University of BRISTOL

Automated Market-Making under Inventory Risk: A Stochastic Optimal Control Framework

J. G. Acton

Supervised by Professor Nick Whiteley
Level H/6
20 Credit Points

April 29th, 2024

Acknowledgement of Sources

For all ideas taken from other sources (books, articles, internet), the source of the ideas is mentioned in the main text and fully referenced at the end of the report.

All material which is quoted essentially word-for-word from other sources is given in quotation marks and referenced.

Pictures and diagrams copied from the internet or other sources are labelled with a reference to the web page or book, article etc.

Signed ____________________

Date ____________________

Abstract

This project presents a review of the mathematical theory that attempts to model the dynamics of an automated market-maker under inventory risk in financial markets. We begin by outlining financial markets, their participants and their microstructure, before discussing the requisite mathematical tools from probability theory, stochastic analysis, stochastic calculus and stochastic control. Next, we investigate the seminal 2008 paper "High-Frequency Trading in a Limit Order Book" by Avellaneda and Stoikov (2008), which formalises the approach of a market-maker trading through limit orders and utilises the dynamic programming principle to solve for the market-maker's optimal bid and ask quotes. We provide proofs of all of the main results presented in the paper, before going on to replicate their numerical simulations of the strategies performance, providing code and results.

Contents

List of Figures

Chapter 1

Introduction and background

1.1 Introduction

This section aims to equip the reader with both the motivation and mathematical tools to begin to formalise problems in mathematical finance. In section 1.2, we will introduce the concepts of financial markets, order books, and dealers, and qualitatively describe how a dealer may wish to behave to optimise their revenue. We will then use section 1.3 to briefly recap some basic measure and probability theory, before turning to stochastic processes in continuous time in section 1.4. In sections 1.5 and 1.6, we introduce stochastic calculus, looking at stochastic integration, Itô's lemma, and finally stochastic differential equations and the existence and uniqueness of their solutions.

1.2 Financial Markets

A market is simply some social structure that attempts to match those who want to sell a good or service to those who want to buy it. Modern financial markets, thanks to recent innovations such as the internet, satellite communication, and fibre-optic cables, are perhaps the most interconnected and widespread markets in human history.

Most people may have heard of the New York Stock Exchange, London Stock Exchange, or NASDAQ, but these are only one type of exchange for one type of financial asset, namely equity (part ownership of a corporate entity, individually called "stocks" or "shares").There are also markets for commodities (oil, gas, industrial metals, precious metals, live cattle and more), bonds (pieces of government or corporate debt, where the holder receives fixed interest payments), currencies (including cryptocurrencies), and derivatives which are legal contracts whose value is some function of the price of a specified underlying asset. In total, on an average day, tens of trillions of US dollars worth of assets change hands.

All markets, whatever the good or service being exchanged, have something in common: Every seller needs a buyer, and every buyer needs a seller. But this raises some natural questions: What happens if no-one wants to sell (or buy)? What happens if the only prices at which people are willing to sell is far out of reach of those who want to buy? Enter the *dealer*: An entity who provides *liquidity* (ease of exchange) to market participants. A dealer does this by simultaneously offering to both buy and sell the particular asset, offering to buy at a slightly lower price than they offer to sell. This known as "making a market", and dealers in modern parlance may also be called "market-makers".

Dealers

Dealers provide a crucial service in financial markets: By providing these quotes, they narrow the *spread* - the difference between the prices at which one can buy or sell an asset in the market. Hence, entities who may need to trade even in adverse market scenarios (such as companies needing to buy foreign currency to pay workers abroad, or oil producers seeking to hedge their production) know that they can reliably find a buyer or seller, regardless of the uncertainty of other market participants such as *speculators* - those believe that a certain asset is under or overvalued, and trade it with the sole motive of making money buy selling it for more than they bought it or vice-versa.

Of course, there is no free lunch. Dealers do not provide this service to the market out of the goodness of their own hearts - they too have a profit motive. While the presence of dealers in the market narrows the spread, it does not eliminate it. The dealers aim is to be constantly selling the asset for a slightly higher price than it is buying it, and taking the spread as profit. In modern electronic markets with very high trading volumes, even in heavily traded assets with very narrow spreads, a spread of only 0.01$ multiplied across millions or billions of trades can be very lucrative for the dealers who are fast enough.

The Limit Orderbook

So far we have discussed markets as an abstract concept, but in order to build a mathematical model of the dealer, we need to specify the framework under which the market operates. Most modern electronic exchanges, including those mentioned above, operate some version of a *limit orderbook* where participants can place two types of orders: a *limit order* or a *market order* depending on their needs. Limit orders specify a side (bid or ask, buying or selling), a quantity (how many units of the asset to buy/sell), and a price at which the order should be executed. These enter a queue of limit orders at the particular price level. Market orders specify a side and a quantity, but not a price: The exchange operates a *matching engine* which takes incoming market orders and attempts to match them to the existing limit orders, and if two orders match, they are executed and a trade occurs.

For an example, consider the orderbook illustrated by figure 1.1, and suppose that individual limit orders may only be placed for 1-share lots. If a market order is placed to buy 10 lots, then the trade will occur at \$1.01, the dealer/s will sell and the placer of the market order will buy, and both the market order and the 10 lowest limit ask orders will be removed from the market. So immediately after this trade, there will be 20 shares left available to be sold at the \$1.01 price level. However, suppose that a market order is placed to buy 30 units. In this case, the orders will still be matched, the buyer will buy 30 units for \$1.01 apiece but all of the limit orders at \$1.01 will be taken off the exchange, and the market mid-point price has now moved up from \$1.00 to \$1.005. If a market order is placed to buy 100 shares, since there are only 80 shares available to be sold, only these 80 will be bought for an average price of $\frac{30\times 1.01\$+50\times 1.02\$}{80} = 1.01625\$$. On the other hand, if a market order is placed and there are no limit orders to match it against, the market order would not be executed at all and be voided.

