Automated Market-Making under Inventory Risk: A Stochastic Optimal Control Framework

Joshua Acton

Supervised by Professor Nick Whiteley
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Declaration

I declare that the work contained herein is my own, unless explicitly stated otherwise.

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To mum

Thank you for everything.

Abstract

This project presents a review of the mathematical theory that attempts to model the dynamics of an automated market-maker under inventory risk in financial markets. We begin by outlining financial markets, their participants and their microstructure, before discussing the requisite mathematical tools from probability theory, stochastic analysis, stochastic calculus and stochastic control. Next, we investigate the seminal 2008 paper "High-Frequency Trading in a Limit Order Book" by Avellaneda and Stoikov (2008), which formalises the approach of a market-maker trading through limit orders and utilises the dynamic programming principle to solve for the market-maker's optimal bid and ask quotes. We then present an extension to this model which extends the process governing the underlying stock price from a simple Brownian Motion to the more standard Geometric Brownian Motion. Finally, we implement the model in Python and present it's empirical results when back-tested on real order book data from a cryptocurrency exchange.

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Chapter 1

Introduction and background

1.1 Introduction

This section aims to equip the reader with both the motivation and mathematical tools to begin to formalise problems in stochastic optimal control and mathematical finance, by first providing some background material on financial markets, their participants, and their structure, and then setting up the basic elements of measure-theoretic probability theory and stochastic processes which we will use to describe them. We will end with a discussion of the market-making problem and how we might go about formalising it.

1.2 Financial Markets

A market is simply some social structure that attempts to match those who want to sell a good or service to those who want to buy it. Modern financial markets, thanks to recent innovations such as the internet, satellite communication, and fibre-optic cables, are perhaps the most interconnected and widespread markets in human history.

Most people may have heard of the New York Stock Exchange, London Stock Exchange, or NASDAQ, but these are only one type of exchange for one type of financial asset, namely equity (part ownership of a corporate entity, individually called “stocks” or “shares”).There are also markets for commodities (oil, gas, industrial metals, precious metals, live cattle and more), bonds (pieces of government or corporate debt, where the holder receives fixed interest payments), currencies (including cryptocurrencies), and derivatives which are legal contracts whose value is some function of the price of a specified underlying asset. In total, on an average day, tens of trillions of US dollars worth of assets change hands.

All markets, whatever the good or service being exchanged, have something in common: Every seller needs a buyer, and every buyer needs a seller. But this raises some natural questions: What happens if no-one wants to sell (or buy)? What happens if the only prices at which people are willing to sell is far out of reach of those who want to buy? Enter the *dealer*: An entity who provides *liquidity* (ease of exchange) to market participants. A dealer does this by simultaneously offering to both buy and sell the particular asset, offering to buy at a slightly lower price than they offer to sell. This known as “making a market”, and dealers in modern parlance may also be called “market-makers”.

Dealers

Dealers provide a crucial service in financial markets: By providing these quotes, they narrow the *spread* - the difference between the prices at which one can buy or sell an asset in the market. Hence, entities who may need to trade even in adverse market scenarios (such as companies needing to buy foreign currency to pay workers abroad, or oil producers seeking to hedge their production) know that they can reliably find a buyer or seller, regardless of the uncertainty of other market participants such as *speculators* - those believe that a certain asset is under or overvalued, and trade it with the sole motive of making money buy selling it for more than they bought it or vice-versa.

Of course, there is no free lunch. Dealers do not provide this service to the market out of the goodness of their own hearts - they too have a profit motive. While the presence of dealers in the market narrows the spread, it does not eliminate it. The dealers aim is to be constantly selling the asset for a slightly higher price than it is buying it, and taking the spread as profit. In modern electronic markets with very high trading volumes, even in heavily traded assets with very narrow spreads, a spread of only 0.01$ multiplied across millions or billions of trades can be very lucrative for the dealers who are fast enough.

The Limit Orderbook

So far we have discussed markets as an abstract concept, but in order to build a mathematical model of the dealer, we need to specify the framework under which the market operates. Most modern electronic exchanges, including those mentioned above, operate some version of a *limit orderbook* where participants can place two types of orders: a *limit order* or a *market order* depending on their needs. Limit orders specify a side (bid or ask, buying or selling), a quantity (how many units of the asset to buy/sell), and a price at which the order should be executed. These enter a queue of limit orders at the particular price level. Market orders specify a side and a quantity, but not a price: The exchange operates a *matching engine* which takes incoming market orders and attempts to match them to the existing limit orders, and if two orders match, they are executed and a trade occurs.

For an example, consider the orderbook illustrated by figure 1.1, and suppose that individual limit orders may only be placed for 1-share lots. If a market order is placed to buy 10 lots, then the trade will occur at \$1.01, the dealer/s will sell and the placer of the market order will buy, and both the market order and the 10 lowest limit ask orders will be removed from the market. So immediately after this trade, there will be 20 shares left available to be sold at the \$1.01 price level. However, suppose that a market order is placed to buy 30 units. In this case, the orders will still be matched, the buyer will buy 30 units for \$1.01 apiece but all of the limit orders at \$1.01 will be taken off the exchange, and the market mid-point price has now moved up from \$1.00 to \$1.005. If a market order is placed to buy 100 shares, since there are only 80 shares available to be sold, only these 80 will be bought for an average price of $\frac{30\times 1.01\$ + 50\times 1.02\$}{80} = 1.01625\$$. On the other hand, if a market order is placed and there are no limit orders to match it against, the market order would not be executed at all and be voided.

Finally, suppose we place a limit order into this market to buy 10 shares for \$0.90. Thus, for our order to ever be executed, a market order or sequence of market orders would have to come in and move the market mid-price by $\approx 10\%$ in order for our order to be touched. Hence in a given (small) interval of time, it is intuitively very unlikely that our order will be executed, especially when you consider that a move of 10% is roughly how much you might expect a stock to move over a year, let alone over a fraction of a trading day. This is one of the fundamental ideas that we will employ to model our dealing agent:

Side	Price /$	Volume
A	1.02	50
A	1.01	30
N/A	1.00	0
B	0.99	25
B	0.98	45

Figure 1.1: An example orderbook

The probability that a limit order will execute is a decreasing function of its distance from the mid-price.

We have also seen the key difference between market and limit orders in action: Limit orders guarantee price, but do not guarantee that all or any of the order will be filled. Market orders guarantee that as much of the order as possible will be filled, but they do not guarantee the price at which the trade will occur.

We can also observe that the market provides us with a way to estimate the true value of the asset. Classical economic theory dictates that in aggregate, market participants react quickly and rationally to new information about a particular asset, meaning that market prices reflect the consensus opinion of market participants about the value of traded assets. The spread exists because people would only want to sell for slightly more than something is worth, and buy it for slightly less. Hence, if you really want to buy an asset you have to pay a premium to "*cross the spread*" to acquire it. From this we can determine that the true price of the asset at a point in time lies somewhere in between the maximum bid price and the minimum ask price for the asset at that time. The most common estimator in the literature and in practice is simply the average of these two values - the mid-market price, but other estimators do exist such as the volume-weighted average price (VWAP) which takes into account the volume of the bids vs asks. For the rest of this report we will use the mid-price as our estimator for the "true" value of an asset.

The aim for the rest of this report is to build up a model of how a dealer should behave to maximise their returns in the presence of uncertainty: namely, uncertainty about the path that the true value of the stock might take. In order to do this, we will need to make use of some basic results from measure/probability theory and stochastic processes, which we will summarise below. We will also briefly introduce some tools from stochastic calculus. Familiarity with standard results from a first-year undergraduate level course in real analysis, probability, and statistics is assumed.

1.3 Measure Theory and Probability

Definition 1.3.1 (σ-algebra)**.** A family $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ of sets is called a σ-algebra if

- $\Omega \in \mathcal{A}$,
- for every countable collection of sets $A_1, A_2, ... \in \mathcal{F}$, $\bigcup_n A_n \in \mathcal{F}$,
- for every $A \in \mathcal{F}, A^c \in \mathcal{F}$.

Remark. The pair $(\Omega, \mathcal{F})$ is called a *measurable space.* Any set $A \in \mathcal{F}$ is called $\mathcal{F}$-*measurable* or simply *measurable.*

Definition 1.3.2. A *measure* μ on a σ-algebra $\mathcal{F}$ is a set function $\mu : \mathcal{F} \to [0, \infty]$ such that $\forall$ mutually disjoint sets $A_1, A_2, ... \in \mathcal{A}$ with $\bigcup_n A_n \in \mathcal{A}$,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \tag{1.1}$$

Remark. If $\mu(\Omega) = 1$ then we call μ a *probability measure*, and often use $\mathbb{P}$ instead. In this case the triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space.*

Lemma 1.3.1. Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ Then $\exists$ a smallest σ-algebra $\sigma(\mathcal{A})$ that contains all sets from $\mathcal{A}$.

Proof. The intersection of σ-algebras is a σ-algebra, so to find the smallest containing some collection of sets, take the intersection of all σ- algebras containing those sets. $\square$

Remark. The above $\sigma(\mathcal{A})$ is usually called the σ-algebra *generated* by $\mathcal{A}$.

Definition 1.3.3 (The Borel σ-algebra)**.** Consider the collection

$$\mathcal{A} = \{(a, b) : a, b \in \mathbb{R} \cup \{-\infty, \infty\}, a < b\}$$

Then define $\mathcal{B}(\mathbb{R}) := \sigma(\mathcal{A})$ the *Borel σ-algebra.* This is the smallest σ-algebra containing all open sets in $\mathbb{R}$. A set $B \in \mathcal{B}$ is a *Borel set.*

Definition 1.3.4 (Measurable functions)**.** Let $(\Omega, \mathcal{F})$ be a measurable space. A function $f : \Omega \to \mathbb{R}$ is *measurable* if for any $B \in \mathcal{B}$,

$$f^{-1}(B) \in \mathcal{F}.$$

Definition 1.3.5 (Simple functions)**.** A *simple function* is a finite linear combination of characteristic (or indicator) functions of measurable sets:

$$\phi = \sum_{i=1}^{n} c_i \chi_{A_i} \tag{1.2}$$

where $c_i \in \mathbb{R}$ and $A_i \in \mathbb{X}$. It is in standard representation if $X = \cup_{i=1}^{n} A_i$, the sets A_i are pairwise disjoint, and the numbers c_i are distinct.

Definition 1.3.6 (Integral of a simple function)**.** Consider a non-negative simple function written in standard form as given above. Then the *integral* of ϕ *with respect to* μ is

$$\int \phi \mathrm{d}\mu := \sum_{i=1}^{n} c_i \mu(A_i) \tag{1.3}$$

which takes values in $\bar{\mathbb{R}}$.

Lemma 1.3.2 (Approximation by simple functions)**.** Let $f \in M(X, \mathbb{X})$, $f \geq 0$. Then there exists a sequence (ϕ_n) in $M(X, \mathbb{X})$ such that

- $0 \leq \phi_n(x) \leq \phi_{n+1}(x) \; \forall \; x \in X, n \in \mathbb{N}$,
- $\lim_{n\to\infty} \phi_n(x) = f(x)$,
- Each ϕ_n is a simple function.

Definition 1.3.7 (Integral of a non-negative measurable function)**.** Let $f \in M^+(X, \mathbb{X})$. Then the *integral* of f *with respect to* μ is

$$\int f \mathrm{d}\mu := \sup\left\{\int \phi \mathrm{d}\mu : 0 \leq \phi \leq f, \phi \text{ is a simple measurable function}\right\} \in \bar{\mathbb{R}}.$$

Definition 1.3.8 (Integral of a non-negative measurable function over a set)**.** Let $f \in M^+(X, \mathbb{X})$. Then the *integral* of f *with respect to* μ *over set* $A \in \mathbb{X}$ is

$$\int_A f \mathrm{d}\mu := \int f \chi_A \mathrm{d}\mu \tag{1.4}$$

Definition 1.3.9 (Integrable functions)**.** Let $(X, \mathbb{X}, \mu)$ be a measure space. $f : X \to \mathbb{R}$ is *integrable* iff

$$\int f^+ \mathrm{d}\mu < +\infty \text{ and } \int f^- \mathrm{d}\mu < +\infty \tag{1.5}$$

where $f^+ := \max\{f, 0\}$ and $f^- := -\min\{f, 0\}$. We then define

$$\int f \mathrm{d}\mu := \int f^+ \mathrm{d}\mu - \int f^- \mathrm{d}\mu \tag{1.6}$$

and for $A \in \mathbb{X}$

$$\int_A f \mathrm{d}\mu := \int_A f^+ \mathrm{d}\mu - \int_A f^- \mathrm{d}\mu \tag{1.7}$$

Remark. All of the standard properties of integrals that one would expect to hold such as linearity are also true for the Lebesgue integral defined above. The Lebesgue integral also coincides with the Riemann and Regulated integrals for all Riemann-integrable and regulated functions respectively.

