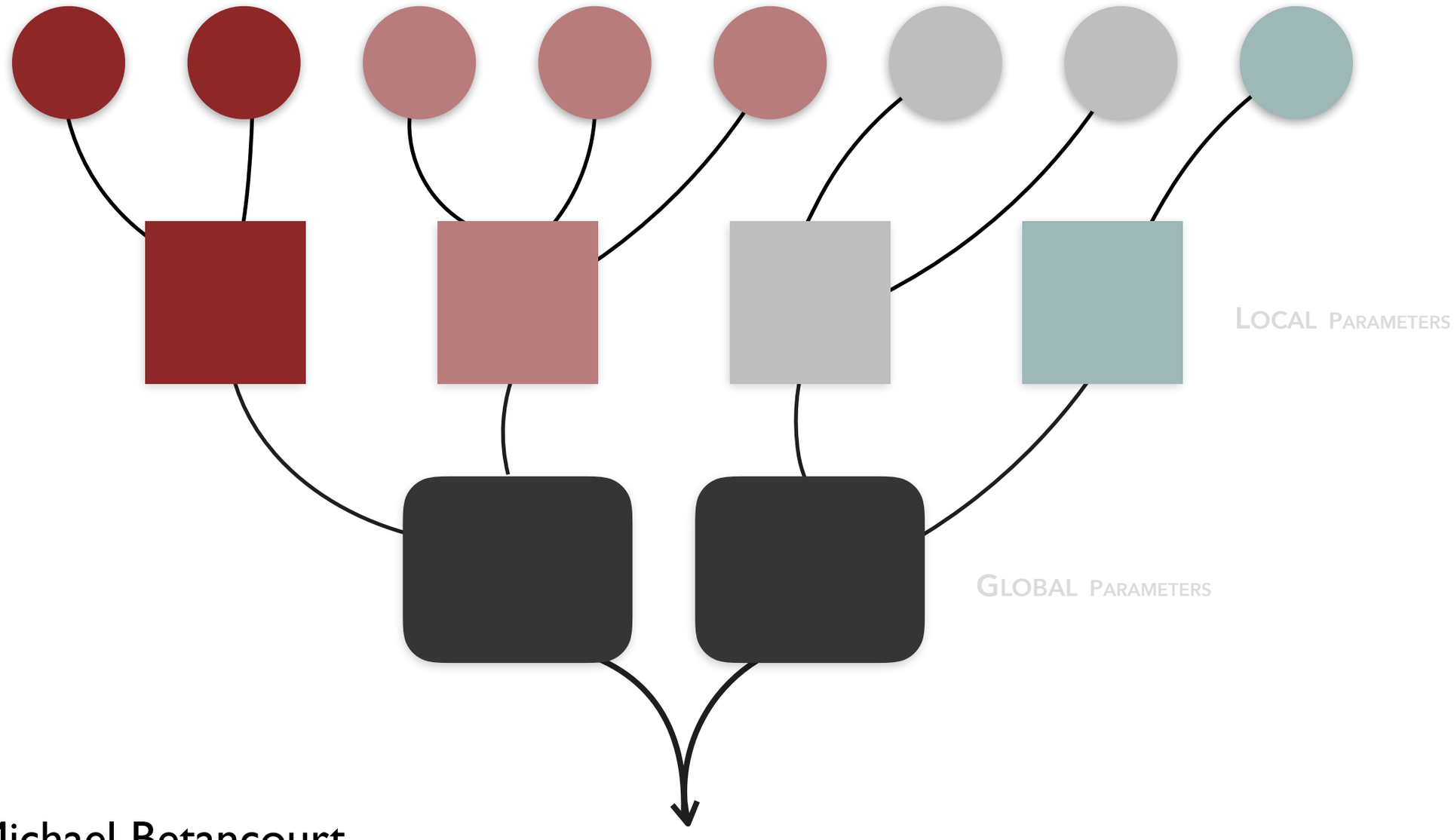
Hierarchical and Multilevel Models



Thanks to Michael Betancourt

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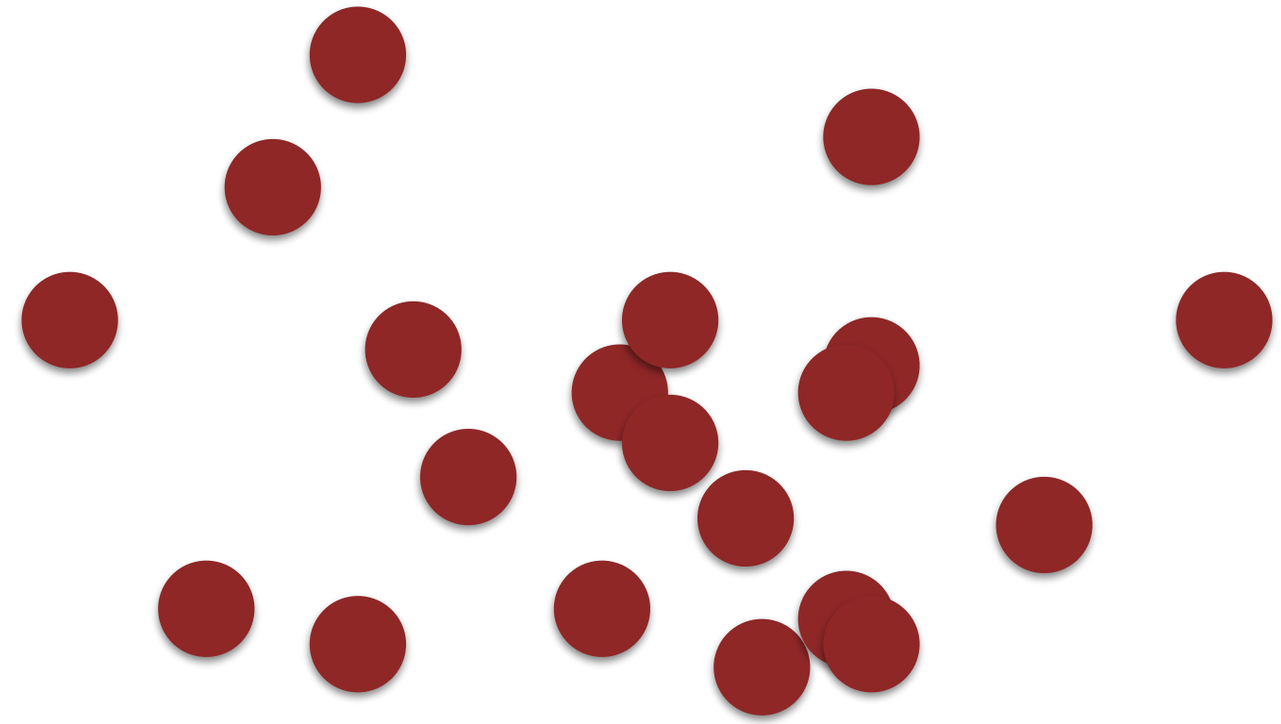
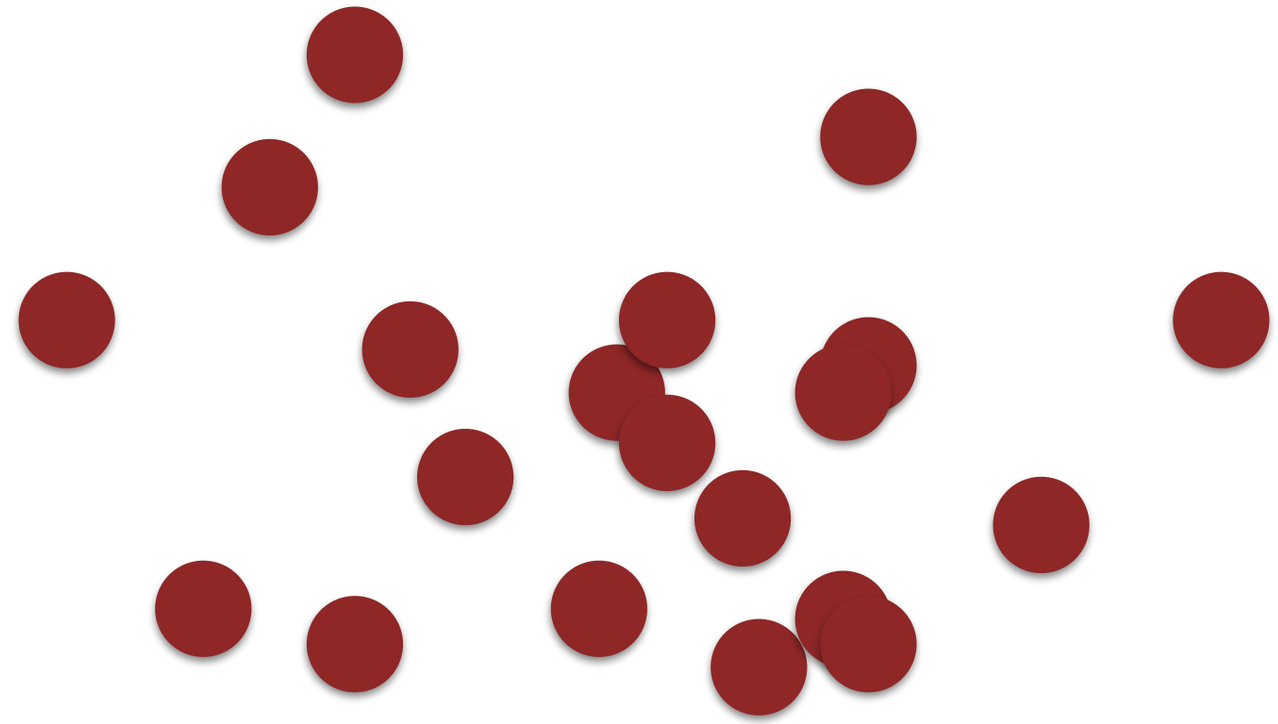


$$y \sim \mathcal{N}(\mathbf{X}^T \boldsymbol{\beta} + \alpha, \sigma)$$

If we assume that every individual is equivalent then we can pool the data, but only at the expense of bias.



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$$y_n \sim \mathcal{N}\left(\mathbf{X}^T \boldsymbol{\beta} + \alpha, \sigma\right)$$

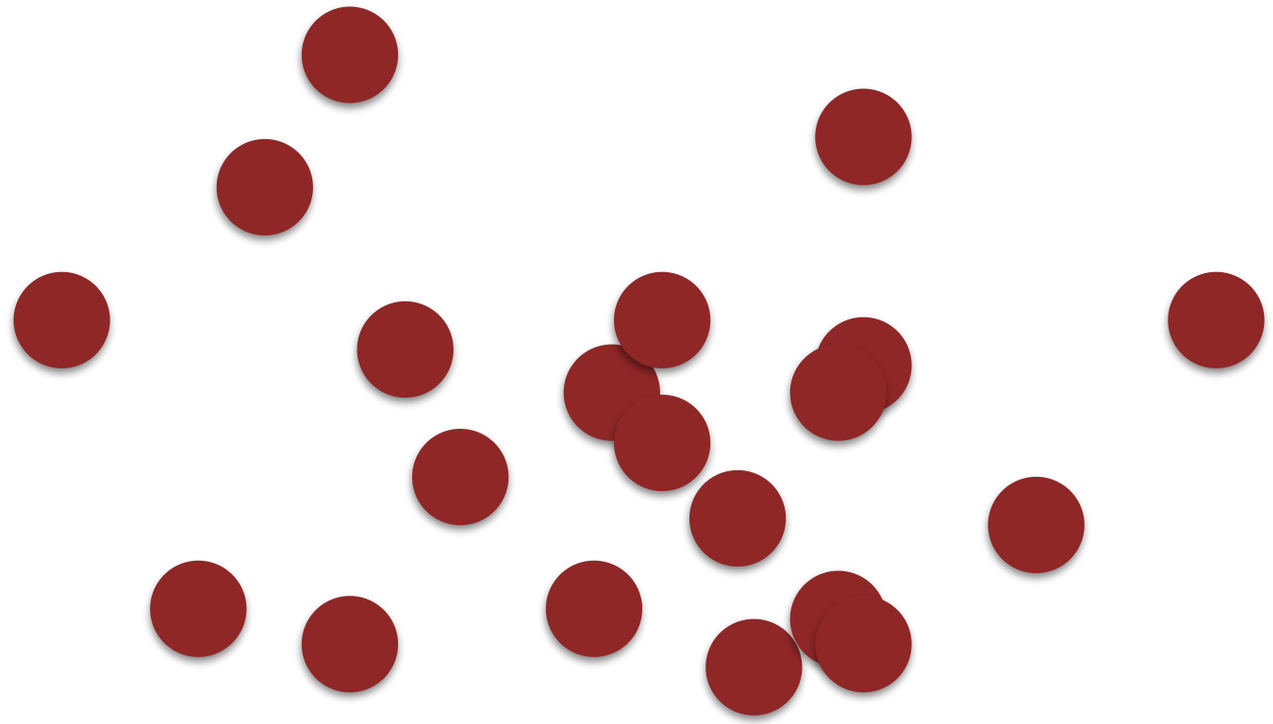
Complete pooling model

```
data {
  int N;
  int K;
  int<lower=0, upper=1> y[N];
  vector[N] x;
}

parameters {
  real beta;
  real alpha;
}

model {
   beta ~ normal(0, 10);
   alpha ~ normal(0, 10);
   y ~ bernoulli_logit(x * beta + alpha);
}
```

Modeling every individual separately avoids any bias, but then the data becomes very sparse and inferences weak.



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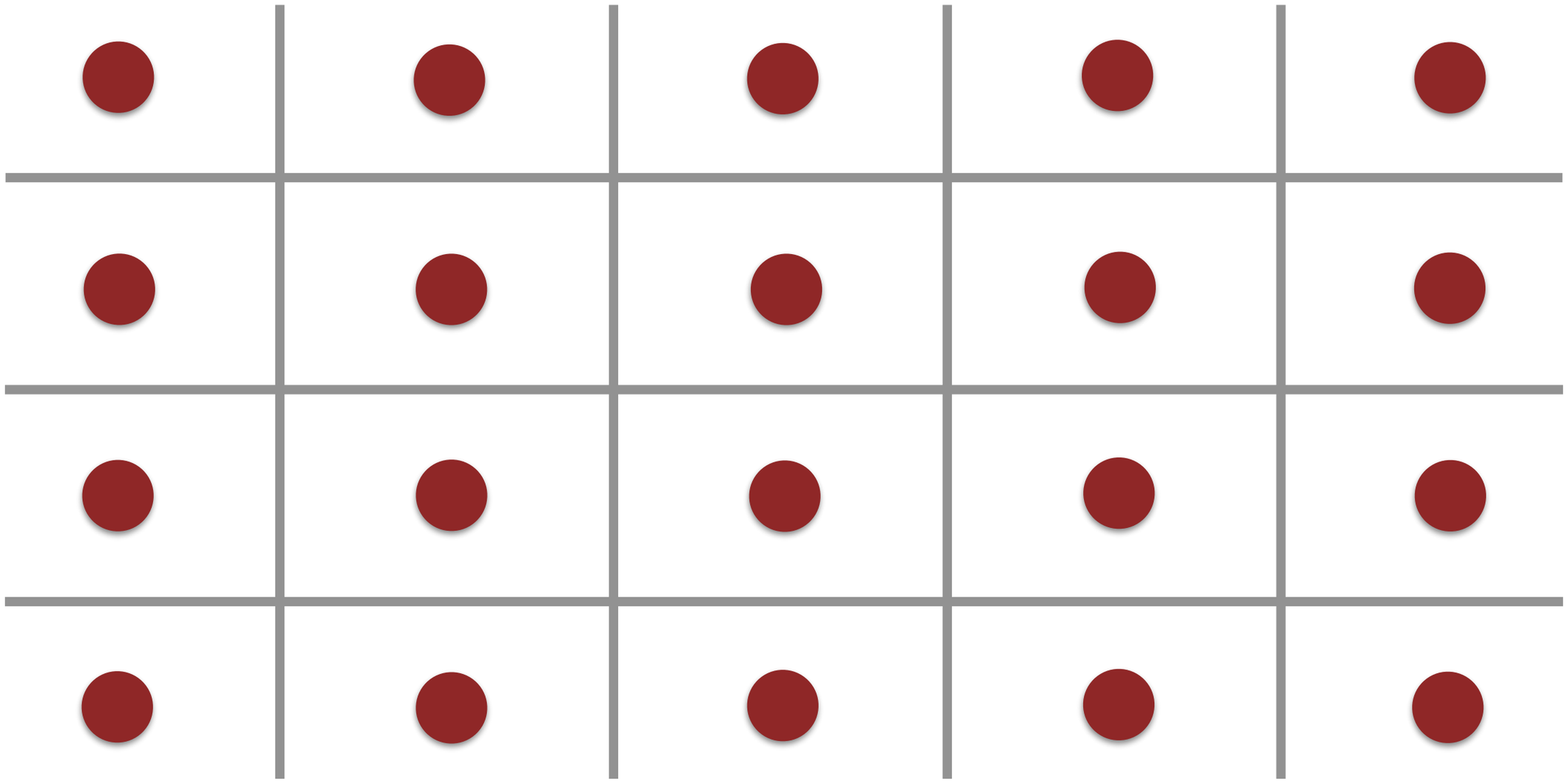

$$y_n \sim \mathcal{N}\left(\mathbf{X}^T \boldsymbol{\beta}_n + \alpha_n, \sigma\right)$$

No pooling model

```
data {
  int N;
  int K;
  int<lower=0, upper=1> y[N];
  vector[N] x;
}

parameters {
  // different slope and intercept for each indiv
  vector[N] beta;
  vector[N] alpha;
}

model {
   beta ~ normal(0, 10);
   alpha ~ normal(0, 10);
   y ~ bernoulli_logit(x .* beta + alpha);
}
```

A compromise between complete pooling and no pooling that could balance bias and variance would be ideal.

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- **Simultaneously** estimate individual (local) parameters and population (global) parameters

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- **Simultaneously** estimate individual (local) parameters and population (global) parameters

- But can be helpful to imagine it **iteratively**

Start by fitting all individuals separately

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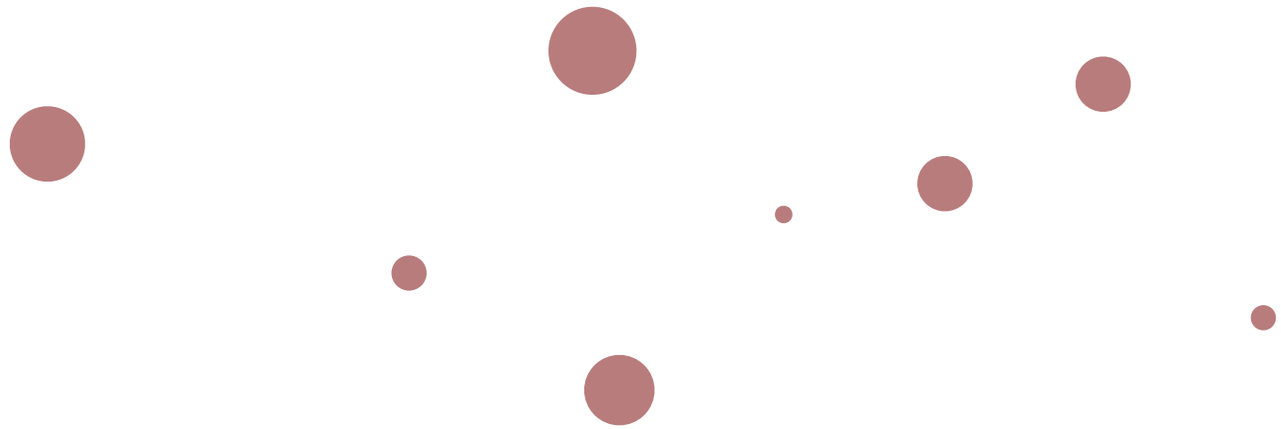
- Smaller circle = smaller uncertainty
  - more data