Finally, suppose we place a limit order into this market to buy 10 shares for \$0.90. Thus, for our order to ever be executed, a market order or sequence of market orders would have to come in and move the market mid-price by $\approx 10\%$ in order for our order to be touched. Hence in a given (small) interval of time, it is intuitively very unlikely that our order will be executed, especially when you consider that a move of 10% is roughly how much you might expect a stock to move over a year, let alone over a fraction of a trading day. This is one of the fundamental ideas that we will employ to model our dealing agent:

Side	Price /\$	Volume
A	1.02	50
A	1.01	30
N/A	1.00	0
B	0.99	25
B	0.98	45

Figure 1.1: An example orderbook

The probability that a limit order will execute is a decreasing function of its distance from the mid-price.

We have also seen the key difference between market and limit orders in action: Limit orders guarantee price, but do not guarantee that all or any of the order will be filled. Market orders guarantee that as much of the order as possible will be filled, but they do not guarantee the price at which the trade will occur.

We can also observe that the market provides us with a way to estimate the true value of the asset. Classical economic theory dictates that in aggregate, market participants react quickly and rationally to new information about a particular asset, meaning that market prices reflect the consensus opinion of market participants about the value of traded assets. The spread exists because people would only want to sell for slightly more than something is worth, and buy it for slightly less. Hence, if you really want to buy an asset you have to pay a premium to "*cross the spread*" to acquire it. From this we can determine that the true price of the asset at a point in time lies somewhere in between the maximum bid price and the minimum ask price for the asset at that time. The most common estimator in the literature and in practice is simply the average of these two values - the mid-market price, but other estimators do exist such as the volume-weighted average price (VWAP) which takes into account the volume of the bids vs asks. For the rest of this report we will use the mid-price as our estimator for the "true" value of an asset.

The aim for the rest of this report is to build up a model of how a dealer should behave to maximise their returns in the presence of uncertainty: namely, uncertainty about the path that the true value of the stock might take. In order to do this, we will need to make use of some basic results from measure/probability theory and stochastic processes, which we will summarise below. We will also briefly introduce some tools from stochastic calculus. Familiarity with standard results from a first-year undergraduate level course in real analysis, probability, and statistics is assumed. All of the results in section 1.3 come from the third year Measure Theory and Integration or Martingale Theory with Applications courses. Section 1.4 recaps content from Probability 2 and Martingale Theory, and also brings in some new material which is cited. Sections 1.5 and 1.6 contain mostly new material, although some was covered towards the end of the Financial Mathematics unit. We skip the vast majority of proofs in the interest of brevity and readability, they are not the point of this project.

1.3 Measure Theory and Probability

We begin by formalising the notion of what "events" in a probability space are. In elementary treatments of probability theory, the set of events is given no particular structure, but, especially with uncountable sample spaces, this can lead to inconsistencies.

Definition 1.3.1 (σ-algebra)**.** A family $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ of sets is called a σ-algebra if

- $\Omega \in \mathcal{F}$,
- for every countable collection of sets $A_1, A_2, ... \in \mathcal{F}$, $\bigcup_n A_n \in \mathcal{F}$,
- for every $A \in \mathcal{F}, A^c \in \mathcal{F}$.

Remark. The pair $(\Omega, \mathcal{F})$ is called a *measurable space*. Any set $A \in \mathcal{F}$ is called $\mathcal{F}$-*measurable* or simply *measurable*.

A measure, which we define next, is simply a function that assigns a magnitude to a measurable set. In $\mathbb{R}^n$, the canonical measure is the Lebesgue measure, which in one dimension is also called length, in two area, in three volume, and so on. In the context of probability theory, the "magnitude" of an event is simply the likelihood that it will occur.

Definition 1.3.2. A *measure* μ on a σ-algebra $\mathcal{F}$ is a set function $\mu : \mathcal{F} \to [0, \infty]$ such that $\forall$ mutually disjoint sets $A_1, A_2, ... \in \mathcal{A}$ with $\bigcup_n A_n \in \mathcal{A}$,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \tag{1.1}$$

Remark. The triplet $(\Omega, \mathcal{F}, \mu)$ is called a measure space. If $\mu(\Omega) = 1$ then we call μ a *probability measure*, and often use $\mathbb{P}$ instead. In this case the triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space.*

Next we see that we can generate σ-algebras from sets of sets, and use this to construct the Borel σ-algebra on $\mathbb{R}$. This is the canonical σ-algebra for use over $\mathbb{R}$ as it contains all Lebesgue-measurable sets, and hence almost all interesting sets, except those that are interesting precisely because they are not Lebesgue-measurable, such as the Vitali set.

Lemma 1.3.1. Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ Then $\exists$ a smallest σ-algebra $\sigma(\mathcal{A})$ that contains all sets from $\mathcal{A}$.

Proof. The intersection of σ-algebras is a σ-algebra, so to find the smallest containing some collection of sets, take the intersection of all σ- algebras containing those sets. □

Remark. The above $\sigma(\mathcal{A})$ is usually called the σ-algebra *generated* by $\mathcal{A}$.

Definition 1.3.3 (The Borel σ-algebra)**.** Consider the collection

$$\mathcal{A} = \{(a, b) : a, b \in \mathbb{R} \cup \{-\infty, \infty\}, a < b\}$$

Then define $\mathcal{B}(\mathbb{R}) := \sigma(\mathcal{A})$ the *Borel σ-algebra.* This is the smallest σ-algebra containing all open sets in $\mathbb{R}$. A set $B \in \mathcal{B}$ is a *Borel set.*

Next we turn to functions whose domain is a measurable space. We also look at simple functions, define the integral of a simple function, approximate non-negative measurable functions by simple functions and arrive at the notion of the Lebesgue integral, a slightly stronger integral than the Riemann or Regulated integral studied in a first year course.

Definition 1.3.4 (Measurable functions)**.** Let $(\Omega, \mathcal{F})$ be a measurable space. A function $f : \Omega \to \mathbb{R}$ is *measurable* if for any $B \in \mathcal{B}$,

$$f^{-1}(B) \in \mathcal{F}.$$

Definition 1.3.5 (Simple functions)**.** A *simple function* is a finite linear combination of characteristic (or indicator) functions of measurable sets:

$$\phi = \sum_{i=1}^{n} c_i \chi_{A_i} \tag{1.2}$$

where $c_i \in \mathbb{R}$ and $A_i \in \mathbb{X}$. It is in standard representation if $X = \cup_{i=1}^{n} A_i$, the sets A_i are pairwise disjoint, and the numbers c_i are distinct.

