Automated Market-Making under Inventory Risk: A Stochastic Optimal Control Framework

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Declaration

I declare that the work contained herein is my own, unless explicitly stated otherwise.

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To mum

Thank you for everything.

Abstract

This project presents a review of the mathematical theory that attempts to model the dynamics of an automated market-maker under inventory risk in financial markets. We begin by outlining financial markets, their participants and their microstructure, before discussing the requisite mathematical tools from probability theory, stochastic analysis, stochastic calculus and stochastic control. Next, we investigate the seminal 2008 paper "High-Frequency Trading in a Limit Order Book" by Avellaneda and Stoikov (2008), which formalises the approach of a market-maker trading through limit orders and utilises the dynamic programming principle to solve for the market-maker's optimal bid and ask quotes. We then present an extension to this model which extends the process governing the underlying stock price from a simple Brownian Motion to the more standard Geometric Brownian Motion. Finally, we implement the model in Python and present it's empirical results when back-tested on real order book data from a cryptocurrency exchange.

Contents

List of Figures

Chapter 1

Introduction and background

1.1 Introduction

This section aims to equip the reader with both the motivation and mathematical tools to begin to formalise problems in stochastic optimal control and mathematical finance, by first providing some background material on financial markets, their participants, and their structure, and then setting up the basic elements of measure-theoretic probability theory and stochastic processes which we will use to describe them. We will end with a discussion of the market-making problem and how we might go about formalising it.

1.2 Financial Markets

A market is simply some social structure that attempts to match those who want to sell a good or service to those who want to buy it. Modern financial markets, thanks to recent innovations such as the internet, satellite communication, and fibre-optic cables, are perhaps the most interconnected and widespread markets in human history.

Most people may have heard of the New York Stock Exchange, London Stock Exchange, or NASDAQ, but these are only one type of exchange for one type of financial asset, namely equity (part ownership of a corporate entity, individually called "stocks" or "shares"). There are also markets for commodities (oil, gas, industrial metals, precious metals, live cattle and more), bonds (pieces of government or corporate debt, where the holder receives fixed interest payments), currencies (including cryptocurrencies), and derivatives which are legal contracts whose value is some function of the price of a specified underlying asset. In total, on an average day, tens of trillions of US dollars worth of assets change hands.

All markets, whatever the good or service being exchanged, have something in common: Every seller needs a buyer, and every buyer needs a seller. But this raises some natural questions: What happens if no-one wants to sell (or buy)? What happens if the only prices at which people are willing to sell is far out of reach of those who want to buy? Enter the *dealer*: An entity who provides *liquidity* (ease of exchange) to market participants. A dealer does this by simultaneously offering to both buy and sell the particular asset, offering to buy at a slightly lower price than they offer to sell. This known as "making a market", and dealers in modern parlance may also be called "market-makers".

Dealers provide a crucial service in financial markets: By providing these quotes, they narrow the *spread* - the difference between the prices at which one can buy or sell an asset in the market. Hence, entities who may need to trade even in adverse market scenarios (such as companies needing to buy foreign currency to pay workers abroad, or

Side	Price /$	Volume
A	1.02	50
A	1.01	30
N/A	1.00	0
B	0.99	25
B	0.98	45

Figure 1.1: An example orderbook

oil producers seeking to hedge their production) know that they can reliably find a buyer or seller, regardless of the uncertainty of other market participants such as *speculators* - those believe that a certain asset is under or overvalued, and trade it with the sole motive of making money buy selling it for more than they bought it or vice-versa.

Of course, there is no free lunch. Dealers do not provide this service to the market out of the goodness of their own hearts - they too have a profit motive. While the presence of dealers in the market narrows the spread, it does not eliminate it. The dealers aim is to be constantly selling the asset for a slightly higher price than it is buying it, and taking the spread as profit. In modern electronic markets with very high trading volumes, even in heavily traded assets with very narrow spreads, a spread of only 0.01$ multiplied across millions or billions of trades can be very lucrative for the dealers who are fast enough.

So far we have discussed markets as an abstract concept, but in order to build a mathematical model of the dealer, we need to specify the framework under which the market operates. Most modern electronic exchanges, including those mentioned above, operate some version of a *limit orderbook* where participants can place two types of orders: a *limit order* or a *market order* depending on their needs. Limit orders specify a side (bid or ask, buying or selling), a quantity (how many units of the asset to buy/sell), and a price at which the order should be executed. These enter a queue of limit orders at the particular price level. Market orders specify the same information, but they do not enter a queue: The exchange operates a *matching engine* which takes incoming market orders and attempts to match them to existing limit orders, and if two orders match, they are executed and a trade occurs.

For an example, consider the orderbook illustrated by If a market order is placed to sell 25 units, then the trade will occur at \$0.99, the dealer will buy and the placer of the market order will sell, and both orders will be removed from the market. However, suppose that a market order is placed to only sell 5 units. In this case, the orders will still be matched, the seller will sell 5 units for \$0.99 apiece but the limit order at \$0.99 will remain on the exchange, only now for 20 units instead of 25. Correspondingly, if the market order is placed for 100 units, the limit orders at both price levels will be hit and taken off the market, and the seller will only sell 70 units for an average price of $\frac{25\times0.99+45\times0.98}{70} = \$\frac{1377}{1400}$ per asset. If a market order is placed and there are no limit orders to match it against, the market order would not be executed at all and be voided.

Here we can see the key difference between market and limit orders in action: Limit orders guarantee price, but do not guarantee that all or any of the order will be filled. Market orders guarantee that as much of the order as possible will be filled, but they do not guarantee the price at which the trade will occur.

We can also observe that the market provides us with a way to estimate the true value of the asset. Classical economic theory dictates that in aggregate, market participants react quickly and rationally to new information about a particular asset, meaning that market prices reflect the consensus opinion of market participants about the value of traded assets.

The spread exists because people would only want to sell for slightly more than something is worth, and buy it for slightly less. Hence, if you really want to buy an asset you have to pay a premium to "*cross the spread*" to acquire it. From this we can determine that the true price of the asset at a point in time lies somewhere in between the maximum bid price and the minimum ask price for the asset at that time. The most common estimator in the literature is simply the average of these two values, but other estimators do exist such as the volume-weighted average price (VWAP) which takes into account the volume of the bids vs asks. For the rest of this paper we will use the midpoint price as our estimator for the "true" value of an asset.

The aim for the rest of this paper is to build up a model of how a dealer should behave to maximise their returns in the presence of uncertainty: namely, uncertainty about the path that the true value of the stock might take. In order to do this, we will need to make use of some basic results from measure/probability theory and stochastic processes, which we will summarise below. We will also briefly introduce some tools from stochastic calculus. Familiarity with standard results from a first-year undergraduate level course in real analysis, probability, and statistics is assumed.

1.3 Measure Theory and Probability

Definition 1.3.1 (σ-algebra)**.** A family $\mathcal{F} \subset\subseteq \mathcal{P}(\Omega)$ of sets is called a σ-algebra if

- $\Omega \in \mathcal{A}$,
- for every countable collection of sets $A_1, A_2, ... \in \mathcal{F}$, $\bigcup_n A_n \in \mathcal{F}$,
- for every $A \in \mathcal{F}, A^c \in \mathcal{F}$.

Remark. The pair $(\Omega, \mathcal{F})$ is called a *measurable space.* Any set $A \in \mathcal{F}$ is called $\mathcal{F}$-*measurable* or simply *measurable.*

Definition 1.3.2. A *measure* μ on a σ-algebra $\mathcal{F}$ is a set function $\mu : \mathcal{F} \to [0, \infty]$ such that $\forall$ mutually disjoint sets $A_1, A_2, ... \in \mathcal{A}$ with $\bigcup_n A_n \in \mathcal{A}$,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \tag{1.1}$$

Remark. If $\mu(\Omega) = 1$ then we call μ a *probability measure*, and often use $\mathbb{P}$ instead. In this case the triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space.*

Lemma 1.3.1. Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ Then $\exists$ a smallest σ-algebra $\sigma(\mathcal{A})$ that contains all sets from $\mathcal{A}$.

Proof. The intersection of σ-algebras is a σ-algebra, so to find the smallest containing some collection of sets, take the intersection of all σ- algebras containing those sets. $\square$

Remark. The above $\sigma(\mathcal{A})$ is usually called the σ-algebra *generated* by $\mathcal{A}$.

