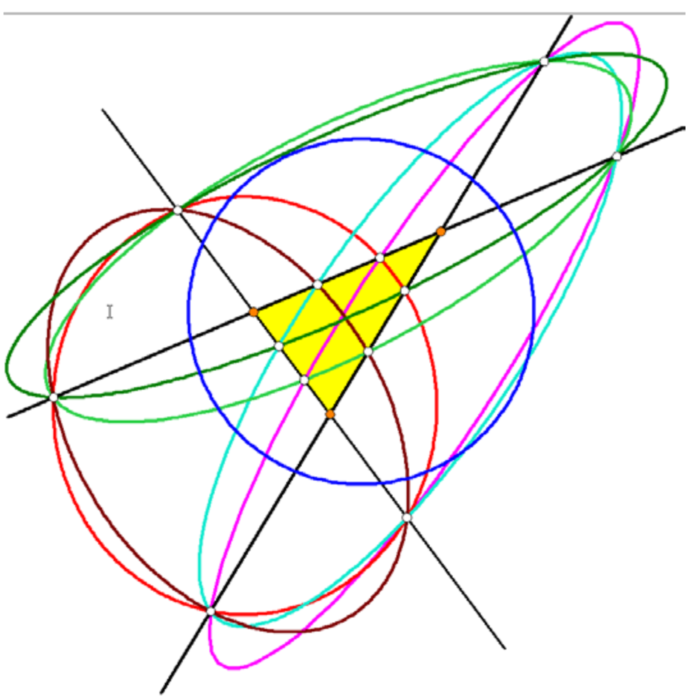
UNIVERSAL HYPERBOLIC GEOMETRY A

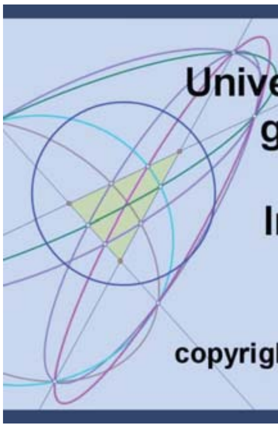
SCREENSHOTS OF VIDEOS UHG 1-32



N J WILDBERGER

wild egg

INDEX: UHG 1-32 VIDEOS



Universal hyperbolic geometry 0:

Introduction

Universal hyperbolic geometry 0: Introduction

This is a new series presenting a simpler, more logical, more general & more beautiful approach to hyperbolic geometry.

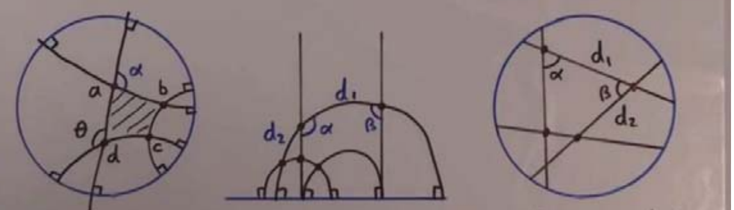
Norman Wildberger (Assoc. Prof. UNSW)

YouTube channel: njwildberger

- WildTrig
- MathFoundations
- Seminars
- WildLinAlg
- AlgTop
- MathHistory

①

Usual story: Euclid's parallel postulate... Bolyai... Lobachevsky... Gauss... Beltrami... Klein... Poincaré... Coxeter... Escher... Thurston... Einstein... distance... angle ②



Poincaré disk model — Upper half plane model — Beltrami Klein projective model

Usual formulas ③

$$ds^2 = \frac{4}{(1-r^2)^2}\,dx^2$$

$$d(a,b) = |\log(ab,pq)|$$

$$\cos c = -\cos a \cos b + \sin a \sin b \cosh C$$

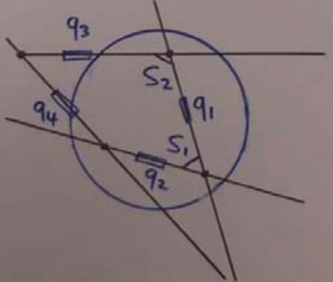
$$e^{-d} = \tan \frac{\Pi(d)}{2}$$

$$(ab,pq) = \frac{a_1b_1+a_2b_2-1-\sqrt{(a_1-b_1)^2+(a_2-b_2)^2-(a_1b_2-a_2b_1)^2}}{a_1b_1+a_2b_2-1+\sqrt{(a_1-b_1)^2+(a_2-b_2)^2-(a_1b_2-a_2b_1)^2}}$$

$$\sin\gamma = \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta+1} \qquad \frac{\sinh A}{\sin a} = \frac{\sinh B}{\sin b} = \frac{\sinh C}{\sin c}.$$

etc.

New story Apollonius... conics... polarity... cross ratio... projective geometry... Bolyai... Lobachevsky... Gauss... Beltrami... Einstein... Minkowski... rational trigonometry...



projective plane + distinguished circle

quadrance + spread

④

New formulas ⑤

$$q(a,b) = 1 - \frac{(a_1b_1+a_2b_2-1)^2}{(a_1^2+a_2^2-1)(b_1^2+b_2^2-1)}$$

$$\frac{S_1}{q_1} = \frac{S_2}{q_2} = \frac{S_3}{q_3}$$

$$q = \frac{S-1}{S} \qquad q_3 = q_1+q_2-q_1q_2$$

$$(q_1q_2S_3-(q_1+q_2+q_3)+2)^2 = 4(1-q_1)(1-q_2)(1-q_3)$$

$$(q_1+q_2+q_3)^2 = 2(q_1^2+q_2^2+q_3^2)+4q_1q_2q_3$$

$$S_2 = 4S(1-S)$$

Advantages of Universal Hyperbolic Geometry

- **simpler**: lines are straight, formulas are algebraic (polynomial), only high school algebra needed, no calculus or transcendental fns.
- **more logical**: no "real numbers", no "axioms", no circular/hyperbolic fns.
- **more general**: works over rational numbers, formulas extend outside disk, unifies hyperbolic + elliptic (spherical) geometries

⑥

and.... ⑦

it's much more beautiful !!

- connects with special relativity better
- computations are faster + more accurate
- hyperbolic triangle geometry becomes a new + vibrant subject
- fascinating combinatorial aspects
- richer group theory + hypergroups
- many new insights into algebraic geometry

This is a self-contained (almost!) course. ⑧

Intended audience: college math students, high school maths teachers, retired engineers, bright high school students, interested laypeople.

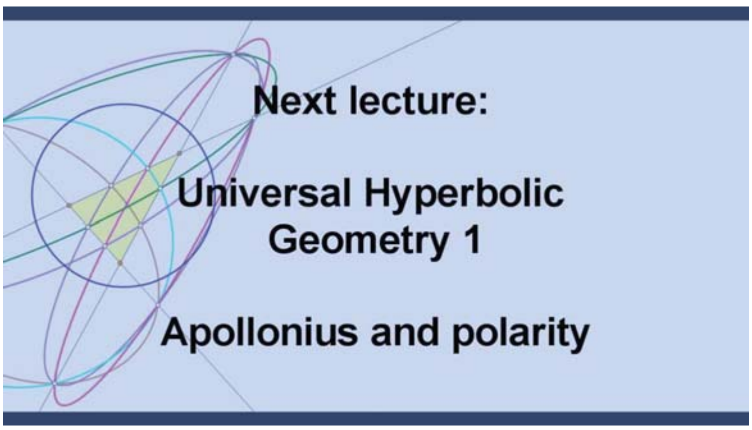
Requirements:
- internet connection
- straight-edge + compass
- algebraic competence:

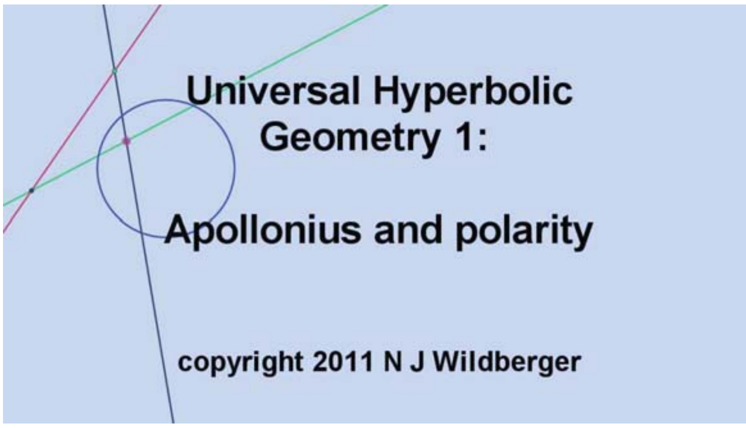
$$(q+q+r)^2=2(q^2+q^2+r^2)+4q^2r \Rightarrow r=0 \text{ or } r=4q(1-q)$$

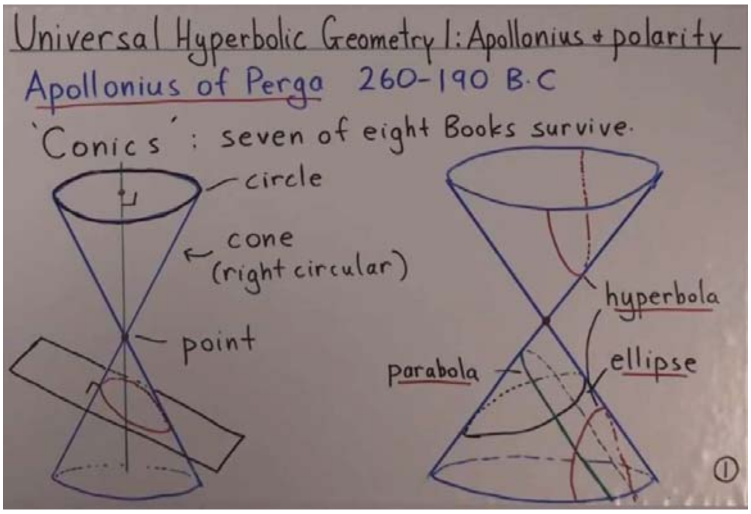
- willingness to work
- open mind (grad students + ...)

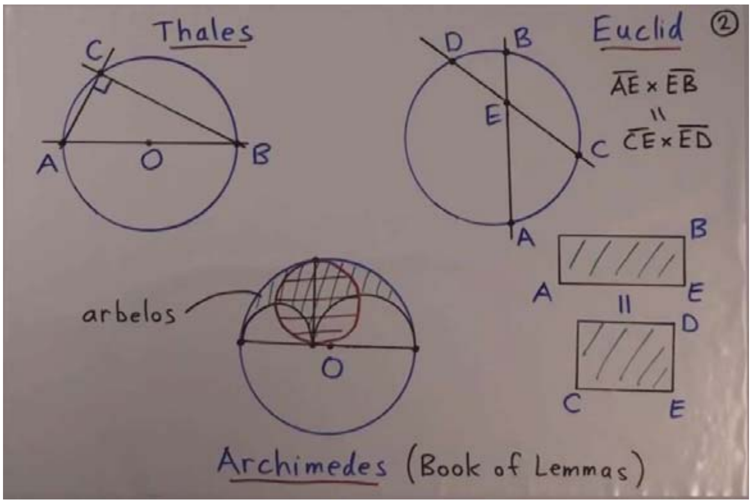
Recommended: a computer geometry program: ⑨
- Geometer's Sketchpad (GSP)
- Cabri
- Cinderella
- Geogebra
- C.a.R. [free!]

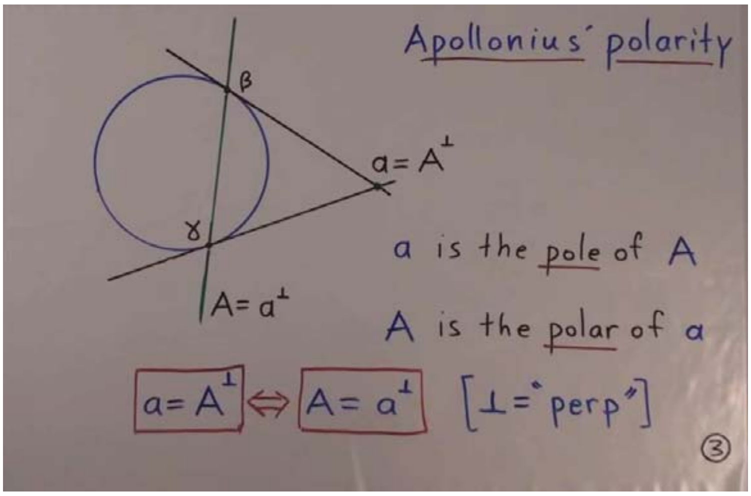
- lectures will be ~ 30-40 minutes
- eventually I hope to post notes somewhere...
- practise problems + research problems will accompany lectures. $\boxed{\text{Good luck!}}$

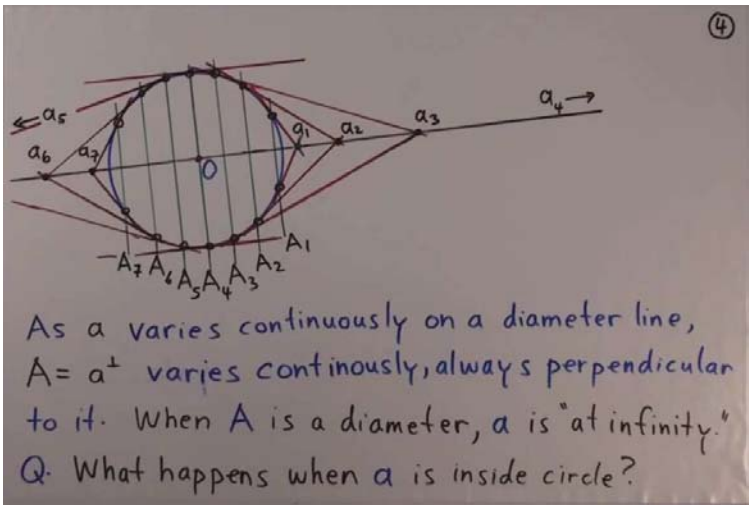


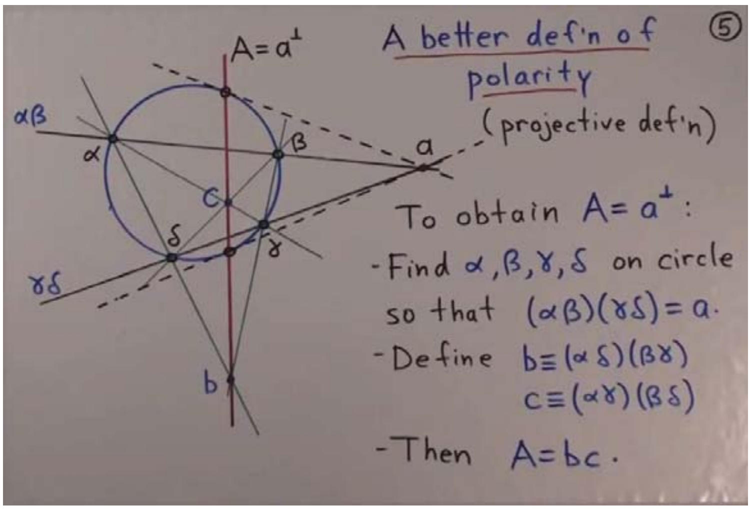

Universal Hyperbolic Geometry 1:
Apollonius and polarity
copyright 2011 N J Wildberger

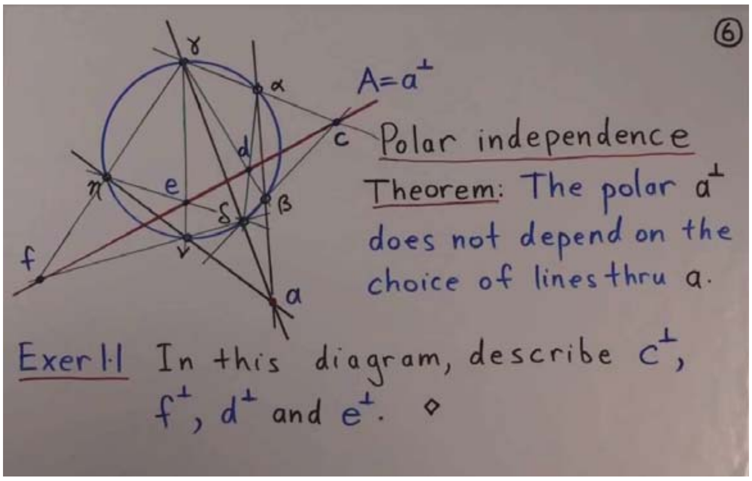

Universal Hyperbolic Geometry 1: Apollonius + polarity
Apollonius of Perga 260-190 B.C
'Conics': seven of eight Books survive.
circle
cone
(right circular)
point
hyperbola
parabola
ellipse
①

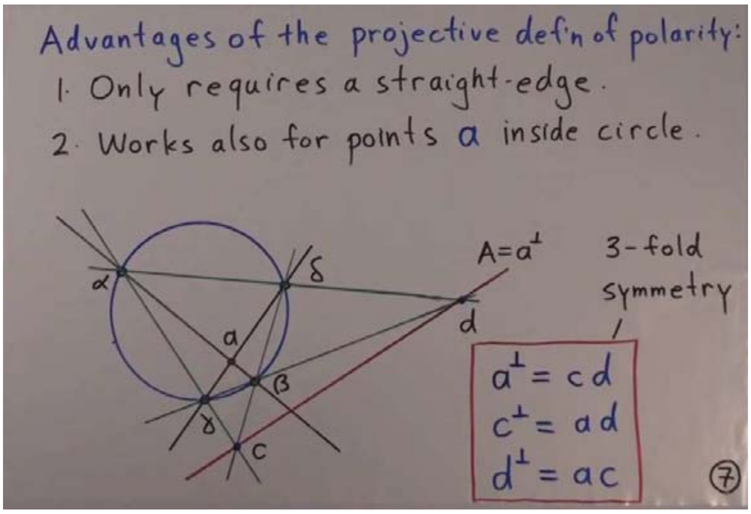

Thales
C
A
O
B
D
B
Euclid
②
E
C
A
$\overline{AE} \times \overline{EB}$
$\overline{CE} \times \overline{ED}$
arbelos
O
B
A
E
D
C
E
Archimedes (Book of Lemmas)

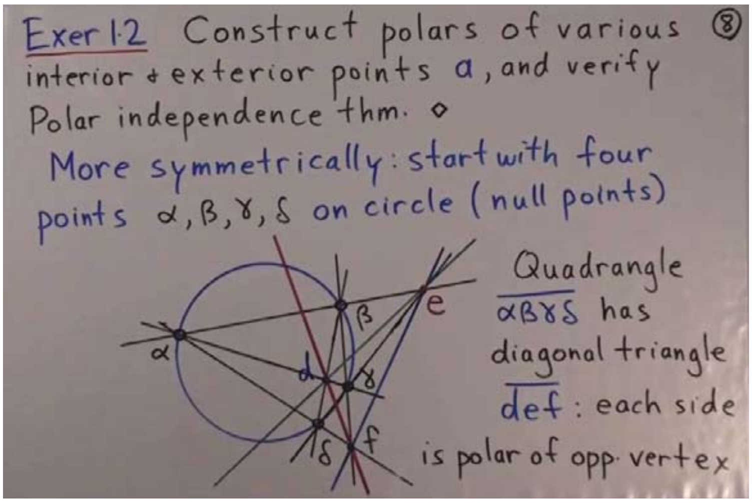

Apollonius' polarity
β
γ
$a = A^\perp$
$A = a^\perp$
a is the pole of A
A is the polar of a
$a = A^\perp \Leftrightarrow A = a^\perp$
[⊥ = "perp"]
③

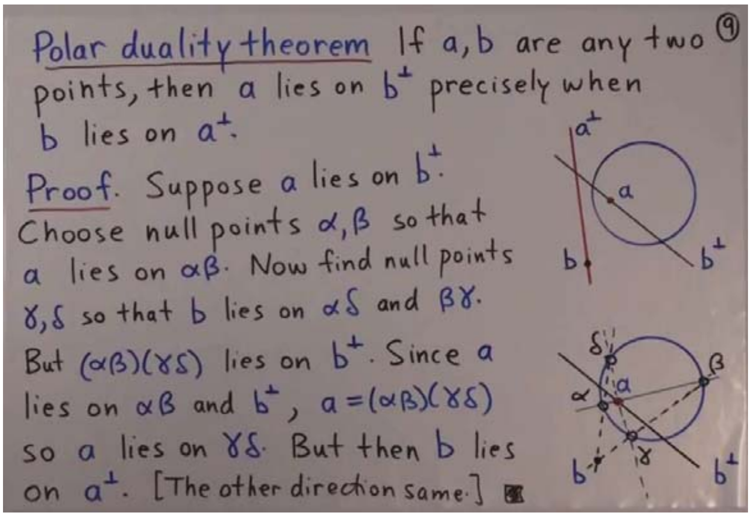

④
a_5
a_6
a_7
O
a_1
a_2
a_3
a_4
A_7 A_6 A_5 A_4 A_3 A_2 A_1
As a varies continuously on a diameter line,
$A = a^\perp$ varies continously, always perpendicular
to it. When A is a diameter, a is "at infinity."
Q. What happens when a is inside circle?

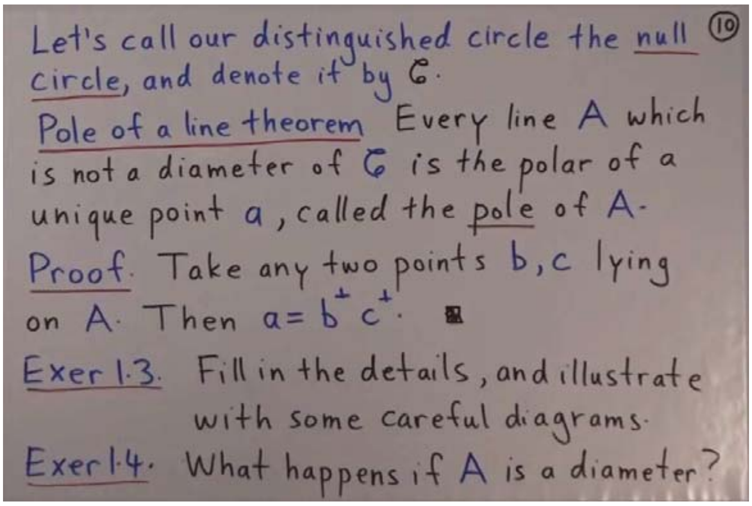

$A = a^\perp$
A better def'n of
polarity
(projective def'n)
⑤
αβ
α
β
a
c
δ
γ
γδ
b
To obtain $A = a^\perp$:
- Find α, β, γ, δ on circle
so that $(\alpha\beta)(\gamma\delta) = a$.
- Define $b \equiv (\alpha\delta)(\beta\gamma)$
$c \equiv (\alpha\gamma)(\beta\delta)$
- Then $A = bc$.

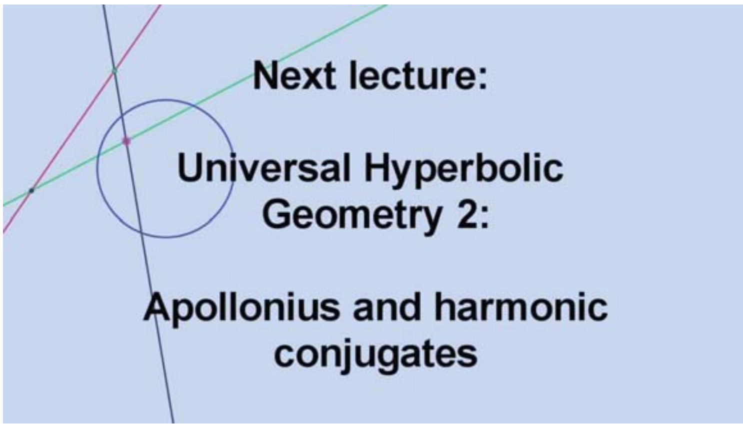

6
A=a⊥
Polar independence
Theorem: The polar a⊥ does not depend on the choice of lines thru a.
Exer 1.1 In this diagram, describe c⊥, f⊥, d⊥ and e⊥. ◇

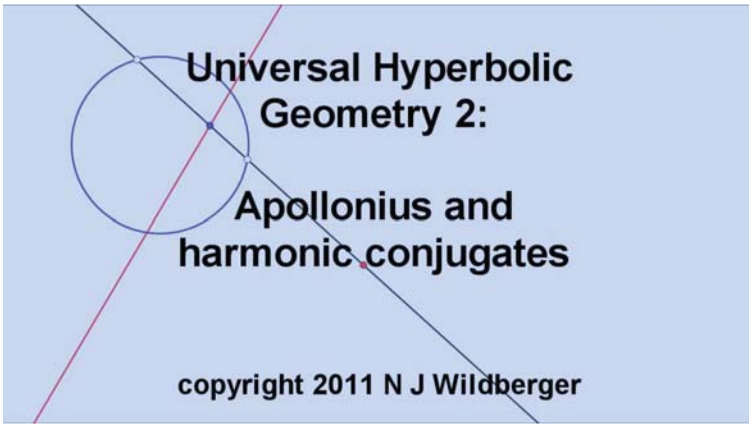

Advantages of the projective def'n of polarity:
1. Only requires a straight-edge.
2. Works also for points a inside circle.
A=a⊥
3-fold symmetry
a⊥ = cd
c⊥ = ad
d⊥ = ac
7

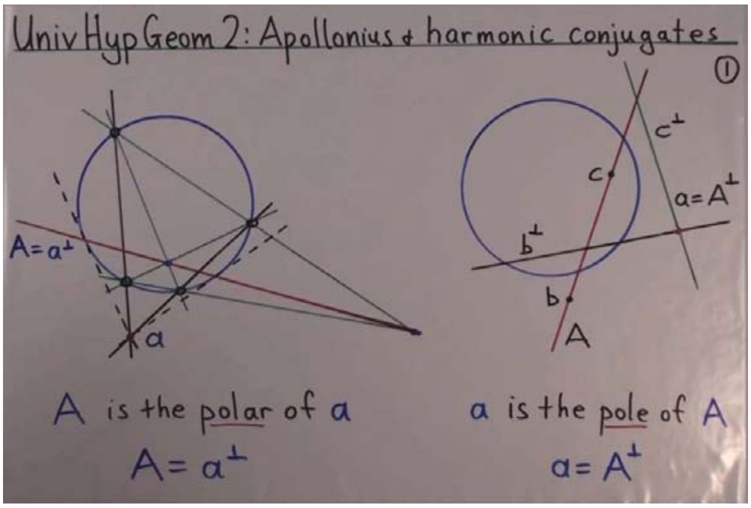

Exer 1.2 Construct polars of various interior + exterior points a, and verify Polar independence thm. ◇
More symmetrically: start with four points α, β, γ, δ on circle (null points)
Quadrangle αβγδ has diagonal triangle def: each side is polar of opp. vertex
8

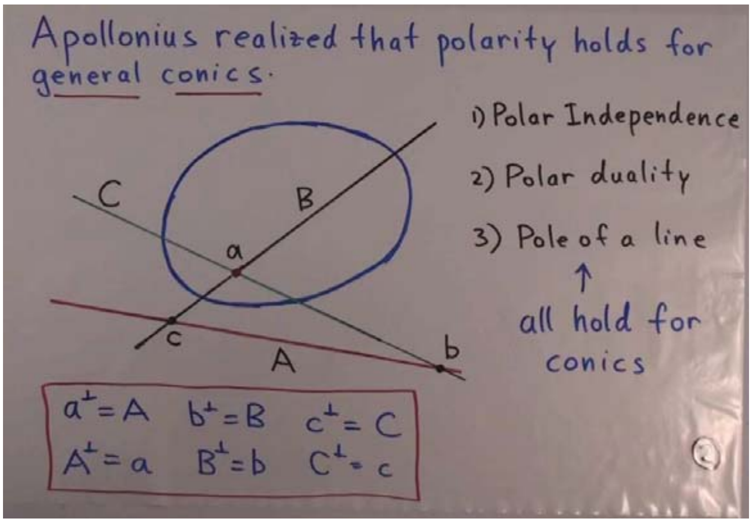

Polar duality theorem If a, b are any two points, then a lies on b⊥ precisely when b lies on a⊥.
Proof. Suppose a lies on b⊥. Choose null points α, β so that a lies on αβ. Now find null points γ, δ so that b lies on αδ and βγ.
But (αβ)(γδ) lies on b⊥. Since a lies on αβ and b⊥, a = (αβ)(γδ) so a lies on γδ. But then b lies on a⊥. [The other direction same.] ■
9

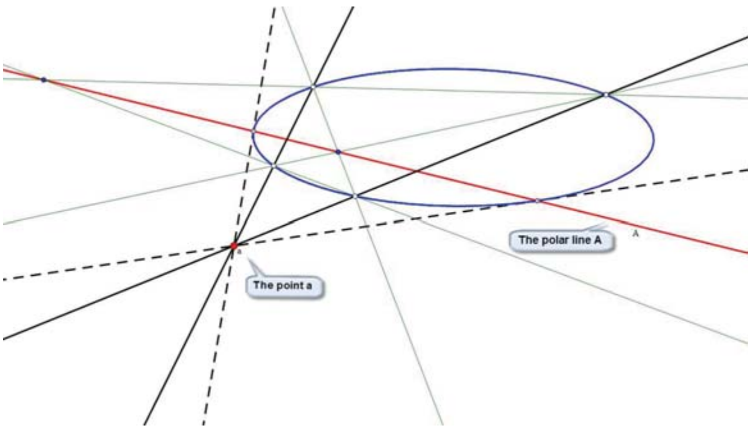

Let's call our distinguished circle the null circle, and denote it by C.
Pole of a line theorem Every line A which is not a diameter of C is the polar of a unique point a, called the pole of A.
Proof. Take any two points b, c lying on A. Then a = b⊥c⊥. ■
Exer 1.3. Fill in the details, and illustrate with some careful diagrams.
Exer 1.4. What happens if A is a diameter?
10

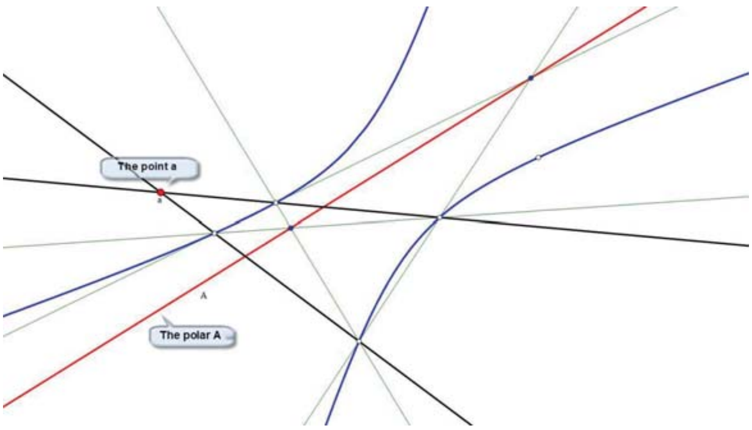

Next lecture:
Universal Hyperbolic Geometry 2:
Apollonius and harmonic conjugates

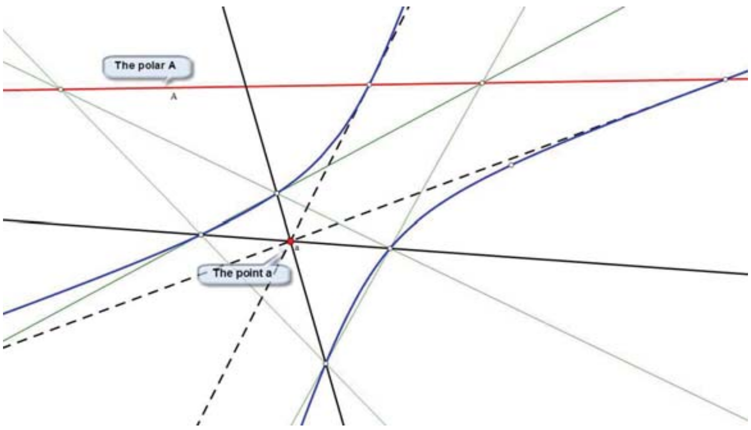

Universal Hyperbolic Geometry 2:
Apollonius and harmonic conjugates
copyright 2011 N J Wildberger


Univ Hyp Geom 2: Apollonius & harmonic conjugates
1
A=a⊥
a
c⊥
c
a=A⊥
b⊥
b
A
A is the polar of a
A = a⊥
a is the pole of A
a = A⊥

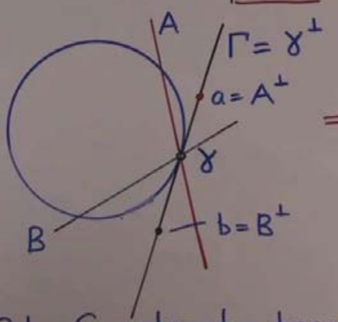

Apollonius realized that polarity holds for general conics.
C
B
a
c
A
b
1) Polar Independence
2) Polar duality
3) Pole of a line
all hold for conics
a⊥=A b⊥=B c⊥=C
A⊥=a B⊥=b C⊥=c
2


The polar line A
A
The point a


The point a
A
The polar A


The polar A
A
The point a

Q. How to find the polar of a null point γ?



A limiting argument $\Rightarrow$ should be tangent to $\mathcal{C}$ at γ

$\gamma^{\perp} = \Gamma$

Exer 2.1 Construct a tangent to a null point using this technique. ◇

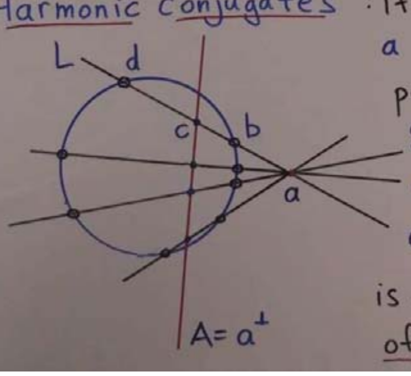
③

---

④

Exer 2.2* How to construct the center of $\mathcal{C}$ using only a straight-edge? ◇

Harmonic conjugates: If a line L through a meets $\mathcal{C}$ at two points $\beta = b$ and $\delta = d$, and $A = a^{\perp}$ at c, then a, b, c, d is a harmonic range of points



The notion of a harmonic range belongs to affine geometry, and (remarkably) even to projective geometry. ⑤

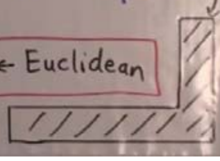
Projective geometry: straight edge constructions

Affine geometry: straight edge + parallelism

Euclidean geometry: straight edge + compass.

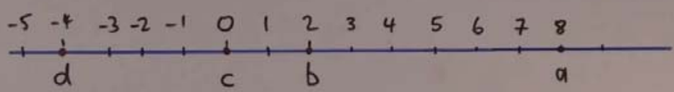
Carpenter's geometry: straight edge + carpenters square

$\boxed{\text{Projective} \leftarrow \text{Affine} \leftarrow \text{Carpenters} \leftarrow \text{Euclidean}}$



---

⑥



With affine geometry, a scale may be constructed on any line [but unit is variable]

$$\frac{\vec{ab}}{\vec{ad}} = \frac{-6}{-12} = \frac{1}{2} \qquad \frac{\vec{cb}}{\vec{cd}} = \frac{2}{-4} = \frac{-1}{2}$$

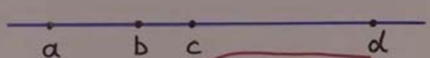
same ratios, one +, one −.

We say a and c are harmonic conjugates with respect to b and d; or

a, b, c, d is a harmonic range

---

⑦



Harmonic condition: $\boxed{\dfrac{\vec{ab}}{\vec{ad}} = -\dfrac{\vec{cb}}{\vec{cd}}}$ is independent of choice of scale

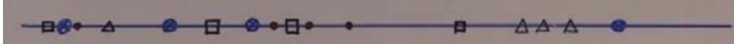
It can be rewritten as

$$\boxed{\frac{\vec{ba}}{\vec{bc}} = -\frac{\vec{da}}{\vec{dc}}}$$

so a, c are harmonic conjugates w.r.t. b, d

$\Leftrightarrow$ b, d are harmonic conjugates w.r.t. a, c.

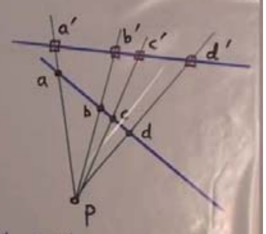
Examples of harmonic ranges ⑧



Exer 2.3 If a, b, c, d have respective co-ods 0, x, 1, y, then what is the relation between x & y? ◇

Harmonic ranges theorem

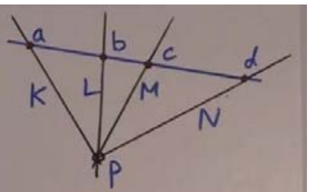
The image of a harmonic range under a projection is another harmonic range.



[i.e. a, b, c, d harmonic range $\Rightarrow$ a′, b′, c′, d′ harmonic range

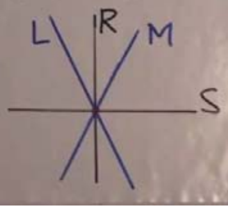
Harmonic pencils

If a,b,c,d are a harmonic range, then K,L,M,N are a harmonic pencil.



Harmonic bisectors theorem

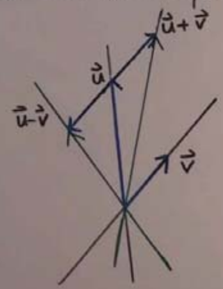
If R, S are bisectors of the vertex formed by intersecting lines L, M, then L, R, M, S is a harmonic pencil.



9

Harmonic vector combinations theorem

If $\vec{u}$ and $\vec{v}$ are linearly independent vectors, then the lines spanned by $\vec{u}$, $\vec{v}$ are harmonic conjugates wrt the lines spanned by $\vec{u}-\vec{v}$, $\vec{u}+\vec{v}$.

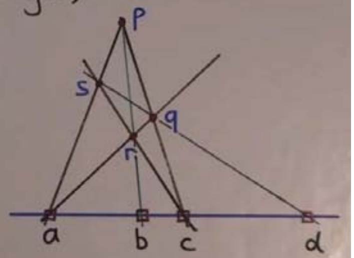


10

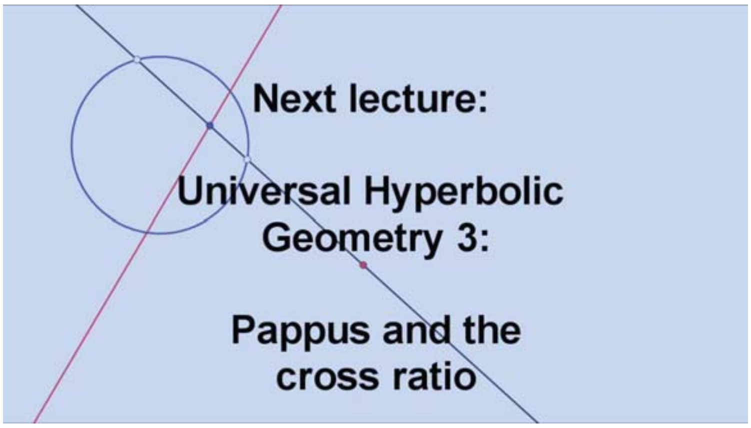
Harmonic quadrangle theorem

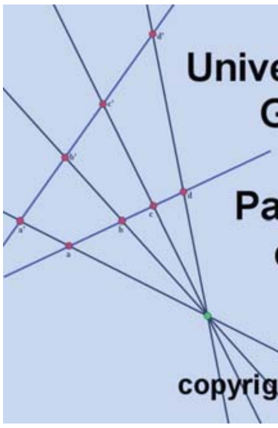
11

If $\overline{pqrs}$ is a quadrangle, then b, d are harmonic conjugates wrt a, c.



Exer 2·4 Verify each of the last four theorems in special cases. ◇





Universal Hyperbolic Geometry 3:

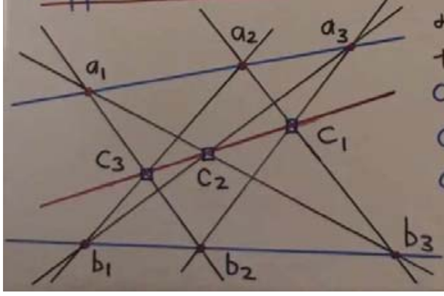
Pappus and the cross ratio

UniHypGeom3: Pappus + the cross ratio ①

Pappus of Alexandria (300 A.D.)

'Collection': eight books (seven survive)

Pappus' theorem



If a_1, a_2, a_3 collinear + b_1, b_2, b_3 collinear then

$c_1 \equiv (a_2 b_3)(a_3 b_2)$

$c_2 \equiv (a_1 b_3)(a_3 b_1)$

$c_3 \equiv (a_1 b_2)(a_2 b_1)$

are collinear.

---

②

Exer 3.1 Verify Pappus' theorem for special cases. What happens if $a_2 b_3$ and $a_3 b_2$ are parallel? ◇

Cross ratio

a   c   b   d

a, c, b, d is a harmonic range* $\Leftrightarrow$ $\boxed{\dfrac{\vec{ac}}{\vec{ad}} = -\dfrac{\vec{bc}}{\vec{bd}}}$ $\Leftrightarrow$ $\boxed{\dfrac{\vec{ac}}{\vec{ad}} \Big/ \dfrac{\vec{bc}}{\vec{bd}} = -1}$

$\Leftrightarrow$ $\boxed{R(a,b:c,d) = -1}$

* Note: different order from WT39

---

③

c   a   d   b

Def: If a, b, c, d are four collinear points,

$$R(a,b:c,d) \equiv \frac{\vec{ac}}{\vec{ad}} \Big/ \frac{\vec{bc}}{\vec{bd}}$$

is a cross ratio of those points.

In terms of co-ordinates:

$$\boxed{R(a,b:c,d) = \frac{a-c}{a-d} \Big/ \frac{b-c}{b-d}}$$

Four points determine 24 cross ratios, but only 6 are generally different.

---

④

Ex. b   a   c   d

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9$

i) $R(a,b:c,d) = \dfrac{a-c}{a-d} \Big/ \dfrac{b-c}{b-d} = \dfrac{1-3}{1-5} \Big/ \dfrac{-2-3}{-2-5}$
$= \dfrac{-2}{-4} \Big/ \dfrac{-5}{-7} = \dfrac{7}{10}$

ii) $R(b,d:c,a) = \dfrac{b-c}{b-a} \Big/ \dfrac{d-c}{d-a} = \dfrac{-2-3}{-2-1} \Big/ \dfrac{5-3}{5-1}$
$= \dfrac{-5}{-3} \Big/ \dfrac{2}{4} = \dfrac{10}{3}$

iii) $R(d,a:b,c) = \dfrac{d-b}{d-c} \Big/ \dfrac{a-b}{a-c} = \dfrac{5-(-2)}{5-3} \Big/ \dfrac{1-(-2)}{1-3} = \dfrac{7}{2} \Big/ \dfrac{3}{-2} = \dfrac{-7}{3}$

---

⑤

Cross ratio transformation theorem

If $R(a,b:c,d) = \lambda$ then

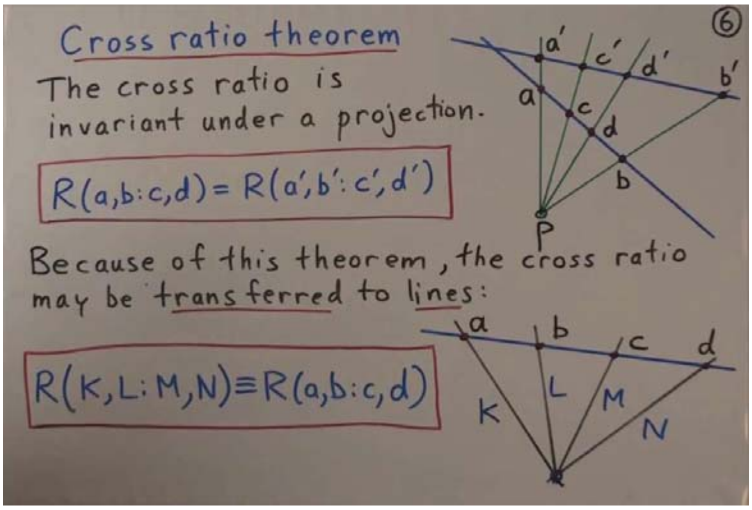
i) $R(b,a:c,d) = R(a,b:d,c) = \dfrac{1}{\lambda}$

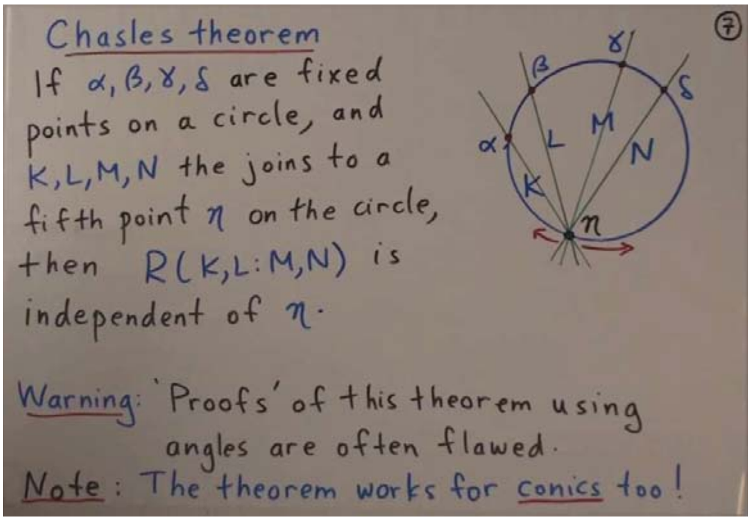
ii) $R(a,c:b,d) = R(d,b:c,a) = 1-\lambda$.

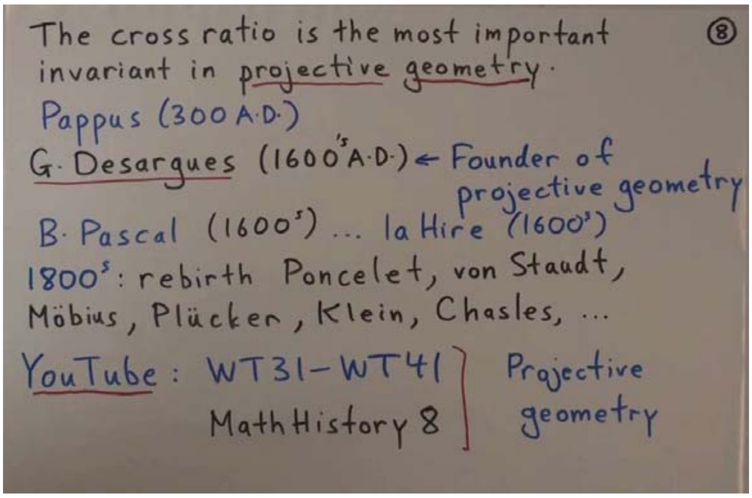
Proof. An exercise using $\lambda = \dfrac{a-c}{a-d} \Big/ \dfrac{b-c}{b-d}$. ■

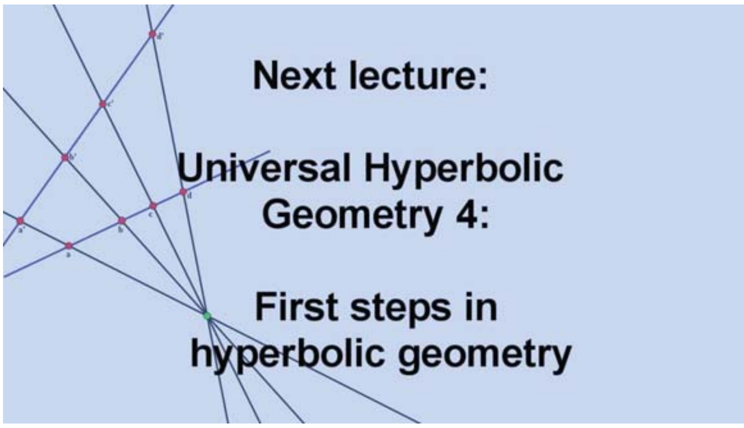
Corollary. Any permutation of a, b, c, d gives a value for the cross ratio one of:

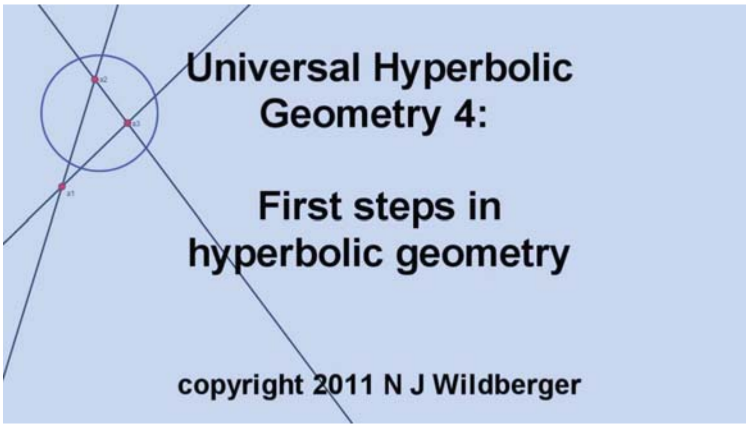
$$\lambda, \quad \frac{1}{\lambda}, \quad 1-\lambda, \quad \frac{1}{1-\lambda}, \quad \frac{\lambda-1}{\lambda}, \quad \frac{\lambda}{\lambda-1}.$$

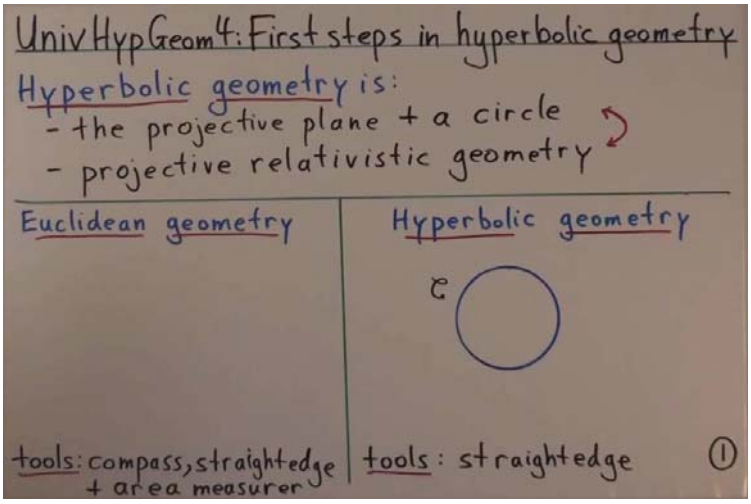

Cross ratio theorem
6
The cross ratio is invariant under a projection.
R(a,b:c,d) = R(a',b':c',d')
a' c' d' b'
a c d b
P
Because of this theorem, the cross ratio may be transferred to lines:
R(K,L:M,N) ≡ R(a,b:c,d)
a b c d
K L M N

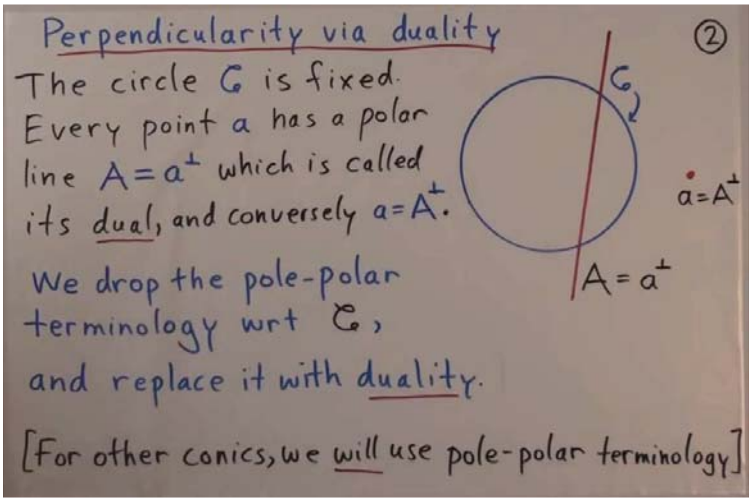

Chasles theorem
7
If α, β, γ, δ are fixed points on a circle, and K,L,M,N the joins to a fifth point η on the circle, then R(K,L:M,N) is independent of η.
α β γ δ
K L M N
η
Warning: 'Proofs' of this theorem using angles are often flawed.
Note: The theorem works for conics too!

