coefficients)

$$\sum_{k=0}^n \binom{n}{k} = 2^n \quad \text{and} \quad \sum_{k=0}^n (-1)^k \binom{n}{k} = 0$$

Proof 1. Take $x = y = 1$ and $x = 1, y = -1$ in the Binomial Theorem. \square

We may also take a more combinatorial approach to proving these identities.

Proof 2. There are 2^n subsets of $[n]$ by counting directly (see exercise 1.2) and by counting each of the subsets size k in groups we get $\sum_{k=0}^n \binom{n}{k}$, giving the first equivalence. For the second, note there are an equal number of odd and even subsets of $[n]$ (also exercise 1.2) so we have

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = \sum_{k=0}^n \sum_{\substack{k=0, \\ 2|k}}^n \binom{n}{k} - \sum_{k=0, \\ 2 \nmid k}^n \binom{n}{k} = 0$$

giving the latter equivalence. \square

Let us recall the following lemma from (elementary) complex analysis, concerning sums of powers of roots of unity.

Lemma 4 (sums of powers of roots of unity)

Let $1, \xi_1, \dots, \xi_{n-1}$ be the n -roots of unity, that is the solutions to $z^n = 1, z \in \mathbb{C}$. Then, one has

$$1 + \sum_{j=1}^{n-1} \xi_j^\ell = \begin{cases} n & : \ell \equiv 0 \pmod{n} \\ 0 & : \ell \not\equiv 0 \pmod{n} \end{cases}$$

Proof. Write $\xi_j = \exp(2\pi i j/n)$. If $\ell = mn, m \in \mathbb{Z}$, then $\xi_j^\ell = \exp(2\pi i m) = 1$ and hence

$$1 + \sum_{j=1}^{n-1} \xi_j^\ell = n$$

Conversely, if $\ell = mn + r, m \in \mathbb{Z}, 0 < r < n$ then $\xi_j^\ell = \exp(2\pi i(mn + r)j/n) = \exp(2\pi i r j/n)$ and

$$1 + \sum_{j=1}^{n-1} \xi_j^\ell = \sum_{j=0}^{n-1} (\exp(2\pi i r/n))^j = \frac{1 - \exp(2\pi i r)}{1 - \exp(2\pi i r/n)} = 0$$

where we used the standard result for the sum of a geometric series in the second equality. \square

With this we can solve the following problem, a generalisation of corollary 1.2.3.

Problem 5 (sums of binomial multiples)

Compute the following quantity.

$$\sum_{k=0}^n \binom{mn}{mk}$$

Solution. Let ξ_0, \dots, ξ_{m-1} be the m^{th} roots of unity, with $\xi_0 = 1$. Then, by our previous lemma and the Binomial theorem, we can deduce

$$\sum_{k=0}^n \binom{mn}{mk} = \sum_{k=0}^{mk} \binom{mn}{k} \left(\frac{1}{m} \sum_{\ell=0}^{m-1} \xi_\ell^k \right) = \frac{1}{m} \sum_{\ell=0}^{m-1} \sum_{k=0}^{mn} \binom{mn}{k} \xi_\ell^k = \frac{1}{m} \sum_{\ell=0}^{m-1} (1 + \xi_\ell)^{mn}$$

The RHS can be further simplified, I elect to leave it here. \square

Proposition 6 (Vandermonde's Identity)

Fix $n, a, b, \in \mathbb{N}$. Then

$$\sum_{k=0}^n \binom{a}{k} \binom{b}{n-k} = \binom{a+b}{n}$$

We observe two proofs, one algebraic and one via double counting.

The motivation for this first proof comes from the fact we are taking a sum of the form $\sum_{k=0}^n a_k b_{n-k}$, which should remind us of the multiplication of polynomials.

Proof 1. We work by comparing the coefficients of $(1+x)^{a+b} = (1+x)^a(1+x)^b$. By the Binomial theorem, one has

$$(1+x)^a(1+x)^b = \left(\sum_{k_1=0}^a \binom{a}{k_1} x^{k_1} \right) \left(\sum_{k_2=0}^b \binom{b}{n-k_2} x^{k_2} \right) = \sum_{n=0}^{a+b} \left(\sum_{k=0}^n \binom{a}{k} \binom{b}{n-k} \right) x^n$$

where the last equality comes from computing the polynomial multiplication. Now, applying the Binomial theorem again, see

$$(1+x)^{a+b} = \sum_{n=0}^{a+b} \binom{a+b}{n} x^n$$

so by the comparing coefficients of these two polynomials we deduce the result. \square

A nice way to think about this double counting proof is to count the ways of forming a committee of n people from a boys and b girls, choosing k boys at a time.