Take individual fits and estimate population distribution



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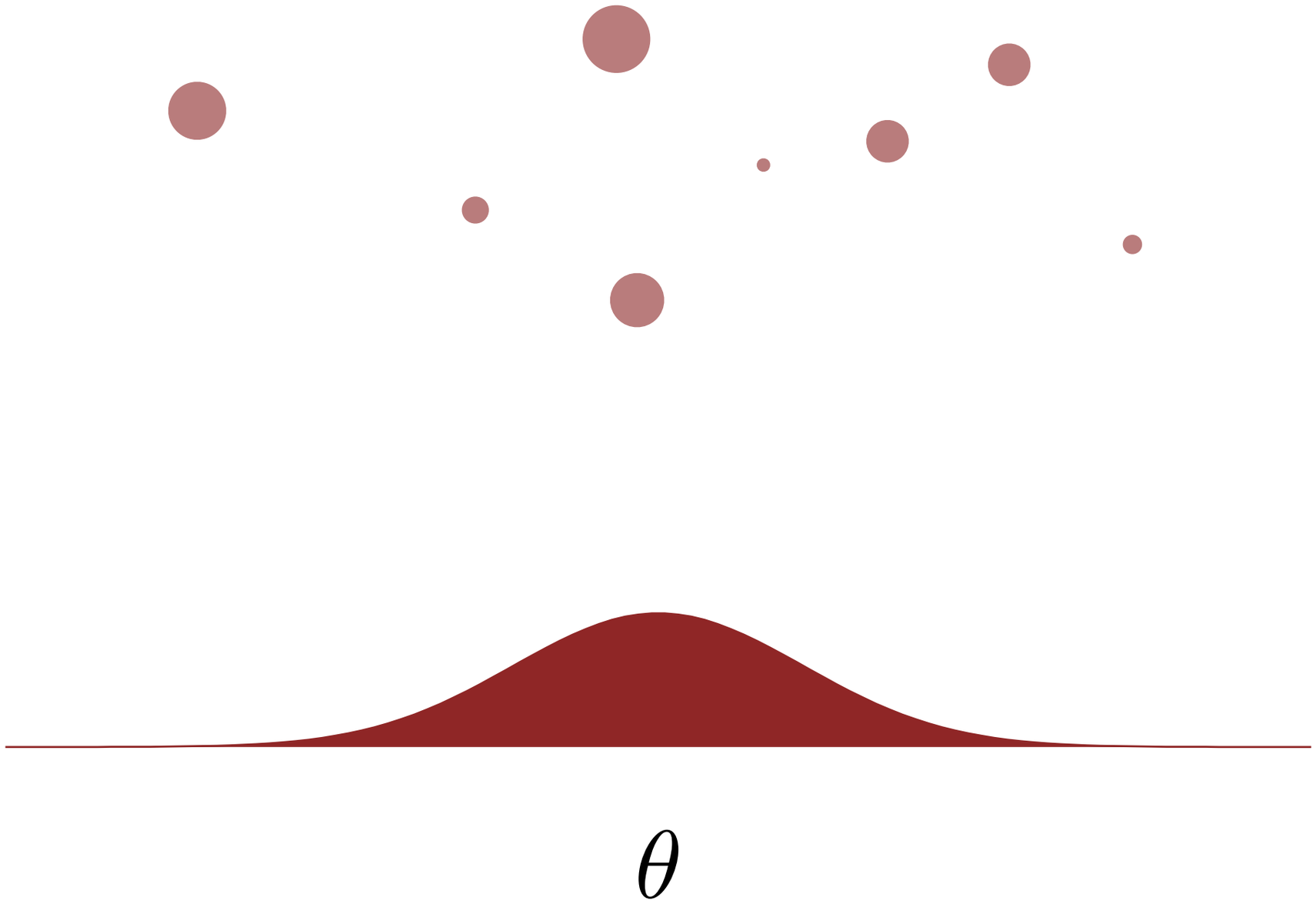
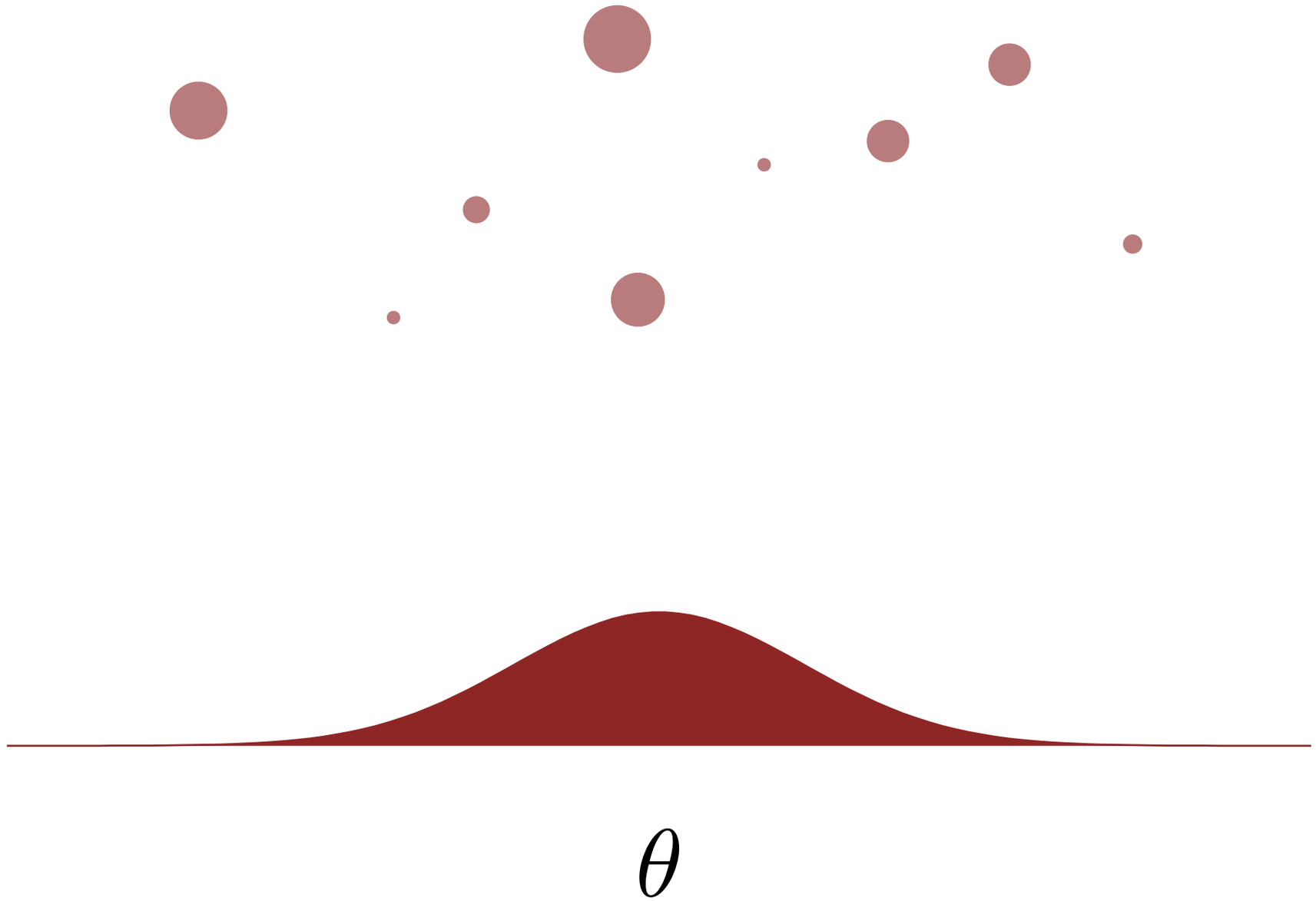


Use population distribution as prior for individuals



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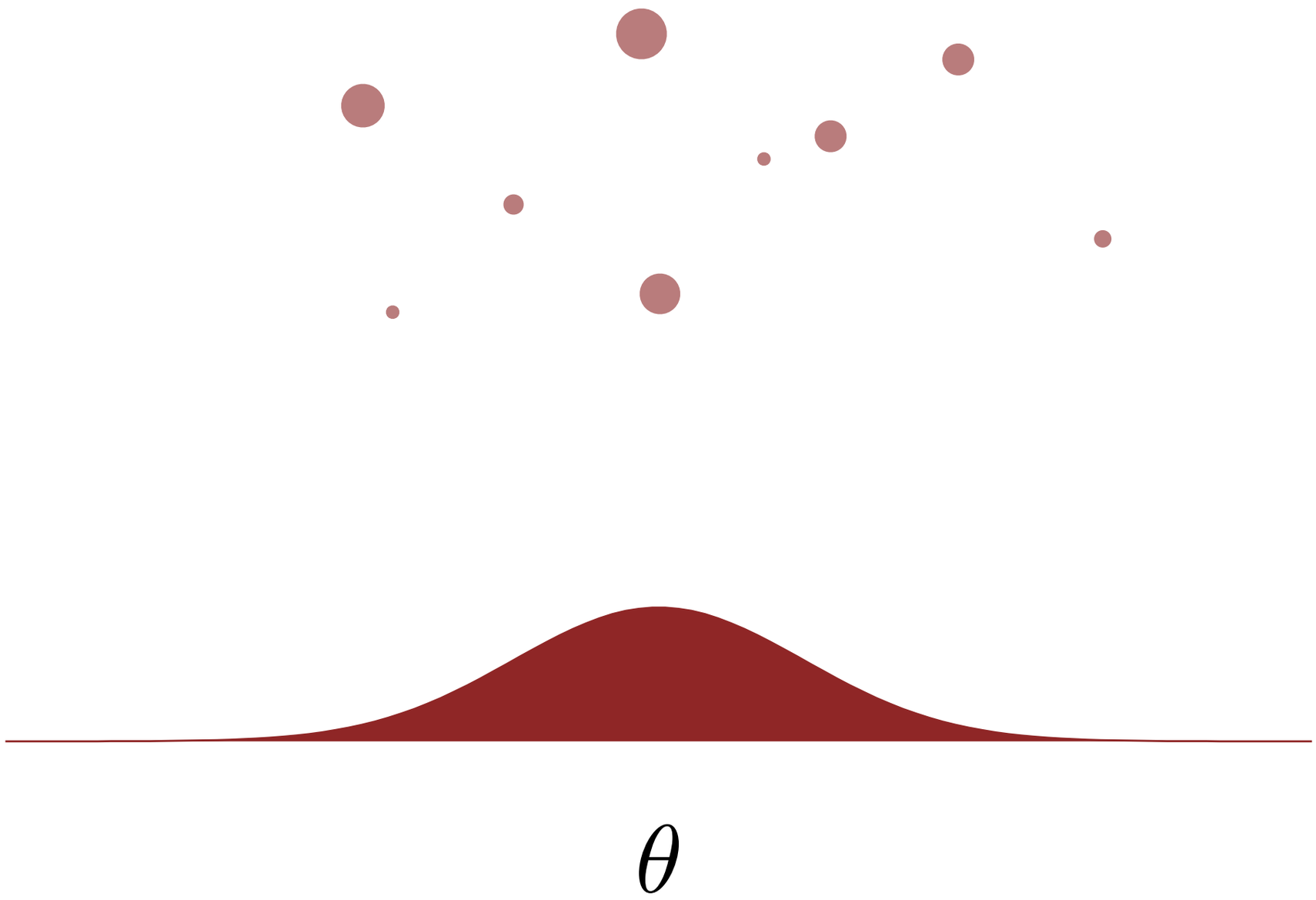
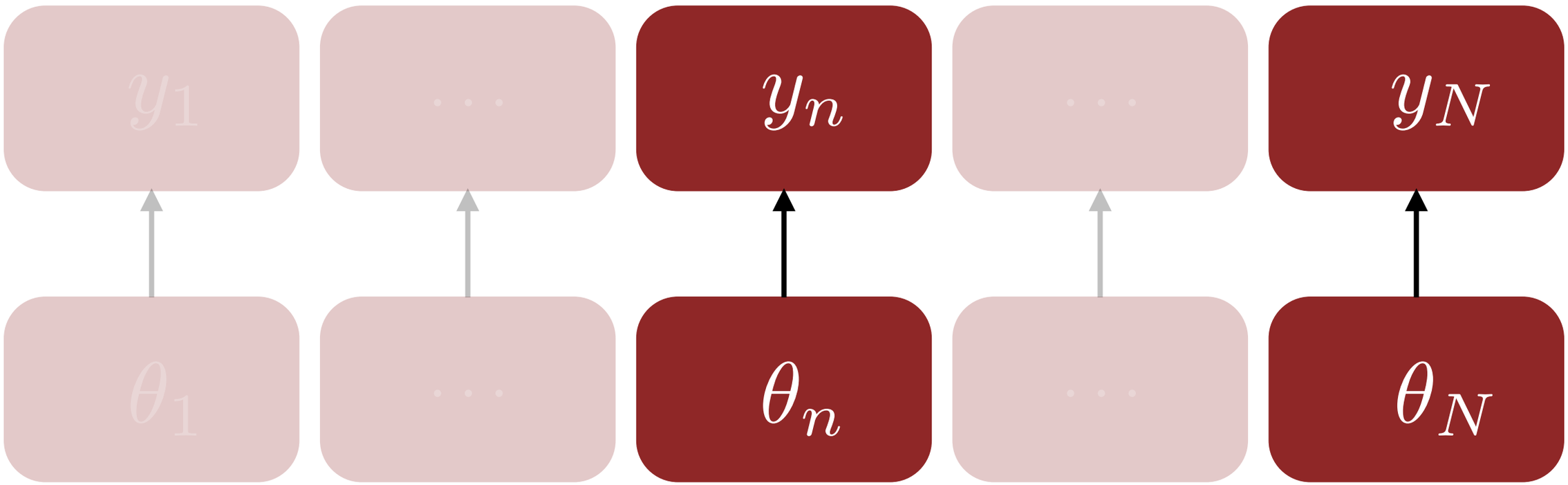


The hyperparameters couple all of the groups together, partially pooling the data and balancing bias and variance.



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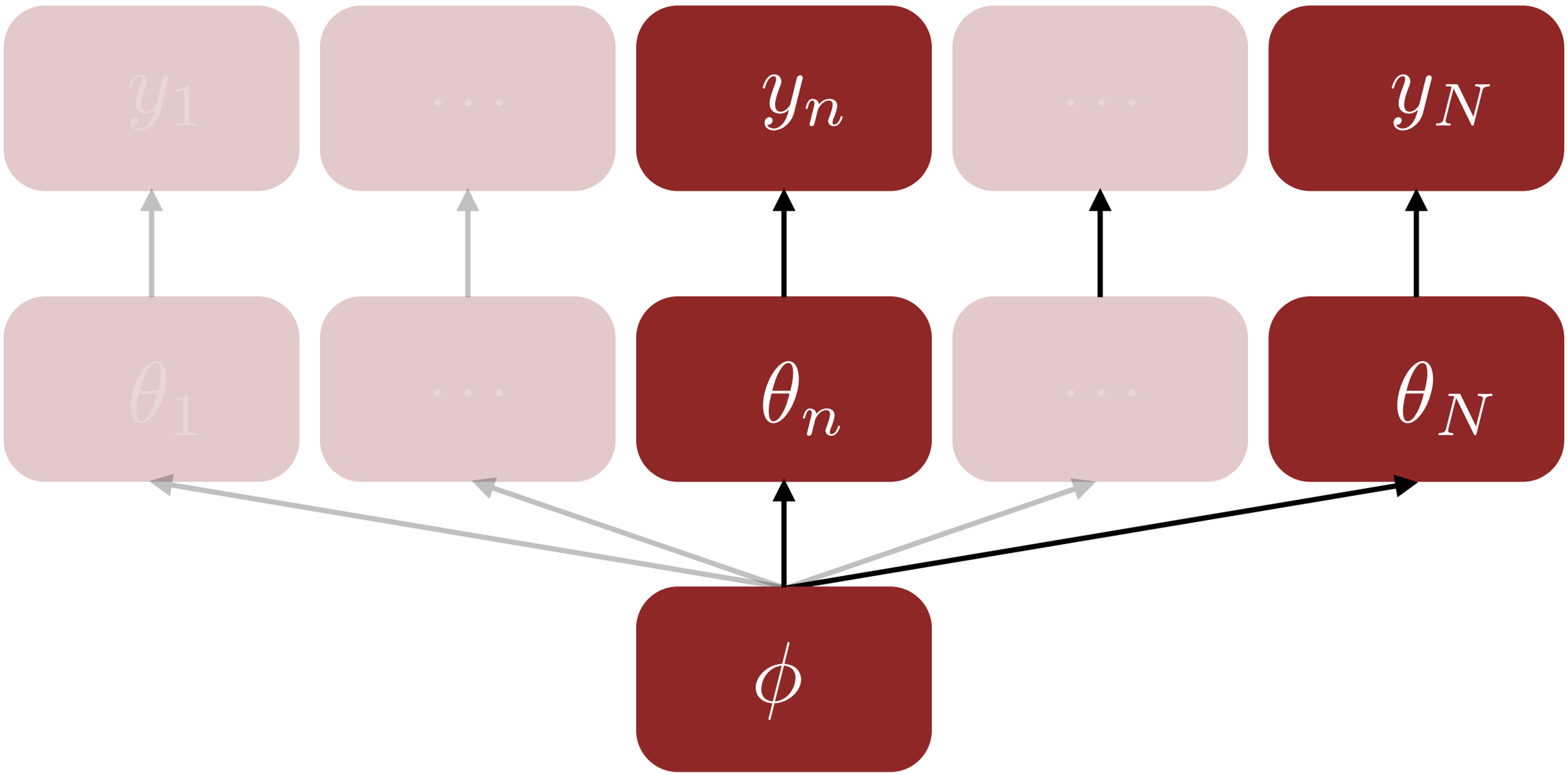

The most common population model is a Gaussian, where we model the population mean and variance.

$$\prod_{n=1}^{N} p(y_n|\theta_n)\, p(\theta_n|\phi)\, p(\phi)$$

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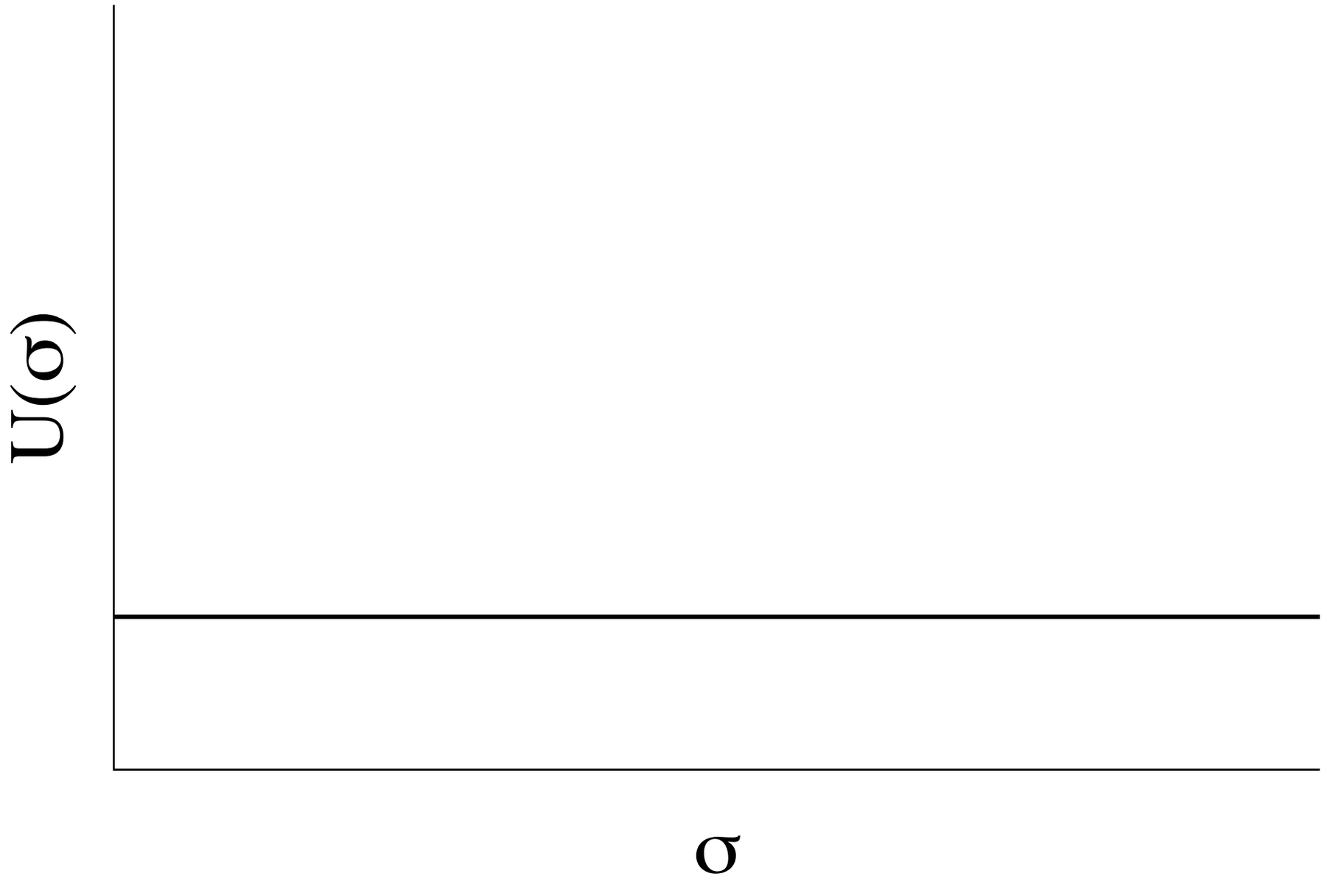
$$\prod_{n=1}^{N} p(y_n|\theta_n)\, \mathcal{N}(\theta_n|\mu,\sigma)\, p(\mu)\, p(\sigma)$$

Informative prior distributions on the population hyperparameters are incredibly important.

$$p(\mu)\, p(\sigma)$$

A uniform prior on the population deviance places too much probability at high values and no pooling.