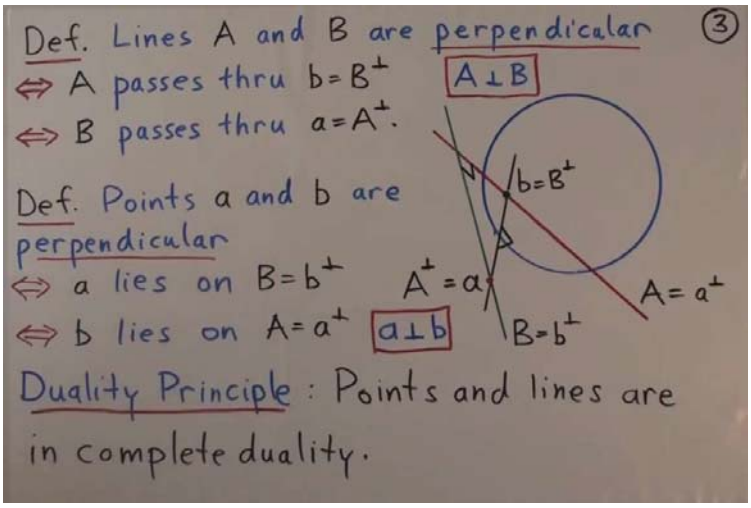

8
The cross ratio is the most important invariant in projective geometry.
Pappus (300 A.D.)
G. Desargues (1600's A.D.) ← Founder of projective geometry
B. Pascal (1600's) ... la Hire (1600's)
1800's: rebirth Poncelet, von Staudt,
Möbius, Plücker, Klein, Chasles, ...
YouTube: WT31–WT41
MathHistory 8
Projective geometry

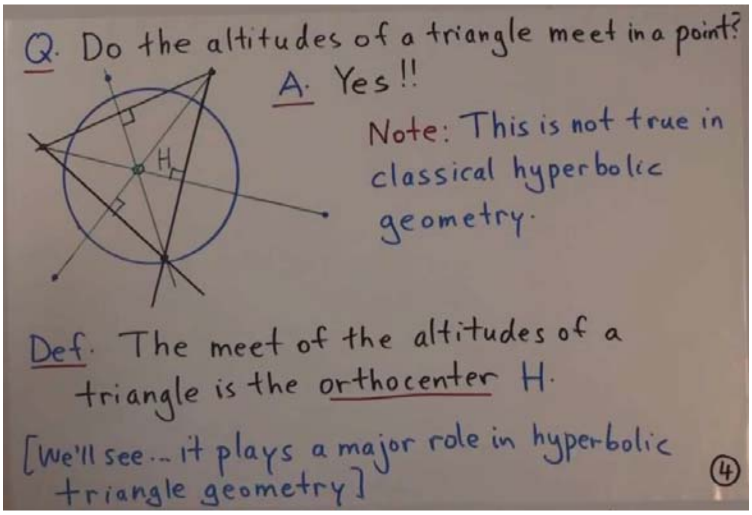

Next lecture:
Universal Hyperbolic Geometry 4:
First steps in hyperbolic geometry

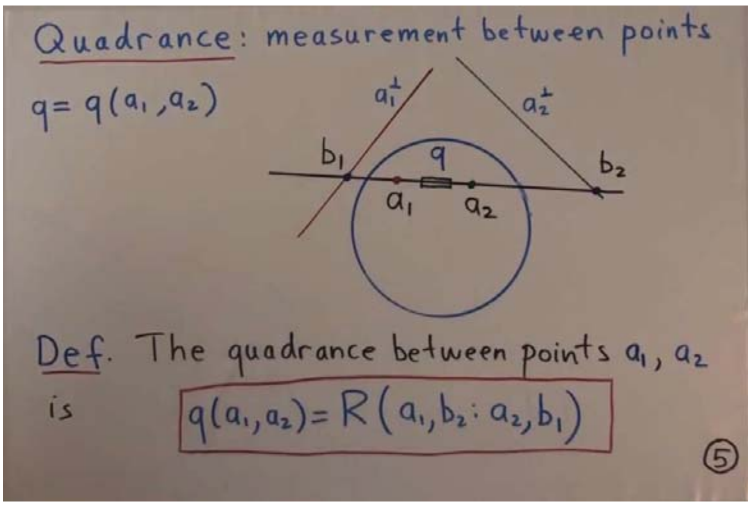

Universal Hyperbolic Geometry 4:
First steps in hyperbolic geometry
copyright 2011 N J Wildberger


UnivHypGeom4: First steps in hyperbolic geometry
Hyperbolic geometry is:
- the projective plane + a circle
- projective relativistic geometry
Euclidean geometry
Hyperbolic geometry
$\mathcal{C}$
tools: compass, straightedge + area measurer
tools: straightedge
1


Perpendicularity via duality
2
The circle $\mathcal{C}$ is fixed.
Every point a has a polar line $A=a^\perp$ which is called its dual, and conversely $a=A^\perp$.
$\mathcal{C}$
$a=A^\perp$
$A=a^\perp$
We drop the pole-polar terminology wrt $\mathcal{C}$, and replace it with duality.
[For other conics, we will use pole-polar terminology]


Def. Lines A and B are perpendicular
3
$\Leftrightarrow$ A passes thru $b=B^\perp$
$\boxed{A \perp B}$
$\Leftrightarrow$ B passes thru $a=A^\perp$.
Def. Points a and b are perpendicular
$\Leftrightarrow$ a lies on $B=b^\perp$
$\Leftrightarrow$ b lies on $A=a^\perp$
$\boxed{a \perp b}$
$b=B^\perp$
$A^\perp=a$
$A=a^\perp$
$B=b^\perp$
Duality Principle: Points and lines are in complete duality.


Q. Do the altitudes of a triangle meet in a point?
A. Yes!!
Note: This is not true in classical hyperbolic geometry.
H
Def. The meet of the altitudes of a triangle is the orthocenter H.
[We'll see... it plays a major role in hyperbolic triangle geometry]
4


Quadrance: measurement between points
$q=q(a_1,a_2)$
$a_1^\perp$
$a_2^\perp$
b_1
q
b_2
a_1
a_2
Def. The quadrance between points a_1, a_2 is
$\boxed{q(a_1,a_2)=R(a_1,b_2 : a_2,b_1)}$
5

(6)

Exer 4.1 i) Show that $q(a_1,a_1)=0$.

ii) Show that if $a_1 \perp a_2$ then $q(a_1,a_2)=1$.

iii) Show that if $q(a_1,a_2)=0$ then $a_1=a_2$ or $b_1=b_2$. Give an example of the latter occurring ◇

Note: In the Beltrami-Klein model, if a_1 and a_2 are inside $\mathcal{C}$, then

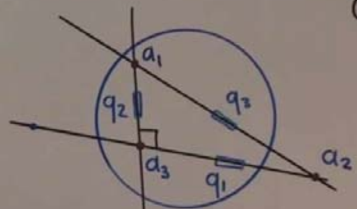
$$\boxed{q(a_1,a_2) = -\sinh^2(d(a_1,a_2))}$$

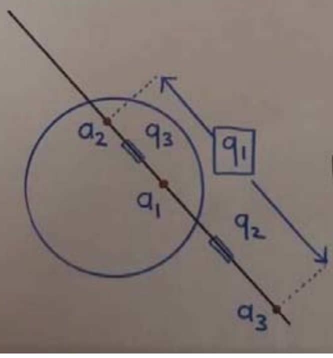
where $d(a_1,a_2)$ = 'distance' between a_1, a_2.

(7)

Pythagoras' theorem

If $a_1a_3 \perp a_2a_3$ then $\boxed{q_3 = q_1+q_2-q_1q_2}$.





Triple quad formula

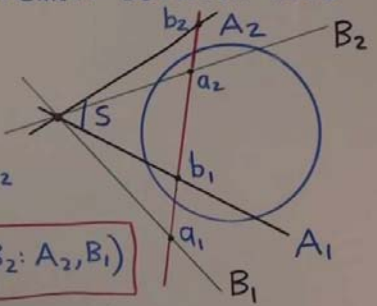
If a_1, a_2, a_3 collinear, then

$$\boxed{(q_1+q_2+q_3)^2 = 2(q_1^2+q_2^2+q_3^2)+4q_1q_2q_3}$$

(8)

Spread: measurement between lines

$S = S(A_1,A_2)$



Def The spread between lines A_1, A_2 is

$$\boxed{S(A_1,A_2) = R(A_1,B_2 : A_2,B_1)}$$

Quadrance spread duality theorem

$a_1 = A_1^{\perp} \quad a_2 = A_2^{\perp} \Rightarrow \boxed{q(a_1,a_2) = S(A_1,A_2)}$

(9)

Note: In the Beltrami-Klein model, if A_1 and A_2 are meeting inside $\mathcal{C}$, then

$$S(A_1,A_2) = \sin^2(\theta(A_1,A_2))$$

where $\theta(A_1,A_2)$ is the 'angle' between A_1, A_2.

Note: In Rational trigonometry $0 \le s \le 1$, but here S can take on any value (even ∞).

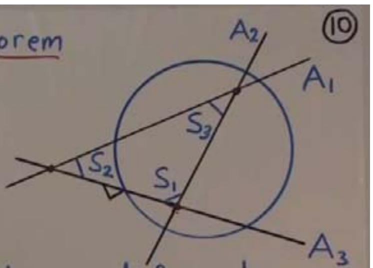
Note: In hyperbolic geometry, the quadrance q and spread S are dual [this is an extension of the Duality Principle.]

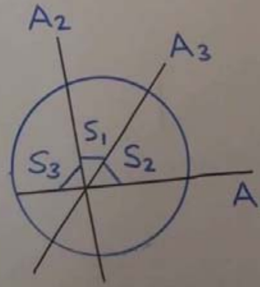
(10)

Pythagoras' dual theorem

If $A_1A_3 \perp A_2A_3$ then

$$\boxed{S_3 = S_1+S_2-S_1S_2}$$





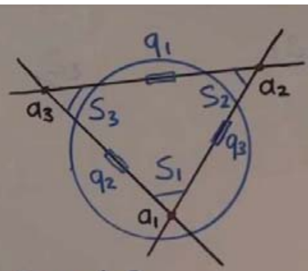
Triple spread formula

If A_1, A_2, A_3 concurrent, then

$$\boxed{(S_1+S_2+S_3)^2 = 2(S_1^2+S_2^2+S_3^2)+4S_1S_2S_3}$$

Spread law

$$\frac{S_1}{q_1} = \frac{S_2}{q_2} = \frac{S_3}{q_3}$$



Cross law

$$(q_1q_2S_3 - (q_1+q_2+q_3)+2)^2 = 4(1-q_1)(1-q_2)(1-q_3)$$

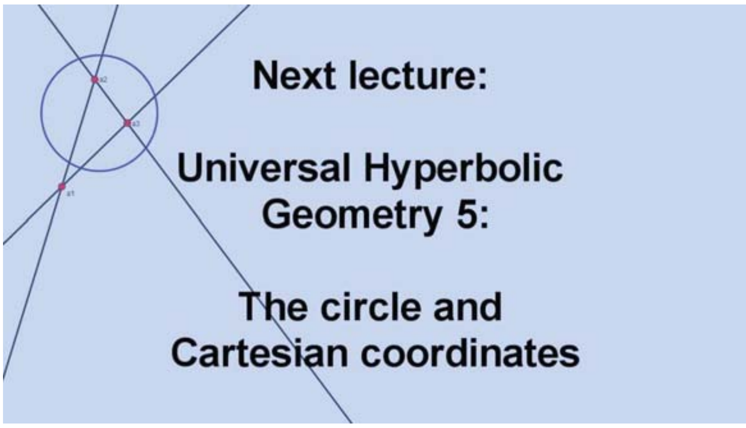
Cross dual law

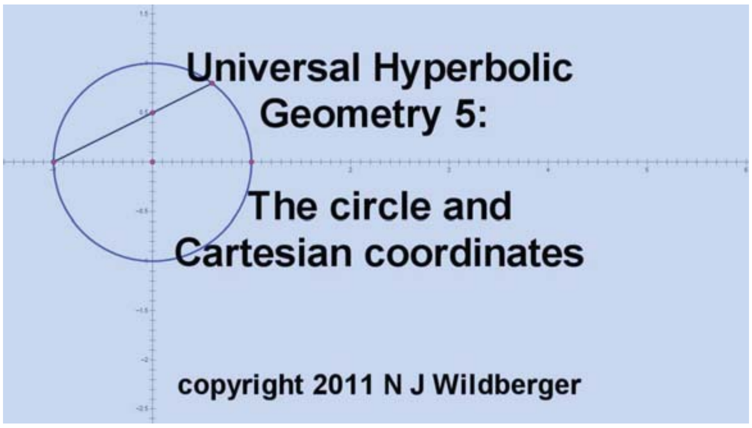
$$(S_1S_2q_3 - (S_1+S_2+S_3)+2)^2 = 4(1-S_1)(1-S_2)(1-S_3)$$

Exer. 4.2

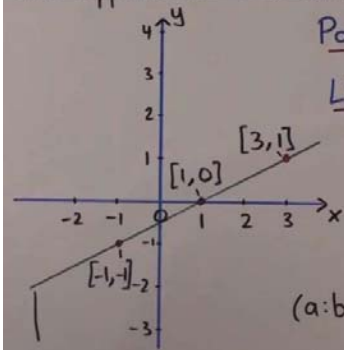
Verify all of these theorems!! [in special cases]

(11)


Next lecture:
Universal Hyperbolic Geometry 5:
The circle and Cartesian coordinates



UnivHypGeom5: The circle + Cartesian co-ordinates ①



Point: $A \equiv [x,y]$

Line: $\ell \equiv (a:b:c)$ representing the equation $\boxed{ax+by=c}$. *

$(a:b:c) = (\lambda a:\lambda b:\lambda c)$ for $\lambda \neq 0$

$(1:-2:1)$ representing $x-2y=1$. $(a:b:c)$ $=$

*Note: different convention than in RT: $\langle a:b:-c\rangle$

Line through two points theorem. For any two points $A_1 \equiv [x_1,y_1]$ and $A_2 \equiv [x_2,y_2]$, there is a unique line ℓ that passes through them:

$$\boxed{\ell = (y_1-y_2 : x_2-x_1 : x_2y_1-x_1y_2)}$$

Collinear points theorem. The points $A_1 \equiv [x_1,y_1]$ $A_2 \equiv [x_2,y_2]$ and $A_3 \equiv [x_3,y_3]$ are collinear precisely when

$$\boxed{x_1y_2-x_1y_3+x_2y_3-x_3y_2+x_3y_1-x_2y_1=0}$$

Exer 5.1 Write down proofs of these! ◇ ②

Determinants   det: matrix $\rightarrow$ number ③

1) $\det(a) \equiv a$ WT70 Det's in geom.

2) $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv ad-bc$ WLA4 Area + volume

3) $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei-afh+bfg-bdi+cdh-ceg$
$= a(ei-fh)-b(di-fg)+c(dh-eg)$
$= a(ei-fh)-d(bi-ch)+g(bf-ec)$

Exer 5.2 Show
$$\det\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix} = x_1y_2-x_1y_3+x_2y_3 - x_3y_2+x_3y_1-x_2y_1$$ ◇

Q. What number system are we using here? ④

A. The **rational numbers**:

$\frac{a}{b}$: a,b integers, $b \neq 0$

$\frac{a}{b} = \frac{c}{d} \iff ad-bc=0$

[But in fact works for any field, char $\neq 2$]

Operations: $+, \times, -, \div$

Note: The usual lax attitude to 'real numbers' is a big mistake. This theory is at least a million times more complicated!!

Concurrent lines theorem. If lines $\ell_1 \equiv (a_1:b_1:c_1)$ $\ell_2 \equiv (a_2:b_2:c_2)$ and $\ell_3 \equiv (a_3:b_3:c_3)$ are concurrent, then

$$\det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = a_1b_2c_3-a_1b_3c_2+a_2b_3c_1-a_3b_2c_1 + a_3b_1c_2-a_2b_1c_3 = 0.$$

Proof. If $[x,y]$ lies on all 3 lines, then the system

$$\boxed{\begin{aligned} a_1x+b_1y+c_1z&=0 \\ a_2x+b_2y+c_2z&=0 \\ a_3x+b_3y+c_3z&=0 \end{aligned}}$$

has the non-zero solution $[x,y,-1]$. ∎

⑤

Def The lines $\ell_1 \equiv (a_1 : b_1 : c_1)$ and $\ell_2 \equiv (a_2 : b_2 : c_2)$ are affinely parallel $\Leftrightarrow$ $a_1 b_2 - a_2 b_1 = 0$.

Point on two lines theorem If the lines ℓ_1 and ℓ_2 are not parallel, then there is a unique point A which lies on them both:

$$\boxed{A = \ell_1 \ell_2 = \left[\frac{b_2 c_1 - b_1 c_2}{a_1 b_2 - a_2 b_1}, \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1} \right]}.$$

Exer 5·3 Prove this by solving

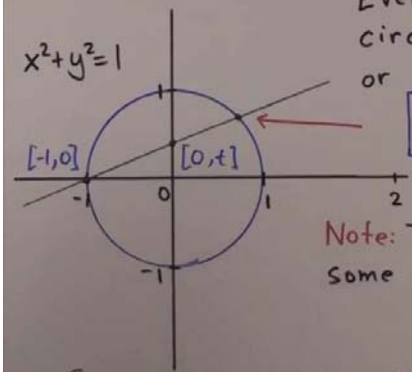
$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2 . \quad \diamond$$

⑥

Parametrization of unit circle theorem ⑦

Every point on the unit circle is either $[-1,0]$ or

$$\left[\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right] \equiv e(t).$$

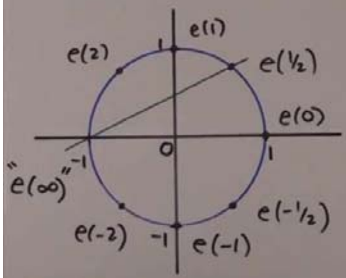


Note: The point $[-1,0]$ is in some sense "$e(\infty)$".

See: WildTrig 14: Rational parameters for circles.

⑧

$$e(t) = \left[\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right]$$



Exer 5·3 Show that from $e(t)$:

i) $e(-t)$ is reflection in x-axis

ii) $e(1/t)$ is reflection in y-axis

iii) $e\left(\frac{1+t}{1-t}\right)$ is rotation by a quarter turn $\diamond$

Meets of line + circle theorem A line ℓ meets the unit circle C in either 1, 2 or 0 points. ⑨

Proof. We need find solutions to

$$\boxed{\begin{array}{ll} ① & ax + by = c \\ ② & x^2 + y^2 = 1 \end{array}}$$

where at least one of a, b is non-zero.

If $a \neq 0$, substitute $x = \frac{c}{a} - \frac{b}{a} y$ into ②:

$$\left(\frac{c}{a} - \frac{b}{a}y\right)^2 + y^2 = 1 \quad \text{or} \quad y^2\left(1 + \frac{b^2}{a^2}\right) - \frac{2bc}{a^2} y + \frac{c^2}{a^2} - 1 = 0$$

or $y^2(a^2+b^2) - 2bcy + c^2 - a^2 = 0$. This has 1, 2 or 0 solutions depending on $\boxed{\Delta = (2bc)^2 - 4(a^2+b^2)(c^2-a^2) = 4a^2(a^2+b^2-c^2)}$ being 0, square, or non-square resp. ∎

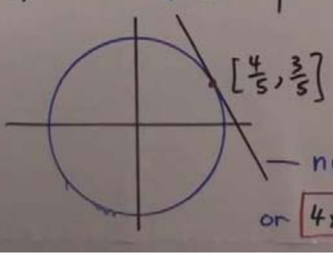
Exer 5·4 Show that the same conclusion holds in case $b \neq 0$. $\diamond$ ⑩

Corollary The line $ax + by = c$

i) is tangent to C $\Leftrightarrow$ $a^2 + b^2 = c^2$

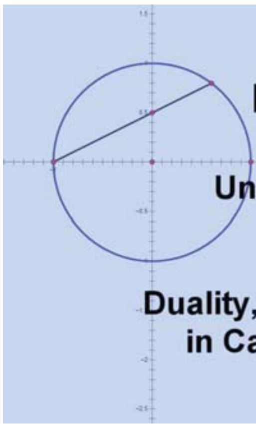
ii) meets C at two points $\Leftrightarrow$ $a^2+b^2-c^2$ is a non-zero square.

iii) meets C at no points $\Leftrightarrow$ $a^2+b^2-c^2$ is a non-square.



— null line $\frac{4}{5}x + \frac{3}{5}y = 1$

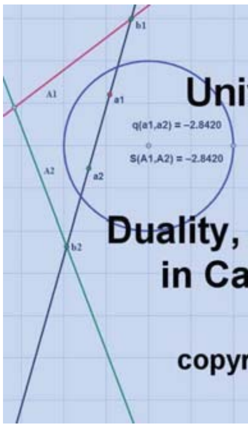
or $\boxed{4x + 3y = 5}$



Next lecture:

Universal Hyperbolic Geometry 6:

Duality, quadrance and spread in Cartesian coordinates



Universal Hyperbolic Geometry 6:

Duality, quadrance and spread in Cartesian coordinates

UnivHypGeom6: Duality, quadrance + spread in Cartesian co-ordinates

①

Duality in co-ods thm. If $a \equiv [x_0, y_0]$ then

$$\boxed{a^\perp = (x_0 : y_0 : 1)}.$$

Quadrance in co-ods thm. If $a_1 \equiv [x_1, y_1]$ + $a_2 \equiv [x_2, y_2]$ then

$$\boxed{q(a_1,a_2) = 1 - \frac{(x_1x_2+y_1y_2-1)^2}{(x_1^2+y_1^2-1)(x_2^2+y_2^2-1)}}.$$

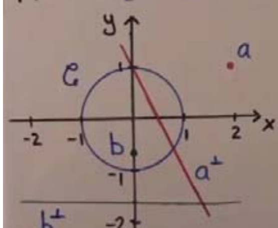
Spread in co-ods thm If $L_1 \equiv (l_1 : m_1 : n_1)$ + $L_2 \equiv (l_2 : m_2 : n_2)$ then

$$\boxed{S(L_1,L_2) = 1 - \frac{(l_1l_2+m_1m_2-n_1n_2)^2}{(l_1^2+m_1^2-n_1^2)(l_2^2+m_2^2-n_2^2)}}.$$

Duality in co-ordinates theorem

②

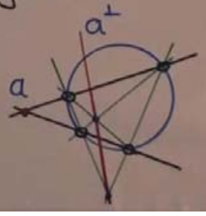
If $a \equiv [x_0, y_0]$ then $a^\perp = (x_0 : y_0 : 1)$.



Ex. $a \equiv [2,1]$ $\quad a^\perp = (2:1:1)$ with eq'n $2x + y = 1$.

Ex. $b \equiv [0, -\frac{2}{3}]$ $\quad b^\perp = (0 : -\frac{2}{3} : 1)$ with eq'n $-2y = 3$.

The main challenge is to show that $a^\perp$ is actually well-defined.



Line through two null points thm. The line L through $e(t_1)$ and $e(t_2)$ is $\boxed{(1-t_1t_2 : t_1+t_2 : 1+t_1t_2)}$.

Proof. Recall $e(t_1) = \left(\frac{1-t_1^2}{1+t_1^2}, \frac{2t_1}{1+t_1^2}\right)$ + $e(t_2) = \left(\frac{1-t_2^2}{1+t_2^2}, \frac{2t_2}{1+t_2^2}\right)$.

So

$$L = \left(\frac{2t_1}{1+t_1^2} - \frac{2t_2}{1+t_2^2} : \frac{1-t_2^2}{1+t_2^2} - \frac{1-t_1^2}{1+t_1^2} : \left(\frac{1-t_2^2}{1+t_2^2}\right)\left(\frac{2t_1}{1+t_1^2}\right) - \left(\frac{1-t_1^2}{1+t_1^2}\right)\left(\frac{2t_2}{1+t_2^2}\right)\right)$$

$$= \left(\begin{matrix}2t_1(1+t_2^2) \\ -2t_2(1+t_1^2)\end{matrix} : \begin{matrix}(1-t_2^2)(1+t_1^2) \\ -(1-t_1^2)(1+t_2^2)\end{matrix} : \begin{matrix}(1-t_2^2)2t_1 \\ -(1-t_1^2)2t_2\end{matrix}\right)$$

③

$$= (1-t_1t_2 : t_1+t_2 : 1+t_1t_2)$$ [after removing common factor $2(t_1-t_2)$] ■

Define $L(t_1,t_2) \equiv (1-t_1t_2 : t_1+t_2 : 1+t_1t_2)$ ④

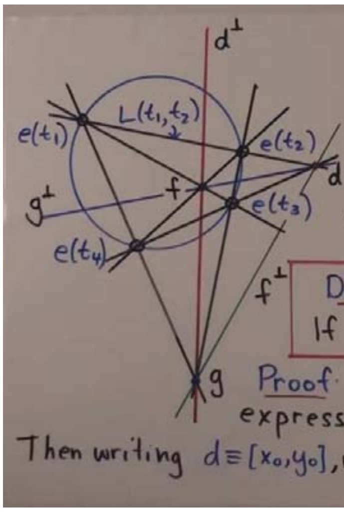
$L(t_3,t_4) \equiv (1-t_3t_4 : t_3+t_4 : 1+t_3t_4)$

Meet of interior lines thm The lines $L(t_1,t_2)$ $L(t_3,t_4)$ are parallel $\Leftrightarrow (1-t_1t_2)(t_3+t_4) = (1-t_3t_4)(t_1+t_2)$. Otherwise they meet at $m(t_1,t_2 : t_3,t_4) \equiv$

$$\left[\frac{(1+t_1t_2)(t_3+t_4)-(1+t_3t_4)(t_1+t_2)}{(1-t_1t_2)(t_3+t_4)-(1-t_3t_4)(t_1+t_2)}, \frac{2(t_3t_4-t_1t_2)}{(1-t_1t_2)(t_3+t_4)-(1-t_3t_4)(t_1+t_2)}\right]$$

Proof: Use Point on two lines theorem with

$$(1-t_1t_2)(1+t_3t_4) - (1-t_3t_4)(1+t_1t_2) = 2(t_3t_4 - t_1t_2).$$ ■



$d = m(t_1,t_2 : t_3,t_4)$ ⑤

$f = m(t_1,t_3 : t_2,t_4)$

$g = m(t_1,t_4 : t_2,t_3)$

Good example of "threeness from fourness."

> Duality in co-ods theorem
> If $d \equiv [x_0, y_0]$ then $d^\perp = (x_0 : y_0 : 1)$.

Proof [A calculation]. We express $d, f + g$ in terms of t_1, t_2, t_3, t_4. Then writing $d \equiv [x_0, y_0]$, we show f, g lie on $(x_0 : y_0 : 1)$.

$d \equiv [x_0, y_0] = m(t_1, t_2 : t_3, t_4)$ So from Meet of int. lines thm

$$x_0 = \frac{(1+t_1t_2)(t_3+t_4)-(1+t_3t_4)(t_1+t_2)}{(1-t_1t_2)(t_3+t_4)-(1-t_3t_4)(t_1+t_2)}$$

$$y_0 = \frac{2(t_3t_4-t_1t_2)}{(1-t_1t_2)(t_3+t_4)-(1-t_3t_4)(t_1+t_2)}$$

$f \equiv [x_1, y_1] = m(t_1, t_3 : t_2, t_4)$

$$x_1 = \frac{(1+t_1t_3)(t_2+t_4)-(1+t_2t_4)(t_1+t_3)}{(1-t_1t_3)(t_2+t_4)-(1-t_2t_4)(t_1+t_3)}$$

$$y_1 = \frac{2(t_2t_4-t_1t_3)}{(1-t_1t_3)(t_2+t_4)-(1-t_2t_4)(t_1+t_3)}$$

We need check that f lies on $(x_0 : y_0 : 1)$ i.e.

$$\boxed{x_0x_1 + y_0y_1 = 1}$$

An identity!

$$[(1+t_1t_2)(t_3+t_4)-(1+t_3t_4)(t_1+t_2)] \times [(1+t_1t_3)(t_2+t_4)-(1+t_2t_4)(t_1+t_3)]$$
$$+\ 2(t_3t_4-t_1t_2) \times 2(t_2t_4-t_1t_3)$$
$$= [(1-t_1t_2)(t_3+t_4)-(1-t_3t_4)(t_1+t_2)] \times [(1-t_1t_3)(t_2+t_4)-(1-t_2t_4)(t_1+t_3)]$$

Same for g. ■

6

Perpendicularity in co-ordinates thm. 7

i) The points $a_1 \equiv [x_1, y_1]$ and $a_2 \equiv [x_2, y_2]$ are perpendicular $\Leftrightarrow$ $\boxed{x_1x_2 + y_1y_2 - 1 = 0}$.

ii) The lines $L_1 \equiv (l_1 : m_1 : n_1)$ and $L_2 \equiv (l_2 : m_2 : n_2)$ are perpendicular $\Leftrightarrow$ $\boxed{l_1l_2 + m_1m_2 - n_1n_2 = 0}$.

Proof. i) $a_1 \perp a_2 \Leftrightarrow a_1$ lies on $a_2^\perp = (x_2 : y_2 : 1)$

$\Leftrightarrow x_1x_2 + y_1y_2 = 1$.

ii) $L_1 \perp L_2 \Leftrightarrow L_1$ passes through $L_2^\perp = \left[\frac{l_2}{n_2}, \frac{m_2}{n_2}\right]$

$\Leftrightarrow l_1\frac{l_2}{n_2} + m_1\frac{m_2}{n_2} = n_1 \Leftrightarrow l_1l_2 + m_1m_2 = n_1n_2$. ■

Exer. 6.1 Prove the case $n_1 = n_2 = 0$ seperately.

Quadrance in co-ods thm If $a_1 \equiv [x_1, y_1]$ & $a_2 \equiv [x_2, y_2]$ then

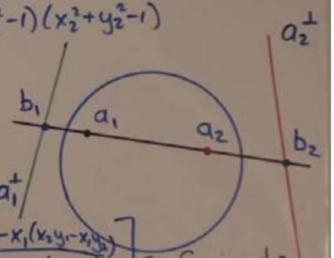
$$q(a_1, a_2) = 1 - \frac{(x_1x_2 + y_1y_2 - 1)^2}{(x_1^2+y_1^2-1)(x_2^2+y_2^2-1)}$$

Proof Recall def'n:

$q(a_1, a_2) \equiv R(a_1, b_2 : a_2, b_1)$

$a_1a_2 = (y_1 - y_2 : x_2 - x_1 : x_2y_1 - x_1y_2)$

$a_1^\perp = (x_1 : y_1 : 1)$ $a_2^\perp = (x_2 : y_2 : 1)$



$$b_1 = \left[\frac{(x_2y_1 - x_1y_2)y_1 - 1(x_2 - x_1)}{(y_1-y_2)y_1 - x_1(x_2-x_1)}, \frac{(y_1-y_2)1 - x_1(x_2y_1 - x_1y_2)}{(y_1-y_2)y_1 - x_1(x_2-x_1)}\right]$$

Conjugate points

$$b_2 = \left[\frac{(x_2y_1 - x_1y_2)y_2 - 1(x_2 - x_1)}{(y_1-y_2)y_2 - x_2(x_2-x_1)}, \frac{(y_1-y_2)1 - x_2(x_2y_1 - x_1y_2)}{(y_1-y_2)y_2 - x_2(x_2-x_1)}\right]$$

8

9

$$q(a_1, a_2) = R(a_1, b_2 : a_2, b_1) = \frac{\overrightarrow{a_1a_2}}{\overrightarrow{a_1b_1}} \Big/ \frac{\overrightarrow{b_2a_2}}{\overrightarrow{b_2b_1}}$$

To evaluate numerator, suppose $x_2 - x_1 \neq 0$:

$$\frac{\overrightarrow{a_1a_2}}{\overrightarrow{a_1b_1}} = \frac{x_2 - x_1}{\left[\frac{(x_2y_1 - x_1y_2)y_1 - (x_2-x_1)}{(y_1-y_2)y_1 - x_1(x_2-x_1)}\right] - x_1}$$

$$= \frac{(x_2-x_1)(x_1^2 + y_1^2 - x_1x_2 - y_1y_2)}{x_2y_1^2 - x_1y_1^2 - (x_2-x_1) + x_1^2(x_2-x_1)}$$

$$= \frac{x_1^2 + y_1^2 - x_1x_2 - y_1y_2}{x_1^2 + y_1^2 - 1}$$ ☺

Exer. 6.1 Do the case $y_2 - y_1 \neq 0$

10

$$\frac{\overrightarrow{b_2a_2}}{\overrightarrow{b_2b_1}} = \frac{\left[\frac{(x_2y_1 - x_1y_2)y_2 - (x_2-x_1)}{(y_1-y_2)y_2 - x_2(x_2-x_1)}\right] - x_2}{\left[\frac{(x_2y_1 - x_1y_2)y_2 - (x_2-x_1)}{(y_1-y_2)y_2 - x_2(x_2-x_1)}\right] - \left[\frac{(x_2y_1 - x_1y_2)y_1 - (x_2-x_1)}{(y_1-y_2)y_1 - x_1(x_2-x_1)}\right]}$$

$$= \frac{(x_2-x_1)(x_2^2+y_2^2-1)(x_1^2+y_1^2-x_1x_2-y_1y_2)}{\left[\begin{array}{l}[(x_2y_1 - x_1y_2)y_2 - (x_2-x_1)][(y_1-y_2)y_1 - x_1(x_2-x_1)] \\ -[(x_2y_1-x_1y_2)y_1 - (x_2-x_1)][(y_1-y_2)y_2 - x_2(x_2-x_1)]\end{array}\right]}$$

$$= \frac{(x_2^2+y_2^2-1)(x_1^2+y_1^2-x_1x_2-y_1y_2)}{-x_1^2-x_2^2-y_1^2-y_2^2+x_1^2y_2^2+x_2^2y_1^2+2x_1x_2+2y_1y_2-2x_1x_2y_1y_2}$$

$$= \frac{(x_2^2+y_2^2-1)(x_1^2+y_1^2-x_1x_2-y_1y_2)}{(x_1^2+y_1^2-1)(x_2^2+y_2^2-1)-(x_1x_2+y_1y_2-1)^2}.$$ ☺ ■

Spread in co-ordinates thm If $L_1 \equiv (l_1 : m_1 : n_1)$ and $L_2 \equiv (l_2 : m_2 : n_2)$ then 11

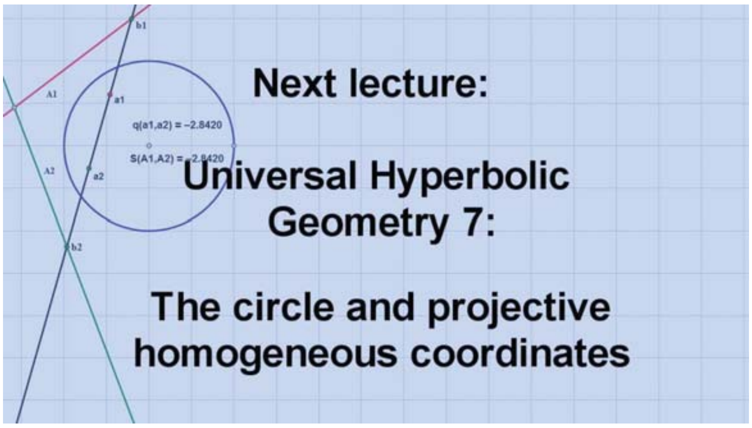
$$S(L_1, L_2) = 1 - \frac{(l_1l_2 + m_1m_2 - n_1n_2)^2}{(l_1^2+m_1^2-n_1^2)(l_2^2+m_2^2-n_2^2)}$$

Proof. Assume n_1, n_2 are non-zero. In that case $L_1^\perp = \left[\frac{l_1}{n_1}, \frac{m_1}{n_1}\right]$ $L_2^\perp = \left[\frac{l_2}{n_2}, \frac{m_2}{n_2}\right]$, so that

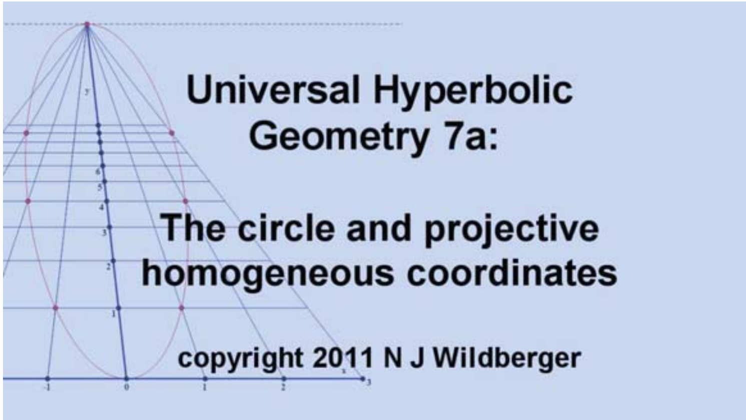
$$S(L_1, L_2) = q(L_1^\perp, L_2^\perp) = 1 - \frac{\left(\frac{l_1}{n_1}\frac{l_2}{n_2} + \frac{m_1}{n_1}\frac{m_2}{n_2} - 1\right)^2}{\left(\left(\frac{l_1}{n_1}\right)^2 + \left(\frac{m_1}{n_1}\right)^2 - 1\right)\left(\frac{l_2^2}{n_2^2} + \frac{m_2^2}{n_2^2} - 1\right)}$$

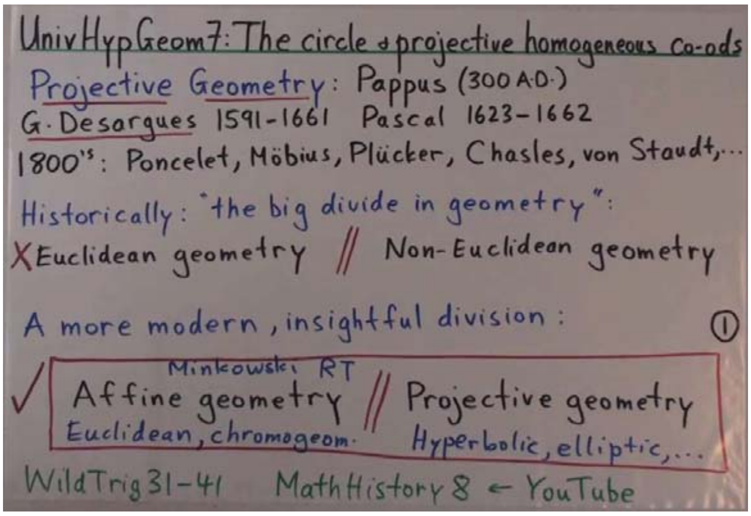
$$= 1 - \frac{(l_1l_2 + m_1m_2 - n_1n_2)^2}{(l_1^2+m_1^2-n_1^2)(l_2^2+m_2^2-n_2^2)}$$

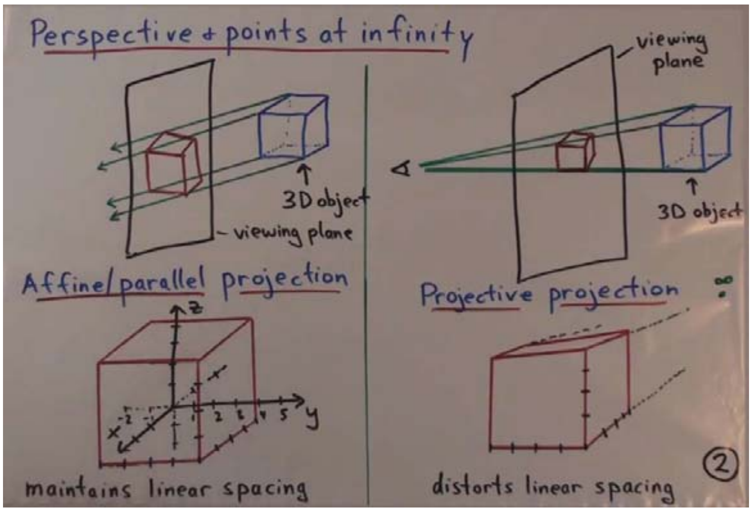
Exer 6.3 Find an argument for cases $n_1, n_2 = 0$. ■

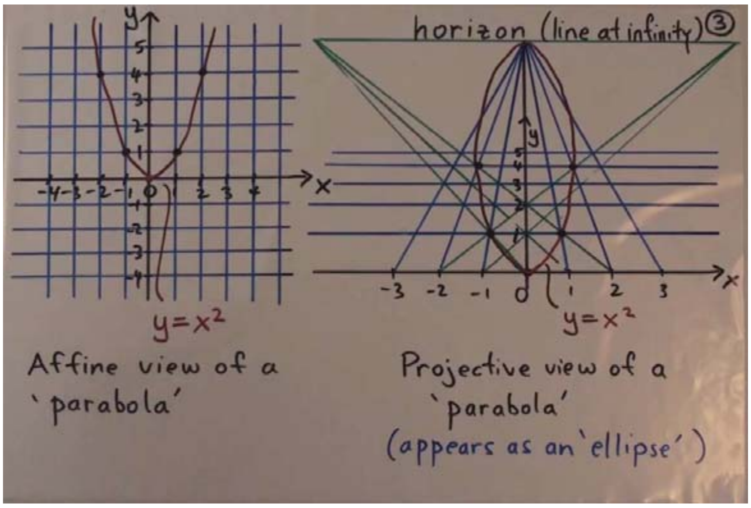


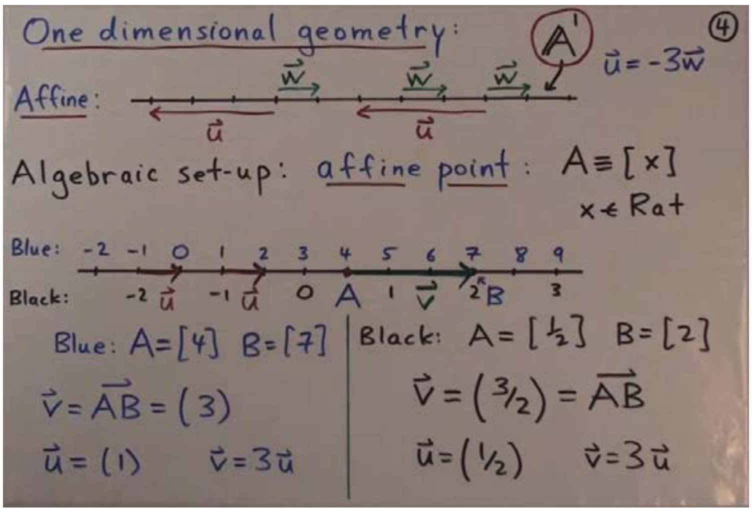
Next lecture:
Universal Hyperbolic Geometry 7:
The circle and projective homogeneous coordinates
b1
A1
a1
q(a1,a2) = −2.8420
S(A1,A2) = −2.8420
A2
a2
b2

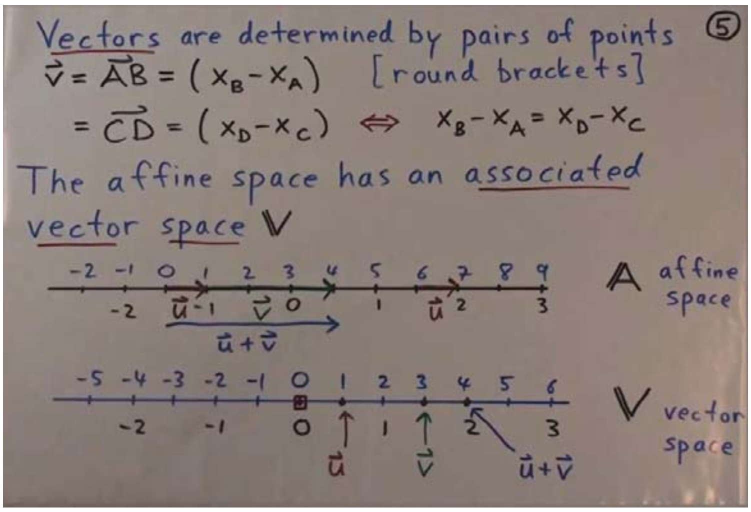

Universal Hyperbolic Geometry 7a:
The circle and projective homogeneous coordinates
copyright 2011 N J Wildberger

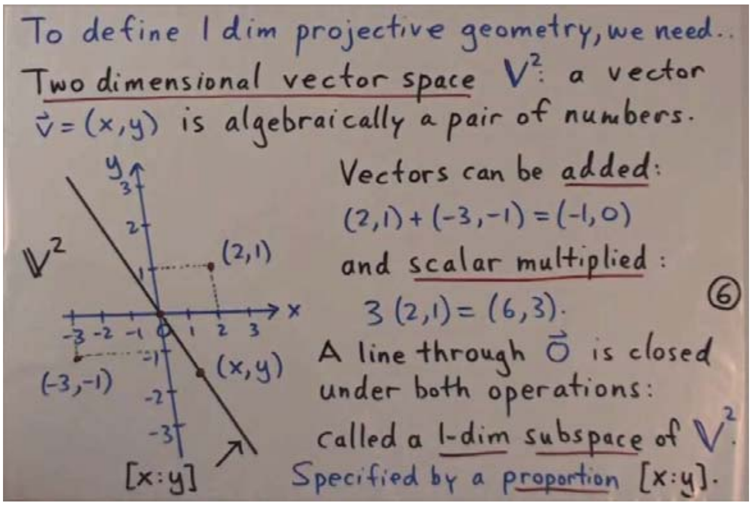

UnivHypGeom7: The circle & projective homogeneous co-ords
Projective Geometry: Pappus (300 A.D.)
G. Desargues 1591-1661 Pascal 1623-1662
1800's: Poncelet, Möbius, Plücker, Chasles, von Staudt, ...
Historically: "the big divide in geometry":
Euclidean geometry // Non-Euclidean geometry
A more modern, insightful division: 1
Minkowski RT
Affine geometry // Projective geometry
Euclidean, chromogeom. Hyperbolic, elliptic, ...
WildTrig 31-41 MathHistory 8 ← YouTube

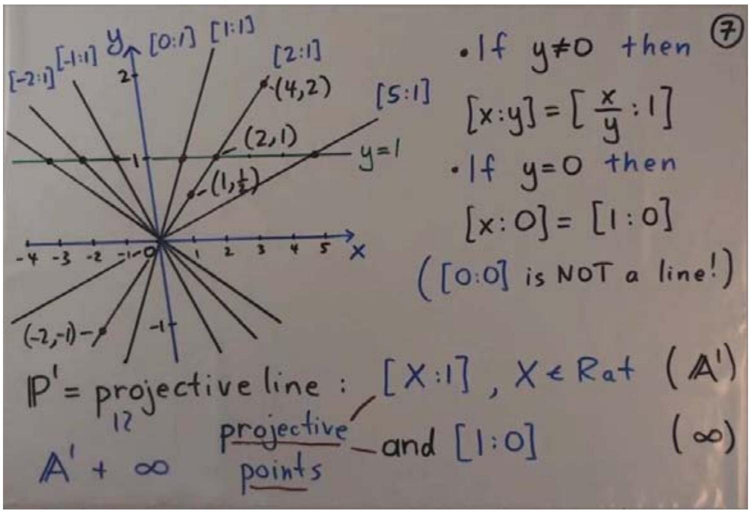

Perspective & points at infinity
viewing plane
3D object
viewing plane
3D object
Affine/parallel projection
Projective projection
maintains linear spacing
distorts linear spacing
2

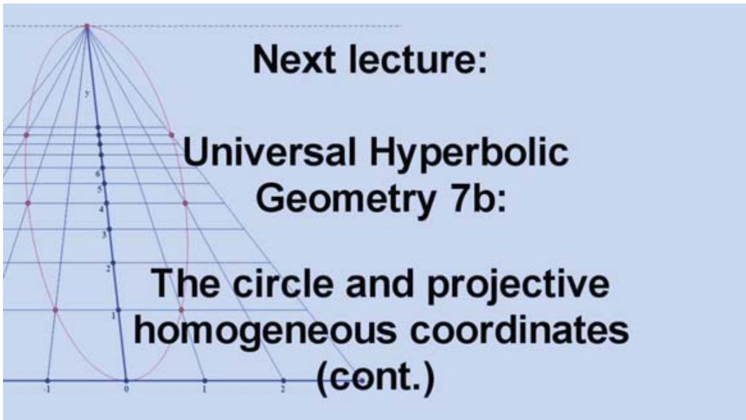

horizon (line at infinity) 3
y = x²
y = x²
Affine view of a 'parabola'
Projective view of a 'parabola'
(appears as an 'ellipse')

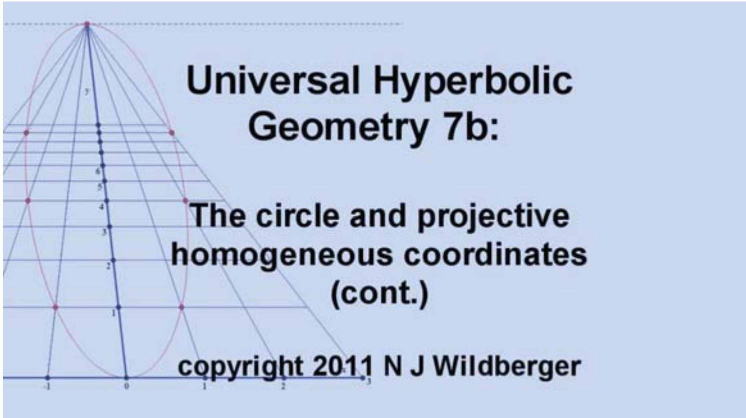

One dimensional geometry: 4
Affine:
u = -3w
Algebraic set-up: affine point: A ≡ [x]
x ∈ Rat
Blue:
Black:
Blue: A=[4] B=[7]
Black: A=[½] B=[2]
v = AB = (3)
v = (3/2) = AB
u = (1) v = 3u
u = (½) v = 3u

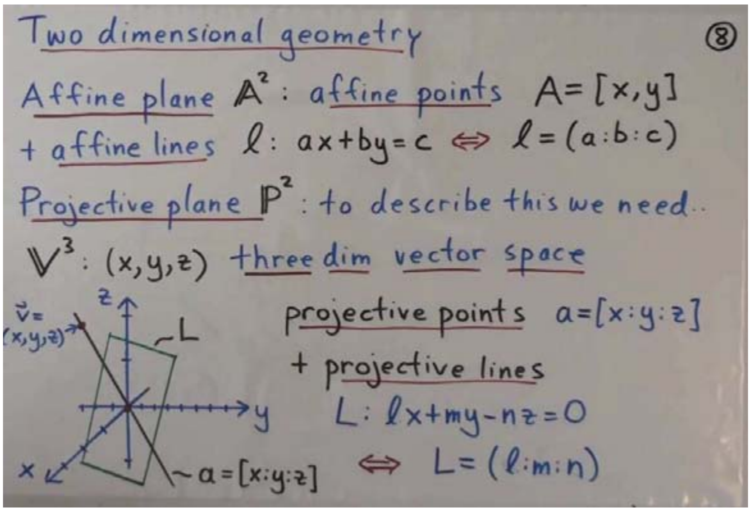

Vectors are determined by pairs of points 5
v = AB = (x_B - x_A) [round brackets]
= CD = (x_D - x_C) ⇔ x_B - x_A = x_D - x_C
The affine space has an associated vector space V
A affine space
V vector space

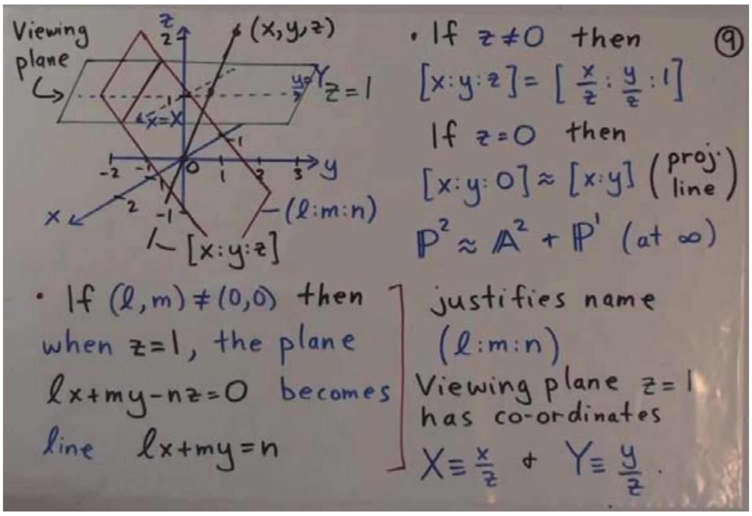

To define 1 dim projective geometry, we need..
Two dimensional vector space V^2: a vector
$\vec{v} = (x,y)$ is algebraically a pair of numbers.
Vectors can be added:
$(2,1) + (-3,-1) = (-1,0)$
and scalar multiplied:
$3(2,1) = (6,3)$.
A line through $\vec{O}$ is closed
under both operations:
called a 1-dim subspace of V^2.
Specified by a proportion $[x:y]$.
$\mathbb{V}^2$
(2,1)
(-3,-1)
(x,y)
[x:y]
6

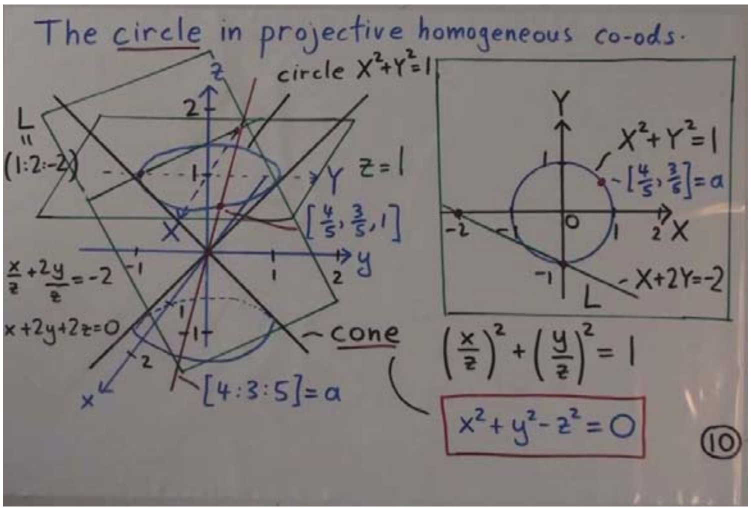

• If $y \neq 0$ then
$[x:y] = [\frac{x}{y} : 1]$
• If $y = 0$ then
$[x:0] = [1:0]$
($[0:0]$ is NOT a line!)
$\mathbb{P}^1$ = projective line : $[X:1]$, $X \in$ Rat ($\mathbb{A}^1$)
and $[1:0]$ (∞)
$\mathbb{A}^1 + \infty$
projective points
[-2:1]
[-1:1]
[0:1]
[1:1]
[2:1]
[5:1]
(4,2)
(2,1)
(1,½)
(-2,-1)
y=1
7

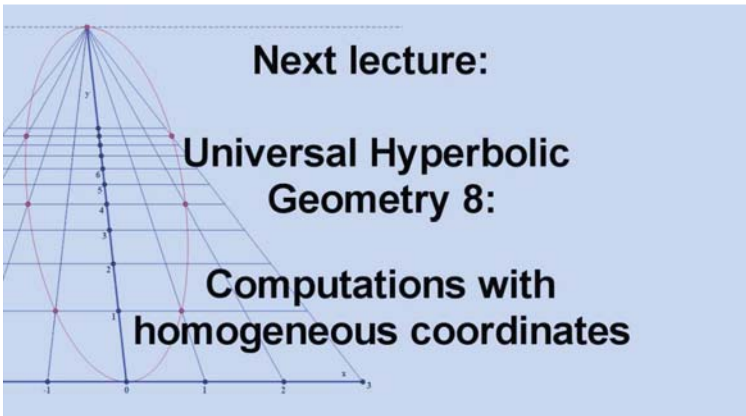

Next lecture:
Universal Hyperbolic Geometry 7b:
The circle and projective homogeneous coordinates (cont.)