Proof 2. We count the number of ways to choose subsets size n from $[a+b]$ in two ways. The first, directly, gives $\binom{a+b}{n}$ choices. The second, is by counting the number of ways to choose such a subset with exactly $0 \leq k \leq n$ elements in $[a]$. We have $\binom{a}{k}$ ways of choosing the k elements in $[a]$, and $\binom{b}{n-k}$ ways of choosing the remaining $n-k$ elements from $[a+b] \setminus [a]$. Thus, summing over $0 \leq k \leq n$ we obtain the desired equality. \square

Proposition 7 (hockeystick lemma)

For $n, r \in \mathbb{N}$ with $n \geq r$, one has

$$\sum_{k=r}^n \binom{k}{r} = \binom{n+1}{r+1}$$

As usual, there are plenty of ways of proving this. We'll do one algebraic and one combinatorial.

Proof 1. We use the Binomial addition formula to transform our sum into a telescoping series. Observe,

$$\sum_{k=r}^n \binom{k}{r} = \sum_{k=r}^n \left[\binom{k+1}{r+1} - \binom{k}{r+1} \right] = \sum_{k=r+1}^{n+1} \binom{k}{r+1} - \sum_{k=r}^n \binom{k}{r+1} = \binom{n+1}{r+1} - \binom{r}{r+1} = \binom{n+1}{r+1}$$

where the first equality is a consequence of Lemma 1.2.1. \square

This next proof uses the so called “stars and bars” method that we encountered last chapter to count the arrangements of n indistinguishable balls into k distinguishable boxes.

Proof 2. We work by double counting the arrangements of n indistinguishable red balls into k indistinguishable blue balls into k distinguishable boxes. By “stars and bars” we have

$$\binom{n+k-1}{k-1}$$

such arrangements. Now instead, label the boxes $1, \dots, k$ and count the arrangements with $0 \leq i \leq k$ blue balls in box 1. There are

$$\binom{(n+k-2)-i}{k-2}$$

such arrangements for each i and hence

$$\binom{n+k-1}{k-1} = \sum_{i=0}^k \binom{(n+k-2)-i}{k-2}$$

Taking $n' = n + k - 2$ and $r = k - 2$ we recover

$$\binom{n'+1}{r+1} = \sum_{i=r}^{n'} \binom{i}{r}$$

by reversing the order of summation, which as k and n were arbitrary concludes the proof. \square

Exercises

Exercise 8

How many words length n can be formed from an alphabet of ℓ letters $\mathcal{A} = \{A_1, \dots, A_\ell\}$ such that the first letter A_1 occurs an even number of times?

Exercise 9

- (i) Using induction, find another proof for the hockeystick lemma.
- (ii) By assigning $n - k + 1$ labels to the elements of $[n + 1]$ or otherwise, find another double counting proof of the hockeystick lemma.

Exercise 10

Compute

$$\sum_{k=0}^n k \binom{n}{k} \quad \text{and} \quad \sum_{k=0}^n k^2 \binom{n}{k}$$

algebraically and combinatorially. Can you find a probabilistic proof?

1.3 The Pigeonhole Principle

If I have 5 pigeons and 4 containers, each only able to fit one pigeon, can I fit all the pigeons into my containers? Of course not! It turns out this common-sense principle allows us to discover many, many, combinatorial facts...

Theorem 1 (pigeonhole principle)

Given a set X of size n , any partition of X into $m < n$ subsets X_1, \dots, X_m must have at least one X_i with $|X_i| > 1$.

We prove this with the following simple contradiction.

Proof. Suppose each $|X_i| \leq 1$, then $|X| = |\cup_{i=1}^m X_i| = \sum_{i=1}^m |X_i| \leq m < n$. Absurd! \square

We can do slightly better than this. What if my containers can fit 2 pigeons and I only have 2 this time. Then I still can't fit my pigeons into my containers. Formally,

Theorem 2 (full pigeonhole principle)

Given a set X of size $n = km + 1$, any partition of X into m subsets X_1, \dots, X_m must have at least one X_i with $|X_i| > k$.