Definition 1.3.3 (The Borel σ-algebra)**.** Consider the collection

$$\mathcal{A} = \{(a, b) : a, b \in \mathbb{R} \cup \{-\infty, \infty\}, a < b\}$$

Then define $\mathcal{B}(\mathbb{R}) := \sigma(\mathcal{A})$ the *Borel σ-algebra.* This is the smallest σ-algebra containing all open sets in $\mathbb{R}$. A set $B \in \mathcal{B}$ is a *Borel set.*

Definition 1.3.4 (Measurable functions)**.** Let $(\Omega, \mathcal{F})$ be a measurable space. A function $f : \Omega \to \mathbb{R}$ is *measurable* if for any $B \in \mathcal{B}$,

$$f^{-1}(B) \in \mathcal{F}.$$

Definition 1.3.5 (Simple functions)**.** A *simple function* is a finite linear combination of characteristic (or indicator) functions of measurable sets:

$$\phi = \sum_{i=1}^{n} c_i \chi_{A_i} \tag{1.2}$$

where $c_i \in \mathbb{R}$ and $A_i \in \mathbb{X}$. It is in standard representation if $X = \cup_{i=1}^{n} A_i$, the sets A_i are pairwise disjoint, and the numbers c_i are distinct.

Definition 1.3.6 (Integral of a simple function)**.** Consider a non-negative simple function written in standard form as given above. Then the *integral* of ϕ *with respect to* μ is

$$\int \phi \mathrm{d}\mu := \sum_{i=1}^{n} c_i \mu(A_i) \tag{1.3}$$

which takes values in $\bar{\mathbb{R}}$.

Lemma 1.3.2 (Approximation by simple functions)**.** Let $f \in M(X, \mathbb{X})$, $f \geq 0$. Then there exists a sequence (ϕ_n) in $M(X, \mathbb{X})$ such that

- $0 \leq \phi_n(x) \leq \phi_{n+1}(x) \; \forall \, x \in X, n \in \mathbb{N}$,
- $\lim_{n\to\infty} \phi_n(x) = f(x)$,
- Each ϕ_n is a simple function.

Definition 1.3.7 (Integral of a non-negative measurable function)**.** Let $f \in M^+(X, \mathbb{X})$. Then the *integral* of f *with respect to* μ is

$$\int f \mathrm{d}\mu := \sup \left\{ \int \phi \mathrm{d}\mu : 0 \leq \phi \leq f, \phi \text{ is a simple measurable function} \right\} \in \bar{\mathbb{R}}.$$

Definition 1.3.8 (Integral of a non-negative measurable function over a set)**.** Let $f \in M^+(X, \mathbb{X})$. Then the *integral* of f *with respect to* μ *over set* $A \in \mathbb{X}$ is

$$\int_A f \mathrm{d}\mu := \int f \chi_A \mathrm{d}\mu \tag{1.4}$$

Definition 1.3.9 (Integrable functions)**.** Let $(X, \mathbb{X}, \mu)$ be a measure space. $f : X \to \mathbb{R}$ is *integrable* iff

$$\int f^+ \mathrm{d}\mu < +\infty \text{ and } \int f^- \mathrm{d}\mu < +\infty \tag{1.5}$$

where $f^+ := \max\{f, 0\}$ and $f^- := -\min\{f, 0\}$. We then define

$$\int f \mathrm{d}\mu := \int f^+ \mathrm{d}\mu - \int f^- \mathrm{d}\mu \tag{1.6}$$

and for $A \in \mathbb{X}$

$$\int_A f \mathrm{d}\mu := \int_A f^+ \mathrm{d}\mu - \int_A f^- \mathrm{d}\mu \tag{1.7}$$

Remark. All of the standard properties of integrals that one would expect to hold such as linearity are also true for the Lebesgue integral defined above. The Lebesgue integral also coincides with the Riemann and Regulated integrals for all Riemann-integrable and regulated functions respectively.

Definition 1.3.10 (Random variables)**.** Recall from above that a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is simply a measure space $(X, \mathbb{X}, \mu)$ where $\mu(X) = 1$. In this case, a measurable function $X : \Omega \to \mathbb{R}$ can be called a random variable.

Definition 1.3.11 (Expectation)**.** The notion of the *expectation* of a **random variable** is exactly equivalent to the notion of the *integral* of a **measurable function**. To be precise,

$$\mathbb{E}[X] := \int X \mathrm{d}\mathbb{P} \tag{1.8}$$

Definition 1.3.12 ($\sigma-$algebra generated by a random variable)**.** The σ-algebra generated by a random variable is $\sigma(Y) := \sigma(Y^{-1}(\mathcal{B}(\mathbb{R})))$

Definition 1.3.13 (Conditional Expectation)**.** Suppose $\mathcal{H} \subseteq \mathcal{F}$ is a sub-σ-algebra of $\mathcal{F}$. Then a *conditional expectation* of random variable X given $\mathcal{H}$ is any $\mathcal{H}$-measurable function $\Omega \to \mathbb{R}$ which satisfies

$$\int_H \mathbb{E}[X|H]\mathrm{d}\mathbb{P} = \int_H X\mathrm{d}\mathbb{P} \tag{1.9}$$

for any $H \in \mathcal{H}$. We define the conditional expectation with respect to a random variable as

$$\mathbb{E}[X|Y] := \mathbb{E}[X|\sigma(Y)] \tag{1.10}$$

where $\sigma(Y)$ is the σ-algebra generated by Y.

Theorem 1.3.3 (Tonelli)**.** Let $X_n \geq 0$ be random variables. Then

$$\mathbb{E}\left[\sum_{k=1}^{\infty} X_k\right] = \sum_{k=1}^{\infty} \mathbb{E}[X_k] \tag{1.11}$$

and the statement also holds with an integral instead of a sum.

Theorem 1.3.4 (Fubini)**.** Let X_n be random variables with $\mathbb{E}\left[\sum_{k=1}^{\infty} |X_k|\right] < \infty$. Then

$$\mathbb{E}\left[\sum_{k=1}^{\infty} X_k\right] = \sum_{k=1}^{\infty} \mathbb{E}[X_k] \tag{1.12}$$

and the statement also holds with an integral instead of a sum.

Definition 1.3.14 (Moment Generating Functions)**.** The *Moment Generating Function* (MGF) of a random variable X is defined as follows

$$M_X(t) := \mathbb{E}[e^{tX}] \tag{1.13}$$

Remark. The MGF of the normal distribution is a commonly used tool when dealing with Brownian Motion and functions of Brownian Motion as we will do throughout this report. Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$. Then

$$M_X(t) = \mathbb{E}[e^{tX}] = e^{t\mu + \frac{t^2\sigma^2}{2}} \tag{1.14}$$

1.4 Stochastic Processes

Definition 1.4.1 (Filtrations & Adaptedness)**.** A *filtered space* is $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^{\infty}, \mathbb{P})$ where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}$ are σ-algebras, jointly called a *filtration.* We also define $\mathcal{F}_\infty := \sigma(\cup_n \mathcal{F}_n) \subseteq \mathcal{F}$. We say a stochastic process or sequence of random variables X_n is adapted to the filtration $(\mathcal{F}_n)_{n\geq 0}$ if for every n, X_n is $\mathcal{F}_n$ -measurable.

Definition 1.4.2 (Martingales)**.** A process $(M_n)_{n\geq 0}$ in a filtered probability space is a *martingale with respect to a filtration* $(\mathcal{F}_n)_{n\geq 0}$ if

- M_n is adapted to $\mathcal{F}_n$,
- $\mathbb{E}[M_n] < \infty \; \forall \, n \geq 0$,
- $\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n$ a.s. $\forall \, n \geq 0$.

Definition 1.4.3 (Poisson Process)**.** Let $\lambda : \mathbb{R}^d \to [0, \infty)$ be a measurable and integrable function such that for every bounded region B the d-dimensional volume integral of λ is finite:

$$\Lambda(B) = \int_B \lambda(x) \mathrm{d}x < \infty \tag{1.15}$$

Then for every collection of disjoint bounded Borel-measurable sets $B_1, \ldots, B_k$, an inhomogeneous *Poisson Point Process* with *intensity function* λ has distribution

$$\mathbb{P}\{N(B_i) = n_i, i = 1, \ldots, k\} = \prod_{i=1}^{k} \frac{(\Lambda(B_i))^{n_i}}{n_i!} e^{-\Lambda(B_i)}. \tag{1.16}$$

Moreover,

$$\mathbb{E}[N(B)] = \Lambda(B). \tag{1.17}$$

Definition 1.4.4 (Brownian Motion)**.** Let $\mathcal{F}_t$ be a filtration. A stochastic process $W = (W_t)_{t\geq 0}$ is a standard one-dimensional *Brownian Motion* or *Wiener Process* if it satisfies the following:

- $W_0 = 0$ a.s.,
- Independent increments: $W_{t+s} - W_t$ is independent of $\mathcal{F}_t \; \forall \, t, s \geq 0$,
- W has stationary Gaussian increments: $W_{t+s} - W_t \sim \mathcal{N}(0, s)$,
- W has continuous sample paths: $W_t(\omega)$ is a continuous function of $t \; \forall \, \omega \in \Omega$.