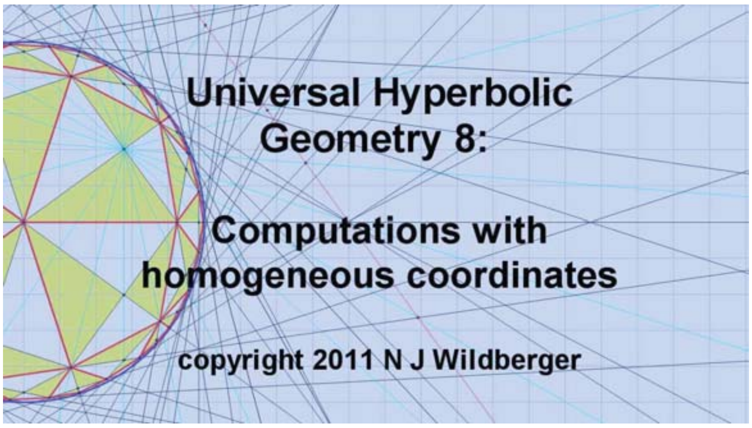

Universal Hyperbolic Geometry 7b:
The circle and projective homogeneous coordinates (cont.)
copyright 2011 N J Wildberger

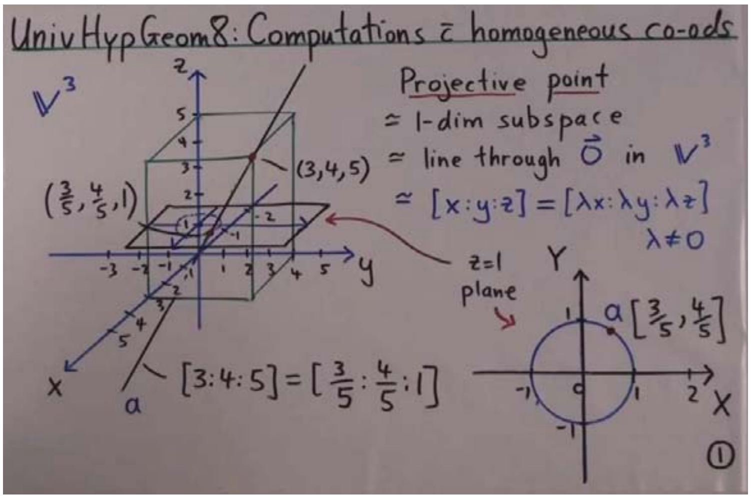

Two dimensional geometry
8
Affine plane $\mathbb{A}^2$: affine points $A=[x,y]$
+ affine lines $\ell: ax+by=c \Leftrightarrow \ell=(a:b:c)$
Projective plane $\mathbb{P}^2$: to describe this we need..
$\mathbb{V}^3$: (x,y,z) three dim vector space
$\vec{v}=(x,y,z)$
L
$a=[x:y:z]$
projective points $a=[x:y:z]$
+ projective lines
$L: \ell x+my-nz=0$
$\Leftrightarrow L=(\ell:m:n)$

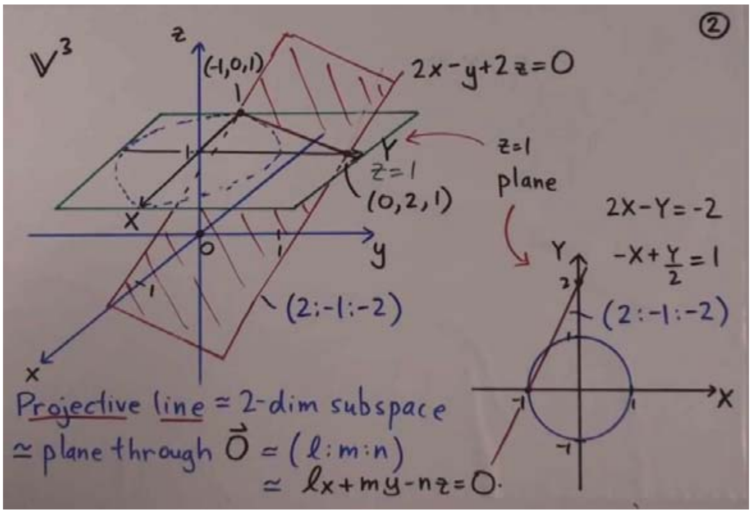

Viewing plane
(x,y,z)
$z=1$
$(\ell:m:n)$
$[x:y:z]$
9
• If $z\neq 0$ then
$[x:y:z]=\left[\frac{x}{z}:\frac{y}{z}:1\right]$
If $z=0$ then
$[x:y:0]\approx[x:y]$ (proj. line)
$\mathbb{P}^2\approx\mathbb{A}^2+\mathbb{P}^1$ (at ∞)
• If $(\ell,m)\neq(0,0)$ then
when $z=1$, the plane
$\ell x+my-nz=0$ becomes
line $\ell x+my=n$
justifies name $(\ell:m:n)$
Viewing plane $z=1$ has co-ordinates
$X\equiv\frac{x}{z}$ + $Y\equiv\frac{y}{z}$.


The circle in projective homogeneous co-ods.
circle $X^2+Y^2=1$
L = (1:2:-2)
$z=1$
$\left[\frac{4}{5},\frac{3}{5},1\right]$
$\frac{x}{z}+2\frac{y}{z}=-2$
$x+2y+2z=0$
cone
$[4:3:5]=a$
$X^2+Y^2=1$
$\left[\frac{4}{5},\frac{3}{5}\right]=a$
$X+2Y=-2$
L
$\left(\frac{x}{z}\right)^2+\left(\frac{y}{z}\right)^2=1$
$\boxed{x^2+y^2-z^2=0}$
10


Next lecture:
Universal Hyperbolic Geometry 8:
Computations with homogeneous coordinates





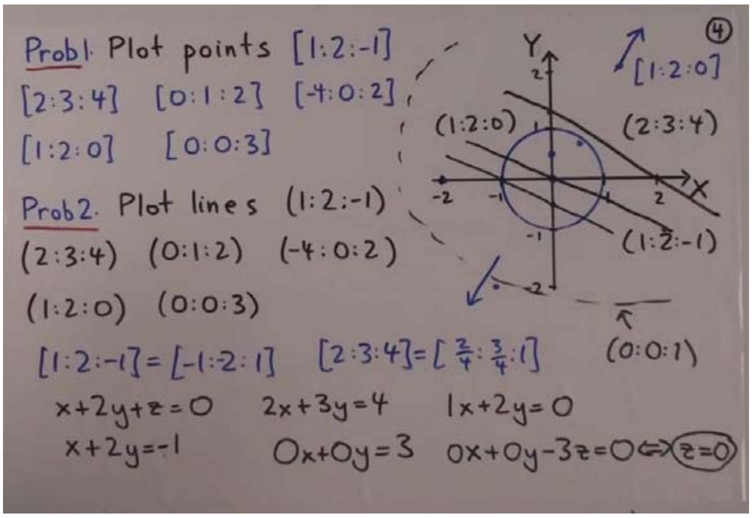


Official definitions ③

Def. A (hyperbolic) point is a proportion $a\equiv[x:y:z]$ of (rational) numbers in square brackets.

Def. A (hyperbolic) line is a proportion $L\equiv(l:m:n)$ of (rational) numbers in round brackets.

Def. The point $a\equiv[x:y:z]$ lies on the line $L\equiv(l:m:n)$ $\Leftrightarrow$ L passes through a $\Leftrightarrow$ $\boxed{lx+my-nz=0}$



⑤

Prob3. Find the line L passing through $a=[2:1:3]$ $b=[1:-1:2]$

Solution. If $L\equiv(l:m:n)$ then we need

$$2l+m-3n=0 \qquad l-m-2n=0 \qquad \begin{pmatrix}2&1&-3\\1&-1&-2\end{pmatrix}\begin{pmatrix}l\\m\\n\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Row reduce!

$$\begin{pmatrix}2&1&-3\\1&-1&-2\end{pmatrix}\to\begin{pmatrix}1&-1&-2\\2&1&-3\end{pmatrix}\to\begin{pmatrix}1&-1&-2\\0&3&1\end{pmatrix}$$

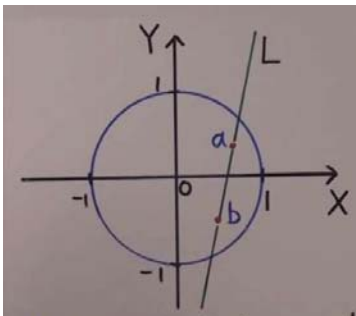
$$\to\begin{pmatrix}1&-1&-2\\0&1&\frac{1}{3}\end{pmatrix}\to\begin{array}{c}\begin{pmatrix}1&0&-\frac{5}{3}\\0&1&\frac{1}{3}\end{pmatrix}\\ l\ \ m\ \ n\end{array}$$

$$l=\frac{5}{3}\lambda \quad m=-\frac{1}{3}\lambda \quad n=\lambda$$

$$L = (l:m:n) = (5:-1:3)$$

◇

(6)



$a=[2:1:3]=[\frac{2}{3}:\frac{1}{3}:1]$

$b=[1:-1:2]=[\frac{1}{2}:-\frac{1}{2}:1]$

$L=(5:-1:3)$

$5X-Y=3$

Prob 4. Find the point a lying on the lines $L=(-2:3:4)$ and $M=(3:1:-2)$.

This is exactly the same kind of problem!

(7)

Solution If $a\equiv[x:y:z]$ then we need

$$-2x+3y-4z=0 \qquad \begin{pmatrix}-2 & 3 & -4\\ 3 & 1 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$
$$3x+y+2z=0$$

$$\begin{pmatrix}-2 & 3 & -4\\ 3 & 1 & 2\end{pmatrix}\xrightarrow[3①+2②]{}\begin{pmatrix}-2 & 3 & -4\\ 0 & 11 & -8\end{pmatrix}\xrightarrow{11①-3②}\begin{pmatrix}-22 & 0 & -20\\ 0 & 11 & -8\end{pmatrix}$$

$$\rightarrow \begin{array}{c}\begin{pmatrix}11 & 0 & 10\\ 0 & 11 & -8\end{pmatrix}\\ x \quad y \quad z\end{array} \qquad \begin{aligned}x&=-\tfrac{10}{11}\lambda\\ y&=\tfrac{8}{11}\lambda\\ z&=\lambda\end{aligned} \qquad \begin{pmatrix}x\\ y\\ z\end{pmatrix}=\lambda\begin{pmatrix}-\frac{10}{11}\\ \frac{8}{11}\\ 1\end{pmatrix}=\lambda'\begin{pmatrix}-10\\ 8\\ 11\end{pmatrix}$$

so $a=[-10:8:11]$ ◇

(8)

Join of two points theorem If $a_1\equiv[x_1:y_1:z_1]$ and $a_2\equiv[x_2:y_2:z_2]$ are distinct points, then there is exactly one line L which passes through them both, namely

$$L\equiv a_1a_2=(y_1z_2-y_2z_1 : z_1x_2-z_2x_1 : x_2y_1-x_1y_2).$$

Proof. The system $\boxed{\begin{aligned}lx_1+my_1-nz_1=0\\ lx_2+my_2-nz_2=0\end{aligned}}$

has solution

$$\begin{pmatrix}l\\ m\\ n\end{pmatrix}=\lambda\begin{pmatrix}y_1z_2-y_2z_1\\ z_1x_2-z_2x_1\\ x_2y_1-x_1y_2\end{pmatrix}.$$

■

Meet of two lines theorem If $L_1\equiv(l_1:m_1:n_1)$ and $L_2\equiv(l_2:m_2:n_2)$ are distinct lines, then there is exactly one point a which lies on them both, namely

$$a=L_1L_2=[m_1n_2-m_2n_1 : n_1l_2-n_2l_1 : l_2m_1-l_1m_2].$$

Proof. The system $\boxed{\begin{aligned}l_1x+m_1y-n_1z=0\\ l_2x+m_2y-n_2z=0\end{aligned}}$

has solution

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}m_1n_2-m_2n_1\\ n_1l_2-n_2l_1\\ l_2m_1-l_1m_2\end{pmatrix}.$$

(9)

■

(10)

Duality principle Points & lines are treated symmetrically.

Useful function(s):

$$J(x_1,y_1,z_1,x_2,y_2,z_2)\equiv(y_1z_2-y_2z_1 : z_1x_2-z_2x_1 : x_2y_1-x_1y_2)$$

$$j(x_1,y_1,z_1,x_2,y_2,z_2)\equiv[y_1z_2-y_2z_1 : z_1x_2-z_2x_1 : x_2y_1-x_1y_2]$$

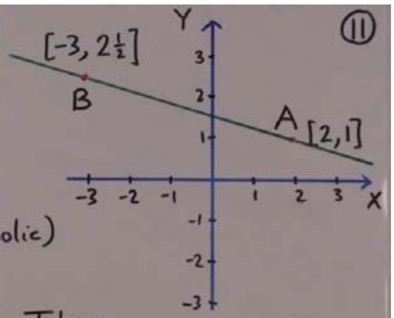
Ex. If $a_1\equiv[3:5:-1]$ and $a_2=[4:1:2]$ then

$$a_1a_2=J(3,5,-1,4,1,2)=(5\times2-1\times(-1) : (-1)\times4-2\times3 : 4\times5-3\times1)$$
$$=(11:-10:17)$$

◇

(11)

Application to Cartesian geometry



Prob 5. Find equation of line through $A\equiv[2,1]$, $B=[-3,2\frac{1}{2}]$

Solution Define (hyperbolic) points $a\equiv[2:1:1]$ and $b\equiv[-3:\frac{5}{2}:1]=[-6:5:2]$. Then

$$ab=J(2,1,1,-6,5,2)=(1\times2-5\times1 : 1\times(-6)-2\times2 : (-6)\times1-2\times5)$$
$$=(-3:-10:-16)=(3:10:16) \;:\; \boxed{3X+10Y=16}$$

Prob 6: Find the meet of the lines $l: x-3y=2$

$m: 4x+y=3$

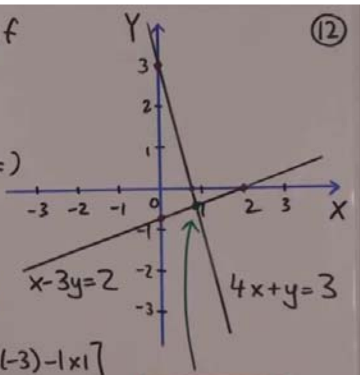
Solution Define (hyperbolic) lines $L \equiv (1:-3:2)$

$M \equiv (4:1:3)$

Then $LM = j(1,-3,2,4,1,3)$

$= [-3\times3-1\times2 : 2\times4-3\times1 : 4\times(-3)-1\times1]$

$= [-11:5:-13] = \left[\frac{11}{13} : \frac{-5}{13} : 1\right]$



$\boxed{\left[\frac{11}{13}, \frac{-5}{13}\right]}$

12

Exer 8.1 From the points $a\equiv[-1:4:2]$ $b\equiv[3:5:1]$ and lines $L\equiv(3:-1:0)$ $M\equiv(6:-2:3)$ find

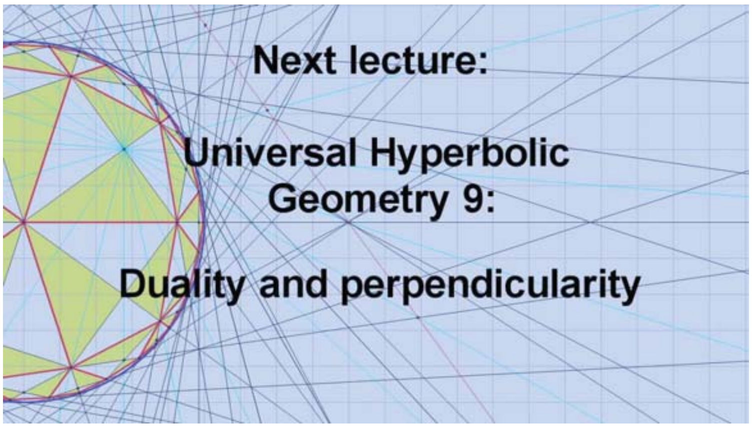
i) ab ii) LM iii) $(ab)L$ iv) $a(LM)$ v) $(ML)b$ ◇

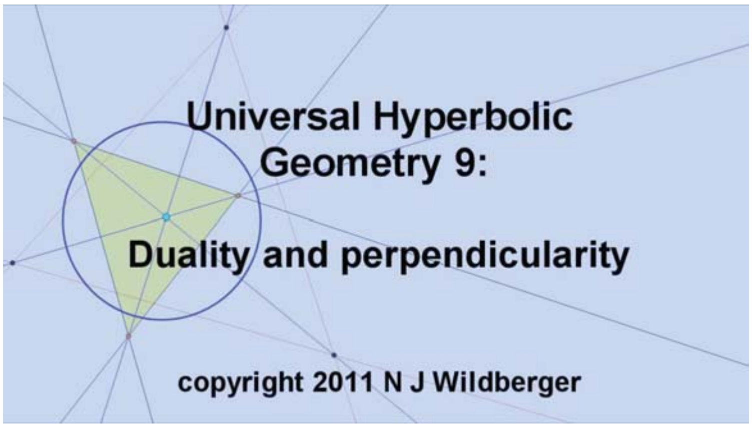
Exer 8.2 Show that for any 3 points a,b,c, not collinear, $(ab)(ac)=a$. ◇

Exer 8.3 Show that for any 3 lines L,M,N, not concurrent, $(LM)(LN)=L$. ◇

13

Exer 8.4 What is $J(1,3,2,4,12,8)$? Explain. ◇





UnivHypGeom 9: Duality + perpendicularity ①

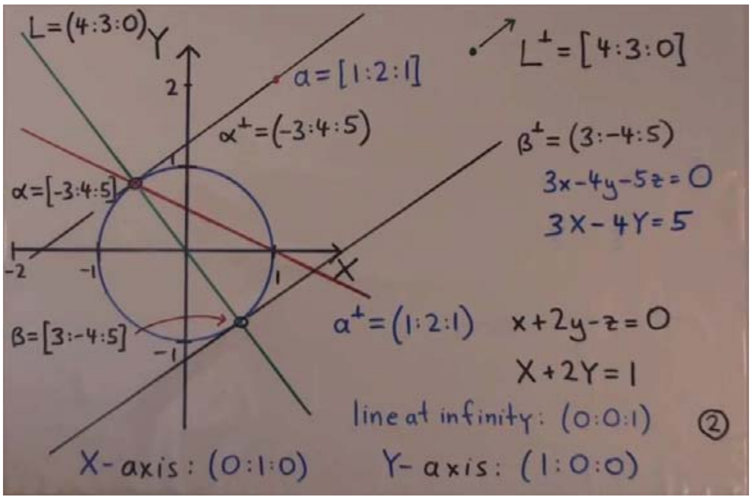
(Hyperbolic) point: $a \equiv [x:y:z]$

(Hyperbolic) line: $L \equiv (l:m:n)$

a lies on L ⇔ L passes through a ⇔ $lx+my-nz=0$

Def. The point $a \equiv [x:y:z]$ is dual to the line $a^{\perp} \equiv (x:y:z)$.

Def. The line $L \equiv (l:m:n)$ is dual to the point $L^{\perp} \equiv [l:m:n]$.



Exer. 9.1 Plot $a \equiv [2:3:4]$ $b \equiv [3:2:3]$ $c \equiv [1:6:5]$ and their duals ◇

Exer. 9.2 Plot $L \equiv (1:3:0)$ $M \equiv (1:3:1)$ $N \equiv (2:-1:4)$ and their duals. ◇

Point duality theorem If a and b are points, then a lies on $b^{\perp}$ precisely when b lies on $a^{\perp}$.

Line duality theorem If L and M are lines, then L passes through $M^{\perp}$ precisely when M passes through $L^{\perp}$. ③

Proof (of Point duality theorem) Suppose that $a \equiv [x:y:z]$ and $b \equiv [u:v:w]$. Then $a^{\perp} = (x:y:z)$ and $b^{\perp} = (u:v:w)$. So a lies on $b^{\perp}$ ⇔ $ux+vy-wz=0$ ⇔ b lies on $a^{\perp}$. ∎ ④

Compare with the Polar duality thm (pg 9) of UnivHypGeom1.

Proof of Line duality theorem the same. ∎

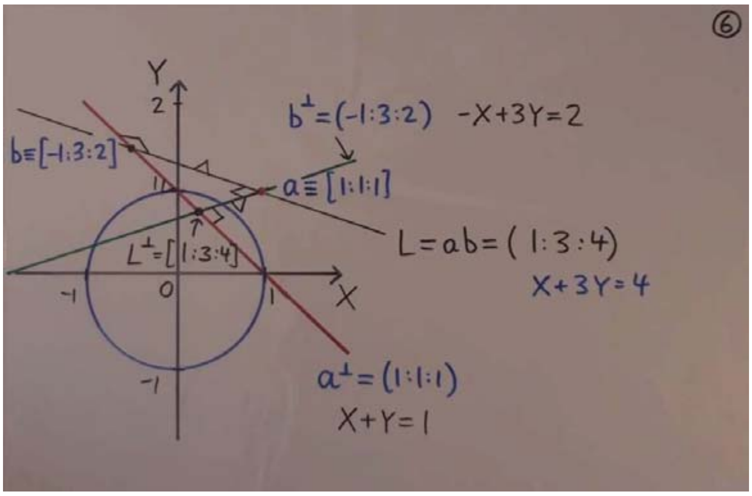
Perpendicularity ⑤

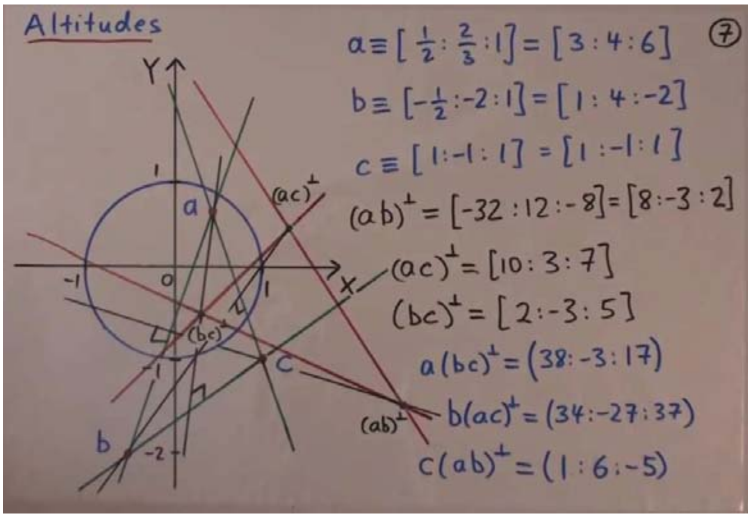
Def The points a and b are perpendicular ⇔ a lies on $b^{\perp}$ ⇔ b lies on $a^{\perp}$

We write $\boxed{a \perp b}$

Def The lines L and M are perpendicular ⇔ L passes through $M^{\perp}$ ⇔ M passes through $L^{\perp}$

We write $\boxed{L \perp M}$





8

Concurrent lines theorem The lines $L_1 \equiv (l_1:m_1:n_1)$, $L_2 \equiv (l_2:m_2:n_2)$ $L_3 \equiv (l_3:m_3:n_3)$ are concurrent precisely when

$$\det\begin{pmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{pmatrix} = 0.$$

Proof. The main case is L_1, L_2, L_3 distinct: then $L_1L_2 = [m_1n_2 - m_2n_1 : n_1l_2 - n_2l_1 : m_1l_2 - m_2l_1]$

L_3 lies on this $\Leftrightarrow$

$$l_3(m_1n_2 - m_2n_1) + m_3(n_1l_2 - n_2l_1) - n_3(m_1l_2 - m_2l_1) = 0.$$

9

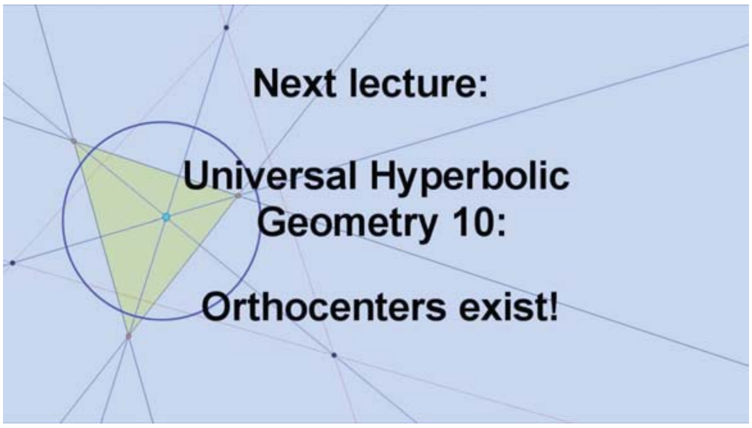
Back to example:

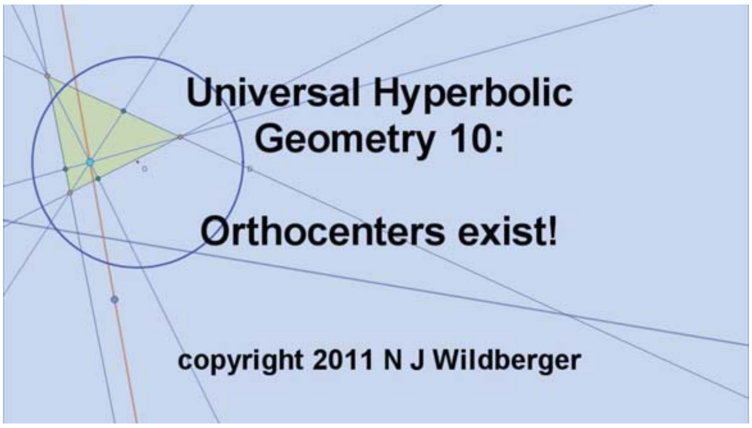
$$\det\begin{pmatrix} 38 & -3 & 17 \\ 34 & -27 & 37 \\ 1 & 6 & -5 \end{pmatrix} = 0$$

so 3 altitudes meet. ◇

Collinear points theorem The points $a_1 \equiv [x_1:y_1:z_1]$ $a_2 \equiv [x_2:y_2:z_2]$ $a_3 \equiv [x_3:y_3:z_3]$ are collinear precisely when

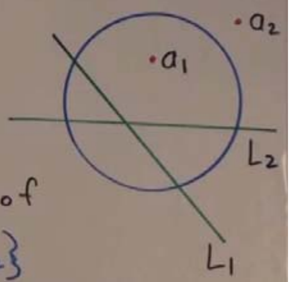
$$\det\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = 0.$$





UnivHypGeom10: Orthocenters exist! ①

Sides, vertices, triangles, altitudes, orthocenter, ortholine, orthostar, + ortho-axis*



Def. A side $\overline{a_1a_2}$ is a set of two points: $\overline{a_1a_2} \equiv \{a_1, a_2\}$

Def. A vertex $\overline{L_1L_2}$ is a set of two lines: $\overline{L_1L_2} \equiv \{L_1, L_2\}$

Def. A couple $\overline{aL}$ is a set consisting of a point + a line: $\overline{aL} \equiv \{a, L\}$

Def. A triangle $\overline{a_1a_2a_3}$ is a set of three non-collinear points: $\overline{a_1a_2a_3} \equiv \{a_1, a_2, a_3\}$

Def. A trilateral $\overline{L_1L_2L_3}$ is a set of three non-concurrent lines:

$$\overline{L_1L_2L_3} \equiv \{L_1, L_2, L_3\}$$

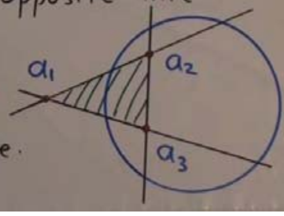
②

A triangle $\overline{a_1a_2a_3}$ has points a_1, a_2, a_3; lines a_1a_2, a_1a_3, a_2a_3; sides $\overline{a_1a_2}, \overline{a_1a_3}, \overline{a_2a_3}$; vertices $\overline{(a_1a_2)(a_1a_3)}$, $\overline{(a_2a_1)(a_2a_3)}$, $\overline{(a_3a_1)(a_3a_2)}$. ③

[Similar def'ns for a trilateral $\overline{L_1L_2L_3}$.]

Def. A triangle $\overline{a_1a_2a_3}$ is dual $\Leftrightarrow$ one of its points is dual to the opposite line.

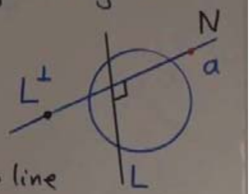
Ex. $\overline{a_1a_2a_3}$ → is a dual triangle.



◇

Def. A couple $\overline{aL}$ is dual $\Leftrightarrow$ $a^\perp = L$. ④

Altitude line theorem For any non-dual couple $\overline{aL}$ there is a unique line N passing through a and perpendicular to L.

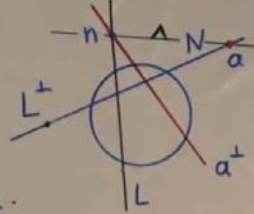


Proof Any line N perpendicular to L must pass through $L^\perp$. Since $a \neq L^\perp$, there is only one line through both a and $L^\perp$:

$$N = aL^\perp \qquad \blacksquare$$

$N \equiv$ altitude line to L through a, or just altitude

Altitude point theorem For any non-dual couple $\overline{aL}$ there is a unique point n which lies on L and is perpendicular to a.



Proof Dual to previous thm:

$$n = a^\perp L \qquad \blacksquare$$

$n \equiv$ altitude point to a on L.

Exer 10.1 Show that for any points a, b: $(ab)^\perp = a^\perp b^\perp$ + similarly for lines ◇

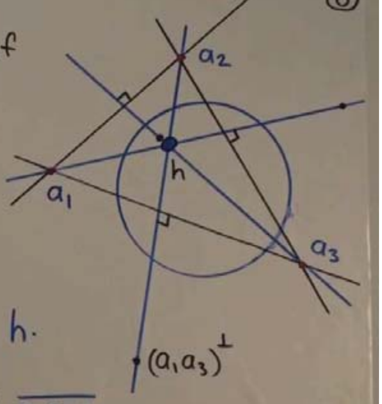
Exer 10.2 Show that $n = N^\perp$. ◇ ⑤

Orthocenter theorem

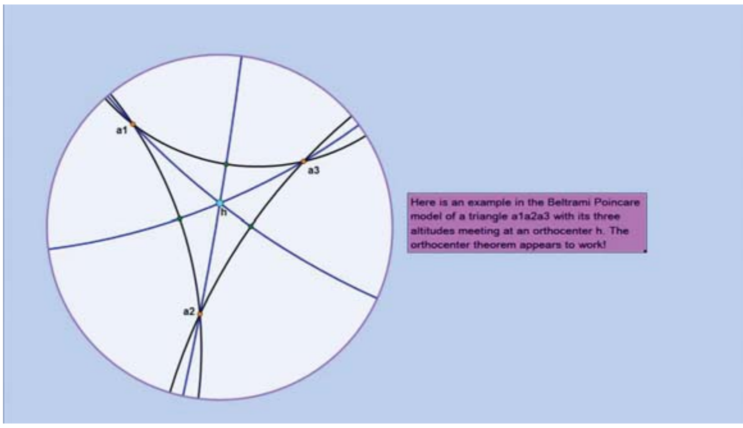
The three altitudes of a non-dual triangle are concurrent.

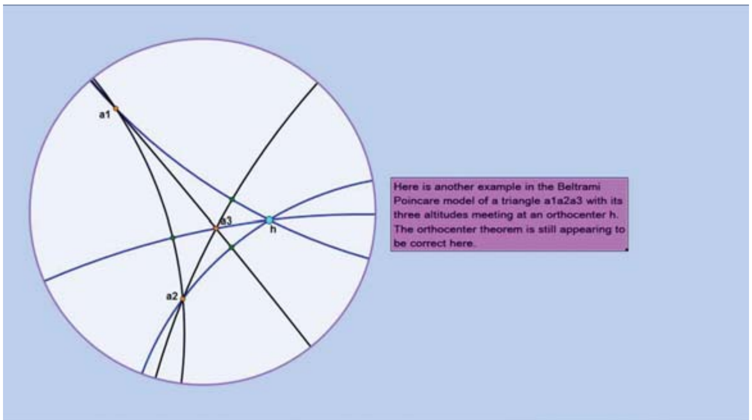
Def. The meet h of the altitudes is the orthocenter of a triangle, denoted by h.

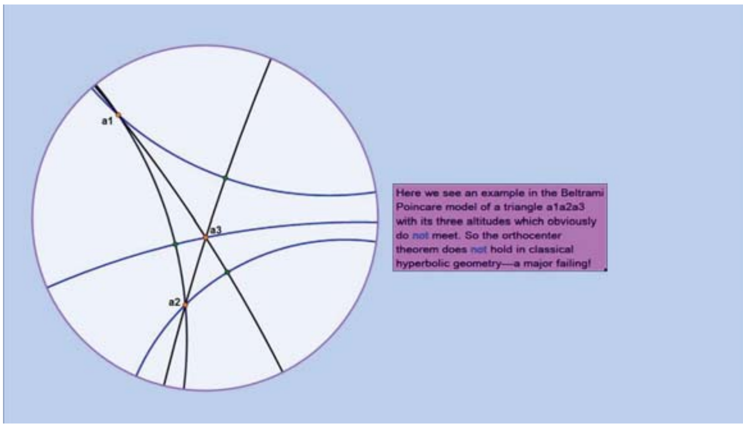
$$h \equiv \text{orthocenter of } \overline{a_1a_2a_3}.$$



⑥







Proof Suppose $a_1 \equiv [x_1 : y_1 : z_1]$, $a_2 \equiv [x_2 : y_2 : z_2]$, $a_3 \equiv [x_3 : y_3 : z_3]$. Then $a_2a_3 = (y_2z_3 - y_3z_2 : z_2x_3 - z_3x_2 : x_3y_2 - x_2y_3)$ and so

$$N_1 \equiv a_1(a_2a_3)^\perp = \big(y_1(x_3y_2 - x_2y_3) - z_1(z_2x_3 - z_3x_2) : z_1(y_2z_3 - y_3z_2) - x_1(x_3y_2 - x_2y_3) : y_1(y_2z_3 - y_3z_2) - x_1(z_2x_3 - z_3x_2)\big)$$

$$\boxed{\begin{matrix} y_1x_3y_2 \\ \shortparallel \\ y_1y_2x_3 \\ \equiv \\ yyx \end{matrix}} \;!!$$

$$\det\begin{pmatrix} yyx - yxy - zzx + zxz & zyz - zzy - xyx + xxy & yyz - yzy - xzx + xxz \\ xyy - yyx - xzz + zzx & zzy - yzz - xxy + yxx & zyy - yyz - xxz + zxx \\ yxy - xyy - zxz + xzz & yzz - zyz - yxx + xyx & yzy - zyy - zxx + xzx \end{pmatrix}$$

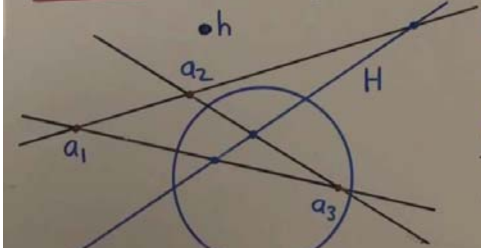
$= 0$, since the sum of all 3 rows is $(0\,0\,0)$.

⑦

Ortholine theorem

The three altitude points of a non-dual triangle are collinear.

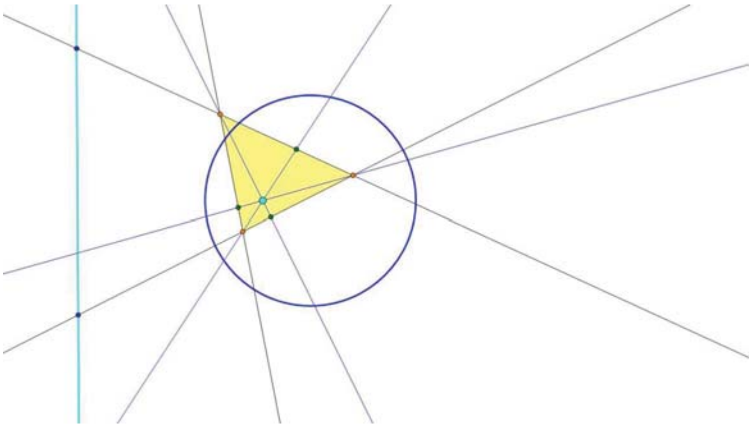
Def. The join of the altitude points is the ortholine of a triangle, denoted by H.

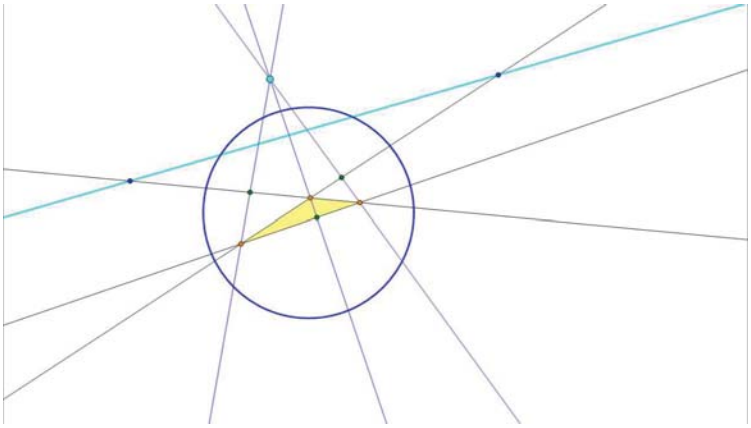


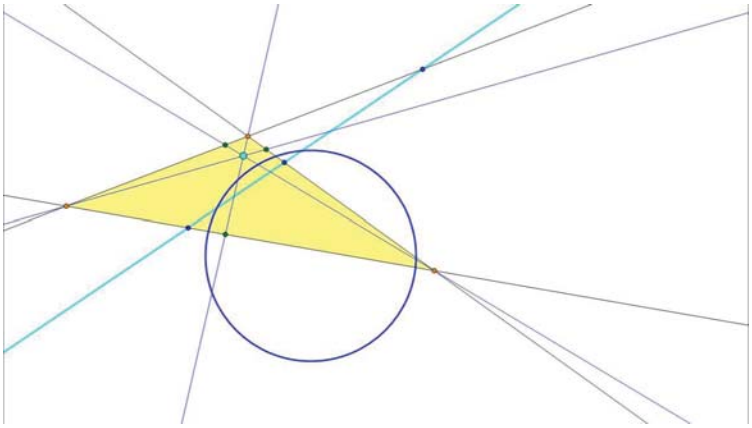
Exer 10.3 Write out a proof.

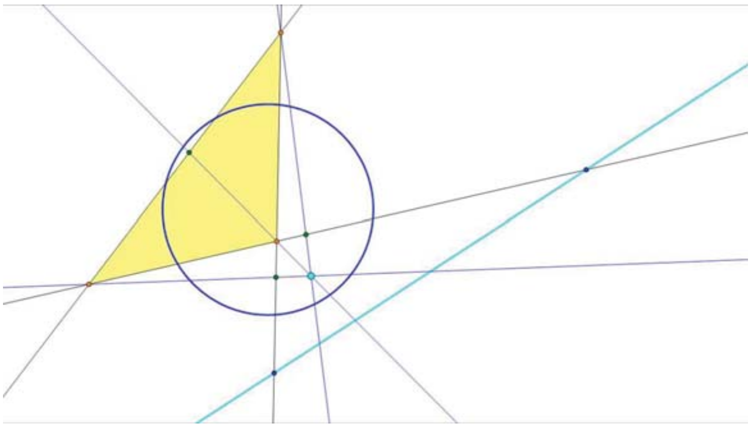
Exer 10.4 Show that the orthocenter & ortholine are dual.

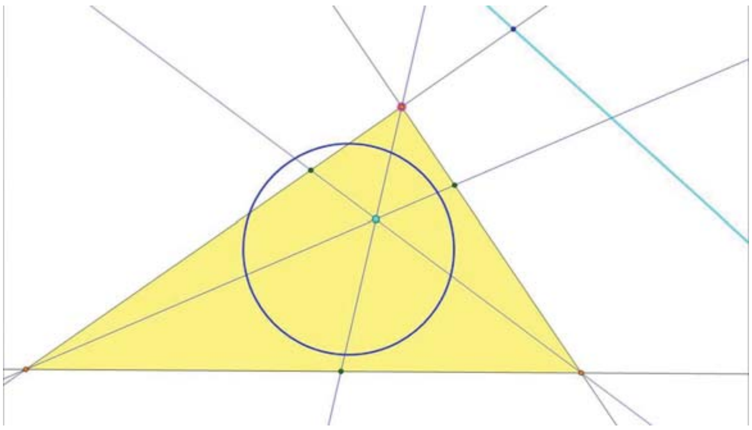
⑧

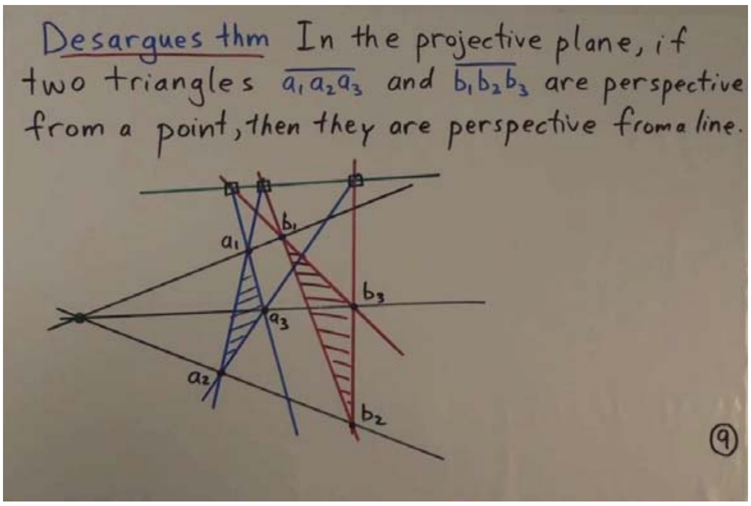


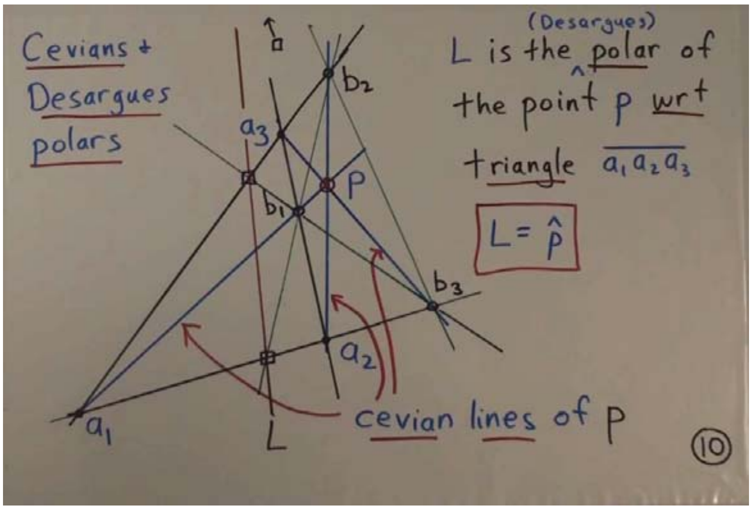


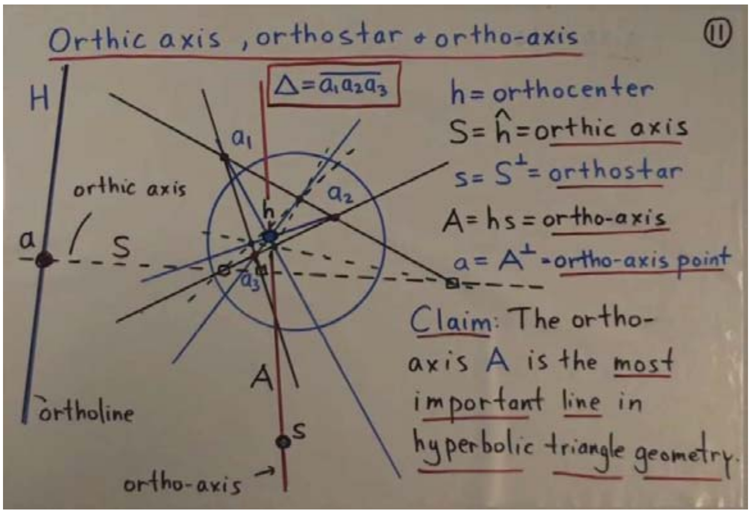


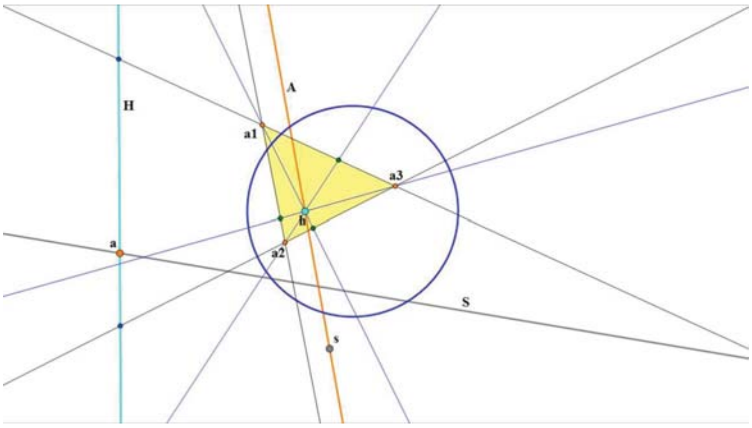


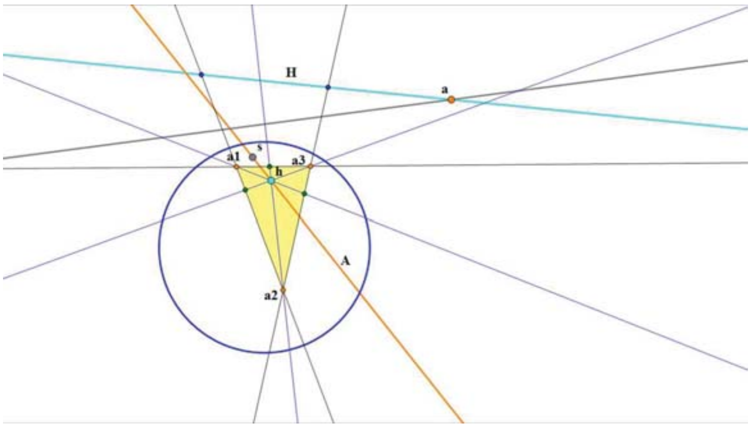


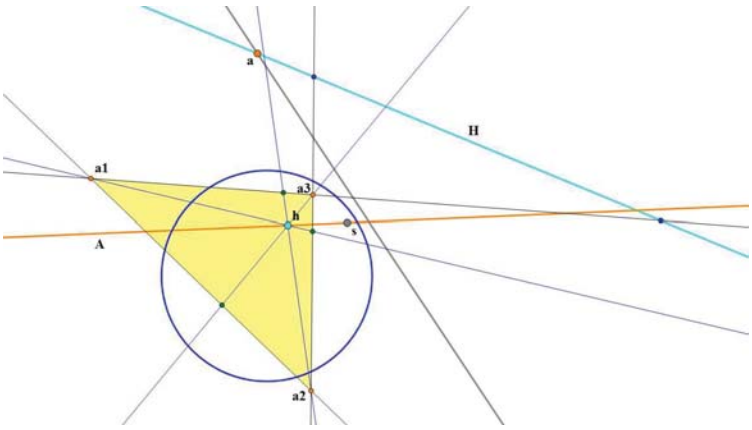

Desargues thm In the projective plane, if two triangles a1a2a3 and b1b2b3 are perspective from a point, then they are perspective from a line.
a1
b1
b3
a3
a2
b2
9

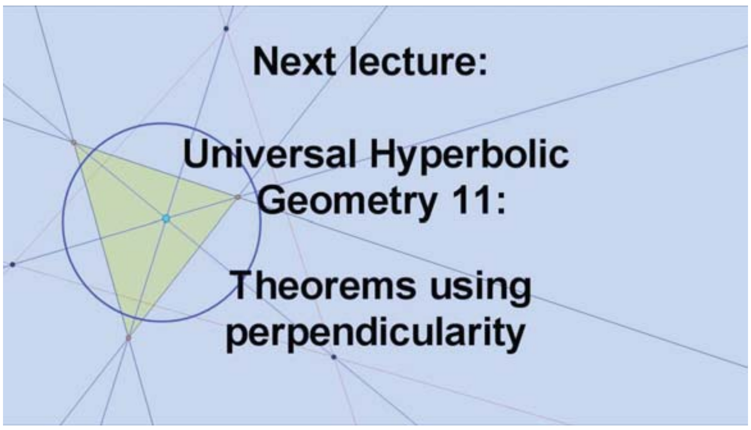

Cevians + Desargues polars
(Desargues)
L is the polar of the point p wrt triangle $\overline{a_1a_2a_3}$
$L = \hat{p}$
cevian lines of p
a_1 a_2 a_3 b_1 b_2 b_3 p L
10

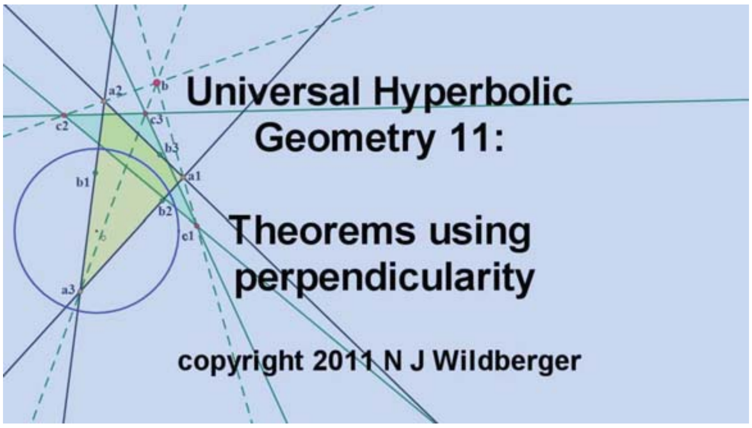

Orthic axis, orthostar + ortho-axis
11
$\Delta = \overline{a_1a_2a_3}$
h = orthocenter
$S = \hat{h}$ = orthic axis
$s = S^{\perp}$ = orthostar
A = hs = ortho-axis
$a = A^{\perp}$ = ortho-axis point
Claim: The ortho-axis A is the most important line in hyperbolic triangle geometry.
orthic axis
ortholine
ortho-axis
H a S A s h a_1 a_2 a_3