Proof. Suppose each $|X_i| \leq k$, then $|X| = |\cup_{i=1}^m X_i| = \sum_{i=1}^m |X_i| \leq km < n$. Absurd! \square

We also have a infinite pigeonhole principle. If I can only fit a finite number of pigeons into each container then, assuming I only have finitely many containers, I cannot fit an infinite number of pigeons into my containers.

Theorem 3 (infinite pigeonhole principle)

Given a set X of infinite cardinality, any partition of X into finitely many sets X_1, \dots, X_m must have some X_i also of infinite cardinality.

Proof. Suppose each $|X_i| = n_i < \infty$. Then $|X| = |\cup_i^m X_i| = \sum_{i=1}^m |X_i| = \prod_{i=1}^m n_i < \infty$. Absurd! \square

Problem 4 (monotone subsequences)

How large must a sequence of distinct real numbers be to guarantee the existence of a monotone subsequence size $n + 1$, $n \in \mathbb{N}$?

Solution. \square

Exercises

1.4 The Principle of Inclusion-Exclusion

It is often much easier to count how many objects have properties A AND B than A OR B . Thankfully we have the following principle to relate the two.

Proposition 1 (2 variable inclusion exclusion principle)

Let A and B be finite sets. Then,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

For an intuitive argument, just draw a venn diagram. To find the count $A \cup B$ we can count the entire circles of A and B individually, but when doing this we count the $A \cap B$ section twice so we must subtract off one lot of it.

We prove this by showing for each $x \in A \cup B$, x is counted exactly once in the RHS.

Proof. Write $|A \cup B| = \sum_{x \in A \cup B} 1$. WLOG take $x \in A$, then either $x \in B$ or $x \notin B$. In the former case, $x \in A \cap B$ gives a count of exactly 1 on the RHS for x and in the latter case $x \notin A \cap B$ affirms the same result. Hence $|A| + |B| - |A \cap B| = \sum_{x \in A \cup B} 1 = |A \cup B|$. \square

It turns out this property can be generalised for n finite sets.

Theorem 2 (inclusion exclusion principle)

Let $n \in \mathbb{N}$ and A_1, \dots, A_n be finite sets. Then,

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{\ell=1}^n (-1)^{\ell-1} \sum_{I \subseteq [n], |I|=\ell} \left| \bigcap_{i \in I} A_i \right|$$

We have two proofs for this. A bashy induction will work, but there is also a clever little trick we can do with indicator functions.

Proof 1. We work via induction... \square

Proof 2. Suppose that all of our subsets lie in a space Ω and denote $A^c := \Omega \setminus A$ for $A \subseteq \Omega$. Then,

$$\mathbf{1} \left[\bigcup_{i=1}^n A_i \right] = \mathbf{1} \left[\left(\bigcap_{i=1}^n A_i^c \right)^c \right] = 1 - \mathbf{1} \left[\bigcap_{i=1}^n A_i^c \right] = 1 - \prod_{i=1}^n (1 - \mathbf{1}[A_i])$$

Where $\mathbf{1}[A] = 1$ if $x \in A$ and 0 elsewhere (we call this the indicator function). Now, expanding the product we see

$$\mathbf{1} \left[\bigcup_{i=1}^n A_i \right] = \sum_{\ell=1}^n \sum_{I \subseteq [n], |I|=\ell} \prod_{i \in I} (-\mathbf{1}[A_i])$$

which, summing both sides over $x \in \Omega$, gives the result. \square

We may count, for n divisible by a prime p , the number of integers less than or equal to n divisible by p as n/p . Recall that for primes p, q and $n \in \mathbb{N}$, one has $p, q \mid n \Leftrightarrow pq \mid n$. Thus, it is easy to count how many naturals below n are divisible by p and q (just biject), and by the inclusion exclusion principle it must also be easy to count how many naturals below n are divisible by p or q .

Recall that *Euler's totient function* $\phi : \mathbb{N} \rightarrow \mathbb{N}$ has $\phi(n) = |\{1 \leq i \leq n : \gcd(i, n) = 1\}|$. That is, ϕ counts the number of integers at most n which are coprime to n . We will count this quantity directly, working with the reasoning outlined above.