Definition 1.3.10 (Random variables)**.** Recall from above that a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is simply a measure space $(X, \mathbb{X}, \mu)$ where $\mu(X) = 1$. In this case, a measurable function $X : \Omega \to \mathbb{R}$ can be called a random variable.

Definition 1.3.11 (Expectation)**.** The notion of the *expectation* of a **random variable** is exactly equivalent to the notion of the *integral* of a **measurable function**. To be precise,

$$\mathbb{E}[X] := \int X \mathrm{d}\mathbb{P} \tag{1.8}$$

Definition 1.3.12 ($\sigma-$algebra generated by a random variable)**.** The σ-algebra generated by a random variable is $\sigma(Y) := \sigma(Y^{-1}(\mathcal{B}(\mathbb{R})))$

Definition 1.3.13 (Conditional Expectation)**.** Suppose $\mathcal{H} \subseteq \mathcal{F}$ is a sub-σ-algebra of $\mathcal{F}$. Then a *conditional expectation* of random variable X given $\mathcal{H}$ is any $\mathcal{H}$-measurable function $\Omega \to \mathbb{R}$ which satisfies

$$\int_H \mathbb{E}[X|H] \mathrm{d}\mathbb{P} = \int_H X \mathrm{d}\mathbb{P} \tag{1.9}$$

for any $H \in \mathcal{H}$. We define the conditional expectation with respect to a random variable as

$$\mathbb{E}[X|Y] := \mathbb{E}[X|\sigma(Y)] \tag{1.10}$$

where $\sigma(Y)$ is the σ-algebra generated by Y.

Theorem 1.3.3 (Tonelli)**.** Let $X_n \geq 0$ be random variables. Then

$$\mathbb{E}\left[\sum_{k=1}^{\infty} X_k\right] = \sum_{k=1}^{\infty} \mathbb{E}[X_k] \tag{1.11}$$

and the statement also holds with an integral instead of a sum.

Theorem 1.3.4 (Fubini)**.** Let X_n be random variables with $\mathbb{E}\left[\sum_{k=1}^{\infty}|X_k|\right] < \infty$. Then

$$\mathbb{E}\left[\sum_{k=1}^{\infty} X_k\right] = \sum_{k=1}^{\infty} \mathbb{E}[X_k] \tag{1.12}$$

and the statement also holds with an integral instead of a sum.

Definition 1.3.14 (Moment Generating Functions)**.** The *Moment Generating Function* (MGF) of a random variable X is defined as follows

$$M_X(t) := \mathbb{E}[e^{tX}] \tag{1.13}$$

Remark. The MGF of the normal distribution is a commonly used tool when dealing with Brownian Motion and functions of Brownian Motion as we will do throughout this report. Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$. Then

$$M_X(t) = \mathbb{E}[e^{tX}] = e^{t\mu + \frac{t^2\sigma^2}{2}} \tag{1.14}$$

1.4 Stochastic Processes

Definition 1.4.1 (Filtrations & Adaptedness)**.** A *filtered space* is $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^{\infty}, \mathbb{P})$ where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}$ are σ-algebras, jointly called a *filtration.* We also define $\mathcal{F}_\infty := \sigma(\cup_n \mathcal{F}_n) \subseteq \mathcal{F}$. We say a stochastic process or sequence of random variables X_n is adapted to the filtration $(\mathcal{F}_n)_{n\geq 0}$ if for every n, X_n is $\mathcal{F}_n$ -measurable.

Definition 1.4.2 (Martingales)**.** A process $(M_n)_{n\geq 0}$ in a filtered probability space is a *martingale with respect to a filtration* $(\mathcal{F}_n)_{n\geq 0}$ if

- M_n is adapted to $\mathcal{F}_n$,
- $\mathbb{E}[M_n] < \infty \; \forall \, n \geq 0$,
- $\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n$ a.s. $\forall \, n \geq 0$.

Definition 1.4.3 (Poisson Process)**.** Let $\lambda : \mathbb{R}^d \to [0, \infty)$ be a measurable and integrable function such that for every bounded region B the d-dimensional volume integral of λ is finite:

$$\Lambda(B) = \int_B \lambda(x)\mathrm{d}x < \infty \tag{1.15}$$

Then for every collection of disjoint bounded Borel-measurable sets $B_1, \ldots, B_k$, an inhomogeneous *Poisson Point Process* with *intensity function* λ has distribution

$$\mathbb{P}\{N(B_i) = n_i, i = 1, \ldots, k\} = \prod_{i=1}^{k} \frac{(\Lambda(B_i))^{n_i}}{n_i!} e^{-\Lambda(B_i)}. \tag{1.16}$$

Moreover,

$$\mathbb{E}[N(B)] = \Lambda(B). \tag{1.17}$$

Definition 1.4.4 (Brownian Motion)**.** Let $\mathcal{F}_t$ be a filtration. A stochastic process $W = (W_t)_{t\geq 0}$ is a standard one-dimensional *Brownian Motion* or *Wiener Process* if it satisfies the following:

- $W_0 = 0$ a.s.,

- Independent increments: $W_{t+s} - W_t$ is independent of $\mathcal{F}_t \; \forall \, t, s \geq 0$,
- W has stationary Gaussian increments: $W_{t+s} - W_t \sim \mathcal{N}(0, s)$,
- W has continuous sample paths: $W_t(\omega)$ is a continuous function of $t \; \forall \, \omega \in \Omega$.

Definition 1.4.5 (Predictable Processes)**.** A stochastic process X_t is *predictable* (in the discrete sense) if X_{t+1} is $\mathcal{F}_t$ measurable for all t. If X_t is a continous stochastic process, then it is predictable if it is measurable with respect to the σ-algebra generated by all left-continuous adapted processes.

Definition 1.4.6 (Progressive Measurability)**.** A continuous-time stochastic process (X_t) is progressively measurable if for every time t, the map $[0, t] \times \Omega \to \mathbb{R}$ defined by $(s, \omega) \to X_s(\omega)$ is $\mathcal{B}(\mathbb{R}) \times \mathcal{F}_t$-measurable. This is a slightly stronger condition than adaptedness, indeed, all progressively measurable processes are adapted but the converse is not true.

Definition 1.4.7 (Càdlàg processes)**.** Right continuous with left limits. Acronym from the french "continue à droite, limite à gauche".

Definition 1.4.8 (Semimartingales)**.** A real-valued process X is called a *semimartingale* if it can be decomposed as

$$X_t = M_t + A_t \tag{1.18}$$

where M is a local martingale and A is a càdlàg, adapted process of locally bounded variation.

1.5 Stochastic Integration

Definition 1.5.1 (Itô (Stochastic) Integral)**.** Let W be a standard Wiener process as defined above, and let H be a càdlàg, adapted (to W), and locally bounded process. If $\{\pi_n\}$ is a sequence of partitions on $[0, t]$ with mesh width decreasing to 0, then the *Itô integral* of H w.r.t. W is the random variable

$$\int_0^t H \mathrm{d}W := \lim_{n \to \infty} \sum_{[t_{i-1}, t_i] \in \pi_n} H_{t_{i-1}} (W_{t_i} - W_{t_{i-1}}) \tag{1.19}$$

Definition 1.5.2 (Itô Processes)**.** An Itô process is any adapted stochastic process that can be written as the sum of a deterministic integral w.r.t. time and a stochastic integral w.r.t. Brownian motion:

$$X_t = X_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W \tag{1.20}$$

where W is a standard Wiener process, σ is predictable and integrable w.r.t. W, and μ is predictable and Lebesgue integrable. Equivalently, in differential form, we may also write

$$\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t \tag{1.21}$$

Lemma 1.5.1 (Itô's Lemma)**.** Probably most important result in stochastic calculus. Provides an analogue of the chain rule, allowing us to find differentials for functions of Itô processes. Suppose X is an Itô process satisfying the differential form given above. Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be continuously differentiable at least once in the first argument and twice in the second. Then we have that

$$\mathrm{d}f = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2} \frac{\partial^2 f}{\partial x^2} \right) \mathrm{d}t + \sigma_t \frac{\partial f}{\partial x} \mathrm{d}W_t \tag{1.22}$$

Proof. Taylor expansions □

1.6 Stochastic Differential Equations

- From PDE to SDE
- Brownian motion as the solution to an SDE
- Geometric Brownian Motion

Overview for sake of brevity. Good introductory text is "Pragmatic Introduction to Stochastic Differential Equations" by cam univ press. Briefly discuss move from ODEs to SDEs given by Einstein and formulate in terms of Itô calculus as given by the more modern theory. Examples including BM, GBM. Motivate controlled diffusion.

Definition 1.6.1 (Strong Solution)**.** A strong solution to this SDE starting at time t is a progressively measurable process X such that for $s \leq t$:

$$X_s = X_t + \int_t^s b(X_u, \alpha_u)du + \int_t^s \sigma(X_u, \alpha_u)dW_u$$

and

$$\int_t^s |b(X_u, \alpha_u)|du + \int_t^s |\sigma(X_u, \alpha_u)|^2 du < \infty$$

a.s.

Definition 1.6.2 (Geometric Brownian Motion)**.** A *geometric brownian motion* is an adapted stochastic process which solves the following stochastic differential equation

$$dS_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t \tag{1.23}$$

for $\mu, \sigma \in \mathbb{R}$ and where W is a standard Wiender process. By Itô's formula with $f(S_t) = \log S_t$, we can write

$$\begin{aligned}
\mathrm{d}f(S_t) &= \frac{1}{S_t}\mathrm{d}S_t - \frac{1}{2S_t^2}(\mathrm{d}S_t)^2 \\
&= \mu \mathrm{d}t + \sigma \mathrm{d}W_t - \frac{1}{2}\sigma^2 \mathrm{d}t \\
\implies \log S_t &= \log S_0 + \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t \\
\implies S_t &= S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}
\end{aligned}$$

and we arrive at the canonical formula for the GBM, where S_0 is the initial value of the process.

Chapter 2

Stochastic Optimal Control

2.1 Introduction

In this chapter we introduce the idea of a stochastic control problem in one dimension, and construct a theoretical framework for the resolution of a regular solution, provided that such a solution exists (which is not guarunteed). We Primarily follow the text of Pham 2009. In section 2.2 we introduce the notion of a controlled diffusion process and its solution. In section 2.3 we consider a stochastic control problem over a finite time horizon, before introducing the dynamic programming principle and Hamilton-Jacobi-Bellman equation in section 2.4 and section 2.5 respectively. Finally, in section 2.6, we consider the verification theorem which allows us to validate the optimality of a candidate solution. We then procede in 2.7 to put these tools to use through a worked example in a financial context, setting us up to tackle the Avellaneda-Stoikov model in Chapter 3.

2.2 Controlled Diffusion Processes

In the previous chapter we have considered Itô processes that are governed either by constants or by functions of time and/or state. Using this, we could for example model a stock price, the movement of a particle, or any other system with the kinds of properties that we study above. If we have a portfolio of cash and an asset, we can model our wealth through time as a stochastic differential equation governed by the risk-free rate at which we earn returns on our cash, and the random fluctuations of the stock price.

For the market-maker however, this is insufficient. We described intuitively at the beginning of chapter 1 how a market maker might be able to influence the flow of orders they recieve over time, and hence their cash flow over time, by adjusting the limit bid and ask quotes that they send to the market. Hence, the market makers portfolio value is governed by not only the fluctuations of the stock and the risk-free rate, but also (stochastically) by the spread that they set. We thus need a model that allows our diffusion process to be governed by not only functions of time and state, but also of some other process which we will call α.

Throughout this chapter we will assume the background of a standard continuous and filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ as defined above in chapter 1.

Definition 2.2.1 (Controlled Diffusion Process)**.** We consider a control model where the state of the system is governed by an $\mathbb{R}$-valued SDE:

$$\mathrm{d}X_t = b(t, X_t, \alpha_t)\mathrm{d}s + \sigma(t, X_t, \alpha_t)\mathrm{d}W_t \tag{2.1}$$

where W is a standard Wiener process. The control $\alpha = (\alpha_t)$ is a *progressively measurable* process valued in $A \subseteq \mathbb{R}^m$.

The functions $b : \mathbb{R}^+ \times \mathbb{R} \times A \to \mathbb{R}$ and $\sigma : \mathbb{R}^+ \times \mathbb{R} \times A \to \mathbb{R}$ are measurable in all of their arguments and satisfy a uniform Lipschitz condition in A: There exists a $K \geq 0$ such that $\forall\, x, y \in \mathbb{R}, \forall\, a \in A$,

$$|b(x,a) - b(y,a)| + |\sigma(x,a) - \sigma(y,a)| \leq K|x-y|. \tag{2.2}$$

In what follows, for $0 \leq t \leq T < \infty$, we denote by $\mathcal{T}_{t,T}$ the set of *stopping times* valued in $[t, T]$.

We can see that we now have a diffusion process...