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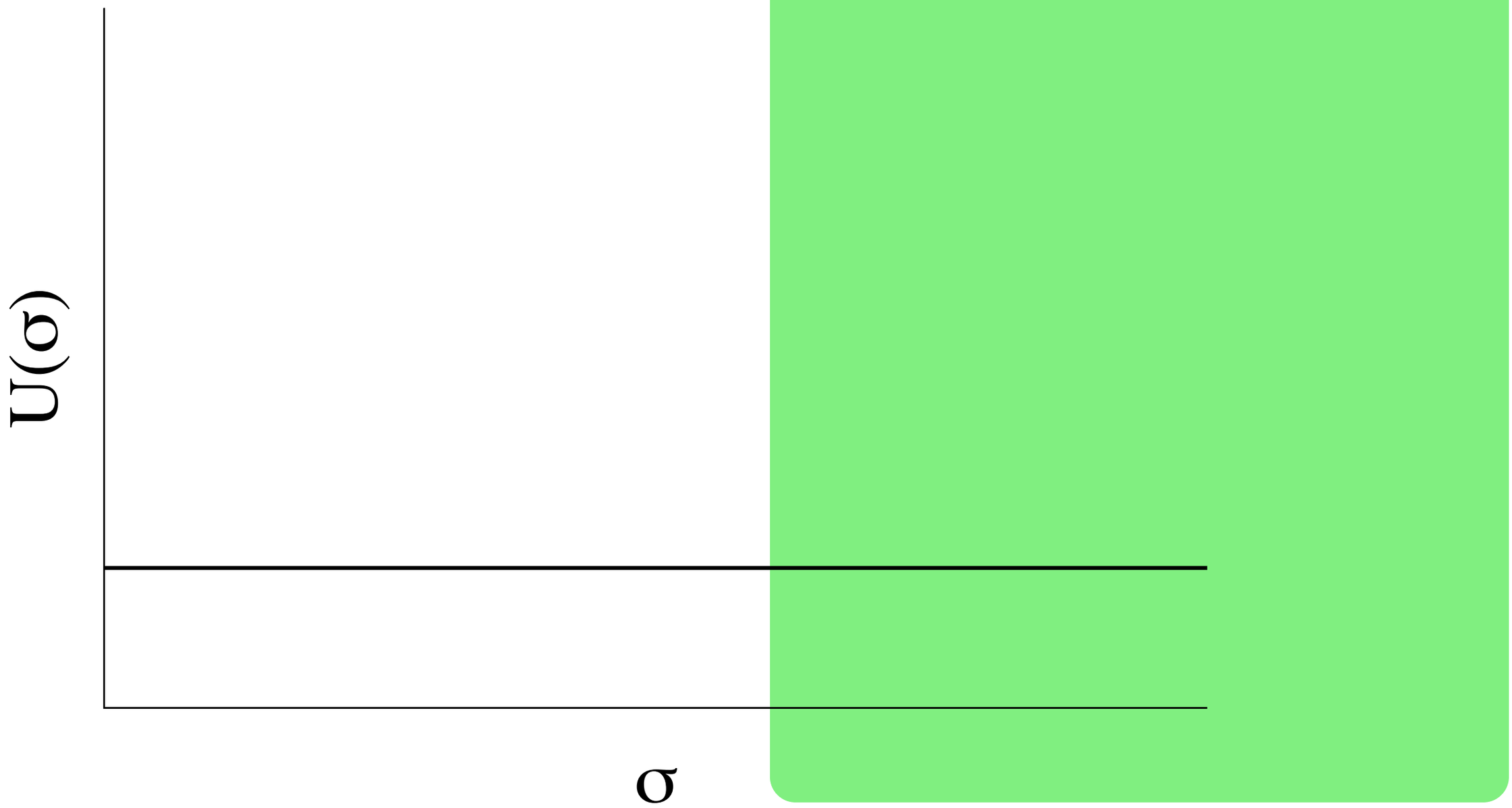
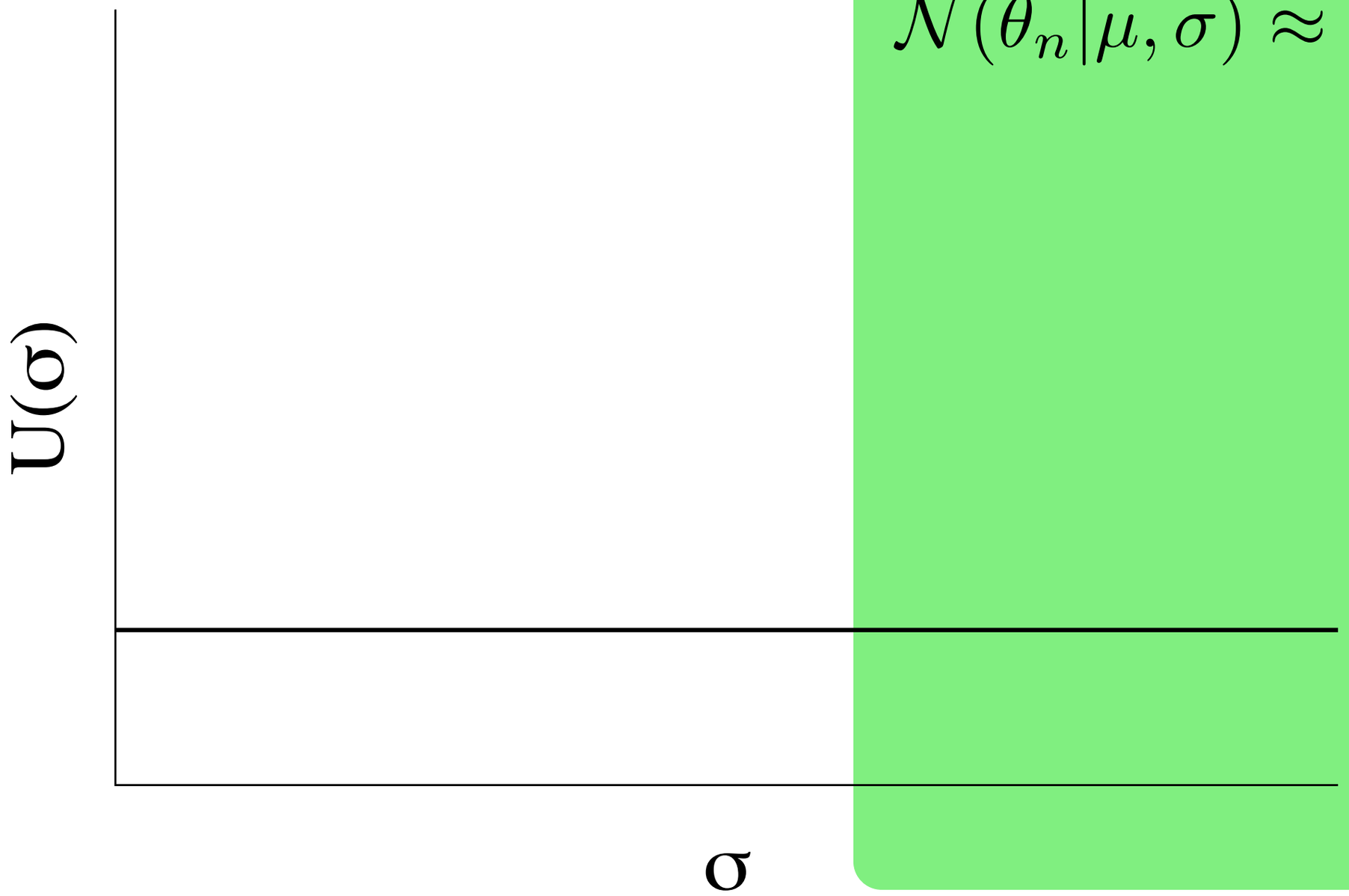


A uniform prior on the population deviance places too much probability at high values and no pooling.

$$\mathcal{N}(\theta_n|\mu,\sigma) \approx U(\theta_n)$$



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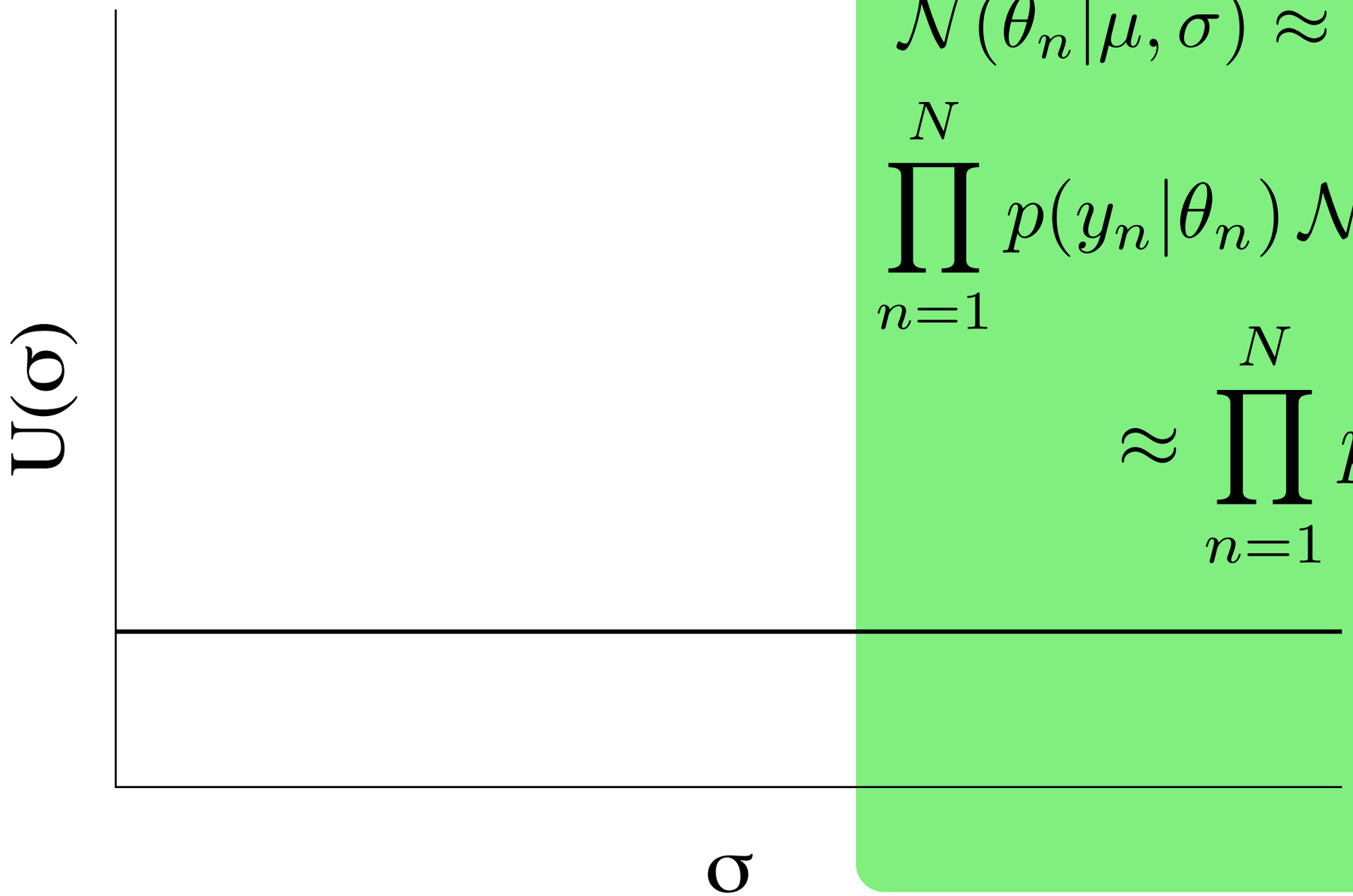

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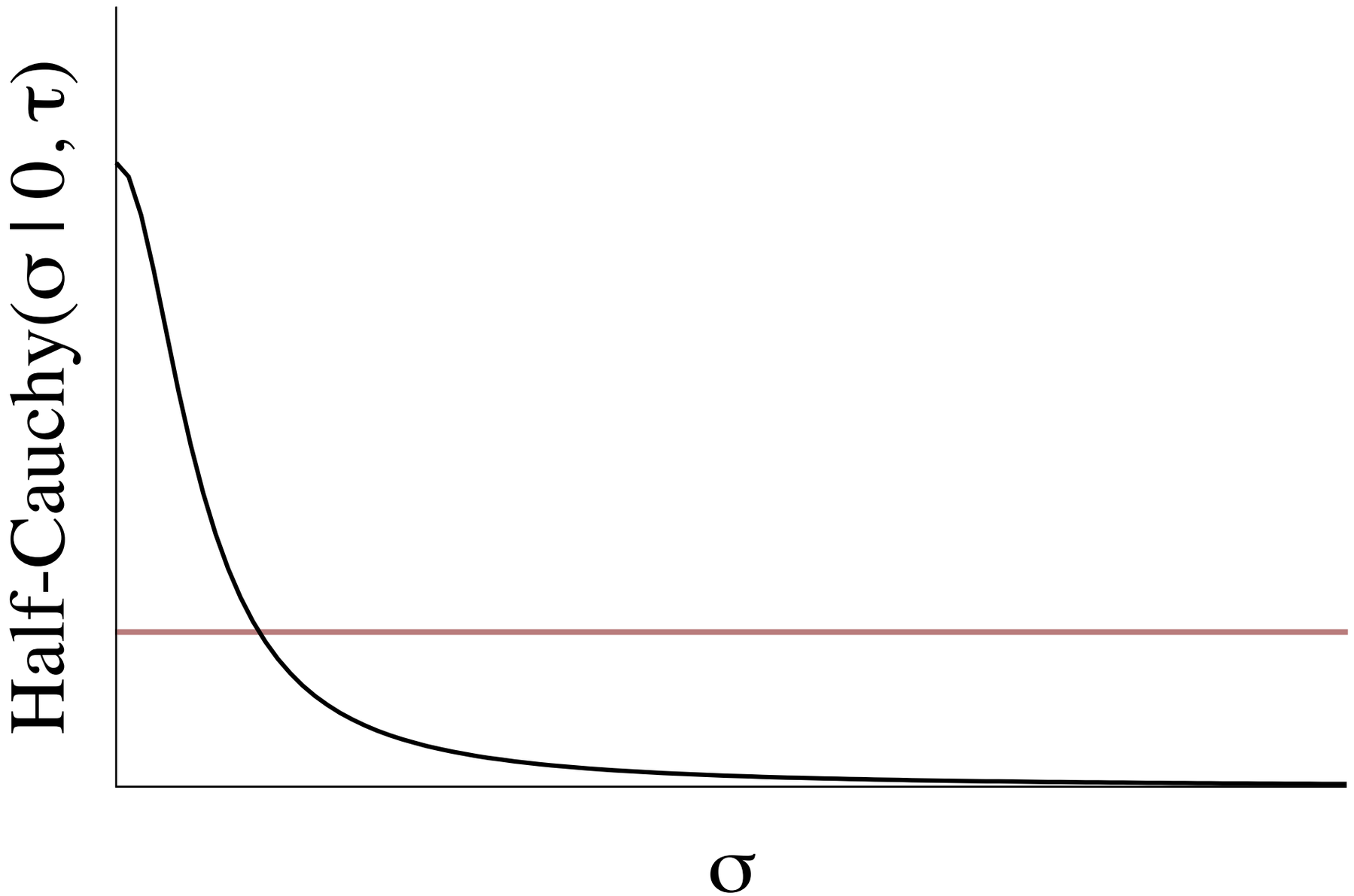
$$\prod_{n=1}^{N} p(y_n|\theta_n)\,\mathcal{N}(\theta_n|\mu,\sigma)$$

$$\approx \prod_{n=1}^{N} p(y_n|\theta_n)$$

For the model to be able to partially pool we need a weakly informative prior that concentrates around zero.



In a *multilevel* model the linear parameters vary across multiple hierarchies.

$$p(y_n|g(X_n^T\boldsymbol{\beta}_n + \alpha_n), \theta)$$

$$\alpha_n = \sum_{k=1}^{N_{\text{groups}}} \alpha_{k,j(n)}$$

$$\alpha_{k,j} \sim \mathcal{N}(\mu_{\alpha_k}, \sigma_{\alpha_k})$$

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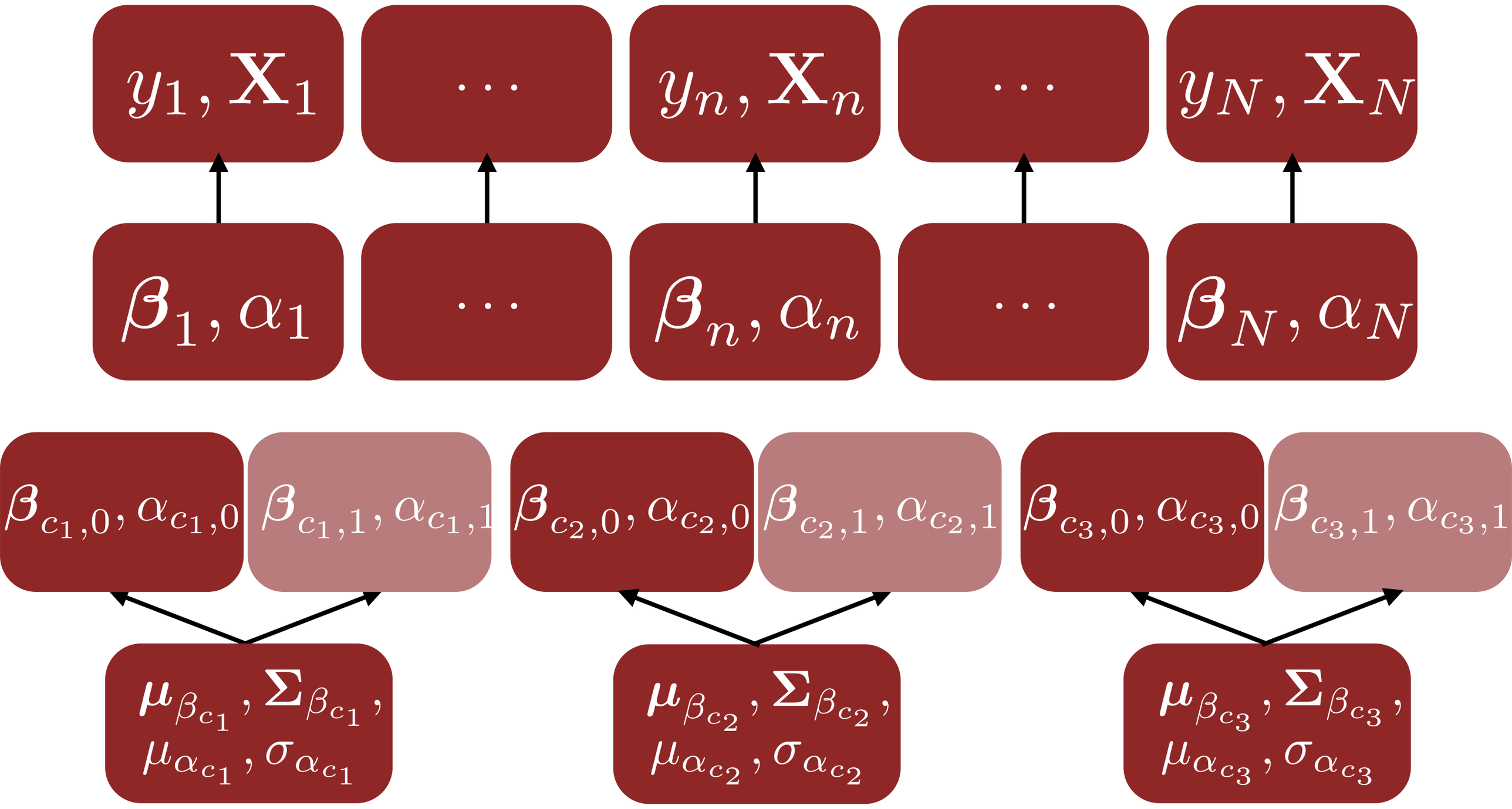