Definition 1.4.5 (Predictable Processes)**.** A stochastic process X_t is *predictable* (in the discrete sense) if X_{t+1} is $\mathcal{F}_t$ measurable for all t. If X_t is a continous stochastic process, then it is predictable if it is measurable with respect to the σ-algebra generated by all left-continuous adapted processes.

Definition 1.4.6 (Progressive Measurability)**.** A continuous-time stochastic process (X_t) is progressively measurable if for every time t, the map $[0, t] \times \Omega \to \mathbb{R}$ defined by $(s, \omega) \to X_s(\omega)$ is $\mathcal{B}(\mathbb{R}) \times \mathcal{F}_t$-measurable. This is a slightly stronger condition than adaptedness, indeed, all progressively measurable processes are adapted but the converse is not true.

1.5 Stochastic Integration

- Stochastic integral
- Itos lemma

1.6 Stochastic Differential Equations

- From PDE to SDE
- Brownian motion as the solution to an SDE
- Geometric Brownian Motion

Definition 1.6.1 (Strong Solution)**.** A strong solution to this SDE starting at time t is a progressively measurable process X such that for $s \leq t$:

$$X_s = X_t + \int_t^s b(X_u, \alpha_u)du + \int_t^s \sigma(X_u, \alpha_u)dW_u$$

and

$$\int_t^s |b(X_u, \alpha_u)|du + \int_t^s |\sigma(X_u, \alpha_u)|^2 du < \infty$$

a.s.

Definition 1.6.2 (Geometric Brownian Motion)**.** A *geometric brownian motion* is an adapted stochastic process which solves the following stochastic differential equation

$$dS_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t \tag{1.18}$$

for $\mu, \sigma \in \mathbb{R}$ and where W is a standard Wiender process. By Itô's formula with $f(S_t) = \log S_t$, we can write

$$\begin{aligned}
\mathrm{d}f(S_t) &= \frac{1}{S_t}\mathrm{d}S_t - \frac{1}{2S_t^2}(\mathrm{d}S_t)^2 \\
&= \mu \mathrm{d}t + \sigma \mathrm{d}W_t - \frac{1}{2}\sigma^2 \mathrm{d}t \\
\implies \log S_t &= \log S_0 + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t \\
\implies S_t &= S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}
\end{aligned}$$

and we arrive at the canonical formula for the GBM, where S_0 is the initial value of the process.

Chapter 2

Stochastic Optimal Control

2.1 Introduction

In this chapter we introduce the idea of a stochastic control problem in one dimension, and construct a theoretical framework for the resolution of a regular solution, provided that such a solution exists (which is not guarunteed). We Primarily follow the text of Pham 2009. In section 2.2 we introduce the notion of a controlled diffusion process and its solution. In section 2.3 we consider a stochastic control problem over a finite time horizon, before introducing the dynamic programming principle and Hamilton-Jacobi-Bellman equation in section 2.4 and section 2.5 respectively. Finally, in section 2.6, we consider the verification theorem which allows us to validate the optimality of a candidate solution. We then procede in 2.7 to put these tools to use through a worked example in a financial context, setting us up to tackle the Avellaneda-Stoikov model in Chapter 3.

2.2 Controlled Diffusion Processes

The following all will rely on the existence of a standard filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^{\infty}, \mathbb{P})$ as defined above in chapter 1.

Definition 2.2.1 (Controlled Diffusion Process)**.** We consider a control model where the state of the system is governed by an $\mathbb{R}$-valued SDE:

$$dX_t = b(t, X_t, \alpha_t)ds + \sigma(t, X_t, \alpha_t)dW_t \tag{2.1}$$

where W is a standard Wiener process. The control $\alpha = (\alpha_t)$ is a progressively measurable process valued in $A \subseteq \mathbb{R}^m$.

The functions $b : \mathbb{R}^+ \times \mathbb{R} \times A \to \mathbb{R}$ and $\sigma : \mathbb{R}^+ \times \mathbb{R} \times A \to \mathbb{R}$ are measurable in all of their arguments and satisfy a uniform Lipschitz condition in A: There exists a $K \geq 0$ such that $\forall\, x, y \in \mathbb{R}, \forall\, a \in A$,

$$|b(x, a) - b(y, a)| + |\sigma(x, a) - \sigma(y, a)| \leq K|x - y|. \tag{2.2}$$

In what follows, for $0 \leq t \leq T < \infty$, we denote by $\mathcal{T}_{t,T}$ the set of stopping times valued in $[t, T]$.

2.3 The Finite-Horizon Problem

Fix a finite horizon $0 < T < \infty$. We denote by $\mathcal{A}$ the set of control processes α such that for any arbitrary $x \in \mathbb{R}$,

$$\mathbb{E}\left[\int_0^T |b(x,\alpha_t)|^2 + |\sigma(x,\alpha_t)|^2 \mathrm{d}t\right] < \infty. \tag{2.3}$$

From Chapter 1, conditions (2.2) and (2.3) ensure the existence and uniqueness of a strong solution to the SDE (2.1) starting from any initial condition $(t,x) \in [0,T] \times \mathbb{R}$ and with any control process $\alpha \in \mathcal{A}$. We denote this unique strong solution with almost surely continuous sample paths by $\{X_s^{t,x}, t \le s \le T\}$.
Pham includes extra technical properties of the strong solution here which I may not require.
Next we set out our functional objective. Let $f : [0,T] \times \mathbb{R} \times A \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two measurable functions. We suppose that:

- g is lower-bounded **or**
- g satisfies a quadratic growth condition: $|g(x)| \le C(1 + |x|^2)\ \forall\ x \in \mathbb{R}$ for some constant C independent of x.

We also denote by $\mathcal{A}(t,x)$ the subset of controls $\alpha \in \mathcal{A}$ such that

$$\mathbb{E}\left[\int_t^T |f(s, X_x^{t,x}, \alpha_s)| \mathrm{d}s\right] < \infty \tag{2.4}$$

for $(t,x) \in [0,T] \times \mathbb{R}$, and we assume that this set is not empty for all $(t,x) \in [0,T] \times \mathbb{R}$. We now define the *gain function:*

Definition 2.3.1 (Gain Function)**.**

$$J(t,x,\alpha) := \mathbb{E}\left[\int_t^T f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + g(X_T^{t,x})\right] \tag{2.5}$$

for all $(t,x) \in [0,T] \times \mathbb{R}$ and $\alpha \in \mathcal{A}(t,x)$.

Our objective is thus to maximise over possible control processes the gain function J, and to do this we introduce the associated *value function*:

Definition 2.3.2 (Value Function)**.**

$$v(t,x) := \sup_{\alpha \in \mathcal{A}(t,x)} J(t,x,\alpha). \tag{2.6}$$

Definition 2.3.3 (Optimal control)**.** Given an initial condition $(t,x) \in [0,T] \times \mathbb{R}$, we say that $\hat{\alpha} \in \mathcal{A}(t,x)$ is an optimal control if

$$v(t,x) = J(t,x,\hat{\alpha}). \tag{2.7}$$

Remark. A control process α of the form $\alpha_s = a(s, X_s^{t,x})$ for some measurable function $a : [0,T] \times \mathbb{R} \to A$ is called a *Markovian* control.

Pham includes a remark here about constant controls and conditions for the equivalence of $\mathcal{A}$ and $\mathcal{A}(t,x)$ which I probably don't need.
Include words on the interpretation and intuition of the f, g, the gain and value functions

2.4 The Dynamic Programming Principle

The Dynamic Programming Principle (DPP) is the fundamental tool upon which much of the theory of stochastic control relies. We formulate it as follows, considering only the context of the finite-horizon problem described above.