2.3 The Finite-Horizon Problem

Fix a finite horizon $0 < T < \infty$. We denote by $\mathcal{A}$ the set of control processes α such that for any arbitrary $x \in \mathbb{R}$,

$$\mathbb{E}\left[\int_0^T |b(x,\alpha_t)|^2 + |\sigma(x,\alpha_t)|^2 \mathrm{d}t\right] < \infty. \tag{2.3}$$

From Chapter 1, conditions (2.2) and (2.3) ensure the existence and uniqueness of a strong solution to the SDE (2.1) starting from any initial condition $(t,x) \in [0,T] \times \mathbb{R}$ and with any control process $\alpha \in \mathcal{A}$. We denote this unique strong solution with almost surely continuous sample paths by $\{X_s^{t,x}, t \leq s \leq T\}$.
Pham includes extra technical properties of the strong solution here which I may not require.
Next we set out our functional objective. Let $f : [0,T] \times \mathbb{R} \times A \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two measurable functions. We suppose that:

- g is lower-bounded **or**
- g satisfies a quadratic growth condition: $|g(x)| \leq C(1 + |x|^2)\ \forall\ x \in \mathbb{R}$ for some constant C independent of x.

We also denote by $\mathcal{A}(t,x)$ the subset of controls $\alpha \in \mathcal{A}$ such that

$$\mathbb{E}\left[\int_t^T |f(s, X_x^{t,x}, \alpha_s)|\mathrm{d}s\right] < \infty \tag{2.4}$$

for $(t,x) \in [0,T] \times \mathbb{R}$, and we assume that this set is not empty for all $(t,x) \in [0,T] \times \mathbb{R}$. We now define the *gain function:*

Definition 2.3.1 (Gain Function)**.**

$$J(t,x,\alpha) := \mathbb{E}\left[\int_t^T f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + g(X_T^{t,x})\right] \tag{2.5}$$

for all $(t,x) \in [0,T] \times \mathbb{R}$ and $\alpha \in \mathcal{A}(t,x)$.

Our objective is thus to maximise over possible control processes the gain function J, and to do this we introduce the associated *value function*:

Definition 2.3.2 (Value Function)**.**

$$v(t,x) := \sup_{\alpha \in \mathcal{A}(t,x)} J(t,x,\alpha). \tag{2.6}$$

Definition 2.3.3 (Optimal control)**.** Given an initial condition $(t, x) \in [0, T] \times \mathbb{R}$, we say that $\hat{\alpha} \in \mathcal{A}(t, x)$ is an optimal control if

$$v(t, x) = J(t, x, \hat{\alpha}). \tag{2.7}$$

Remark. A control process α of the form $\alpha_s = a(s, X_s^{t,x})$ for some measurable function $a : [0, T] \times \mathbb{R} \to A$ is called a *Markovian* control.

Pham includes a remark here about constant controls and conditions for the equivalence of $\mathcal{A}$ and $\mathcal{A}(t, x)$ which I probably don't need.
Include words on the interpretation and intuition of the f, g, the gain and value functions

2.4 The Dynamic Programming Principle

The Dynamic Programming Principle (DPP) is the fundamental tool upon which much of the theory of stochastic control relies. We formulate it as follows, considering only the context of the finite-horizon problem described above.

Theorem 2.4.1 (Dynamic Programming Principle)**.** Let $(t, x) \in [0, T] \times \mathbb{R}$. Then we have

$$\begin{aligned} v(t, x) &= \sup_{\alpha \in \mathcal{A}(t,x)} \sup_{\theta \in \mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right] && (2.8) \\ &= \sup_{\alpha \in \mathcal{A}(t,x)} \inf_{\theta \in \mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right] && (2.9) \\ & && (2.10) \end{aligned}$$

Proof of the DPP. By pathwise uniqueness of the SDE for X, for any admissible control $\alpha \in \mathcal{A}(t, x)$, for any $\theta \in \mathcal{T}_{t,T}$ and for all $s \geq \theta$

$$X_s^{t,x} = X_s^{\theta, X_\theta^{t,x}}. \tag{2.11}$$

By the law of iterated expectations we then have

$$\begin{aligned} J(t, x, \alpha) &= \mathbb{E}\left[\int_t^T f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + g(X_T^{t,x})\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\int_t^T f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + g(X_T^{t,x}) | \mathcal{F}_\theta\right]\right] \\ &= \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + \mathbb{E}\left[\int_\theta^T f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + g(X_T^{t,x}) | \mathcal{F}_\theta\right]\right] \\ &= \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + \mathbb{E}\left[\int_\theta^T f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + g(X_T^{t,x})\right]\right] \\ &= \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + J(\theta, X_\theta^{t,x}, \alpha)\right] \end{aligned}$$

and since $J(., ., \alpha) \leq v$ and θ is arbitrary in $\mathcal{T}_{t,T}$ we obtain

$$\begin{aligned} J(t, x, \alpha) &\leq \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right] \\ &\leq \inf_{\theta \in \mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right] \\ &\leq \sup_{\alpha \in \mathcal{A}(t,x)} \inf_{\theta \in \mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right] \end{aligned}$$

and by taking the supremum over α in the left hand side, we obtain the second of the desired inequalities:

$$v(t,x) \le \sup_{\alpha\in\mathcal{A}(t,x)} \inf_{\theta\in\mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},\alpha_s)\mathrm{d}s + v(\theta,X_\theta^{t,x})\right]. \tag{2.12}$$

Next we fix an arbitrary control $\alpha \in \mathcal{A}(t,x)$ and $\theta \in \mathcal{T}_{t,T}$. By the definition of the value function, for any $\epsilon > 0$ and $\omega \in \Omega$ there exists an $\alpha^{\epsilon,\omega} \in \mathcal{A}(\theta(\omega), X_{\theta(\omega)}^{t,x}(\omega))$ that is an ϵ-optimal control for $v(\theta, X_{\theta(\omega)}^{t,x}(\omega))$, i.e.

$$v(\theta, X_{\theta(\omega)}^{t,x}(\omega)) - \epsilon \le J(\theta(\omega), X_{\theta(\omega)}^{t,x}(\omega), \alpha^{\epsilon,\omega}). \tag{2.13}$$

Maybe include some more intuition behind this point?
We now define the process

$$\hat{\alpha}_s(\omega) = \begin{cases} \alpha_s(\omega), s\in[0,\theta(\omega)], \\ \alpha_s^{\epsilon,\omega}(\omega), s\in[\theta(\omega),T]. \end{cases} \tag{2.14}$$

It can be shown by the measurable selection theorem
do this in the appendix
that the process $\hat{\alpha}$ is progressively measurable, and so lies in $\mathcal{A}(t,x)$. Again by the law of iterated expectations and (2.13) we get

$$\begin{aligned} v(t,x) \ge J(t,x,\hat{\alpha}) &= \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},\alpha_s)\mathrm{d}s + J(\theta,X_\theta^{t,x},\alpha^\epsilon)\right] \\ &\ge \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},\alpha_s)\mathrm{d}s + v(\theta,X_\theta^{t,x})\right] - \epsilon. \end{aligned}$$

Finally, by the fact that $\alpha \in \mathcal{A}(t,x), \theta \in \mathcal{T}_{t,T}$ and $\epsilon > 0$ are all arbitrary, we obtain the first inequality:

$$v(t,x) = \sup_{\alpha\in\mathcal{A}(t,x)} \sup_{\theta\in\mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},\alpha_s)\mathrm{d}s + v(\theta,X_\theta^{t,x})\right]. \tag{2.15}$$

□

Remark (Equivalent Formulations). We normally write the DPP as

$$v(t,x) = \sup_{\alpha\in\mathcal{A}(t,x)} \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},\alpha_s)\mathrm{d}s + v(\theta,X_\theta^{t,x})\right], \tag{2.16}$$

however it is sometimes useful to use the following equivalent formulation of the DPP:
(i) For all $\alpha \in \mathcal{A}(t,x)$ and $\theta \in \mathcal{T}_{t,T}$:

$$v(t,x) \ge \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},\alpha_s)\mathrm{d}s + v(\theta,X_\theta^{t,x})\right]. \tag{2.17}$$

(ii) For all $\epsilon > 0$, there exists $\alpha \in \mathcal{A}(t,x)$ such that for all $\theta \in \mathcal{T}_{t,T}$:

$$v(t,x) - \epsilon \le \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},\alpha_s)\mathrm{d}s + v(\theta,X_\theta^{t,x})\right]. \tag{2.18}$$

Include words on the interpretation and intuition of the DPP.

2.5 Hamilton-Jacobi-Bellman Equation

The Dynamic Programming Principle tells us that we can consider a stochastic control problem as a sequence of smaller sub-problems defined over intervals of $[0,T]$ characterised by stopping times, i.e., $[0,T]=[0,\theta_1]\cup(\theta_1,\theta_2]\cup\cdots\cup(\theta_n,T]$ where $\theta_1\leq\cdots\leq\theta_n\in\mathcal{T}_{t,T}$. Thus, a natural thing to consider is the following: What happens as $n\to\infty$ and correspondingly $\theta_{i+1}-\theta_i\to 0$? What we obtain is the Hamilton-Jacobi-Bellman equation (HJB) which describes the dynamics of the value function over small increments of time. In this chapter and what follows, we will use the HJB equation as follows:

- Provide a formal derivation of the HJB equation.
- Obtain or try to show the existence of a smooth solution.
- Verification step: Show that the smooth solution is the value function.
- As a byproduct, we obtain an optimal feedback control.

Theorem 2.5.1 (Hamilton-Jacobi-Bellman Equation)**.** The dynamics of the value function $v(t,x)$ satisfy the following non-linear second-order partial differential equation:

$$\begin{cases}\dfrac{\partial v}{\partial t}(t,x)+\sup\limits_{a\in A}\left[\mathcal{L}^a v(t,x)+f(t,x,a)\right]=0\ \forall\ (t,x)\in[0,T)\times\mathbb{R}\\ v(T,x)=g(x)\ \forall\ x\in\mathbb{R}.\end{cases}\tag{2.19}$$

where $\mathcal{L}^a$ is the operator associated to the diffusion (2.1) and defined by (see Section 1.6)

$$\mathcal{L}^a v=b(t,x,a)v_x+\frac{1}{2}\sigma(t,x,a)^2 v_{xx}.\tag{2.20}$$

Proof. Let us consider time $\theta=t+h$ and a constant control $\alpha_s=a$ for some arbitrary $a\in A$, in our slightly stronger variant of the DPP (2.17):

$$v(t,x)\geq\mathbb{E}\left[\int_t^{t+h}f(s,X_s^{t,x},a)\mathrm{d}s+v(t+h,X_{t+h}^{t,x})\right].\tag{2.21}$$

By assuming that v is smooth enough, we can apply Itô's formula between t and $t+h$:

$$v(t+h,X_{t+h}^{t,x})=v(t,x)+\int_t^{t+h}\left(\frac{\partial v}{\partial t}+\mathcal{L}^a v\right)(s,X_s^{t,x})\mathrm{d}s+\text{(local) martingale.}\tag{2.22}$$

We can then substitute back into (2.21) to obtain

$$0\geq\mathbb{E}\left[\int_t^{t+h}\left(\frac{\partial v}{\partial t}+\mathcal{L}^a v\right)(s,X_s^{t,x})+f(s,X_s^{t,x},a)\mathrm{d}s\right]\tag{2.23}$$

which if we divide by h and send $h\to 0$ we yield

$$0\geq\frac{\partial v}{\partial t}(t,x)+\mathcal{L}^a v(t,x)+f(t,x,a)\tag{2.24}$$

by the mean-value theorem. Since this holds true for any $a\in A$, we obtain the inequality

$$-\frac{\partial v}{\partial t}(t,x)-\sup_{a\in A}[\mathcal{L}^a v(t,x)+f(t,x,a)]\geq 0.\tag{2.25}$$

On the other hand, suppose that α^* is an optimal control. Then in (2.16) we have

$$v(t,x)=\mathbb{E}\left[\int_t^{t+h}f(s,X_s^*,\alpha_s^*)\mathrm{d}s+v(t+h,X_{t+h}^*)\right],\tag{2.26}$$

where X^* is the solution to (2.1) starting from state x at time t with control α^*. Again by Itô's formula we have that

$$v(t+h, X^*_{t+h}) = v(t,x) + \int_t^{t+h} \left(\frac{\partial v}{\partial t} + \mathcal{L}^a v \right)(s, X^*_s)\mathrm{d}s + \text{(local) martingale} \tag{2.27}$$

which we can again substitute back into (2.26) to obtain

$$0 = \mathbb{E}\left[\int_t^{t+h} \left(\frac{\partial v}{\partial t} + \mathcal{L}^a v \right)(s, X^*_s) + f(s, X_s*, a)\mathrm{d}s \right] \tag{2.28}$$

and hence once again we divide by h and send $h \to 0$ yielding

$$-\frac{\partial v}{\partial t}(t,x) - \mathcal{L}^{\alpha^*_t} v(t,x) - f(t,x,\alpha^*_t) = 0. \tag{2.29}$$