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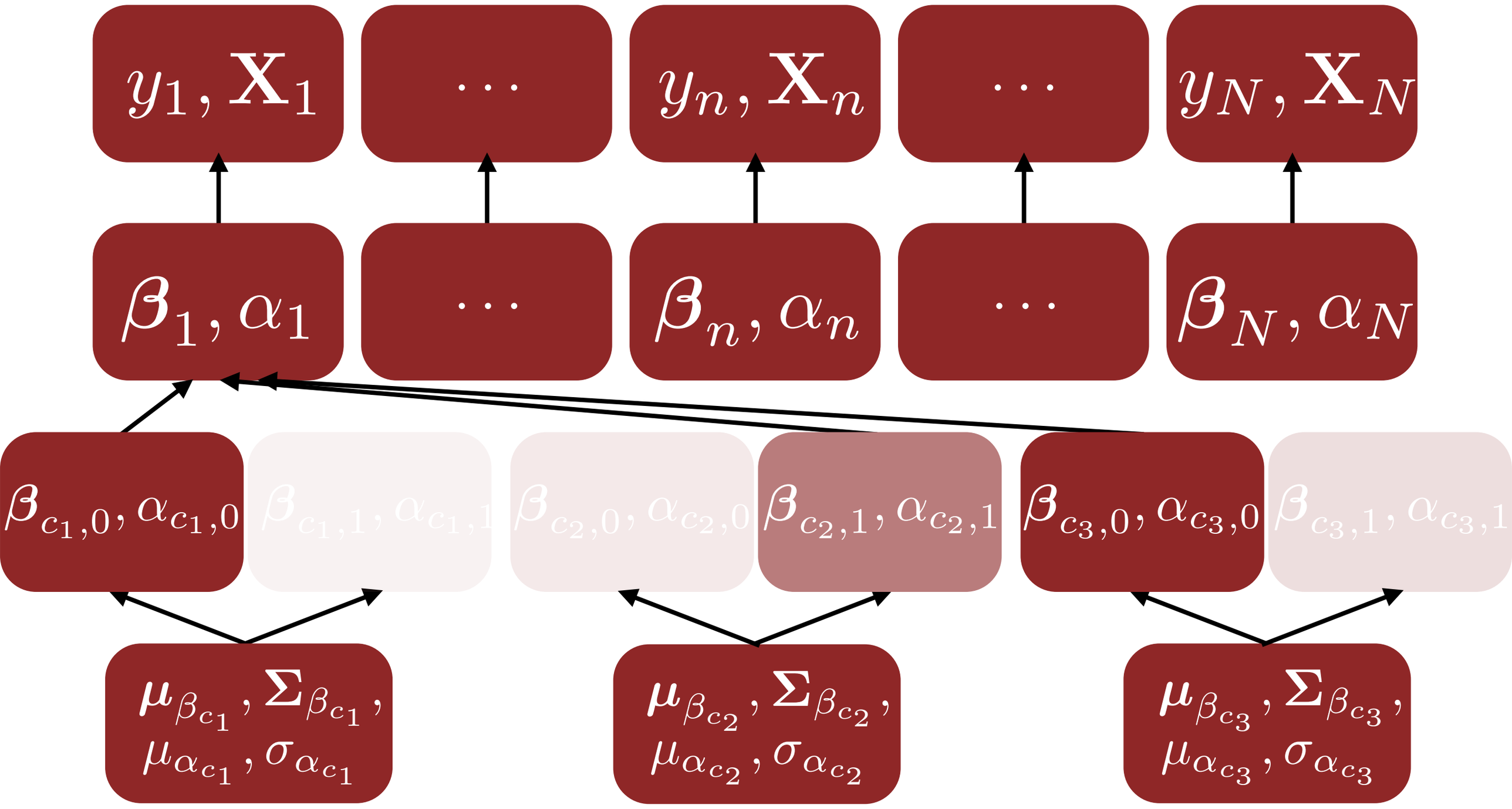

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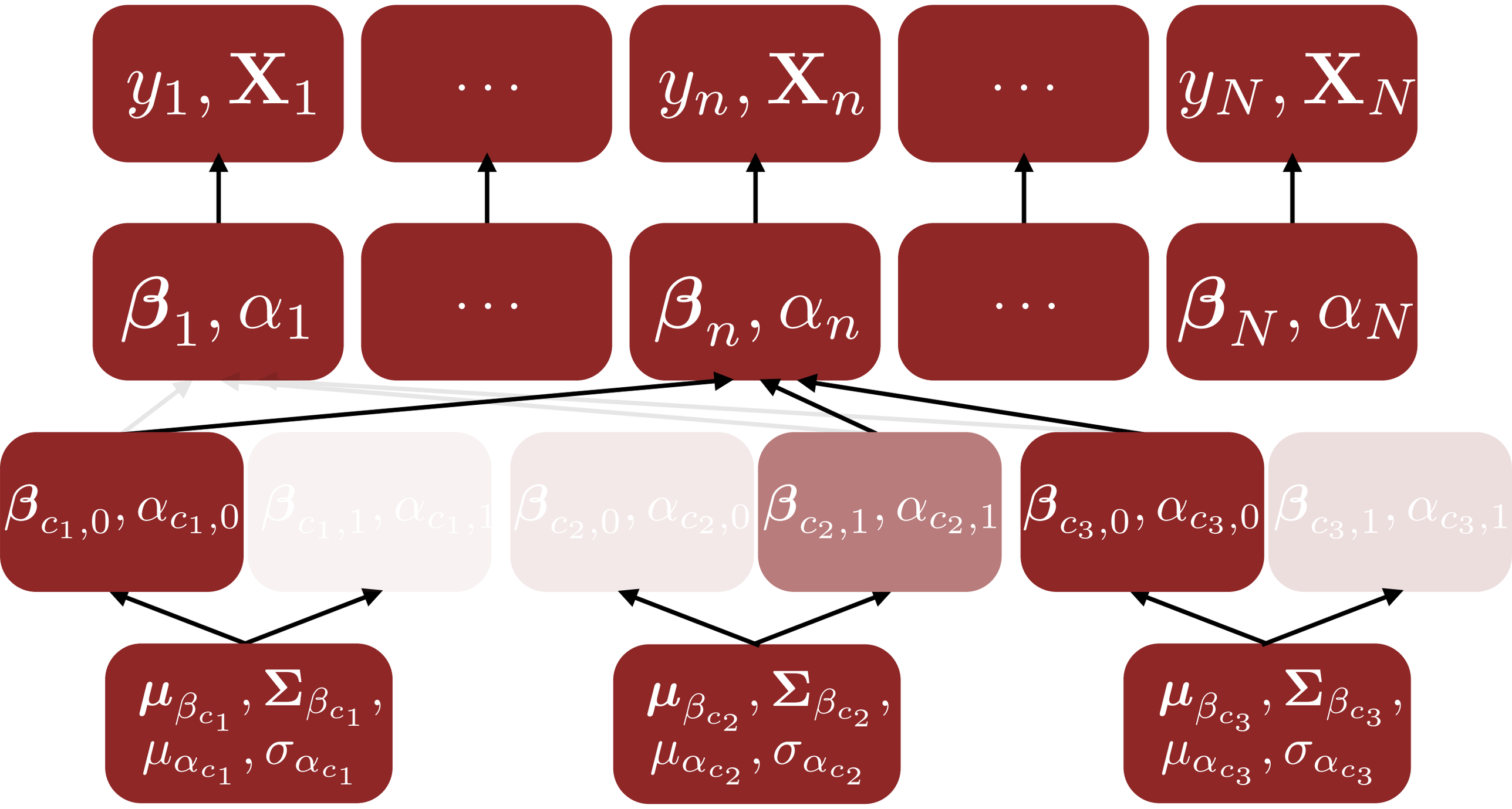

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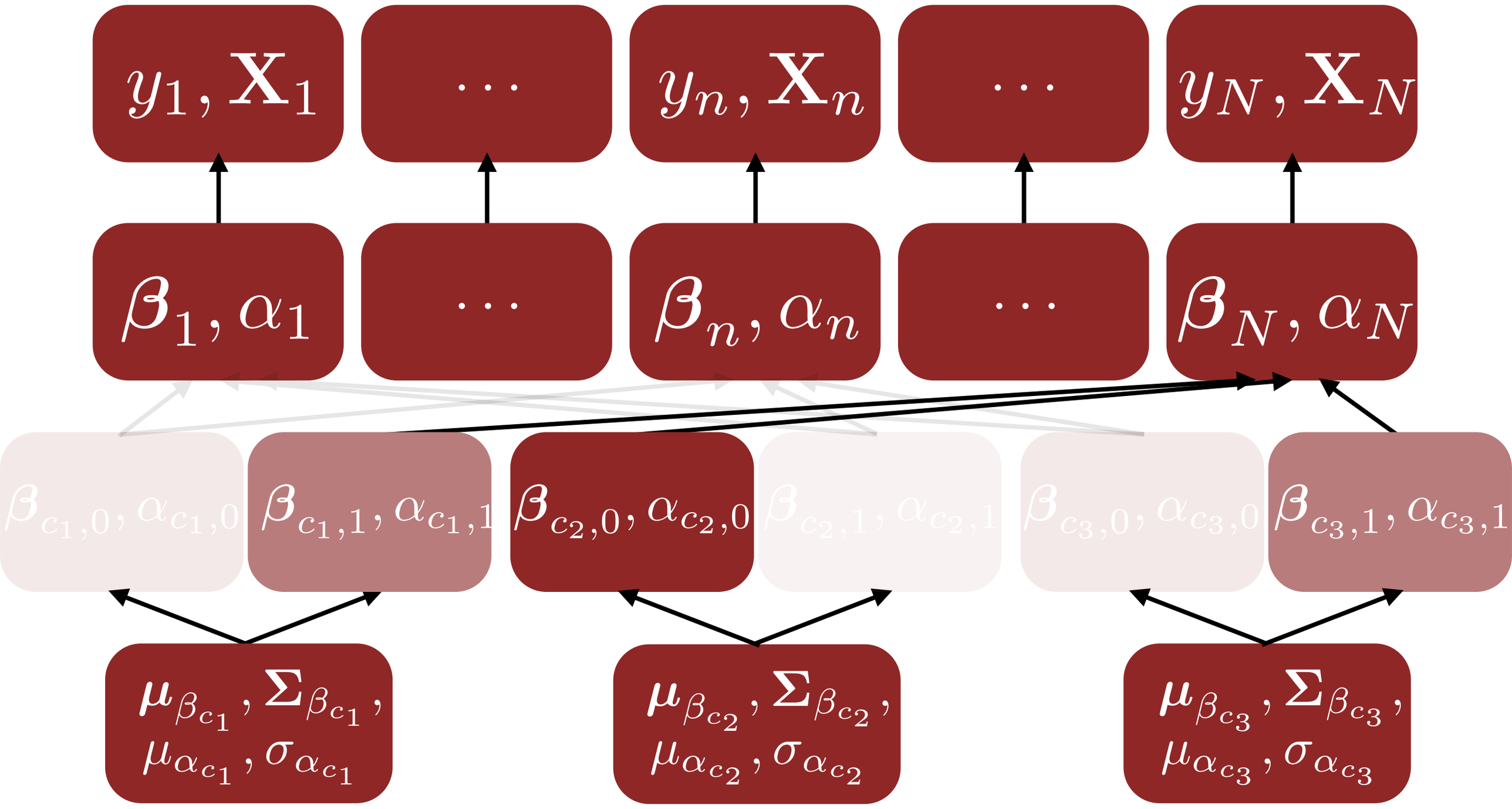
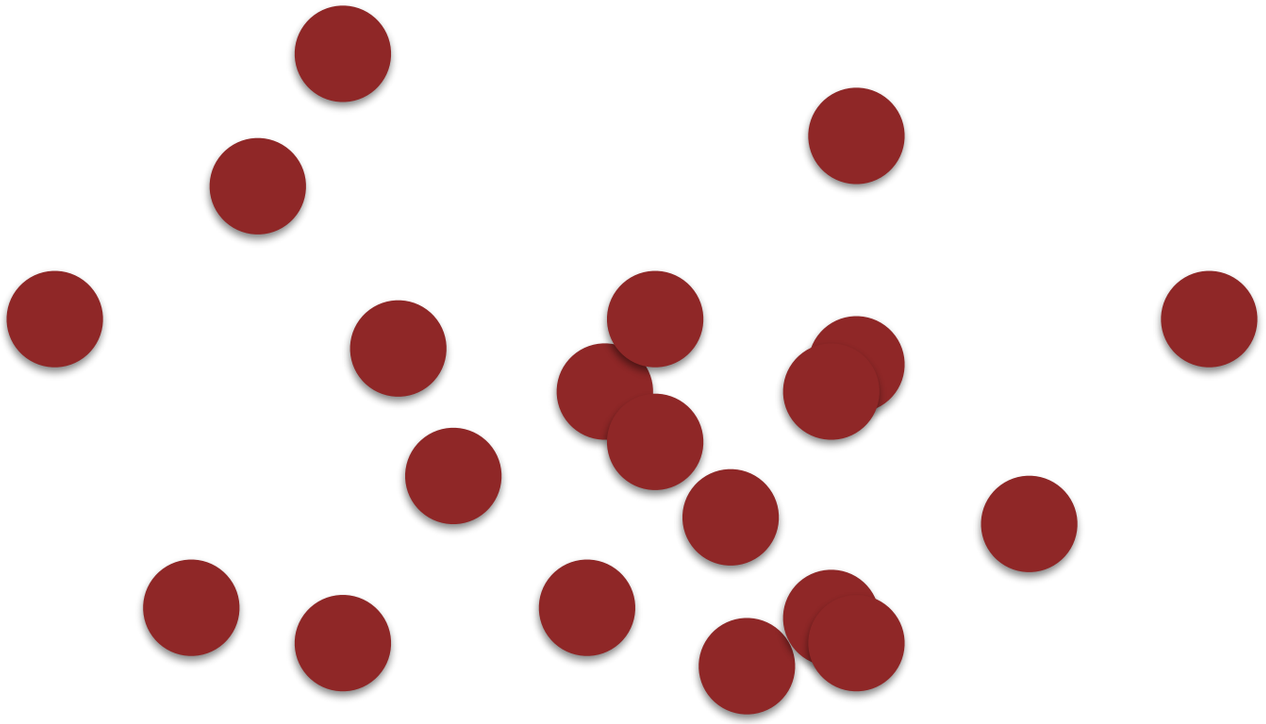


Each level contributes its own slope and/or intercept.



$$\left\{c_n^1, c_n^2, c_n^3\right\} \in \{0, 1\}$$

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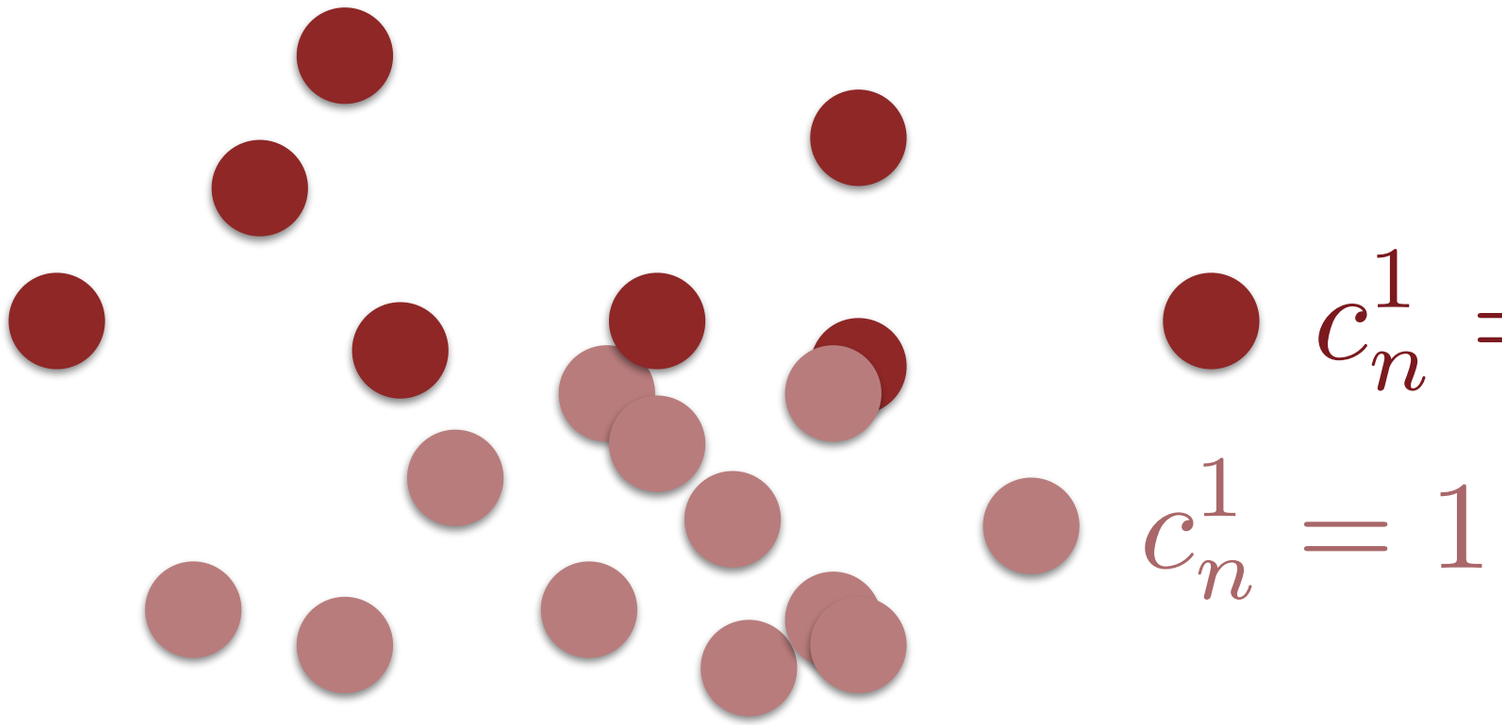
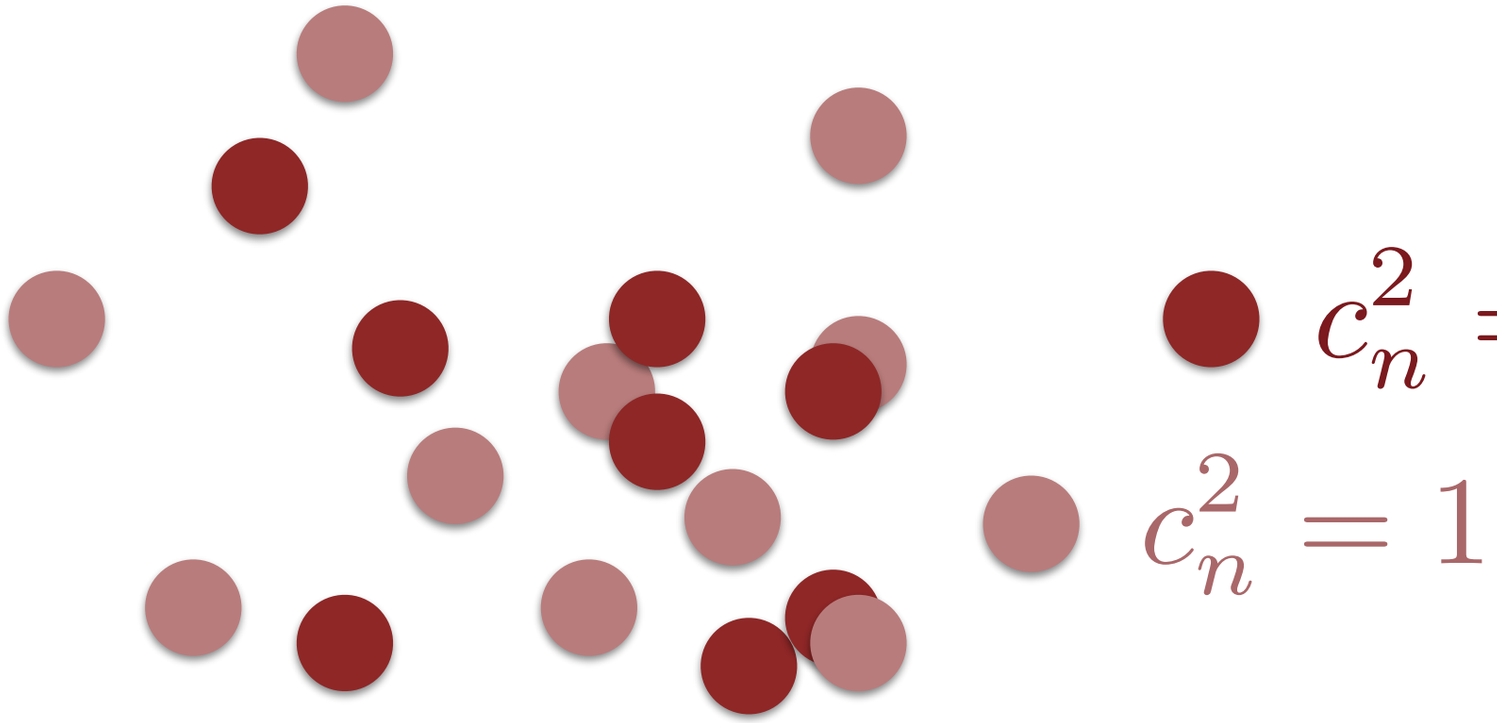


$$\alpha_{c_1,0} \quad \alpha_{c_1,1}$$

$$\alpha_{c_1,j} \sim \mathcal{N}\left(\mu_{\alpha_{c_1}}, \sigma_{\alpha_{c_1}}\right)$$

Each level contributes its own slope and/or intercept.



$$\alpha_{c_2,0} \quad \alpha_{c_2,1}$$

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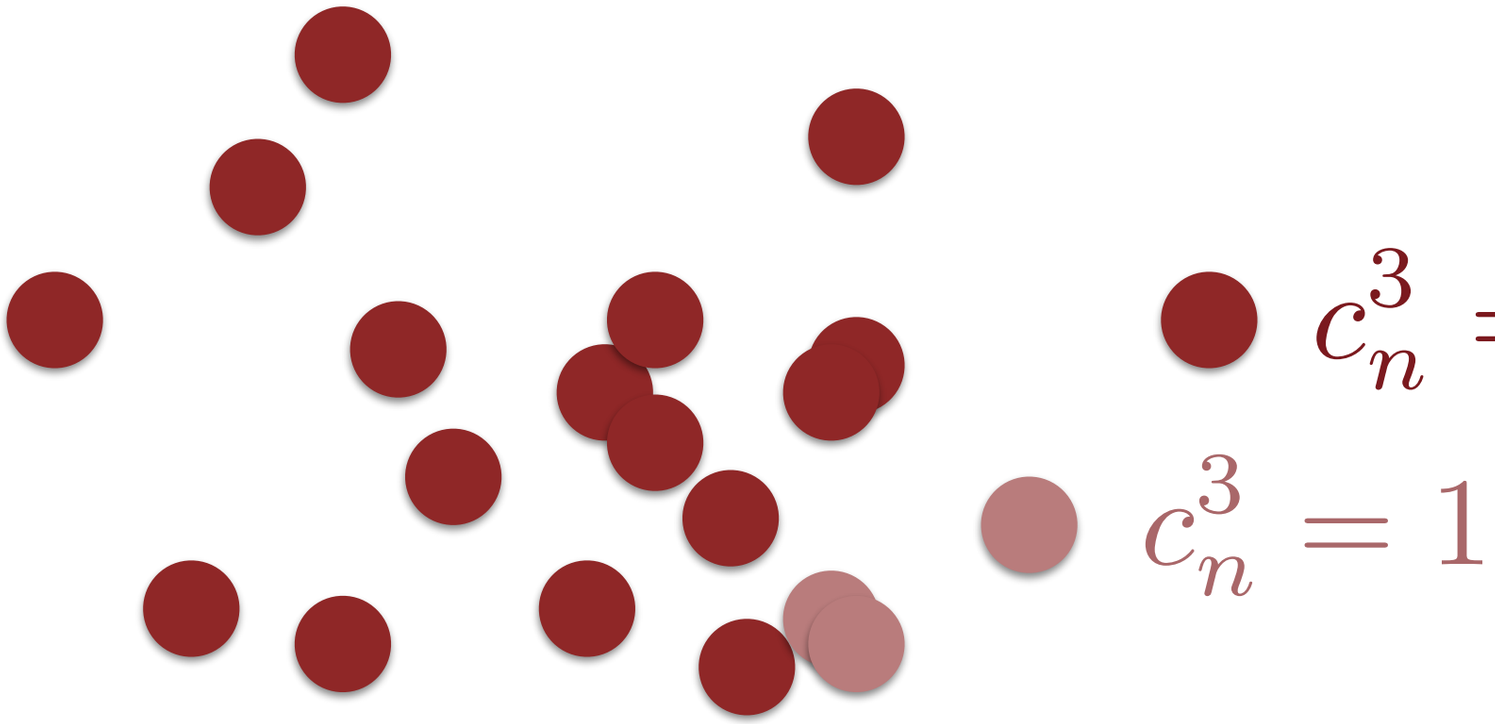
Each level contributes its own slope and/or intercept.