Definition 1.3.6 (Integral of a simple function)**.** Consider a non-negative simple function written in standard form as given above. Then the *integral* of ϕ *with respect to* μ is

$$\int \phi \mathrm{d}\mu := \sum_{i=1}^{n} c_i \mu(A_i) \tag{1.3}$$

which takes values in $\bar{\mathbb{R}}$.

Lemma 1.3.2 (Approximation by simple functions)**.** Let $f \in M(X, \mathbb{X})$, $f \geq 0$. Then there exists a sequence (ϕ_n) in $M(X, \mathbb{X})$ such that

- $0 \leq \phi_n(x) \leq \phi_{n+1}(x)\ \forall\, x \in X, n \in \mathbb{N}$,
- $\lim_{n\to\infty} \phi_n(x) = f(x)$,
- Each ϕ_n is a simple function.

Definition 1.3.7 (Integral of a non-negative measurable function)**.** Let $f \in M^+(X, \mathbb{X})$. Then the *integral* of f *with respect to* μ is

$$\int f \mathrm{d}\mu := \sup \left\{ \int \phi \mathrm{d}\mu : 0 \leq \phi \leq f, \phi \text{ is a simple measurable function} \right\} \in \bar{\mathbb{R}}.$$

Definition 1.3.8 (Integral of a non-negative measurable function over a set)**.** Let $f \in M^+(X, \mathbb{X})$. Then the *integral* of f *with respect to* μ *over set* $A \in \mathbb{X}$ is

$$\int_A f \mathrm{d}\mu := \int f \chi_A \mathrm{d}\mu \tag{1.4}$$

Definition 1.3.9 (Integrable functions)**.** Let $(X, \mathbb{X}, \mu)$ be a measure space. $f : X \to \mathbb{R}$ is *integrable* iff

$$\int f^+ \mathrm{d}\mu < +\infty \text{ and } \int f^- \mathrm{d}\mu < +\infty \tag{1.5}$$

where $f^+ := \max\{f, 0\}$ and $f^- := -\min\{f, 0\}$. We then define

$$\int f \mathrm{d}\mu := \int f^+ \mathrm{d}\mu - \int f^- \mathrm{d}\mu \tag{1.6}$$

and for $A \in \mathbb{X}$

$$\int_A f \mathrm{d}\mu := \int_A f^+ \mathrm{d}\mu - \int_A f^- \mathrm{d}\mu \tag{1.7}$$

Remark. All of the standard properties of integrals that one would expect to hold such as linearity are also true for the Lebesgue integral defined above. The Lebesgue integral also coincides with the Riemann and Regulated integrals for all Riemann-integrable and regulated functions respectively.

Now we have seen a brief introduction to measure theory and integration, we turn to probability theory. As it turns out, probability theory is almost entirely a special case of measure theory, where, as mentioned above, the measure of the full space is one.

Definition 1.3.10 (Random variables)**.** Recall from above that a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is simply a measure space $(X, \mathbb{X}, \mu)$ where $\mu(X) = 1$. In this case, a measurable function $X : \Omega \to \mathbb{R}$ can be called a random variable.

Definition 1.3.11 (Expectation)**.** The notion of the *expectation* of a **random variable** is exactly equivalent to the notion of the *integral* of a **measurable function**. To be precise,

$$\mathbb{E}[X] := \int X \mathrm{d}\mathbb{P} \tag{1.8}$$

Definition 1.3.12 ($\sigma-$algebra generated by a random variable)**.** The σ-algebra generated by a random variable $Y : \Omega \to \mathbb{R}$ is $\sigma(Y) := \sigma(Y^{-1}(\mathcal{B}(\mathbb{R})))$.

Definition 1.3.13 (Conditional Expectation)**.** Suppose $\mathcal{H} \subseteq \mathcal{F}$ is a sub-σ-algebra of $\mathcal{F}$. Then a *conditional expectation* of random variable X given $\mathcal{H}$, is any $\mathcal{H}$-measurable function $V : \Omega \to \mathbb{R}$ with $\mathbb{E}[V] < \infty$ which satisfies

$$\int_H V \mathrm{d}\mathbb{P} = \int_H X \mathrm{d}\mathbb{P} \tag{1.9}$$

for any $H \in \mathcal{H}$. We then define the notational convenience

$$\mathbb{E}[X|\mathcal{H}] := V. \tag{1.10}$$

The conditional expectation with respect to a random variable is defined according to

$$\mathbb{E}[X|Y] := \mathbb{E}[X|\sigma(Y)] \tag{1.11}$$

where $\sigma(Y)$ is the σ-algebra generated by Y.

Theorem 1.3.3 (Tower rule / Law of iterated conditional expectation)**.** Let X be a random variable on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}[|X|] < \infty$. Let $\mathcal{G} \subset \mathcal{H}$ be sub σ-algebras, $\mathcal{G}$ the courser and $\mathcal{H}$ the finer. Then:

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\mathcal{G}] = \mathbb{E}[X|\mathcal{G}] \tag{1.12}$$

All of these results are slight generalisations of results encountered in school or a first year course. We now turn to a useful theorem from measure theory concering the swapping of integrals (or integral and expectation, since expectation is equivalent to integration in the probability space).

Theorem 1.3.4 (Fubini-Tonelli)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathbb{R}, \mathcal{B}, \lambda)$ be two σ-finite measure spaces (this is a technical condition beyond the scope of this introduction, all measure spaces encountered from here on out will be σ-finite). Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a measurable function w.r.t. both measure spaces. Then

$$\int\int f(x,y)\mathrm{d}\mathbb{P}\mathrm{d}\lambda = \int \mathbb{E}[f(x,y)]\mathrm{d}\lambda = \mathbb{E}\left[\int f(x,y)\mathrm{d}\lambda\right] = \int\int f(x,y)\mathrm{d}\lambda\mathrm{d}\mathbb{P} \tag{1.13}$$

Definition 1.3.14 (Moment Generating Functions)**.** The *Moment Generating Function* (MGF) of a random variable X is defined as follows

$$M_X(t) := \mathbb{E}[e^{tX}] \tag{1.14}$$

Remark. The MGF of the normal distribution is a commonly used tool when dealing with Brownian Motion and functions of Brownian Motion as we will do throughout this report. Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$. Then

$$M_X(t) = \mathbb{E}[e^{tX}] = e^{t\mu + \frac{t^2\sigma^2}{2}} \tag{1.15}$$

1.4 Stochastic Processes

Next we move on to look at Stochasic Processes, which are nothing more than sequences of random variables, typically viewed as moving through time. Going forward, we will need a few properties regarding measurability and some common examples of widely used stochastic processes for modelling real-world phenomena. Firstly, we need to introduce some extra structure to our probability space, representing the information about the process that we recieve over time.