Combining this with (2.25), v should satisfy

$$-\frac{\partial v}{\partial t}(t,x) - \sup_{a \in A}[\mathcal{L}^a v(t,x) + f(t,x,a)] = 0 \;\forall\; (t,x) \in [0,T) \times \mathbb{R}, \tag{2.30}$$

if the above supremum in a is finite. This may arise when the control space A is unbounded, and we will see how to deal with this later on. We can also obtain the terminal condition associated to this PDE:

$$v(T,x) = g(x) \;\forall\; x \in \mathbb{R} \tag{2.31}$$

which results immediately from the definition in (2.6) of the value function considered at the horizon time T. □

2.6 Verification Theorem

Theorem 2.6.1 (Verification Theorem)**.** Let $w : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function which is continuously differentiable at least once in its first argument and twice in its second. Let w also satisfy a quadratic growth condition, i.e. there exists a constant C such that

$$|w(t,x)| \le C(1+x^2) \;\forall\; (t,x) \in [0,T] \times \mathbb{R}.$$

(i) Suppose that

$$\frac{\partial w}{\partial t}(t,x) + \sup_{a \in A}[\mathcal{L}^a w(t,x) + f(t,x,a)] \ge 0,\; (t,x) \in [0,T) \times \mathbb{R} \tag{2.32}$$

$$w(T,x) \ge g(x),\; x \in \mathbb{R}. \tag{2.33}$$

Then $w \ge v$ on $[0,T] \times \mathbb{R}$.
(ii) Suppose further that $w(T,.) = g$ and that there exists a measurable function $\hat{\alpha} : [0,T) \times \mathbb{R} \to A$ such that

$$\frac{\partial w}{\partial t}(t,x) + \sup_{a \in A}[\mathcal{L}^a w(t,x) + f(t,x,a)] = \frac{\partial w}{\partial t}(t,x) + \mathcal{L}^{\hat{\alpha}(t,x)} w(t,x) + f(t,x,\hat{\alpha}(t,x)) = 0,$$

the SDE

$$\mathrm{d}X_t = b(s, X_s, \hat{\alpha}(s, X_s))\mathrm{d}s + \sigma(s, X_s, \hat{\alpha}(s, X_s))\mathrm{d}W_s$$

admits a unique solution denoted by $\hat{X}^{t,x}_s$ given an initial condition $X_t = x$, and the process $\{\hat{\alpha}(s, \hat{X}^{t,x}_s) : t \le s \le T\}$ lies in $\mathcal{A}(t,x)$. Then

$$w = v \text{ on } [0,T] \times \mathbb{R} \tag{2.34}$$

and $\hat{\alpha}$ is an optimal Markovian control.

Proof. □

2.7 A Worked Example

Finite-Horizon Merton Portfolio Allocation Problem

Chapter 3

The Avellaneda-Stoikov Model

3.1 Introduction

In this chapter, we can finally return to the market-making problem introduced in Chapter 1, fully armed with the theory of stochastic optimal control that we have built up in Chapter 2. We will formulate the problem and our assumptions in the framework of Avellaneda and Stoikov 2008, and walk through their methodology and theoretical results.

We begin in section 3.2 by setting out our assumptions our assumptions about the dynamics of the market mid-point price and our agents utility function. In section 3.3, we introduce the concept of an indifference or reservation price in the context of a passive agent with constant inventory, and derive some expressions. In 3.4, we briefly analyse the infinite time horizon case, showing that analagous reservation prices exist, which may be of greater interest to dealers in markets that trade 24/7 such as FX and crypto. In section 3.5 we return to the finite horizon setting and define concepts such as market impact, arriving at the objective function of the agent who can set limit orders and thus influence the dynamics of their wealth over time.

Of crucial importance to this agent are the statistical properties of market orders: Their arrival frequency, the distribution of their size, and how they impact prices, which we discuss in section 3.6. Next we derive the Hamilton-Jacobi-Bellman equation in section 3.7, and introduce an ansatz which allows us to simplify our problem and derive some useful relations between the agents reservation price and optimal bid-ask spread. Finally, we introduce some analytical approximations in section 3.8 that enable us to derive an approximate solution in terms of our model parameters.

The main result, which we summarise in section 3.9, is that optimal bid and ask quotes can be computed through an intuitive two-step procedure: First, the agent computes a personal reservation price for the asset, given her current inventory. Second, she calibrates her bid and ask quotes to the limit order book, by considering the probability with which her quotes will be executed as a function of their distance from the midpoint price.

3.2 Model assumptions

The paper of Avellaneda and Stoikov 2008 is closely related to that of Ho and Stoll 1981, with the crucial difference being that while Ho and Stoll consider a monopolistic dealer, Avellaneda and Stoikov consider a dealer who is potentially one of many dealers and many other market participants who may set limit orders.

In Ho and Stoll (1981), the authors specify a 'true' price for the asset, and then allow the dealer to set quotes around this price. This may be more applicable to OTC markets in illiquid products where there is no openly accessible limit orderbook, but Avellaneda and Stoikov consider a dealer operating in an openly accessible limit orderbook, and hence it makes sense to view the mid-point price in the orderbook as the true price of the security.

Another point to make here is that Avellaneda and Stoikov consider a dealer who is concerned only with inventory risk, not asymmetric information, and so assuming that other market participants are better informed and reacting to fundamental arbitrage opportunities, by the efficient market hypothesis we would have that the market mid price is the best available measure of the true price of the asset given all the information available up to a particular point in time.

We will assume that the market mid-point price evolves according to the SDE

$$\mathrm{d}S_u = \sigma \mathrm{d}W_u \tag{3.1}$$

with initial value $S_t = s$. W_t is a standard one-dimensional Brownian motion, and $\sigma > 0$ is constant. Underlying this model is an implicit assumption that the agent has no opinion on the drift or any autocorrelation or stochasticity of volatility for the stock.

We also assume for simplicity that the money market pays no interest. Moreover, the limit orders set by the agent can be continuously updated at no cost. In reality, the cost of trading will differ depending on the exchange in question, as most charge a small percentage fee of every executed trade and some only charge market orders, while providing rebates to dealers' trades for the liquidity they provide. Finally, we assume that the lot sizes our limit orders are constant at one share per order, and that the overall arrival frequency of market orders is constant.

We summarise our assumptions in the list below:

- The dealer being modelled is one of many players in the market
- The 'true' price is given by the market mid-price
- The mid-price evolves according to a brownian motion with constant volatility σ
- The agent has no opinion on drift or autocorrelation of the stock price
- The money-market pays no interest
- Limit orders can be continuously updated at no cost
- Limit orders are of fixed size 1
- The arrival frequency of market orders to the market is constant

3.3 Modelling an inactive trader

Our agents objective will be to maximise the expected utility of their wealth at a terminal time T. Avellaneda and Stoikov's choice of exponential utility is convenient since its convexity allows us to define reservation prices that are indepenent of the agents current wealth.

The utility function

Initially, we consider an inactive trader who holds a fixed inventory of q stocks until the terminal time T. The agent's value function is

$$v(x, s, q, t) = \mathbb{E}\left[-e^{-\gamma(x+qS_T)}|\mathcal{F}_t\right] \tag{3.2}$$

where x is the initial wealth in dollars, t is the present time and γ is a personal pre-defined risk-aversion parameter. By some simple manipulations, we can write this in a more convenient form as follows:

$$\begin{aligned} v(x, s, q, t) &= \mathbb{E}\left[-e^{-\gamma(x+qS_T)}|\mathcal{F}_t\right] \\ &= -e^{-\gamma x}\mathbb{E}\left[e^{-\gamma q S_T}|\mathcal{F}_t\right] \\ &= -e^{-\gamma x}e^{-\gamma q s+\frac{\gamma^2 q^2 \sigma^2 (T-t)}{2}} \\ &= -e^{-\gamma x}e^{-\gamma q s}e^{\frac{\gamma^2 q^2 \sigma^2 (T-t)}{2}} \end{aligned}$$

Reservation prices

Following Avellaneda and Stoikov 2008, we can now use our value function to define the agents reservation bid and ask prices. The reservation bid and reservation ask prices are simply the prices at which the agent is indifferent between buying/selling and doing nothing. In other words, the reservation bid (ask) is the price at which the agent is indifferent between her current portfolio and her current portfolio $\pm$ one stock and $\mp$ the cash price.

Definition 3.3.1 (Reservation bid price)**.** Let v be the value function of the agent. Its reservation bid price r^b is given implicitly by the relation

$$v(x - r^b(s, q, t), s, q + 1, t) = v(x, s, q, t) \tag{3.3}$$

and the corresponding reservation ask price r^a is similarly implicit in the relation

$$v(x + r^a(s, q, t), s, q - 1, t) = v(x, s, q, t). \tag{3.4}$$

We can determine an exact expression for $r^b(s, q, t)$ by plugging our prior definition for the value function, (3.2), in to our relation (3.3) as follows:

$$\begin{aligned} v(x - r^b(s, q, t), s, q + 1, t) &= v(x, s, q, t) \\ -e^{-\gamma(x-r^b(s,q,t))}e^{-\gamma s(q+1)}e^{\frac{\gamma^2(q+1)^2\sigma^2(T-t)}{2}} &= -e^{-\gamma x}e^{-\gamma q s}e^{\frac{\gamma^2 q^2 \sigma^2(T-t)}{2}} \\ -\gamma(x - r^b(s, q, t)) - \gamma s(q + 1) + \frac{\gamma^2(q + 1)^2\sigma^2(T - t)}{2} &= -\gamma x - \gamma q s + \frac{\gamma^2 q^2 \sigma^2(T - t)}{2} \\ \gamma r^b(s, q, t) - \gamma s + \frac{\gamma^2(1 + 2q)\sigma^2(T - t)}{2} &= 0, \end{aligned}$$

dividing by γ and rearranging to obtain

$$r^b(s, q, t) = s + (-1 - 2q)\frac{\gamma\sigma^2(T - t)}{2} \tag{3.5}$$

Similarly for $r^a(s,q,t)$:

$$\begin{aligned}
v(x + r^a(s,q,t), s, q-1, t) &= v(x,s,q,t) \\
-e^{-\gamma(x+r^a(s,q,t))}e^{-\gamma s(q-1)}e^{\frac{\gamma^2(q-1)^2\sigma^2(T-t)}{2}} &= -e^{-\gamma x}e^{-\gamma qs}e^{\frac{\gamma^2q^2\sigma^2(T-t)}{2}} \\
-\gamma(x + r^a(s,q,t)) - \gamma s(q-1) + \frac{\gamma^2(q-1)^2\sigma^2(T-t)}{2} &= -\gamma x - \gamma qs + \frac{\gamma^2q^2\sigma^2(T-t)}{2} \\
-\gamma r^a(s,q,t) + \gamma s + \frac{\gamma^2(1-2q)\sigma^2(T-t)}{2} &= 0,
\end{aligned}$$

again dividing by γ and rearranging to obtain

$$r^a(s,q,t) = s + (1-2q)\frac{\gamma\sigma^2(T-t)}{2} \tag{3.6}$$

We define the *reservation* or *indifference* price to be the average of these two *given* that the agent currently holds q stocks:

$$\begin{aligned}
r(s,q,t) &= \frac{r^a(s,q,t) + r^b(s,q,t)}{2} \\
&= \frac{s + (1-2q)\frac{\gamma\sigma^2(T-t)}{2} + s + (-1-2q)\frac{\gamma\sigma^2(T-t)}{2}}{2} \\
&= \frac{2s - 2q\gamma\sigma^2(T-t)}{2} \\
&= s - q\gamma\sigma^2(T-t)
\end{aligned}$$

This price is nothing more than an adjustment to the mid-price which accounts for the effect of the inventory held by the agent on the agents preference to buy or sell. It is easy to see that if the agent is long stock ($q > 0$), the reservation price will be lower than the mid-price, reflecting the agents willingness to sell at a discount in order to reduce its inventory. Conversely, if the agent is short stock ($q < 0$), its reservation price will be greater than the mid-price, indicating the agents preference to buy at a premium to the market in order to return to a market-neutral position.

We note that the expressions derived above for r^a and r^b (and consequently r) exist in the setting where q is a fixed constant, and therefore it is not so simple to derive these expressions when our agent is permitted to set limit orders. However, they are important both as an illustrative example and because when we introduce our approximate solution in 3.8, we will arrive at a very similar reservation price.

3.4 The Optimising Agent with Infinite Horizon

We will now briefly analyse the infinite horizon variant of the dealer problem, showing that we can derive a stationary version of the reservation price through defining an infinite horizon variant of our value function including a discount factor. This is necessary since in our finite horizon case discussed above, our reservation price is dependent upon the time interval $T - t$. The intuition for this is that at or close to T, the agent may liquidate any remaining inventory for (or at least close to) S_T, hence the closer time is to T, the less risk there is in the dealer's position.