H
A
a1
a3
a
h
a2
S
s

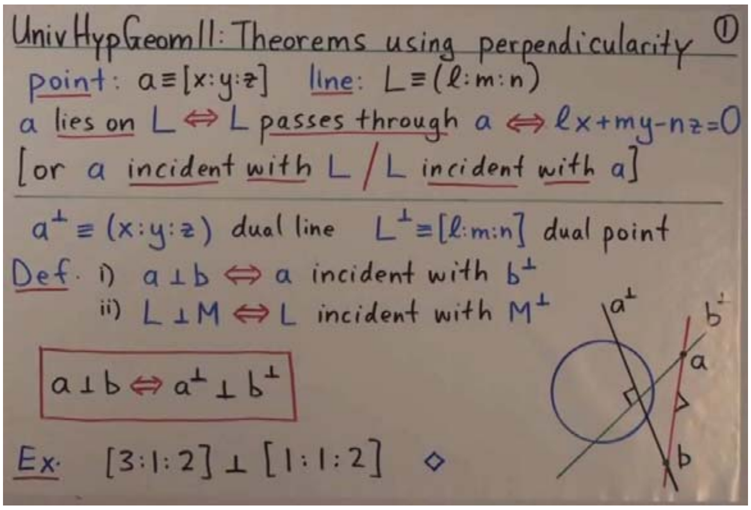

H
a
a1
s
a3
h
A
a2


a
H
a1
a3
h
s
A
a2

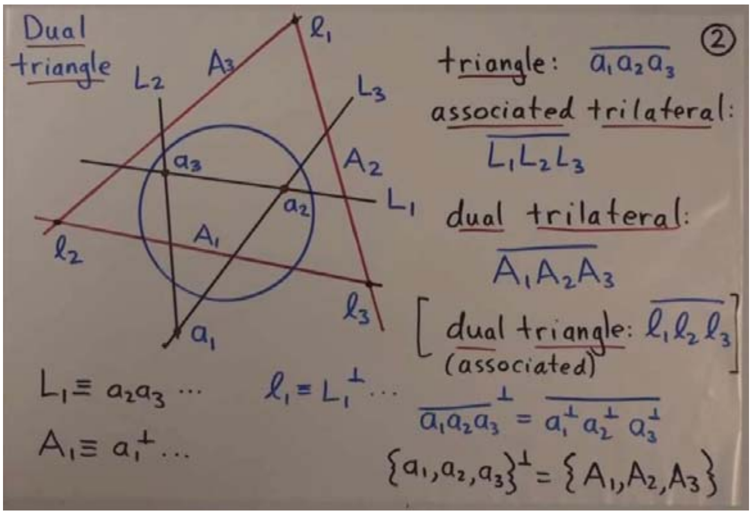

Next lecture:
Universal Hyperbolic Geometry 11:
Theorems using perpendicularity



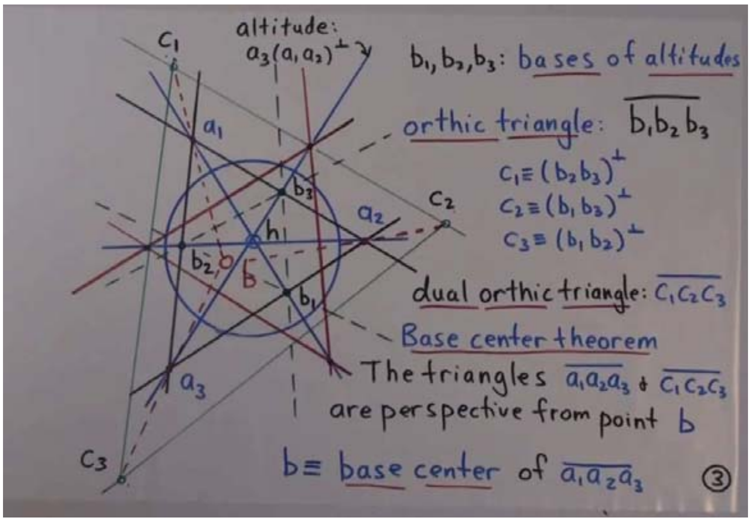
Universal Hyperbolic Geometry 11:
Theorems using perpendicularity
copyright 2011 N J Wildberger



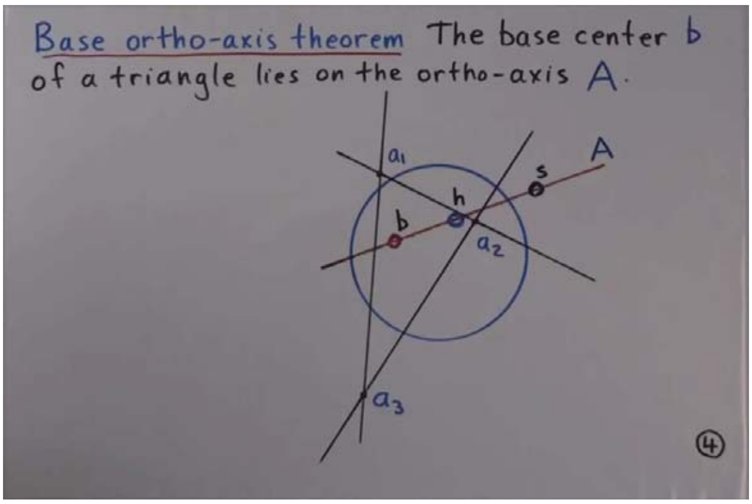
UnivHypGeom11: Theorems using perpendicularity ①
point: $a \equiv [x:y:z]$ line: $L \equiv (l:m:n)$
a lies on L ⇔ L passes through a ⇔ $lx+my-nz=0$
[or a incident with L / L incident with a]
$a^\perp \equiv (x:y:z)$ dual line $L^\perp \equiv [l:m:n]$ dual point
Def. i) $a \perp b$ ⇔ a incident with $b^\perp$
ii) $L \perp M$ ⇔ L incident with $M^\perp$
$a \perp b \Leftrightarrow a^\perp \perp b^\perp$
Ex. $[3:1:2] \perp [1:1:2]$ ◇



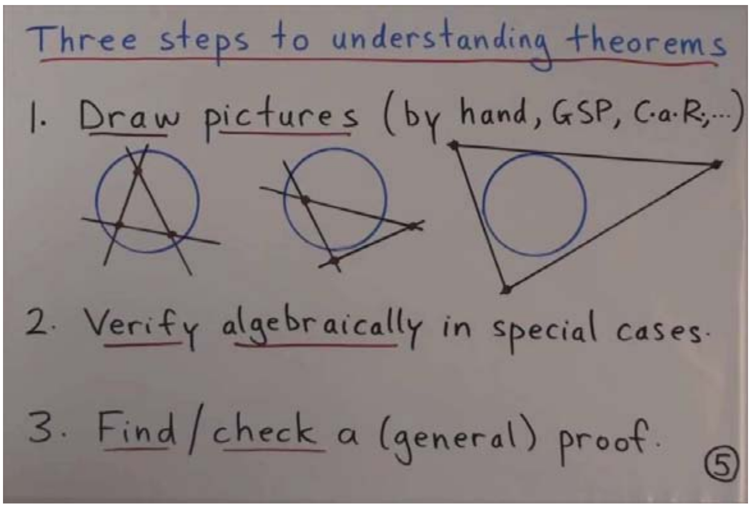
Dual triangle ②
triangle: $\overline{a_1a_2a_3}$
associated trilateral: $\overline{L_1L_2L_3}$
dual trilateral: $\overline{A_1A_2A_3}$
[dual triangle: $\overline{l_1l_2l_3}$ (associated)]
$L_1 \equiv a_2a_3 \ldots$ $l_1 = L_1^\perp \ldots$
$A_1 \equiv a_1^\perp \ldots$
$\overline{a_1a_2a_3}^\perp = \overline{a_1^\perp a_2^\perp a_3^\perp}$
$\{a_1,a_2,a_3\}^\perp = \{A_1,A_2,A_3\}$



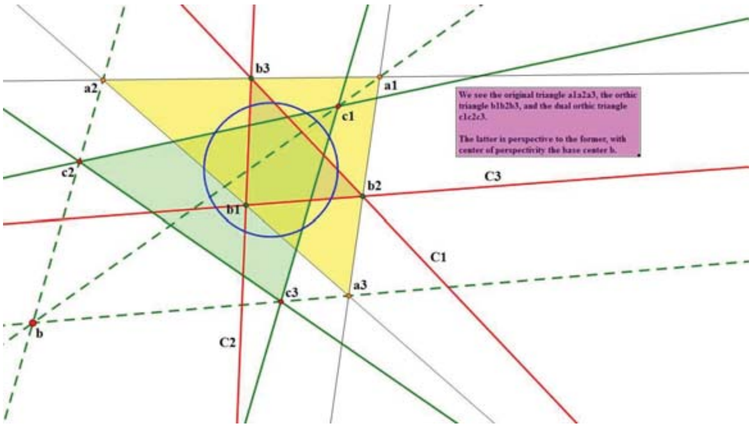
altitude: $a_3(a_1a_2)^\perp$
b_1,b_2,b_3: bases of altitudes
orthic triangle: $\overline{b_1b_2b_3}$
$c_1 \equiv (b_2b_3)^\perp$
$c_2 \equiv (b_1b_3)^\perp$
$c_3 \equiv (b_1b_2)^\perp$
dual orthic triangle: $\overline{c_1c_2c_3}$
Base center theorem
The triangles $\overline{a_1a_2a_3}$ + $\overline{c_1c_2c_3}$ are perspective from point b
b ≡ base center of $\overline{a_1a_2a_3}$ ③

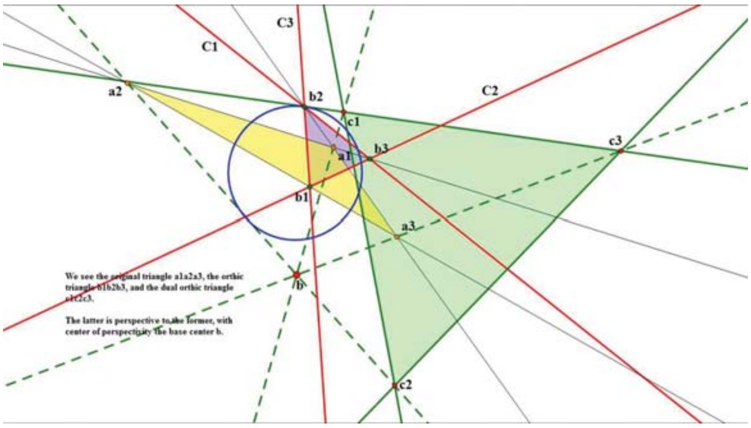


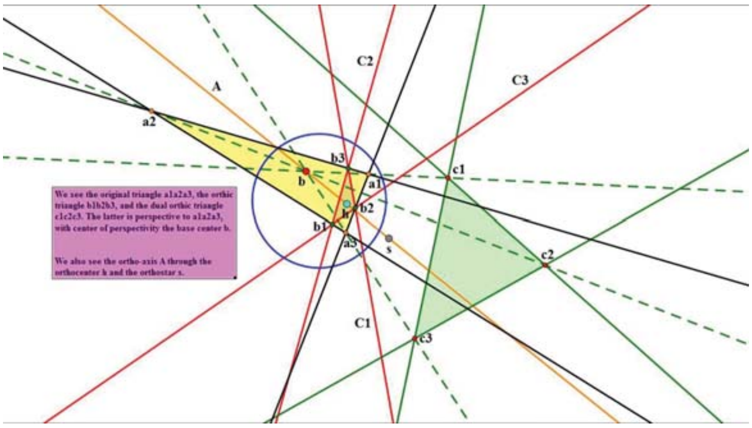
Base ortho-axis theorem The base center b of a triangle lies on the ortho-axis A.
④

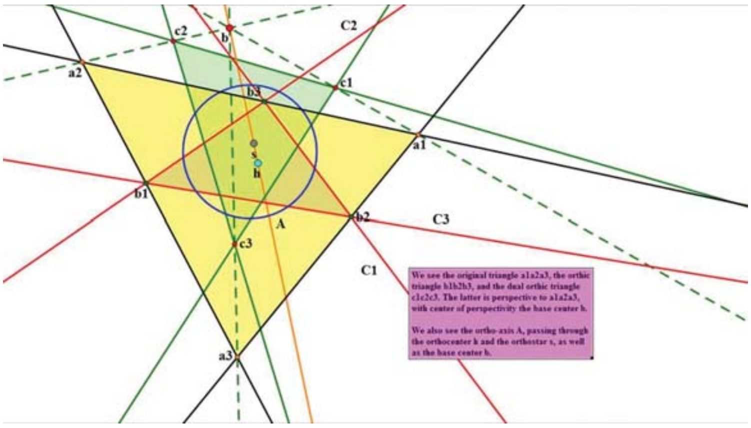


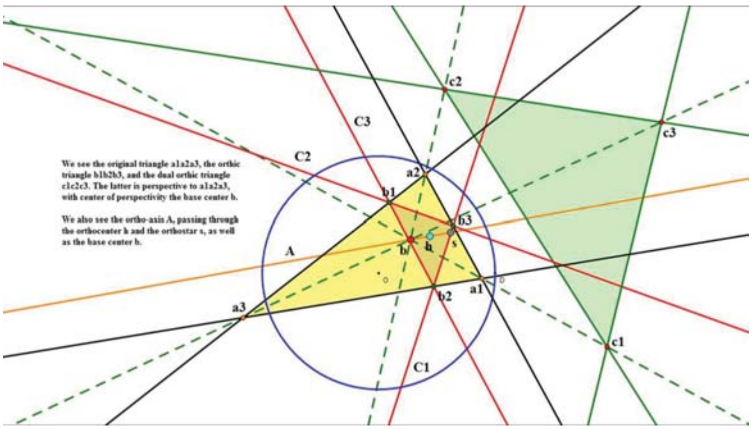
Three steps to understanding theorems
1. Draw pictures (by hand, GSP, C.a.R.,...)
2. Verify algebraically in special cases.
3. Find / check a (general) proof.
⑤

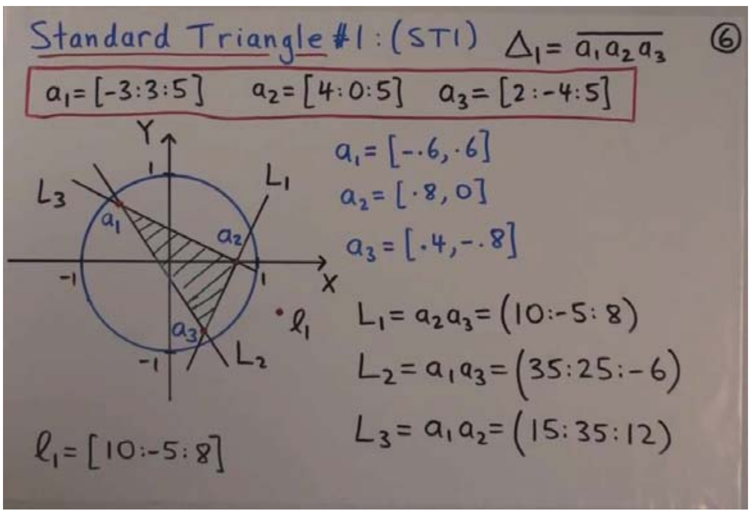

We see the original triangle a1a2a3, the orthic triangle b1b2b3, and the dual orthic triangle c1c2c3.
The latter is perspective to the former, with center of perspectivity the base center b.
a1
a2
a3
b1
b2
b3
c1
c2
c3
b
C1
C2
C3


We see the original triangle a1a2a3, the orthic triangle b1b2b3, and the dual orthic triangle c1c2c3.
The latter is perspective to the former, with center of perspectivity the base center b.
a1
a2
a3
b1
b2
b3
c1
c2
c3
b
C1
C2
C3


We see the original triangle a1a2a3, the orthic triangle b1b2b3, and the dual orthic triangle c1c2c3. The latter is perspective to a1a2a3, with center of perspectivity the base center b.
We also see the ortho-axis A through the orthocenter h and the orthostar s.
A
a1
a2
a3
b1
b2
b3
c1
c2
c3
b
h
s
C1
C2
C3


We see the original triangle a1a2a3, the orthic triangle b1b2b3, and the dual orthic triangle c1c2c3. The latter is perspective to a1a2a3, with center of perspectivity the base center b.
We also see the ortho-axis A, passing through the orthocenter h and the orthostar s, as well as the base center b.
A
a1
a2
a3
b1
b2
b3
c1
c2
c3
b
h
s
C1
C2
C3


We see the original triangle a1a2a3, the orthic triangle b1b2b3, and the dual orthic triangle c1c2c3. The latter in perspective to a1a2a3, with center of perspectivity the base center b.
We also see the ortho-axis A, passing through the orthocenter h and the orthostar s, as well as the base center b.
A
a1
a2
a3
b1
b2
b3
c1
c2
c3
b
h
s
C1
C2
C3

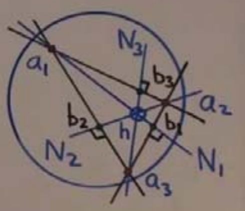

Standard Triangle #1: (ST1) $\Delta_1 = \overline{a_1 a_2 a_3}$
6
$a_1 = [-3:3:5]$ $a_2 = [4:0:5]$ $a_3 = [2:-4:5]$
$a_1 = [-.6, .6]$
$a_2 = [.8, 0]$
$a_3 = [.4, -.8]$
$L_1 = a_2 a_3 = (10:-5:8)$
$L_2 = a_1 a_3 = (35:25:-6)$
$L_3 = a_1 a_2 = (15:35:12)$
$\ell_1 = [10:-5:8]$
Y
X
1
-1
L_1
L_2
L_3
a_1
a_2
a_3
ℓ_1

Altitudes ⑦

$N_1 = a_1(a_2a_3)^\perp = a_1 \ell_1 = (49:74:15)$

$N_2 = a_2(a_1a_3)^\perp = (125:-199:100)$

$N_3 = a_3(a_1a_2)^\perp = (223:-51:130)$



Orthocenter

$h = N_1N_2 = N_2N_3 = N_3N_1$

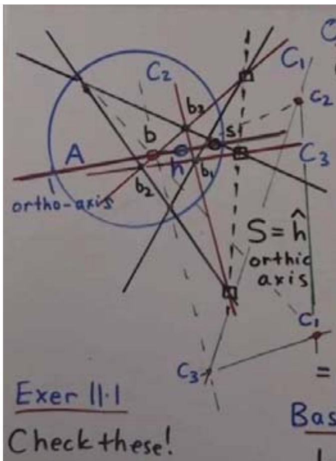
$= [10{,}385 : -3025 : 19{,}001]$

Base points

$b_1 = N_1L_1 = [667:-242:985]$

$b_2 = N_2L_2 = [653:-2125:5045]$

$b_3 = N_3L_3 = [2581:363:4285]$



Orthic lines ⑧

$C_1 = (7430:-6945:3887)$

$C_2 = (11{,}525:2610:7163)$

$C_3 = (3195:-9970:4613)$

Orthic axis

$S = \hat{h} = (7290:-415:7889)$

Ortho-axis

$A = hs =$

$= (187{,}986:-665{,}765:208{,}735)$

Base center

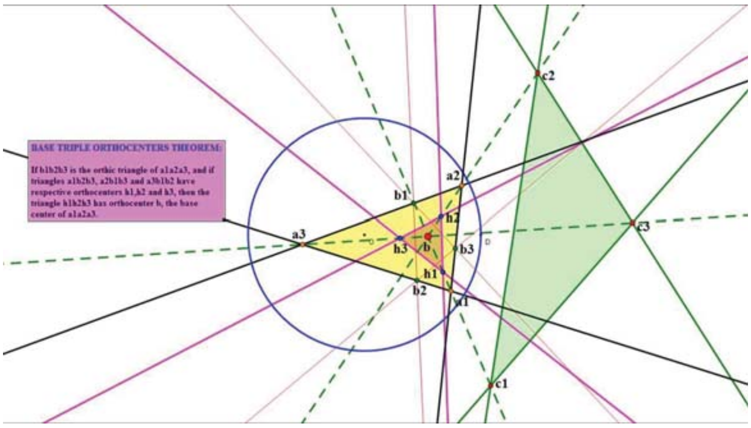
$b = [3095:-2610:11{,}112]$

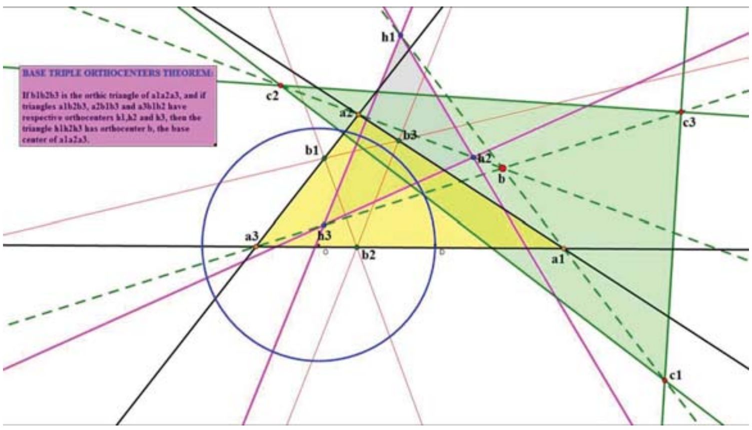
Exer 11·1

Check these! ◇

Base triple orthocenter theorem ⑨

Suppose that the triangle $\overline{a_1a_2a_3}$ has orthic triangle $\overline{b_1b_2b_3}$. Suppose that h_1, h_2, h_3 are the respective orthocenters of $\overline{a_1b_2b_3}$, $\overline{a_2b_1b_3}$ and $\overline{a_3b_1b_2}$. Then the orthocenter of $\overline{h_1h_2h_3}$ is the base center b of $\overline{a_1a_2a_3}$. Furthermore b is the center of perspectivity between $\overline{a_1a_2a_3}$ and $\overline{h_1h_2h_3}$.





Ex (ST1) ⑩

$h_1 = \text{orthocenter}(\overline{a_1b_2b_3})$

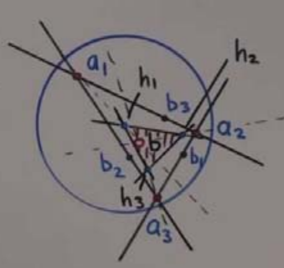
$= [2770131775 : -2244686655 : 13010262679]$

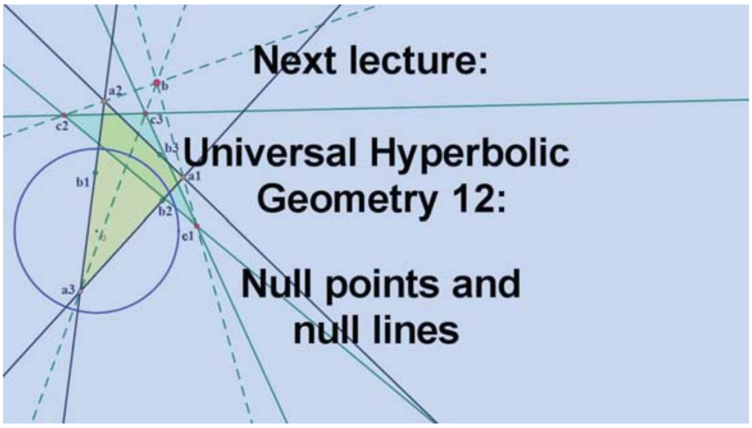
$h_2 = \text{orthocenter}(\overline{a_2b_1b_3})$

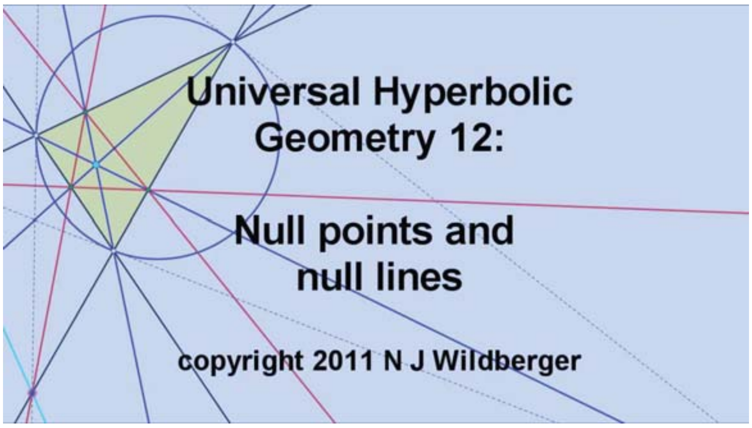
$= [710857435 : -38213010 : 994620217]$

$h_3 = \text{orthocenter}(\overline{a_3b_1b_2})$

$= [445198925 : -623584030 : 1364495423]$ ◇




Next lecture:
Universal Hyperbolic
Geometry 12:
Null points and
null lines
a1
a2
a3
b
b1
b2
b3
c1
c2
c3



UnivHypGeom12: Null points & null lines ①

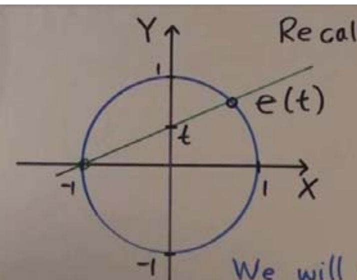
$$[x:y:z]^{\perp} = (x:y:z)$$

Def. A point $a \equiv [x:y:z]$ is null $\Leftrightarrow$ a is incident with $a^{\perp}$ $\Leftrightarrow$ $\boxed{x^2+y^2-z^2=0.}$

Def. A line $L \equiv (\ell:m:n)$ is null $\Leftrightarrow$ L is incident with $L^{\perp}$ $\Leftrightarrow$ $\boxed{\ell^2+m^2-n^2=0.}$

Note: a is a null point $\Leftrightarrow$ $a^{\perp}$ is a null line

Ex. $a \equiv [3:4:5]$ $\quad L=(5:12:13)$ $\diamond$



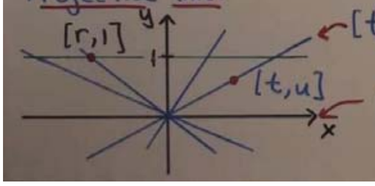
Recall: Rational parametrization of unit (null) circle

$$\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) \equiv e(t)$$

$[-1,0] \approx e(\infty)$??

We will now improve this!

Projective line:



$[t:u] = [\frac{t}{u}:1]$ if $u \neq 0$

$[1:0]$ ②

Projective parametrization of the unit circle. ③

Every null point α is of the form

$$\alpha = e(t:u) \equiv [u^2-t^2 : 2ut : u^2+t^2]$$

for a unique proportion $t:u$.

Proof: Note $(u^2-t^2)^2+(2ut)^2 = (u^2+t^2)^2$;
also $e(t:1) = e(t)$ and $e(1:0) = [-1:0:1]$ ■

Projective parametrization of null lines.

Every null line Φ is of the form

$$\Phi = E(t:u) = (u^2-t^2 : 2ut : u^2+t^2)$$

for a unique proportion $t:u$. ■

Note: The three quadratic forms ④

$Q_r(u,t) \equiv u^2-t^2$ $\quad Q_g(u,t) \equiv 2ut$ $\quad Q_b(u,t) \equiv u^2+t^2$

are also the basis for Chromogeometry

[see YouTube: WildTrig 53-57]

Note: The projective approach via homogeneous co-ords establishes hyperbolic geometry as a theory over the integers!!

Note: Each quad. form has a bilinear form:

$$\begin{pmatrix}u_1\\t_1\end{pmatrix} \cdot_r \begin{pmatrix}u_2\\t_2\end{pmatrix} \equiv u_1u_2 - t_1t_2 \qquad \begin{pmatrix}u_1\\t_1\end{pmatrix} \cdot_g \begin{pmatrix}u_2\\t_2\end{pmatrix} \equiv t_1u_2 + t_2u_1$$

Join of null points theorem The line L passing through $\alpha_1 \equiv e(t_1:u_1)$ and $\alpha_2 \equiv e(t_2:u_2)$ is ⑤

$$\boxed{L \equiv \alpha_1\alpha_2 = (u_1u_2 - t_1t_2 : t_1u_2 + t_2u_1 : u_1u_2 + t_1t_2)}$$

Proof 1. First check that L is a valid proportion. If $u_1u_2 - t_1t_2 = t_1u_2 + t_2u_1 = u_1u_2 + t_1t_2 = 0$ then $0 = u_1u_2 = t_1t_2$. But since $t_1:u_1$ & $t_2:u_2$ are valid, either $u_1 = t_2 = 0$ & $u_2, t_1 \neq 0$ or $u_2 = t_1 = 0$ & $u_1, t_2 \neq 0$. So $t_1u_2 + t_2u_1 \neq 0$, contradiction. Now just observe

$$(u_1^2 - t_1^2)(u_1u_2 - t_1t_2) + 2u_1t_1(t_1u_2 + t_2u_1) - (u_1^2 + t_1^2)(u_1u_2 + t_1t_2) = 0.$$

So α_1 lies on L. The same hold for α_2, so $L = \alpha_1\alpha_2$ ■

Proof 2 $e(t_1:u_1)=[u_1^2-t_1^2 : 2u_1t_1 : u_1^2+t_1^2]\equiv\alpha_1$ ⑥

$e(t_2:u_2)=[u_2^2-t_2^2 : 2u_2t_2 : u_2^2+t_2^2]\equiv\alpha_2$

So
$$\alpha_1\alpha_2=\big(2u_1t_1(u_2^2+t_2^2)-2u_2t_2(u_1^2+t_1^2) : (u_1^2+t_1^2)(u_2^2-t_2^2)-(u_2^2+t_2^2)(u_1^2-t_1^2) : 2u_1t_1(u_2^2-t_2^2)-2u_2t_2(u_1^2-t_1^2)\big)$$
$$=\big((t_1u_2-t_2u_1)(u_1u_2-t_1t_2) : (t_1u_2-t_2u_1)(t_1u_2+t_2u_1) : (t_1u_2-t_2u_1)(u_1u_2+t_1t_2)\big)$$
$$=(u_1u_2-t_1t_2 : t_1u_2+t_2u_1 : u_1u_2+t_1t_2)$$
since $t_1u_2-t_2u_1\neq 0$ [α_1 and α_2 are distinct!] ■

Meet of null lines theorem The point a lying on the null lines $\Phi_1\equiv E(t_1:u_1)$ and $\Phi_2\equiv E(t_2:u_2)$ is ⑦

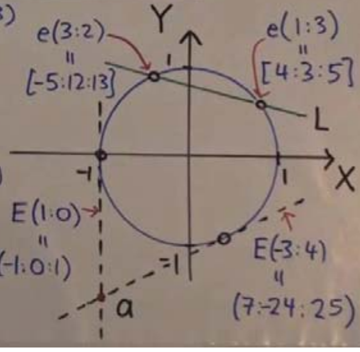
$$\boxed{a\equiv\Phi_1\Phi_2=[u_1u_2-t_1t_2 : t_1u_2+t_2u_1 : u_1u_2+t_1t_2]}$$

Ex. The join of $e(3:2)$ & $e(1:3)$ is $L=(6-3:9+2:6+3)$
$=(3:11:9)$

$[3X+11Y=9]$ ◇

Ex. The meet of $E(1:0)$ & $E(-3:4)$ is $a=[0+3:4+0:0-3]$
$=[-3:-4:3]$

(or $[-1,-\frac{4}{3}]$) ◇

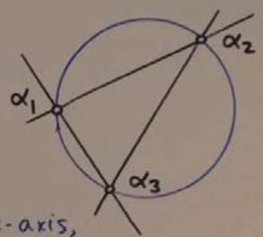


A triply nill triangle (ST 2) ⑧

$\alpha_1\equiv e(1:0)=[-1:0:1]$

$\alpha_2\equiv e(1:2)=[3:4:5]$

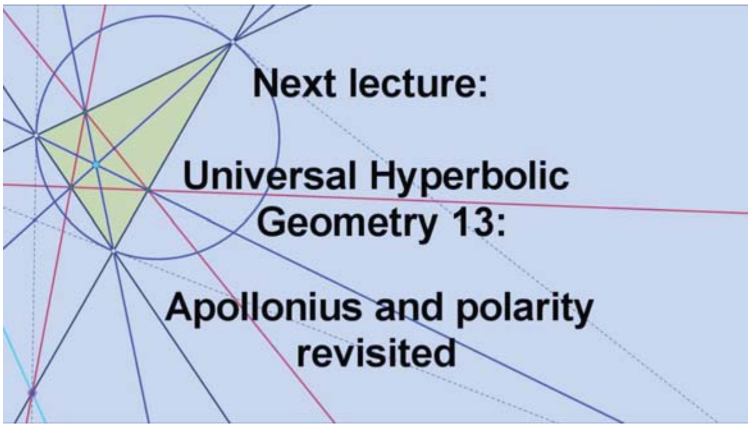
$\alpha_3\equiv e(-4:3)=[-7:-24:25]$

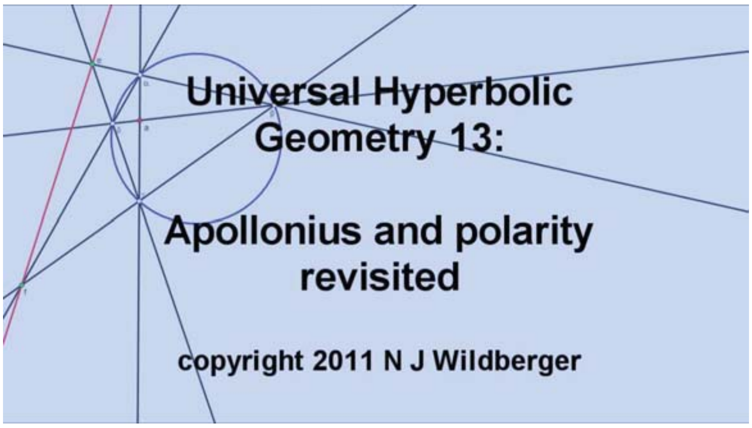


Exer 12·1 Compute & draw altitudes, orthocenter, orthic-axis, orthic triangle, orthostar, base center... ◇

Exer 12·2 Show that the altitudes of $\overline{\alpha_1\alpha_2\alpha_3}$ & its orthic triangle coincide. ◇

Exer 12·3 Try to generalize discoveries. ◇





UnivHypGeom13: Apollonius + polarity revisited ①

Null point: $\alpha = e(t:u) \equiv [u^2-t^2 : 2ut : u^2+t^2]$

Null line: $\Phi = E(t:u) \equiv (u^2-t^2 : 2ut : u^2+t^2)$

Join of null points: $\alpha_1\alpha_2 = F(t_1:u_1 \mid t_2:u_2)$

$$\equiv (u_1u_2 - t_1t_2 : t_1u_2 + t_2u_1 : u_1u_2 + t_1t_2)$$

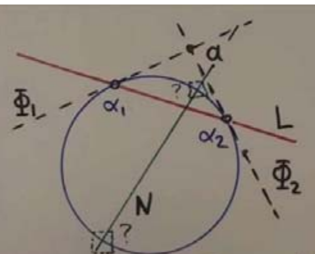
Meet of null lines: $\Phi_1\Phi_2 = f(t_1:u_1 \mid t_2:u_2)$

$$\equiv [u_1u_2 - t_1t_2 : t_1u_2 + t_2u_1 : u_1u_2 + t_1t_2]$$

A line L is interior $\Leftrightarrow$ $L = \alpha_1\alpha_2$ where α_1, α_2 null

A point a is exterior $\Leftrightarrow$ $a = \Phi_1\Phi_2$ where Φ_1, Φ_2 null

②



If α is a null point, then $\alpha^\perp$ is a null line, and conversely.

So $L = \alpha_1\alpha_2$ is interior $\Leftrightarrow$ $a = L^\perp = \alpha_1^\perp\alpha_2^\perp$ is exterior.

WARNING! The notions of interior/exterior are more subtle than they appear.

A line may look like it is interior, but in fact it isn't! A point may look like it is exterior, but in fact it isn't! Ex. $N = (1:-1:0)$

A quadrangle $\overline{abcd}$ is a set of four points, no three of which are collinear:

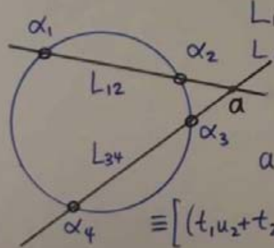
$$\overline{abcd} = \{a, b, c, d\}$$

A quadrilateral $\overline{PQRS}$ is a set of four lines, no three of which are concurrent:

$$\overline{PQRS} = \{P, Q, R, S\}$$

Apollonius, polarity $\Rightarrow$ we should consider quadrangles of null points, and quadrilaterals of null lines. ③

④



$L_{12} = (u_1u_2 - t_1t_2 : t_1u_2 + t_2u_1 : u_1u_2 + t_1t_2)$

$L_{34} = (u_3u_4 - t_3t_4 : t_3u_4 + t_4u_3 : u_3u_4 + t_3t_4)$

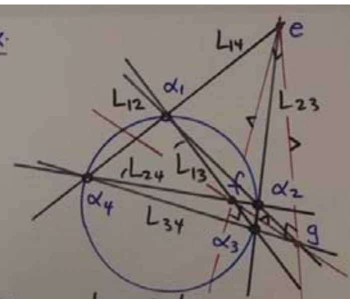
These two lines meet at a:

$$a = g(t_1:u_1 \mid t_2:u_2 \,\|\, t_3:u_3 \mid t_4:u_4)$$

$$\equiv [(t_1u_2+t_2u_1)(u_3u_4+t_3t_4) - (t_3u_4+t_4u_3)(u_1u_2+t_1t_2) : (u_1u_2+t_1t_2)(u_3u_4-t_3t_4) - (u_3u_4+t_3t_4)(u_1u_2-t_1t_2) : (t_1u_2+t_2u_1)(u_3u_4-t_3t_4) - (t_3u_4+t_4u_3)(u_1u_2-t_1t_2)]$$

$$= [(t_1u_2+t_2u_1)(u_3u_4+t_3t_4) - (t_3u_4+t_4u_3)(u_1u_2+t_1t_2) : 2(t_1t_2u_3u_4 - t_3t_4u_1u_2) : (t_1u_2+t_2u_1)(u_3u_4-t_3t_4) - (t_3u_4+t_4u_3)(u_1u_2-t_1t_2)]$$

Ex. ⑤



$\alpha_1 \equiv e(1:1) = [0:1:1]$

$\alpha_2 \equiv e(0:1) = [1:0:1]$

$\alpha_3 \equiv e(-1:5) = [12:-5:13]$

$\alpha_4 \equiv e(6:1) = [-35:12:37]$

$L_{12} = \alpha_1\alpha_2 = (1:1:1)$

$L_{34} = (-11:-29:1)$

$L_{13} = (3:2:2)$

$L_{24} = (1:6:1)$

$L_{14} = (-5:7:7)$

$L_{23} = (5:-1:5)$

Diagonal points

$e = L_{14}L_{23} = [7:10:5]$

$f = L_{13}L_{24} = [10:1:16]$

$g = L_{12}L_{34} = [5:-2:3]$

$$\boxed{ef = g^\perp \quad eg = f^\perp \quad fg = e^\perp}$$

◇

<u>Nil quadrangle diagonals theorem</u>. If $\overline{\alpha_1\alpha_2\alpha_3\alpha_4}$ is a quadrangle of null points, then the diagonal points

$$e \equiv (\alpha_1\alpha_4)(\alpha_2\alpha_3) \qquad f \equiv (\alpha_1\alpha_3)(\alpha_2\alpha_4) \qquad g \equiv (\alpha_1\alpha_2)(\alpha_3\alpha_4)$$

are mutually perpendicular, and distinct.

So $e^\perp = fg \quad f^\perp = eg \quad g^\perp = ef$.

<u>Proof</u>. First observe that e, f, g are distinct; for otherwise if say $e=f$ then e lies on both $\alpha_1\alpha_4$ and $\alpha_1\alpha_3$, and since $\alpha_1, \alpha_3, \alpha_4$ are not collinear, $e=f=\alpha_1$ and similary $e=f=\alpha_2$, impossible.

6

$\boxed{f \perp g}$ $[(t_1u_3+t_3u_1)(u_2u_4+t_2t_4)-(t_2u_4+t_4u_2)(u_1u_3+t_1t_3)]$

$\times[(t_1u_2+t_2u_1)(u_3u_4+t_3t_4)-(t_3u_4+t_4u_3)(u_1u_2+t_1t_2)]$

$+2(t_1t_3u_2u_4-t_2t_4u_1u_3)\times 2(t_1t_2u_3u_4-t_3t_4u_1u_2)$

$-[(t_1u_3+t_3u_1)(u_2u_4-t_2t_4)-(t_2u_4+t_4u_2)(u_1u_3-t_1t_3)]$

$\times[(t_1u_2+t_2u_1)(u_3u_4-t_3t_4)-(t_3u_4+t_4u_3)(u_1u_2-t_1t_2)]$

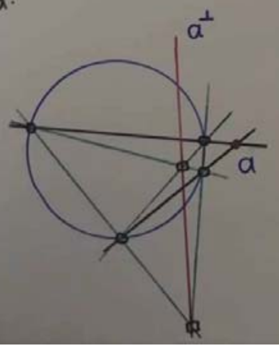
$=0$.

By symmetry, $g \perp e \quad e \perp f$. Now since g, f are distinct & both perpendicular to e,

$e^\perp = fg$. Similarly $f^\perp = eg$, $g^\perp = ef$. ∎

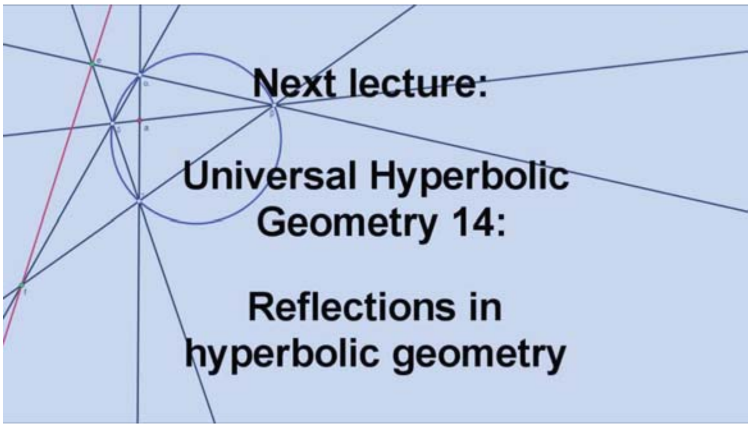
7

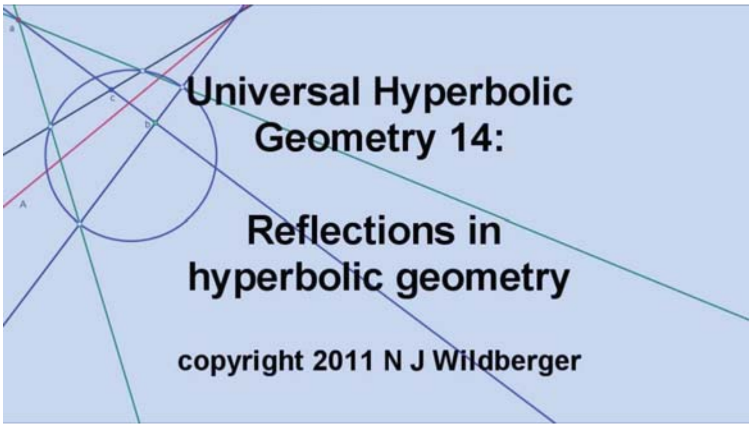
As a Corollary, we deduce that the dual of a point may indeed be constructed as a polar, and that is independent of the nil quadrangle used.



Since $[a:b:c] \sim \left[\frac{a}{c}, \frac{b}{c}\right]$ and $(a:b:c) \sim aX+bY=c$ we deduce that the dual/polar of $[r,s]$ is $rX+sY=1$.

8





Universal Hyperbolic Geometry 14:

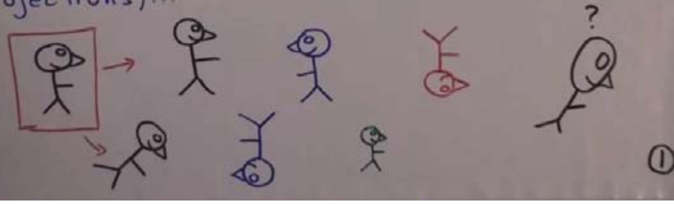
Reflections in hyperbolic geometry

UnivHypGeom14: Reflections in hyperbolic geom.

Symmetries of a geometry ⇒ group theory

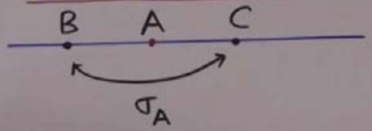
F. Klein's Erlangen program: "Geometry is a group of transformations acting on a space." (1872)

Types of symmetries: translations, rotations, reflections, glide reflections, ..., dilations, ... projections, ...



①

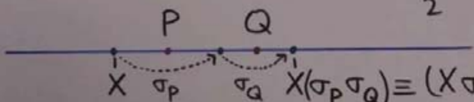
Reflections in Euclidean geometry: 1 dim ②



reflection in A: σ_A

$\boxed{B = C\sigma_A}$ $\boxed{C = B\sigma_A}$

$\frac{b+c}{2} = a \Leftrightarrow c = 2a - b$

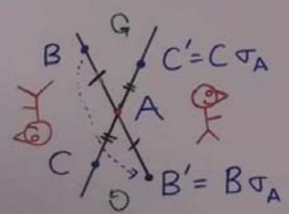


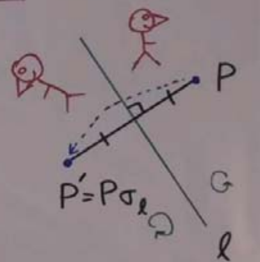
$X(\sigma_P\sigma_Q) \equiv (X\sigma_P)\sigma_Q$

$\sigma_P\sigma_Q$ is translation (by twice $\overrightarrow{PQ}$)

Exer 14.1 Check this geometrically & using co-ordinates. ◇

Reflections in Eucl. geom: 2 dims ③

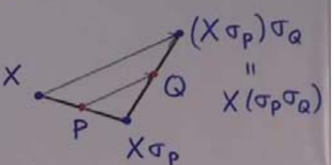




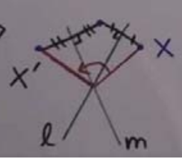
reflection in a point A: σ_A (requires compass) preserves orientation

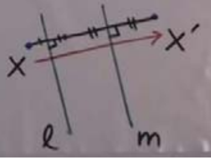
reflection in a line ℓ: σ_ℓ (requires compass) reverses orientation

④ Exer 14.2 Show that if P, Q are points in Euclidean plane, then $\sigma_P\sigma_Q$ is translation by $2\overrightarrow{PQ}$. ◇



Exer 14.3 Show that if ℓ, m are lines in Eucl. plane, then $\sigma_\ell\sigma_m$ is either a rotation or a translation. ◇





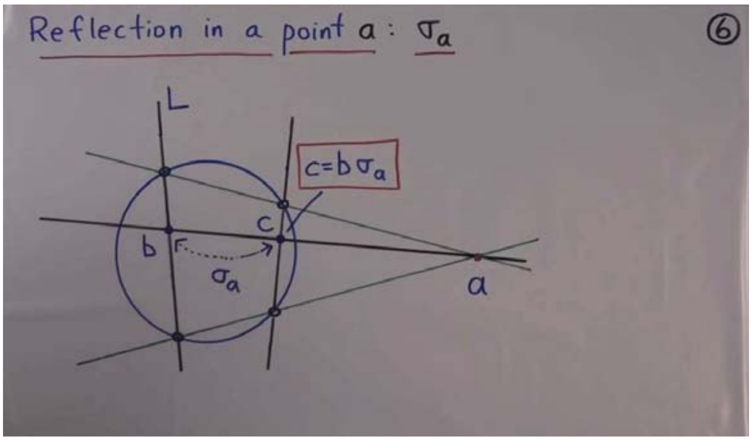
Reflections in Universal Hyperbolic Geometry

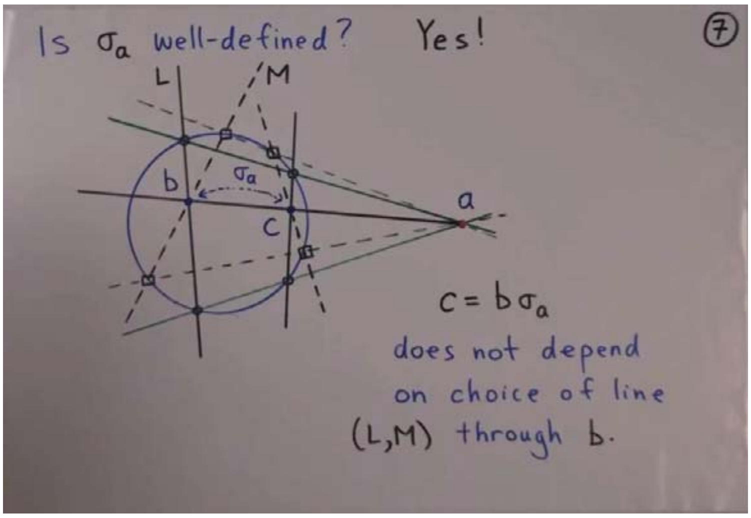
- reflection in a point a: σ_a
- reflection in a line L: σ_L

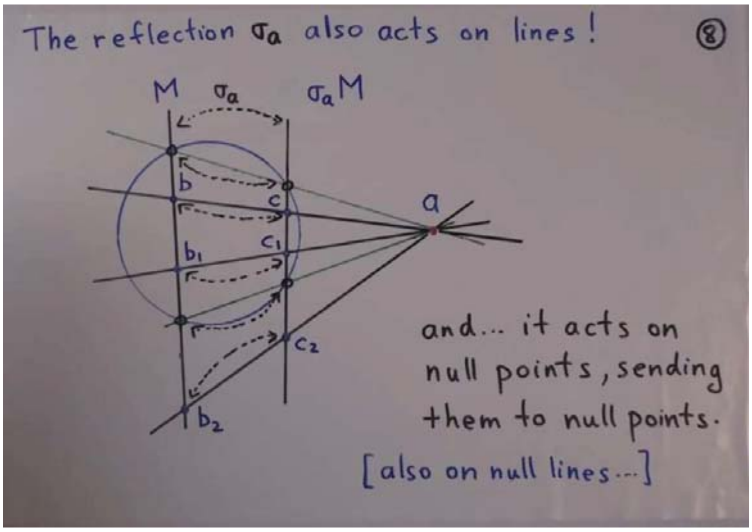
Theorem If $L = a^\perp$ then $\boxed{\sigma_a = \sigma_L}$.

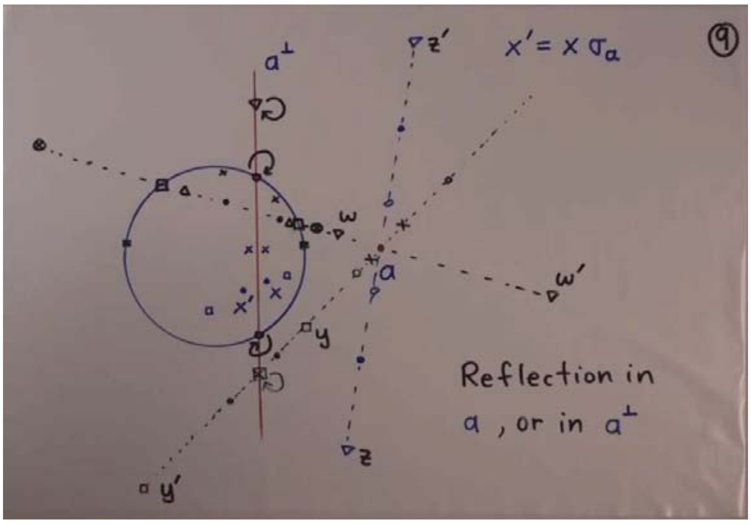
Another big difference: the projective plane is not orientable.

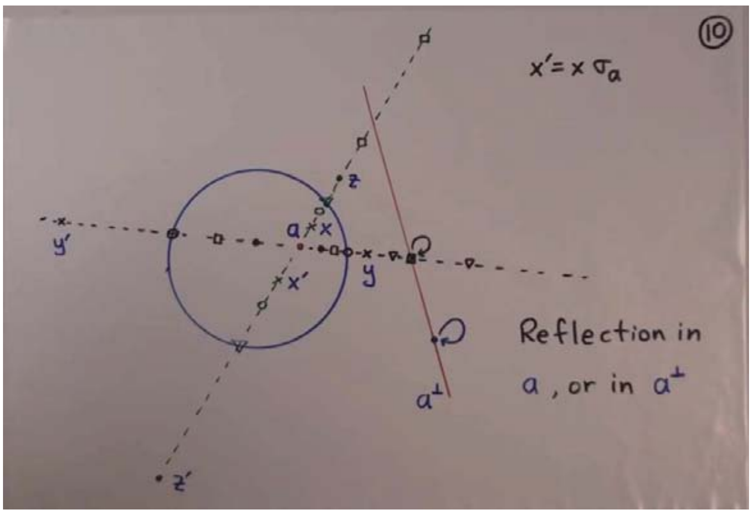
The symmetry group of the (universal) hyperbolic plane is very rich! However the generators σ_a (or σ_L) are simple & elegant. ⑤

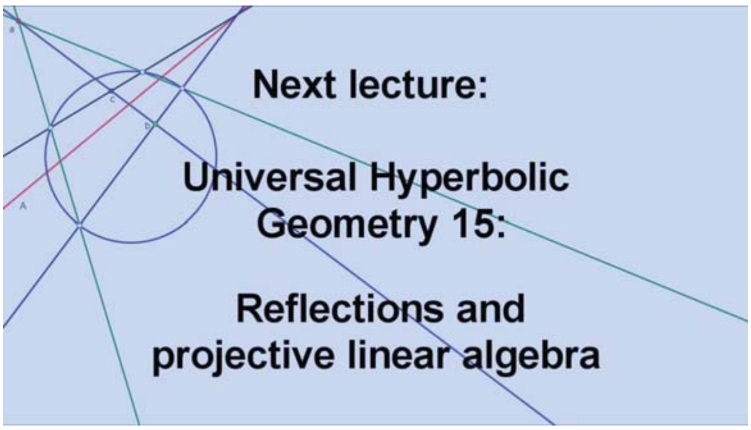

Reflection in a point a : σa
⑥
L
c=bσa
c
b
σa
a

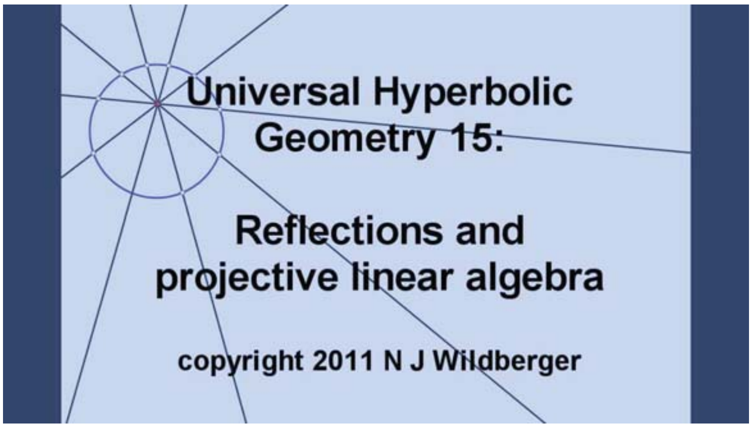

Is σa well-defined? Yes!
⑦
L
M
σa
b
a
c
c = bσa
does not depend
on choice of line
(L,M) through b.