Definition 1.4.1 (Filtrations & Adaptedness)**.** A *filtered space* is $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^{\infty}, \mathbb{P})$ where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}$ are σ-algebras, jointly called a *filtration.* We also define $\mathcal{F}_\infty := \sigma(\cup_n \mathcal{F}_n) \subseteq \mathcal{F}$.

We say a stochastic process or sequence of random variables X_n is adapted to the filtration $(\mathcal{F}_n)_{n\geq 0}$ if for every n, X_n is $\mathcal{F}_n$ -measurable.

Definition 1.4.2 (Martingales)**.** A process $(M_n)_{n\geq 0}$ in a filtered probability space is a *martingale with respect to a filtration* $(\mathcal{F}_n)_{n\geq 0}$ if

- M_n is adapted to $\mathcal{F}_n$,
- $\mathbb{E}[M_n] < \infty \; \forall \, n \geq 0$,
- $\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n$ a.s. $\forall \, n \geq 0$.

Definition 1.4.3 ((Inhomogeneous) Poisson Process)**.** Let $\lambda : \mathbb{R} \to [0, \infty)$ be a measurable and integrable function such that for every bounded set B the integral of λ is finite:

$$\Lambda(B) = \int_B \lambda(x)\mathrm{d}x < \infty \tag{1.16}$$

In particular, the function Λ is a measure. Then for every collection of disjoint bounded Borel-measurable sets $B_1, \ldots, B_k$, an inhomogeneous *Poisson Point Process* with *intensity function* λ has distribution

$$\mathbb{P}\{N(B_i) = n_i, i = 1, \ldots, k\} = \prod_{i=1}^{k} \frac{(\Lambda(B_i))^{n_i}}{n_i!} e^{-\Lambda(B_i)}. \tag{1.17}$$

Moreover,

$$\mathbb{E}[N(B)] = \Lambda(B). \tag{1.18}$$

This comes from Daley and Vere-Jones 2008.

Definition 1.4.4 (Brownian Motion)**.** Let $\mathcal{F}_t$ be a filtration. A stochastic process $W = (W_t)_{t\geq 0}$ is a standard one-dimensional *Brownian Motion* or *Wiener Process* if it satisfies the following (Karatzas and S. E. Shreve 1998):

- $W_0 = 0$ a.s.,
- Independent increments: $W_{t+s} - W_t$ is independent of $\mathcal{F}_t \; \forall \, t, s \geq 0$,
- W has stationary Gaussian increments: $W_{t+s} - W_t \sim \mathcal{N}(0, s)$,
- W has continuous sample paths: $W_t(\omega)$ is a continuous function of $t \; \forall \, \omega \in \Omega$.

Definition 1.4.5 (Predictable Processes)**.** A stochastic process X_t is *predictable* (in the discrete sense) if X_{t+1} is $\mathcal{F}_t$ measurable for all t.

If X_t is a continous stochastic process, then it is predictable if it is measurable with respect to the σ-algebra generated by all left-continuous adapted processes.

This includes all left-continuous stochastic processes, since we can find X_t by finding $\lim_{s\to t^-} X_s$ without needing to observe X at time t.

Definition 1.4.6 (Stopping Times)**.** A random variable $\tau : \Omega \to [0, \infty]$ is a stopping time (with respect to the filtration $\mathcal{F}$) if $\forall\, t \in [0, T]$

$$\{\tau \leq t\} := \{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t. \tag{1.19}$$

This concept will be very useful later on when we look at the dynamic programming principle. In particular, any random time equal to a positive constant t is a stopping time. We conclude this section with a few more technical definitions which we will use only in the following Chapter on stochastic control.

Definition 1.4.7 (Progressive Measurability)**.** We use the definition given by Pham 2009 for this and the last three definitions in this section. A continuous-time stochastic process (X_t) is progressively measurable if for every time t, the map $[0, t] \times \Omega \to \mathbb{R}$ defined by $(s, \omega) \to X_s(\omega)$ is $\mathcal{B}(\mathbb{R}) \times \mathcal{F}_t$-measurable. This is a slightly stronger condition than adaptedness, indeed, all progressively measurable processes are adapted but the converse is not true.

Definition 1.4.8 (Càdlàg process)**.** A stochastic process is called càdlàg if it is right continuous with left limits. Acronym from the french "continue à droite, limite à gauche".

Definition 1.4.9 (Local Martingale)**.** Let X be a càdlàg and adapted process. We say that X is a local martingale if there exists a sequence of stopping times $(\tau_n)_{n\geq 1}$ such that $\lim_{n\to\infty} \tau_n = \infty$ a.s. and the process $X_{\min\{t,\tau_n\}}$ is a martingale for all n.

Definition 1.4.10 (Semimartingale)**.** A continuous real-valued process X is called a *semimartingale* if it can be decomposed as

$$X_t = X_0 + M_t + A_t \tag{1.20}$$

where M is a (continuous) local martingale and A is a càdlàg, adapted process of locally bounded variation: $\forall\, \omega, t$

$$\sup \sum_{i=1}^{n} |A_{t_i}(\omega) - A_{t_i-1}(\omega)| < \infty$$

where the supremum is taken over all subdivisions $0 = t_0 < t_1 < \cdots < t_n = t$ of $[0, t]$.

Now that we have seen an overview of many key definitions of stochastic processes that will be of great utility when modelling financial markets, we turn to the theory of stochastic calculus. We specifically make use of Itô calculus, which focuses on integration, as this will be most useful when we come to looking at Stochastic Differential Equations and controlled diffusion.

