1 Introduction and Background

- Order books, order types, inventory risk, market participants, why market making is important
- Probability spaces, martingales
- Continuous-time stochastic processes
- Brownian motion
- Stochastic integration
- Itô's formula
- Girsanov's theorem?
- SDEs and Feynman-Kac?

2 Stochastic Optimal Control

- Controlled diffusion
- The Dynamic Programming Principle
- Hamilton-Jacobi-Bellman
- Verification theorem
- Maybe a simple worked example? e.g. Merton's portfolio optimisation

3 Formalising the Market-Making Problem

- What are we actually trying to optimise? Define the value function
- Reservation prices accounting for inventory risk
- Market impact of trades
- Trading intensity models

4 The Avellaneda-Stoikov Model

- Solve the HJB equation for $r^b(s, q, t)$ and $r^a(s, q, t)$
- Obtain the optimal distances $\delta^b(s, q, t)$ and $\delta^a(s, q, t)$
- Derive approximate and more computationally tractable solutions by asymptotic expansion in q
- Possible extensions: Options, interest, drift in the stock price, stochastic volatility model/autocorrelation, transaction costs, order sizes > 1,

5 Implementation in Python

- Python code and discussion of implementation methodology e.g. discrete time steps, correction for floating point computation errors with small values

6 Empirical Results

- Simulate paths of stock price, plot optimal bids/asks
- Simulate a limit orderbook with orders being placed, can plot inventory over time of the agent, PnL
- Maybe download orderbook data from cryptocurrency exchange API and backtest strategy?

7 Main Results

7.1 Controlled Diffusion

$$dX_s = b(X_s, \alpha_s)ds + \sigma(X_s, \alpha_s)dW_s$$

A strong solution to this SDE starting at time t is a progressively measurable process X such that for $s \leq t$:

$$X_s = X_t + \int_t^s b(X_u, \alpha_u)du + \int_t^s \sigma(X_u, \alpha_u)dW_u$$

and

$$\int_t^s |b(X_u, \alpha_u)|du + \int_t^s |\sigma(X_u, \alpha_u)|^2 du < \infty$$

a.s.

7.2 Finite-Horizon Problem

We say that $\hat{\alpha}$ is an optimal control for a given initial condition $(t, x) \in [0, T) \times \mathbb{R}^n$ if

$$v(t, x) = J(t, x, \hat{\alpha})$$

where J is the gain function and v is the associated value function

$$v(t, x) = \sup_{\alpha \in \mathcal{A}(t,x)} J(t, x, \alpha)$$

and

$$J(t, x, \alpha) = \mathbb{E}\left[\int_t^T f(s, X_s^{t,x}, \alpha_s)ds + g(X_T^{t,x})\right]$$

and $\mathcal{A}(t, x) \subseteq \mathcal{A}$ such that

$$\mathbb{E}\left[\int_t^T |f(s, X_s^{t,x}, \alpha_s)|ds\right] < \infty$$

where $\mathcal{A}$ is the set of control processes such that

$$\mathbb{E}\left[\int_0^T |b(x, \alpha_t)|^2 + |\sigma(x, \alpha_t)|^2 dt\right] < \infty$$

and

$$f : [0, T] \times \mathbb{R}^n \times A \to \mathbb{R}$$

is a rolling reward function and

$$g : \mathbb{R}^n \to \mathbb{R}$$

is the terminal payoff function.

7.3 Dynamic Programming Principle

$$v(t, x) = \sup_{\alpha \in \mathcal{A}(t,x)} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)ds + v(\theta, X_\theta^{t,x})\right]$$

for any $\theta \in \mathcal{T}_{t,T}$ where for $0 \leq t \leq T \leq \infty$ we denote the set of stopping times valued in $[t, T]$ by $\mathcal{T}_{t,T}$

7.4 Hamilton-Jacobi-Bellman Equation

$$-\frac{\partial v}{\partial t}(t, x) - H(t, x, D_x v(t, x), D_x^2 v(t, x)) = 0$$

where

$$H(t, x, p, M) = \sup_{a \in A}\left[b(x, a) \cdot p + \frac{1}{2}\mathrm{tr}(\sigma\sigma'(x, a)M) + f(t, x, a)\right]$$

is the Hamiltonian of the associated control problem. We have the regular terminal condition of our PDE:

$$v(T, x) = g(x)$$

8 Avellaneda-Stoikov Model

8.1 Assumptions

- The dealer being modelled is one of many players in the market
- The 'true' price is given by the market mid-price
- The money-market pays no interest
- The agent has no opinion on drift or autocorrelation of the stock price
- Limit orders can be continuously updated at no cost
- The arrival frequency of market orders to the market is constant
- Limit orders are of fixed size 1

8.2 Stock Price Model

Assume the stock evolves according to a standard Wiener process with some variance σ^2:

$$dS_u = \sigma dW_u$$

8.3 Value Function

Consider an inactive trader who holds an inventory of q stocks until the terminal time T. The agent's value function is

$$v(x, s, q, t) = \mathbb{E}_t \left(-e^{-\gamma(x+qS_T)} \right)$$

where x is the initial wealth in dollars, t is the present time and γ is a user-defined risk-aversion parameter.