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$c_n^1 = 0$

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$\alpha_n =$

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$\alpha_n = \quad \alpha_{c_1,0}$

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$$c_n^1 = 0 \qquad c_n^2 = 1 \qquad c_n^3 = 0$$

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$$\alpha_n = \quad \alpha_{c_1,0} \quad + \quad \alpha_{c_2,1} \quad + \quad \alpha_{c_3,0}$$

Partial pooling model

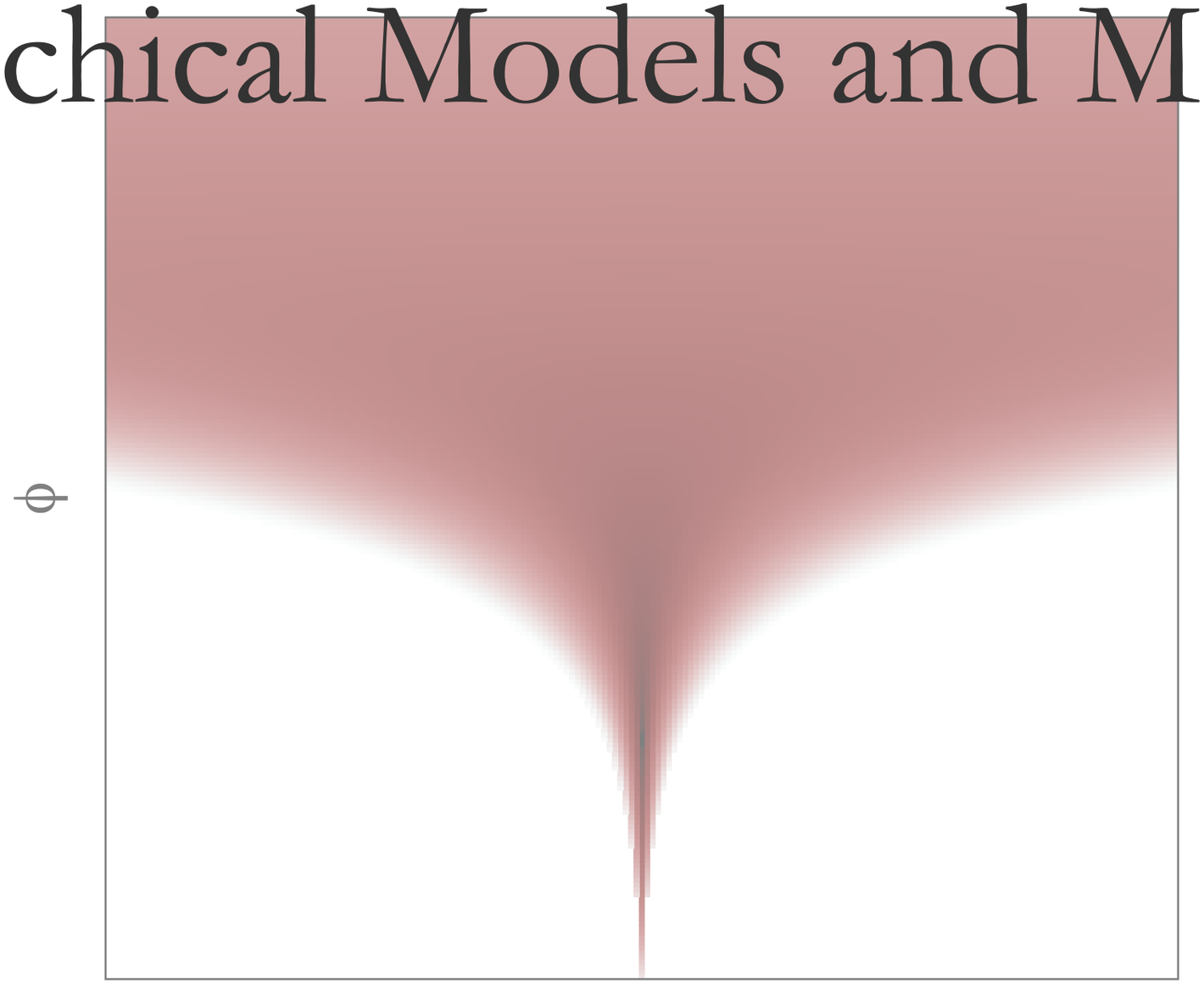
```
data {
  int N;
  int K;
  int<lower=1> N_groups;
  int<lower=1> group[N];
  int<lower=0, upper=1> y[N];
  vector[N] x;
}

parameters {
  vector[N_groups] alpha; // Intercept for every group
  real beta; // could also let beta vary by group
  real mu_alpha; // Hierarchical mean
  real<lower=0> sigma_alpha; // Hierarchical sd
}

model {
  y ~ bernoulli_logit(x * beta + alpha[group]);

  beta ~ normal(0, 10);
  alpha ~ normal(mu_alpha, sigma_alpha);
  mu_alpha ~ normal(0, 10);
  sigma_alpha ~ cauchy(0, 10);
}
```

Generalized Linear Hierarchical Models and MCMC



Neal's funnel distribution illustrates some of the problems with sampling from hierarchical distributions.

$$p(\theta|\phi) = \prod_{n=1}^{N} \mathcal{N}\left(\theta_n|0, e^{\frac{\phi}{2}}\right) \mathcal{N}(\phi|0,3)$$

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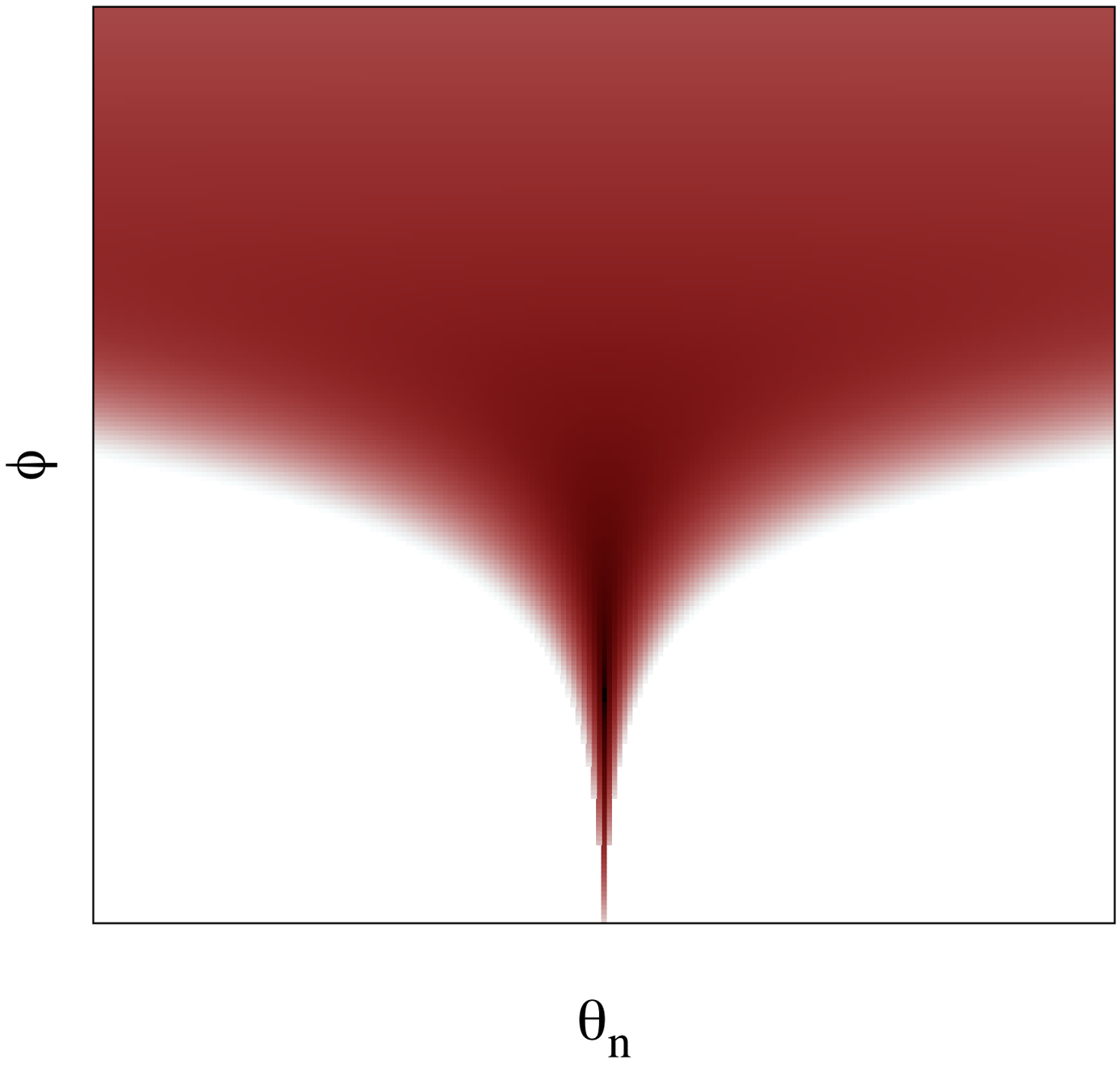

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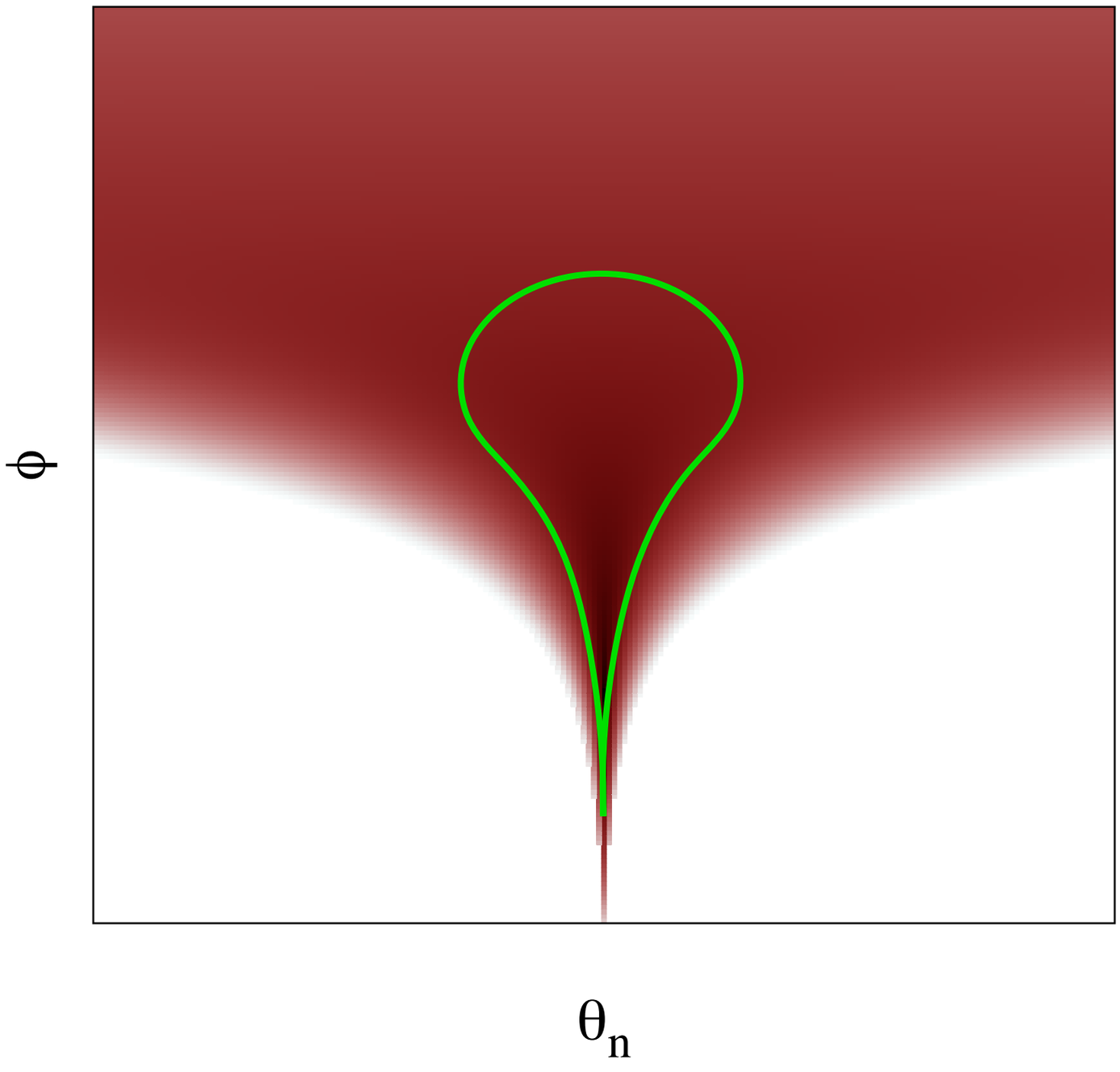

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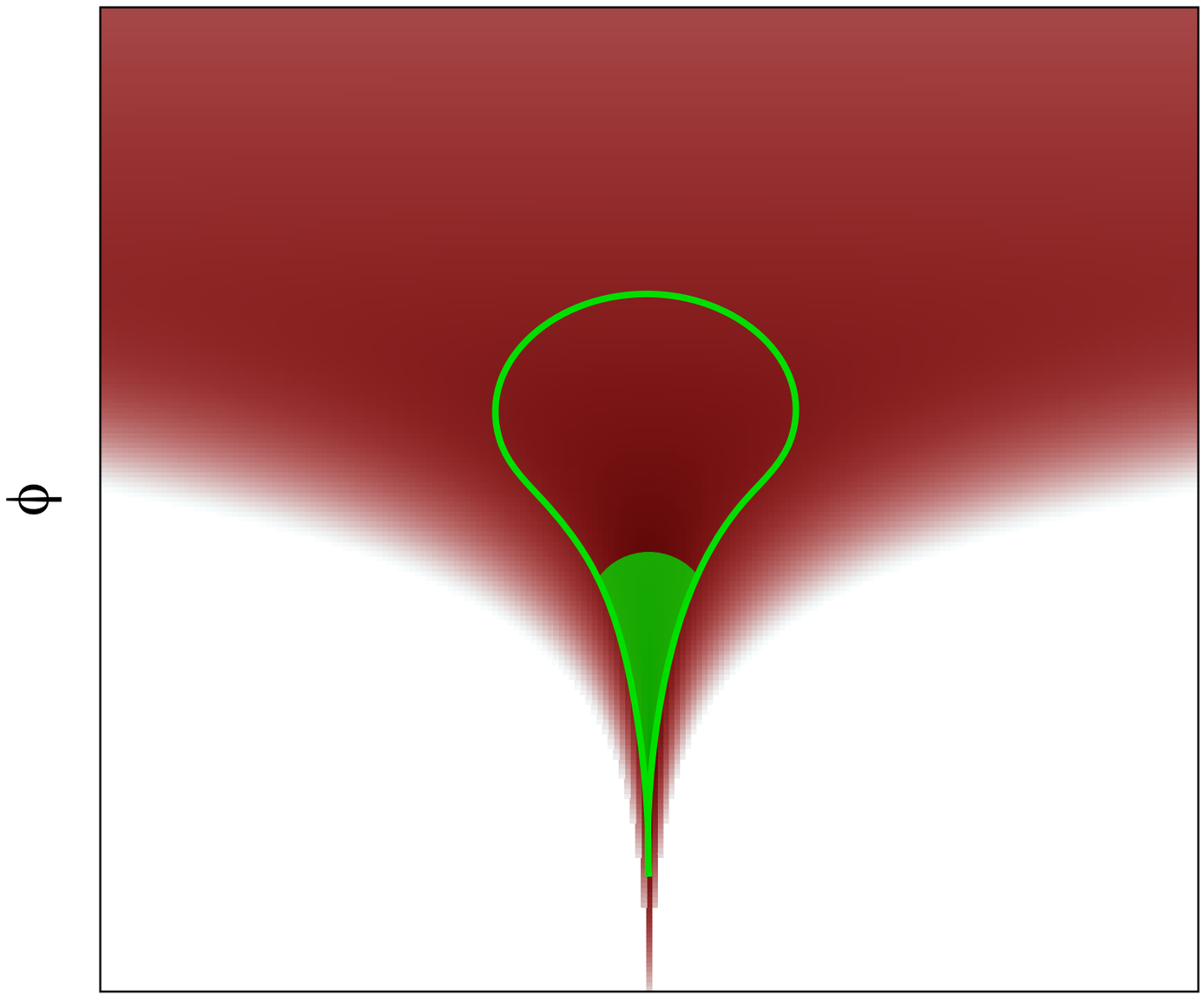