1.5 Stochastic Integration

Throughout this section we follow S. Shreve 2008 in our construction of the Itô integral. In order to make sense of the expression

$$\int H \mathrm{d}W_t$$

where H is an adapted stochastic process and W is a standard Brownian Motion or Wiener Process, we turn to the idea we saw in section 1.3 for the construction of the Lebesgue integral: Approximation by simple functions.

Definition 1.5.1 (Stochastic Integral for Simple Processes)**.** Suppose that H is an adapted, simple process, in the sense that $\exists$ a partition of $[0,T]$, $\Pi_n = \{[t_0,t_1),[t_1,t_2),\ldots,[t_{n-1},t_n),[t_n,T]\}$ where $H_t = c_i \ \forall \ t_i \le t < t_{i+1}, i < n$ or $H_t = c_n$ for $t \in [t_n, T]$. Then we can define the stochastic integral as a sum:

$$I(t) := \sum_{j=0}^{n-1} H_{t_j}(W_{t_{j+1}} - W_{t_j}) + H_{t_n}(W_T - W_{t_n}) \tag{1.21}$$

Now we can approximate a more general H by a simple stochastic process and take the limit of the above summation to arrive at Itô's notion of a stochastic integral.

Definition 1.5.2 (Itô (Stochastic) Integral)**.** Let W be a standard Wiener process as defined above, and let H be adapted (to W), and square-integrable:

$$\int H_t^2 \mathrm{d}t < \infty.$$

If $\{\Pi_n\}$ is a sequence of partitions on $[0,t]$ with mesh width decreasing to 0, $t_0 = 0$, and $t_n = t$, then the *Itô integral* of H w.r.t. W is the random variable

$$I(t) = \int_0^t H \mathrm{d}W := \lim_{n\to\infty} \sum_{[t_{i-1},t_i]\in\pi_n} H_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}) \tag{1.22}$$

It is possible to generalise this notion of stochastic integration to integration w.r.t. a continuous semimartingale, however this is not necessary for the content of this report. We continue by stating some of the properties of the Itô integral which we may need later on.

Remark (Properties of the Itô Integral)*.* Below we list some elementary properties of the stochastic integral we have constructed:

- Continuity: As a function of the upper limit of integration t, the paths of $I(t)$ are continuous.
- Adaptivity: For each t, $I(t)$ is $\mathcal{F}_t$-measurable.
- Linearity: Summation of the integrals of processes is equivalent to integrating the summation. Ditto multiplication by a constant.
- Martingale: $I(t)$ is a martingale.

Definition 1.5.3 (Itô Processes)**.** An Itô process is any adapted stochastic process that can be written as the sum of a deterministic integral w.r.t. time and a stochastic integral w.r.t. Brownian motion:

$$X_t = X_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W \tag{1.23}$$

where W is a standard Wiener process, σ is predictable and integrable w.r.t. W, and μ is predictable and Lebesgue integrable. Equivalently, in differential form, we may also write

$$\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t \tag{1.24}$$

Remark. All Itô processes are continuous semimartingales.

Lemma 1.5.1 (Itô's Lemma)**.** This is probably most important result in stochastic calculus, and we will use it several times in the rest of this report. It provides an analogue of the chain rule, allowing us to find differentials for functions of Itô processes. Suppose X is an Itô process satisfying the differential form given above. Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be continuously differentiable at least once in the first argument and twice in the second. Then we have that

$$\mathrm{d}f = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2}\right)\mathrm{d}t + \sigma_t \frac{\partial f}{\partial x}\mathrm{d}W_t \tag{1.25}$$

We can also state this result in the integral form, which is more mathematically meaningful thanks to our definition of the stochastic integral given above:

$$f(t, X_t) - f(0, x_0) = \int_0^t f_t(s, W_s) + \mu_s f_x(s, X_s) + \frac{\sigma_s^2}{2} f_{xx}(s, W_s)\mathrm{d}s + \int_0^t \sigma_s f_x(s, W_s)\mathrm{d}W_s \tag{1.26}$$

Proof. We present here a brief and informal proof. Suppose X_t is an Itô process as previously defined. If $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a twice differentiable scalar function, then it has the Taylor expansion

$$\mathrm{d}f = \frac{\partial f}{\partial t}\mathrm{d}t + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}\mathrm{d}t^2 + \cdots + \frac{\partial f}{\partial x}\mathrm{d}x + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\mathrm{d}x^2 + \ldots. \tag{1.27}$$

Substituting X_t for x and therefore $\mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t$ for $\mathrm{d}x$ gives

$$\begin{aligned}\mathrm{d}f = &\frac{\partial f}{\partial t}\mathrm{d}t + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}\mathrm{d}t^2 + \ldots \\ &+ \frac{\partial f}{\partial x}(\mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mu_t^2(\mathrm{d}t)^2 + 2\mu_t\sigma_t \mathrm{d}t\mathrm{d}W_t + \sigma_t^2(\mathrm{d}W_t)^2) + \ldots.\end{aligned}$$

As $\mathrm{d}t \to 0$, the terms $\mathrm{d}t^2$ and $\mathrm{d}t\mathrm{d}W_t$ tend to 0 faster than $\mathrm{d}W_t^2$ which is $\wr(\mathrm{d}t)$ due to the quadratic variation of the Wiener process. Thus setting the $\mathrm{d}t^2$ and $\mathrm{d}t\mathrm{d}W_t$ terms as well as terms with an order > 2 to 0 and setting $\mathrm{d}W_t^2 = \mathrm{d}t$ we obtain the required expression. ☐

1.6 Stochastic Differential Equations

Next, we investigate the concept of a Stochastic Differential Equation. Again, as with this entire chapter, the content here is a short overview largely without proof. A good introductory textbook for this content is Sarkka and Solin 2019. From a financial perspective, chapter 6 of S. Shreve 2008 provides a useful introduction, and in the context of stochastic control in finance, Pham 2009 provides a recap in chapter 1.