8.4 Reservation Prices

The reservation price is the price that would make the agent indifferent between his current portfolio and his current portfolio plus one stock. So r^b can be determined from the relation

$$v(x - r^b(s, q, t), s, q + 1, t) = v(x, s, q, t)$$

and r^a solves

$$v(x + r^a(s, q, t), s, q - 1, t) = v(x, s, q, t).$$

We solve these to obtain

$$r^a(s, q, t) = s + (1 - 2q)\frac{\gamma\sigma^2(T - t)}{2}$$

and

$$r^b(s, q, t) = s + (-1 - 2q)\frac{\gamma\sigma^2(T - t)}{2}.$$

We define the "reserve" or "indifference" price to be the average of these two *given* that the agent currently holds q stocks:

$$r(s, q, t) = s - q\gamma\sigma^2(T - t)$$

8.5 Limit Orders

The agent quotes the bid price p^b and the ask price p^a, and the current shape of the limit orderbook as well as the distances $\delta^b = s - p^b$ and $\delta^a = p^a - s$ determine the priority of execution when large market orders are placed. E.g. when a market order to buy Q shares arrives, the Q limit orders with the lowest ask prices will be lifted. Let p^Q be the price of the highest limit order executed in this trade. Then $\Delta p = p^Q - s$ is the temporary market impact of the trade of size Q. Then we have that if our $\delta^a < \Delta p$, our agents limit order will be executed. We assume that market orders will fill our limit orders at Poisson rates $\lambda^a(\delta^a)$ and $\lambda^b(\delta^b)$, decreasing functions of δ^a and δ^b resp. (further away from midpoint $\rightarrow$ orders hit less often).

8.6 Wealth Process

We now have stochastic wealth and inventory: Let N_t^b and N_t^a be Poisson processes with intensities λ^b and λ^a representing the amount of stocks bought/sold by the agent at time t. The inventory at time t is $q_t = N_t^b - N_t^a$ and the wealth process evolves according to

$$dXt = p^a dN_t^a - p^b dN_t^b.$$

The objective of the agent who sets limit orders is

$$u(s, x, q, t) = \max_{\delta^a, \delta^b} \mathbb{E}_t \left[-e^{-\gamma(X_T + q_T S_T)} \right]$$

8.7 Trading Intensity

Assume constant frequency Λ of market orders. We want to determine some realistic functional forms for the relationship between the Poisson intensity λ and distance to mid-price δ. To do this we need information on: (i) the overall frequency of market orders, (ii) the distribution of their size, (iii) the temporary impact of a large market order. The distribution of size of market orders has been found to obey a power law:

$$f^Q(x) \propto x^{-1-\alpha} \tag{1}$$

for large x, with $\alpha \in [1.4, 1.6]$. Less consensus on size distribution. Some find change in price Δp after market order size Q given by

$$\Delta p \propto Q^\beta, \beta \in [0.5, 0.8] \tag{2}$$

while others find

$$\Delta p \propto \log(Q) \tag{3}$$

Using (1) and (3) we can derive the poisson intensity as

$$\lambda(\delta) = \frac{\Lambda}{\alpha} e^{-\alpha K \delta}$$

while (1) and (2) yield

$$\lambda(\delta) = B \delta^{\frac{-\alpha}{\beta}}.$$

[Need to figure out what B is]. Other methods exist i.e. integrating the density of the orderbook, potentially better since we only care abt short-term liquidity?

8.8 The Solution

Ho and stoll use the dynamic programming principle to show that a function u must solve the HJB:

$$u_t + \frac{1}{2}\sigma^2 u_{ss} + \max_{\delta^b} \lambda^b(\delta^b)[u(s, x - s + \delta^b, q + 1, t) - u(s, x, q, t)] + \max_{\delta^a} \lambda^a(\delta^a)[u(s, x + s + \delta^a, q - 1, t) - u(s, x, q, t)] = 0,$$

$$u(s, x, q, T) = -e^{-\gamma(x+qs)}$$

but due to our choice of exponential utility we can simplify the problem with the ansatz:

$$u(s, x, q, t) = -e^{-\gamma x} e^{-\gamma \theta(s,q,t)}$$

and by substitution we find the following equation for θ :

$$\theta_t + \frac{1}{2}\sigma^2 \theta_{ss} - \frac{1}{2}\sigma^2 \gamma \theta_{ss}^2 + \max_{\delta^b} \left[\frac{\lambda^b(\delta^b)}{\gamma} (1 - e^{\gamma(s - \delta^b - r^b)}) \right] + \max_{\delta^a} \left[\frac{\lambda^a(\delta^a)}{\gamma} (1 - e^{-\gamma(s + \delta^a + r^a)}) \right] = 0,$$

$$\theta(s, q, T) = qs.$$

By the definitions of the reserve bid and ask prices we obtain

$$r^b(s, q, t) = \theta(s, q + 1, t) - \theta(s, q, t)$$

and

$$r^a(s, q, t) = \theta(s, q, t) - \theta(s, q - 1, t)$$

and then through the following implicit relation we can obtain the optimal distances δ^b and δ^a:

$$s - r^b(s, q, t) = \delta^b - \frac{1}{\gamma} \log \left(1 - \gamma \frac{\lambda^b(\delta^b)}{\frac{\partial \lambda^b}{\partial \delta}(\delta^b)} \right)$$

and

$$r^a(s, q, t) - s = \delta^a - \frac{1}{\gamma} \log \left(1 - \gamma \frac{\lambda^a(\delta^a)}{\frac{\partial \lambda^a}{\partial \delta}(\delta^a)} \right).$$

8.9 Asymptotic expansion