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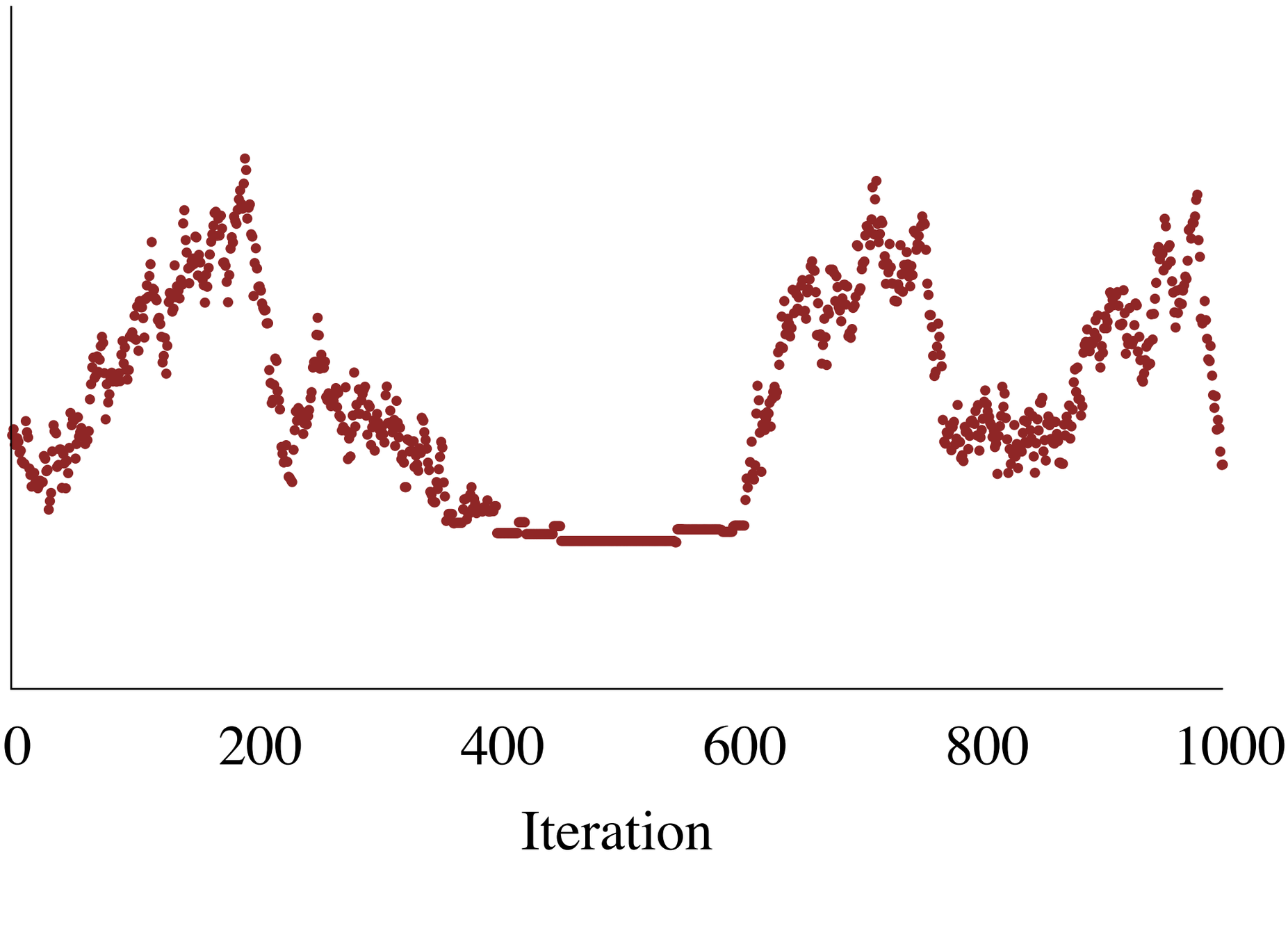

The geometry of the funnel, however, dramatically improves when we sample from auxiliary parameters.

$$p(\theta|\phi) = \prod_{n=1}^{N} \mathcal{N}\left(\theta_n|0, e^{\frac{\phi}{2}}\right) \mathcal{N}(\phi|0, 3)$$

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$$\tilde{\theta}_n \sim \mathcal{N}(0, 1)$$

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$$\theta_n = 0 + \tilde{\theta}_n e^{\frac{\phi}{2}}$$

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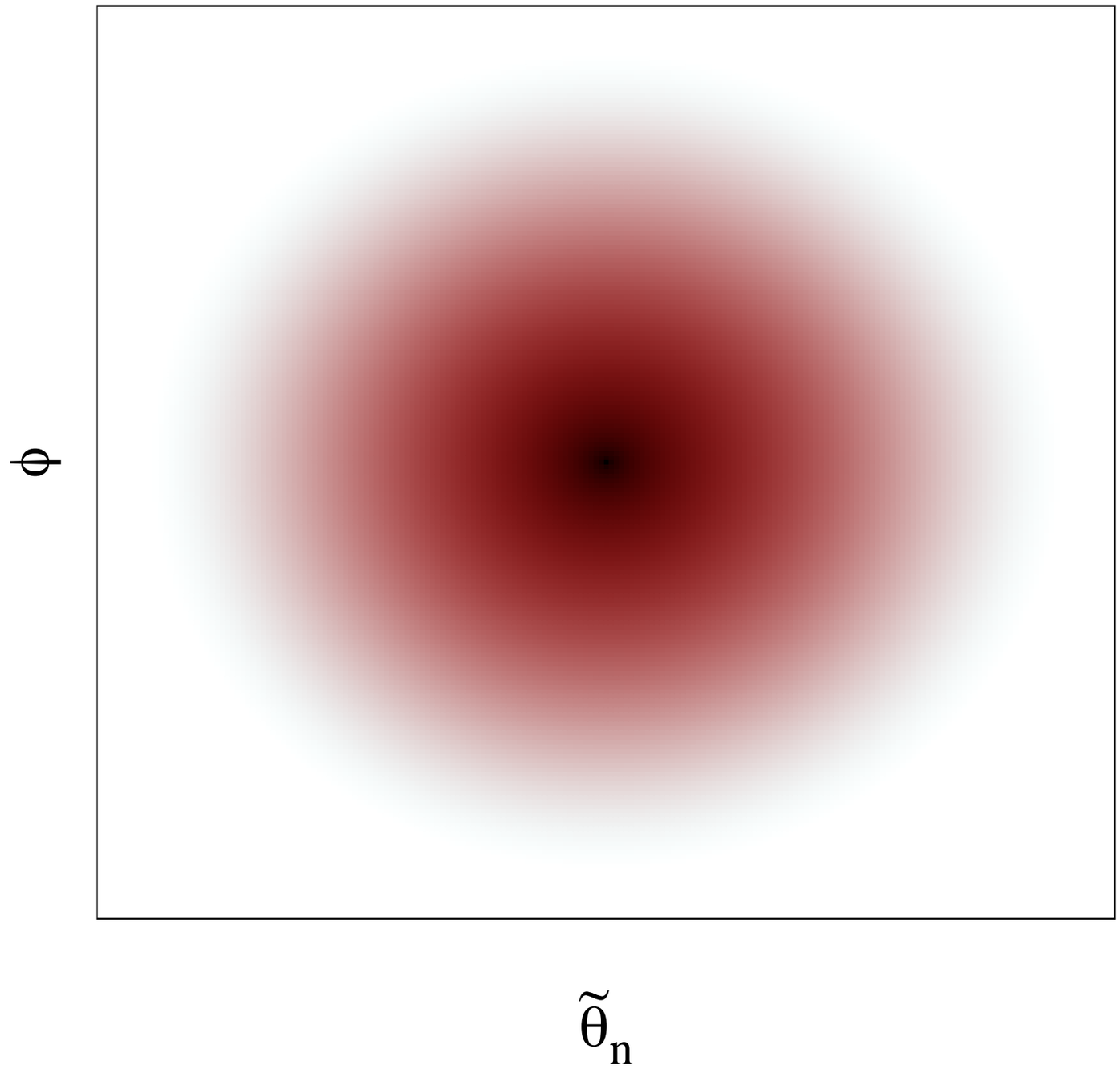


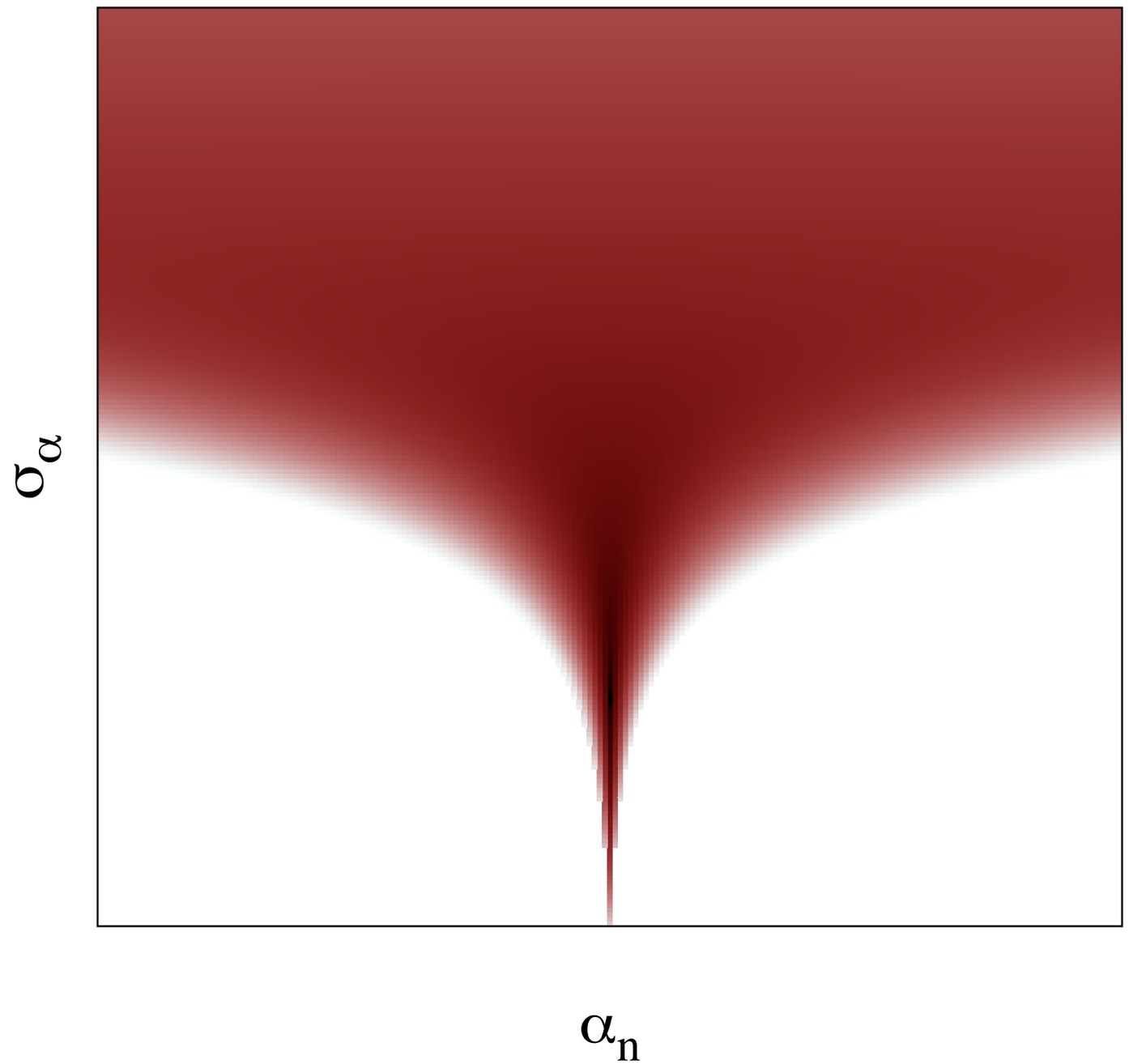
The most natural representation of a general linear hierarchical model is a *centered parameterization*.

$$p(y_n|g(\mathbf{X}_n^T\boldsymbol{\beta}_n + \alpha_n, \theta)$$

$$\boldsymbol{\beta}_n \sim \mathcal{N}(\boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta)$$

$$\alpha_n \sim \mathcal{N}(\mu_\alpha, \sigma_\alpha)$$

When data are sparse and uninformative, the centered parameterization of the posterior resembles the funnel.



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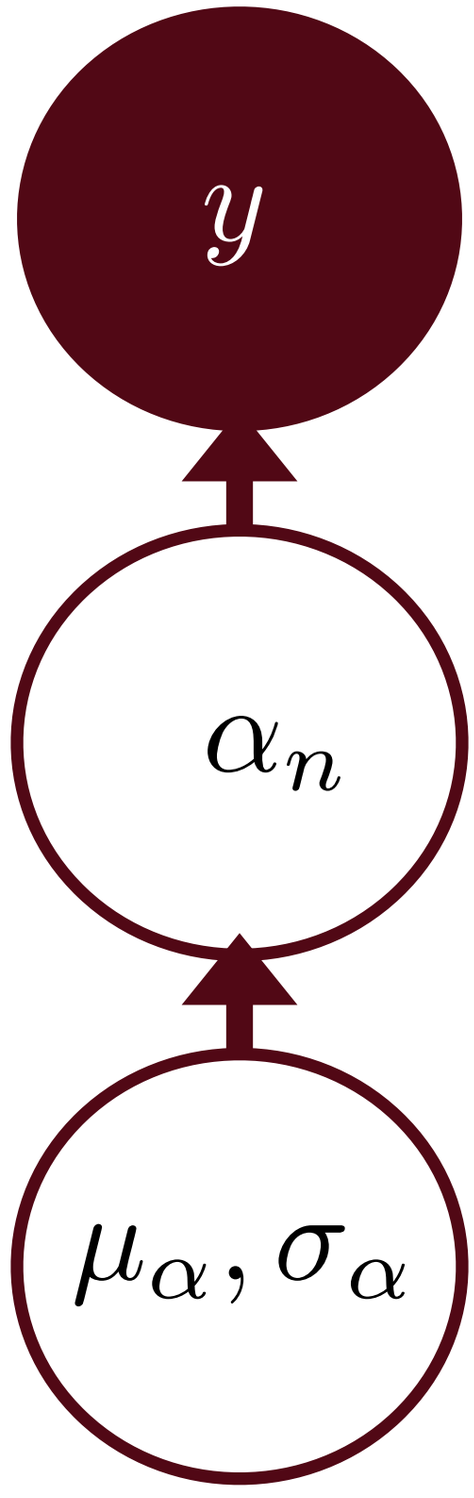
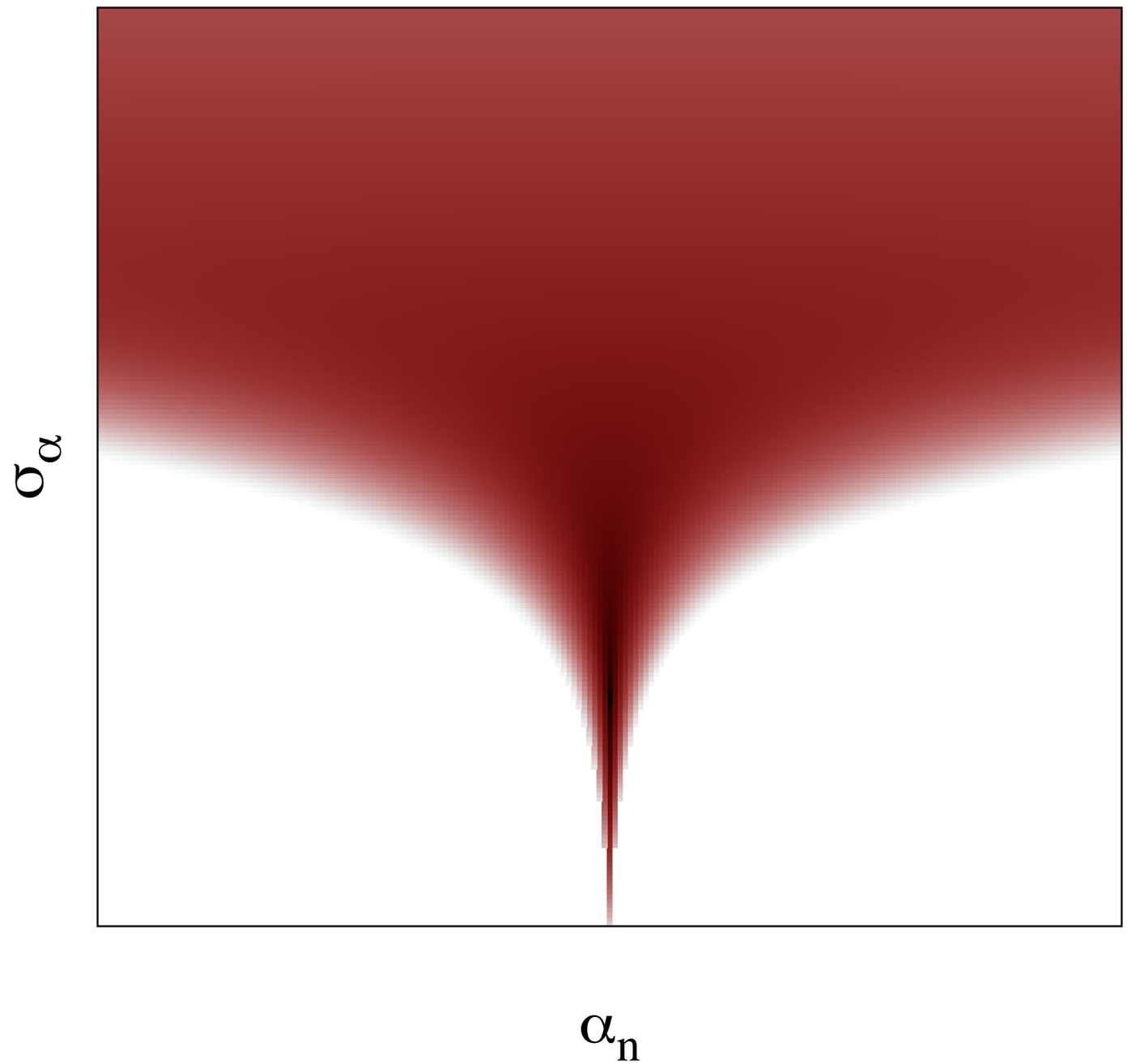


But when the data are more informative
the posterior geometry is much nicer.



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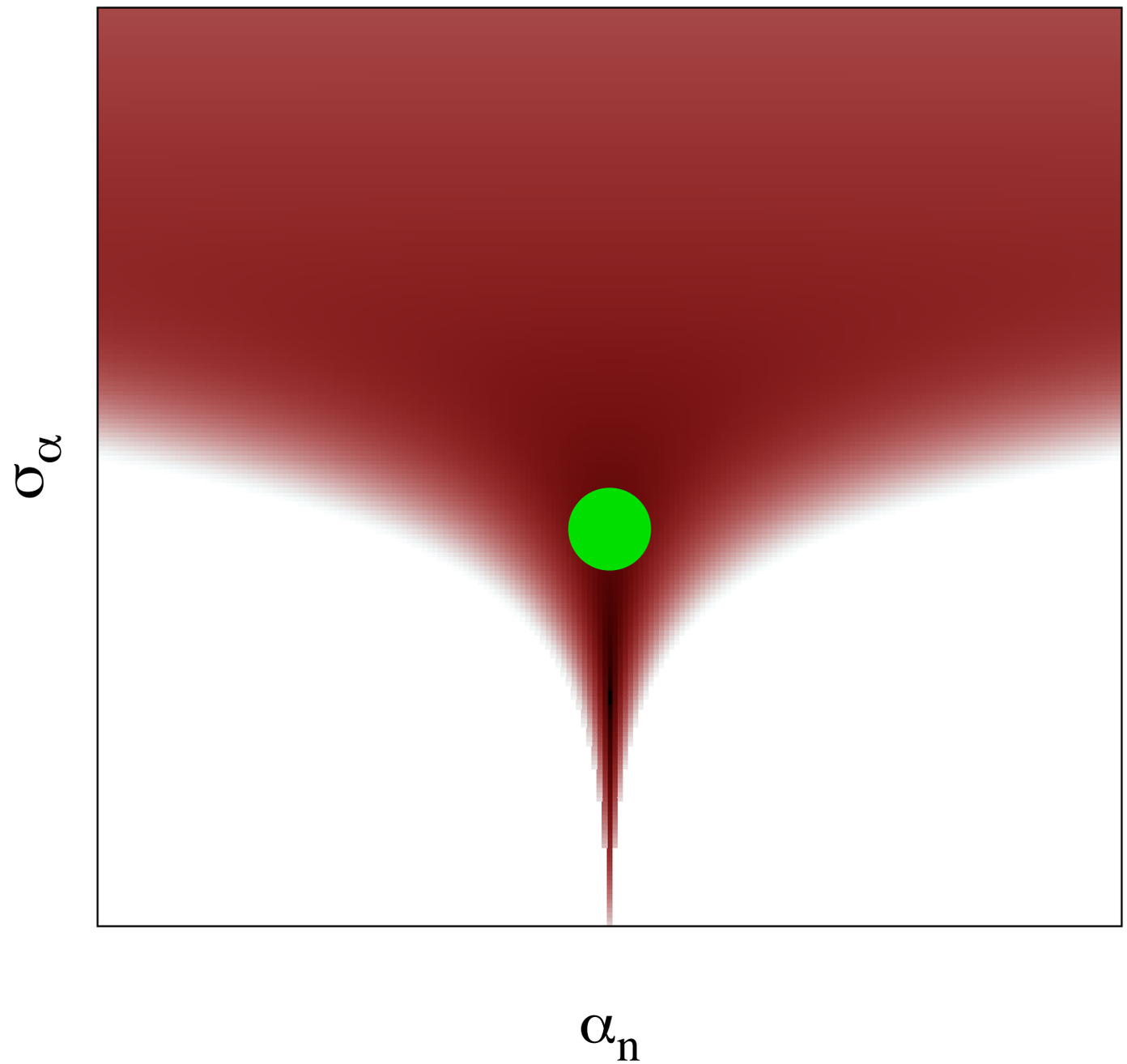