Theorem 2.4.1 (Dynamic Programming Principle)**.** Let $(t, x) \in [0, T] \times \mathbb{R}$. Then we have

$$v(t,x) = \sup_{\alpha \in \mathcal{A}(t,x)} \sup_{\theta \in \mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right] \tag{2.8}$$

$$= \sup_{\alpha \in \mathcal{A}(t,x)} \inf_{\theta \in \mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right] \tag{2.9}$$

$$\tag{2.10}$$

Proof of the DPP. By pathwise uniqueness of the SDE for X, for any admissible control $\alpha \in \mathcal{A}(t, x)$, for any $\theta \in \mathcal{T}_{t,T}$ and for all $s \geq \theta$

$$X_s^{t,x} = X_s^{\theta, X_\theta^{t,x}}. \tag{2.11}$$

By the law of iterated expectations we then have

$$\begin{aligned}
J(t,x,\alpha) &= \mathbb{E}\left[\int_t^T f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + g(X_T^{t,x})\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\int_t^T f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + g(X_T^{t,x}) | \mathcal{F}_\theta\right]\right] \\
&= \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + \mathbb{E}\left[\int_\theta^T f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + g(X_T^{t,x}) | \mathcal{F}_\theta\right]\right] \\
&= \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + \mathbb{E}\left[\int_\theta^T f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + g(X_T^{t,x})\right]\right] \\
&= \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + J(\theta, X_\theta^{t,x}, \alpha)\right]
\end{aligned}$$

and since $J(.,.,\alpha) \leq v$ and θ is arbitrary in $\mathcal{T}_{t,T}$ we obtain

$$\begin{aligned}
J(t,x,\alpha) &\leq \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right] \\
&\leq \inf_{\theta \in \mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right] \\
&\leq \sup_{\alpha \in \mathcal{A}(t,x)} \inf_{\theta \in \mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right]
\end{aligned}$$

and by taking the supremum over α in the left hand side, we obtain the second of the desired inequalities:

$$v(t,x) \leq \sup_{\alpha \in \mathcal{A}(t,x)} \inf_{\theta \in \mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right]. \tag{2.12}$$

Next we fix an arbitrary control $\alpha \in \mathcal{A}(t, x)$ and $\theta \in \mathcal{T}_{t,T}$. By the definition of the value function, for any $\epsilon > 0$ and $\omega \in \Omega$ there exists an $\alpha^{\epsilon,\omega} \in \mathcal{A}(\theta(\omega), X_{\theta(\omega)}^{t,x}(\omega))$ that is an ϵ-optimal control for $v(\theta, X_{\theta(\omega)}^{t,x}(\omega))$, i.e.

$$v(\theta, X_{\theta(\omega)}^{t,x}(\omega)) - \epsilon \leq J(\theta(\omega), X_{\theta(\omega)}^{t,x}(\omega), \alpha^{\epsilon,\omega}). \tag{2.13}$$

Maybe include some more intuition behind this point?
We now define the process

$$\hat{\alpha}_s(\omega) = \begin{cases} \alpha_s(\omega), s \in [0, \theta(\omega)], \\ \alpha_s^{\epsilon,\omega}(\omega), s \in [\theta(\omega), T]. \end{cases} \tag{2.14}$$

It can be shown by the measurable selection theorem
do this in the appendix
that the process $\hat{\alpha}$ is progressively measurable, and so lies in $\mathcal{A}(t,x)$. Again by the law of iterated expectations and (2.13) we get

$$\begin{aligned} v(t,x) \geq J(t,x,\hat{\alpha}) &= \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + J(\theta, X_\theta^{t,x}, \alpha^\epsilon)\right] \\ &\geq \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right] - \epsilon. \end{aligned}$$

Finally, by the fact that $\alpha \in \mathcal{A}(t,x), \theta \in \mathcal{T}_{t,T}$ and $\epsilon > 0$ are all arbitrary, we obtain the first inequality:

$$v(t,x) = \sup_{\alpha \in \mathcal{A}(t,x)} \sup_{\theta \in \mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right]. \tag{2.15}$$

□

Remark (Equivalent Formulations). We normally write the DPP as

$$v(t,x) = \sup_{\alpha \in \mathcal{A}(t,x)} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right], \tag{2.16}$$

however it is sometimes useful to use the following equivalent formulation of the DPP:
(i) For all $\alpha \in \mathcal{A}(t,x)$ and $\theta \in \mathcal{T}_{t,T}$:

$$v(t,x) \geq \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right]. \tag{2.17}$$

(ii) For all $\epsilon > 0$, there exists $\alpha \in \mathcal{A}(t,x)$ such that for all $\theta \in \mathcal{T}_{t,T}$:

$$v(t,x) - \epsilon \leq \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right]. \tag{2.18}$$

Include words on the interpretation and intuition of the DPP.

2.5 Hamilton-Jacobi-Bellman Equation

The Dynamic Programming Principle tells us that we can consider a stochastic control problem as a sequence of smaller sub-problems defined over intervals of $[0,T]$ characterised by stopping times, i.e., $[0,T] = [0,\theta_1] \cup (\theta_1, \theta_2] \cup \cdots \cup (\theta_n, T]$ where $\theta_1 \leq \cdots \leq \theta_n \in \mathcal{T}_{t,T}$. Thus, a natural thing to consider is the following: What happens as $n \to \infty$ and correspondingly $\theta_{i+1} - \theta_i \to 0$? What we obtain is the Hamilton-Jacobi-Bellman equation (HJB) which describes the dynamics of the value function over small increments of time. In this chapter and what follows, we will use the HJB equation as follows:

- Provide a formal derivation of the HJB equation.

- Obtain or try to show the existence of a smooth solution.
- Verification step: Show that the smooth solution is the value function.
- As a byproduct, we obtain an optimal feedback control.

Theorem 2.5.1 (Hamilton-Jacobi-Bellman Equation)**.** The dynamics of the value function $v(t, x)$ satisfy the following non-linear second-order partial differential equation:

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) + \sup\limits_{a \in A} \left[\mathcal{L}^a v(t,x) + f(t,x,a)\right] = 0 \ \forall \ (t,x) \in [0,T) \times \mathbb{R} \\ v(T,x) = g(x) \ \forall \ x \in \mathbb{R}. \end{cases} \tag{2.19}$$

where $\mathcal{L}^a$ is the operator associated to the diffusion (2.1) and defined by (see Section 1.6)

$$\mathcal{L}^a v = b(t,x,a) v_x + \frac{1}{2}\sigma(t,x,a)^2 v_{xx}. \tag{2.20}$$

Proof. Let us consider time $\theta = t + h$ and a constant control $\alpha_s = a$ for some arbitrary $a \in A$, in our slightly stronger variant of the DPP (2.17):

$$v(t,x) \geq \mathbb{E}\left[\int_t^{t+h} f(s, X_s^{t,x}, a)\mathrm{d}s + v(t+h, X_{t+h}^{t,x})\right]. \tag{2.21}$$

By assuming that v is smooth enough, we can apply Itô's formula between t and $t+h$:

$$v(t+h, X_{t+h}^{t,x}) = v(t,x) + \int_t^{t+h} \left(\frac{\partial v}{\partial t} + \mathcal{L}^a v\right)(s, X_s^{t,x})\mathrm{d}s + \text{(local) martingale}. \tag{2.22}$$

We can then substitute back into (2.21) to obtain

$$0 \geq \mathbb{E}\left[\int_t^{t+h} \left(\frac{\partial v}{\partial t} + \mathcal{L}^a v\right)(s, X_s^{t,x}) + f(s, X_s^{t,x}, a)\mathrm{d}s\right] \tag{2.23}$$

which if we divide by h and send $h \to 0$ we yield

$$0 \geq \frac{\partial v}{\partial t}(t,x) + \mathcal{L}^a v(t,x) + f(t,x,a) \tag{2.24}$$

by the mean-value theorem. Since this holds true for any $a \in A$, we obtain the inequality

$$-\frac{\partial v}{\partial t}(t,x) - \sup_{a \in A}[\mathcal{L}^a v(t,x) + f(t,x,a)] \geq 0. \tag{2.25}$$

On the other hand, suppose that α^* is an optimal control. Then in (2.16) we have

$$v(t,x) = \mathbb{E}\left[\int_t^{t+h} f(s, X_s^*, \alpha_s^*)\mathrm{d}s + v(t+h, X_{t+h}^*)\right], \tag{2.26}$$

where X^* is the solution to (2.1) starting from state x at time t with control α^*. Again by Itô's formula we have that

$$v(t+h, X_{t+h}^*) = v(t,x) + \int_t^{t+h} \left(\frac{\partial v}{\partial t} + \mathcal{L}^a v\right)(s, X_s^*)\mathrm{d}s + \text{(local) martingale} \tag{2.27}$$

which we can again substitute back into (2.26) to obtain

$$0 = \mathbb{E}\left[\int_t^{t+h} \left(\frac{\partial v}{\partial t} + \mathcal{L}^a v\right)(s, X_s^*) + f(s, X_s*, a)\mathrm{d}s\right] \tag{2.28}$$

and hence once again we divide by h and send $h \to 0$ yielding

$$-\frac{\partial v}{\partial t}(t,x) - \mathcal{L}^{\alpha_t^*} v(t,x) - f(t,x,\alpha_t^*) = 0. \tag{2.29}$$

Combining this with (2.25), v should satisfy

$$-\frac{\partial v}{\partial t}(t,x) - \sup_{a \in A} \left[\mathcal{L}^a v(t,x) + f(t,x,a)\right] = 0 \;\forall\; (t,x) \in [0,T) \times \mathbb{R}, \tag{2.30}$$

if the above supremum in a is finite. This may arise when the control space A is unbounded, and we will see how to deal with this later on. We can also obtain the terminal condition associated to this PDE:

$$v(T,x) = g(x) \;\forall\; x \in \mathbb{R} \tag{2.31}$$

which results immediately from the definition in (2.6) of the value function considered at the horizon time T. □