We consider an infinite-horizon value function of the form

$$\bar{v}(x,s,q) = \mathbb{E}\left[\int_0^\infty -e^{-\omega t}e^{-\gamma(x+qS_t)}\mathrm{d}t\right]$$

where ω is our discount factor. An interpretation of ω is that it may represent an upper bound on the absolute inventory position that the agent is allowed to build up. A natural choice is to take $\omega = \frac{1}{2}\gamma^2\sigma^2(q_{\max}+1)^2$, this will be justified shortly.

Using the definition of reservation bid and ask prices given above in section 3.3, we can attain stationary versions of the reservation prices r^b and r^a with much the same method as before, only relying on slightly more advanced theory, appealing to Tonelli's theorem (1.11) which allows us to swap the expectation and integral in the value function. For r^b, we have the following:

$$
\begin{aligned}
\bar{v}(x-\bar{r}^b(s,q),s,q+1) &= \bar{v}(x,s,q) \\
\mathbb{E}\left[\int_0^\infty -e^{-\omega t}e^{-\gamma(x-\bar{r}^b(s,q)+(q+1)S_t)}\mathrm{d}t\right] &= \mathbb{E}\left[\int_0^\infty -e^{-\omega t}e^{-\gamma(x+qS_t)}\mathrm{d}t\right] \\
\int_0^\infty e^{-\omega t}e^{-\gamma(x-\bar{r}^b(s,q))}\mathbb{E}\left[e^{-\gamma(q+1)S_t}\right]\mathrm{d}t &= \int_0^\infty e^{-\omega t}e^{-\gamma x}\mathbb{E}\left[e^{-\gamma q S_t}\right]\mathrm{d}t \text{ (by Tonelli)} \\
e^{-\gamma(x-\bar{r}^b(s,q))}\int_0^\infty e^{-\omega t}e^{-\gamma(q+1)s+\frac{\gamma^2(q-1)^2\sigma^2 t}{2}}\mathrm{d}t &= e^{-\gamma x}\int_0^\infty e^{-\omega t}e^{-\gamma q s+\frac{\gamma^2 q^2\sigma^2 t}{2}}\mathrm{d}t \\
e^{-\gamma(x-\bar{r}^b(s,q))}e^{-\gamma(q+1)s}\int_0^\infty e^{-\omega t}e^{\frac{\gamma^2(q+1)^2\sigma^2 t}{2}}\mathrm{d}t &= e^{-\gamma x}e^{-\gamma q s}\int_0^\infty e^{-\omega t}e^{\frac{\gamma^2 q^2\sigma^2 t}{2}}\mathrm{d}t \\
e^{\gamma\bar{r}^b(s,q)}e^{-\gamma s}\int_0^\infty e^{\left(\frac{\gamma^2(q+1)^2\sigma^2-2\omega}{2}\right)t}\mathrm{d}t &= \int_0^\infty e^{\left(\frac{\gamma^2 q^2\sigma^2-2\omega}{2}\right)t}\mathrm{d}t \\
e^{\gamma\bar{r}^b(s,q)}e^{-\gamma s}\left(\frac{2}{2\omega-\gamma^2(q+1)^2\sigma^2}\right) &= \left(\frac{2}{2\omega-\gamma^2 q^2\sigma^2}\right) \\
e^{\gamma(\bar{r}^b(s,q)-s)} &= \frac{2\omega-\gamma^2(q+1)^2\sigma^2}{2\omega-\gamma^2 q^2\sigma^2} \\
e^{\gamma(\bar{r}^b(s,q)-s)} &= 1-\frac{(1+2q)\gamma^2\sigma^2}{2\omega-\gamma^2 q^2\sigma^2} \\
\gamma\bar{r}^b(s,q)-\gamma s &= \log\left(1+\frac{(-1-2q)\gamma^2\sigma^2}{2\omega-\gamma^2 q^2\sigma^2}\right) \\
\bar{r}^b(s,q) &= s+\frac{1}{\gamma}\log\left(1+\frac{(-1-2q)\gamma^2\sigma^2}{2\omega-\gamma^2 q^2\sigma^2}\right)
\end{aligned}
$$

which is valid for $\omega > \frac{1}{2}\gamma^2\sigma^2 q^2$ and agrees exactly with the result presented in Avellaneda and Stoikov 2008. We can now perform the same procedure for the reservation ask price

r^a:

$$\begin{aligned}
\bar{v}(x + r^a(s,q), s, q-1) &= \bar{v}(x,s,q) \\
\mathbb{E}\left[\int_0^\infty -e^{-\omega t}e^{-\gamma(x+r^a(s,q)+(q-1)S_t)}\mathrm{d}t\right] &= \mathbb{E}\left[\int_0^\infty -e^{-\omega t}e^{-\gamma(x+qS_t)}\mathrm{d}t\right] \\
\int_0^\infty e^{-\omega t}e^{-\gamma(x+r^a(s,q))}\mathbb{E}\left[e^{-\gamma(q-1)S_t}\right]\mathrm{d}t &= \int_0^\infty e^{-\omega t}e^{-\gamma x}\mathbb{E}\left[e^{-\gamma q S_t}\right]\mathrm{d}t \text{ (by Tonelli)} \\
e^{-\gamma(x+r^a(s,q))}\int_0^\infty e^{-\omega t}e^{-\gamma(q-1)s+\frac{\gamma^2(q-1)^2\sigma^2 t}{2}}\mathrm{d}t &= e^{-\gamma x}\int_0^\infty e^{-\omega t}e^{-\gamma q s+\frac{\gamma^2 q^2\sigma^2 t}{2}}\mathrm{d}t \\
e^{-\gamma(x+r^a(s,q))}e^{-\gamma(q-1)s}\int_0^\infty e^{-\omega t}e^{\frac{\gamma^2(q-1)^2\sigma^2 t}{2}}\mathrm{d}t &= e^{-\gamma x}e^{-\gamma q s}\int_0^\infty e^{-\omega t}e^{\frac{\gamma^2 q^2\sigma^2 t}{2}}\mathrm{d}t \\
e^{-\gamma r^a(s,q)}e^{\gamma s}\int_0^\infty e^{\left(\frac{\gamma^2(q-1)^2\sigma^2-2\omega}{2}\right)t}\mathrm{d}t &= \int_0^\infty e^{\left(\frac{\gamma^2 q^2\sigma^2-2\omega}{2}\right)t}\mathrm{d}t \\
e^{-\gamma r^a(s,q)}e^{\gamma s}\left(\frac{2}{2\omega-\gamma^2(q-1)^2\sigma^2}\right) &= \left(\frac{2}{2\omega-\gamma^2 q^2\sigma^2}\right) \\
e^{\gamma(s-r^a(s,q))} &= \frac{2\omega-\gamma^2(q-1)^2\sigma^2}{2\omega-\gamma^2q^2\sigma^2} \\
e^{\gamma(s-r^a(s,q))} &= 1-\frac{(1-2q)\gamma^2\sigma^2}{2\omega-\gamma^2q^2\sigma^2} \\
\gamma s-\gamma r^a(s,q) &= \log\left(1-\frac{(1-2q)\gamma^2\sigma^2}{2\omega-\gamma^2q^2\sigma^2}\right) \\
r^a(s,q) &= s-\frac{1}{\gamma}\log\left(1-\frac{(1-2q)\gamma^2\sigma^2}{2\omega-\gamma^2q^2\sigma^2}\right)
\end{aligned}$$

which is again valid for $\omega > \frac{1}{2}\gamma^2\sigma^2q^2$. However, this differs from the result presented in Avellaneda and Stoikov 2008, which they give to be:

$$r^a(s,q) = s+\frac{1}{\gamma}\log\left(1+\frac{(1-2q)\gamma^2\sigma^2}{2\omega-\gamma^2q^2\sigma^2}\right)$$

Probably don't include this last bit
Using the correct values of the reservation prices obtained above, we can derive an expression for the reservation price as we did before.

$$\begin{aligned}
r(s,q) :=& \frac{r^a(s,q)+r^b(s,q)}{2} \\
=& \frac{s-\frac{1}{\gamma}\log\left(1-\frac{(1-2q)\gamma^2\sigma^2}{2\omega-\gamma^2q^2\sigma^2}\right)+s+\frac{1}{\gamma}\log\left(1+\frac{(-1-2q)\gamma^2\sigma^2}{2\omega-\gamma^2q^2\sigma^2}\right)}{2} \\
=& s+\frac{1}{2\gamma}\left(\log\left(1+\frac{(-1-2q)\gamma^2\sigma^2}{2\omega-\gamma^2q^2\sigma^2}\right)-\log\left(1-\frac{(1-2q)\gamma^2\sigma^2}{2\omega-\gamma^2q^2\sigma^2}\right)\right) \\
=& s+\frac{1}{2\gamma}\log\left(\frac{1-\frac{(1+2q)\gamma^2\sigma^2}{2\omega-\gamma^2q^2\sigma^2}}{1-\frac{(1-2q)\gamma^2\sigma^2}{2\omega-\gamma^2q^2\sigma^2}}\right) \\
=& s+\frac{1}{2\gamma}\log\left(\frac{2\omega-\gamma^2q^2\sigma^2-(1+2q)\gamma^2\sigma^2}{2\omega-\gamma^2q^2\sigma^2-(1-2q)\gamma^2\sigma^2}\right) \\
=& s+\frac{1}{2\gamma}\log\left(1+\frac{4q\gamma^2\sigma^2}{2\omega-\gamma^2q^2\sigma^2-(1-2q)\gamma^2\sigma^2}\right)
\end{aligned}$$

3.5 Modelling Limit Orders

Now that we have defined and discussed the idea of a reservation price for the dealer, we should move on to considering the case of the dealer who can indirectly influence both their inventory and cash flow through the setting of limit orders.

As mentioned in section 3.2, the agent quotes bid and ask limit orders in lot sizes of 1 only. We denote the agent's quotes p^a and p^b for the ask and bid respectively, and note that the agent is comitted to sell or buy 1 unit of stock respectively should these orders be "hit" or "lifted" by an incoming market order. These quotes can also be updated continuously at no cost. The distances

$$\delta^a := p^a - s \tag{3.7}$$

and

$$\delta^b := s - p^b \tag{3.8}$$

as well as the current shape of the orderbook determine the priority of execution when large market orders are placed.

For example, when a market order to buy Q shares arrives, the Q limit orders with the lowest ask prices will be lifted automatically by the exchanges matching engine. If Q is greater than the number of shares available at the lowest ask level in the orderbook, the order causes a temporary market impact since transactions will occur at a price not only higher than the mid-price, but higher than the best ask.

Definition 3.5.1 (Temporary market impact)**.** Let p^Q be the price of the highest (most expensive) limit order executed in this trade. Then

$$\Delta p := p^Q - s \tag{3.9}$$

is the temporary market impact of the trade of size Q.

Then we have that if our agent's $\delta^a < \Delta p$, our agents limit order will be executed. We will assume that market buy orders will lift our agent's sell limit orders with a Poisson intensity function denoted $\lambda^a(\delta^a)$ which is a decreasing function of δ^a. Likewise, we assume that market sell orders will hit our agent's bid limit orders with Poisson intensity $\lambda^b(\delta^b)$, decreasing in δ^b. Intuitively, this encapsulates the fact that further away from the mid-price the agent places her quotes, the less often she will receive market orders.

Now, our cash wealth and portfolio of stock is stochastic and depends on the incoming flow of market buy and sell orders. Naturally, both our cash flow and inventory jump every time a market order executes one of our agent's limit orders. Let N_t^a and N_t^b be Poisson point processes with intensities λ^a and λ^b, representing the amount of stocks sold or bought by the agent up to time t respectively. Our inventory at time t is thus

$$q_t := N_t^b - N_t^a \tag{3.10}$$

and our wealth process evolves according to

$$\mathrm{d}Xt = p^a \mathrm{d}N_t^a - p^b \mathrm{d}N_t^b. \tag{3.11}$$

Finally, we can reformulate our value function from section 3.3. The goal we set for our agent is still to maximise the expected expontential utility of terminal wealth, however now the cash and inventory components of our terminal portfolio are stochastic as well as the mid-price itself. Hence, our value function becomes the following:

Definition 3.5.2 (Value function of Market-Making Agent)**.**

$$u(s,x,q,t) := \max_{\delta^a,\delta^b} \mathbb{E}\left[-e^{-\gamma(X_T+q_T S_T)}|\mathcal{F}_t\right] \tag{3.12}$$

Notice that our agent choses its quote spreads δ^a and δ^b, and hence controls its quotes p^a and p^b. This means that the agent therefore indirectly influences the flow of orders she receives.

In the next section we will consider some realistic forms for the functions λ^a and λ^b based on results in the econophysics literature exploring the statistical properties of the limit orderbook, before turning to the application of Stochastic Control and solution to the above problem in section 3.7.

3.6 Modelling Trading Intensity

Here, we will focus on deriving a realistic form for the Poisson intensity λ with which a limit order will be executed as a function of its distance δ to the mid-price. In order to quantify this, we need to infer some statistics regarding

- The overall frequency of market orders
- The distribution of the size of market orders
- The temporary price impact of a large market order

For simplicity, we will assume a constant frequency Λ of market buy or sell orders. In practice, this could be estimated by simply dividing the total volume bought or sold in a given time interval by the average volume of market buy/sell orders in that interval.