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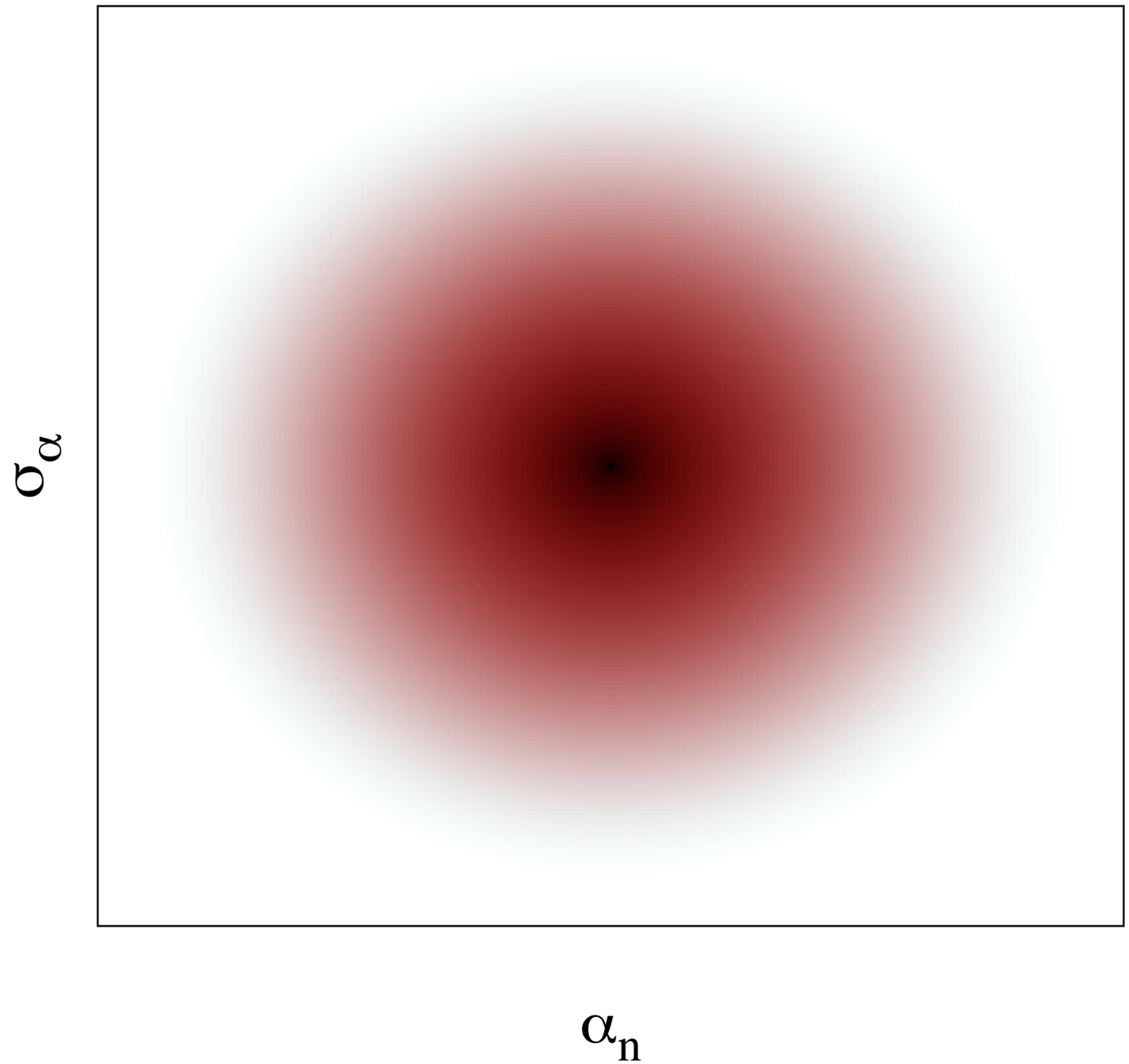


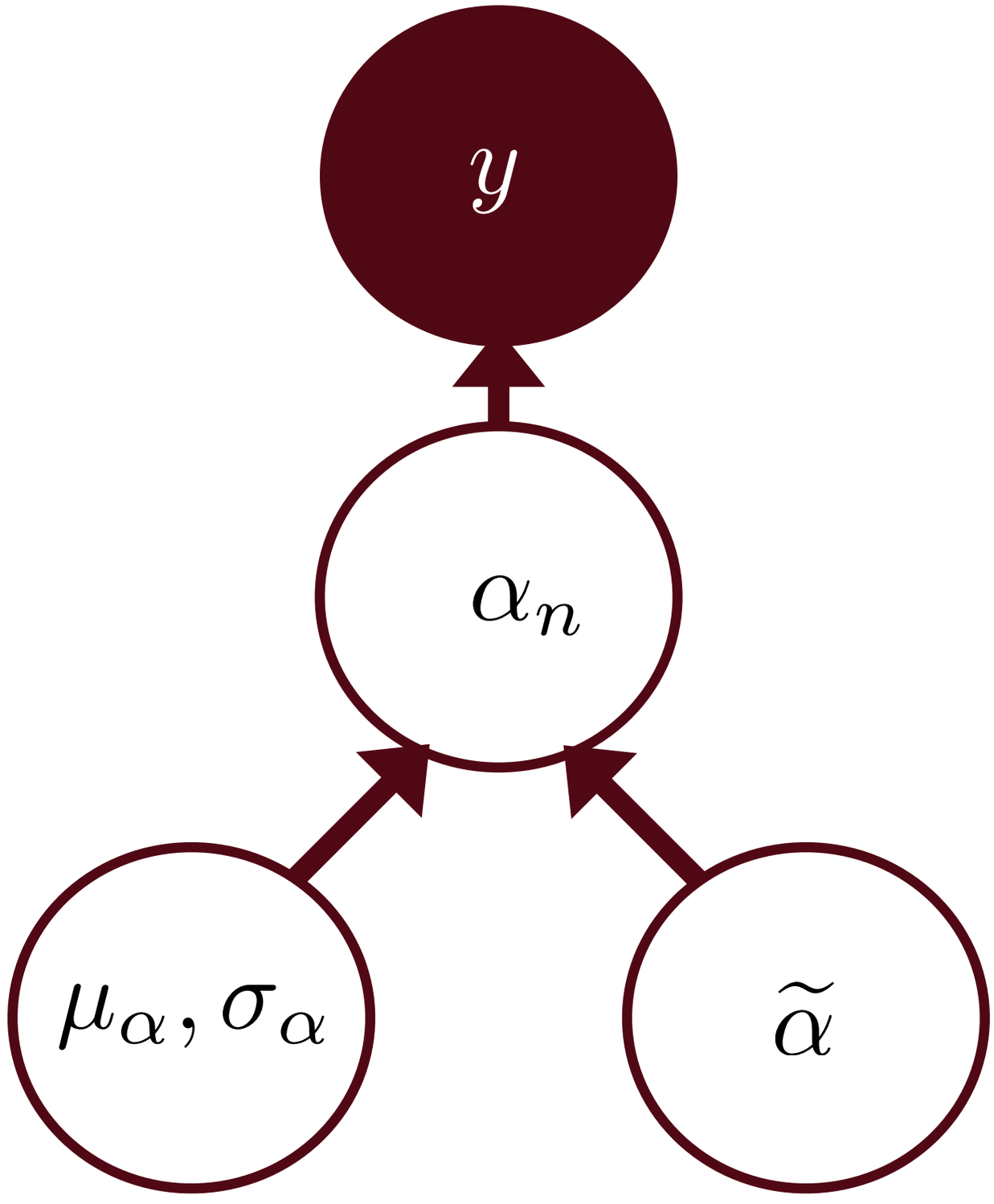
The same general linear hierarchical model can also be implemented with a *non-centered parameterization*.

$$p(y_n|g(\mathbf{X}_n^T\boldsymbol{\beta}_n + \alpha_n, \theta)$$

$$\tilde{\boldsymbol{\beta}}_n \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$$
$$\boldsymbol{\beta}_n = \boldsymbol{\mu}_\beta + \mathbf{L}\tilde{\boldsymbol{\beta}}_n,\ \mathbf{L}\mathbf{L}^T = \boldsymbol{\Sigma}_\beta$$

$$\tilde{\alpha}_n \sim \mathcal{N}(0, 1)$$
$$\alpha_n = \mu_\alpha + \sigma_\alpha\tilde{\alpha}_n$$

When data are sparse, the non-centered parameterization of the posterior yields a favorable geometry.



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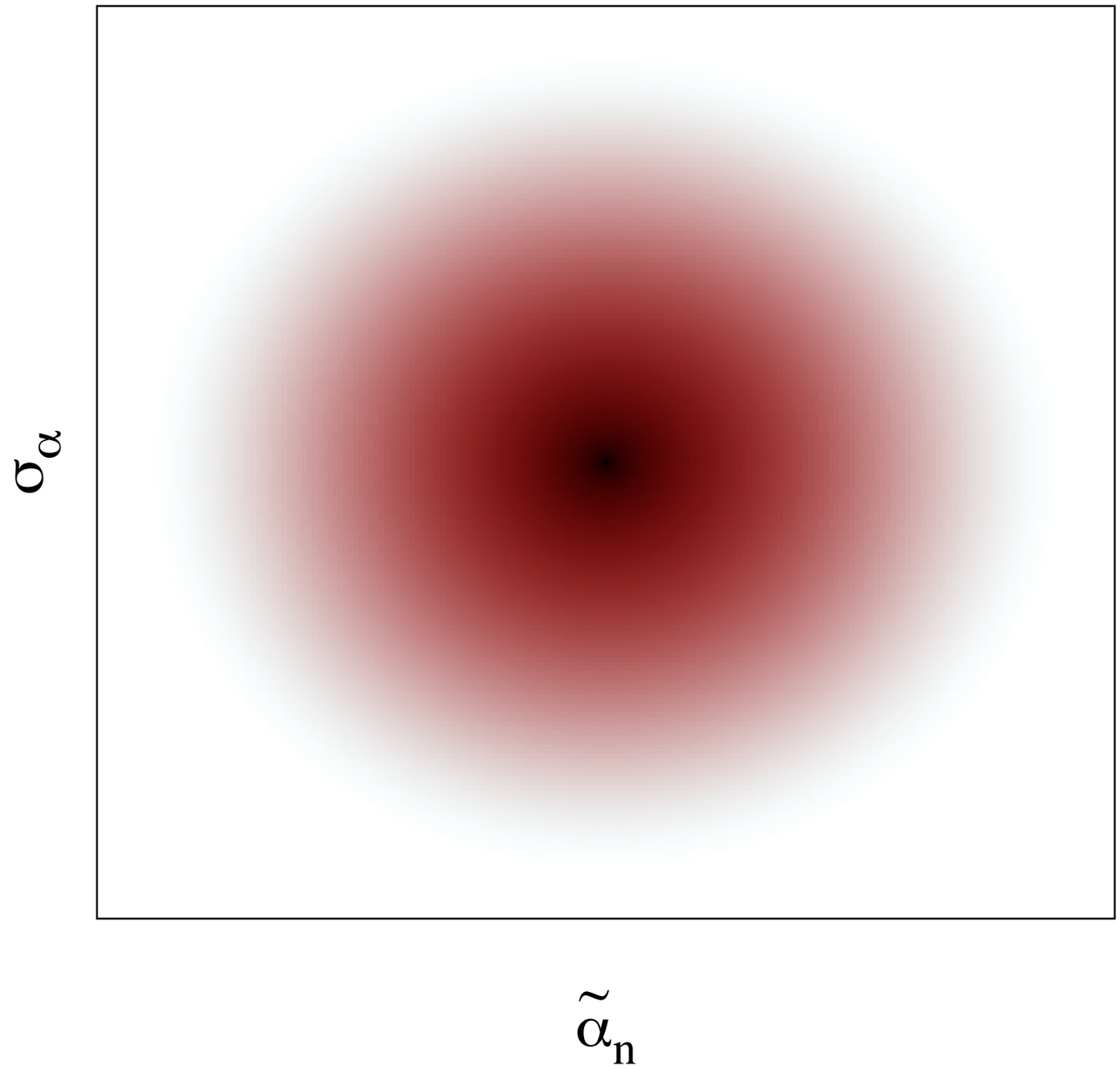
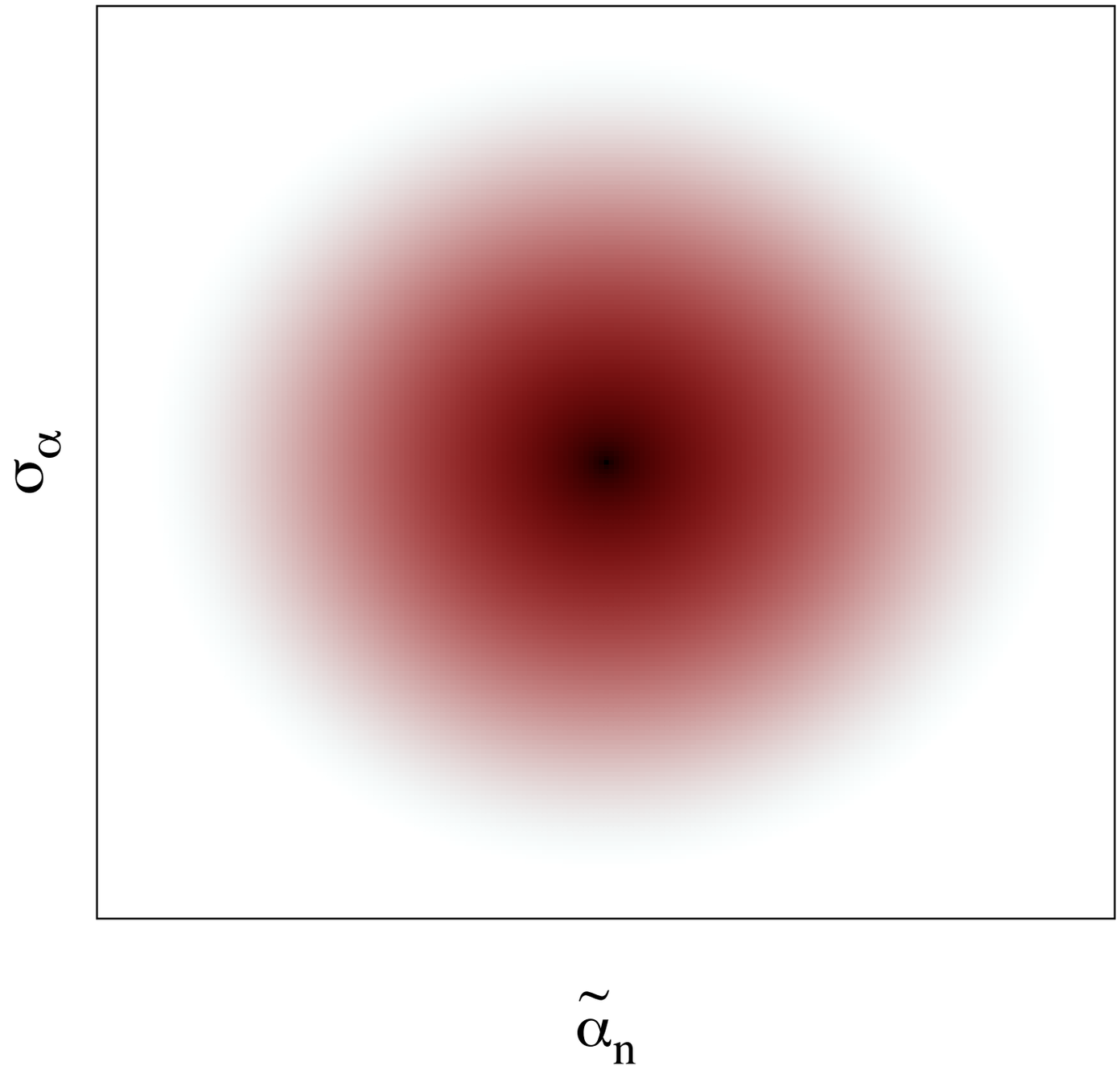


The constraints imposed by the data, however, are more complex, yielding pathological posteriors with more data.



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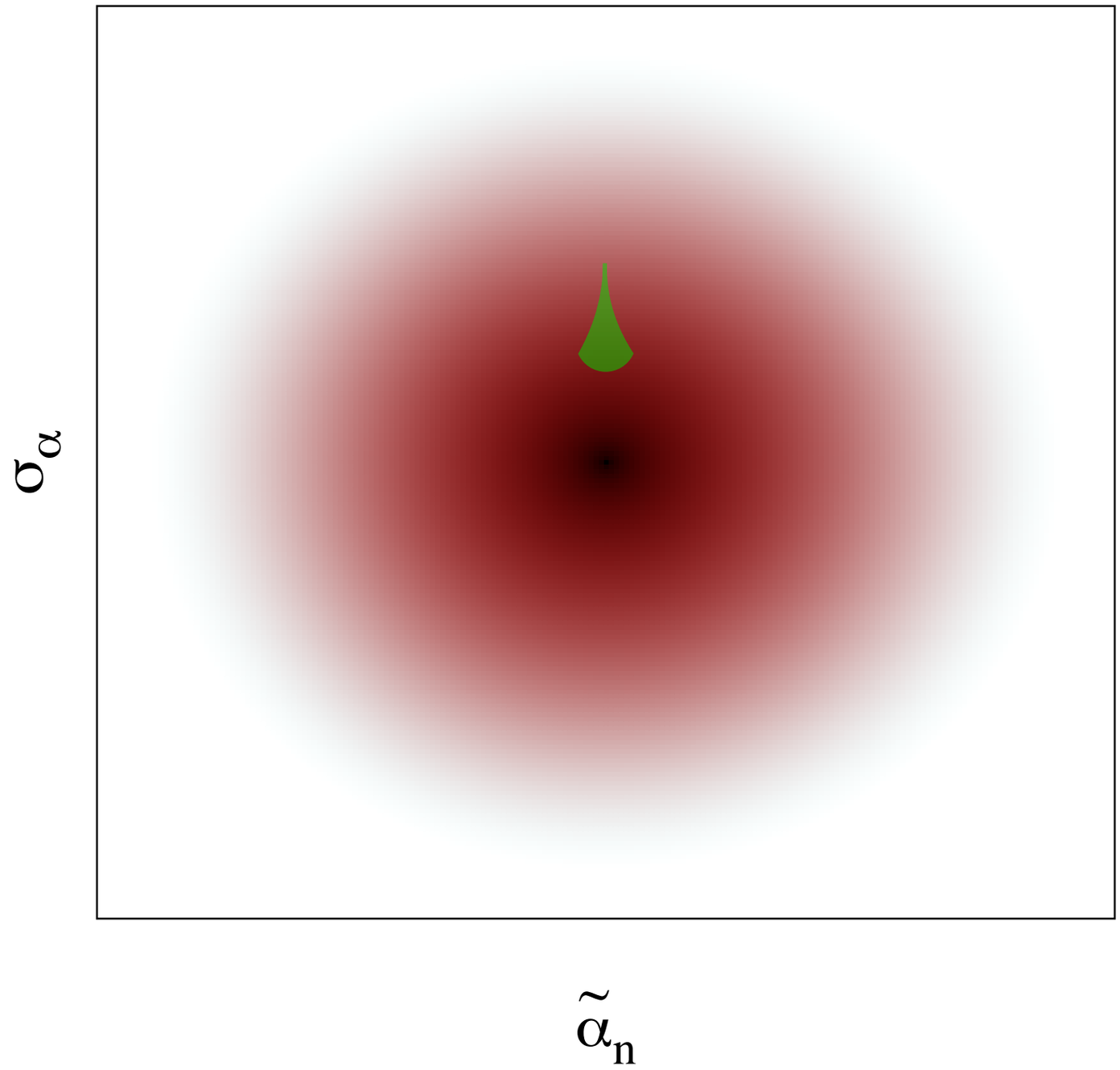