The reflection σa also acts on lines!
⑧
M σa σaM
a
b
c
b1
c1
c2
b2
and... it acts on
null points, sending
them to null points.
[also on null lines...]


a⊥
z'
x' = x σa
⑨
w
a
w'
x
x'
y
y'
z
Reflection in
a , or in a⊥

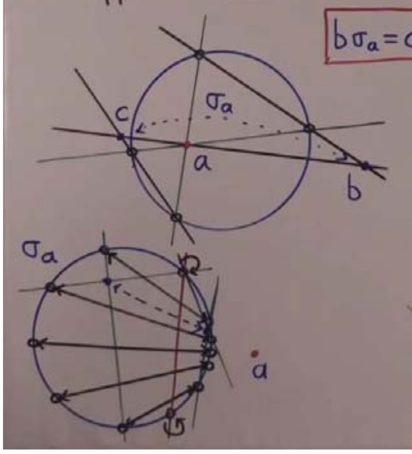

⑩
x' = x σa
z
a
x
y'
x'
y
Reflection in
a , or in a⊥
a⊥
z'

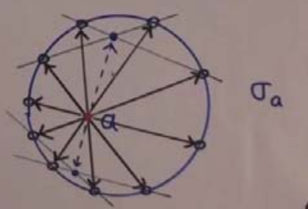

Next lecture:
Universal Hyperbolic
Geometry 15:
Reflections and
projective linear algebra



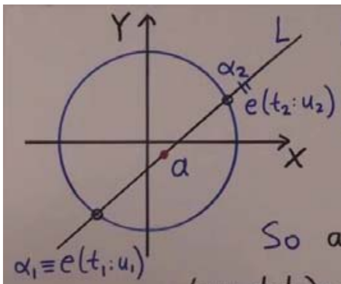
UnivHypGeom15: Reflections + projective linear algebra

$\boxed{b\sigma_a = c}$ Reflection in a is determined by its action on null points





①



②

Null point: $e(t:u) \equiv [u^2-t^2 : 2ut : u^2+t^2]$

Join of null points: $L = \alpha_1\alpha_2$

$= (u_1u_2 - t_1t_2 : t_1u_2 + t_2u_1 : u_1u_2 + t_1t_2)$

So $a \equiv [x:y:z]$ lies on $L \Leftrightarrow$

$(u_1u_2 - t_1t_2)x + (t_1u_2 + t_2u_1)y - (u_1u_2 + t_1t_2)z = 0$

$\Leftrightarrow t_2(u_1y - t_1x - t_1z) + u_2(u_1x + t_1y - u_1z) = 0$

$\Leftrightarrow \boxed{t_2 : u_2 = (t_1y + u_1(x-z)) : (t_1(x+z) - u_1y)} \quad (*)$

Linear algebra (in 2 dim's) [WildLinAlg 6+7] ③

row vector: $(x\ y)$ ← 1x2 column vector: $\begin{pmatrix} l \\ m \end{pmatrix}$ ← 2x1

matrix: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ← 2x2

Multiplication: $(x\ y)\begin{pmatrix} l \\ m \end{pmatrix} \equiv (xl + ym) = xl + ym$

$(x\ y)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (xa + yc \quad xb + yd)$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} l \\ m \end{pmatrix} \equiv \begin{pmatrix} al + bm \\ cl + dm \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{pmatrix}$

Associative!!

Projective linear algebra (in 2-dim's) ④

projective row vector: $[x\ y]$

the square brackets here mean invariance under scaling: $\boxed{[x\ y] = [\lambda x\ \lambda y]}$ if $\lambda \neq 0$

Similarly: projective column vector: $\begin{bmatrix} l \\ m \end{bmatrix}$

projective matrix: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Addition not defined, but multiplication is.

$[1\ 2]\begin{bmatrix} 3 \\ 4 \end{bmatrix} = [11] = [1]$

$\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

⑤

Now lets rewrite (*):

$[t_2\ u_2] = [t_1y + u_1(x-z) \quad t_1(x+z) - u_1y]$

$= \underset{\text{(projective)}}{[t_1\ u_1]}\begin{bmatrix} y & x+z \\ x-z & -y \end{bmatrix} \quad (**)$

Def The matrix of the point $a \equiv [x:y:z]$ is the projective matrix

$$m_a \equiv \begin{bmatrix} y & x+z \\ x-z & -y \end{bmatrix}$$

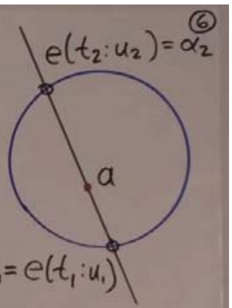
$\operatorname{tr}\begin{pmatrix} y & x+z \\ x-z & -y \end{pmatrix} = 0$

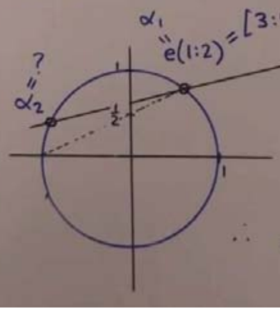
$\det\begin{pmatrix} y & x+z \\ x-z & -y \end{pmatrix} = -x^2 - y^2 + z^2 = -(x^2+y^2-z^2)$

(6)

Reflection matrix theorem

If $[t_2\ u_2]=[t_1\ u_1]\,m_a$ then a lies on $\alpha_1\alpha_2$, where $\alpha_1=e(t_1:u_1)$ and $\alpha_2=e(t_2:u_2)$. ■





$$[t_2\ u_2]=[1\ 2]\begin{bmatrix}1 & 3\\ 1 & -1\end{bmatrix}=[3:1]$$

$\therefore\ \alpha_2=e(3:1)=[-8:6:10]=[-4:3:5]$

Point/matrix correspondence (projective)

$$a\equiv[x:y:z]\ \longleftrightarrow\ m_a\equiv\begin{bmatrix}y & x+z\\ x-z & -y\end{bmatrix}$$

$$m_b\equiv\begin{bmatrix}p & q\\ r & -p\end{bmatrix}\ \longleftrightarrow\ b\equiv\left[\tfrac{r+q}{2}:p:\tfrac{q-r}{2}\right]=[q+r:2p:q-r]$$

This is a complete correspondence between points (hyperbolic) and non-zero projective matrices of trace zero.

Exer 15.1 a is a null point $\Leftrightarrow \det m_a=0$. ◇

(7)

(8)

Exer 15.2 If a is a non-null point, then $m_a^2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ (so $m_a=m_a^{-1}$). ◇

Reflection matrix conjugation theorem.

If $\alpha_1\equiv e(t_1:u_1)$ and $\alpha_2\equiv e(t_2:u_2)$ are null points, and if $a\equiv[x:y:z]$ has the property that $[t_2\ u_2]=[t_1\ u_1]\,m_a$, then

$$\boxed{m_{\alpha_2}=m_a\,m_{\alpha_1}\,m_a}.$$

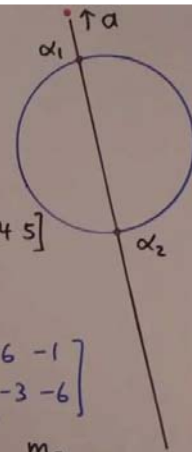
(9)

Ex. $\alpha_1=e(3:2)=[-5:12:13]$

$\alpha_2=e(4:5)=[9:-40:41]$

$\alpha_1\alpha_2=(-22:-7:2)$

Then $a=[1:-3:-\frac{1}{2}]=[-2:6:1]$ lies on $\alpha_1\alpha_2$.



$$[3\ 2]\begin{bmatrix}6 & -1\\ -3 & -6\end{bmatrix}=[12\ \ -15]=[-4\ 5]$$

$$\underbrace{\begin{bmatrix}-40 & 50\\ -32 & 40\end{bmatrix}}_{m_{\alpha_2}}=\underbrace{\begin{bmatrix}6 & -1\\ -3 & -6\end{bmatrix}}_{m_a}\underbrace{\begin{bmatrix}12 & 8\\ -18 & -12\end{bmatrix}}_{m_{\alpha_1}}\underbrace{\begin{bmatrix}6 & -1\\ -3 & -6\end{bmatrix}}_{m_a}$$

◇

(10)

Proof. $a\equiv[x:y:z]$ $\quad [t_2\ u_2]=[t_1\ u_1]\begin{bmatrix}y & x+z\\ x-z & -y\end{bmatrix}$

$\alpha_1=e(t_1:u_1)=[u_1^2-t_1^2:2u_1t_1:u_1^2+t_1^2]$

$$m_{\alpha_1}=\begin{bmatrix}2u_1t_1 & 2u_1^2\\ -2t_1^2 & -2u_1t_1\end{bmatrix}=\begin{bmatrix}u_1t_1 & u_1^2\\ -t_1^2 & -u_1t_1\end{bmatrix}=\begin{bmatrix}u_1\\ -t_1\end{bmatrix}[t_1\ u_1]$$

$$=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}t_1\\ u_1\end{bmatrix}[t_1\ u_1]$$

$$m_{\alpha_2}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}t_2\\ u_2\end{bmatrix}[t_2\ u_2]=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}m_a^T\begin{bmatrix}t_1\\ u_1\end{bmatrix}[t_1\ u_1]\,m_a$$

$$=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}m_a^T\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}m_{\alpha_1}m_a \qquad \text{But}$$

$$\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}m_a^T\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}y & x-z\\ x+z & -y\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}=\begin{bmatrix}y & x+z\\ x-z & -y\end{bmatrix}=m_a$$ ■

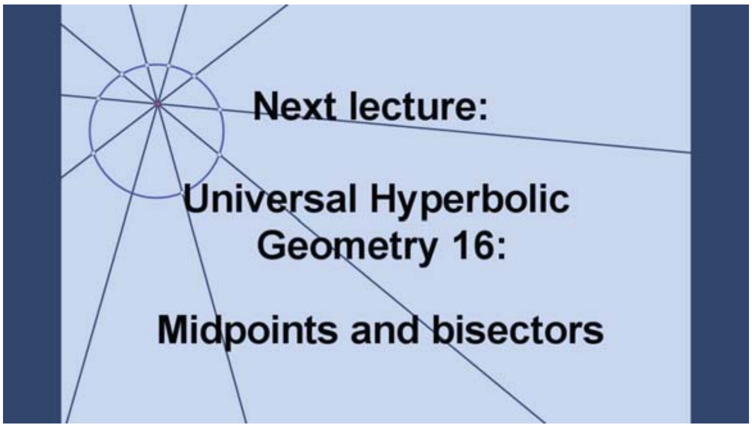
(11)

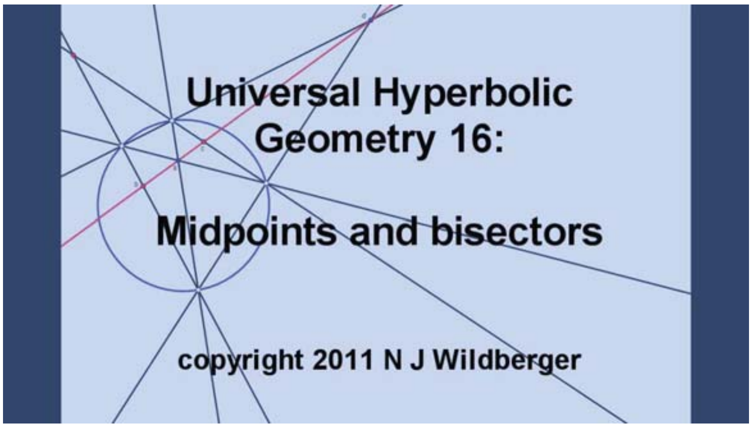
Corollary If $\alpha=e(t:u)$ then $\boxed{m_\alpha=\begin{bmatrix}u\\ -t\end{bmatrix}[t\ u]}$.

Corollary The reflection σ_a sends lines to lines: i.e. if b,c,d are collinear points then so are $b\sigma_a, c\sigma_a, d\sigma_a$.

Proof The action of σ_a on null points, defined by $\alpha_2=\alpha_1\sigma_a \Leftrightarrow [t_2\ u_2]=[t_1\ u_1]\,m_a$ is linear in homogeneous co-ordinates since

$$m_{\alpha_2}=m_a\,m_{\alpha_1}\,m_a.$$ ■


Next lecture:
Universal Hyperbolic Geometry 16:
Midpoints and bisectors



UnivHypGeom16: Midpoints and bisectors ①

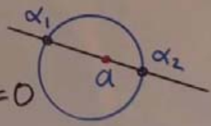
point/matrix correspondence:

$$a \equiv [x:y:z] \longleftrightarrow m_a \equiv \begin{bmatrix} y & x+z \\ x-z & -y \end{bmatrix}$$

Reflection matrix (conjugation) theorem

$[t_2 \ u_2] = [t_1 \ u_1]\, m_a \Leftrightarrow a$ lies on $\alpha_1 \alpha_2$ where $\alpha_1 \equiv e(t_1:u_1)$ & $\alpha_2 \equiv e(t_2:u_2)$.

$(\Leftrightarrow \ m_{\alpha_2} = m_a m_{\alpha_1} m_a)$.



Exer 16·1 Show that $\operatorname{tr} m_a m_b m_a = 0$ for any points a, b. ◇

Def. The point c is the reflection of the point b in the point a precisely when

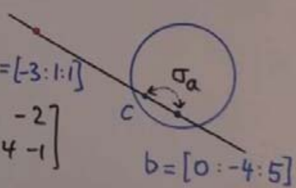
$$m_c = m_a m_b m_a.$$

In this case we write $\boxed{c = b\sigma_a}$. ②

Ex. If $a = [-3:1:1]$ and $b = [0:-4:5]$ then

$$m_a m_b m_a = \begin{bmatrix} 1 & -2 \\ -4 & -1 \end{bmatrix}\begin{bmatrix} -4 & 5 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -4 & -1 \end{bmatrix}$$

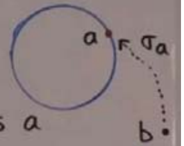
$$= \begin{bmatrix} 18 & -9 \\ 117 & -18 \end{bmatrix} \text{ so } b\sigma_a = [108:36:-126] = [-54:-18:63] = c \ ◇$$



Ex. If $a = [4:3:5]$ and $b = [4:-3:2]$ then

$$m_a m_b m_a = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} -3 & 6 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} -18 & -54 \\ 6 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} = m_a. \quad ◇$$



Null reflection theorem. If a is a null point, then $b\sigma_a = a$ for any point b.

Proof: Exer 16·2 [You need show $m_a m_b m_a = m_a$.] ■ ③

Matrix perpendicularity theorem For any points a, b:

$$a \perp b \Leftrightarrow \operatorname{tr} m_a m_b = 0$$

Proof. If $a = [x:y:z]$ and $b = [u:v:w]$ then $\operatorname{tr} m_a m_b = \operatorname{tr} \begin{bmatrix} y & x+z \\ x-z & -y \end{bmatrix}\begin{bmatrix} v & u+w \\ u-w & -v \end{bmatrix}$

$$= \operatorname{tr} \begin{bmatrix} yv+(x+z)(u-w) & * \\ * & (x-z)(u+w)+yv \end{bmatrix}$$

$$= 2yv + 2xu - 2zw = 2(xu+yv-zw) = 0$$

$$\Leftrightarrow [x:y:z] \perp [u:v:w].$$ ■ ④

Reflection (preserves) perpendicalarity theorem

If b & c are any points, and a is non-null, then $\boxed{b \perp c \Leftrightarrow b\sigma_a \perp c\sigma_a}$. ⑤

Proof. Since a is non-null, $m_a^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

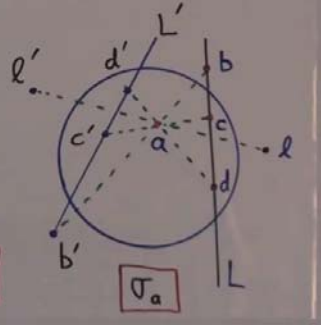
We need the following: $\operatorname{tr} AB = \operatorname{tr} BA$.

Then
$$\begin{aligned} b \perp c &\Leftrightarrow \operatorname{tr} m_b m_c = 0 \\ &\Leftrightarrow \operatorname{tr} m_b m_c m_a^2 = 0 \\ &\Leftrightarrow \operatorname{tr} m_a m_b m_c m_a = 0 \\ &\Leftrightarrow \operatorname{tr} m_a m_b m_a m_a m_c m_a = 0 \\ &\Leftrightarrow \operatorname{tr} m_{b\sigma(a)} m_{c\sigma(a)} = 0 \Leftrightarrow b\sigma_a \perp c\sigma_a. \end{aligned}$$ ■

Reflection (preserves) lines theorem (6)

If a is a non-null point, then b,c,d are collinear $\Leftrightarrow$ $b\sigma_a$, $c\sigma_a$, $d\sigma_a$ are collinear.

Proof If b,c,d collinear, they lie on a line L, so that they are all perpendicular to $\ell \equiv L^\perp$. But then $b' \equiv b\sigma_a$, $c' \equiv c\sigma_a$, $d' \equiv d\sigma_a$ are all $\perp$ to $\ell' \equiv \ell\sigma_a$, so they all lie on $L' \equiv \ell'^\perp$. ■ $\boxed{L' = \sigma_a L}$



Exer 16.3 (7)

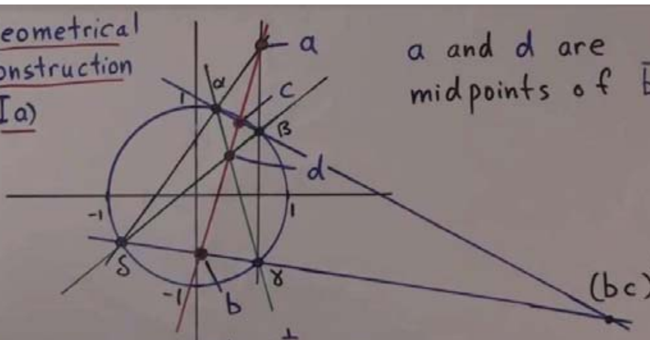
Check that $b \equiv [3:2:4]$, $c \equiv [1:-1:5]$, $d \equiv [10:5:17]$ are collinear. Verify that if $a \equiv [-1:2:3]$ then $b\sigma_a$, $c\sigma_a$, $d\sigma_a$ are collinear. Graph all. Now repeat with $a \equiv [-4:-1:2]$. ◇

Midpoints

Def The non-null point a is a midpoint of the side $\overline{bc}$ $\Leftrightarrow$ $\boxed{b\sigma_a = c}$.

Ex. $a \equiv [2:5:3]$ $b \equiv [-1:4:7]$ $c = b\sigma_a = [-4:55:79]$ so a is a midpoint of $\overline{bc}$: but so is $d \equiv [-14:95:149]$. Since $c = b\sigma_d$. [Check!] ◇

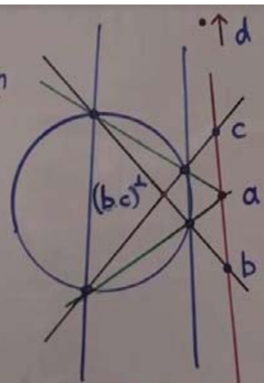
Geometrical Construction I a) (8)



a and d are midpoints of $\overline{bc}$

1. Construct $(bc)^\perp$
2. Joins $b(bc)^\perp$, $c(bc)^\perp$. Assume these interior.
3. Other two diagonal points of $\overline{\alpha\beta\gamma\delta}$ are midpoints a and d of $\overline{bc}$.

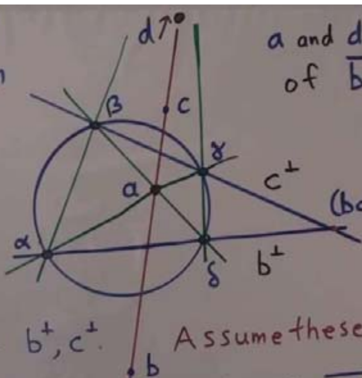
Geometrical construction I b) (9)



a and d are midpoints of $\overline{bc}$.

Harmonic quadrangle thm: $\Rightarrow$ a and d are harmonic conjugates wrt b and c (UnivHypGeom2)

Geometrical construction II.

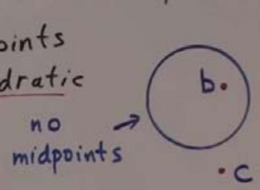


a and d are midpoints of $\overline{bc}$

1. Construct $b^\perp, c^\perp$. Assume these are interior.
2. Other two diagonal points of $\overline{\alpha\beta\gamma\delta}$ are midpoints a and d of $\overline{bc}$. (10)

Note: Not all sides $\overline{bc}$ have midpoints (11)

The existence of midpoints depends on solving a quadratic equation

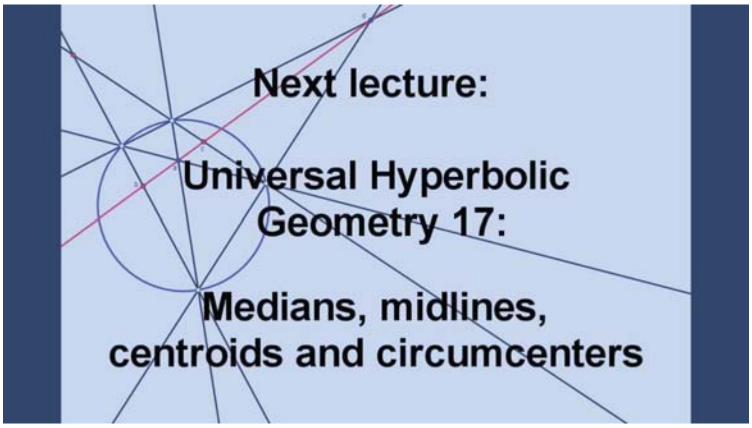


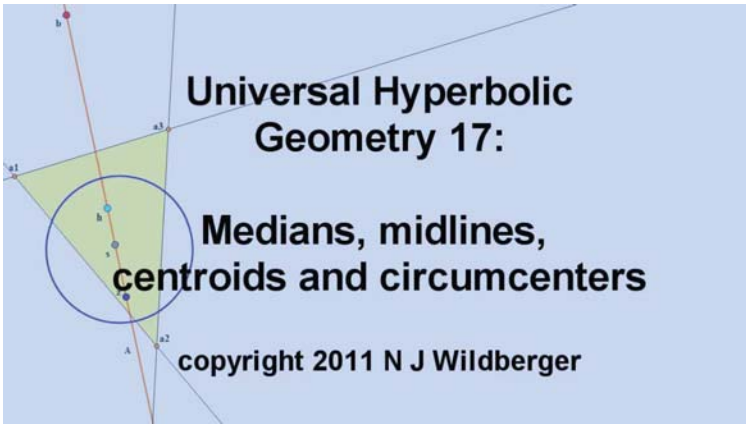
Bisectors

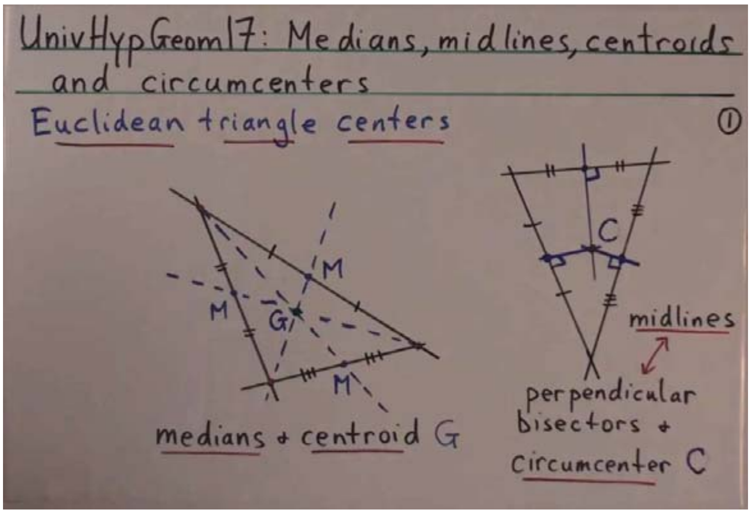
side $\overline{bc}$ $\longleftrightarrow$ vertex $\overline{BC}$

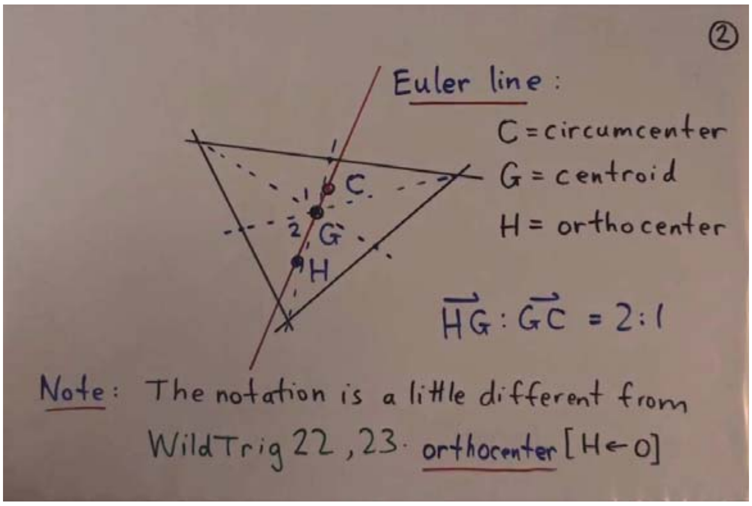
midpoints a,d $\longleftrightarrow$ bisectors A,D

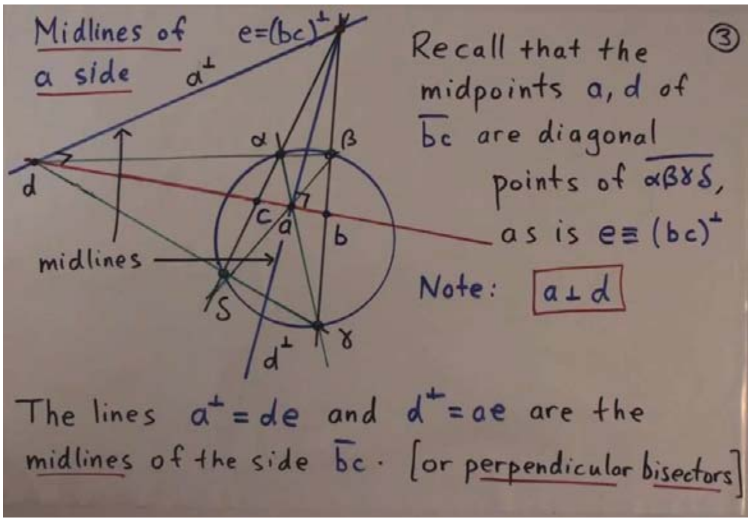
Def A is a bisector of the vertex $\overline{BC}$ $\Leftrightarrow$ $a = A^\perp$ is a midpoint of side $\overline{bc} = \overline{B^\perp C^\perp}$.

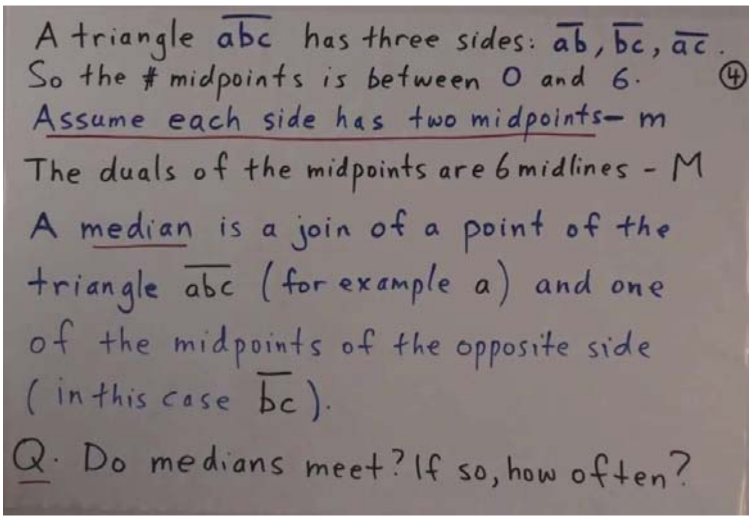

Next lecture:
Universal Hyperbolic Geometry 17:
Medians, midlines, centroids and circumcenters

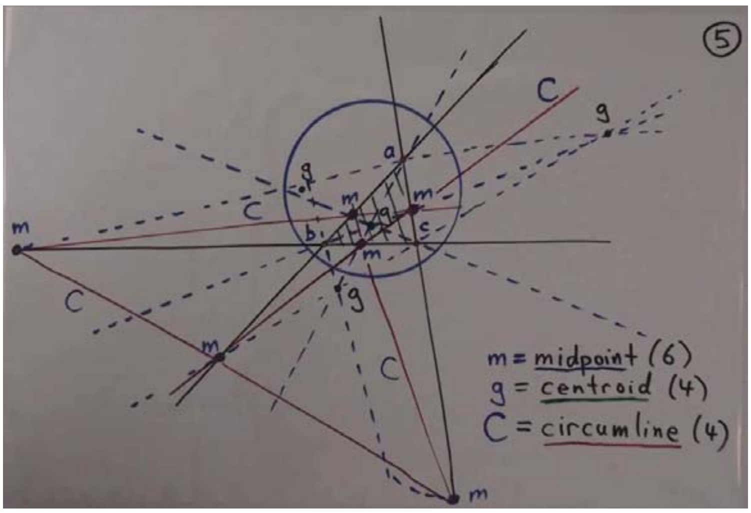

Universal Hyperbolic
Geometry 17:
Medians, midlines,
centroids and circumcenters
copyright 2011 N J Wildberger

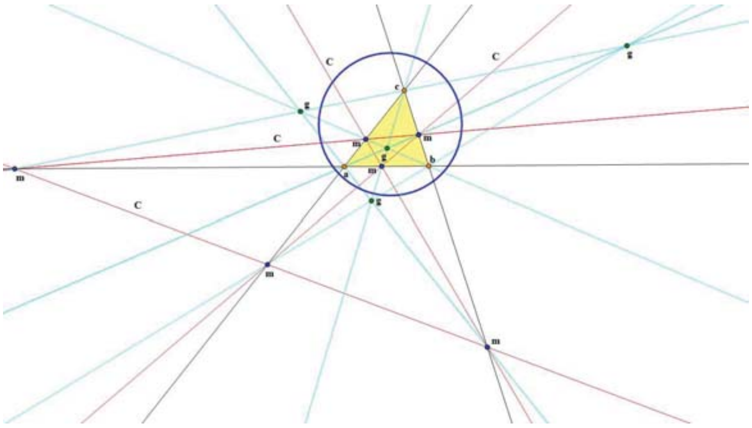

UnivHypGeom17: Medians, midlines, centroids
and circumcenters
Euclidean triangle centers
1
M
M
G
M
medians + centroid G
C
midlines
perpendicular
bisectors +
circumcenter C

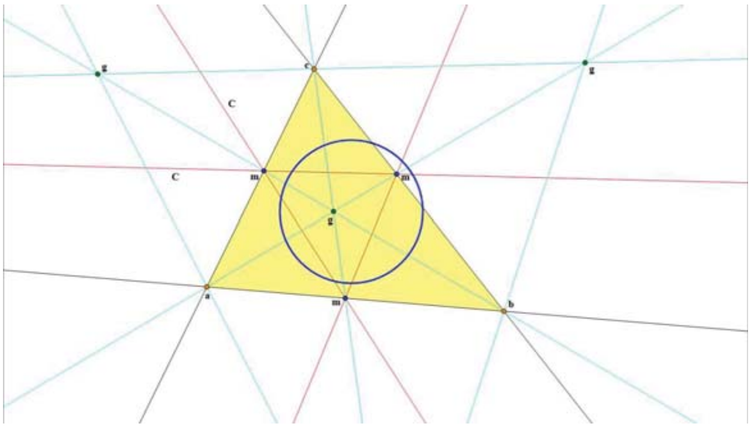

2
Euler line:
C = circumcenter
G = centroid
H = orthocenter
C
G
H
1
2
$\overrightarrow{HG} : \overrightarrow{GC} = 2:1$
Note: The notation is a little different from
WildTrig 22, 23. orthocenter [H ← O]

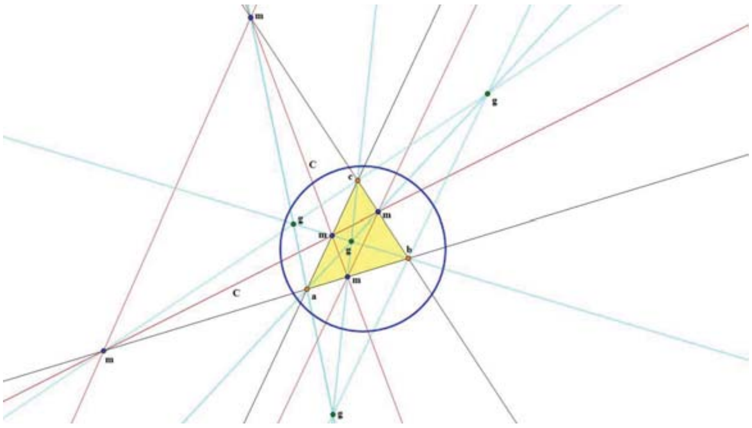

Midlines of
a side
$e=(bc)^\perp$
$a^\perp$
d
α
β
c
a
b
midlines
δ
γ
$d^\perp$
Recall that the
midpoints a, d of
$\overline{bc}$ are diagonal
points of αβγδ,
as is $e \equiv (bc)^\perp$
3
Note: $\boxed{a \perp d}$
The lines $a^\perp = de$ and $d^\perp = ae$ are the
midlines of the side $\overline{bc}$. [or perpendicular bisectors]


A triangle $\overline{abc}$ has three sides: $\overline{ab}, \overline{bc}, \overline{ac}$.
So the # midpoints is between 0 and 6.
4
Assume each side has two midpoints – m
The duals of the midpoints are 6 midlines – M
A median is a join of a point of the
triangle $\overline{abc}$ (for example a) and one
of the midpoints of the opposite side
(in this case $\overline{bc}$).
Q. Do medians meet? If so, how often?


5
C
g
a
g
C
m
m
m
b
c
m
C
g
C
m
m
m = midpoint (6)
g = centroid (4)
C = circumline (4)





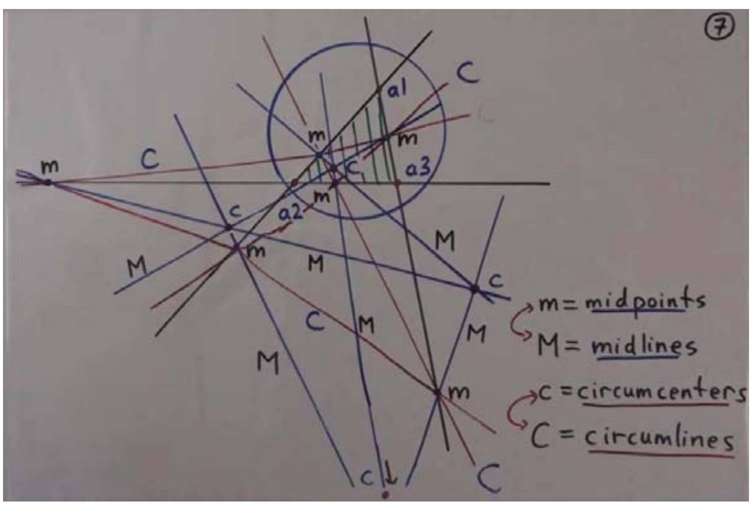


Meets of medians theorem. If a triangle has six midpoints, then the six medians meet 3 at a time at 4 centroids: g. ⑥

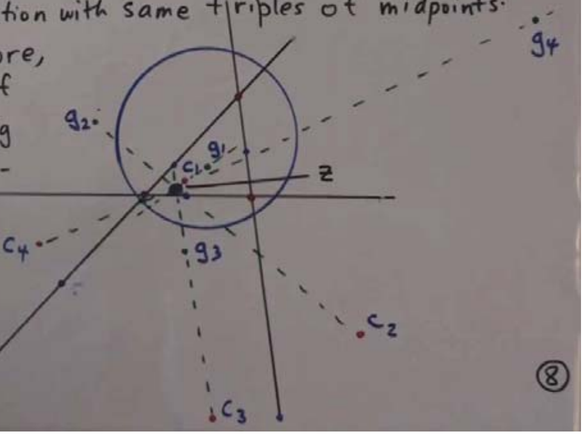
Joins of midpoints theorem. If a triangle has six midpoints, then they are collinear 3 at a time on 4 circumlines: C.

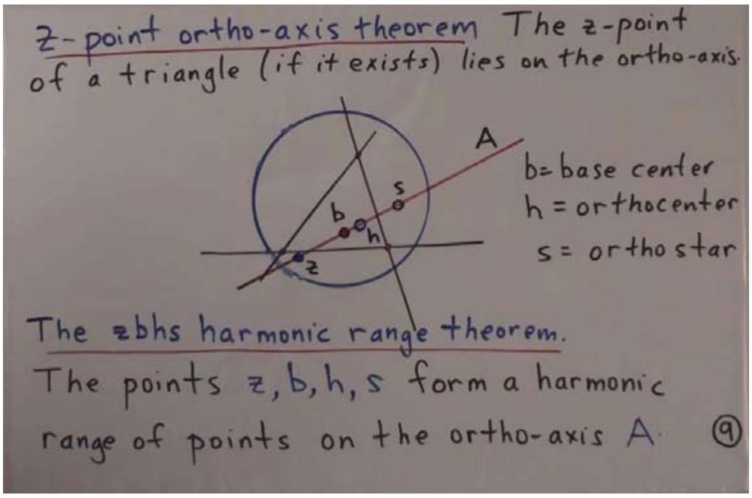
dual

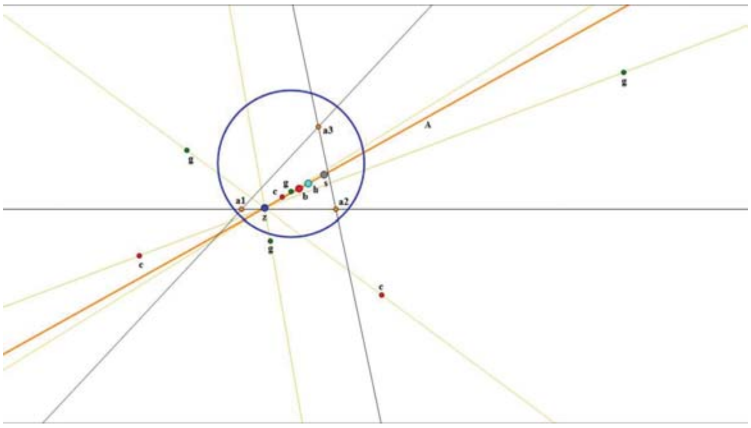
Meets of midlines theorem. If a triangle has six midpoints, then the six midlines are concurrent 3 at a time at 4 circumcenters: c.

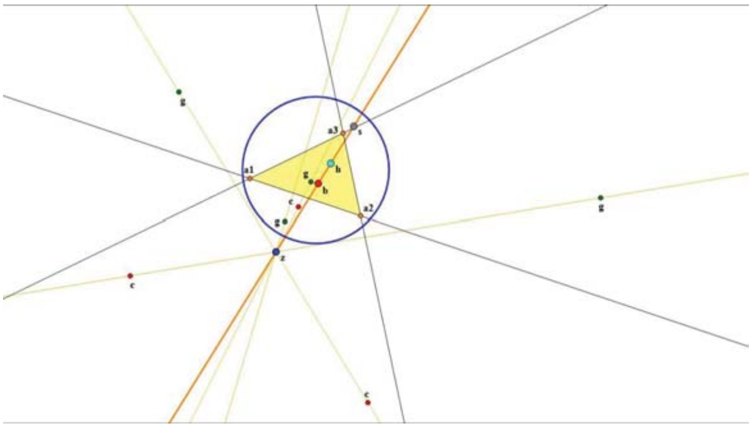


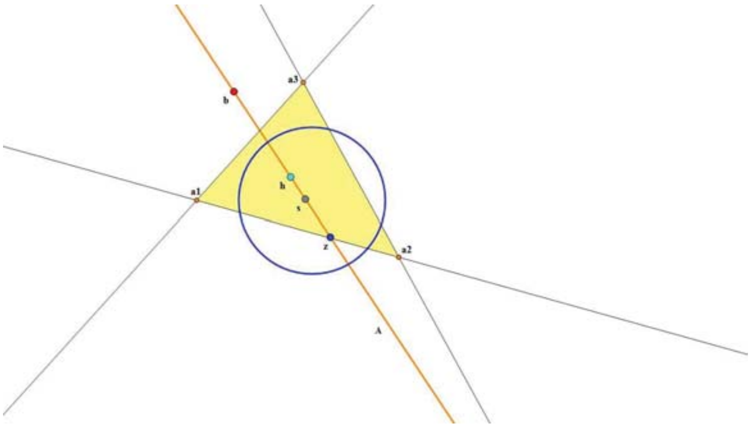
Centroid circumcenter correspondence theorem: The 4 centroids + 4 circumcenters correspond by association with same triples of midpoints. Furthermore, the joins of corresponding pairs meet in the z-point of the triangle.

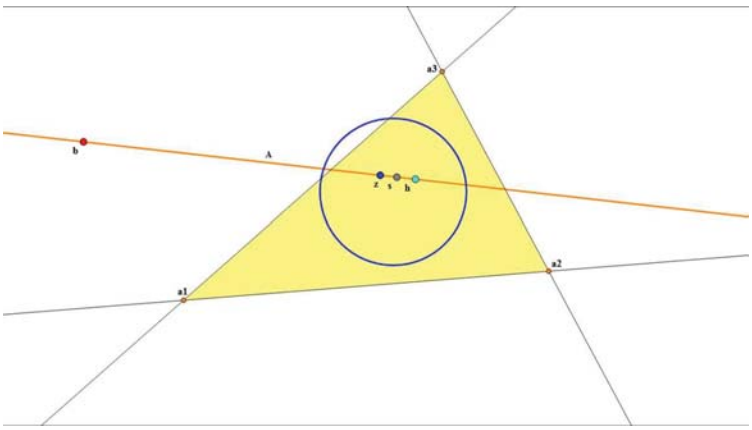


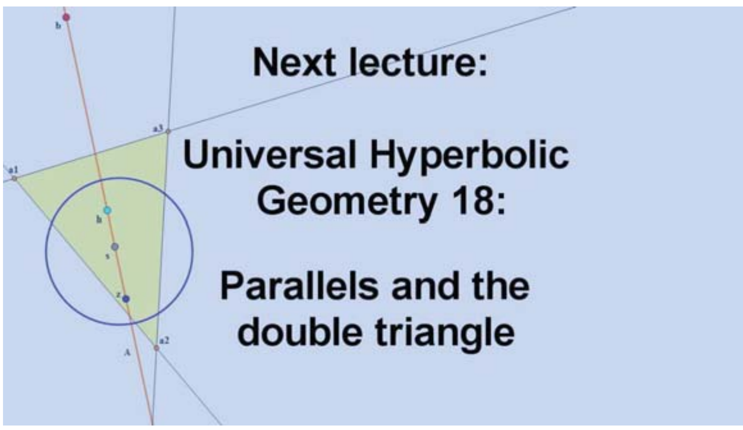

Z-point ortho-axis theorem The z-point of a triangle (if it exists) lies on the ortho-axis.
A
b
s
h
z
b = base center
h = orthocenter
s = orthostar
The zbhs harmonic range theorem.
The points z, b, h, s form a harmonic range of points on the ortho-axis A.
9

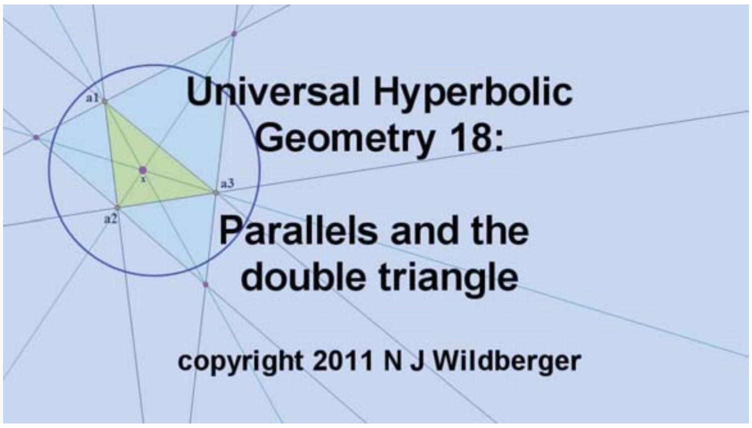


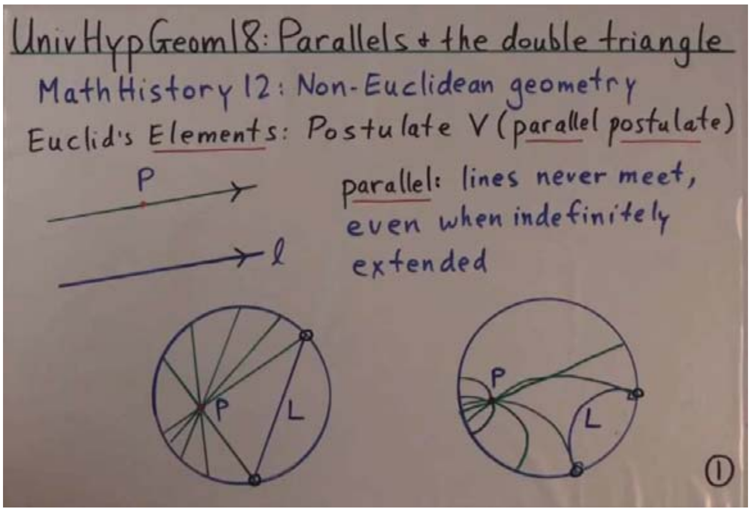


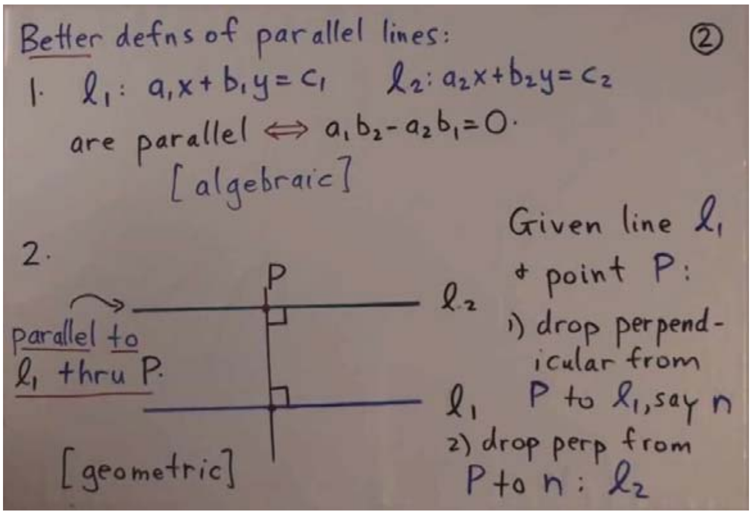

a3
a1
a2
b
h
s
z
A

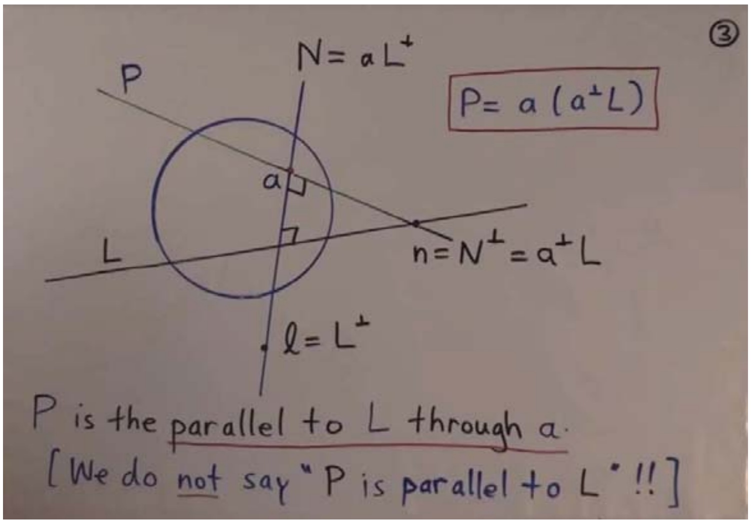

a3
b
A
z
s
h
a1
a2

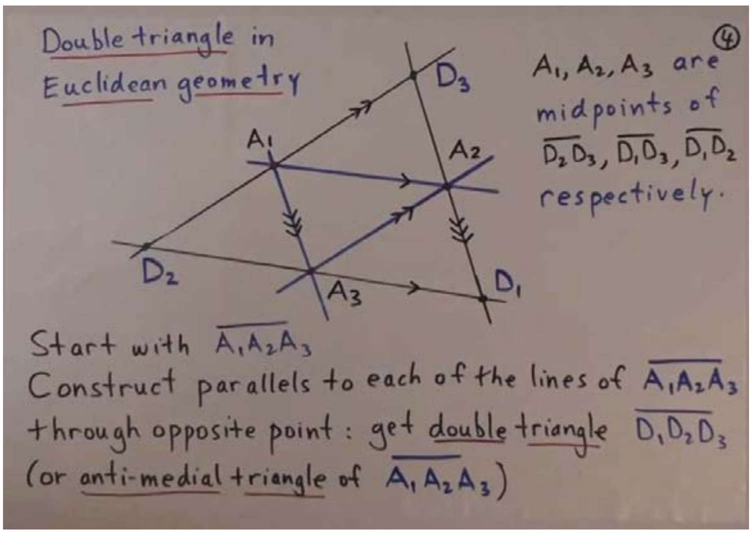

Next lecture:
Universal Hyperbolic Geometry 18:
Parallels and the double triangle

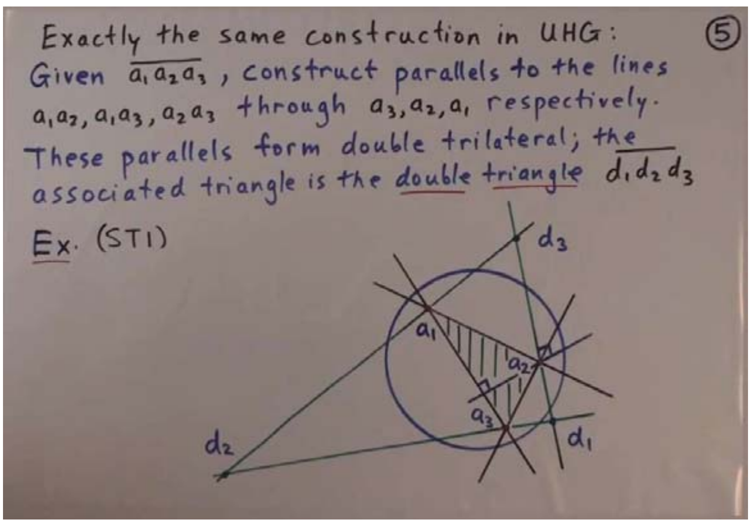

Universal Hyperbolic Geometry 18:
Parallels and the double triangle
copyright 2011 N J Wildberger
a1
a2
a3
x


UnivHypGeom18: Parallels + the double triangle
MathHistory 12: Non-Euclidean geometry
Euclid's Elements: Postulate V (parallel postulate)
P
l
parallel: lines never meet, even when indefinitely extended
P
L
P
L
1


Better defns of parallel lines:
2
1. $l_1: a_1x+b_1y=c_1$ $l_2: a_2x+b_2y=c_2$
are parallel $\Leftrightarrow a_1b_2-a_2b_1=0$.
[algebraic]
2.
P
l_2
l_1
parallel to l_1 thru P.
[geometric]
Given line l_1 + point P:
1) drop perpendicular from P to l_1, say n
2) drop perp from P to n: l_2


3
$N = aL^\perp$
P
$\boxed{P = a(a^\perp L)}$
a
L
$n = N^\perp = a^\perp L$
$l = L^\perp$
P is the parallel to L through a.
[We do not say "P is parallel to L" !!]