Distribution of the size of market orders

The distribution of size of market orders has been found to obey a power law:

Theorem 3.6.1 (Density of Market Order Size)**.** The distribution of size of market orders has been found to obey a power law:

$$f^Q(x) \propto x^{-1-\alpha} \tag{3.13}$$

for large x, with $\alpha = 1.53$ in Gopikrishnan et al. 2000 for US stocks, $\alpha = 1.4$ in Maslov and Mills 2001 for shares traded on the NASDAQ and $\alpha = 1.5$ in Gabaix et al. 2006 for shares on the Paris Bourse.

Modelling market impact

Here there is much less consensus on market impact, due to lack of agreement on how to define it and how to measure it. Some papers find that the change in price Δp after a market order of size Q is described well by

$$\Delta p \propto Q^\beta \tag{3.14}$$

with $\beta = 0.5$ in Gabaix et al. 2006 and $\beta = 0.76$ in Weber and Rosenow 2005, while Potters and Bouchaud 2003 find a better fit to the relationship

$$\Delta p \propto \log(Q). \tag{3.15}$$

Using (3.13) and (3.15) we can derive the poisson intensity as follows:

$$\begin{aligned}
\lambda(\delta) &= \Lambda\mathbb{P}(\delta < \Delta p) \\
&= \Lambda\mathbb{P}\left(\delta < \frac{\log Q}{K}\right) \\
&= \Lambda\mathbb{P}(K\delta < \log Q) \\
&= \Lambda\mathbb{P}\left(e^{K\delta} < Q\right) \\
&= \Lambda\int_{e^{K\delta}}^{\infty} x^{-1-\alpha}dx \\
&= \Lambda\left[\frac{-x^{-\alpha}}{\alpha}\right]_{e^{K\delta}}^{\infty} \\
&= \Lambda\left(\lim_{t\to\infty}\frac{-t^{-\alpha}}{\alpha} + \frac{e^{-K\delta\alpha}}{\alpha}\right) \\
&= \frac{\Lambda}{\alpha}\left(e^{-K\delta\alpha} - \lim_{t\to\infty}\frac{1}{t^{\alpha}}\right) \\
&= \frac{\Lambda}{\alpha}e^{-\alpha K\delta} \\
&= Ae^{-k\delta}
\end{aligned}$$

where $A = \frac{\Lambda}{\alpha}$ and $k = \alpha K$. On the other hand, (3.13) and (3.14) yield:

$$\begin{aligned}
\lambda(\delta) &= \Lambda\mathbb{P}(\delta < \Delta p) \\
&= \Lambda\mathbb{P}(\delta < kQ^{\beta}) \\
&= \Lambda\mathbb{P}\left(Q > \left(\frac{\delta}{k}\right)^{-\beta}\right) \\
&= \Lambda\int_{\left(\frac{\delta}{k}\right)^{-\beta}}^{\infty} x^{-1-\alpha}dx \\
&= \Lambda\left[\lim_{t\to\infty}\frac{-t^{-\alpha}}{\alpha} + \frac{\left(\frac{\delta}{k}\right)^{-\frac{\alpha}{\beta}}}{\alpha}\right] \\
&= \frac{\Lambda\left(\frac{\delta}{k}\right)^{-\frac{\alpha}{\beta}}}{\alpha} \\
&= B\delta^{-\frac{\alpha}{\beta}}
\end{aligned}$$

where $B = \frac{\Lambda}{k\alpha}$. Other methods do exist i.e. directly integrating the density of the orderbook, which is potentially better since we only care about short-term liquidity. This procedure is described in Weber and Rosenow 2005 and Smith et al. 2003.

Include discussion and explanation of this method and derived form of λ

3.7 The Hamilton-Jacobi-Bellman Equation

Now that we have formulated our agent's value function, and discussed some empirical results on the form of the Poisson intensity λ, we turn to the solution of the problem at hand. Following on from our discussion of the theory of stochastic control in Chapter 2, our first goal will be to formulate the Hamilton-Jacobi-Bellman PDE associated to our value function which we defined in (3.12). Recall that this is given by

$$v(s,x,q,t) = \max_{\delta^a,\delta^b} \mathbb{E}\left[-e^{-\gamma(X_T+q_T S_T)}|\mathcal{F}_t\right] \tag{3.16}$$

where our optimal control processes δ^a and δ^b will turn out to be time and state dependent. This type of optimal dealer problem was first studied by Ho and Stoll 1981, who use the Dynamic Programming Principle to show that v satisfies the following HJB:

Theorem 3.7.1 (Ho and Stoll (1981) HJB)**.**

$$\begin{cases} v_t + \dfrac{1}{2}\sigma^2 v_{ss} + \max\limits_{\delta^b} \lambda^b(\delta^b)[v(s,x-s+\delta^b,q+1,t) - v(s,x,q,t)] \\ \qquad\qquad + \max\limits_{\delta^a} \lambda^a(\delta^a)[v(s,x+s+\delta^a,q-1,t) - v(s,x,q,t)] = 0, \\ \qquad\qquad v(s,x,q,T) = -e^{-\gamma(x+qs)}. \end{cases} \tag{3.17}$$

Proof. □

However, Avellaneda and Stoikov 2008 argue that due to our choice of exponential utility, we can simplify the problem with the following ansatz:

$$v(s,x,q,t) = -e^{-\gamma x}e^{-\gamma\theta(s,q,t)} \tag{3.18}$$

The intuition behind this is that considering our value function at time t, our current wealth X_t is a predetermined constant and thus measurable w.r.t $\mathcal{F}_t$. Hence we can take $-e^{-\gamma x}$ from the expectation. The remainder, being our future cash flow, future inventory and terminal portfolio value are all time and state dependent and hence encapsulated by some function θ of s, q and t. Moreover, thanks to the properties of the exponential function, the expectation (integral) of the utility of our future wealth can also be written in an exponential form. Finally, we also assume that the function θ factors in our optimal control $\alpha^* = \begin{pmatrix}\delta^{a*} \\ \delta^{b*}\end{pmatrix}$

By substitution of Avellaneda and Stoikov's ansatz (3.18) into Ho and Stoll's HJB (3.17) we obtain the following HJB equation for θ:

Theorem 3.7.2 (Avellaneda and Stoikov (2008) HJB)**.**

$$\begin{cases} \theta_t + \dfrac{1}{2}\sigma^2\theta_{ss} - \dfrac{1}{2}\sigma^2\gamma\theta_s^2 + \max\limits_{\delta^b}\left[\dfrac{\lambda^b(\delta^b)}{\gamma}(1-e^{\gamma(s-\delta^b-r^b)})\right] \\ \qquad\qquad + \max\limits_{\delta^a}\left[\dfrac{\lambda^a(\delta^a)}{\gamma}(1-e^{-\gamma(s+\delta^a-r^a)})\right] = 0, \\ \qquad\qquad \theta(s,q,T) = qs. \end{cases} \tag{3.19}$$

Relations for the reserve prices

Before we can prove that this subsitution provides an equivalent formulation of our Hamilton-Jacobi-Bellman equation, we need a lemma relating the definitions we gave of the dealer's reservation bid and ask prices in section 3.3 to our new function θ. We find that we can express r^b and r^a directly in terms of θ as follows:

Lemma 3.7.3. We have using the ansatz 3.18 that the reservation bid and ask prices defined in definition 3.3.1 are given by

$$r^b(s,q,t) = \theta(s,q+1,t) - \theta(s,q,t) \tag{3.20}$$

and

$$r^a(s,q,t) = \theta(s,q,t) - \theta(s,q-1,t). \tag{3.21}$$

Proof. We prove the above directly from the definition of the reserve bid and ask respectively:

$$\begin{aligned} u(s,x-r^b(s,q,t),q+1,t) &= u(s,x,q,t) \\ -e^{-\gamma(x-r^b(s,q,t))}e^{-\gamma\theta(s,q+1,t)} &= -e^{-\gamma x}e^{-\gamma\theta(s,q,t)} \\ -\gamma(x-r^b(s,q,t)) - \gamma\theta(s,q+1,t) &= -\gamma x - \gamma\theta(s,q,t) \\ x - r^b(s,q,t) + \theta(s,q+1,t) &= x + \theta(s,q,t) \\ r^b(s,q,t) &= \theta(s,q+1,t) - \theta(s,q,t) \end{aligned}$$

and

$$\begin{aligned} u(s,x+r^a(s,q,t),q-1,t) &= u(s,x,q,t) \\ -e^{-\gamma(x+r^a(s,q,t))}e^{-\gamma\theta(s,q-1,t)} &= -e^{-\gamma x}e^{-\gamma\theta(s,q,t)} \\ -\gamma(x+r^a(s,q,t)) - \gamma\theta(s,q-1,t) &= -\gamma x - \gamma\theta(s,q,t) \\ x + r^a(s,q,t) + \theta(s,q-1,t) &= x + \gamma\theta(s,q,t) \\ r^a(s,q,t) &= \theta(s,q,t) - \theta(s,q-1,t). \end{aligned}$$

□

Using this result, we can check that the ansatz 3.18 allows us to derive the HJB equation given in theorem 3.7.2.

Proof of Theorem 3.7.2. First we check the terminal condition:

$$\begin{aligned} v(s,x,q,T) &= -e^{-\gamma(x+qs)} \text{ from (3.17)} \\ &= -e^{-\gamma x}e^{-\gamma\theta(s,q,T)} \text{ from (3.18)} \\ &= -e^{-\gamma(x+\theta(s,q,T))} \\ \implies \theta(s,q,T) &= qs. \end{aligned}$$

which is what we expected. Next, by direct substitution, note that

$$v_t = -\frac{\partial}{\partial t}e^{-\gamma x}e^{-\gamma\theta} = -e^{-\gamma x} \times -\gamma\theta_t e^{-\gamma\theta} = \gamma e^{-\gamma x}\theta_t e^{-\gamma\theta}$$

and

$$\begin{aligned} \frac{1}{2}\sigma^2 v_{ss} &= -e^{-\gamma x}\frac{1}{2}\sigma^2\frac{\partial^2}{\partial s^2}e^{-\gamma\theta} \\ &= \gamma e^{-\gamma x}\frac{1}{2}\sigma^2\frac{\partial}{\partial s}\theta_s e^{-\gamma\theta} \\ &= \gamma e^{-\gamma x}\frac{1}{2}\sigma^2\left(\theta_{ss}e^{-\gamma\theta} - \gamma\theta_s^2 e^{-\gamma\theta}\right). \end{aligned}$$

Next we consider the maximised terms:

$$\begin{aligned}\lambda^b(\delta^b)[v(s,x-s+\delta^b,q+1,t)-v(s,x,q,t)] &= \lambda^b(\delta^b)[-e^{-\gamma(x-s+\delta^b)}e^{-\gamma\theta(s,q+1,t)}+e^{-\gamma x}e^{-\gamma\theta(s,q,t)}] \\ &= \lambda^b(\delta^b)[e^{-\gamma x}e^{-\gamma\theta(s,q,t)}-e^{-\gamma x}e^{\gamma s}e^{-\gamma\delta^b}e^{-\gamma\theta(s,q+1,t)}]\end{aligned}$$

and

$$\begin{aligned}\lambda^a(\delta^a)[v(s,x+s+\delta^a,q-1,t)-v(s,x,q,t)] &= \lambda^a(\delta^a)[-e^{-\gamma(x+s+\delta^a)}e^{-\gamma\theta(s,q-1,t)}+e^{-\gamma x}e^{-\gamma\theta(s,q,t)}] \\ &= \lambda^a(\delta^a)[e^{-\gamma x}e^{-\gamma\theta(s,q,t)}-e^{-\gamma x}e^{-\gamma s}e^{-\gamma\delta^a}e^{-\gamma\theta(s,q-1,t)}].\end{aligned}$$

We note that since the R.H.S. of our equation is 0 and all of our expressions contain $e^{-\gamma x}$, we can mutliply by this term. We can remove all $e^{-\gamma\theta}$ terms similarly. Dividing all expressions by γ and substituting into (3.17) yields a L.H.S. of

$$\begin{aligned}\theta_t+\frac{1}{2}\sigma^2\theta_{ss}-\frac{1}{2}\sigma^2\gamma\theta_s^2+\max_{\delta^b}\frac{\lambda^b(\delta^b)}{\gamma}[1-e^{\gamma s}e^{-\gamma\delta^b}e^{-\gamma(\theta(s,q+1,t)-\theta(s,q,t))}] \\ +\max_{\delta^a}\frac{\lambda^a(\delta^a)}{\gamma}[1-e^{-\gamma s}e^{-\gamma\delta^a}e^{\gamma(\theta(s,q,t)-\theta(s,q-1,t))}]\end{aligned}$$

which by lemma 3.7.3 simplifies to

$$\begin{aligned}&\theta_t+\frac{1}{2}\sigma^2\theta_{ss}-\frac{1}{2}\sigma^2\gamma\theta_s^2+\max_{\delta^b}\frac{\lambda^b(\delta^b)}{\gamma}[1-e^{\gamma s}e^{-\gamma\delta^b}e^{-\gamma r^b(s,q,t)}] \\ &\qquad+\max_{\delta^a}\frac{\lambda^a(\delta^a)}{\gamma}[1-e^{-\gamma s}e^{-\gamma\delta^a}e^{\gamma r^a(s,q,t)}] \\ &=\theta_t+\frac{1}{2}\sigma^2\theta_{ss}-\frac{1}{2}\sigma^2\gamma\theta_s^2+\max_{\delta^b}\frac{\lambda^b(\delta^b)}{\gamma}[1-e^{\gamma(s-\delta^b-r^b(s,q,t))}] \\ &\qquad+\max_{\delta^a}\frac{\lambda^a(\delta^a)}{\gamma}[1-e^{-\gamma(s+\delta^a-r^a(s,q,t))}]\end{aligned}$$

□

Implicit relations for the optimal spreads δ^a and δ^b

We can derive some relations for the optimal distances δ^a and δ^b that are implicit in our slightly simplified HJB equation (3.19). Inspecting the maximised terms in the HJB, we can invoke a first-order optimality condition to find an expression involving the optimal spreads, the reservation prices, and our Poisson intensity λ.