Definition 1.6.1 (Stochastic Differential Equation)**.** A *stochastic differential equation* (SDE) is an equation of the form

$$\mathrm{d}X_u = b(u, X_u)\mathrm{d}u + \sigma(u, X_u)\mathrm{d}W_u. \tag{1.28}$$

b and σ are given, Borel-measurable functions, called the *drift* and *diffusion* respectively. We also specify an initial condition of the form $X_t = x$, where $t \geq 0$ and $x \in \mathbb{R}$ are specified. The problem then is to find a stochastic process X_s defined for $s \geq t$, such that

$$\begin{aligned}X_t &= x \\ X_s &= X_t + \int_t^s b(u, X_u)\mathrm{d}u + \int_t^s \sigma(u, X_u)\mathrm{d}W_u.\end{aligned}$$

Definition 1.6.2 (Strong Solution)**.** A strong solution to this SDE starting at time t is a progressively measurable process X such that for $t \leq s$:

$$X_s = X_t + \int_t^s b(X_u, \alpha_u)\mathrm{d}u + \int_t^s \sigma(X_u, \alpha_u)\mathrm{d}W_u$$

and

$$\int_t^s |b(X_u, \alpha_u)|\mathrm{d}u + \int_t^s |\sigma(X_u, \alpha_u)|^2\mathrm{d}u < \infty$$

a.s.

Theorem 1.6.1. Suppose there exists a deterministic constant K and an $\mathbb{R}$-valued process k such that for every $t \in [0, T], \omega \in \Omega, x, y \in \mathbb{R}$:

$$|b(t, x, \omega) - b(t, y, \omega)| + |\sigma(t, x, \omega) - \sigma(t, y, \omega)| \leq K|x - y|, \tag{1.29}$$

$$|b(t, x, \omega) + \sigma(t, x, \omega)| \leq k_t(\omega) + K|x|, \text{ with} \tag{1.30}$$

$$\mathbb{E}\left[\int_0^t k_u^2\mathrm{d}u\right] < \infty \; \forall \, t \in [0, T]. \tag{1.31}$$

Then under these conditions, there exists for all $t \in [0, T]$ a strong solution to the SDE 1.28 starting at time t with initial condition $X_t = x$.

Moreover, this solution is pathwise unique, meaning that if X and Y are two such strong solutions, we have $\mathbb{P}(X_s = Y_s \; \forall \, t \leq s) = 1$. Calling the solution X, we also have that X is square-integrable: For all $T > t$, there exists a constant C_T such that

$$\mathbb{E}\left[\sup_{t \leq s \leq T} |X_s|^p\right] \leq C_t(1 + |x|^p). \tag{1.32}$$

The proof of this is pages long and quite dry, so we skip it here. It can be found in Chapter 6 of Krylov 1980.

Definition 1.6.3 (Infinitessimal Generator)**.** A useful concept is that of the generator of a diffusion process governed by an SDE. We define it as follows:

$$\mathcal{L}_t(\omega)f(t, x) := b(t, x, \omega)\frac{\partial f}{\partial x}(t, x) + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}(t, x) \tag{1.33}$$

which we can recognise as making up part of the $\mathrm{d}t$ term in Itô's lemma.

Theorem 1.6.2. Given a strong solution X to the SDE (1.28), and a function f of class $C^{1,2}$ on $[0, T] \times \mathbb{R}$ (continuously differentiable at least once in the first argument and twice in the second), we can write Itô's lemma (1.26) for $s \geq t$ as

$$f(s, X_s) = f(t, X_t) + \int_t^s f_t(u, X_u) + \mathcal{L}_u f(u, X_u)\mathrm{d}u + \int_t^s \sigma(u, X_u)f_x(u, X_u)\mathrm{d}W_u \tag{1.34}$$

Definition 1.6.4 (Geometric Brownian Motion)**.** A *geometric brownian motion* is an adapted stochastic process which solves the following stochastic differential equation

$$\mathrm{d}S_t = \mu S_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t \tag{1.35}$$

for $\mu, \sigma \in \mathbb{R}$ and where W is a standard Wiener process. This is an example of an SDE with an explicit (strong) solution: In general, this is not the case, but one-dimensional linear SDEs all have this property. By Itô's formula (1.25) with $f(t, S_t) = \log S_t$, we can write

$$
\begin{aligned}
\mathrm{d}f &= \left(\mu S_t \frac{\partial f}{\partial x} + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 f}{\partial x^2}\right) \mathrm{d}t + \sigma S_t \frac{\partial f}{\partial x} \mathrm{d}S_t \\
&= \left(\mu S_t \frac{1}{S_t} - \frac{\sigma^2 S_t^2}{2} \frac{1}{S_t^2}\right) \mathrm{d}t + \frac{\sigma S_t}{S_t} \mathrm{d}W_t \\
&= \left(\mu - \frac{\sigma^2}{2}\right) \mathrm{d}t + \sigma \mathrm{d}W_t,
\end{aligned}
$$

hence, in the integral form,

$$
\begin{aligned}
\log S_t &= \log S_0 + \int_0^t \mu - \frac{\sigma^2}{2} \mathrm{d}s + \int_0^t \sigma \mathrm{d}W_s \\
&= \log S_0 + \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma(W_t - W_0) \\
S_t &= S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t}
\end{aligned}
$$

and we arrive at the canonical formula for the GBM, where S_0 is the initial value of the process.

Now imagine again the position of a dealer in a financial market. Considering a dealer who is not trading and has a fixed position, and assuming that asset prices evolve according to an Itô process S, their wealth will evolve according to the simple SDE $\mathrm{d}X_t = q\mathrm{d}S_t$ where q is the inventory. However, the dealer can continuously update their bid and ask quotes, hence influencing the flow of orders they receive. The dealer "controls" the process $\alpha = \begin{pmatrix} p^a \\ p^b \end{pmatrix}$ where p^a and p^b are the ask and bid quotes respectively, and this process α should not be determined in advance, but be a function of arguments such as time and the dealers current inventory, which itself is random as it depends on the flow of market orders received by the dealer. Hence the dealers wealth evolves according to the *controlled diffusion*

$$
\mathrm{d}X_t = \beta(t, \alpha_t) \mathrm{d}S_t.
$$

In chapter 2, we will build up a general theory of how to deal with such processes, with the goal of determining the *optimal control* α^* with respect to some particular criteria, such as maximising a particular expected utility.