Double triangle in Euclidean geometry
4
D_3
A_1
A_2
D_2
A_3
D_1
A_1, A_2, A_3 are midpoints of $\overline{D_2D_3}, \overline{D_1D_3}, \overline{D_1D_2}$ respectively.
Start with $\overline{A_1A_2A_3}$
Construct parallels to each of the lines of $\overline{A_1A_2A_3}$ through opposite point: get double triangle $\overline{D_1D_2D_3}$
(or anti-medial triangle of $\overline{A_1A_2A_3}$)


Exactly the same construction in UHG:
5
Given $\overline{a_1a_2a_3}$, construct parallels to the lines a_1a_2, a_1a_3, a_2a_3 through a_3, a_2, a_1 respectively.
These parallels form double trilateral; the associated triangle is the double triangle $d_1d_2d_3$
Ex. (STI)
d_3
a_1
a_2
a_3
d_2
d_1

STI $a_1 \equiv [-3:3:5]$ $a_2 \equiv [4:0:5]$ $a_3 \equiv [2:-4:5]$ ⑥

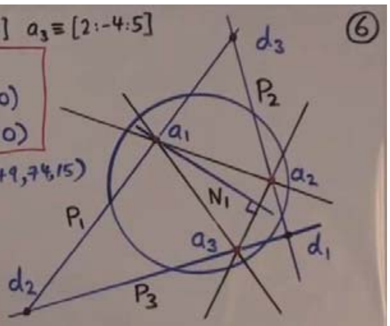
Altitudes: $\boxed{N_1 = (49:74:15) \quad N_2 = (125:-199:100) \quad N_3 = (223:-51:130)}$

Parallels $P_1 = a_1 n_1 = J(-3,3,5,49,74,15)$

$\boxed{P_1 = (-325:290:369) \quad P_2 = (995:225:796) \quad P_3 = (53:-171:158)}$

Double triangle $d_1 = P_2 P_3 = J(995,225,796,53,-171,158)$

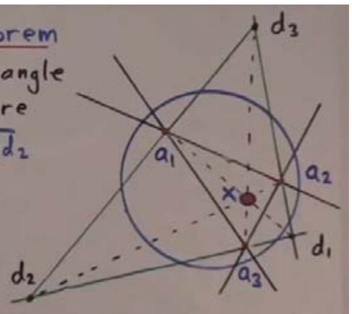
$\boxed{d_1 = [297:-199:315] \quad d_2 = [-149:-97:55] \quad d_3 = [47:199:115]}$



Double triangle midpoint theorem If $\overline{d_1 d_2 d_3}$ is the double triangle of $\overline{a_1 a_2 a_3}$, then a_1, a_2, a_3 are midpoints of $\overline{d_2 d_3}$, $\overline{d_1 d_3}$ & $\overline{d_1 d_2}$ respectively.

Double triangle perspective theorem. The double triangle $\overline{d_1 d_2 d_3}$ is perspective with the original triangle $\overline{a_1 a_2 a_3}$.

Def. The center of perspectivity is the x-point of $\overline{a_1 a_2 a_3}$, or the double point. ⑦



Ex (STI) $x = [195:-97:485]$ (Exer 18.1) ⑧

ortho-axis:

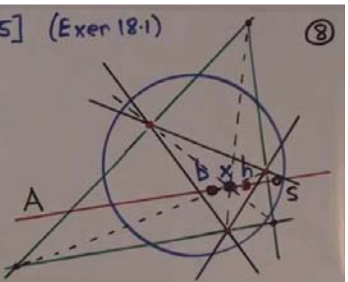
$A = (187986:-665765:208735)$

Then x lies on A.

x-point ortho-axis theorem The x-point of a triangle lies on the ortho-axis

shxb cross-ratio theorem

$$R(s,h:x,b) = \frac{3}{2}$$



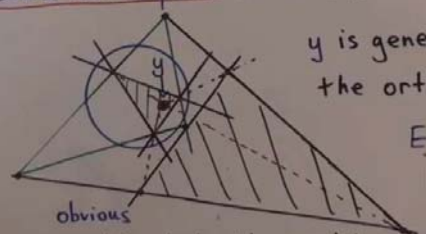
Second double triangle perspective theorem ⑨

The double triangle of the double triangle is perspective with the original triangle $\overline{a_1 a_2 a_3}$.

Def. The center of perspectivity is the y-point or second double point of $\overline{a_1 a_2 a_3}$.

y is generally not on the ortho-axis!

Exer 18.2 Find y for STI.



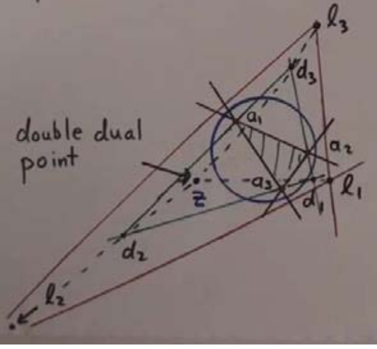
There is no obvious third double point!

Double dual triangle perspective theorem ⑩

The double triangle $d_1 d_2 d_3$ and the dual triangle $\overline{l_1 l_2 l_3}$ of a triangle $\overline{a_1 a_2 a_3}$ are perspective from the z-point, or double dual point of $\overline{a_1 a_2 a_3}$.

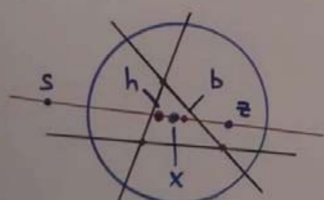
This is the same z-point as in centroid circumcenter

It lies on ortho-axis!

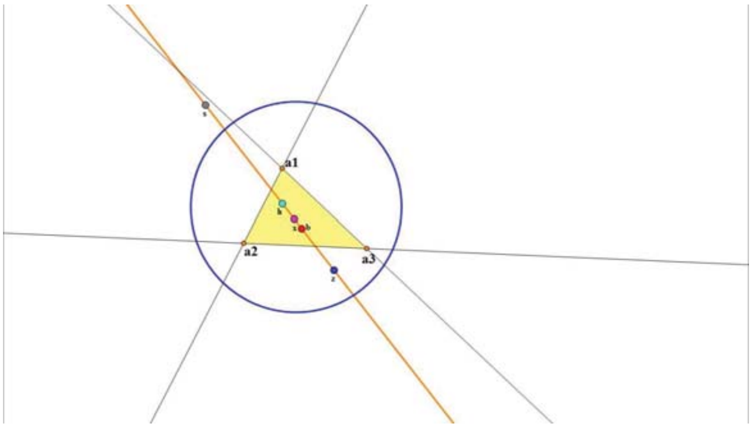


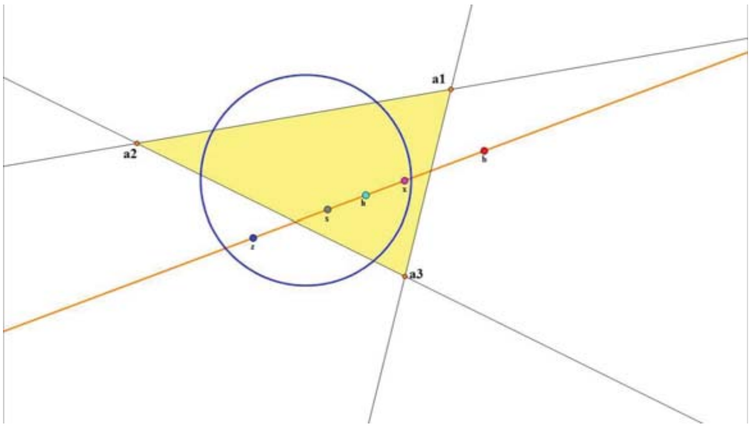
zbhs harmonic range theorem The points z, b, h, s form a harmonic range of points on the ortho-axis

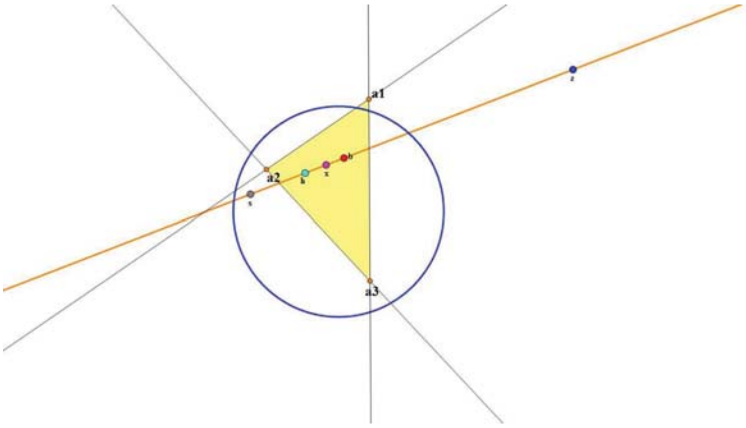
zbxh harmonic range theorem The points z, b, x, h form a harmonic range of points on the ortho-axis.

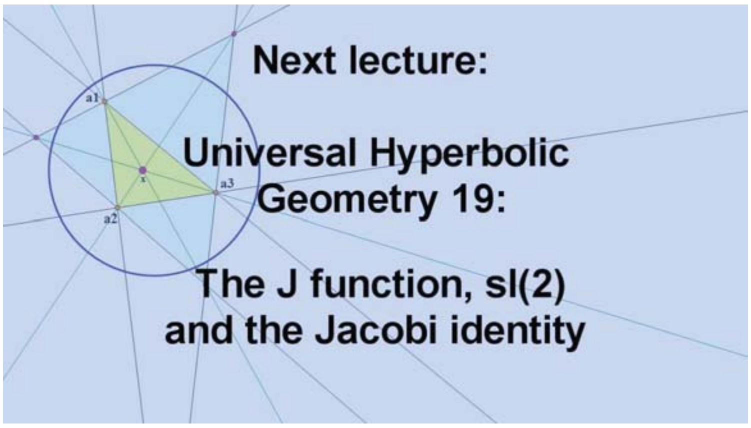


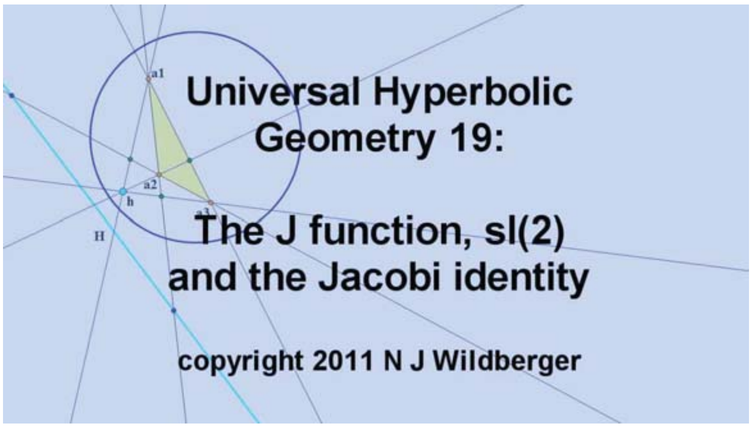
⑪


a1
a2
a3


a1
a2
a3


a1
a2
a3


Next lecture:
Universal Hyperbolic Geometry 19:
The J function, sl(2) and the Jacobi identity
a1
a2
a3



UnivHypGeom19: The J function, sl(2), and the Jacobi identity ①

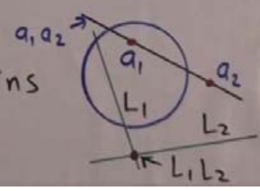
(hyperbolic) point: $a \equiv [x:y:z]$ $\quad \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix}$

(hyperbolic) line: $L \equiv (l:m:n)$

a incident with $L \Leftrightarrow \boxed{lx+my-nz=0}$

$$a_1a_2 = (y_1z_2-y_2z_1 : z_1x_2-z_2x_1 : y_1x_2-y_2x_1)$$
$$\equiv J(x_1,y_1,z_1,x_2,y_2,z_2)$$

The J function computes joins of points / meets of lines



sl(2): $A \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \operatorname{tr} A \equiv a+d = 0$ ②

So sl(2) represents a type of matrix

This is a Lie algebra: space of matrices (allowing linear combinations) with a bracket operation $[\ ,\ ]$:

$\boxed{[A,B] \equiv AB-BA}$

symmetry — Sophus Lie — quantum mechanics

This is a closed operation since

$$\operatorname{tr}(AB-BA) = \operatorname{tr} AB - \operatorname{tr} BA = 0.$$

Jacobi Identity $\quad A,B,C \quad 2\times2$ matrices ③

$$\boxed{[[A,B],C]+[[B,C],A]+[[C,A],B]=0}$$

Proof. $(AB-BA)C - C(AB-BA)$
$+ (BC-CB)A - A(BC-CB)$
$+ (CA-AC)B - B(CA-AC) = 0$ ■

Simpler identity: $\boxed{[A,B] = -[B,A]}$.

Q. Can we define brackets for projective matrices?

Projective algebra of matrices. ④

If A, B are 2×2 matrices, we can form $3A, 2B, 3A+2B$ as well as $AB, A^2, B^3, \ldots$

If m, n are 2×2 projective matrices, we can form mn, while $3m=m$ and $2n=n$. An expression like $m+n$ or $3m+2n$ is not well-defined.

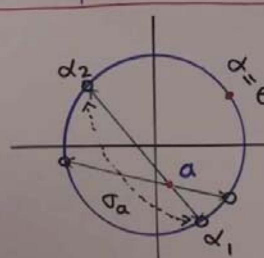
But... $\boxed{[m,n] \equiv mn-nm}$ is well-defined!!

Ex. $m \equiv \begin{bmatrix} 3 & 2 \\ -8 & -3 \end{bmatrix}$ $\quad n \equiv \begin{bmatrix} -4 & 7 \\ -3 & 4 \end{bmatrix}$ $\quad \begin{bmatrix} 25 & 29 \\ 41 & -25 \end{bmatrix}$ ◇

$$[m,n] = \begin{bmatrix} 3 & 2 \\ -8 & -3 \end{bmatrix}\begin{bmatrix} -4 & 7 \\ -3 & 4 \end{bmatrix} - \begin{bmatrix} -4 & 7 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -8 & -3 \end{bmatrix} = \begin{bmatrix} -18 & 29 \\ 41 & -68 \end{bmatrix} - \begin{bmatrix} -68 & -29 \\ -41 & -18 \end{bmatrix}$$

Projective parametrization of null circle: ⑤

$$x^2+y^2-z^2=0.$$



$\alpha = e(t:u) \equiv [u^2-t^2 : 2ut : u^2+t^2]$

$a \equiv [x:y:z]$

σ_a = reflection in a

$\alpha_1\sigma_a = \alpha_2$

$$[t_1\ u_1]\begin{bmatrix} y & x+z \\ x-z & -y \end{bmatrix} = [t_2\ u_2]$$

$\equiv m_a$ $\quad$ 2×2 projective matrix, trace zero, sl(2)

General formula for reflection: ⑥

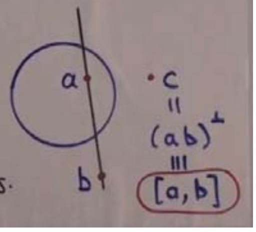
$c = b\sigma_a \Leftrightarrow \boxed{m_c = m_a m_b m_a}$ a non-null $\Leftrightarrow m_a = m_a^{-1}$

So conjugation of matrices $\longleftrightarrow$ reflection

Bracket theorem If a and b are distinct points, then $[m_a, m_b] = m_c$ where $c = (ab)^\perp$.

So the bracket computes the J function!

· gives a "multiplication" of points.



Proof Suppose that $a \equiv [x:y:z]$ $b \equiv [u:v:w]$ ⑦

Then $c = (ab)^\perp = J(x,y,z,u,v,w)$

$= [yw - zv : zu - xw : yu - xv]$

But $m_a = \begin{bmatrix} y & x+z \\ x-z & -y \end{bmatrix}$ $m_b = \begin{bmatrix} v & u+w \\ u-w & -v \end{bmatrix}$ so

$$[m_a, m_b] = \begin{bmatrix} y & x+z \\ x-z & -y \end{bmatrix}\begin{bmatrix} v & u+w \\ u-w & -v \end{bmatrix} - \begin{bmatrix} v & u+w \\ u-w & -v \end{bmatrix}\begin{bmatrix} y & x+z \\ x-z & -y \end{bmatrix}$$

$$= \begin{bmatrix} (x+z)(u-w)-(u+w)(x-z) & 2y(u+w)-2v(x+z) \\ 2v(x-z)-2y(u-w) & (x-z)(u+w)-(u-w)(x+z) \end{bmatrix}$$

$$= \begin{bmatrix} zu-xw & (yw-zv)+(yu-xv) \\ (yw-zv)-(yu-xv) & -(zu-xw) \end{bmatrix} = m_c. \blacksquare$$

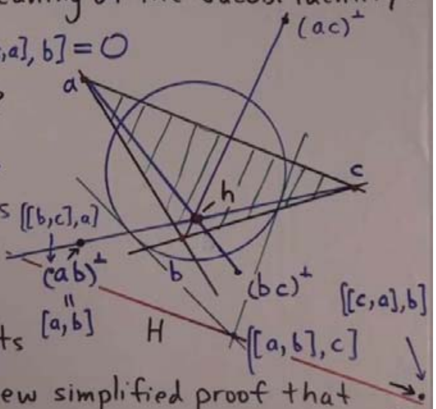
Q. What is the meaning of the Jacobi identity?

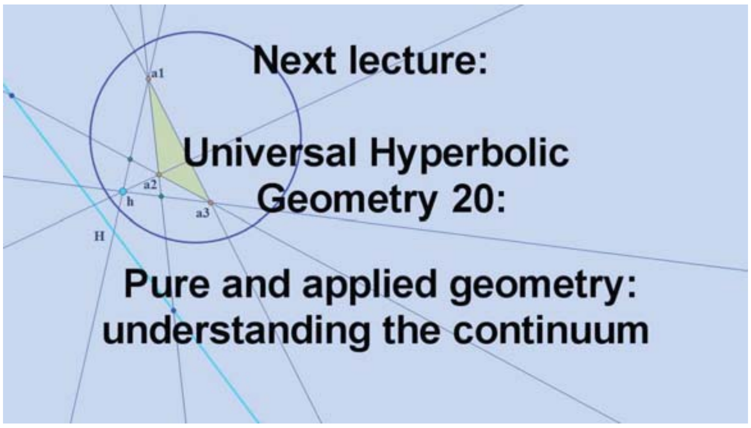
$[[a,b],c] + [[b,c],a] + [[c,a],b] = 0$

What is $[[a,b],c]$?

The matrices of the three altitude points sum to 0 ∴ are linearly dependent.

∴ three altitude points are collinear! A new simplified proof that ortholine/orthocenter exists! ⑧

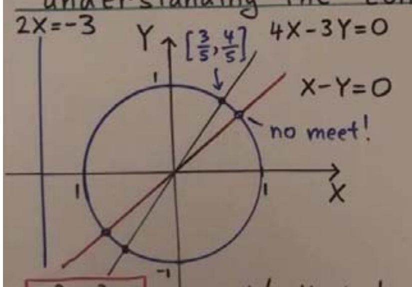




Universal Hyperbolic Geometry 20:

Pure and applied geometry--understanding the continuum

UnivHypGeom20: Pure and applied geometry — understanding the continuum ①



$\boxed{X^2+Y^2=1}$ ← unit/null circle → $\boxed{x^2+y^2-z^2=0}$

$X=\frac{x}{z}$ $\quad Y=\frac{y}{z}$

Over the rational numbers, there is no "$\sqrt{2}$", so the line $X-Y=0$ does not meet the unit circle $X^2+Y^2=1$.

But it appears to!! ??

"Meaningful distinctions deserve to be maintained." Errett Bishop 1973 ②

Pure Geometry ↔	Applied Geometry
the theory in this course on UHG	Some of the pictures shown in this course [GSP]
- requires complete precision + accuracy	- compromise between theoretical correctness + practical application
- everything is black + white!	- shades of grey!
- works over rational numbers	- works over approximate rational numbers [decimals]

Pure Geometry ⇕ Rational numbers $\qquad$ Applied Geometry ⇕ Decimal numbers

Rat (rational numbers): $\frac{a}{b}$ where $a,b \in \mathrm{Int}$, $b \neq 0$

with following notion of equality:
$$\frac{a}{b}=\frac{c}{d} \iff ad-bc=0$$

operations: $\boxed{\frac{a}{b}\times\frac{c}{d}\equiv\frac{ac}{bd}}$ $\quad \boxed{\frac{a}{b}\pm\frac{c}{d}\equiv\frac{ad\pm bc}{bd}}$ ③

Dec (decimal numbers) ④

$12.387 \equiv 12\frac{387}{1000}$

$$\begin{array}{r} 32.54 \\ +\ 41.072 \\ \hline 73.612 \end{array}$$ addition

$$\begin{array}{r} 3.59 \\ \times\ 4.2 \\ \hline .718 \\ 14.36 \\ \hline 15.078 \end{array}$$ multiplication

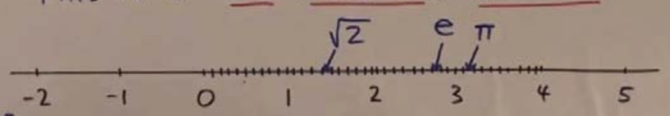
This is a variant of rational arithmetic working for only special rational numbers (denominators powers of 10)

It requires some considerable work + care to formulate this theory (sadly neglected!!)

Extension to "repeated decimals": $0.\overline{571428}=\frac{4}{7}$

The theory of decimal numbers is usually "extended" to "infinite decimals" → "real numbers". ⑤

This is a BIG LOGICAL MISTAKE!

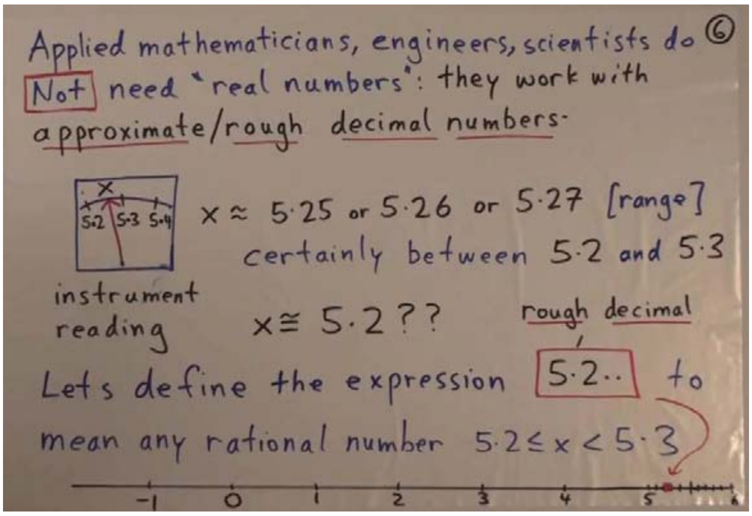


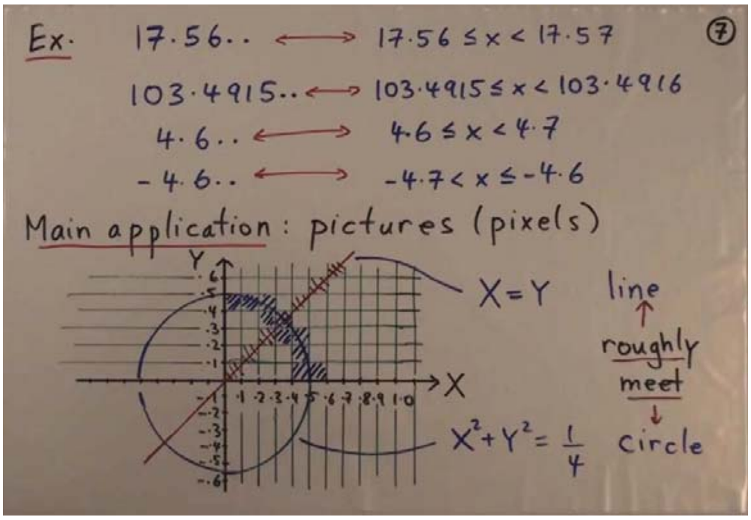
the "real number line" or "the continuum".

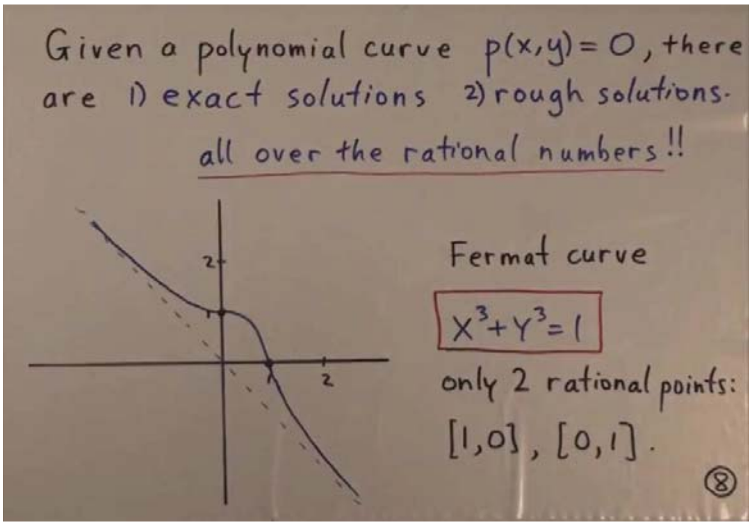
$\sqrt{2} = 1.4142135623730950488\ldots$?

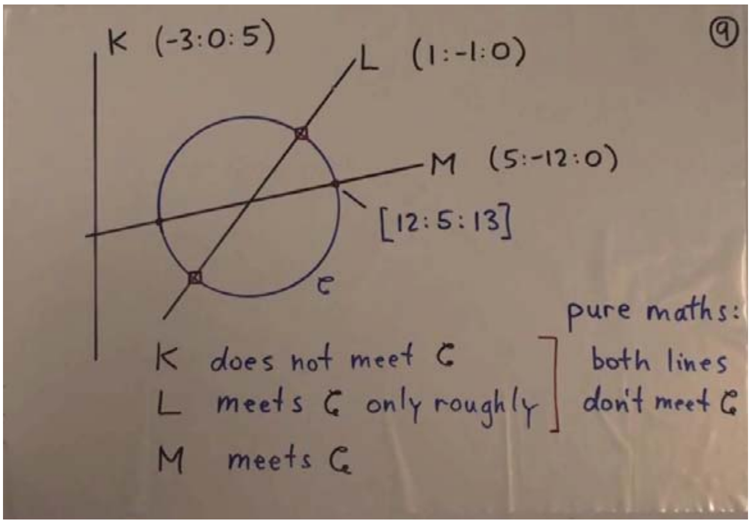
$\pi = 3.1415926535897932385\ldots$?

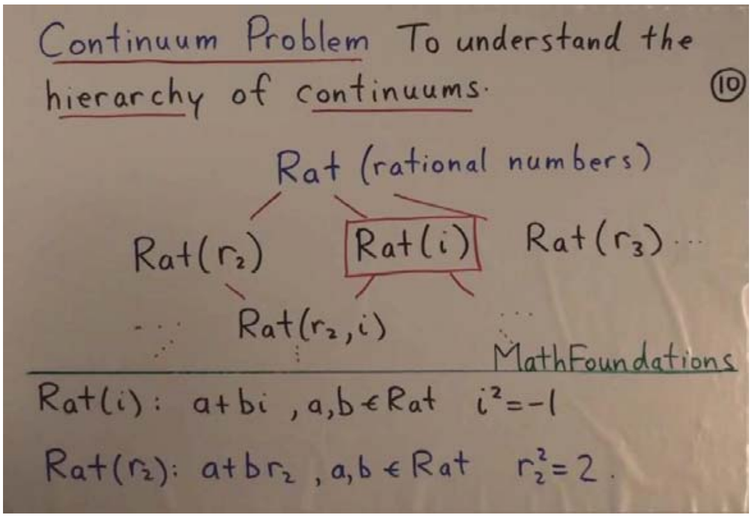
$e = 2.7182818284590452354\ldots$?

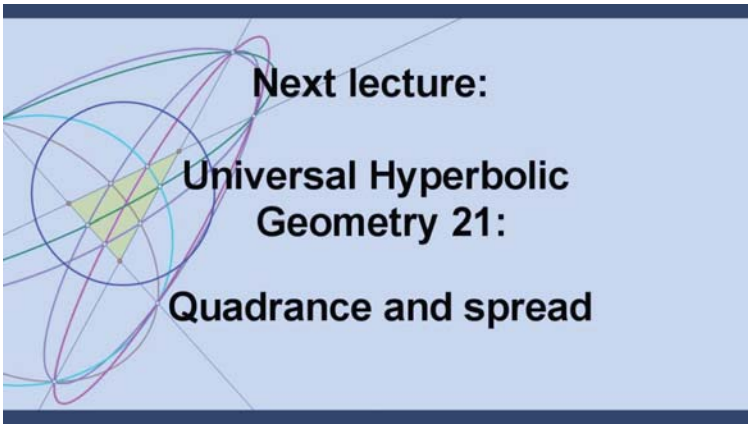

Applied mathematicians, engineers, scientists do ⑥
Not need "real numbers": they work with
approximate/rough decimal numbers.
5.2 5.3 5.4
x ≈ 5.25 or 5.26 or 5.27 [range]
certainly between 5.2 and 5.3
instrument reading
x ≅ 5.2 ??
rough decimal
Lets define the expression 5.2.. to
mean any rational number 5.2 ≤ x < 5.3
-1 0 1 2 3 4 5


Ex. 17.56.. ⟷ 17.56 ≤ x < 17.57 ⑦
103.4915.. ⟷ 103.4915 ≤ x < 103.4916
4.6.. ⟷ 4.6 ≤ x < 4.7
-4.6.. ⟷ -4.7 < x ≤ -4.6
Main application: pictures (pixels)
X=Y
line
roughly meet
circle
X² + Y² = 1/4


Given a polynomial curve p(x,y) = 0, there
are 1) exact solutions 2) rough solutions.
all over the rational numbers !!
Fermat curve
X³ + Y³ = 1
only 2 rational points:
[1,0], [0,1].
⑧


K (-3:0:5)
L (1:-1:0)
⑨
M (5:-12:0)
[12:5:13]
C
pure maths:
K does not meet C
L meets C only roughly
M meets C
both lines don't meet C


Continuum Problem To understand the
hierarchy of continuums. ⑩
Rat (rational numbers)
Rat(r₂) Rat(i) Rat(r₃) ...
Rat(r₂, i)
MathFoundations
Rat(i): a+bi, a,b ∈ Rat i² = -1
Rat(r₂): a+br₂, a,b ∈ Rat r₂² = 2.


Next lecture:
Universal Hyperbolic
Geometry 21:
Quadrance and spread

Universal Hyperbolic Geometry 21:

Quadrance and spread

UnivHypGeom21: Quadrance and spread ①

Metrical notions (over rational numbers!)

classical hyp·geom ⟷ distance + angle ✗

universal hyp·geom ⟷ quadrance + spread ✓

$q(a_1,a_2)$: quadrance between points a_1, a_2

$S(L_1,L_2)$: spread between lines L_1, L_2

These are hyperbolic analogs of corresponding notions in Rational Trigonometry: $Q(A_1,A_2)$ and $s(\ell_1,\ell_2)$.

②

Affine geometry	Projective geometry
- Euclidean	- hyperbolic
- Relativistic	- elliptic (spherical)
$\boxed{\text{Rational Trigonometry}}$	$\boxed{\text{Universal Hyperbolic Geometry}}$
'Divine Proportions: Rational Trigonometry to Universal Geometry'	'Universal Hyperbolic Geometry I: Trigonometry'
points: A_1, A_2	points: a_1, a_2
lines: ℓ_1, ℓ_2	lines: L_1, L_2
quadrance: $Q(A_1,A_2)$	quadrance: $q(a_1,a_2)$
spread: $s(\ell_1,\ell_2)$	spread: $S(L_1,L_2)$

DUALITY

Preliminary: Rational Trigonometry in Euclidean geo. ③

Point: $A \equiv [x,y]$ Line: $\ell \equiv \langle a:b:c \rangle$ $[a,b] \neq [0,0]$

Def. A is incident with $\ell \Leftrightarrow ax+by+c=0$

Quadrance: $A_1 \equiv [x_1,y_1]$ $A_2 \equiv [x_2,y_2]$

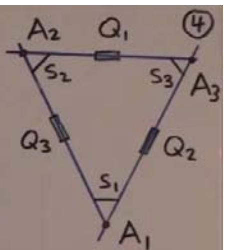
$$\boxed{Q(A_1,A_2) \equiv (x_2-x_1)^2 + (y_2-y_1)^2}$$

Spread: $\ell_1 \equiv \langle a_1:b_1:c_1 \rangle$ $\ell_2 \equiv \langle a_2:b_2:c_2 \rangle$

$$\boxed{s(\ell_1,\ell_2) \equiv 1 - \frac{(a_1a_2+b_1b_2)^2}{(a_1^2+b_1^2)(a_2^2+b_2^2)} = \frac{(a_1b_2-a_2b_1)^2}{(a_1^2+b_1^2)(a_2^2+b_2^2)}}$$

Def. i) $\ell_1 \perp \ell_2 \Leftrightarrow s(\ell_1,\ell_2)=1$ ii) $\ell_1 \parallel \ell_2 \Leftrightarrow s(\ell_1,\ell_2)=0$

Main laws of RT ④



Distinct points A_1, A_2, A_3

Pythagoras $A_1A_3 \perp A_2A_3 \Leftrightarrow Q_1+Q_2=Q_3$

Triple quad formula $A_1A_3 \parallel A_2A_3 \Leftrightarrow$

$$(Q_1+Q_2+Q_3)^2 = 2(Q_1^2+Q_2^2+Q_3^2)$$

Spread law $\dfrac{s_1}{Q_1} = \dfrac{s_2}{Q_2} = \dfrac{s_3}{Q_3}$

Cross law $(Q_1+Q_2-Q_3)^2 = 4Q_1Q_2(1-s_3)$

Triple spread formula

$$(s_1+s_2+s_3)^2 = 2(s_1^2+s_2^2+s_3^2) + 4s_1s_2s_3.$$

Trigonometry in Universal Hyperbolic Geometry

Point: $a \equiv [x:y:z]$ Line: $L = (\ell:m:n)$

Def: a is incident with $L \Leftrightarrow \ell x + my - nz = 0$

Quadrance: $a_1 \equiv [x_1:y_1:z_1]$ $a_2 \equiv [x_2:y_2:z_2]$

$$\boxed{q(a_1,a_2) \equiv 1 - \frac{(x_1x_2+y_1y_2-z_1z_2)^2}{(x_1^2+y_1^2-z_1^2)(x_2^2+y_2^2-z_2^2)}}$$

$a_1 \perp a_2 \Updownarrow q(a_1,a_2)=1$

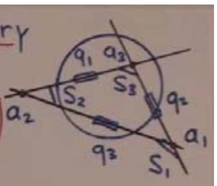
Spread: $L_1 \equiv (\ell_1:m_1:n_1)$ $L_2 \equiv (\ell_2:m_2:n_2)$

$$\boxed{S(L_1,L_2) \equiv 1 - \frac{(\ell_1\ell_2+m_1m_2-n_1n_2)^2}{(\ell_1^2+m_1^2-n_1^2)(\ell_2^2+m_2^2-n_2^2)}}$$

$L_1 \perp L_2 \Updownarrow S(L_1,L_2)=1$

⑤

Main laws of Hyperbolic trigonometry



Distinct points a_1, a_2, a_3

Pythagoras $a_1a_3 \perp a_2a_3 \Rightarrow \boxed{q_3 = q_1 + q_2 - q_1q_2}$

Triple quad formula a_1, a_2, a_3 collinear

$$\Rightarrow \boxed{(q_1+q_2+q_3)^2 = 2(q_1^2+q_2^2+q_3^2) + 4q_1q_2q_3}$$

Spread law $\boxed{\dfrac{S_1}{q_1} = \dfrac{S_2}{q_2} = \dfrac{S_3}{q_3}}$

Cross law

$$\boxed{(q_1q_2S_3 - q_1 - q_2 - q_3 + 2)^2 = 4(1-q_1)(1-q_2)(1-q_3)}$$

*plus Dual formulas! *

⑥

Exer. 21.1 Show that $1 - \dfrac{(x_1x_2+y_1y_2-z_1z_2)^2}{(x_1^2+y_1^2-z_1^2)(x_2^2+y_2^2-z_2^2)}$

$$= -\frac{(y_1z_2-y_2z_1)^2 + (x_1z_2-x_2z_1)^2 - (x_1y_2-x_2y_1)^2}{(x_1^2+y_1^2-z_1^2)(x_2^2+y_2^2-z_2^2)}$$ ◆

Exer. 21.2 Show that $q_3 = q_1 + q_2 - q_1q_2$ (Pyth.) is equivalent to $1 - q_3 = (1-q_1)(1-q_2)$. ◆

Exer. 21.3 Show Cross law $\Rightarrow$ Pythagoras. ◆

Exer. 21.4 Show Cross law $\Rightarrow$ Triple quad formula. ◆

Exer. 21.5 Show Cross law $\Rightarrow$ Spread law. ◆

⑦

Exer. 21.6 Compute the quadrances + spreads for ST#1: $a_1 = [-3:3:5]$ $a_2 = [4:0:5]$ $a_3 = [2:-4:5]$ ◆

Exer. 21.7 Find a right triangle (two lines perpendicular), and verify Pythagoras' thm. ◆

Exer. 21.8 Write down the Dual laws, and verify them in some cases. ◆

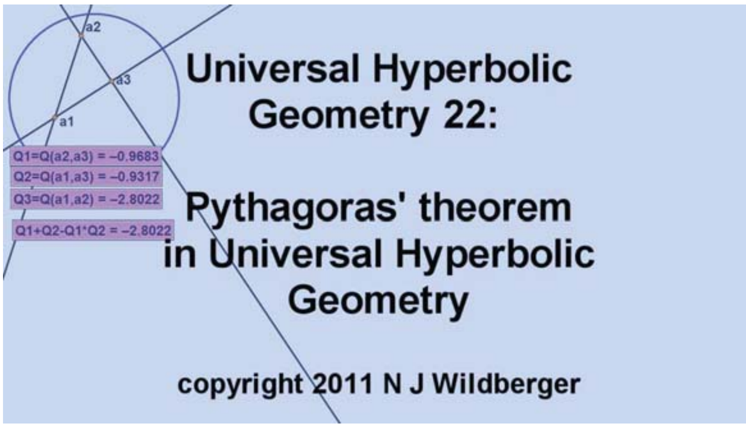
Exer. 21.9 Verify the Spread law + Cross laws for ST#1. ◆

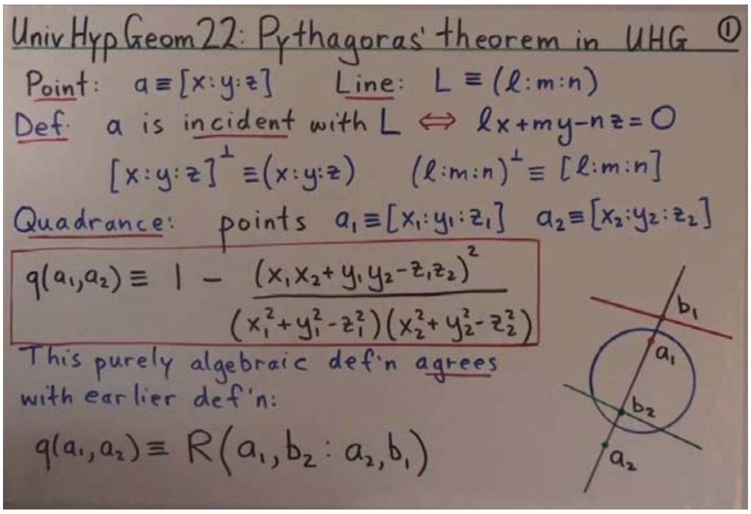
⑧

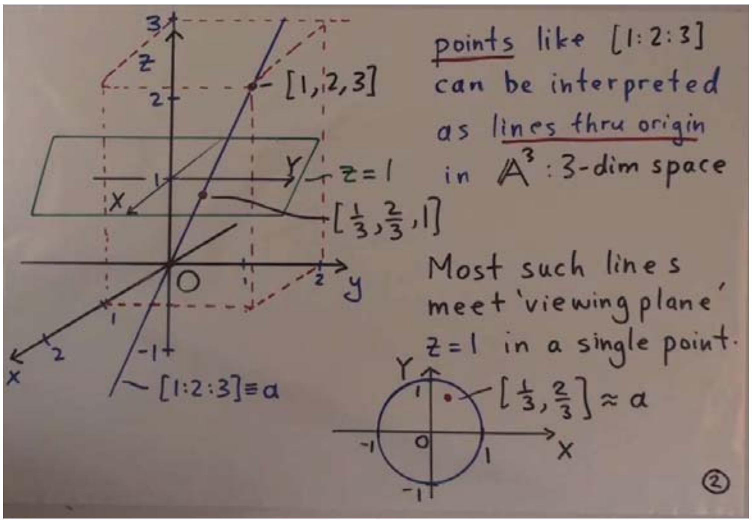
Next lecture:

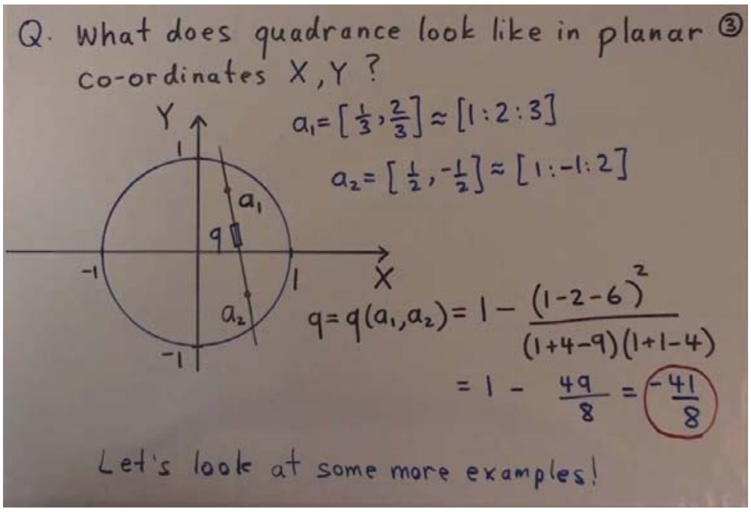
Universal Hyperbolic Geometry 22:

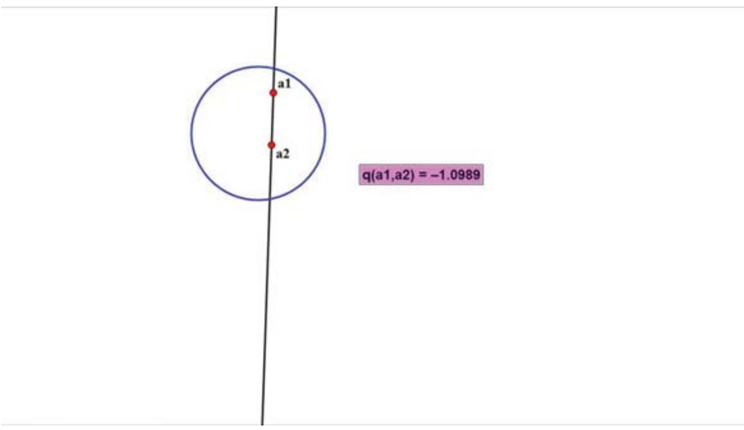
Pythagoras' theorem in universal hyperbolic geometry

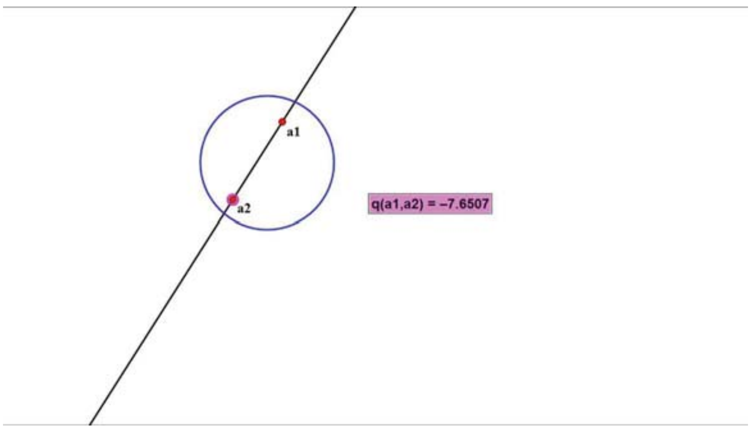

Universal Hyperbolic Geometry 22:
Pythagoras' theorem in Universal Hyperbolic Geometry
copyright 2011 N J Wildberger
a1
a2
a3
Q1=Q(a2,a3) = -0.9683
Q2=Q(a1,a3) = -0.9317
Q3=Q(a1,a2) = -2.8022
Q1+Q2-Q1*Q2 = -2.8022

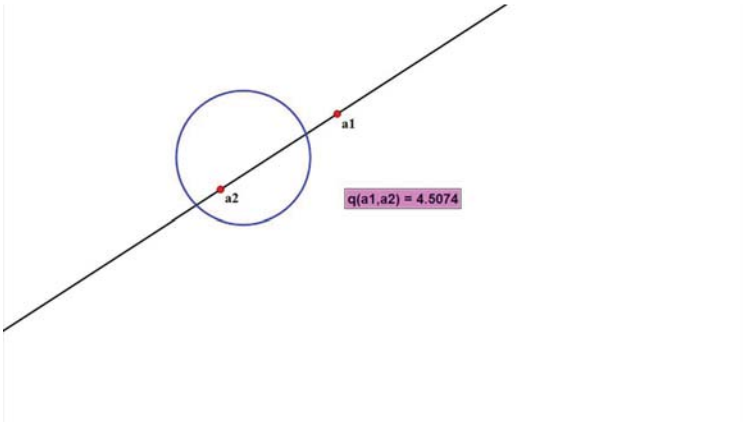

Univ Hyp Geom 22: Pythagoras' theorem in UHG ①
Point: $a \equiv [x:y:z]$ Line: $L \equiv (l:m:n)$
Def. a is incident with L $\Leftrightarrow$ $lx+my-nz=0$
$[x:y:z]^\perp \equiv (x:y:z)$ $(l:m:n)^\perp \equiv [l:m:n]$
Quadrance: points $a_1 \equiv [x_1:y_1:z_1]$ $a_2 \equiv [x_2:y_2:z_2]$
$q(a_1,a_2) \equiv 1 - \frac{(x_1x_2+y_1y_2-z_1z_2)^2}{(x_1^2+y_1^2-z_1^2)(x_2^2+y_2^2-z_2^2)}$
This purely algebraic def'n agrees with earlier def'n:
$q(a_1,a_2) \equiv R(a_1,b_2 : a_2,b_1)$
b_1
a_1
b_2
a_2

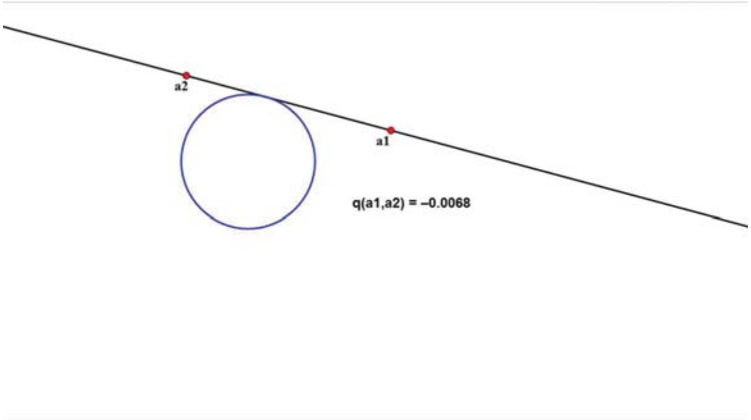

points like [1:2:3] can be interpreted as lines thru origin in $\mathbb{A}^3$: 3-dim space
Most such lines meet 'viewing plane' z=1 in a single point.
[1,2,3]
z=1
$[\frac{1}{3},\frac{2}{3},1]$
[1:2:3]≡a
$[\frac{1}{3},\frac{2}{3}] \approx a$
O
X
Y
②

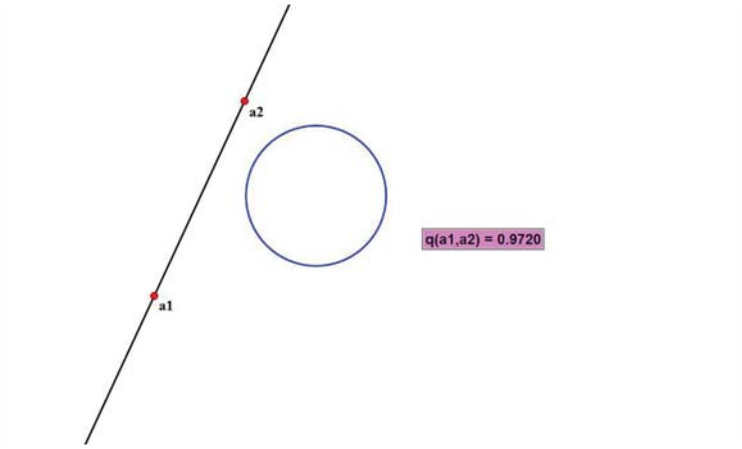

Q. What does quadrance look like in planar co-ordinates X, Y? ③
$a_1 = [\frac{1}{3},\frac{2}{3}] \approx [1:2:3]$
$a_2 = [\frac{1}{2},-\frac{1}{2}] \approx [1:-1:2]$
$q = q(a_1,a_2) = 1 - \frac{(1-2-6)^2}{(1+4-9)(1+1-4)}$
$= 1 - \frac{49}{8} = -\frac{41}{8}$
Let's look at some more examples!
a_1
a_2
q
X
Y

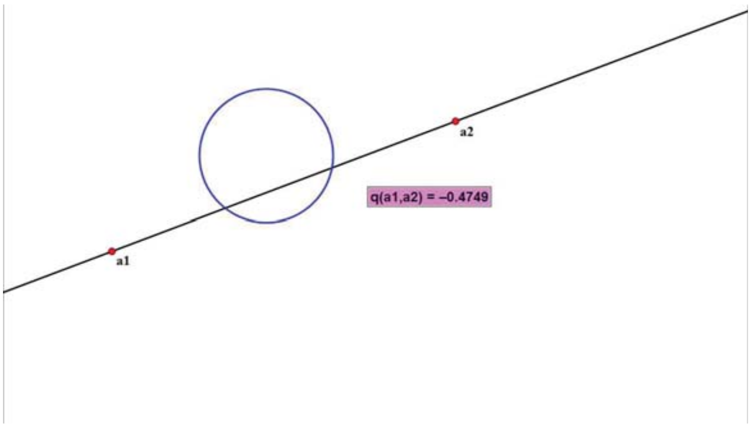

a1
a2
q(a1,a2) = -1.0989


a1
a2
q(a1,a2) = -7.6507









Planar formula: If $a_1 \equiv [X_1:Y_1:1]$ & $a_2 \equiv [X_2:Y_2:1]$ ④

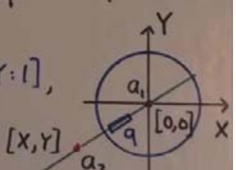
then $$q(a_1,a_2) \equiv 1 - \frac{(X_1X_2+Y_1Y_2-1)^2}{(X_1^2+Y_1^2-1)(X_2^2+Y_2^2-1)}$$

Note: This is undefined if one or both of a_1, a_2 are null points (i.e. lie on null circle)

Zero denominator convention: A zero in the denominator of a fraction makes equation/statement empty.

Ex. If $a_1 = [0:0:1]$ and $a_2 = [X:Y:1]$,

$$q(a_1,a_2) = \frac{X^2+Y^2}{X^2+Y^2-1} = q. \blacklozenge$$



Pythagoras' theorem (hyperbolic version) ⑤

Suppose a_1, a_2, a_3 are distinct points, with $q_1 \equiv q(a_2,a_3)$, $q_2 \equiv q(a_1,a_3)$ & $q_3 \equiv q(a_1,a_2)$. If $a_1a_3 \perp a_2a_3$ then $\boxed{q_3 = q_1+q_2-q_1q_2}$.