Theorem 3.7.4 (Implicit relations for the optimal spreads δ^a and δ^b)**.** For δ^b, we obtain the following:

$$s-r^b(s,q,t)=\delta^b-\frac{1}{\gamma}\log\left(1-\gamma\frac{\lambda^b(\delta^b)}{\frac{\partial\lambda^b}{\partial\delta}(\delta^b)}\right) \tag{3.22}$$

while for δ^a we have

$$r^a(s,q,t)-s=\delta^a-\frac{1}{\gamma}\log\left(1-\gamma\frac{\lambda^a(\delta^a)}{\frac{\partial\lambda^a}{\partial\delta}(\delta^a)}\right) \tag{3.23}$$

Proof. Taking the derivative of the term in 3.19 that we maximise w.r.t. δ^b and setting it equal to 0, we find:

$$\begin{aligned}
\frac{\partial}{\partial \delta}\left[\frac{\lambda^b(\delta)}{\gamma}(1-e^{\gamma(s-\delta-r^b(s,q,t))})\right](\delta^b) &= 0 \\
\frac{1}{\gamma}\left[\frac{\partial \lambda^b}{\partial \delta}(\delta^b) - \frac{\partial}{\partial \delta}\lambda^b(\delta^b)e^{\gamma(s-\delta^b-r^b(s,q,t))}\right] &= 0 \\
\frac{\partial \lambda^b}{\partial \delta}(\delta^b) - \frac{\partial \lambda^b}{\partial \delta}(\delta^b)e^{\gamma(s-\delta^b-r^b(s,q,t))} + \gamma\lambda^b(\delta^b)e^{\gamma(s-\delta^b-r^b(s,q,t))} &= 0 \\
\left(\gamma\lambda^b(\delta^b) - \frac{\partial \lambda^b}{\partial \delta}(\delta^b)\right)e^{\gamma(s-\delta^b-r^b(s,q,t))} &= -\frac{\partial \lambda^b}{\partial \delta}(\delta^b) \\
-\left(\frac{\partial \lambda^b}{\partial \delta}(\delta^b)\right)e^{-\gamma(s-\delta^b-r^b(s,q,t))} &= \gamma\lambda^b(\delta^b) - \frac{\partial \lambda^b}{\partial \delta}(\delta^b) \\
e^{-\gamma(s-\delta^b-r^b(s,q,t))} &= 1-\gamma\frac{\lambda^b(\delta^b)}{\frac{\partial \lambda^b}{\partial \delta}(\delta^b)} \\
-\gamma(s-\delta^b-r^b(s,q,t)) &= \log\left(1-\gamma\frac{\lambda^b(\delta^b)}{\frac{\partial \lambda^b}{\partial \delta}(\delta^b)}\right) \\
s-\delta^b-r^b(s,q,t) &= -\frac{1}{\gamma}\log\left(1-\gamma\frac{\lambda^b(\delta^b)}{\frac{\partial \lambda^b}{\partial \delta}(\delta^b)}\right) \\
s-r^b(s,q,t) &= \delta^b-\frac{1}{\gamma}\log\left(1-\gamma\frac{\lambda^b(\delta^b)}{\frac{\partial \lambda^b}{\partial \delta}(\delta^b)}\right)
\end{aligned}$$

while a similar procedure for δ^a yields:

$$\begin{aligned}
\frac{\partial}{\partial \delta}\left[\frac{\lambda^a(\delta)}{\gamma}(1-e^{-\gamma(s+\delta-r^a(s,q,t))})\right](\delta^a) &= 0 \\
\frac{1}{\gamma}\left[\frac{\partial \lambda^a}{\partial \delta}(\delta^a) - \frac{\partial}{\partial \delta}\lambda^a(\delta^a)e^{-\gamma(s+\delta^a-r^a(s,q,t))}\right] &= 0 \\
\frac{\partial \lambda^a}{\partial \delta}(\delta^a) - \frac{\partial \lambda^a}{\partial \delta}(\delta^a)e^{-\gamma(s+\delta^a-r^a(s,q,t))} + \gamma\lambda^a(\delta^a)e^{-\gamma(s+\delta^a-r^a(s,q,t))} &= 0 \\
\left(\gamma\lambda^a(\delta^a) - \frac{\partial \lambda^a}{\partial \delta}(\delta^a)\right)e^{-\gamma(s+\delta^a-r^a(s,q,t))} &= -\frac{\partial \lambda^a}{\partial \delta}(\delta^a) \\
-\left(\frac{\partial \lambda^a}{\partial \delta}(\delta^a)\right)e^{\gamma(s+\delta^a-r^a(s,q,t))} &= \gamma\lambda^a(\delta^a) - \frac{\partial \lambda^a}{\partial \delta}(\delta^a) \\
e^{\gamma(s+\delta^a-r^a(s,q,t))} &= 1-\gamma\frac{\lambda^a(\delta^a)}{\frac{\partial \lambda^a}{\partial \delta}(\delta^a)} \\
\gamma(s+\delta^a-r^a(s,q,t)) &= \log\left(1-\gamma\frac{\lambda^a(\delta^a)}{\frac{\partial \lambda^a}{\partial \delta}(\delta^a)}\right) \\
s+\delta^a-r^a(s,q,t) &= \frac{1}{\gamma}\log\left(1-\gamma\frac{\lambda^a(\delta^a)}{\frac{\partial \lambda^a}{\partial \delta}(\delta^a)}\right) \\
r^a(s,q,t)-s &= \delta^a-\frac{1}{\gamma}\log\left(1-\gamma\frac{\lambda^a(\delta^a)}{\frac{\partial \lambda^a}{\partial \delta}(\delta^a)}\right).
\end{aligned}$$

□

To briefly summarise, the optimal bid and ask spreads δ^a and δ^a are derived in an intuitive two-step procedure. First, we solve the HJB PDE (3.19) to obtain the reservation bid and ask prices. Then, we use the relations (3.22) and (3.23) to find the optimal bid and ask spreads $\delta^b(s,q,t)$ and $\delta^a(s,q,t)$ between the mid-price and obtimal bid and ask quotes respectively.

3.8 Asymptotic Expansion in q

The main computational difficulty in this advance is solving equation 3.19 due to the order-arrival terms (the terms to be maximised) being highly nonlinear and dependent on the inventory variable q. To get around this, Avellaneda and Stoikov 2008 suggest an asymptotic expansion of θ in the inventory variable q, and a linear approximation of the order arrival terms.

Following on from our work in section 3.6, we will assume that our arrival rates are symmetric and exponential given by

$$\lambda^a(\delta) + \lambda^b(\delta) = Ae^{-k\delta} \tag{3.24}$$

in which case, our agents indifference prices $r^a(s,q,t)$ and $r^b(s,q,t)$ coincide with their frozen inventory values given in section 3.3.

With some elementary calculus we can see that our exponential arrival rates satisfy the following property:

$$\frac{\lambda(\delta)}{\frac{\partial\lambda}{\partial\delta}(\delta)} = \frac{Ae^{-k\delta}}{-kAe^{-k\delta}} = -\frac{1}{k}.$$

Hence by plugging in the relations (3.22) and (3.23) into the maximised terms in the HJB equation (3.19) under the assumption of symmetric exponential arrival rates (3.24), we see that

$$\begin{aligned}
&\max_{\delta^b}\left[\frac{\lambda^b(\delta^b)}{\gamma}(1-e^{\gamma(s-\delta^b-r^b)})\right] + \max_{\delta^a}\left[\frac{\lambda^a(\delta^a)}{\gamma}(1-e^{-\gamma(s+\delta^a-r^a)})\right] \\
&= \frac{Ae^{-k\delta^b}}{\gamma}\left(1-e^{\gamma\left(-\frac{1}{\gamma}\log\left(1+\frac{\gamma}{k}\right)\right)}\right) + \frac{Ae^{-k\delta^a}}{\gamma}\left(1-e^{-\gamma\left(\frac{1}{\gamma}\log\left(1+\frac{\gamma}{k}\right)\right)}\right) \\
&= \left[\frac{A}{\gamma}\left(1-e^{-\log\left(1+\frac{\gamma}{k}\right)}\right)\right](e^{-k\delta^b}+e^{-k\delta^a}) \\
&= \left[\frac{A}{\gamma}\left(1-\frac{1}{1+\frac{\gamma}{k}}\right)\right](e^{-k\delta^b}+e^{-k\delta^a}) \\
&= \left(\frac{A}{\gamma}-\frac{A}{\gamma+\frac{\gamma^2}{k}}\right)(e^{-k\delta^b}+e^{-k\delta^a}) \\
&= \left(\frac{A\left(1+\frac{\gamma}{k}\right)-A}{\gamma+\frac{\gamma^2}{k}}\right)(e^{-k\delta^b}+e^{-k\delta^a}) \\
&= \left(\frac{A\frac{\gamma}{k}}{\gamma+\frac{\gamma^2}{k}}\right)(e^{-k\delta^b}+e^{-k\delta^a}) \\
&= \frac{A}{k+\gamma}(e^{-k\delta^b}+e^{-k\delta^a})
\end{aligned}$$

which results in the simplified HJB equation below:

$$\begin{cases}\theta_t + \frac{1}{2}\sigma^2\theta_{ss} - \frac{1}{2}\sigma^2\gamma\theta_s^2 + \frac{A}{k+\gamma}(e^{-k\delta^a}+e^{-k\delta^b}) = 0 \\ \theta(s,q,T) = qs.\end{cases} \tag{3.25}$$

Next, we consider an asymptotic expansion of θ in the inventory variable q:

$$\theta(q,s,t) = \theta^0(s,t) + q\theta^1(s,t) + \frac{1}{2}q^2\theta^2(s,t) + \ldots \tag{3.26}$$

where the superscripts denote different functions, not powers - we do not use subscripts to avoid conflicts with our notation for partial derivatives later on.

The exact relations for the reserve bid and ask prices obtained in lemma 3.7.3 yield

$$r^b(s,q,t) = \theta^1(s,t) + (1+2q)\theta^2(s,t) + \ldots \tag{3.27}$$

$$r^a(s,q,t) = \theta^1(s,t) + (-1-2q)\theta^2(s,t) + \ldots . \tag{3.28}$$

Then our reservation price

$$r(s,q,t) = \frac{r^a(s,q,t) + r^b(s,q,t)}{2} = \theta^1(s,t) + 2q\theta^2(s,t) \tag{3.29}$$

follows immediately. We can interpret this expression nicely: θ^1 is the reserve price when the inventory is 0, and θ^2 is our agent's sensitivity to changes in inventory. We might then expect that $\theta^2 < 0 \ \forall (s,t)$, since then a long position will result in lower quotes (more willing to sell) and vice-versa. We also have that

$$\begin{aligned}
\delta^a + \delta^b &= \frac{1}{\gamma}\log\left(1+\frac{\gamma}{k}\right) + r^a(s,q,t) - s + \frac{1}{\gamma}\log\left(1+\frac{\gamma}{k}\right) + s - r^b(s,q,t) \\
&= r^a(s,q,t) - r^b(s,q,t) + \frac{2}{\gamma}\log\left(1+\frac{\gamma}{k}\right) \\
&= -2\theta^2(s,t) + \frac{2}{\gamma}\log\left(1+\frac{\gamma}{k}\right)
\end{aligned} \tag{3.30}$$

through our approximation and the relations (3.22) and (3.23). Now consider a first-order approximation of the order arrival term:

$$\frac{A}{k+\gamma}(e^{-\gamma\delta^a} + e^{-\gamma\delta^b}) = \frac{A}{k+\gamma}(2 - k(\delta^a + \delta^b) + \ldots) \tag{3.31}$$

where we notice that the linear term does not depend on the inventory q. Therefore, by substituting (3.26) and (3.31) into (3.25) and grouping terms of order q we obtain

$$\begin{cases} \theta^1_t + \frac{1}{2}\sigma^2\theta^1_{ss} = 0 \\ \theta^1(s,T) = s. \end{cases} \tag{3.32}$$

which admits the solution $\theta^1(s,t) = s$ by inspection.