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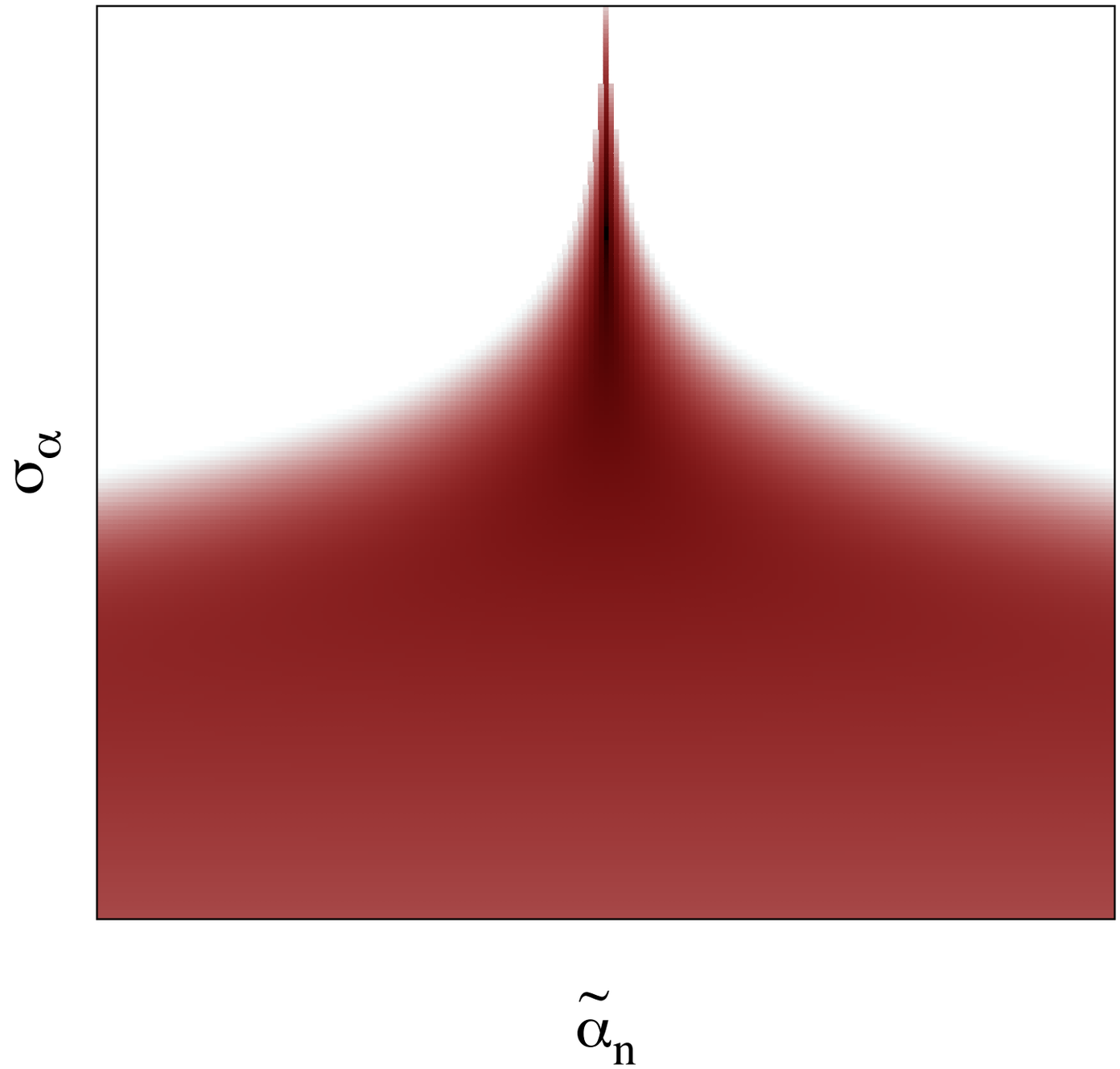

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Centered Parameterization

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Hierarchical
Geometry

Centered
Parameterization

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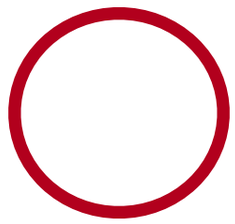
	Hierarchical Geometry	Measurement Geometry
Centered Parameterization		

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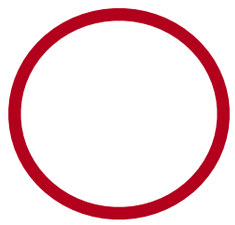
	Hierarchical Geometry	Measurement Geometry	Sparse Data
Centered Parameterization			

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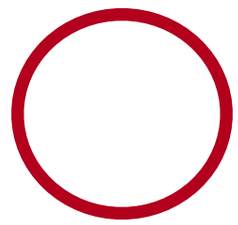
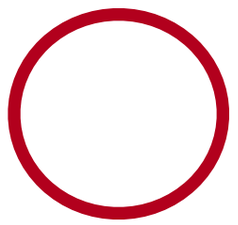
	Hierarchical Geometry	Measurement Geometry	Sparse Data	Dense Data
Centered Parameterization				

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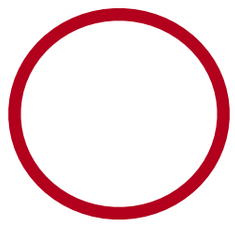
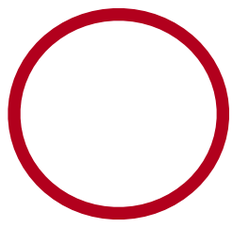
Centered Parameterization				
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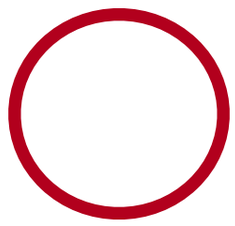
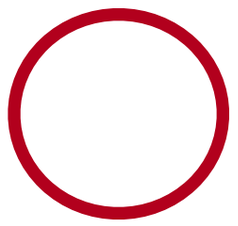
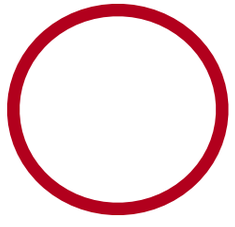
Centered Parameterization				
Non-Centered Parameterization				

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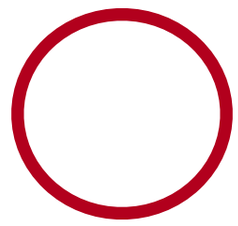
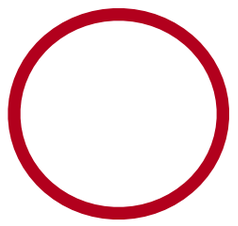
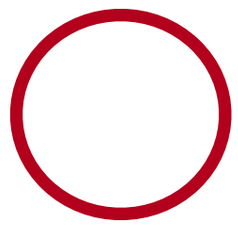
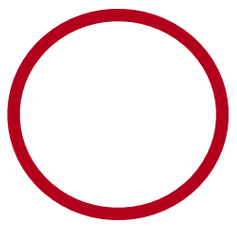
Centered Parameterization				
Non-Centered Parameterization				

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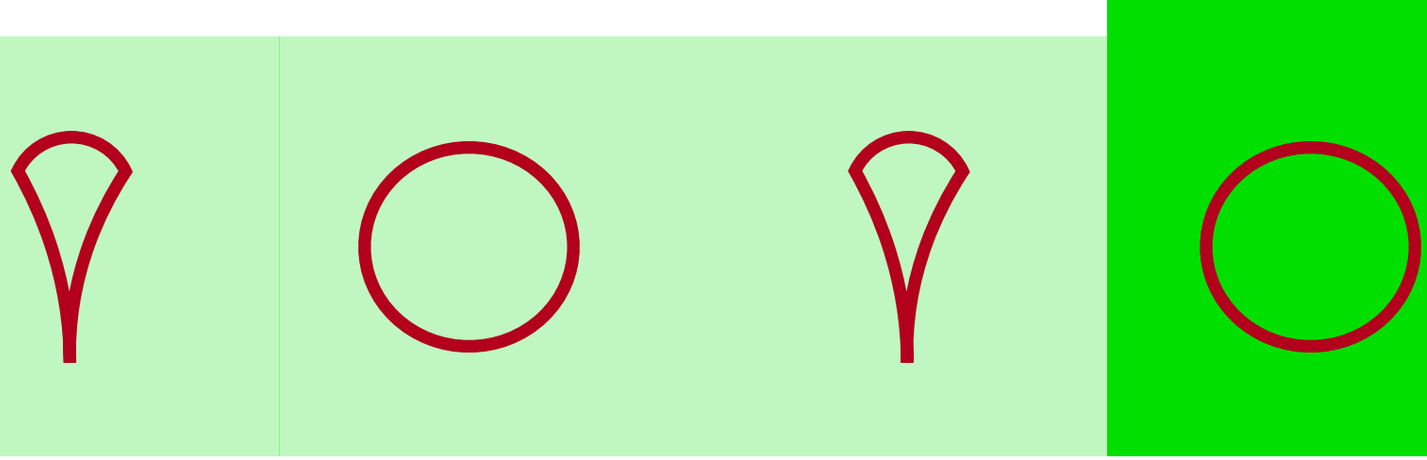
Centered Parameterization			
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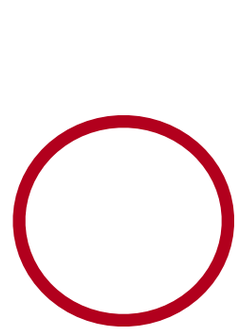
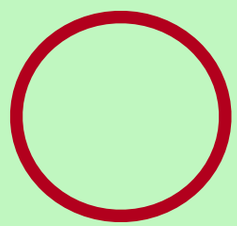
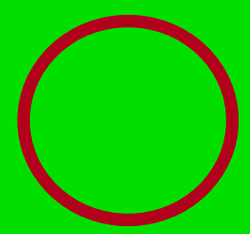
Centered Parameterization				
Non-Centered Parameterization				

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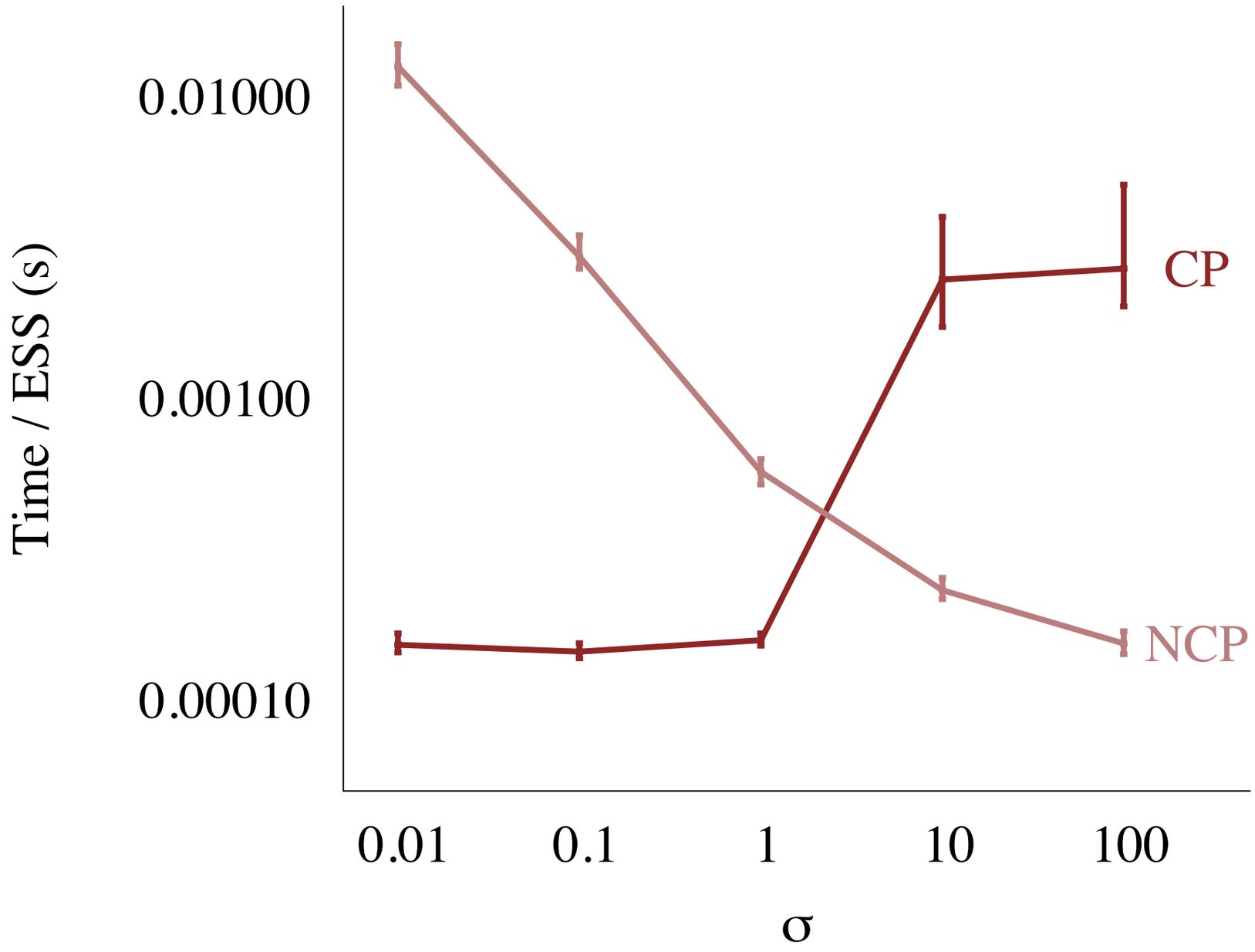
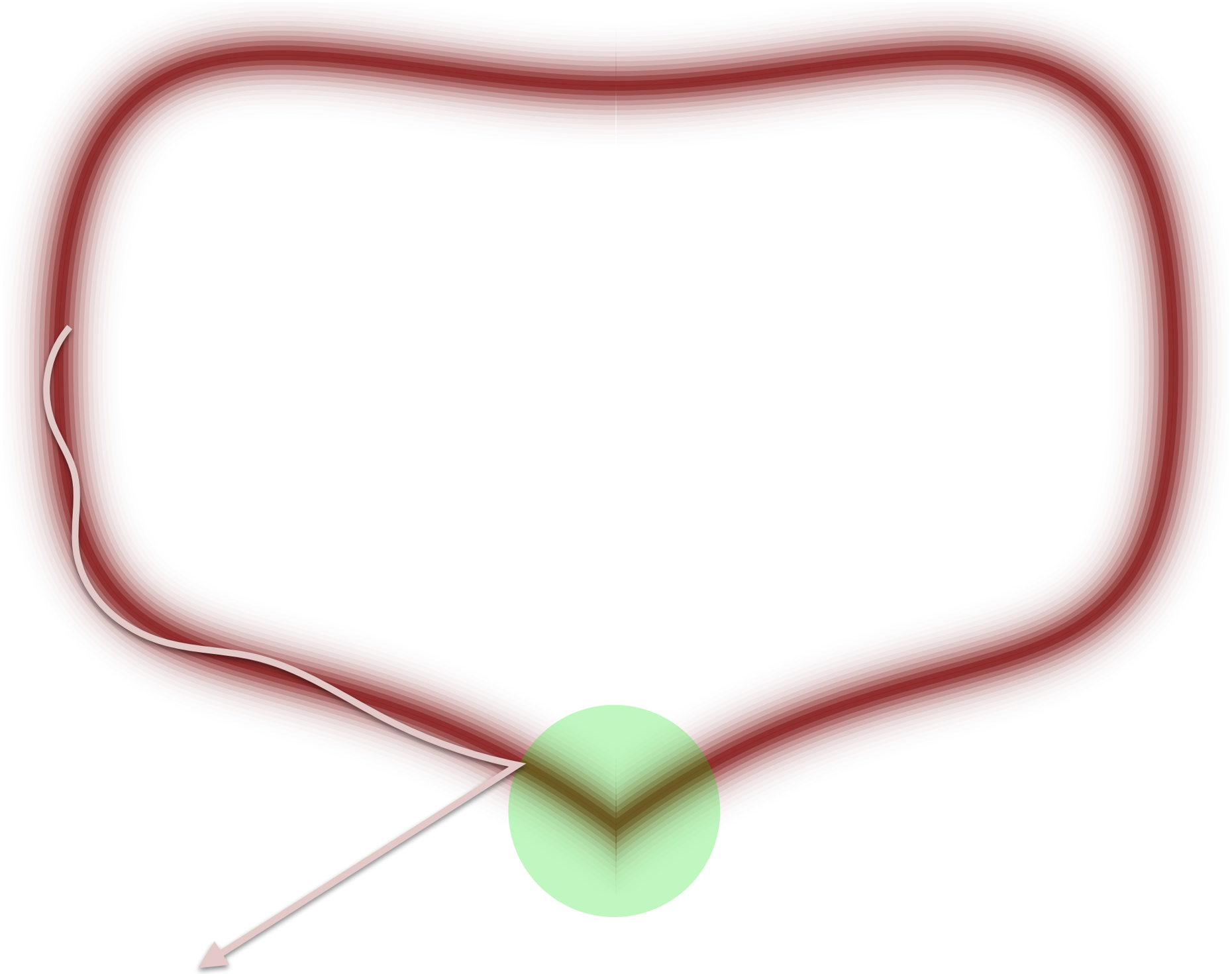
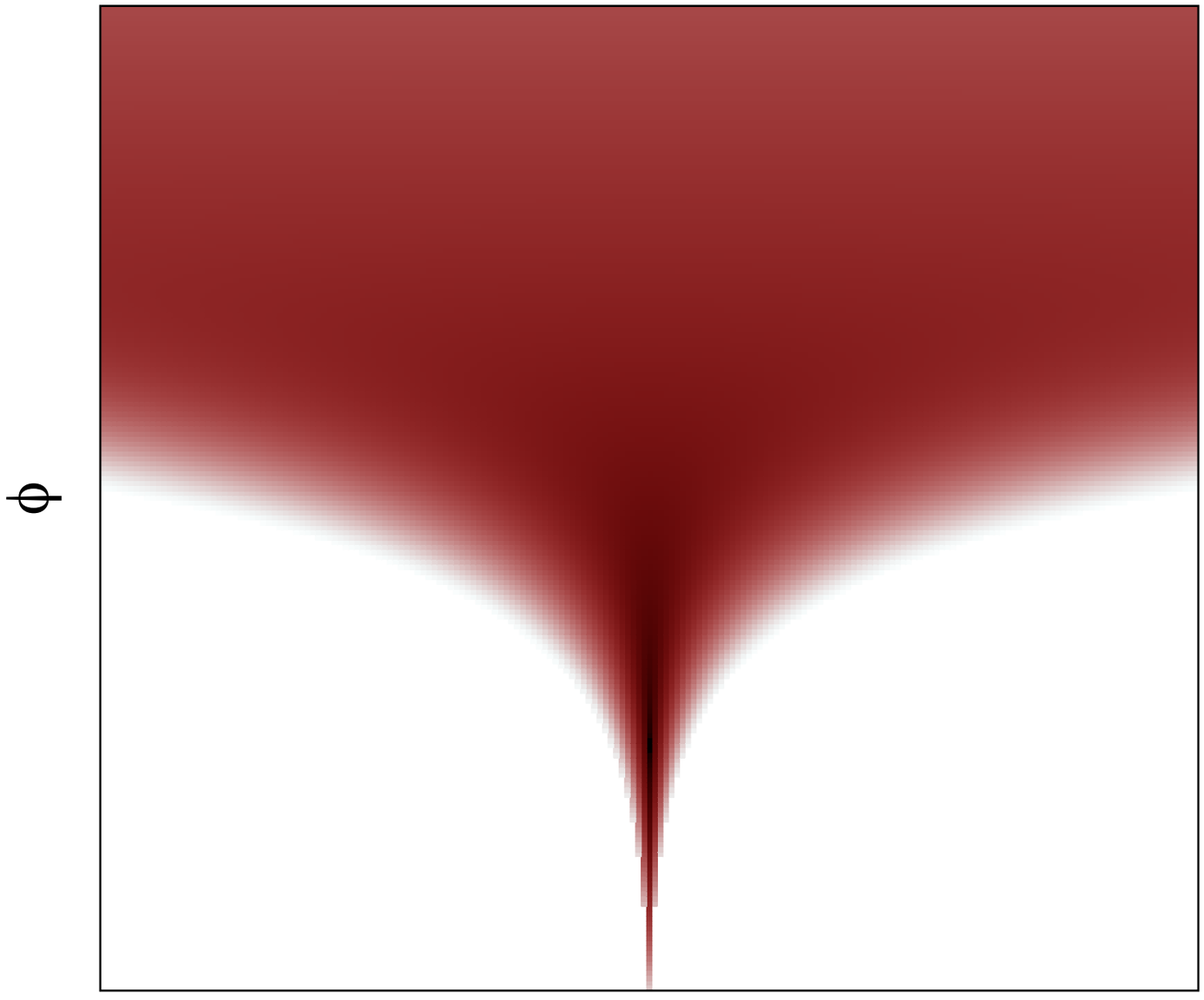


In practice, careful consideration of MCMC diagnostics help to identify poorly chosen parameterizations.





θ_n

Prediction in Hierarchical Models

In hierarchical models the separation between prior and likelihood is ambiguous.

$$\int \mathrm{d}\phi \prod_{n=1}^{N} p(y_n|\theta_n)\, p(\theta_n|\phi)\, p(\phi)$$

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$$\underbrace{\prod_{n=1}^{N} p(y_n|\theta_n)}_{p(\mathbf{y}|\boldsymbol{\theta})} \underbrace{\int \mathrm{d}\phi \prod_{n=1}^{N} p(\theta_n|\phi)\, p(\phi)}_{p(\boldsymbol{\theta})}$$

In hierarchical models the separation between prior and likelihood is ambiguous.

$$\int \mathrm{d}\phi \prod_{n=1}^{N} \underbrace{p(y_n|\theta_n)\, p(\theta_n|\phi)}_{p(y_n,\theta_n|\phi)}\, \underbrace{p(\phi)}_{p(\phi)}$$

Although this doesn't affect the posterior distribution,
it does affect posterior predictive distributions.

Interpretation	**Posterior Predictive Sampling Procedure**
Population as *parameter*	$y_n \sim p(y_n \mid \theta_n)$
Population as *missing data*	$\theta_n \sim p(\theta_n \mid \phi)$ $y_n \sim p(y_n \mid \theta_n)$