2.6 Verification Theorem

2.7 A Worked Example

Chapter 3

The Avellaneda-Stoikov Model

In this section we will walk through the methodology and theoretical results of Avellaneda and Stoikov (Avellaneda and Stoikov 2008)

3.1 Model assumptions

- The dealer being modelled is one of many players in the market
- The 'true' price is given by the market mid-price
- The money-market pays no interest
- The agent has no opinion on drift or autocorrelation of the stock price
- Limit orders can be continuously updated at no cost
- The arrival frequency of market orders to the market is constant
- Limit orders are of fixed size 1

Assume the stock evolves according to a standard Wiener process with some variance σ^2:

$$dS_u = \sigma dW_u$$

3.2 Modelling an inactive trader

The utility function

Consider an inactive trader who holds an inventory of q stocks until the terminal time T. The agent's value function is

$$v(x, s, q, t) = \mathbb{E}_t \left(-e^{-\gamma(x+qS_T)} \right)$$

where x is the initial wealth in dollars, t is the present time and γ is a user-defined risk-aversion parameter. By some simple manipulations, we can write this in a more convenient form as follows:

$$\begin{aligned}
v(x,s,q,t) &= \mathbb{E}_t\left(-e^{-\gamma(x+qS_T)}\right)\\
&= \mathbb{E}\left[-\exp(-\gamma(x+qS_T))|\mathcal{F}_t\right]\\
&= -e^{-\gamma x}\mathbb{E}\left[e^{-\gamma q S_T}\right]\\
&= -e^{-\gamma x}e^{-\gamma q s+\frac{\gamma^2 q^2\sigma^2(T-t)}{2}}\\
&= -e^{-\gamma x}e^{-\gamma q s}e^{\frac{\gamma^2 q^2\sigma^2(T-t)}{2}}
\end{aligned}$$

Reservation prices

The reservation price is the price that would make the agent indifferent between his current portfolio and his current portfolio $\pm$ one stock. So r^b can be determined from the relation

$$v(x-r^b(s,q,t),s,q+1,t)=v(x,s,q,t) \tag{3.1}$$

and r^a solves

$$v(x+r^a(s,q,t),s,q-1,t)=v(x,s,q,t). \tag{3.2}$$

We solve for $r^b(s,q,t)$ by plugging 3.1 in to our expression for the value function as follows:

$$\begin{aligned}
v(x-r^b(s,q,t),s,q+1,t) &= v(x,s,q,t)\\
-e^{-\gamma(x-r^b(s,q,t))}e^{-\gamma s(q+1)}e^{\frac{\gamma^2(q+1)^2\sigma^2(T-t)}{2}} &= -e^{-\gamma x}e^{-\gamma q s}e^{\frac{\gamma^2 q^2\sigma^2(T-t)}{2}}\\
-\gamma(x-r^b(s,q,t))-\gamma s(q+1)+\frac{\gamma^2(q+1)^2\sigma^2(T-t)}{2} &= -\gamma x-\gamma q s+\frac{\gamma^2 q^2\sigma^2(T-t)}{2}\\
\gamma r^b(s,q,t)-\gamma s+\frac{\gamma^2(1+2q)\sigma^2(T-t)}{2} &= 0,
\end{aligned}$$

dividing by γ and rearranging to obtain

$$r^b(s,q,t)=s+(-1-2q)\frac{\gamma\sigma^2(T-t)}{2} \tag{3.3}$$

Similarly for $r^a(s,q,t)$:

$$\begin{aligned}
v(x+r^a(s,q,t),s,q-1,t) &= v(x,s,q,t)\\
-e^{-\gamma(x+r^a(s,q,t))}e^{-\gamma s(q-1)}e^{\frac{\gamma^2(q-1)^2\sigma^2(T-t)}{2}} &= -e^{-\gamma x}e^{-\gamma q s}e^{\frac{\gamma^2 q^2\sigma^2(T-t)}{2}}\\
-\gamma(x+r^a(s,q,t))-\gamma s(q-1)+\frac{\gamma^2(q-1)^2\sigma^2(T-t)}{2} &= -\gamma x-\gamma q s+\frac{\gamma^2 q^2\sigma^2(T-t)}{2}\\
-\gamma r^a(s,q,t)+\gamma s+\frac{\gamma^2(1-2q)\sigma^2(T-t)}{2} &= 0,
\end{aligned}$$

dividing by γ and rearranging to obtain

$$r^a(s,q,t)=s+(1-2q)\frac{\gamma\sigma^2(T-t)}{2} \tag{3.4}$$

We define the "reserve" or "indifference" price to be the average of these two *given* that the agent currently holds q stocks:

$$
\begin{aligned}
r(s,q,t) &= \frac{r^a(s,q,t)+r^b(s,q,t)}{2} \\
&= \frac{s+(1-2q)\frac{\gamma\sigma^2(T-t)}{2}+s+(-1-2q)\frac{\gamma\sigma^2(T-t)}{2}}{2} \\
&= \frac{2s-2q\gamma\sigma^2(T-t)}{2} \\
&= s-q\gamma\sigma^2(T-t)
\end{aligned}
$$

3.3 The Optimising Agent with Infinite Horizon

$$
\bar{v}(x,s,q)=\mathbb{E}\left[\int_0^\infty -e^{-\omega t}e^{-\gamma(x+qS_t)}dt\right]
$$

$$
\begin{aligned}
\bar{v}(x-\bar{r}^b(s,q),s,q+1) &= \bar{v}(x,s,q) \\
\mathbb{E}\left[\int_0^\infty -e^{-\omega t}e^{-\gamma(x-\bar{r}^b(s,q)+(q+1)S_t)}\mathrm{d}t\right] &= \mathbb{E}\left[\int_0^\infty -e^{-\omega t}e^{-\gamma(x+qS_t)}\mathrm{d}t\right] \\
\int_0^\infty e^{-\omega t}e^{-\gamma(x-\bar{r}^b(s,q))}\mathbb{E}\left[e^{-\gamma(q+1)S_t}\right]\mathrm{d}t &= \int_0^\infty e^{-\omega t}e^{-\gamma x}\mathbb{E}\left[e^{-\gamma qS_t}\right]\mathrm{d}t \text{ (by Tonelli)} \\
e^{-\gamma(x-\bar{r}^b(s,q))}\int_0^\infty e^{-\omega t}e^{-\gamma(q+1)s+\frac{\gamma^2(q-1)^2\sigma^2t}{2}}\mathrm{d}t &= e^{-\gamma x}\int_0^\infty e^{-\omega t}e^{-\gamma qs+\frac{\gamma^2q^2\sigma^2t}{2}}\mathrm{d}t \\
e^{-\gamma(x-\bar{r}^b(s,q))}e^{-\gamma(q+1)s}\int_0^\infty e^{-\omega t}e^{\frac{\gamma^2(q+1)^2\sigma^2t}{2}}\mathrm{d}t &= e^{-\gamma x}e^{-\gamma qs}\int_0^\infty e^{-\omega t}e^{\frac{\gamma^2q^2\sigma^2t}{2}}\mathrm{d}t \\
e^{\gamma\bar{r}^b(s,q)}e^{-\gamma s}\int_0^\infty e^{\left(\frac{\gamma^2(q+1)^2\sigma^2-2\omega}{2}\right)t}\mathrm{d}t &= \int_0^\infty e^{\left(\frac{\gamma^2q^2\sigma^2-2\omega}{2}\right)t}\mathrm{d}t \\
e^{\gamma\bar{r}^b(s,q)}e^{-\gamma s}\left(\frac{2}{2\omega-\gamma^2(q+1)^2\sigma^2}\right) &= \left(\frac{2}{2\omega-\gamma^2q^2\sigma^2}\right) \\
e^{\gamma(\bar{r}^b(s,q)-s)} &= \frac{2\omega-\gamma^2(q+1)^2\sigma^2}{2\omega-\gamma^2q^2\sigma^2} \\
e^{\gamma(\bar{r}^b(s,q)-s)} &= 1-\frac{(1+2q)\gamma^2\sigma^2}{2\omega-\gamma^2q^2\sigma^2} \\
\gamma\bar{r}^b(s,q)-\gamma s &= \log\left(1+\frac{(-1-2q)\gamma^2\sigma^2}{2\omega-\gamma^2q^2\sigma^2}\right) \\
\bar{r}^b(s,q) &= s+\frac{1}{\gamma}\log\left(1+\frac{(-1-2q)\gamma^2\sigma^2}{2\omega-\gamma^2q^2\sigma^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\bar{v}(x + r^a(s,q), s, q-1) &= \bar{v}(x,s,q) \\
\mathbb{E}\left[\int_0^\infty -e^{-\omega t} e^{-\gamma(x + r^a(s,q) + (q-1)S_t)} \mathrm{d}t\right] &= \mathbb{E}\left[\int_0^\infty -e^{-\omega t} e^{-\gamma(x+qS_t)} \mathrm{d}t\right] \\
\int_0^\infty e^{-\omega t} e^{-\gamma(x + r^a(s,q))} \mathbb{E}\left[e^{-\gamma(q-1)S_t}\right] \mathrm{d}t &= \int_0^\infty e^{-\omega t} e^{-\gamma x} \mathbb{E}\left[e^{-\gamma q S_t}\right] \mathrm{d}t \text{ (by Tonelli)} \\
e^{-\gamma(x + r^a(s,q))} \int_0^\infty e^{-\omega t} e^{-\gamma(q-1)s + \frac{\gamma^2(q-1)^2\sigma^2 t}{2}} \mathrm{d}t &= e^{-\gamma x} \int_0^\infty e^{-\omega t} e^{-\gamma q s + \frac{\gamma^2 q^2 \sigma^2 t}{2}} \mathrm{d}t \\
e^{-\gamma(x + r^a(s,q))} e^{-\gamma(q-1)s} \int_0^\infty e^{-\omega t} e^{\frac{\gamma^2(q-1)^2\sigma^2 t}{2}} \mathrm{d}t &= e^{-\gamma x} e^{-\gamma q s} \int_0^\infty e^{-\omega t} e^{\frac{\gamma^2 q^2 \sigma^2 t}{2}} \mathrm{d}t \\
e^{-\gamma r^a(s,q)} e^{\gamma s} \int_0^\infty e^{\left(\frac{\gamma^2(q-1)^2\sigma^2 - 2\omega}{2}\right)t} \mathrm{d}t &= \int_0^\infty e^{\left(\frac{\gamma^2 q^2 \sigma^2 - 2\omega}{2}\right)t} \mathrm{d}t \\
e^{-\gamma r^a(s,q)} e^{\gamma s} \left(\frac{2}{2\omega - \gamma^2(q-1)^2\sigma^2}\right) &= \left(\frac{2}{2\omega - \gamma^2 q^2 \sigma^2}\right) \\
e^{\gamma(s - r^a(s,q))} &= \frac{2\omega - \gamma^2(q-1)^2\sigma^2}{2\omega - \gamma^2 q^2 \sigma^2} \\
e^{\gamma(s - r^a(s,q))} &= 1 - \frac{(1-2q)\gamma^2\sigma^2}{2\omega - \gamma^2 q^2 \sigma^2} \\
\gamma s - \gamma r^a(s,q) &= \log\left(1 - \frac{(1-2q)\gamma^2\sigma^2}{2\omega - \gamma^2 q^2 \sigma^2}\right) \\
r^a(s,q) &= s - \frac{1}{\gamma}\log\left(1 - \frac{(1-2q)\gamma^2\sigma^2}{2\omega - \gamma^2 q^2 \sigma^2}\right)
\end{aligned}
$$