Grouping terms of order q^2 yields

$$\begin{cases} \theta^2_t + \frac{1}{2}\sigma^2\theta^2_{ss} - \frac{1}{2}\sigma^2\gamma(\theta^1_s)^2 = 0 \\ \theta^2(s,T) = 0 \end{cases} \tag{3.33}$$

which simplifies by our previous solution to (3.32) to

$$\begin{cases} \theta^2_t + \frac{1}{2}\sigma^2\theta^2_{ss} - \frac{1}{2}\sigma^2\gamma = 0 \\ \theta^2(s,T) = 0 \end{cases}$$

with solution $\theta^2(s,t) = -\frac{1}{2}\sigma^2\gamma(T-t)$ by inspection.

Thus for this linear approximation of the order arrival term, we can substitute our solutions back into (3.29) to obtain the same indifference price

$$r(s,t) = s - q\gamma\sigma^2(T-t) \tag{3.34}$$

as in the case where no trading is allowed. We quote a bid-ask spread that is symmetric about this reservation price and is given by the below expression, which is again acquired through subsituting our solutions for θ^1 and θ^2 back into (3.30).

$$\delta^a + \delta^b = \gamma\sigma^2(T-t) + \frac{2}{\gamma}\log\left(1+\frac{\gamma}{k}\right) \tag{3.35}$$

3.9 Summary

In the above section, we have seen how the problem of finding the optimal behaviour of a dealer in a limit orderbook, which we introduced in chapter 1, can be formulated as a stochastic control problem using the theoretical framework we built up in chapter 2. We have also noted a possible miscalculation or typographical error in the short section of the original paper of Avellaneda and Stoikov 2008 on the infinite-horizon agent, which we address in section 3.4.

We go on to provide full derivations of all of the results presented in Avellaneda and Stoikov 2008, explain the use of the ansatz (3.18) as a key simplifying assumption, and walk through the analytical approximation used to yield the final results that Avellaneda and Stoikov present, and then numerically simulate.

Chapter 4

Extension and Statistical Analysis of Orderbooks

4.1 Extension to Geometric Brownian Motion

Consider now an alternative model for the midprice of the security for which we wish to make a market:

$$dS_u = \sigma S_u dW_u$$

with initial value $S_0 = s$. This, as we have seen in section 1 defines a geometric brownian motion (without drift), and is the canonical stochastic process used to model most asset prices, as in contrast with the standard Brownian Motion, this process is valued in $\mathbb{R}^+$. As we have previously seen, the above SDE gives us the solution

$$S_t = se^{\frac{\sigma^2}{2}t+\sigma^2 W_t} \tag{4.1}$$

where

$$\begin{aligned}
\mathbb{E}[S_t] &= s \\
\mathbb{E}[S_t^2] &= s^2 e^{\sigma^2(T-t)} \\
\mathrm{Var}[S_t] &= s^2\left(e^{\sigma^2 t} - 1\right)
\end{aligned}$$

Consider the mean/variance objective

$$V(s,x,q,t) = \mathbb{E}\left[(x+qS_T) - \frac{\gamma}{2}(qS_T - qs)^2 | \mathcal{F}_t\right]. \tag{4.2}$$

Letting $S_t = s$ and using the above properties of GBM we can simplify as follows:

$$\begin{aligned}
V(s,x,q,t) &= \mathbb{E}\left[(x+qS_T) - \frac{\gamma}{2}(qS_T - qs)^2 | \mathcal{F}_t\right] \\
&= x + q\mathbb{E}[S_T|\mathcal{F}_t] - \frac{\gamma q^2}{2}\mathbb{E}[S_T^2|\mathcal{F}_t] + \gamma q^2 s\mathbb{E}[S_T|\mathcal{F}_t] - \frac{\gamma q^2 s^2}{2} \\
&= x + qs - \frac{\gamma q^2}{2}\left(s^2 e^{\sigma^2(T-t)}\right) + \gamma\sigma^2 s^2 - \frac{\gamma q^2 s^2}{2} \\
&= x + qs - \frac{\gamma q^2 s^2}{2}\left(e^{\sigma^2(T-t)}\right) + \frac{\gamma q^2 s^2}{2} \\
&= x + qs - \frac{\gamma q^2 s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right)
\end{aligned}$$

Using the definition of reserve bid and ask prices we obtain

$$\begin{aligned}
V(s, x + R^a(s,q,t), q-1, t) &= V(s,x,q,t) \\
x + R^a(s,q,t) + (q-1)s - \frac{\gamma(q-1)^2 s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right) &= x + qs - \frac{\gamma q^2 s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right) \\
R^a(s,q,t) + qs - s - \frac{\gamma(q^2 - 2q + 1)s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right) &= qs - \frac{\gamma q^2 s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right) \\
R^a(s,q,t) - s - \frac{\gamma q^2 s^2 + \gamma(1-2q)s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right) &= -\frac{\gamma q^2 s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right) \\
R^a(s,q,t) - \frac{(1-2q)\gamma s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right) &= s \\
R^a(s,q,t) &= s + \frac{(1-2q)\gamma s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right)
\end{aligned}$$

and

$$\begin{aligned}
V(s, x - R^b(s,q,t), q+1, t) &= V(s,x,q,t) \\
x - R^b(s,q,t) + (q+1)s - \frac{\gamma(q+1)^2 s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right) &= x + qs - \frac{\gamma q^2 s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right) \\
-R^b(s,q,t) + qs + s - \frac{\gamma(q^2 + 2q + 1)s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right) &= qs - \frac{\gamma q^2 s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right) \\
-R^b(s,q,t) + s - \frac{(1+2q)\gamma s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right) &= 0 \\
R^b(s,q,t) &= s - \frac{(1+2q)\gamma s^2}{2}\left(e^{\sigma^2(T-t)} - 1\right)
\end{aligned}$$

4.2 Estimating the Frequency of Market Orders

4.3 Estimating the Size Distribution of Market Orders

4.4 Estimating the Temporary Price Impact of Large Market Orders

Chapter 5

Numerical Analysis and Simulations

5.1 Introduction

5.2 Numerical Simulations - Avellaneda & Stoikov

Listing 5.1: Auxiliary Functions

```
import numpy as np

def computeReservePrice(s,q,gamma,sigma,t,T):
    return s-q*gamma*(sigma**2)*(T-t)

def computeSpread(gamma,sigma,t,T,k):
    return (gamma * (sigma ** 2) * (T-t)) + ((2 / gamma) *
        np.log(1 + (gamma / k)))

def computeRate(A,k,delta):
    return A*np.exp(-k*delta)

def computeSamplePath(S0, sigma, dt, T):
    return np.insert(S0 + np.cumsum(sigma * np.sqrt(dt) *
        np.random.choice([1,-1], int(T/dt))), 0, S0)
```

`computeReservePrice` takes the current stock price, current inventory, risk aversion, volatility, current time and end time as arguments and returns the current reservation price as given in (3.34).

`computeSpread` takes the risk aversion, volatility, current time, end time and orderbook parameter k as arguments and computes the current spread as given by (3.35).

`computeRate` computes the function λ in the form assumed by (3.24), taking the orderbook parameters A and k and the current spread δ as arguments.

`computeSamplePath` generates a sample path from a brownian motion from t to T with stepsize $\mathrm{d}t$, volatility σ and starting value S_0.

Listing 5.2: Avellaneda-Stoikov Model

```
def simulateBothStrategies(gamma):
S0=100
T=1
sigma=2
dt=0.005
inv_q=0
sym_q=0
k=1.5
A=140
inv_X=0
sym_X=0
inf_X=0

inv_bids = []
inv_asks = []
inv_wealth = []
inv_adj_wealth = []
inv_inventory = []

sym_bids=[]
sym_asks=[]
sym_wealth = []
sym_adj_wealth = []
sym_inventory = []

price_process = computeSamplePath(S0, sigma, dt, T)

sym_spread = 0
for i in np.arange(0, T, dt):
    sym_spread += computeSpread(gamma, sigma, i, T, k)
av_sym_spread = (sym_spread / (T / dt))
sym_prob = min(A * np.exp(- k * av_sym_spread / 2) * dt, 1)
sym_bids = price_process - av_sym_spread/2
sym_asks = price_process + av_sym_spread/2

for step, s in enumerate(price_process):
    r = computeReservePrice(s,inv_q,gamma,sigma,step*dt,T)
    spread = computeSpread(gamma,sigma,step*dt,T,k)/2
    delta_a = (spread+r)-s
    delta_b = s-(r-spread)

    inv_asks.append(s+delta_a)
    inv_bids.append(s-delta_b)
    inv_wealth.append(inv_X)
    inv_adj_wealth.append(inv_X+inv_q*s)
    inv_inventory.append(inv_q)

    prob_a = min(computeRate(A,k,delta_a)*dt, 1)
    prob_b = min(computeRate(A,k,delta_b)*dt, 1)
    p = np.random.default_rng().uniform(0,1,None)
```

```
        if p <= prob_a:
            inv_q -= 1
            inv_X += (s+delta_a)
        p = np.random.default_rng().uniform(0,1,None)
        if p <= prob_b:
            inv_q += 1
            inv_X -= (s-delta_b)

        sym_wealth.append(sym_X)
        sym_adj_wealth.append(sym_X+sym_q*s)
        sym_inventory.append(sym_q)

        p = np.random.default_rng().uniform(0,1,None)
        if p <= sym_prob:
            sym_q -= 1
            sym_X += (s+av_sym_spread/2)
        p = np.random.default_rng().uniform(0,1,None)
        if p <= sym_prob:
            sym_q += 1
            sym_X -= (s-av_sym_spread/2)

return((inv_wealth[-1], inv_inventory[-1], price_process[-1],
        sym_wealth[-1], sym_inventory[-1], av_sym_spread))
```

Using the code above, we compare the “inventory” strategy that we developed in chapter 3 to a “symmetric” strategy that simply takes the mean spread from the inventory strategy and quotes that spread, symmetrically about the mid-price, for the entire time interval. We are also interested in the PnL profile of the strategies under varying values of γ, the user-defined risk-aversion parameter. We run 10000 simulations and report means and standard deviations for profit and final inventory, as well as the mean quote spread. We also plot the shape of the PnL distributions for the two strategies.

Strategy	μ (Spread)	μ (Profit)	σ (Profit)	μ (Final q)	σ (Final q)
Inventory	1.49	64.9	6.7	0.03	2.9
Symmetric	1.49	68.2	13.1	-0.1	8.3

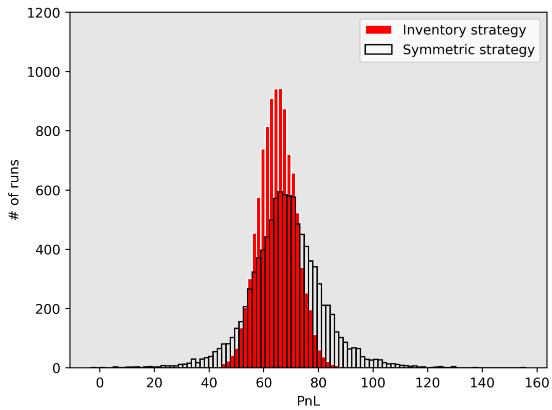

Figure 5.1: Results for $\gamma = 0.1$

Strategy	μ (Spread)	μ (Profit)	σ (Profit)	μ (Final q)	σ (Final q)
Inventory	1.35	68.5	9.1	0.04	5.3
Symmetric	1.35	68.6	13.8	0.04	8.6

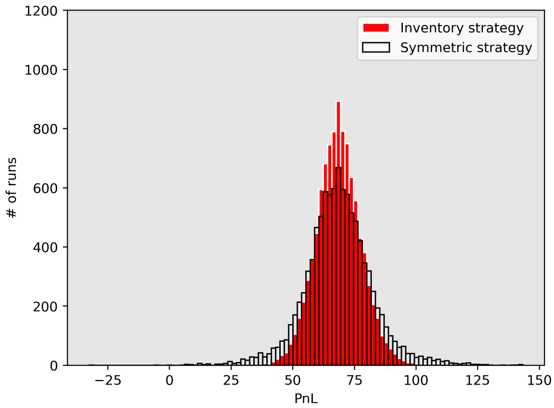

Figure 5.2: Results for $\gamma = 0.01$

5.3 Simulation of Extended Model for GBM

5.4 Estimation of Order Book Parameters for Real-World Data

5.5 Market Making in the Binance Order Book

Chapter 6

Conclusion

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Appendix A

“Delicate Measurability Questions”

A.1 The Measurable Selection Theorem