Chapter 2

Stochastic Optimal Control

2.1 Introduction

In this chapter we introduce the idea of a stochastic control problem in one dimension, and construct a theoretical framework for the resolution of a regular solution, provided that such a solution exists, which is guarunteed under some conditions which we enumerate. We primarily follow the text of Pham 2009. In section 2.2 we introduce the notion of a controlled diffusion process and its solution. In section 2.3 we consider a stochastic control problem over a finite time horizon, before introducing the dynamic programming principle and Hamilton-Jacobi-Bellman equation in sections 2.4 and 2.5 respectively. Finally, in section 2.6, we put these tools to use through a worked example in a financial context, setting us up to tackle the Avellaneda-Stoikov model in Chapter 3.

2.2 Controlled Diffusion Processes

In the previous chapter we have considered Itô processes that are governed either by constants or by functions of time and/or state. Using this, we could for example model a stock price, the movement of a particle, or any other system with the kinds of properties that we study above. If we have a portfolio of cash and an asset, we can model our wealth through time as a stochastic differential equation governed by the risk-free rate at which we earn returns on our cash, and the random fluctuations of the stock price.

For the market-maker however, this is insufficient. We described intuitively at the beginning of chapter 1 how a market maker might be able to influence the flow of orders they recieve over time, and hence their cash flow over time, by adjusting the limit bid and ask quotes that they send to the market. Hence, the market makers portfolio value is governed by not only the fluctuations of the stock and the risk-free rate, but also (stochastically) by the spread that they set. We thus need a model that allows our diffusion process to be governed by not only functions of time and state, but also of some other process which we will call α.

Throughout this chapter we will assume the background of a standard continuous and filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ as defined above in chapter 1.

Definition 2.2.1 (Controlled Diffusion Process)**.** We consider a control model where the state of the system is governed by an $\mathbb{R}$-valued SDE:

$$\mathrm{d}X_t = b(t, X_t, \alpha_t)\mathrm{d}s + \sigma(t, X_t, \alpha_t)\mathrm{d}W_t \tag{2.1}$$

where W is a standard Wiener process. The control $\alpha = (\alpha_t)$ is a *progressively measurable* process valued in $A \subseteq \mathbb{R}^m$.

The functions $b : \mathbb{R}^+ \times \mathbb{R} \times A \to \mathbb{R}$ and $\sigma : \mathbb{R}^+ \times \mathbb{R} \times A \to \mathbb{R}$ are measurable in all of their arguments and satisfy a uniform Lipschitz condition in A: There exists a $K \geq 0$ such that $\forall\, x, y \in \mathbb{R}, \forall\, a \in A$,

$$|b(x,a) - b(y,a)| + |\sigma(x,a) - \sigma(y,a)| \leq K|x-y|. \tag{2.2}$$

In what follows, for $0 \leq t \leq T < \infty$, we denote by $\mathcal{T}_{t,T}$ the set of *stopping times* valued in $[t,T]$.

2.3 The Finite-Horizon Problem

Fix a finite horizon $0 < T < \infty$. We denote by $\mathcal{A}$ the set of control processes α such that for any arbitrary $x \in \mathbb{R}$,

$$\mathbb{E}\left[\int_0^T |b(x,\alpha_t)|^2 + |\sigma(x,\alpha_t)|^2 \mathrm{d}t\right] < \infty. \tag{2.3}$$

From Chapter 1 (theorem 1.6.1), conditions (2.2) and (2.3) ensure the existence and uniqueness of a strong solution to the SDE (2.1) starting from any initial condition $(t,x) \in [0,T] \times \mathbb{R}$ and with any control process $\alpha \in \mathcal{A}$. We denote this unique strong solution with almost surely continuous sample paths by $\{X_s^{t,x}, t \leq s \leq T\}$.
Next we set out our functional objective. Let $f : [0,T] \times \mathbb{R} \times A \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two measurable functions. We suppose that:

- g is lower-bounded **or**
- g satisfies a quadratic growth condition: $|g(x)| \leq C(1 + |x|^2)\ \forall\ x \in \mathbb{R}$ for some constant C independent of x.

In our current context, f will represent a kind of "rolling" reward function, which, when summed or integrated over time allows us to measure the payoff of our actions. The function g represents a kind of terminal reward, for example, some kind of bonus for good performance received at the end of the time period. We also denote by $\mathcal{A}(t,x)$ the subset of controls $\alpha \in \mathcal{A}$ such that

$$\mathbb{E}\left[\int_t^T |f(s, X_x^{t,x}, \alpha_s)| \mathrm{d}s\right] < \infty \tag{2.4}$$

for $(t,x) \in [0,T] \times \mathbb{R}$, and we assume that this set is not empty for all $(t,x) \in [0,T] \times \mathbb{R}$. This integrability condition defines our set of *admissible* controles, control processes which we may find it easy to work with mathematically. We now define the *gain function* to be the expected value of our cumulative rolling reward f and terminal payoff g under a particular control process α:

Definition 2.3.1 (Gain Function)**.**

$$J(t,x,\alpha) := \mathbb{E}\left[\int_t^T f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + g(X_T^{t,x})\right] \tag{2.5}$$

for all $(t,x) \in [0,T] \times \mathbb{R}$ and $\alpha \in \mathcal{A}(t,x)$.

Our objective is thus to maximise over possible control processes the gain function J, and to do this we introduce the associated *value function*:

Definition 2.3.2 (Value Function)**.**

$$v(t,x) := \sup_{\alpha\in\mathcal{A}(t,x)} J(t,x,\alpha). \tag{2.6}$$

Hence, it is easy to see that the value function simply represents the best possible value function that we could achieve under what we will come to call an *optimal* control process α. The concept behind much of stochastic control is that if we can find what this value function v should be, then we can work backwards to determine the optimal α.

Definition 2.3.3 (Optimal control)**.** Given an initial condition $(t,x)\in[0,T]\times\mathbb{R}$, we say that $\hat{\alpha}\in\mathcal{A}(t,x)$ is an optimal control if

$$v(t,x) = J(t,x,\hat{\alpha}). \tag{2.7}$$

Remark. A control process α of the form $\alpha_s = a(s, X_s^{t,x})$ for some measurable function $a:[0,T]\times\mathbb{R}\to A$ is called a *Markovian* control.

2.4 The Dynamic Programming Principle

The Dynamic Programming Principle (DPP) is the fundamental tool upon which much of the theory of stochastic control relies. We formulate it as follows, considering only the context of the finite-horizon problem described above.