3.4 Modelling Limit Orders

The agent quotes the bid price p^b and the ask price p^a, and the current shape of the limit orderbook as well as the distances

$$\delta^b := s - p^b$$

and

$$\delta^a := p^a - s$$

determine the priority of execution when large market orders are placed. E.g. when a market order to buy Q shares arrives, the Q limit orders with the lowest ask prices will be lifted. Let p^Q be the price of the highest limit order executed in this trade. Then

$$\Delta p := p^Q - s$$

is the temporary market impact of the trade of size Q. Then we have that if our $\delta^a < \Delta p$, our agents limit order will be executed. We assume that market orders will fill our limit orders at Poisson rates $\lambda^a(\delta^a)$ and $\lambda^b(\delta^b)$, decreasing functions of δ^a and δ^b resp. (further away from midpoint $\rightarrow$ orders hit less often).

We now have stochastic wealth and inventory: Let N_t^b and N_t^a be Poisson processes with intensities λ^b and λ^a representing the amount of stocks bought/sold by the agent at time t. The inventory at time t is

$$q_t = N_t^b - N_t^a$$

and the wealth process evolves according to

$$dXt = p^a dN_t^a - p^b dN_t^b.$$

The objective of the agent who sets limit orders is

$$u(s,x,q,t) = \max_{\delta^a,\delta^b} \mathbb{E}_t\left[-e^{-\gamma(X_T + q_T S_T)}\right]$$

3.5 Modelling Trading Intensity

Assume constant frequency Λ of market orders. We want to determine some realistic functional forms for the relationship between the Poisson intensity λ and distance to mid-price δ. To do this we need information on: (i) the overall frequency of market orders, (ii) the distribution of their size, (iii) the temporary impact of a large market order.

Distribution of the size of market orders

The distribution of size of market orders has been found to obey a power law:

$$f^Q(x) \propto x^{-1-\alpha} \tag{3.5}$$

for large x, with $\alpha \in [1.4, 1.6]$.

Modelling market impact

Less consensus on market impact. Some find change in price Δp after market order size Q given by

$$\Delta p \propto Q^\beta, \beta \in [0.5, 0.8] \tag{3.6}$$

while others find

$$\Delta p \propto \log(Q) \tag{3.7}$$

Using 3.5 and 3.7 we can derive the poisson intensity as follows:

$$\begin{aligned}
\lambda(\delta) &= \Lambda \mathbb{P}(\delta < \Delta p) \\
&= \Lambda \mathbb{P}\left(\delta < \frac{\log Q}{K}\right) \\
&= \Lambda \mathbb{P}(K\delta < \log Q) \\
&= \Lambda \mathbb{P}\left(e^{K\delta} < Q\right) \qquad\qquad = \Lambda \int_{e^{K\delta}}^{\infty} x^{-1-\alpha} dx \\
&= \Lambda \left[\frac{-x^{-\alpha}}{\alpha}\right]_{e^{K\delta}}^{\infty} \\
&= \Lambda \left(\lim_{t\to\infty} \frac{-t^{-\alpha}}{\alpha} + \frac{e^{-K\delta\alpha}}{\alpha}\right) \\
&= \frac{\Lambda}{\alpha}\left(e^{-K\delta\alpha} - \lim_{t\to\infty} \frac{1}{t^\alpha}\right) \\
&= \frac{\Lambda}{\alpha} e^{-\alpha K \delta}
\end{aligned}$$

while 3.5 and 3.6 yield:

$$\begin{aligned}
\lambda(\delta) &= \Lambda \mathbb{P}(\delta < \Delta p) \\
&= \Lambda \mathbb{P}(\delta < kQ^\beta) \\
&= \Lambda \mathbb{P}\left(Q > \left(\frac{\delta}{k}\right)^{-\beta}\right) \\
&= \Lambda \int_{\left(\frac{\delta}{k}\right)^{-\beta}}^{\infty} x^{-1-\alpha} dx \\
&= \Lambda \left[\lim_{t\to\infty} \frac{-t^{-\alpha}}{\alpha} + \frac{\left(\frac{\delta}{k}\right)^{-\frac{\alpha}{\beta}}}{\alpha}\right] \\
&= \frac{\Lambda \left(\frac{\delta}{k}\right)^{-\frac{\alpha}{\beta}}}{\alpha}
\end{aligned}$$

Other methods exist i.e. integrating the density of the orderbook, potentially better since we only care abt short-term liquidity?

3.6 The Hamilton-Jacobi-Bellman Equation

Ho and stoll use the dynamic programming principle to show that a function u must solve the HJB:

$$\begin{cases} u_t + \frac{1}{2}\sigma^2 u_{ss} + \max_{\delta^b} \lambda^b(\delta^b)[u(s, x - s + \delta^b, q + 1, t) - u(s, x, q, t)] \\ \qquad\qquad + \max_{\delta^a} \lambda^a(\delta^a)[u(s, x + s + \delta^a, q - 1, t) - u(s, x, q, t)] = 0, \\ \qquad u(s, x, q, T) = -e^{-\gamma(x+qs)} \end{cases} \tag{3.8}$$

but due to our choice of exponential utility we can simplify the problem with the ansatz:

$$u(s, x, q, t) = -e^{-\gamma x} e^{-\gamma\theta(s,q,t)}$$

and by substitution we find the following equation for θ:

$$\begin{cases} \theta_t + \frac{1}{2}\sigma^2\theta_{ss} - \frac{1}{2}\sigma^2\gamma\theta_{ss}^2 + \max_{\delta^b}\left[\frac{\lambda^b(\delta^b)}{\gamma}(1 - e^{\gamma(s-\delta^b-r^b)})\right] \\ \qquad\qquad + \max_{\delta^a}\left[\frac{\lambda^a(\delta^a)}{\gamma}(1 - e^{-\gamma(s+\delta^a-r^a)})\right] = 0, \\ \qquad \theta(s, q, T) = qs. \end{cases} \tag{3.9}$$