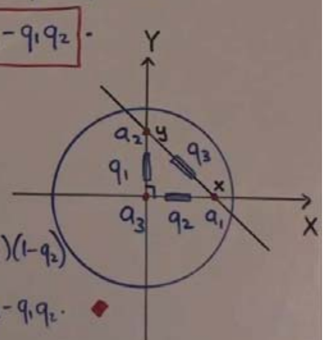
Ex. $a_1 \equiv [x:0:1]$ $a_2 \equiv [0:y:1]$

$a_3 \equiv [0:0:1]$. Then

$$q_1 = \frac{y^2}{y^2-1} \qquad q_2 = \frac{x^2}{x^2-1}$$

$$q_3 = 1 - \frac{1}{(x^2-1)(y^2-1)} \quad \text{so} \quad 1-q_3 = (1-q_1)(1-q_2)$$

$$\Leftrightarrow q_3 = q_1+q_2-q_1q_2. \blacklozenge$$



Exer. 22.1 Show that the lines of previous example are $L_1 \equiv a_2a_3 = (1:0:0)$ $L_2 \equiv a_1a_3 = (0:1:0)$ $L_3 \equiv a_1a_2 = (y:x:xy)$

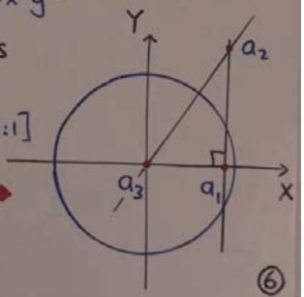
Show that the spreads are

$S_1 = \frac{(1-x^2)y^2}{x^2+y^2-x^2y^2}$ $S_2 = \frac{(1-y^2)x^2}{x^2+y^2-x^2y^2}$ $S_3 = 1$. ◆

Exer 22.2 Work out quadrances + spreads of $\overline{a_1a_2a_3}$ where

$a_1 \equiv [x:0:1]$ $a_2 \equiv [x:y:1]$ $a_3 \equiv [0:0:1]$

Verify Pythagoras' theorem. ◆



⑥

Pythagoras If $a_1a_3 \perp a_2a_3$ then $q_3 = q_1 + q_2 - q_1q_2$. ⑦

Proof. Suppose $a_1 \equiv [x_1:y_1:z_1]$ $a_2 \equiv [x_2:y_2:z_2]$ $a_3 \equiv [x_3:y_3:z_3]$.

Then $a_1a_3 = (y_1z_3 - y_3z_1 : z_1x_3 - z_3x_1 : x_3y_1 - x_1y_3)$

$a_2a_3 = (y_2z_3 - y_3z_2 : z_2x_3 - z_3x_2 : x_3y_2 - x_2y_3)$

These lines are perpendicular $\Leftrightarrow$

$(y_1z_3-y_3z_1)(y_2z_3-y_3z_2)+(z_1x_3-z_3x_1)(z_2x_3-z_3x_2)-(x_3y_1-x_1y_3)(x_3y_2-x_2y_3) = 0$. $\Leftrightarrow$

$(x_3^2+y_3^2-z_3^2)(x_1x_2+y_1y_2-z_1z_2)-(x_2x_3+y_2y_3-z_2z_3)(x_1x_3+y_1y_3-z_1z_3) = 0$. (*) (We multiplied by -1)

Assuming $a_1a_3 \perp a_2a_3$, need prove $q_3 = q_1 + q_2 - q_1q_2$ or equivalently $(1-q_3)-(1-q_1)(1-q_2) = 0$. ⑧

Now $(1-q_3)-(1-q_1)(1-q_2)$

$$= \frac{(x_1x_2+y_1y_2-z_1z_2)^2}{(x_1^2+y_1^2-z_1^2)(x_2^2+y_2^2-z_2^2)} - \frac{(x_2x_3+y_2y_3-z_2z_3)^2}{(x_2^2+y_2^2-z_2^2)(x_3^2+y_3^2-z_3^2)}\frac{(x_1x_3+y_1y_3-z_1z_3)^2}{(x_1^2+y_1^2-z_1^2)(x_3^2+y_3^2-z_3^2)}$$

$$= \frac{(x_3^2+y_3^2-z_3^2)^2(x_1x_2+y_1y_2-z_1z_2)^2 - (x_2x_3+y_2y_3-z_2z_3)^2(x_1x_3+y_1y_3-z_1z_3)^2}{(x_1^2+y_1^2-z_1^2)\,(x_2^2+y_2^2-z_2^2)\,(x_3^2+y_3^2-z_3^2)^2}$$

The numerator is a difference of squares. One of the factors is (*). So $a_1a_3 \perp a_2a_3 \Rightarrow (1-q_3) = (1-q_1)(1-q_2)$. ∎

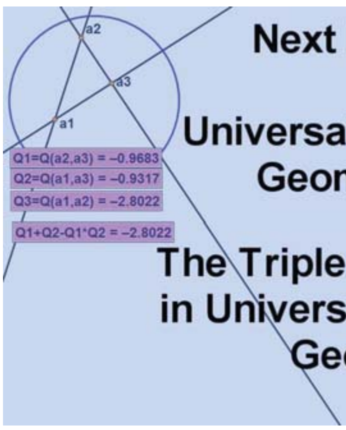
Note: The converse does not follow - because of other factor.

Exer. 22.3 Show that $\overline{a_1a_2a_3}$ is a right triangle if $a_1 \equiv [x:0:1]$ $a_2 = [1:0:0]$ $a_3 = [1:y:0]$. ⑨

Calculate quadrances + spreads, and verify Pythagoras' thm. ◆

Exer. 22.4 Same for $\overline{b_1b_2b_3}$ where $b_1 \equiv [x:1-x:1]$ $b_2 \equiv [y:1-y:1]$ $b_3 \equiv [1:1:1]$. ◆

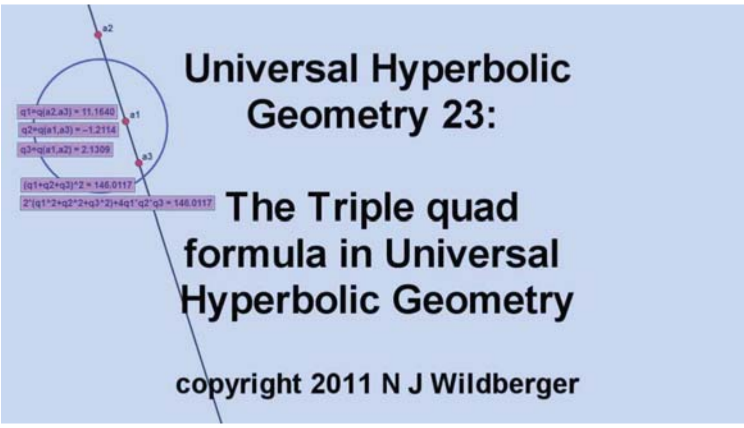
Exer. 22.5 Same for $\overline{c_1c_2c_3}$ where $c_1 \equiv [x:0:1]$ $c_2 \equiv [y:0:1]$ $c_3 \equiv [0:1:0]$. ◆

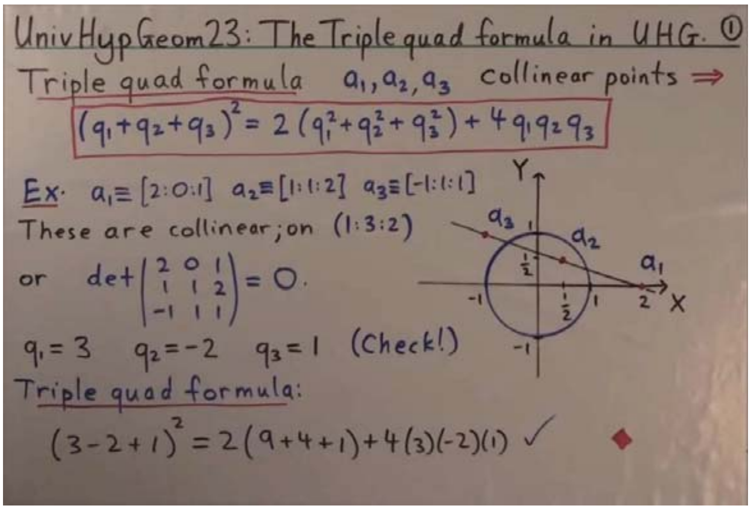


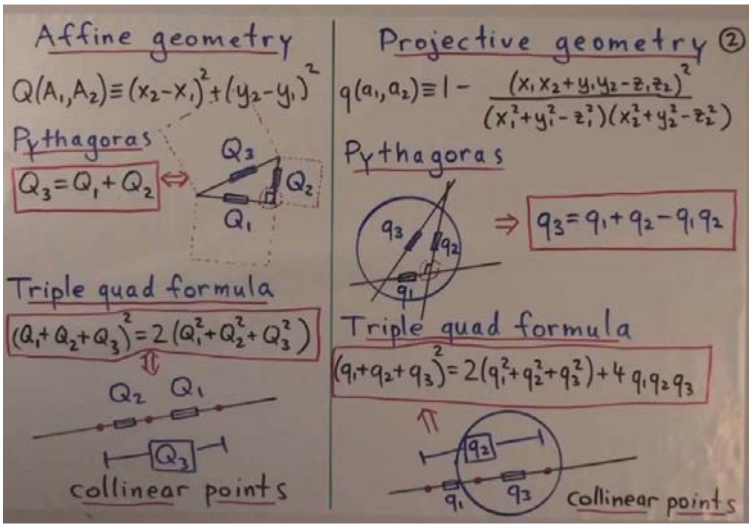
Next lecture:

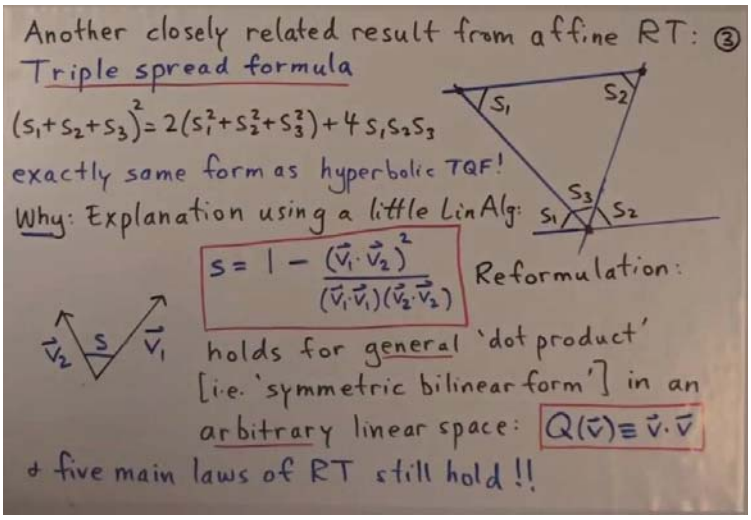
Universal Hyperbolic Geometry 23:

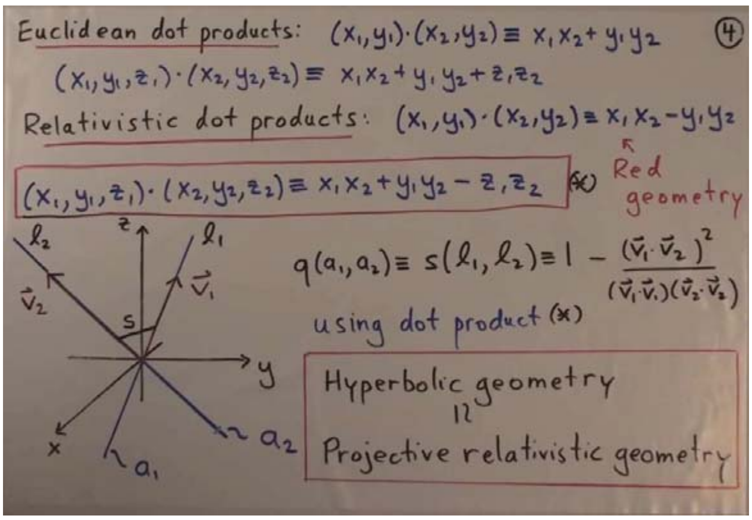
The Triple quad formula in Universal Hyperbolic Geometry

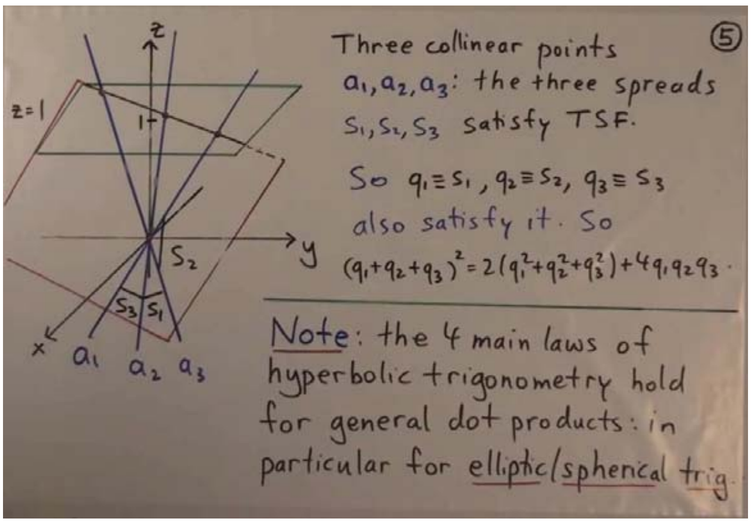

Universal Hyperbolic Geometry 23:
The Triple quad formula in Universal Hyperbolic Geometry
copyright 2011 N J Wildberger


UnivHypGeom23: The Triple quad formula in UHG. ①
Triple quad formula a_1, a_2, a_3 collinear points ⇒
$(q_1+q_2+q_3)^2 = 2(q_1^2+q_2^2+q_3^2)+4q_1q_2q_3$
Ex. $a_1 \equiv [2:0:1]$ $a_2 \equiv [1:1:2]$ $a_3 \equiv [-1:1:1]$
These are collinear; on $(1:3:2)$
or $\det\begin{vmatrix} 2 & 0 & 1 \\ 1 & 1 & 2 \\ -1 & 1 & 1 \end{vmatrix} = 0.$
$q_1 = 3$ $q_2 = -2$ $q_3 = 1$ (Check!)
Triple quad formula:
$(3-2+1)^2 = 2(9+4+1)+4(3)(-2)(1)$ ✓


Affine geometry
$Q(A_1,A_2) \equiv (x_2-x_1)^2+(y_2-y_1)^2$
Pythagoras
$Q_3 = Q_1+Q_2$ ⇔
Triple quad formula
$(Q_1+Q_2+Q_3)^2 = 2(Q_1^2+Q_2^2+Q_3^2)$
collinear points
Projective geometry ②
$q(a_1,a_2) \equiv 1 - \frac{(x_1x_2+y_1y_2-z_1z_2)^2}{(x_1^2+y_1^2-z_1^2)(x_2^2+y_2^2-z_2^2)}$
Pythagoras
⇒ $q_3 = q_1+q_2-q_1q_2$
Triple quad formula
$(q_1+q_2+q_3)^2 = 2(q_1^2+q_2^2+q_3^2)+4q_1q_2q_3$
collinear points


Another closely related result from affine RT: ③
Triple spread formula
$(s_1+s_2+s_3)^2 = 2(s_1^2+s_2^2+s_3^2)+4s_1s_2s_3$
exactly same form as hyperbolic TQF!
Why: Explanation using a little LinAlg:
$s = 1 - \frac{(\vec{v}_1\cdot\vec{v}_2)^2}{(\vec{v}_1\cdot\vec{v}_1)(\vec{v}_2\cdot\vec{v}_2)}$
Reformulation:
holds for general 'dot product'
[i.e. 'symmetric bilinear form'] in an
arbitrary linear space: $Q(\vec{v}) \equiv \vec{v}\cdot\vec{v}$
+ five main laws of RT still hold!!


Euclidean dot products: $(x_1,y_1)\cdot(x_2,y_2) \equiv x_1x_2+y_1y_2$ ④
$(x_1,y_1,z_1)\cdot(x_2,y_2,z_2) \equiv x_1x_2+y_1y_2+z_1z_2$
Relativistic dot products: $(x_1,y_1)\cdot(x_2,y_2) \equiv x_1x_2-y_1y_2$
$(x_1,y_1,z_1)\cdot(x_2,y_2,z_2) \equiv x_1x_2+y_1y_2-z_1z_2$ (*)
Red geometry
$q(a_1,a_2) \equiv s(l_1,l_2) \equiv 1 - \frac{(\vec{v}_1\cdot\vec{v}_2)^2}{(\vec{v}_1\cdot\vec{v}_1)(\vec{v}_2\cdot\vec{v}_2)}$
using dot product (*)
Hyperbolic geometry
≡
Projective relativistic geometry


Three collinear points ⑤
a_1, a_2, a_3: the three spreads
s_1, s_2, s_3 satisfy TSF.
So $q_1 \equiv s_1$, $q_2 \equiv s_2$, $q_3 \equiv s_3$
also satisfy it. So
$(q_1+q_2+q_3)^2 = 2(q_1^2+q_2^2+q_3^2)+4q_1q_2q_3$.
Note: the 4 main laws of
hyperbolic trigonometry hold
for general dot products: in
particular for elliptic/spherical trig.

Triple quad formula If the points a_1, a_2, a_3 are collinear, and $q_1 \equiv q(a_2,a_3)$, $q_2 \equiv q(a_1,a_3)$, $q_3 \equiv q(a_1,a_2)$, then ⑥

$$(q_1+q_2+q_3)^2 = 2(q_1^2+q_2^2+q_3^2) + 4q_1q_2q_3.$$

Proof. First check that

$$(q_1+q_2+q_3)^2 - 2(q_1^2+q_2^2+q_3^2) - 4q_1q_2q_3$$
$$= 4(1-q_1)(1-q_2)(1-q_3) - (q_1+q_2+q_3-2)^2 \equiv S(q_1,q_2,q_3)$$

Suppose $a_1 \equiv [x_1:y_1:z_1]$ $a_2 \equiv [x_2:y_2:z_2]$ $a_3 \equiv [x_3:y_3:z_3]$

Then $1-q_1 = \dfrac{(x_2x_3+y_2y_3-z_2z_3)^2}{(x_2^2+y_2^2-z_2^2)(x_3^2+y_3^2-z_3^2)}$

$1-q_2 = \dfrac{(x_1x_3+y_1y_3-z_1z_3)^2}{(x_1^2+y_1^2-z_1^2)(x_3^2+y_3^2-z_3^2)}$ $\quad 1-q_3 = \dfrac{(x_1x_2+y_1y_2-z_1z_2)^2}{(x_1^2+y_1^2-z_1^2)(x_2^2+y_2^2-z_2^2)}$

Then $S(q_1,q_2,q_3) = 4(1-q_1)(1-q_2)(1-q_3) - (1-q_1+1-q_2+1-q_3-1)^2$ ⑦

is a difference of squares. One factor is

$$\frac{2(x_2x_3+y_2y_3-z_2z_3)(x_1x_3+y_1y_3-z_1z_3)(x_1x_2+y_1y_2-z_1z_2)}{(x_1^2+y_1^2-z_1^2)(x_2^2+y_2^2-z_2^2)(x_3^2+y_3^2-z_3^2)}$$

$$-\frac{(x_2x_3+y_2y_3-z_2z_3)^2}{(x_2^2+y_2^2-z_2^2)(x_3^2+y_3^2-z_3^2)} - \frac{(x_1x_3+y_1y_3-z_1z_3)^2}{(x_1^2+y_1^2-z_1^2)(x_3^2+y_3^2-z_3^2)} - \frac{(x_1x_2+y_1y_2-z_1z_2)^2}{(x_1^2+y_1^2-z_1^2)(x_2^2+y_2^2-z_2^2)} + 1$$

Remarkably, this expression is identically equal to

$$-\frac{(x_1y_2z_3 - x_1y_3z_2 + x_2y_3z_1 - x_3y_2z_1 + x_3y_1z_2 - x_2y_1z_3)^2}{(x_1^2+y_1^2-z_1^2)(x_2^2+y_2^2-z_2^2)(x_3^2+y_3^2-z_3^2)}.$$

But the numerator is $\det^2\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$, which is 0 if a_1, a_2, a_3 collinear. ■

Def. The Triple spread function is defined by ⑧

$$S(a,b,c) \equiv (a+b+c)^2 - 2(a^2+b^2+c^2) - 4abc$$

Exer 23.1 Show that

$$S(a,b,c) = 2ab+2bc+2ac-a^2-b^2-c^2-4abc$$
$$= 4(1-a)(1-b)(1-c) - (a+b+c-2)^2$$
$$= -\det\begin{pmatrix} 0 & a & b & 1 \\ a & 0 & c & 1 \\ b & c & 0 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix} \quad ◆$$

Exer 23.2 Show that the TQF can be rewritten as:

$$q_3^2 - 2q_3(q_1+q_2-2q_1q_2) + (q_1-q_2)^2 = 0. \quad ◆$$

Exer 23.3 Verify that ⑨

$$S(a,b,c) = (1-c)(c+(1-2a)(1-2b)) - (a+b-1)^2.$$

Deduce **Complimentary quadrances theorem:** If a_1, a_2, a_3 are collinear points, and $q_3 = 1$, then $\boxed{q_1+q_2=1}$. ◆

Exer 23.4 Verify that

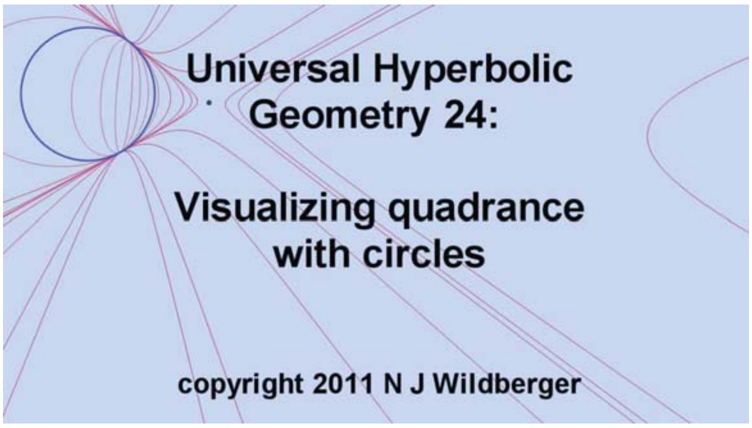
$$S(a,b,c) = (a-b)(b-a-2c+4ac) - c(c-4a+4a^2)$$

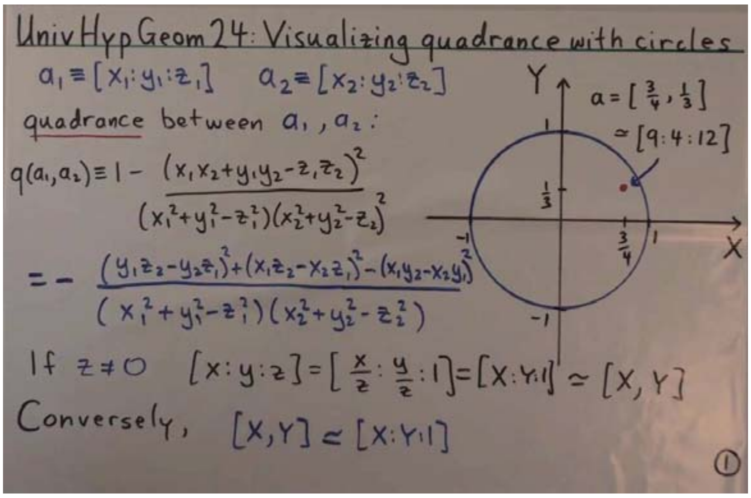
Deduce **Equal quadrances theorem:** If a_1, a_2, a_3 are collinear points, and $q_1 = q_2$, then either $q_3 = 0$ or $\boxed{q_3 = 4q_1(1-q_1)}$. ◆

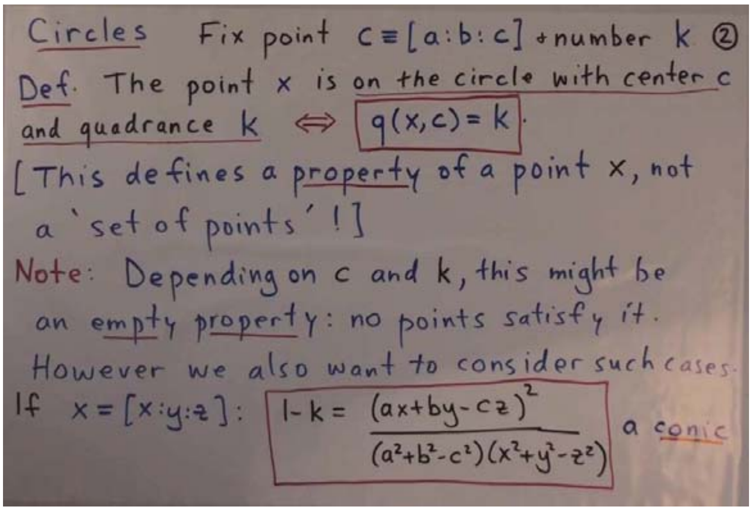
Next lecture:

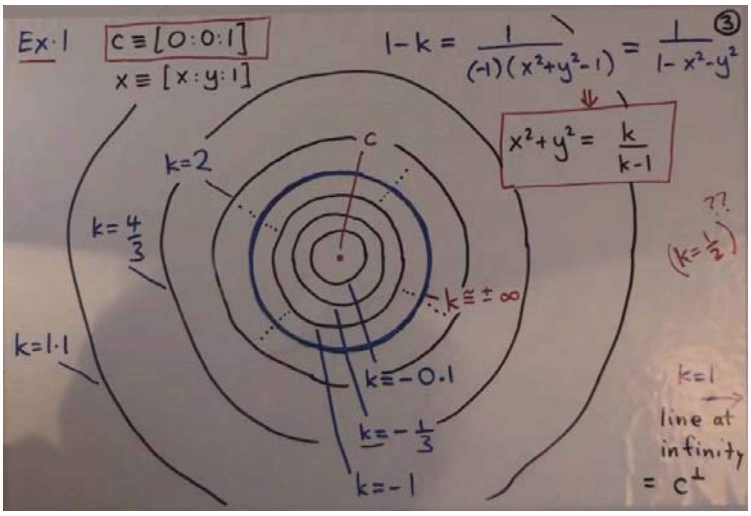
Universal Hyperbolic Geometry 24:

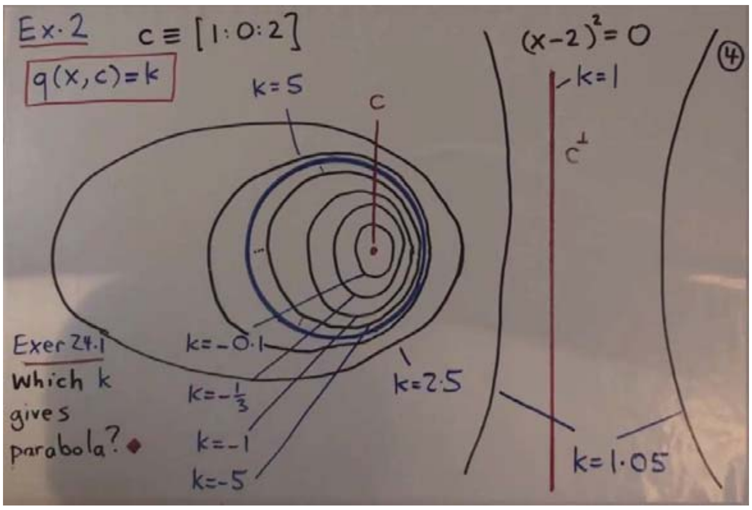
Visualizing qadrance using circles

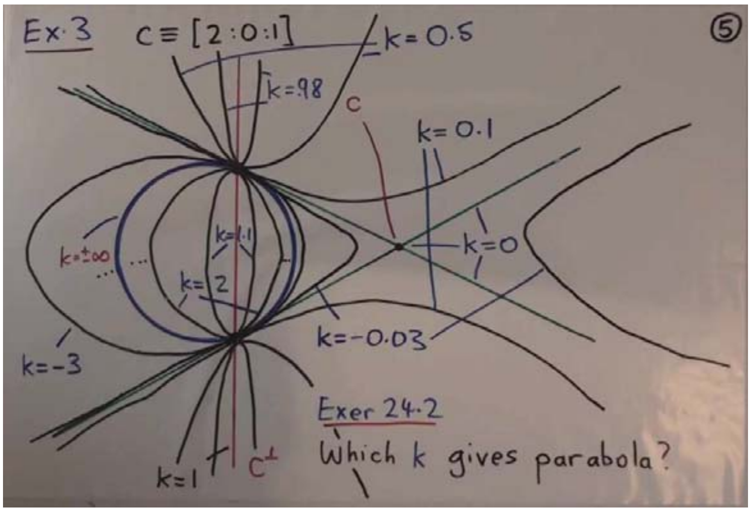

Universal Hyperbolic Geometry 24:
Visualizing quadrance with circles
copyright 2011 N J Wildberger

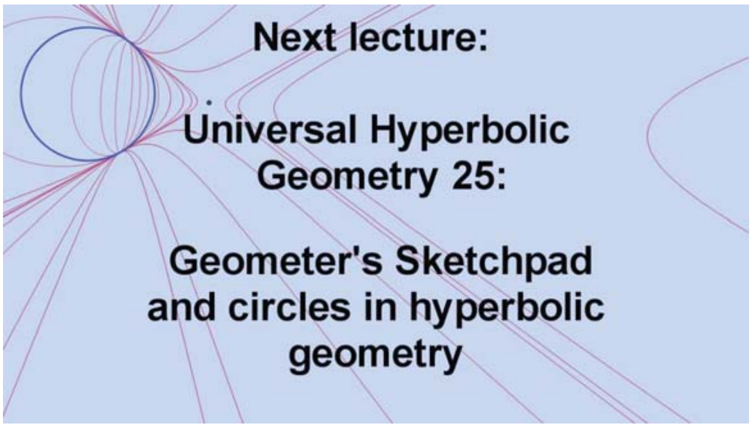

UnivHypGeom 24: Visualizing quadrance with circles
$a_1 \equiv [x_1:y_1:z_1]$ $a_2 \equiv [x_2:y_2:z_2]$
quadrance between a_1, a_2:
$q(a_1,a_2) \equiv 1 - \frac{(x_1x_2+y_1y_2-z_1z_2)^2}{(x_1^2+y_1^2-z_1^2)(x_2^2+y_2^2-z_2^2)}$
$= -\frac{(y_1z_2-y_2z_1)^2+(x_1z_2-x_2z_1)^2-(x_1y_2-x_2y_1)^2}{(x_1^2+y_1^2-z_1^2)(x_2^2+y_2^2-z_2^2)}$
If $z \neq 0$ $[x:y:z]=[\frac{x}{z}:\frac{y}{z}:1]=[X:Y:1] \simeq [X,Y]$
Conversely, $[X,Y] \simeq [X:Y:1]$
Y
X
$a=[\frac{3}{4},\frac{1}{3}]$
$\simeq [9:4:12]$
1
-1
$\frac{1}{3}$
$\frac{3}{4}$
①

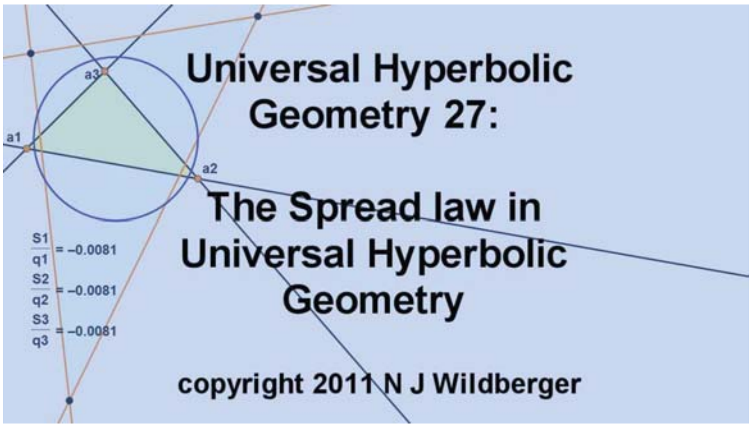

Circles Fix point $c \equiv [a:b:c]$ + number k ②
Def. The point x is on the circle with center c and quadrance k ⟺ $\boxed{q(x,c)=k}$.
[This defines a property of a point x, not a 'set of points'!]
Note: Depending on c and k, this might be an empty property: no points satisfy it.
However we also want to consider such cases.
If $x=[x:y:z]$: $\boxed{1-k=\frac{(ax+by-cz)^2}{(a^2+b^2-c^2)(x^2+y^2-z^2)}}$ a conic


Ex.1 $\boxed{c \equiv [0:0:1]}$
$x \equiv [x:y:1]$
$1-k=\frac{1}{(-1)(x^2+y^2-1)}=\frac{1}{1-x^2-y^2}$
③
$\boxed{x^2+y^2=\frac{k}{k-1}}$
c
k=2
$k=\frac{4}{3}$
k=1.1
k≡±∞
k≡-0.1
$k=-\frac{1}{3}$
k=-1
??
$(k=\frac{1}{2})$
k=1
line at infinity
$= c^\perp$


Ex.2 $c \equiv [1:0:2]$
$\boxed{q(x,c)=k}$
$(x-2)^2=0$
k=1
$c^\perp$
④
k=5
c
k=-0.1
$k=-\frac{1}{3}$
k=-1
k=-5
k=2.5
k=1.05
Exer 24.1
Which k gives parabola?


Ex.3 $c \equiv [2:0:1]$
k=0.5
⑤
k=.98
c
k=0.1
k=0
k=±∞
k=1.1
k=2
k=-0.03
k=-3
k=1
$c^\perp$
Exer 24.2
Which k gives parabola?


Next lecture:
Universal Hyperbolic Geometry 25:
Geometer's Sketchpad and circles in hyperbolic geometry



Universal Hyperbolic Geometry 27:

The Spread law in Universal Hyperbolic Geometry

UnivHypGeom27: The Spread law in UHG ①

Lines $L_1 \equiv (l_1 : m_1 : n_1)$ and $L_2 \equiv (l_2 : m_2 : n_2)$

spread between L_1, L_2:

$$S(L_1,L_2) \equiv 1 - \frac{(l_1 l_2 + m_1 m_2 - n_1 n_2)^2}{(l_1^2+m_1^2-n_1^2)(l_2^2+m_2^2-n_2^2)}$$

$$= -\frac{(m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 - (l_1 m_2 - l_2 m_1)^2}{(l_1^2+m_1^2-n_1^2)(l_2^2+m_2^2-n_2^2)}$$

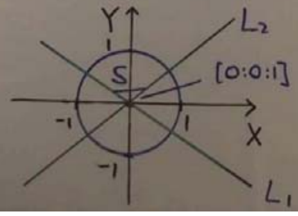
Prop. (Quadrance spread duality)

$$\boxed{q(a_1,a_2) = S(a_1^\perp, a_2^\perp)}$$ ■

Ex. The spread between the lines $L_1 \equiv (l_1 : m_1 : 0)$ and $L_2 \equiv (l_2 : m_2 : 0)$ passing through the point $[0:0:1]$ is ②

$$S(L_1,L_2) = \frac{(l_1 m_2 - l_2 m_1)^2}{(l_1^2+m_1^2)(l_2^2+m_2^2)}$$

This is the same as the Euclidean spread.

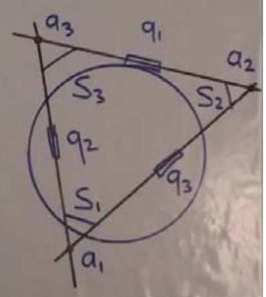


Spread law (hyperbolic version) ③

Suppose that a_1, a_2, a_3 are distinct points, with $q_1 \equiv q(a_2,a_3)$, $q_2 \equiv q(a_1,a_3)$ & $q_3 \equiv q(a_1,a_2)$ and with $S_1 \equiv S(a_1a_2, a_1a_3)$, $S_2 \equiv S(a_2a_1, a_2a_3)$ & $S_3 \equiv S(a_3a_1, a_3a_2)$.

Then

$$\boxed{\frac{S_1}{q_1} = \frac{S_2}{q_2} = \frac{S_3}{q_3}}$$



Proof. Suppose $a_1 \equiv [x_1 : y_1 : z_1]$, $a_2 \equiv [x_2 : y_2 : z_2]$ & $a_3 \equiv [x_3 : y_3 : z_3]$.

Let's find S_1 in terms of $x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3$.

Now $S_1 \equiv S(a_1a_2, a_1a_3) = S(L_2, L_3)$ where ④

$L_3 \equiv a_1a_2 = (l_3 : m_3 : n_3)$ and from Join of points thm; where

$L_2 \equiv a_1a_3 = (l_2 : m_2 : n_2)$

$l_2 \equiv y_3z_1 - y_1z_3 \quad m_2 \equiv z_3x_1 - z_1x_3 \quad n_2 \equiv x_1y_3 - x_3y_1$

$l_3 \equiv y_2z_1 - y_1z_2 \quad m_3 \equiv z_2x_1 - z_1x_2 \quad n_3 \equiv x_1y_2 - x_2y_1$.

So

$$S_1 = \frac{(l_2^2+m_2^2-n_2^2)(l_3^2+m_3^2-n_3^2) - (l_2l_3+m_2m_3-n_2n_3)^2}{(l_2^2+m_2^2-n_2^2)(l_3^2+m_3^2-n_3^2)}$$

$$= -\frac{(x_1^2+y_1^2-z_1^2)(x_1y_2z_3 - x_1y_3z_2 + x_2y_3z_1 - x_3y_2z_1 + x_3y_1z_2 - x_2y_1z_3)^2}{(l_2^2+m_2^2-n_2^2)(l_3^2+m_3^2-n_3^2)} \quad (*) \quad (!!)$$

Now let's find q_1: define also ⑤

$l_1 \equiv y_2z_3 - y_3z_2 \quad m_1 \equiv z_2x_3 - z_3x_2 \quad n_1 \equiv x_3y_2 - x_2y_3$

so

$$q_1 = -\frac{l_1^2+m_1^2-n_1^2}{(x_2^2+y_2^2-z_2^2)(x_3^2+y_3^2-z_3^2)} \quad (**)$$

Now combine $(*)$ and $(**)$:

$$\frac{S_1}{q_1} = \frac{(x_1^2+y_1^2-z_1^2)(x_2^2+y_2^2-z_2^2)(x_3^2+y_3^2-z_3^2)}{(l_1^2+m_1^2-n_1^2)(l_2^2+m_2^2-n_2^2)(l_3^3+m_3^2-n_3^2)} \times D^2 \quad (***)$$

where $D \equiv x_1y_2z_3 - x_1y_3z_2 + x_2y_3z_1 - x_3y_2z_1 + x_3y_1z_2 - x_2y_1z_3$.

This is all symmetric in the 3 indices: so

$$\frac{S_1}{q_1} = \frac{S_2}{q_2} = \frac{S_3}{q_3}.$$ ■

From (**) and symmetry, (6)

$$q_1 q_2 q_3 = -\frac{(l_1^2+m_1^2-n_1^2)(l_2^2+m_2^2-n_2^2)(l_3^2+m_3^2-n_3^2)}{(x_1^2+y_1^2-z_1^2)^2(x_2^2+y_2^2-z_2^2)^2(x_3^2+y_3^2-z_3^2)^2}$$

Now multiply this by (***):

$$q_2 q_3 S_1 = -\frac{(x_1y_2z_3 - x_1y_3z_2 + x_2y_3z_1 - x_3y_2z_1 + x_3y_1z_2 - x_2y_1z_3)^2}{(x_1^2+y_1^2-z_1^2)(x_2^2+y_2^2-z_2^2)(x_3^2+y_3^2-z_3^2)}$$

$= q_1 q_3 S_2 = q_1 q_2 S_3$ by symmetry

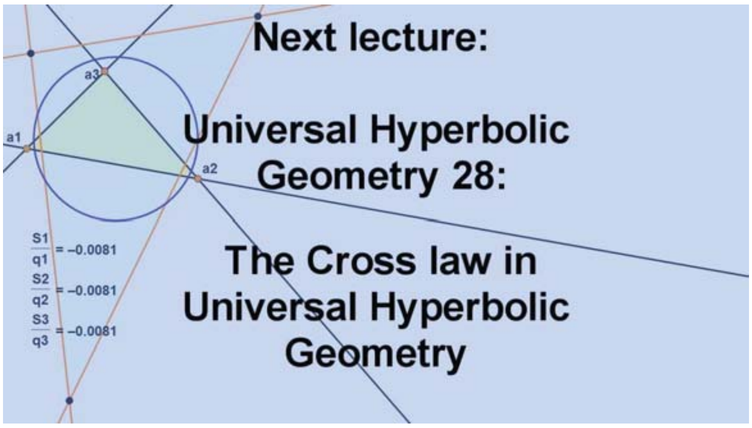
$\equiv A \equiv A(\overline{a_1a_2a_3})$ quadrea of the triangle.

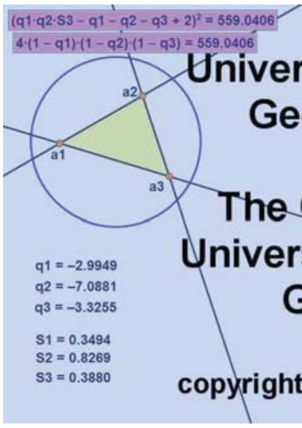
[Turns out to be: most important triangle invariant.]

Exer. 27.1 Compute the quadrances + spreads of $\overline{a_1a_2a_3}$ where $a_1 \equiv [0:0:1]$, $a_2 \equiv [x:y:1]$ $a_3 \equiv [x:-y:1]$. Verify the Spread law + find the quadrea $A(\overline{a_1a_2a_3})$. ◆

Exer. 27.2 Same for $\overline{b_1b_2b_3}$ where $b_1 \equiv [1:1:1]$, $b_2 \equiv [x:y:1]$ + $b_3 \equiv [x:-y:1]$. ◆

Exer. 27.3 Same for $\overline{c_1c_2c_3}$ where (7) $c_1 \equiv [z:0:1]$, $c_2 \equiv [x:1:0]$ + $c_3 \equiv [y:1:0]$. ◆





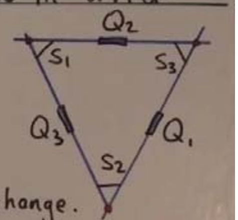
Universal Hyperbolic Geometry 28:

The Cross law in Universal Hyperbolic Geometry

Univ Hyp Geom 28: The Cross law in UHG. ①

Cross law in Euclidean R.T.

$$\boxed{(Q_1+Q_2-Q_3)^2 = 4Q_1Q_2(1-s_3)}$$



$c_3 \equiv 1-s_3$ is the cross

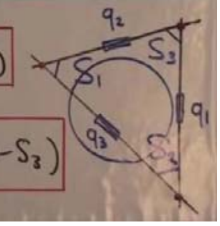
If we scale the Δ, spreads don't change.

Cross law in UHG:

$$\boxed{(q_1q_2S_3-q_1-q_2-q_3+2)^2 = 4(1-q_1)(1-q_2)(1-q_3)}$$

Cross Dual law:

$$\boxed{(S_1S_2q_3-S_1-S_2-S_3+2)^2 = 4(1-S_1)(1-S_2)(1-S_3)}$$



In classical HG: hyperbolic Cosine law: ②

$$\cosh d_3 = \cosh d_1 \cosh d_2 - \sinh d_1 \sinh d_2 \cos\theta_1$$

$$\cosh d_3 = \frac{\cos\theta_1\cos\theta_2+\cos\theta_3}{\sin\theta_1\sin\theta_2}$$

No! No!

Exer. 28.1 Show that

$$(q_1q_2S_3-q_1-q_2-q_3+2)^2 = 4(1-q_1)(1-q_2)(1-q_3)$$

can be rewritten as

$$\boxed{(q_1+q_2-q_3-q_1q_2S_3)^2 = 4q_1q_2(1-q_3)(1-S_3)}.$$

From this follows Pythagoras' thm. Also the Euclidean Cross law is a limiting case as $q_i \to 0$

Cross law (hyperbolic version) ③

Suppose that a_1, a_2, a_3 are distinct points with quadrances $q_1 \equiv q(a_2,a_3)$, $q_2 \equiv q(a_1,a_3)$ & $q_3 \equiv q(a_1,a_2)$, and spread $S_3 = S(a_3a_1, a_3a_2)$. Then

$$\boxed{(q_1q_2S_3-q_1-q_2-q_3+2)^2 = 4(1-q_1)(1-q_2)(1-q_3)}.$$

Proof. Suppose that $a_1 \equiv [x_1:y_1:z_1]$ $a_2 \equiv [x_2:y_2:z_2]$ $a_3 \equiv [x_3:y_3:z_3]$. Recall last lecture we proved:

$$A = q_1q_2S_3 = -\frac{(x_1y_2z_3-x_1y_3z_2+x_2y_3z_1-x_3y_2z_1+x_3y_1z_2-x_2y_1z_3)^2}{(x_1^2+y_1^2-z_1^2)(x_2^2+y_2^2-z_2^2)(x_3^2+y_3^2-z_3^2)}$$

$$(= q_1q_3S_2 = q_2q_3S_1)$$

Behold: $-(x_1y_2z_3-x_1y_3z_2+x_2y_3z_1-x_3y_2z_1+x_3y_1z_2-x_2y_1z_3)^2$

$+(x_1^2+y_1^2-z_1^2)(x_2x_3+y_2y_3-z_2z_3)^2+(x_2^2+y_2^2-z_2^2)(x_1x_3+y_1y_3-z_1z_3)^2$

$+(x_3^2+y_3^2-z_3^2)(x_1x_2+y_1y_2-z_1z_2)^2$

$-(x_1^2+y_1^2-z_1^2)(x_2^2+y_2^2-z_2^2)(x_3^2+y_3^2-z_3^2)$ (!!)

$=2(x_2x_3+y_2y_3-z_2z_3)(x_1x_3+y_1y_3-z_1z_3)(x_1x_2+y_1y_2-z_1z_2)$

Square both sides, and divide by

$$(x_1^2+y_1^2-z_1^2)^2(x_2^2+y_2^2-z_2^2)^2(x_3^2+y_3^2-z_3^2)^2$$

to get ④

$$(A+(1-q_1)+(1-q_2)+(1-q_3)-1)^2 = 4(1-q_1)(1-q_2)(1-q_3).$$

Equilateral Δ theorem If a triangle has three equal non-zero quadrances $q_1=q_2=q_3=q$, then it also has three equal spreads; $S_1=S_2=S_3=S$ ⑤

Furthermore $\boxed{(1-Sq)^2 = 4(1-S)(1-q)}$.

Proof. The first statement follows from the Spread law: $\frac{S_1}{q_1}=\frac{S_2}{q_2}=\frac{S_3}{q_3}$ unless $q_1=q_2=q_3=0$.

Then the Cross law $\Rightarrow$

$(q^2S-3q+2)^2 = 4(1-q)^3 \Leftrightarrow S^2q^2-6Sq+4S+4q-3=0$

$\Leftrightarrow (1-Sq)^2 = 4(1-S)(1-q)$. ■ **Exer. 28.2** Check this.

6

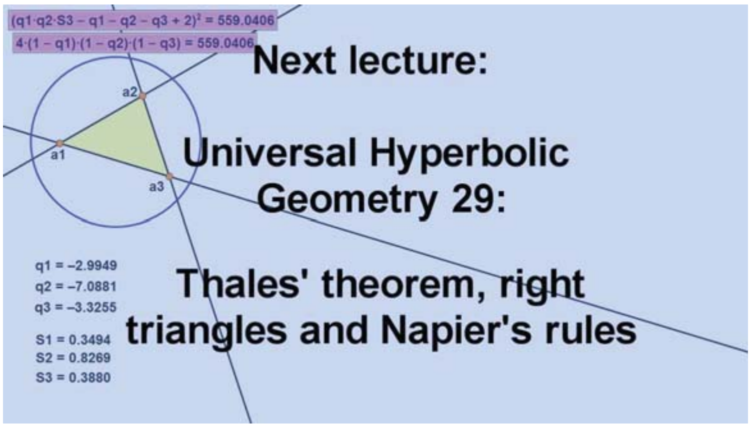
Exer. 28·3 Check the Cross law for particular triangles whose quadrances + spreads you have already calculated.

Exer. 28·4 Show that over field $\mathrm{Rat}(\sqrt{2},\sqrt{3})$

$a_1 \equiv [\sqrt{2}:0:1]$ $\quad a_2 \equiv [-1:\sqrt{3}:\sqrt{2}]$ $\quad a_3 \equiv [-1:-\sqrt{3}:\sqrt{2}]$

defines an equilateral triangle. What are its quadrances + spreads? What is its quadrea?

Exer. 28·5* Are there any equilateral triangles over the rational numbers?



Universal Hyperbolic Geometry 29:

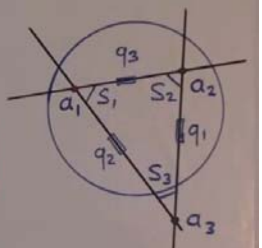
Thales' theorem, right triangles and Napier's rules

copyright 2011 N J Wildberger

$S1 = 0.2183$
$q1 = -4.2395$
$q3 = -19.4221$
$\frac{q1}{q3} = 0.2183$

UHG 29: Thales theorem, right triangles + Napier's rules ①

Four main laws of trigonometry



Pythagoras: If $S_3=1$ then

$$\boxed{q_3 = q_1+q_2-q_1q_2}$$

Triple quad formula: If $S_3=0$ then

$$\boxed{(q_1+q_2+q_3)^2 = 2(q_1^2+q_2^2+q_3^2)+4q_1q_2q_3}$$

Spread law: $\boxed{\frac{S_1}{q_1} = \frac{S_2}{q_2} = \frac{S_3}{q_3}}$

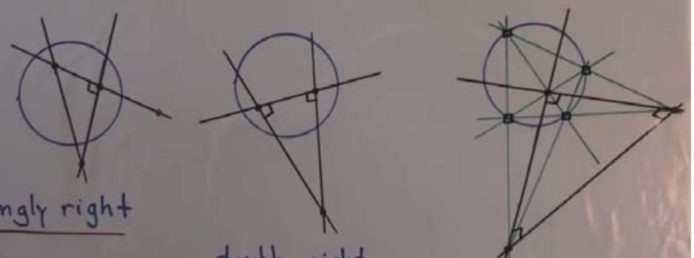
Cross law:

$$\boxed{(q_1q_2S_3-q_1-q_2-q_3+2)^2 = 4(1-q_1)(1-q_2)(1-q_3)}$$

+ duals!

Right triangles ②

Def A triangle $\overline{a_1a_2a_3}$ is right $\Leftrightarrow$ at least one of its spreads S_1, S_2, S_3 equals 1.



Thales theorem ③

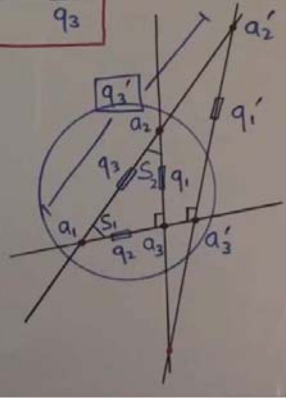
If $S_3=1$ then $\boxed{S_1 = \frac{q_1}{q_3}}$ and $\boxed{S_2 = \frac{q_2}{q_3}}$

Proof: Follows immediately from the Spread law

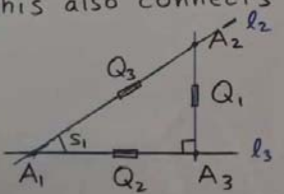
$$\frac{S_1}{q_1} = \frac{S_2}{q_2} = \frac{S_3}{q_3}. \quad \blacksquare$$

A kind of 'similarity' for right triangles:

$$\boxed{\frac{q_1}{q_3} = \frac{q_1'}{q_3'}} \; !$$



This also connects to Euclidean RT: ④



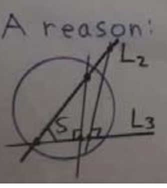
spread between lines: $s(A_1A_2, A_1A_3) = s_1 \equiv \frac{Q_1}{Q_3}$.

But why not consider the

cross $\quad c_1 \equiv \frac{Q_2}{Q_3}$ or

twist $\quad t_1 \equiv \frac{Q_1}{Q_2}$ as primary?

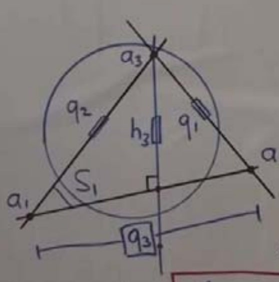
A reason:



In hyperbolic geometry the spread $S(L_2, L_3)$ is well defined; but the cross + twist are not.

Quadrea ⑤

$$\mathcal{A} = \mathcal{A}(\overline{a_1a_2a_3}) = q_1q_2S_3 = q_1q_3S_2 = q_2q_3S_1.$$



From Thales' theorem

$$S_1 = \frac{h_3}{q_2} \quad \text{so} \quad h_3 = q_2S_1$$

It follows that

$$\mathcal{A} = q_3(q_2S_1) = q_3h_3.$$

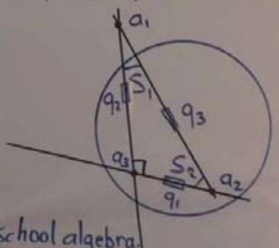
Similarly $\boxed{\mathcal{A} = q_1h_1 = q_2h_2 = q_3h_3}$

So quadrea $\mathcal{A} \approx (\text{area})^2$ [very roughly!]