Proposition 3 (explicit form of Euler's ϕ -function)

Let $n \in \mathbb{N}$. Then, letting ϕ be Euler's totient function,

$$\phi(n) = n \prod_{\substack{p|n, \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right)$$

Proof. Factorise $n = p_1^{\ell_1} \cdots p_k^{\ell_k}$ into primes. Then we have $1 \leq x \leq n$ coprime to n if and only if $p_1, \dots, p_k \nmid x$. Define $A_p := \{1 \leq x \leq n : p \mid x\} \subseteq [n]$. By the inclusion exclusion principle we may compute, where compliments are taken with respect to $[n]$,

$$\phi(n) = \left| \bigcap_{i=1}^k A_{p_i}^c \right| = n - \left| \bigcup_{i=1}^k A_{p_i} \right| \quad (1)$$

$$= n - \sum_{m=1}^k (-1)^{m-1} \sum_{I \subseteq [k], |I|=m} \left| \bigcap_{i \in I} A_{p_i} \right| \quad (2)$$

$$= n - \sum_{m=1}^k (-1)^{m-1} \sum_{I \subseteq [k], |I|=m} \frac{n}{\prod_{i \in I} p_i} \quad (3)$$

$$= n \sum_{m=0}^k (-1)^m \sum_{I \subseteq [k], |I|=m} \frac{1}{\prod_{i \in I} p_i} = n \prod_{p|n} \left(1 - \frac{1}{p}\right) \quad (4)$$

Where in (1) we used DeMorgan's laws, (2) the inclusion exclusion principle, in (3) the fact that $|\bigcap_{p \in P} A_p| = n / \prod_{p \in P} p$ when $p \in P$ are primes dividing n (easily seen with $pq \mid n \Leftrightarrow p, q \mid n$) and in (4) the fact $\prod_{i \in I} (1 - x_i) = \sum_{J \subseteq I} (-1)^{|J|} \prod_{j \in J} x_j$, where the empty product is defined as 1. \square

We can also use the inclusion exclusion principles to count the number of permutations of $[n]$ with no fixed point, the so called *dearrangements* of $[n]$. A fixed point of a permutation $\pi : [n] \rightarrow [n]$ is simply an $x \in [n]$ with $\pi(x) = x$.

Proposition 4 (dearrangements)

The number of permutations of $[n]$ with no fixed point is

$$n! \sum_{k=0}^n \frac{(-1)^k}{k!}$$

Proof. This is another direct application of the inclusion exclusion principle. Let A_i be the set of permutations on $[n]$ with i a fixed point. Then we compute

$$\left| \bigcap_{i=1}^n A_i^c \right| = n! - \left| \bigcup_{i=1}^n A_i \right| = n! - \sum_{k=1}^n \sum_{I \subseteq [n], |I|=k} \left| \bigcap_{i \in I} A_i \right| = n! - \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} (n-k)! = n! \sum_{k=0}^n \frac{(-1)^k}{k!}$$

using the fact there are $\binom{n}{k}$ subsets $I \subseteq [n]$ of size k and $(n-k)!$ permutations with k elements fixed. \square

If we take the limit $n \rightarrow \infty$, we see that the probability of a (uniformly) random permutation of $[n]$ having no fixed points tends to $1/e$.

1.5 Recursion I

Exercises

1.6 Recursion II

1.7 Miscellaneous Topics

Exercises

1.8 Additional Problems

2 Non-constructive Methods in Combinatorics

2.1 Introduction

In these notes we are primarily concerned with the probabilistic method. Towards the end we will touch on some topological and algebraic methods.

Typically, the probabilistic method is used to prove the existence of a construction with desirable properties without actually constructing it. We do this by showing it exists with some positive probability.

A simple tool for doing so is the following.

Lemma 0 (a random variable must take its mean value)

Let X be an integer-valued random variable. Then

$$\mathbb{P}(X \geq \mathbb{E}[X]) > 0$$

Proof. Suppose $\mathbb{P}(X \geq \mathbb{E}[X]) = 0$. Then,

$$\begin{aligned} \mathbb{E}[X] &= \sum_{x=-\infty}^{\mathbb{E}[X]-1} x\mathbb{P}(X=x) + \sum_{x=\mathbb{E}[X]}^{\infty} x\mathbb{P}(X=x) \\ &\leq (\mathbb{E}[X]-1) \sum_{x=-\infty}^{\mathbb{E}[X]-1} \mathbb{P}(X=x) + \sum_{x=\mathbb{E}[X]}^{\infty} x\mathbb{P}(X=x) \\ &= (\mathbb{E}[X]-1)\mathbb{P}(X \leq \mathbb{E}[X]-1) = \mathbb{E}[X]-1 \end{aligned}$$

Absurd! □

Clearly our argument works for any X taking values on a countable set. A continuous analogue also exists.