Relations for the reserve prices

By the definitions of the reserve bid and ask prices we obtain

$$\begin{aligned} u(s, x - r^b(s, q, t), q + 1, t) &= u(s, x, q, t) \\ -e^{-\gamma(x - r^b(s,q,t))} e^{-\gamma\theta(s,q+1,t)} &= -e^{-\gamma x} e^{-\gamma\theta(s,q,t)} \\ -\gamma(x - r^b(s, q, t)) - \gamma\theta(s, q + 1, t) &= -\gamma x - \gamma\theta(s, q, t) \\ x - r^b(s, q, t) + \theta(s, q + 1, t) &= x + \theta(s, q, t) \\ r^b(s, q, t) &= \theta(s, q + 1, t) - \theta(s, q, t) \end{aligned}$$

and

$$\begin{aligned} u(s, x + r^a(s, q, t), q - 1, t) &= u(s, x, q, t) \\ -e^{-\gamma(x + r^a(s,q,t))} e^{-\gamma\theta(s,q-1,t)} &= -e^{-\gamma x} e^{-\gamma\theta(s,q,t)} \\ -\gamma(x + r^a(s, q, t)) - \gamma\theta(s, q - 1, t) &= -\gamma x - \gamma\theta(s, q, t) \\ x + r^a(s, q, t) + \theta(s, q - 1, t) &= x + \gamma\theta(s, q, t) \\ r^a(s, q, t) &= \theta(s, q, t) - \theta(s, q - 1, t) \end{aligned}$$

From the first order optimality condition on the maximised terms in the HJB equation, we may obtain as follows some implicit relations on the optimal bid and ask spreads δ^b

and δ^a:

$$\begin{aligned}
\frac{\partial}{\partial\delta}\left[\frac{\lambda^b(\delta)}{\gamma}(1-e^{\gamma(s-\delta-r^b(s,q,t))})\right](\delta^b) &= 0 \\
\frac{1}{\gamma}\left[\frac{\partial\lambda^b}{\partial\delta}(\delta^b)-\frac{\partial}{\partial\delta}\lambda^b(\delta^b)e^{\gamma(s-\delta^b-r^b(s,q,t))}\right] &= 0 \\
\frac{\partial\lambda^b}{\partial\delta}(\delta^b)-\frac{\partial\lambda^b}{\partial\delta}(\delta^b)e^{\gamma(s-\delta^b-r^b(s,q,t))}+\gamma\lambda^b(\delta^b)e^{\gamma(s-\delta^b-r^b(s,q,t))} &= 0 \\
\left(\gamma\lambda^b(\delta^b)-\frac{\partial\lambda^b}{\partial\delta}(\delta^b)\right)e^{\gamma(s-\delta^b-r^b(s,q,t))} &= -\frac{\partial\lambda^b}{\partial\delta}(\delta^b) \\
-\left(\frac{\partial\lambda^b}{\partial\delta}(\delta^b)\right)e^{-\gamma(s-\delta^b-r^b(s,q,t))} &= \gamma\lambda^b(\delta^b)-\frac{\partial\lambda^b}{\partial\delta}(\delta^b) \\
e^{-\gamma(s-\delta^b-r^b(s,q,t))} &= 1-\gamma\frac{\lambda^b(\delta^b)}{\frac{\partial\lambda^b}{\partial\delta}(\delta^b)} \\
-\gamma(s-\delta^b-r^b(s,q,t)) &= \log\left(1-\gamma\frac{\lambda^b(\delta^b)}{\frac{\partial\lambda^b}{\partial\delta}(\delta^b)}\right) \\
s-\delta^b-r^b(s,q,t) &= -\frac{1}{\gamma}\log\left(1-\gamma\frac{\lambda^b(\delta^b)}{\frac{\partial\lambda^b}{\partial\delta}(\delta^b)}\right) \\
s-r^b(s,q,t) &= \delta^b-\frac{1}{\gamma}\log\left(1-\gamma\frac{\lambda^b(\delta^b)}{\frac{\partial\lambda^b}{\partial\delta}(\delta^b)}\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial\delta}\left[\frac{\lambda^a(\delta)}{\gamma}(1-e^{-\gamma(s+\delta-r^a(s,q,t))})\right](\delta^a) &= 0 \\
\frac{1}{\gamma}\left[\frac{\partial\lambda^a}{\partial\delta}(\delta^a)-\frac{\partial}{\partial\delta}\lambda^a(\delta^a)e^{-\gamma(s+\delta^a-r^a(s,q,t))}\right] &= 0 \\
\frac{\partial\lambda^a}{\partial\delta}(\delta^a)-\frac{\partial\lambda^a}{\partial\delta}(\delta^a)e^{-\gamma(s+\delta^a-r^a(s,q,t))}+\gamma\lambda^a(\delta^a)e^{-\gamma(s+\delta^a-r^a(s,q,t))} &= 0 \\
\left(\gamma\lambda^a(\delta^a)-\frac{\partial\lambda^a}{\partial\delta}(\delta^a)\right)e^{-\gamma(s+\delta^a-r^a(s,q,t))} &= -\frac{\partial\lambda^a}{\partial\delta}(\delta^a) \\
-\left(\frac{\partial\lambda^a}{\partial\delta}(\delta^a)\right)e^{\gamma(s+\delta^a-r^a(s,q,t))} &= \gamma\lambda^a(\delta^a)-\frac{\partial\lambda^a}{\partial\delta}(\delta^a) \\
e^{\gamma(s+\delta^a-r^a(s,q,t))} &= 1-\gamma\frac{\lambda^a(\delta^a)}{\frac{\partial\lambda^a}{\partial\delta}(\delta^a)} \\
\gamma(s+\delta^a-r^a(s,q,t)) &= \log\left(1-\gamma\frac{\lambda^a(\delta^a)}{\frac{\partial\lambda^a}{\partial\delta}(\delta^a)}\right) \\
s+\delta^a-r^a(s,q,t) &= \frac{1}{\gamma}\log\left(1-\gamma\frac{\lambda^a(\delta^a)}{\frac{\partial\lambda^a}{\partial\delta}(\delta^a)}\right) \\
r^a(s,q,t)-s &= \delta^a-\frac{1}{\gamma}\log\left(1-\gamma\frac{\lambda^a(\delta^a)}{\frac{\partial\lambda^a}{\partial\delta}(\delta^a)}\right)
\end{aligned}$$

3.7 Asymptotic Expansion in q

$$\lambda^a(\delta)+\lambda^b(\delta) = Ae^{-k\delta} \tag{3.10}$$

Using exponential arrival rates in our relations above for the optimal bid and ask prices, we can see that

$$\frac{\lambda(\delta)}{\frac{\partial \lambda}{\partial \delta}(\delta)} = \frac{Ae^{-k\delta}}{-kAe^{-k\delta}} = -\frac{1}{k}.$$

Hence by plugging in the aforementioned relations under the assumption of symmetric exponential arrival rates to the maximised terms in the HJB equation 3.9 we see that

$$\begin{aligned}
&\max_{\delta^b}\left[\frac{\lambda^b(\delta^b)}{\gamma}(1-e^{\gamma(s-\delta^b-r^b)})\right] + \max_{\delta^a}\left[\frac{\lambda^a(\delta^a)}{\gamma}(1-e^{-\gamma(s+\delta^a-r^a)})\right] \\
&= \frac{Ae^{-k\delta^b}}{\gamma}\left(1-e^{\gamma\left(-\frac{1}{\gamma}\log\left(1+\frac{\gamma}{k}\right)\right)}\right) + \frac{Ae^{-k\delta^a}}{\gamma}\left(1-e^{-\gamma\left(\frac{1}{\gamma}\log\left(1+\frac{\gamma}{k}\right)\right)}\right) \\
&= \left[\frac{A}{\gamma}\left(1-e^{-\log\left(1+\frac{\gamma}{k}\right)}\right)\right](e^{-k\delta^b}+e^{-k\delta^a}) \\
&= \left[\frac{A}{\gamma}\left(1-\frac{1}{1+\frac{\gamma}{k}}\right)\right](e^{-k\delta^b}+e^{-k\delta^a}) \\
&= \left(\frac{A}{\gamma}-\frac{A}{\gamma+\frac{\gamma^2}{k}}\right)(e^{-k\delta^b}+e^{-k\delta^a}) \\
&= \left(\frac{A\left(1+\frac{\gamma}{k}\right)-A}{\gamma+\frac{\gamma^2}{k}}\right)(e^{-k\delta^b}+e^{-k\delta^a}) \\
&= \left(\frac{A\frac{\gamma}{k}}{\gamma+\frac{\gamma^2}{k}}\right)(e^{-k\delta^b}+e^{-k\delta^a}) \\
&= \frac{A}{k+\gamma}(e^{-k\delta^b}+e^{-k\delta^a})
\end{aligned}$$

which results in the simplified HJB equation below:

$$\begin{cases} \theta_t + \frac{1}{2}\sigma^2\theta_{ss} - \frac{1}{2}\sigma^2\gamma\theta_s^2 + \frac{A}{k+\gamma}(e^{-k\delta^a}+e^{-k\delta^b}) = 0 \\ \theta(s,q,T) = qs. \end{cases} \tag{3.11}$$