Napier's rules: If a right triangle $\overline{a_1a_2a_3}$ has quadrances q_1, q_2, q_3 and spreads $S_1, S_2, S_3=1$ then any two of q_1, q_2, q_3, S_1, S_2 determine the other three, solely by the three basic equations: ⑥

$$S_1 = \frac{q_1}{q_3} \qquad S_2 = \frac{q_2}{q_3} \qquad q_3 = q_1 + q_2 - q_1 q_2.$$

This is much simpler than the classical Napier's rules!



The proof needs only high school algebra.

Proof: $\boxed{\text{Case 1}}$ two of q_1, q_2, q_3 known. ⑦

Solve $\boxed{q_3 = q_1 + q_2 - q_1 q_2}$ $\boxed{q_1 = \frac{q_3 - q_2}{1 - q_2}}$ $\boxed{q_2 = \frac{q_3 - q_1}{1 - q_1}}$

for third quadrance, then for spreads by $\boxed{S_1 = \frac{q_1}{q_3}}$ $\boxed{S_2 = \frac{q_2}{q_3}}$. ■

$\boxed{\text{Case 2}}$ S_1, S_2 known. Since $q_1 = q_3 S_1$ $q_2 = q_3 S_2$, Pythagoras $\Rightarrow$ $1 = S_1 + S_2 - S_1 S_2 q_3 \Rightarrow \boxed{q_3 = \frac{S_1 + S_2 - 1}{S_1 S_2}}$

$\Rightarrow \boxed{q_1 = \frac{S_1 + S_2 - 1}{S_2}}$ and $\boxed{q_2 = \frac{S_1 + S_2 - 1}{S_1}}$ ■

$\boxed{\text{Case 3}}$ S_1, q_3 known. Then $\boxed{q_1 = q_3 S_1}$ and ⑧

$\boxed{q_2 = \frac{q_3 - q_1}{1 - q_1} = \frac{q_3(1 - S_1)}{1 - q_3 S_1}}$ $\boxed{S_2 = \frac{q_2}{q_3} = \frac{1 - S_1}{1 - q_3 S_1}}$. ■

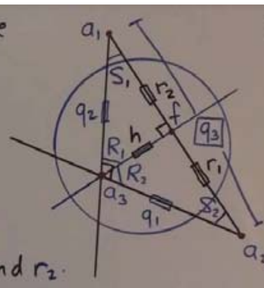
$\boxed{\text{Case 4}}$ S_1, q_1 known. Then $\boxed{q_3 = \frac{q_1}{S_1}}$ and

$\boxed{q_2 = \frac{q_3 - q_1}{1 - q_1} = \frac{q_1(1 - S_1)}{S_1(1 - q_1)}}$ $\boxed{S_2 = \frac{q_2}{q_3} = \frac{1 - S_1}{1 - q_1}}$. ■

$\boxed{\text{Case 5}}$ S_1, q_2 known. Since $q_1 = q_3 S_1$, Pythagoras $\Rightarrow$ $q_3 = q_3 S_1 + q_2 - q_2 q_3 S_1$. So $\boxed{q_3 = \frac{q_2}{1 - S_1(1 - q_2)}}$ and so

$\boxed{q_1 = \frac{q_2 S_1}{1 - S_1(1 - q_2)}}$ and $\boxed{S_2 = \frac{q_2}{q_3} = 1 - S_1(1 - q_2)}$. ■ ■

Exer. 29.1 Let $a_3 f$ be altitude of right triangle $\overline{a_1a_2a_3}$ whose quadrances are q_1, q_2, q_3. Show that ⑨

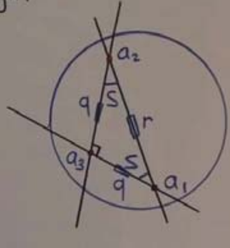


i) $h \equiv q(a_3, f) = \frac{q_1 q_2}{q_3}$

ii) $r_1 = q(a_2, f) = \frac{q_1(q_3 - q_2)}{q_3 - q_1 q_2}$ Find r_2.

iii) $R_1 = \frac{q_2(1 - q_1)}{q_3 - q_1 q_2}$ Find R_2. iv) $\frac{R_1}{R_2} = \frac{q_2(1 - q_1)}{q_1(1 - q_2)}$. ◆

Exer. 29.2 Suppose the right triangle $\overline{a_1a_2a_3}$ is isosceles so that $q(a_1, a_3) = q(a_2, a_3) = q$ and $S_1 = S_2 = S$, while $S_3 = 1$. Show that ⑩



i) $r = q(2 - q)$

ii) $S = \frac{1}{2 - q}$ & $q = 2 - \frac{1}{S}$

iii) If $S = \frac{1}{n}$ then find q, r. ◆

Exer. 29.3 Show that if a triangle $\overline{a_1a_2a_3}$ has $S_1 = S_2 = 1$ then $q_1 = q_2 = 1$, and also $q_3 = S_3$. ◆

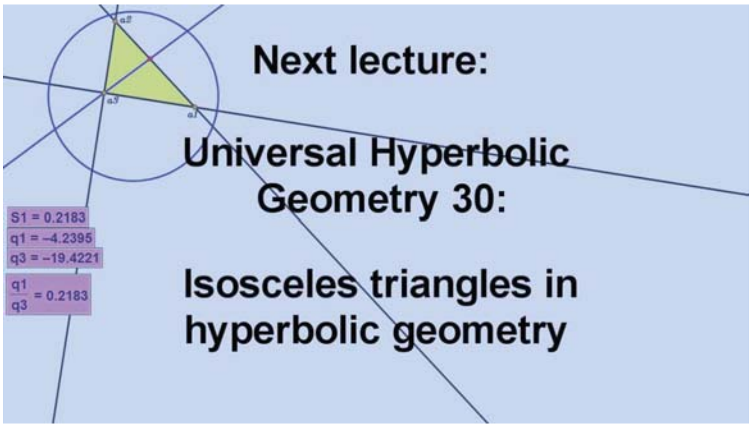
Exer. 29.4 Solve for q_1, q_2, q_3, S_1, S_2 in a right triangle where $S_3 = 1$:

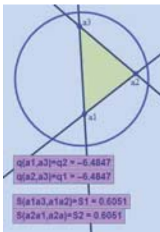
i) $q_1 = -2$ $q_3 = -3$

ii) $q_1 = -\frac{1}{2}$ $S_1 = \frac{1}{2}$

iii) $q_1 = -\frac{1}{2}$ $S_2 = \frac{1}{3}$

iv) $S_1 = \frac{1}{2}$ $S_2 = 3/4$ ◆ ⑪


Next lecture:
Universal Hyperbolic Geometry 30:
Isosceles triangles in hyperbolic geometry
a1
a2
a3
S1 = 0.2183
q1 = −4.2395
q3 = −19.4221
q1/q3 = 0.2183

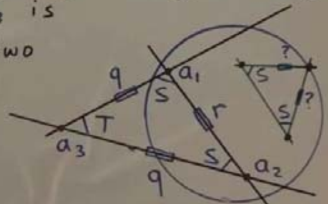


Universal Hyperbolic Geometry 30:

Isosceles triangles in hyperbolic geometry

UHG 30: Isosceles triangles in hyperbolic geom. ①

Def. A triangle $\overline{a_1a_2a_3}$ is isosceles $\Leftrightarrow$ at least two of its quadrances are equal, or two of its spreads are equal.

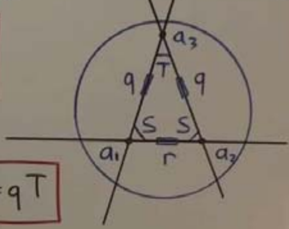


Theorem (Pons Asinorum) If a triangle $\overline{a_1a_2a_3}$ has quadrances q_1, q_2, q_3 + spreads S_1, S_2, S_3 then $q_1 = q_2 \Leftrightarrow S_1 = S_2$.

Proof. Follows from Spread law $\dfrac{S_1}{q_1} = \dfrac{S_2}{q_2} = \dfrac{S_3}{q_3}$. ∎

Notation for isosceles triangle: ②

$\overline{a_1a_2a_3}$ $\quad \boxed{\begin{matrix} q_1 = q_2 \equiv q & q_3 \equiv r \\ S_1 = S_2 \equiv S & S_3 \equiv T \end{matrix}}$



From Spread law:

$$\boxed{\frac{S}{q} = \frac{T}{r}} \quad \text{or} \quad \boxed{rS = qT}$$

Isosceles triangle theorem With this notation,

$$\boxed{r = \frac{4q(1-q)(1-S)}{(1-Sq)^2}} \quad \text{and} \quad \boxed{T = \frac{4S(1-S)(1-q)}{(1-Sq)^2}}$$

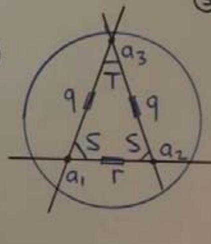
Proof. Apply the Cross law: ③

$$(qrS - q - r - q + 2)^2 = 4(1-q)^2(1-r)$$

$$(r(qS-1) + 2(1-q))^2 = 4(1-q)^2(1-r)$$

$$r^2(qS-1)^2 + 4r(qS-1)(1-q) = -4r(1-q)^2$$

$$r(qS-1)^2 = 4(1-q)[1 - qS + q - 1]$$



$$\boxed{r = \frac{4q(1-q)(1-S)}{(1-Sq)^2}} \quad \text{Since } T = \frac{rS}{q},$$

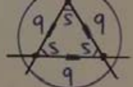
$$\boxed{T = \frac{4S(1-S)(1-q)}{(1-Sq)^2}}$$

∎

Equilateral triangle. If $r = q$ then ④

$$q = \frac{4q(1-q)(1-S)}{(1-Sq)^2} \Rightarrow \boxed{4(1-q)(1-S) = (1-Sq)^2}$$

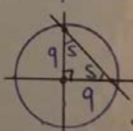
Recovers our earlier result. ◆



Right triangle If $T = 1$ then

$$1 = \frac{4S(1-S)(1-q)}{(1-Sq)^2} \Rightarrow (1-Sq)^2 - 4S(1-S)(1-q) = 0$$

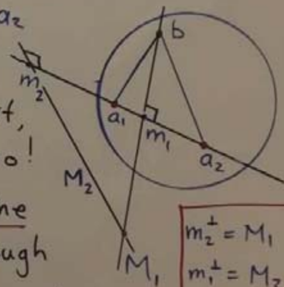
$$\Rightarrow [1 - S(2-q)]^2 = 0 \Rightarrow \boxed{S = \frac{1}{2-q}}$$



◆

Def. The point m is a *midpoint* of $\overline{a_1a_2} \Leftrightarrow m$ lies on a_1a_2 and $q(a_1, m) = q(m, a_2)$. ⑤

Midpoints don't always exist, and if they do, there are two!



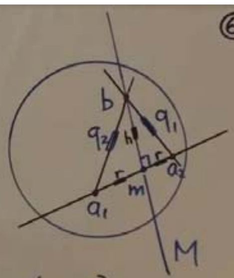
$\boxed{\begin{matrix} m_2^\perp = M_1 \\ m_1^\perp = M_2 \end{matrix}}$

Def. The line M is a *midline* of $\overline{a_1a_2} \Leftrightarrow M$ passes through a midpoint m and is perpendicular to $a_1a_2 \Leftrightarrow M$ is dual to a midpoint.

Midlines are called "perpendicular bisectors" in Eucl. geom.

Midline theorem ⑥

If b lies on a midline M of $\overline{a_1a_2}$, then $q(a_1,b)=q(a_2,b)$.



Proof. Suppose M passes thru the midpoint m of $\overline{a_1a_2}$.
Define $h\equiv q(b,m)$ $\quad r\equiv q(a_1,m)=q(a_2,m)$.
Then by Pythagoras

$$q_2\equiv q(a_1,b)=r+h-rh=q(a_2,b)\equiv q_1 . \blacksquare$$

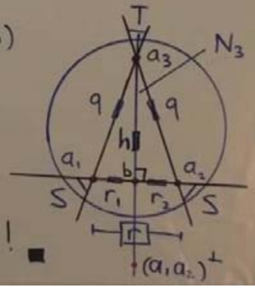
Isosceles mid theorem ⑦

If an isosceles triangle $\overline{a_1a_2a_3}$ with quadrances $q_1=q_2=q$, $q_3=r$ and spreads $S_1=S_2=S$, $S_3=T$ is non-dual, so that $a_3^{\perp}\neq a_1a_2$, then $b\equiv(a_1a_2)N_3$ the foot of the altitude $N_3=a_3(a_1a_2)^{\perp}$ is a midpoint of $\overline{a_1a_2}$.

If $r_1=q(a_1,b)$, $r_2=q(a_2,b)$, $h=q(a_3,b)$ then $\boxed{h=Sq}$ and

$$\boxed{r_1=r_2=\frac{q(1-S)}{1-Sq}}$$

Proof. Use Thales & Pythagoras! ■



Exer 30.1 ⑧

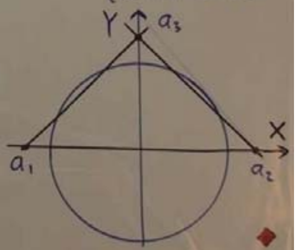
Set $a_1\equiv[a:0:1]$ $\quad a_2\equiv[-a:0:1]$ $\quad a_3\equiv[0:b:1]$.
For the triangle $\overline{a_1a_2a_3}$ compute that

$$q_1=q_2=q=-\frac{a^2+b^2-a^2b^2}{(1-a^2)(1-b^2)} \qquad q_3=r=-\frac{4a^2}{(1-a^2)^2}$$

$$S_1=S_2=S=\frac{b^2(1-a^2)}{a^2+b^2-a^2b^2} \qquad S_3=T=\frac{4a^2b^2(1-b^2)}{(a^2+b^2-a^2b^2)^2}$$

Then check the relations of the Isosceles triangle thm & the Isosceles mid thm.

Easier versions: i) $a=\frac{1}{2}$, $b=2$ ii) $a=2$, $b=\frac{1}{2}$ iii) $a=\frac{4}{3}$, $b=\frac{2}{3}$



Exer 30.2 ⑨

Show that if $r=\dfrac{4q(1-q)(1-S)}{(1-Sq)^2}$ then $1-r=\dfrac{(1-q(2-S))^2}{(1-Sq)^2}$ is a square.

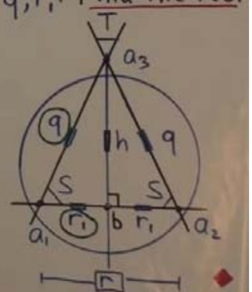
Exer 30.3

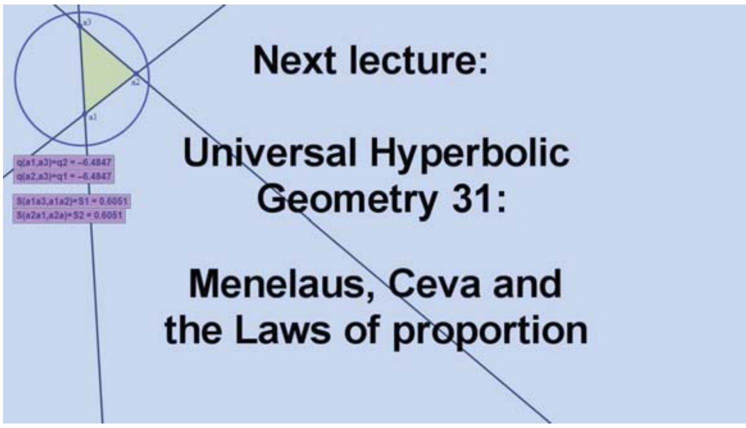
Suppose we know q, r_1. Find the rest.

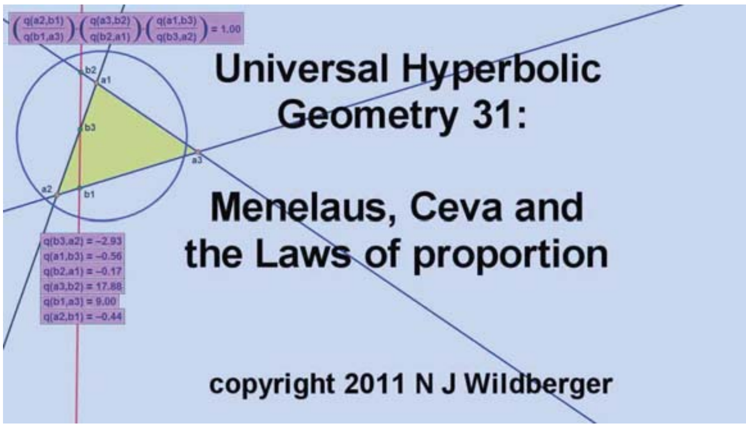
Show that i) $\boxed{r=4r_1(1-r_1)}$

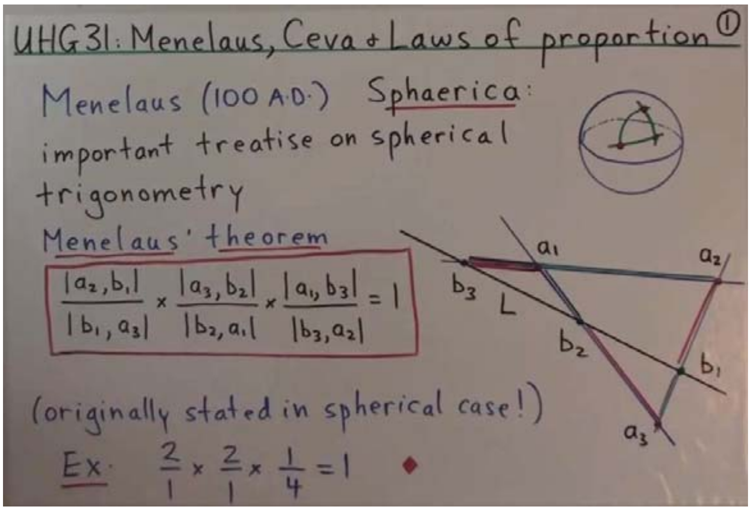
ii) $\boxed{h=\dfrac{q-r_1}{1-r_1}}$ iii) $\boxed{S=\dfrac{q-r_1}{q(1-r_1)}}$

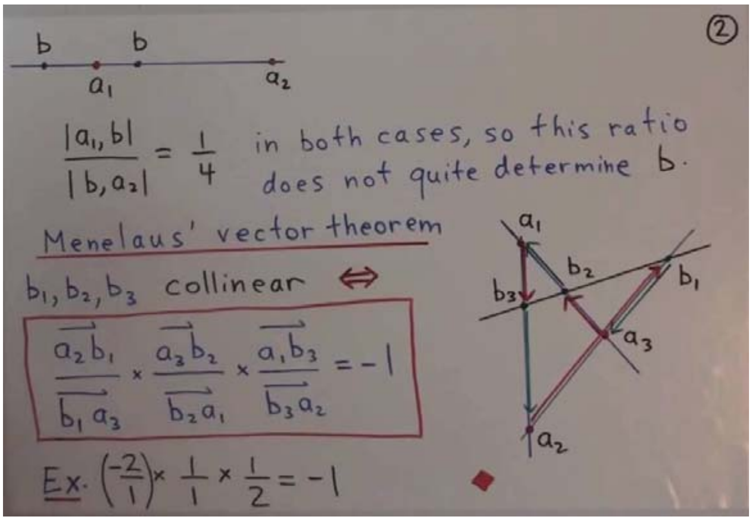
iv) $\boxed{T=\dfrac{4r_1(q-r_1)}{q^2}}$

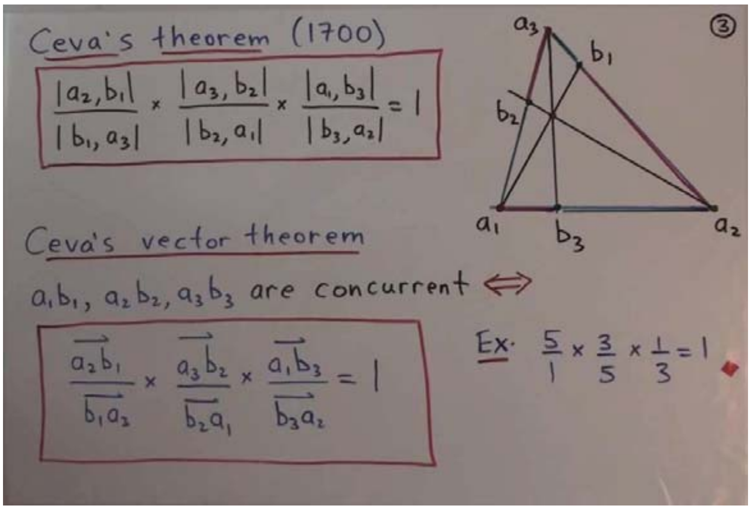


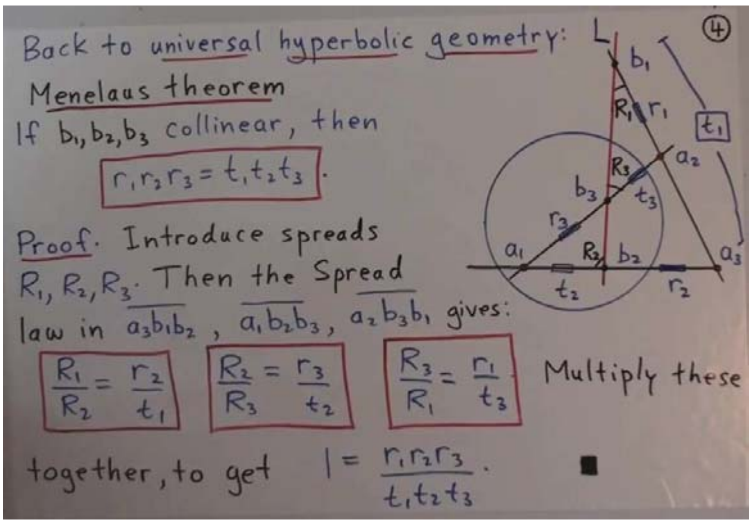


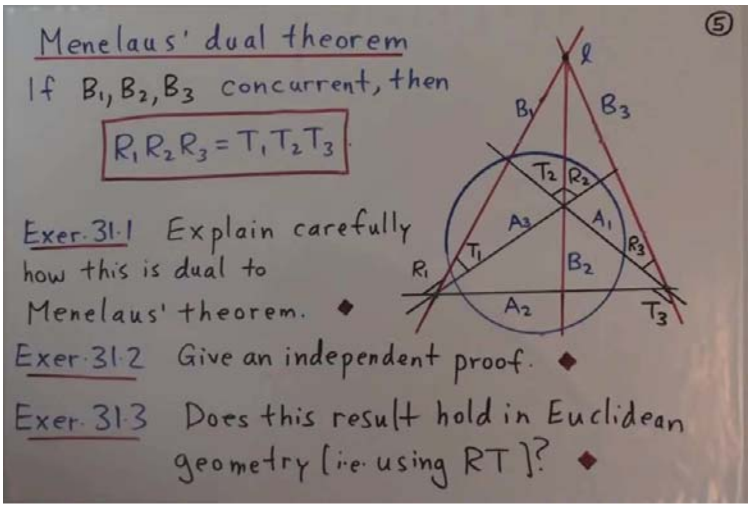

Universal Hyperbolic Geometry 31:
Menelaus, Ceva and the Laws of proportion
copyright 2011 N J Wildberger
$\left(\frac{q(a2,b1)}{q(b1,a3)}\right)\left(\frac{q(a3,b2)}{q(b2,a1)}\right)\left(\frac{q(a1,b3)}{q(b3,a2)}\right) = 1.00$
q(b3,a2) = −2.93
q(a1,b3) = −0.56
q(b2,a1) = −0.17
q(a3,b2) = 17.88
q(b1,a3) = 9.00
q(a2,b1) = −0.44

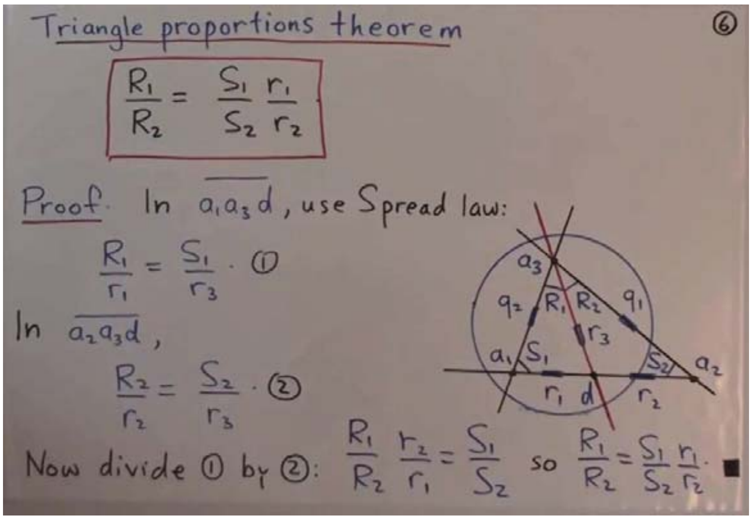

UHG31: Menelaus, Ceva + Laws of proportion ①
Menelaus (100 A.D.) Sphaerica: important treatise on spherical trigonometry
Menelaus' theorem
$\frac{|a_2,b_1|}{|b_1,a_3|} \times \frac{|a_3,b_2|}{|b_2,a_1|} \times \frac{|a_1,b_3|}{|b_3,a_2|} = 1$
(originally stated in spherical case!)
Ex. $\frac{2}{1} \times \frac{2}{1} \times \frac{1}{4} = 1$
a_1 a_2 b_3 L b_2 b_1 a_3

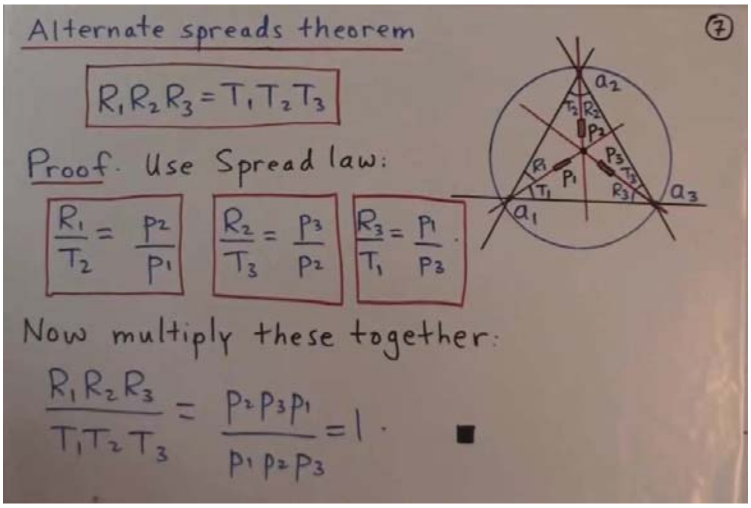

②
b b a_1 a_2
$\frac{|a_1,b|}{|b,a_2|} = \frac{1}{4}$ in both cases, so this ratio does not quite determine b.
Menelaus' vector theorem
b_1, b_2, b_3 collinear $\Leftrightarrow$
$\frac{\overrightarrow{a_2b_1}}{\overrightarrow{b_1a_3}} \times \frac{\overrightarrow{a_3b_2}}{\overrightarrow{b_2a_1}} \times \frac{\overrightarrow{a_1b_3}}{\overrightarrow{b_3a_2}} = -1$
Ex. $\left(\frac{-2}{1}\right) \times \frac{1}{1} \times \frac{1}{2} = -1$
a_1 b_2 b_1 b_3 a_3 a_2

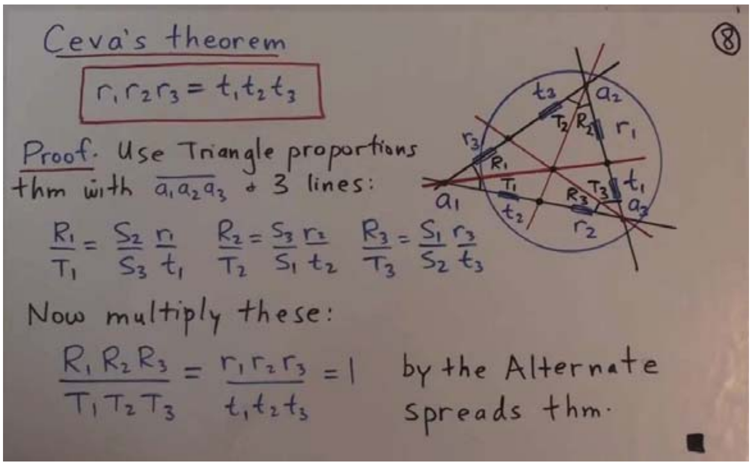

Ceva's theorem (1700) ③
$\frac{|a_2,b_1|}{|b_1,a_3|} \times \frac{|a_3,b_2|}{|b_2,a_1|} \times \frac{|a_1,b_3|}{|b_3,a_2|} = 1$
Ceva's vector theorem
a_1b_1, a_2b_2, a_3b_3 are concurrent $\Leftrightarrow$
$\frac{\overrightarrow{a_2b_1}}{\overrightarrow{b_1a_3}} \times \frac{\overrightarrow{a_3b_2}}{\overrightarrow{b_2a_1}} \times \frac{\overrightarrow{a_1b_3}}{\overrightarrow{b_3a_2}} = 1$
Ex. $\frac{5}{1} \times \frac{3}{5} \times \frac{1}{3} = 1$
a_3 b_1 b_2 a_1 b_3 a_2

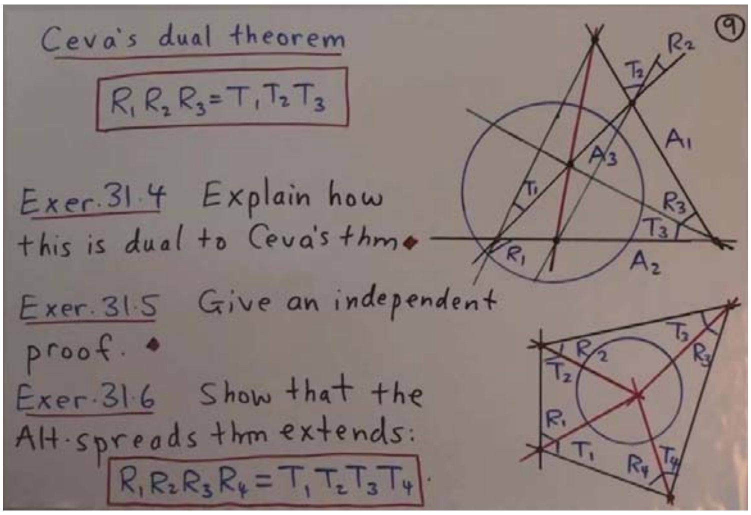

Back to universal hyperbolic geometry: ④
Menelaus theorem
If b_1, b_2, b_3 collinear, then $r_1r_2r_3 = t_1t_2t_3$.
Proof. Introduce spreads R_1, R_2, R_3. Then the Spread law in $\overline{a_3b_1b_2}$, $\overline{a_1b_2b_3}$, $a_2b_3b_1$ gives:
$\frac{R_1}{R_2} = \frac{r_2}{t_1}$ $\frac{R_2}{R_3} = \frac{r_3}{t_2}$ $\frac{R_3}{R_1} = \frac{r_1}{t_3}$ Multiply these together, to get $1 = \frac{r_1r_2r_3}{t_1t_2t_3}$. ■
L b_1 R_1 r_1 t_1 R_3 a_2 b_3 t_3 r_3 a_1 R_2 b_2 a_3 t_2 r_2

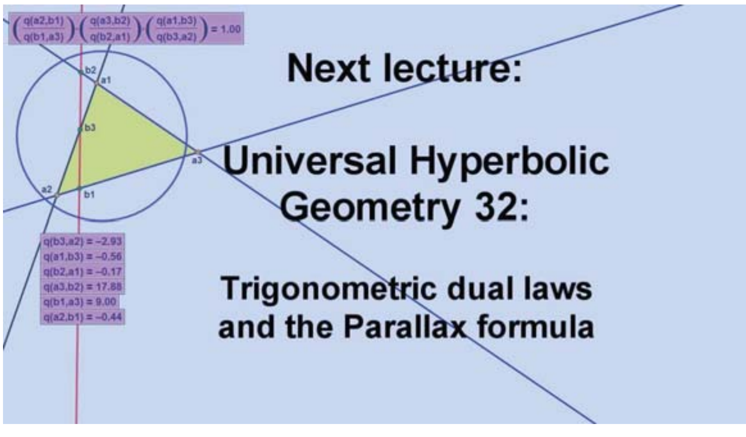

Menelaus' dual theorem ⑤
If B_1, B_2, B_3 concurrent, then $R_1R_2R_3 = T_1T_2T_3$.
Exer. 31.1 Explain carefully how this is dual to Menelaus' theorem.
Exer. 31.2 Give an independent proof.
Exer. 31.3 Does this result hold in Euclidean geometry (i.e. using RT)?
l B_1 B_3 T_2 R_2 A_3 A_1 T_1 R_3 R_1 B_2 A_2 T_3

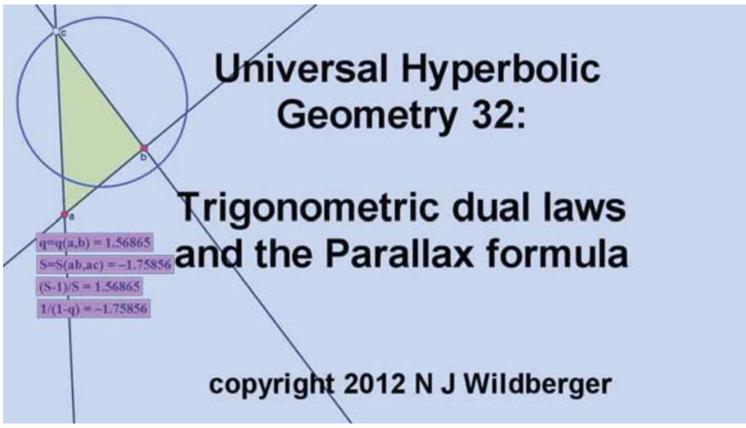

Triangle proportions theorem
6
$\frac{R_1}{R_2} = \frac{S_1}{S_2}\frac{r_1}{r_2}$
Proof. In $\overline{a_1a_3d}$, use Spread law:
$\frac{R_1}{r_1} = \frac{S_1}{r_3}$. ①
In $\overline{a_2a_3d}$,
$\frac{R_2}{r_2} = \frac{S_2}{r_3}$. ②
Now divide ① by ②: $\frac{R_1}{R_2}\frac{r_2}{r_1} = \frac{S_1}{S_2}$ so $\frac{R_1}{R_2} = \frac{S_1}{S_2}\frac{r_1}{r_2}$.
a_3 q_2 R_1 R_2 q_1 r_3 a_1 S_1 S_2 a_2 r_1 d r_2

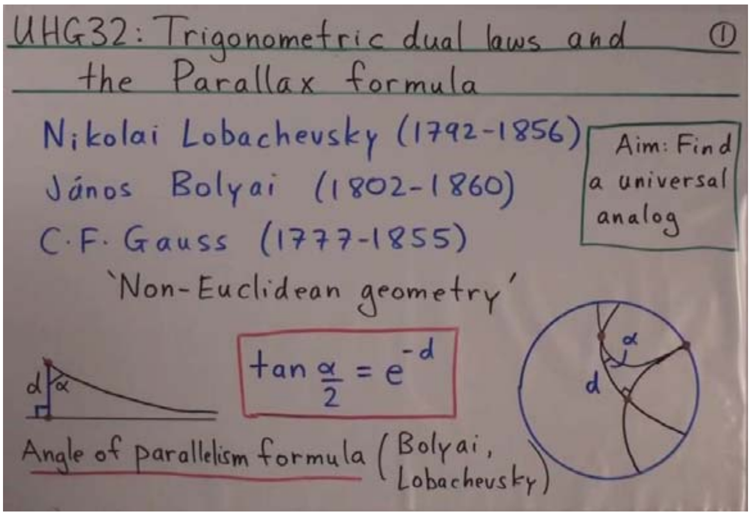

Alternate spreads theorem
7
$R_1R_2R_3 = T_1T_2T_3$
Proof. Use Spread law:
$\frac{R_1}{T_2} = \frac{p_2}{p_1}$ $\frac{R_2}{T_3} = \frac{p_3}{p_2}$ $\frac{R_3}{T_1} = \frac{p_1}{p_3}$.
Now multiply these together:
$\frac{R_1R_2R_3}{T_1T_2T_3} = \frac{p_2p_3p_1}{p_1p_2p_3} = 1$.
a_2 T_2 R_2 p_2 p_3 T_3 R_1 p_1 T_1 R_3 a_1 a_3

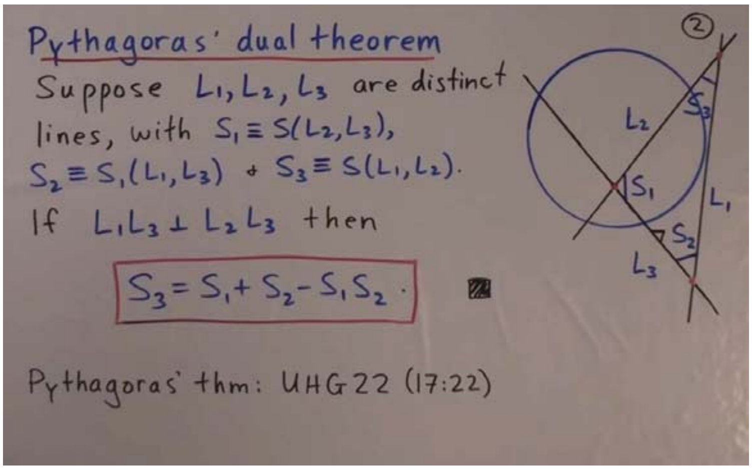

Ceva's theorem
8
$r_1r_2r_3 = t_1t_2t_3$
Proof. Use Triangle proportions thm with $\overline{a_1a_2a_3}$ + 3 lines:
$\frac{R_1}{T_1} = \frac{S_2}{S_3}\frac{r_1}{t_1}$ $\frac{R_2}{T_2} = \frac{S_3}{S_1}\frac{r_2}{t_2}$ $\frac{R_3}{T_3} = \frac{S_1}{S_2}\frac{r_3}{t_3}$
Now multiply these:
$\frac{R_1R_2R_3}{T_1T_2T_3} = \frac{r_1r_2r_3}{t_1t_2t_3} = 1$ by the Alternate spreads thm.
t_3 a_2 T_2 R_2 r_1 r_3 R_1 T_1 a_1 t_2 R_3 T_3 t_1 a_3 r_2

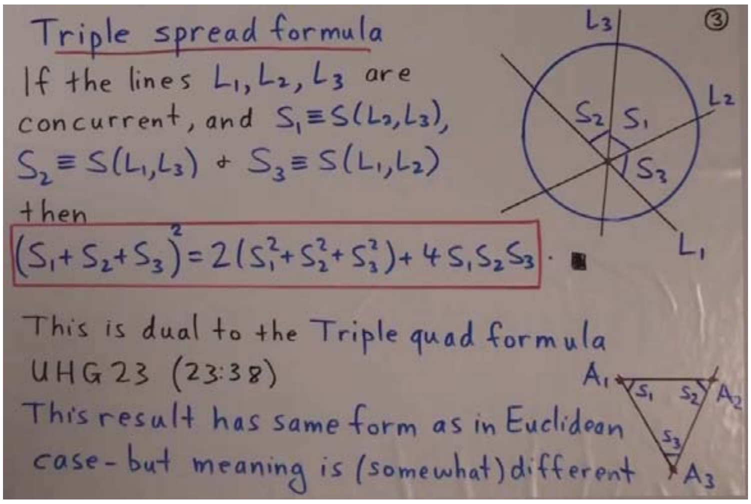

Ceva's dual theorem
9
$R_1R_2R_3 = T_1T_2T_3$
Exer. 31.4 Explain how this is dual to Ceva's thm.
Exer. 31.5 Give an independent proof.
Exer. 31.6 Show that the Alt. spreads thm extends:
$R_1R_2R_3R_4 = T_1T_2T_3T_4$.
R_2 T_2 A_1 A_3 T_1 R_3 T_3 R_1 A_2
T_3 R_2 R_3 T_2 R_1 T_1 R_4 T_4

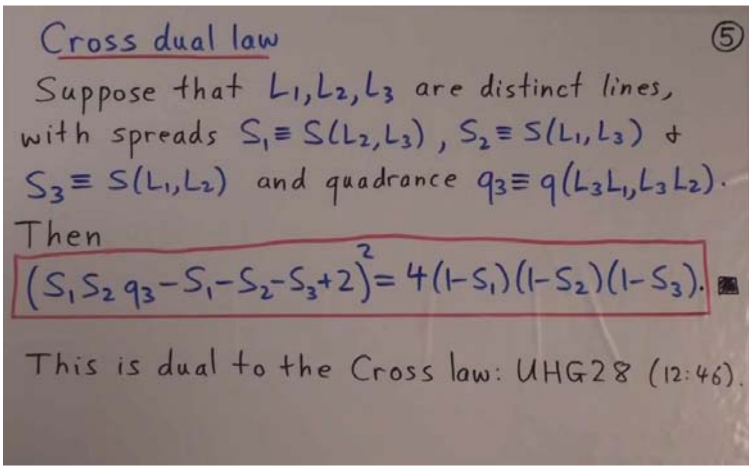

$\left(\frac{q(a2,b1)}{q(b1,a3)}\right)\left(\frac{q(a3,b2)}{q(b2,a1)}\right)\left(\frac{q(a1,b3)}{q(b3,a2)}\right) = 1.00$
b2 a1 b3 a3 a2 b1
q(b3,a2) = −2.93
q(a1,b3) = −0.56
q(b2,a1) = −0.17
q(a3,b2) = 17.88
q(b1,a3) = 9.00
q(a2,b1) = −0.44
Next lecture:
Universal Hyperbolic Geometry 32:
Trigonometric dual laws and the Parallax formula

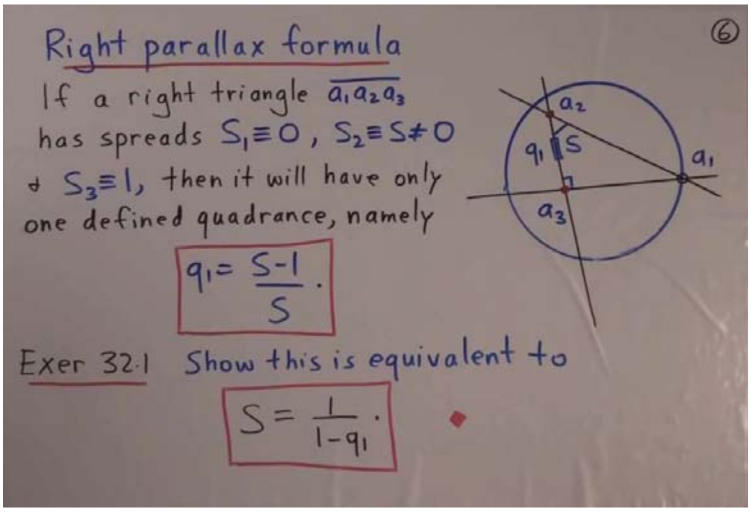

Universal Hyperbolic Geometry 32:
Trigonometric dual laws and the Parallax formula
q=q(a,b) = 1.56865
S=S(ab,ac) = −1.75856
(S-1)/S = 1.56865
1/(1-q) = −1.75856
copyright 2012 N J Wildberger


UHG32: Trigonometric dual laws and the Parallax formula ①
Nikolai Lobachevsky (1792-1856)
János Bolyai (1802-1860)
C.F. Gauss (1777-1855)
'Non-Euclidean geometry'
Aim: Find a universal analog
$\tan \frac{\alpha}{2} = e^{-d}$
d
α
Angle of parallelism formula (Bolyai, Lobachevsky)


Pythagoras' dual theorem ②
Suppose L_1, L_2, L_3 are distinct lines, with $S_1 \equiv S(L_2,L_3)$, $S_2 \equiv S_1(L_1,L_3)$ & $S_3 \equiv S(L_1,L_2)$.
If $L_1L_3 \perp L_2L_3$ then
$S_3 = S_1 + S_2 - S_1S_2$.
Pythagoras' thm: UHG22 (17:22)
L_1 L_2 L_3 S_1 S_2 S_3

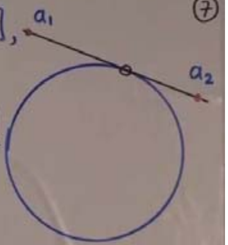

Triple spread formula ③
If the lines L_1, L_2, L_3 are concurrent, and $S_1 \equiv S(L_2,L_3)$, $S_2 \equiv S(L_1,L_3)$ & $S_3 \equiv S(L_1,L_2)$ then
$(S_1+S_2+S_3)^2 = 2(S_1^2+S_2^2+S_3^2) + 4S_1S_2S_3$.
This is dual to the Triple quad formula UHG23 (23:38)
This result has same form as in Euclidean case - but meaning is (somewhat) different
L_1 L_2 L_3 S_1 S_2 S_3 A_1 A_2 A_3


Cross dual law ⑤
Suppose that L_1, L_2, L_3 are distinct lines, with spreads $S_1 \equiv S(L_2,L_3)$, $S_2 \equiv S(L_1,L_3)$ & $S_3 \equiv S(L_1,L_2)$ and quadrance $q_3 \equiv q(L_3L_1, L_3L_2)$.
Then
$(S_1S_2q_3 - S_1 - S_2 - S_3 + 2)^2 = 4(1-S_1)(1-S_2)(1-S_3)$.
This is dual to the Cross law: UHG28 (12:46).


Right parallax formula ⑥
If a right triangle $\overline{a_1a_2a_3}$ has spreads $S_1 \equiv 0$, $S_2 \equiv S \neq 0$ & $S_3 \equiv 1$, then it will have only one defined quadrance, namely
$q_1 = \frac{S-1}{S}$.
Exer 32.1 Show this is equivalent to
$S = \frac{1}{1-q_1}$.
a_1 a_2 a_3 q_1 S

Zero quadrance theorem If a_1, a_2 are distinct points, then $q(a_1,a_2)=0 \Leftrightarrow a_1a_2$ is a null line. (7)

Proof. If $a_1 \equiv [x_1:y_1:z_1]$, $a_2 \equiv [x_2:y_2:z_2]$,



then $q(a_1,a_2) = 1 - \dfrac{(x_1x_2+y_1y_2-z_1z_2)^2}{(x_1^2+y_1^2-z_1^2)(x_2^2+y_2^2-z_2^2)}$

$$= -\frac{(y_1z_2-y_2z_1)^2+(x_1z_2-x_2z_1)^2-(x_1y_2-x_2y_1)^2}{(x_1^2+y_1^2-z_1^2)(x_2^2+y_2^2-z_2^2)}$$

This is zero $\Leftrightarrow$

$a_1a_2 = (y_1z_2-y_2z_1 : z_1x_2-z_2x_1 : x_2y_1-x_1y_2)$ [Join of two points thm UHG 8]

is a null line. ■

The dual result: (8)

Zero spread theorem: If L_1, L_2 are distinct lines, then $S(L_1,L_2)=0 \Leftrightarrow L_1L_2$ is a null point. ■

Right parallax thm: $\overline{a_1a_2a_3}$, $S_1 \equiv 0$, $S_2 \equiv S \neq 0$, $S_3 \equiv 1$

$$q_1 = \frac{S-1}{S}.$$

Proof. If $S_1=0$ the a_1 is a null point, so q_2, q_3 not defined. The Cross dual law applies:

$$(S_2S_3q_1-S_1-S_2-S_3+2)^2 = 4(1-S_1)(1-S_2)(1-S_3)$$

which becomes $(Sq_1-S+1)^2=0$.

So $q_1 = \dfrac{S-1}{S}$. ■

Q. How to connect $\boxed{\tan\frac{\alpha}{2}=e^{-d}}$ with $\boxed{q=\frac{S-1}{S}}$?

Very roughly: For interior points,

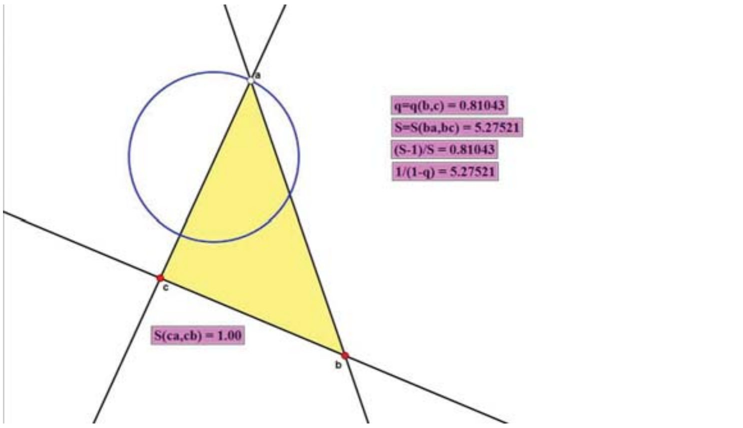
$\boxed{q=-\sinh^2 d}$ $\boxed{S=\sin^2\alpha}$ [many assumptions implicit here!!]

So if $\boxed{q=\frac{S-1}{S}}$ then $\sinh^2 d = \dfrac{1-\sin^2\alpha}{\sin^2\alpha} = \dfrac{\cos^2\alpha}{\sin^2\alpha} = \left(\dfrac{1}{\tan\alpha}\right)^2$

so $\sinh d = \pm\dfrac{1}{\tan\alpha}$. (9)

But $\sinh d = \dfrac{e^d-e^{-d}}{2}$ and $\tan\alpha = \dfrac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}$

$\therefore\ e^d - e^{-d} = \tan^{-1}\frac{\alpha}{2} - \tan\frac{\alpha}{2} \Rightarrow \boxed{\tan\frac{\alpha}{2}=e^{-d}}$

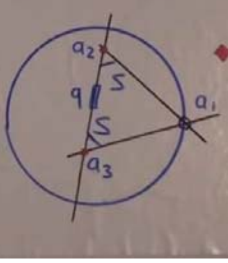


Exer 32·2 Show that i) $q_1=-1 \Leftrightarrow S=\frac{1}{2}$ (10)

ii) $q_1=-3 \Leftrightarrow S=\frac{1}{4}$ iii) $q_1=-\frac{1}{3} \Leftrightarrow S=\frac{3}{4}$

Exer 32·3 Prove the Isosceles parallax theorem: If $\overline{a_1a_2a_3}$ is a non-null isosceles triangle with a_1 a null point, $q_1=q$ and $S_2=S_3\equiv S$, then

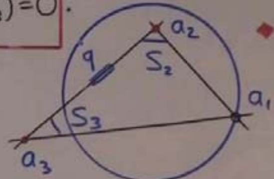
$$\boxed{q=\frac{4(S-1)}{S^2}}.$$



◆

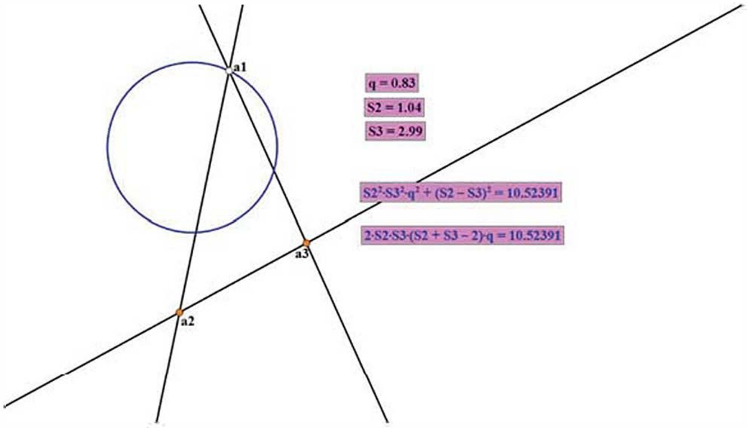
Exer. 32·4 Prove the General parallax formula:

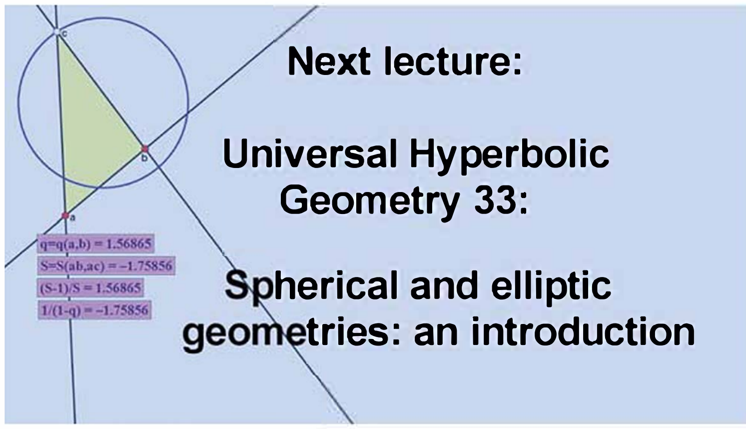
$$\boxed{S_2^2S_3^2q^2-2S_2S_3(S_2+S_3-2)q+(S_2-S_3)^2=0}.$$



◆

Exer. 32·5 [Only for dedicated classical hyperbolic geometers!] Convert the General parallax formula to the associated General angle parallelism formula, or vice versa. ◆ (11)





UNIVERSAL HYPERBOLIC GEOMETRY B

and other screenshot pdfs

available at

http://wildegg.com

wild egg