We can use this to prove any graph has a large bipartite subgraph. The idea will be to 2-colour its vertices uniformly, using the fact a bipartite subgraph is a graph where we can two colour the vertices such that every edge has endpoints with distinct colours.

Proposition 1 (large bipartite subgraph)

Every graph with m edges has a bipartite subgraph with at least $m/2$ edges.

Proof. Let our graph be $G = G(V, E)$ and assign each vertex a colour from $\{1, 2\}$. Let $E' \subseteq E$ be the set of edges with distinctly coloured endpoints. Then, enumerating $E = \{e_1, \dots, e_m\}$ and letting A_i be the event edge e_i has distinctly coloured endpoints,

$$\mathbb{E}[|E'|] = \mathbb{E}\left[\sum_{i=1}^m \mathbf{1}(A_i)\right] = \sum_{i=1}^m \mathbb{P}(A_i) = \frac{1}{2}m$$

via the linearity of expectations and the fact $\mathbb{E}[\mathbf{1}(A)] = \mathbb{P}(A)$ for any event A , where $\mathbf{1}(\cdot)$ is the indicator function⁴. Hence, by our previous lemma

$$\mathbb{P}(|E'| \geq m/2) > 0$$

and we have our large bipartite subgraph. □

Linearity of expectations as a tool is extremely prevalent in probabilistic combinatorics, so much so we dedicate an entire section to its enumerable applications.

⁴That is, $\mathbf{1}(A) = 1$ where A is true and 0 elsewhere

2.1.1 Lower Ramsey Bounds

Definition 2 (Ramsey number)

Let $k, \ell \in \mathbb{N}$. Then the Ramsey number $R(k, \ell)$ is the smallest $n \in \mathbb{N}$ such that in every 2-colouring of K_n we have a monochromatic K_k of colour 1 or a monochromatic K_ℓ of colour 2.

For example, one can prove $R(3, 3) = 6$ with the Pidgeonhole principle. WLOG we can choose a vertex such that 3 of the edges from this vertex have colour 1. Now consider the triangle formed by the edges connecting these 3 endpoints. If any one of the edges in this triangle have colour 1 then we have a monochromatic triangle, if not then we also have a monochromatic triangle, giving $R(3, 3) \leq 6$. Then colour K_5 nicely to prove $R(3, 3) > 5$.

Remark 3 (a nice interpretation)

Proving $R(3, 3) \geq 6$ is a classical olympiad problem, often stated as “At a party with 6 people, there are 3 people that all know each other or 3 people that do not know one another.”

Ramsey first proved that these numbers exist.

Theorem 4 (Ramsey’s Theorem - 1929)

For all $k, \ell \in \mathbb{N}$, $R(k, \ell)$ exists and is finite.

Proof. Omitted. □

Estimating the Ramsey number is a fundamental problem in Ramsey theory, a vast subfield of graph theory. We will prove 3 quantitative lower bounds for the diagonal Ramsey number $R(k, k)$, each serving as an exhibition of an important technique in probabilistic combinatorics.

The first technique we use will be the so called union bound, often referred to by probability theorists as Boole’s inequality.

Lemma 5 (union bound / Boole’s inequality)

Let A_1, \dots, A_n be events on a probability space $(\omega, \mathcal{F}, \mathbb{P})$. Then,

$$\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) \leq \sum_{i=1}^n \mathbb{P}(A_i)$$

We’ll prove the $n = 2$ case, which gives the $n \in \mathbb{N}$ case immediately by induction.

Proof. Write $A_1 \cup A_2 = (A_1 \setminus A_2) \cup (A_2 \setminus A_1)$ as a disjoint union, noting $A_i \setminus A_j \subseteq A_i$ for $i \neq j$. Then,

$$\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1 \setminus A_2) + \mathbb{P}(A_2 \setminus A_1) \leq \mathbb{P}(A_1) + \mathbb{P}(A_2)$$

as required. □

Alternatively, the 2 variable inclusion-exclusion principle gives this immediately.

We may use this to bound unions from above, giving us a way to bound the probability of some “bad” event A_i from a set of “bad” events A_1, \dots, A_n occurring. Erdős used this methodology to obtain the following diagonal Ramsey lower bound, one of the first known applications of the probabilistic method in combinatorics.

Proposition 6 (Ramsey LB via union bound - Erdős 1947)

If $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$ then $R(k, k) > n$.

We will consider random 2-colourings of the edges of K_n , showing with positive probability if k satisfies $\binom{n}{k}2^{1-\binom{k}{2}} < 1$ then K_n has no monochromatic K_k .