Asymptotic expansion in the inventory variable q:

$$\theta(q,s,t) = \theta^0(s,t) + q\theta^1(s,t) + \frac{1}{2}q^2\theta^2(s,t) + ... \tag{3.12}$$

The exact relations for the reserve bid and ask prices obtained above yield

$$r^b(s,q,t) = \theta^1(s,t) + (1+2q)\theta^2(s,t) + ... \tag{3.13}$$

$$r^a(s,q,t) = \theta^1(s,t) + (-1-2q)\theta^2(s,t) + ... \tag{3.14}$$

Then

$$r(s,q,t) = \frac{r^a(s,q,t)+r^b(s,q,t)}{2} = \theta^1(s,t) + 2q\theta^2(s,t) \tag{3.15}$$

follows immediately, and we also have that

$$\begin{aligned}
\delta^a + \delta^b &= \frac{1}{\gamma}\log\left(1+\frac{\gamma}{k}\right) + r^a(s,q,t) - s + \frac{1}{\gamma}\log\left(1+\frac{\gamma}{k}\right) + s - r^b(s,q,t) \\
&= r^a(s,q,t) - r^b(s,q,t) + \frac{2}{\gamma}\log\left(1+\frac{\gamma}{k}\right) \\
&= -2\theta^2(s,t) + \frac{2}{\gamma}\log\left(1+\frac{\gamma}{k}\right)
\end{aligned} \tag{3.16}$$

Now consider a first-order approximation of the order arrival term:

$$\frac{A}{k+\gamma}(e^{-\gamma\delta^a} + e^{-\gamma\delta^b}) = \frac{A}{k+\gamma}(2 - k(\delta^a + \delta^b) + ...) \tag{3.17}$$

The linear term does not depend on the inventory q. Therefore, by substituting 3.12 and 3.17 into 3.11 and grouping terms of order q we obtain

$$\begin{cases} \theta^1_t + \frac{1}{2}\sigma^2\theta^1_{ss} = 0 \\ \theta^1(s,T) = s. \end{cases} \tag{3.18}$$

which admits the solution $\theta^1(s,t) = s$. Grouping terms of order q^2 yields

$$\begin{cases} \theta^2_t + \frac{1}{2}\sigma^2\theta^2_{ss} - \frac{1}{2}\sigma^2\gamma(\theta^1_s)^2 = 0 \\ \theta^2(s,T) = 0 \end{cases} \tag{3.19}$$

with solution $\theta^2(s,t) = -\frac{1}{2}\sigma^2\gamma(T-t)$. Thus for this linear approximation of the order arrival term, we can substitute our solutions back into 3.15 to obtain the same indifference price

$$r(s,t) = s - q\gamma\sigma^2(T-t) \tag{3.20}$$

as in the case where no trading is allowed. We quote a bid-ask spread given by

$$\delta^a + \delta^b = \gamma\sigma^2(T-t) + \frac{2}{\gamma}\log\left(1 + \frac{\gamma}{k}\right) \tag{3.21}$$

which is again aquired by substituting our solutions back into 3.16.

3.8 Summary

The reserve ask price derived by Avellaneda and Stoikov in the infinite horizon case is potentially incorrect, a correction is presented.

Chapter 4

Extension and Statistical Analysis of Orderbooks

4.1 Extension to Geometric Brownian Motion

Consider now an alternative model for the midprice of the security for which we wish to make a market:

$$dS_u = \sigma S_u dW_u$$

with initial value $S_0 = s$. This, as we have seen in section 1 defines a geometric brownian motion (without drift), and is the canonical stochastic process used to model most asset prices, as in contrast with the standard Brownian Motion, this process is valued in $\mathbb{R}^+$. As we have previously seen, the above SDE gives us the solution

$$S_t = se^{\frac{\sigma^2}{2}t+\sigma^2 W_t} \tag{4.1}$$

where

$$\begin{aligned}
\mathbb{E}[S_t] &= s \\
\mathbb{E}[S_t^2] &= s^2 e^{\sigma^2(T-t)} \\
\mathrm{Var}[S_t] &= s^2\left(e^{\sigma^2 t} - 1\right)
\end{aligned}$$

Consider the mean/variance objective

$$V(s,x,q,t) = \mathbb{E}\left[(x+qS_T) - \frac{\gamma}{2}(qS_T - qs)^2|\mathcal{F}_t\right]. \tag{4.2}$$

Letting $S_t = s$ and using the above properties of GBM we can simplify as follows:

$$\begin{aligned}
V(s,x,q,t) &= \mathbb{E}\left[(x+qS_T) - \frac{\gamma}{2}(qS_T - qs)^2|\mathcal{F}_t\right] \\
&= x + q\mathbb{E}[S_T|\mathcal{F}_t] - \frac{\gamma q^2}{2}\mathbb{E}[S_T^2|\mathcal{F}_t] + \gamma q^2 s\mathbb{E}[S_T|\mathcal{F}_t] - \frac{\gamma q^2 s^2}{2} \\
&= x + qs - \frac{\gamma q^2}{2}\left(s^2 e^{\sigma^2(T-t)}\right) + \gamma\sigma^2 s^2 - \frac{\gamma q^2 s^2}{2} \\
&= x + qs - \frac{\gamma q^2 s^2}{2}\left(e^{\sigma^2(T-t)}\right) + \frac{\gamma q^2 s^2}{2} \\
&= x + qs - \frac{\gamma q^2 s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right)
\end{aligned}$$

Using the definition of reserve bid and ask prices we obtain

$$
\begin{aligned}
V(s, x + R^a(s,q,t), q-1, t) &= V(s,x,q,t) \\
x + R^a(s,q,t) + (q-1)s - \frac{\gamma(q-1)^2 s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right) &= x + qs - \frac{\gamma q^2 s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right) \\
R^a(s,q,t) + qs - s - \frac{\gamma(q^2 - 2q + 1)s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right) &= qs - \frac{\gamma q^2 s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right) \\
R^a(s,q,t) - s - \frac{\gamma q^2 s^2 + \gamma(1-2q)s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right) &= -\frac{\gamma q^2 s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right) \\
R^a(s,q,t) - \frac{(1-2q)\gamma s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right) &= s \\
R^a(s,q,t) &= s + \frac{(1-2q)\gamma s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right)
\end{aligned}
$$

and

$$
\begin{aligned}
V(s, x - R^b(s,q,t), q+1, t) &= V(s,x,q,t) \\
x - R^b(s,q,t) + (q+1)s - \frac{\gamma(q+1)^2 s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right) &= x + qs - \frac{\gamma q^2 s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right) \\
-R^b(s,q,t) + qs + s - \frac{\gamma(q^2 + 2q + 1)s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right) &= qs - \frac{\gamma q^2 s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right) \\
-R^b(s,q,t) + s - \frac{(1+2q)\gamma s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right) &= 0 \\
R^b(s,q,t) &= s - \frac{(1+2q)\gamma s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right)
\end{aligned}
$$

4.2 Estimating the Frequency of Market Orders

4.3 Estimating the Size Distribution of Market Orders

4.4 Estimating the Temporary Price Impact of Large Market Orders

Chapter 5

Numerical Analysis and Simulations

5.1 Replication of the Results of Avellaneda and Stoikov

Listing 5.1: Python example

```
import numpy as np

def incmatrix(genl1,genl2):
    m = len(genl1)
    n = len(genl2)
    M = None #to become the incidence matrix
    VT = np.zeros((n*m,1), int)  #dummy variable

    #compute the bitwise xor matrix
    M1 = bitxormatrix(genl1)
    M2 = np.triu(bitxormatrix(genl2),1)

    for i in range(m-1):
        for j in range(i+1, m):
            [r,c] = np.where(M2 == M1[i,j])
            for k in range(len(r)):
                VT[(i)*n + r[k]] = 1;
                VT[(i)*n + c[k]] = 1;
                VT[(j)*n + r[k]] = 1;
                VT[(j)*n + c[k]] = 1;

                if M is None:
                    M = np.copy(VT)
                else:
                    M = np.concatenate((M, VT), 1)

                VT = np.zeros((n*m,1), int)

    return M
```

5.2 Simulation of Extended Model for GBM

5.3 Estimation of Order Book Parameters for Real-World Data

5.4 Market Making in the Binance Order Book

Chapter 6

Conclusion

Appendix A

“Appendix”

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