Proof. Independently and uniformly 2-colour the edges of K_n at random. There are $\binom{n}{k}$ subgraphs isomorphic to K_k , and these are exactly the graphs obtained by taking the subgraph induced by keeping k edges from K_n . Let $m = \binom{n}{k}$ and A_1, \dots, A_m be the events these graphs are monochromatic. Then,

$$\mathbb{P}\left(\bigcup_{i=1}^m A_i\right) \leq \sum_{i=1}^m \mathbb{P}(A_i) = \binom{n}{k} \times (2 \times 2^{-\binom{k}{2}})$$

so that, via DeMorgan's laws,

$$\binom{n}{k}2^{1-\binom{k}{2}} < 1 \implies \mathbb{P}\left(\bigcap_{i=1}^m A_i^c\right) > 0$$

giving us a 2-colouring of the edges of K_n with no monochromatic K_k . \square

This bound doesn't, at first, tell us anything about the size of k relative to n . To deduce this fact we'll need to do some work with asymptotics.

Proposition 7 (quantitative Erdős lower bound)

For $k \in \mathbb{N}$,

$$R(k, k) > \left(\frac{1}{\sqrt{e}} + o(1)\right) k2^{k/2}$$

We will work with Stirling's approximation.

Proof. I can't get it to bloody work :/ \square

The next approach we'll take is via the so called method of *alteration*. In this approach, generally, we start with a random construction, fix the undesirable features of said construction and prove our construction still has an interesting size. This approach is so important we'll dedicate a whole section to it.

In this application, as before, we'll take a 2-colouring of the edges of K_n independently and uniformly. This time though, instead of just analysing these colourings, we'll turn them into graphs K_X that certainly contain no monochromatic K_k and analyse the size of X .

Proposition 8 (Ramsey LB via alteration - unknown)

For all $k, n \in \mathbb{N}$, $R(k, k) > n - \binom{n}{k}2^{1-\binom{k}{2}}$

Proof. 2-colour the edges of the clique K_n independently and uniformly at random. Iteratively delete a vertex from each monochromatic K_k . Let $m = \binom{n}{k}$ and let A_1, \dots, A_m be the events the i^{th} k -clique is monochromatic after the initial colouring (before the deletion of any vertices). Let X be the number of vertices we delete during the deletion process. Then,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^m \mathbf{1}(A_i)\right] = m\mathbb{P}(A_i) = \binom{n}{k}2^{1-\binom{k}{2}}$$

so, as we have $n - X$ vertices remaining after the alteration, we have

$$\mathbb{P}\left(n - X \geq n - \binom{n}{k}2^{1-\binom{k}{2}}\right) > 0$$

i.e. a 2-colouring of the edges of the clique size $n - \binom{n}{k}2^{1-\binom{k}{2}}$ containing no monochromatic K_k . \square

Notice, again we used linearity of expectations here, as well as lemma 2.1. This will often be the case in alteration proofs.

As before, we can optimise n for fixed k to gain a quantitative bound.

Proposition 9 (quantitative Ramsey LB via alteration)

$$R(k, k) > \left(\frac{1}{e} + o(1) \right) k 2^{k/2}$$

Proof. □

Note this improves the Erdős bound by a factor of $\frac{1}{\sqrt{e}}$.

Now we introduce our third method, again, warranting its own section. The idea is that the so called “bad” events A_1, \dots, A_n we want to avoid are often only weakly dependent, and we have a great bound for the probability that none of the “bad” events occur in such a setup (courtesy of Lovász).

Lemma 10 (Lovász Local Lemma - random variable model)

Let X_1, \dots, X_n be independent random variables. Let $B_1, \dots, B_m \subset [n]$ and, for each $1 \leq i \leq m$, let A_i be an event depending only on the random variables $\{X_j, j \in B_i\}$.

If for each i , $B_i \cap B_j \neq \emptyset$ for at most d other B_j ’s and $\mathbb{P}(A_i) \leq 1/e(d+1)$ then with positive probability none of the A_i occur.

We will save the proof of this for our section dedicated to this lemma.

Now lets use this to obtain another improvement on our Ramsey lower bound. As before we’ll take random 2-colourings and consider the events A_1, \dots, A_m where clique i is monochromatic, but, observing the weak dependence between these events (and letting the colour of the edges be our random variables) this makes a perfect candidate for the Lovász Local Lemma.

Proposition 11 (Ramsey LB via LLL - Spencer 1977)

If $\left(\binom{k}{2} \binom{n}{k-2} + 1 \right) 2^{1-\binom{k}{2}} < \frac{1}{e}$ then $R(k, k) > n$.

Proof. 2-Colour the edges of K_n independently and uniformly at random, letting the colour of $1 \leq i \leq \binom{n}{2} =: N$ be the random variable X_i . Enumerate the k -cliques of K_n as G_1, \dots, G_M where $M := \binom{n}{k}$ and let $A_j : 1 \leq j \leq M$ be the event G_i is monochromatic. Let $B_j \subset [N] : 1 \leq j \leq M$ be the edges of G_i . Clearly A_i depends only on the random variables $\{X_j : j \in B_i\}$.

Now $B_i \cap B_j \neq \emptyset$ exactly when G_i and G_j share at least one edge. This is equivalent (edges have two endpoints and two vertices make an edge!) to G_i and G_j sharing at least two vertices. Fixing G_i , we may count the number of G_j satisfying this condition by counting the number of ways to choose 2 vertices from G_i (our shared vertices) and multiplying by the number of ways to choose the remaining $k-2$ vertices from the vertices of K_n (noting we may choose from all n , as B_i can share any number of vertices with B_j) giving us a candidate d of

$$d = \binom{k}{2} \binom{n}{k-2}$$

Now, as we’ve already seen, $\mathbb{P}(A_i) = 2^{1-\binom{k}{2}}$, so to apply the Lovász local lemma we need

$$\binom{k}{2} \binom{n}{k-2} 2^{1-\binom{k}{2}} < \frac{1}{e}$$

If this inequality in n and k holds, we have with positive probability a colouring of the edges of K_n with no monochromatic k -clique, and thus $R(k, k) > n$. □

We may optimise one final time for n in k to get a quantitative bound.

Proposition 12 (quantitative Ramsey LB via LLL)

$$R(k, k) > \left(\frac{\sqrt{2}}{e} + o(1) \right) k 2^{k/2}$$

Proof. □

This improves our alteration bound by a factor of $\sqrt{2}$, and is currently the best known lower bound for the diagonal Ramsey number. Philosophically, it is interesting to note that all three of these different techniques only improve our lower bound by a constant factor. With these methods, we always obtain a $O(k 2^{k/2})$ lower bound.

We are very far from knowing the exact diagonal Ramsey number. Recently... (Campos upper bound exponential dialogue via a kind of alteration).

2.1.2 Set Systems

A *set system* is simply a family \mathcal{F} of subsets of some space X . Often we take $X = [n]$. In *extremal set theory* we are usually concerned with finding the maximal/minimal size of a set system that has some desirable property.

The first type of set system we study are the so called *antichains*. These are families that are pairwise incomparable by containment, that is $\forall A \in \mathcal{F} \nexists B \in \mathcal{F}$ with $A \subseteq B$.

We will write $\binom{[n]}{k}$ for the family of k -element subsets of $[n]$.

Claim: The family $\binom{[n]}{k}$ is an antichain for any $0 \leq k \leq n$

Proof. Suppose there are k element subsets of $A, B \subseteq [n]$ with $A \subseteq B$. Then, as $|A| = k = |B|$ we have $A = B$ forced but, as our family is a set, its elements must be unique. Contradiction! □

The size of $\binom{[n]}{k}$ is $\binom{n}{k}$, which is maximised by taking $k = \lfloor n/2 \rfloor$. It turns out the family $\binom{[n]}{\lfloor n/2 \rfloor}$ is our biggest antichain in $[n]$.

Theorem 13 (Sperner's theorem - 1928)

If \mathcal{F} is an antichain on $[n]$, then $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$

Rather than proving this we will prove a stronger result independently discovered by several mathematicians, often referred to as the LYM inequality.

Theorem 14 (LYM inequality - 1954/1963/1965/1966)

If \mathcal{F} is an antichain on $[n]$ then

$$\sum_{A \in \mathcal{F}} \binom{n}{|A|}^{-1} \leq 1$$

Note this immediately uncovers Sperner's theorem as $\binom{n}{|A|} \leq \binom{n}{\lfloor n/2 \rfloor}$.

The idea will be to consider random permutations of n and the canonical chain generated from this permutation. If we have two events in said canonical chain we no longer have an anti-chain, so an easy disjoint probabilities sum arises.

Proof. Choose a permutation σ of $[n]$ uniformly at random and consider the canonical chain

$$\mathcal{S}(\sigma) = \{\emptyset, \{\sigma(1)\}, \{\sigma(1), \sigma(2)\}, \dots, \{\sigma(1), \dots, \sigma(n)\}\}$$

For subsets $A \subset [n]$ let E_A be the event $A \in \mathcal{S}(\sigma)$. Then we have $A \in \mathcal{S}(\sigma)$ if and only if the elements of A are exactly the first $|A|$ elements in the listed permutation $(\sigma(1), \dots, \sigma(|A|))$ up to reordering, leaving the remaining $n - |A|$ elements of $[n]$ to be the remaining elements $(\sigma(|A|+1), \dots, \sigma(n))$ up to reordering. Thus, as we have $|A|!(n - |A|!)$ such permutations, we have

$$\mathbb{P}(E_A) = \frac{|A|!(n - |A|)!}{n!} = \binom{n}{|A|}^{-1}$$

Now suppose \mathcal{F} is an antichain. Then clearly we have at most one $A \in \mathcal{F}$ with $A \in \mathcal{S}(\sigma)$, hence $\{E_A : A \in \mathcal{F}\}$ are disjoint events and we have

$$1 \geq \mathbb{P}\left(\bigcup_{A \in \mathcal{F}} E_A\right) = \sum_{A \in \mathcal{F}} \mathbb{P}(E_A) = \sum_{A \in \mathcal{F}} \binom{n}{|A|}^{-1}$$

□

Here our probabilistic tool was actually just an axiom of measure, namely the sum of the measure of disjoint subsets is equal the measure of the union of said subsets. One could quite easily rephrase this as a counting argument by using disjointness and the fact we have at most $n!$ permutations, then dividing through by $n!$ at the very end.

Exercises

2.2 Linearity of Expectation

Exercises

2.3 Alteration

Exercises

2.4 Lovas  Local Lemma

Exercises

3 Solutions

Here I list the exercise solutions in a random order. I will store which exercises have a solution in the README of this repository.

Solution 1.1.8. We observe that the table has rotational symmetry as the table is round, so that each $n!$ ways we could arrange the people in a line could sit in n possible ways giving $(n-1)!$ total seatings. \square

Solution 1.1.9. For the first part I give three solutions.

1. We write the set as a binary string. Map $\mu_A(i) = 1$ if $i \in A$ and 0 elsewhere in $[n]$ for each subset $A \subseteq [n]$. Then there as each μ_A can either have $i = 0$ or $i = 1$ for $i \in [n]$ we have 2^n possible choices for the μ_A which bijects with the subsets of $[n]$. \square
2. We induct. Clearly there are 2 subsets of $[1]$, \emptyset and $\{1\}$. Suppose there are 2^{n-1} subsets of $[n-1]$. Then for each subset of $[n]$, either $n \in [n]$ or $n \notin [n]$ so we have $2 \times 2^{n-1} = 2^n$ subsets of $[n]$ and the induction is complete. \square
3. For each $0 \leq k \leq n$ there are $\binom{n}{k}$ subsets of size k so by the binomial theorem there are $\sum_{k=0}^n \binom{n}{k} = 2^n$ such subjects. \square

For the second, two.

1. We instead biject from the odd subsets to the even. Consider the map $\mu(A)$ from the subsets of $[n-1]$ to the even sized subsets of $[n]$ that has $A \mapsto A \cup \{n\}$ iff $|A|$ odd and to itself otherwise. Then $\mu(A)$ has a clear inverse mapping $A \mapsto A \setminus \{n\}$ iff $|A|$ odd and to itself if $|A|$ even. Hence as μ bijects we have 2^{n-1} such subsets from the first part. \square
2. We have

$$\sum_{\substack{A \subseteq [n] \\ 2 \nmid |A|}} \binom{n}{|A|} - \sum_{\substack{A \subseteq [n] \\ 2 \nmid |A|}} \binom{n}{|A|} = \sum_{A \subseteq [n]} \binom{n}{|A|} (-1)^{|A|} = 0$$

from the binomial theorem and hence an equal number of odd and even sized subsets, giving $2^n/2 = 2^{n-1}$ odd/even subsets. \square

I'm sure more proofs of these facts exist. Try to come up with some! \square

Solution 1.1.20. Select the pairs sequentially, giving $(n \times (n-1)) \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$ possible ways of selecting the people, and $(n!)^2$ \square