Theorem 2.4.1 (Dynamic Programming Principle)**.** Let $(t,x)\in[0,T]\times\mathbb{R}$. Then we have

$$\begin{aligned} v(t,x) &= \sup_{\alpha\in\mathcal{A}(t,x)} \sup_{\theta\in\mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},\alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right] && (2.8)\\ &= \sup_{\alpha\in\mathcal{A}(t,x)} \inf_{\theta\in\mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},\alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right] && (2.9)\\ & && (2.10)\end{aligned}$$

Proof of the DPP. By pathwise uniqueness of the SDE for X, for any admissible control $\alpha\in\mathcal{A}(t,x)$, for any $\theta\in\mathcal{T}_{t,T}$ and for all $s\geq\theta$

$$X_s^{t,x} = X_s^{\theta, X_\theta^{t,x}}. \tag{2.11}$$

By the law of iterated expectations we then have

$$\begin{aligned} J(t,x,\alpha) &= \mathbb{E}\left[\int_t^T f(s,X_s^{t,x},\alpha_s)\mathrm{d}s + g(X_T^{t,x})\right]\\ &= \mathbb{E}\left[\mathbb{E}\left[\int_t^T f(s,X_s^{t,x},\alpha_s)\mathrm{d}s + g(X_T^{t,x})|\mathcal{F}_\theta\right]\right]\\ &= \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},\alpha_s)\mathrm{d}s + \mathbb{E}\left[\int_\theta^T f(s,X_s^{t,x},\alpha_s)\mathrm{d}s + g(X_T^{t,x})|\mathcal{F}_\theta\right]\right]\\ &= \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},\alpha_s)\mathrm{d}s + \mathbb{E}\left[\int_\theta^T f(s,X_s^{t,x},\alpha_s)\mathrm{d}s + g(X_T^{t,x})\right]\right]\\ &= \mathbb{E}\left[\int_t^\theta f(s,X_s^{t,x},\alpha_s)\mathrm{d}s + J(\theta, X_\theta^{t,x},\alpha)\right]\end{aligned}$$

and since $J(.,.,\alpha) \leq v$ and θ is arbitrary in $\mathcal{T}_{t,T}$ we obtain

$$\begin{aligned}
J(t,x,\alpha) &\leq \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right] \\
&\leq \inf_{\theta\in\mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right] \\
&\leq \sup_{\alpha\in\mathcal{A}(t,x)} \inf_{\theta\in\mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right]
\end{aligned}$$

and by taking the supremum over α in the left hand side, we obtain the second of the desired inequalities:

$$v(t,x) \leq \sup_{\alpha\in\mathcal{A}(t,x)} \inf_{\theta\in\mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right]. \tag{2.12}$$

Next we fix an arbitrary control $\alpha \in \mathcal{A}(t,x)$ and $\theta \in \mathcal{T}_{t,T}$. By the definition of the value function, and the properties of the supremum and of continuity, for any $\epsilon > 0$ and $\omega \in \Omega$ there exists an $\alpha^{\epsilon,\omega} \in \mathcal{A}(\theta(\omega), X_{\theta(\omega)}^{t,x}(\omega))$ that is an ϵ-optimal control for $v(\theta, X_{\theta(\omega)}^{t,x}(\omega))$, i.e.

$$v(\theta, X_{\theta(\omega)}^{t,x}(\omega)) - \epsilon \leq J(\theta(\omega), X_{\theta(\omega)}^{t,x}(\omega), \alpha^{\epsilon,\omega}). \tag{2.13}$$

We now define the process

$$\hat{\alpha}_s(\omega) = \begin{cases} \alpha_s(\omega), s \in [0, \theta(\omega)], \\ \alpha_s^{\epsilon,\omega}(\omega), s \in [\theta(\omega), T]. \end{cases} \tag{2.14}$$

It can be shown by the measurable selection theorem that the process $\hat{\alpha}$ is progressively measurable, and so lies in $\mathcal{A}(t,x)$. This is well beyond the scope of this project, and so will not be done here, however one example of this can be found in Chapter 7 of Bertsekas and S. Shreve 1978. Again by the law of iterated expectations and (2.13) we get

$$\begin{aligned}
v(t,x) \geq J(t,x,\hat{\alpha}) &= \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + J(\theta, X_\theta^{t,x}, \alpha^\epsilon)\right] \\
&\geq \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right] - \epsilon.
\end{aligned}$$

Finally, by the fact that $\alpha \in \mathcal{A}(t,x), \theta \in \mathcal{T}_{t,T}$ and $\epsilon > 0$ are all arbitrary, we obtain the first inequality:

$$v(t,x) = \sup_{\alpha\in\mathcal{A}(t,x)} \sup_{\theta\in\mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right]. \tag{2.15}$$

□

Remark (Equivalent Formulations). We normally write the DPP as

$$v(t,x) = \sup_{\alpha\in\mathcal{A}(t,x)} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right], \tag{2.16}$$

however it is sometimes useful to use the following equivalent formulation of the DPP:
(i) For all $\alpha \in \mathcal{A}(t,x)$ and $\theta \in \mathcal{T}_{t,T}$:

$$v(t,x) \geq \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right]. \tag{2.17}$$

(ii) For all $\epsilon > 0$, there exists $\alpha \in \mathcal{A}(t,x)$ such that for all $\theta \in \mathcal{T}_{t,T}$:

$$v(t,x) - \epsilon \leq \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right]. \tag{2.18}$$

The idea behind the DPP is that we do not have to solve for the optimal control over the entire time interval in one go: We can split the problem into two, first finding an optimal control from θ to T, and finding the optimal control from t up to θ, as given in (2.16). The magic in this is that we are not limited to two sub-problems: We can continue splitting up our interval of time into increasingly small chunks, hopefully making each subproblem more tractable.

2.5 Hamilton-Jacobi-Bellman Equation

The Dynamic Programming Principle tells us that we can consider a stochastic control problem as a sequence of smaller sub-problems defined over intervals of $[0,T]$ characterised by stopping times, i.e., $[0,T] = [0,\theta_1] \cup (\theta_1,\theta_2] \cup \cdots \cup (\theta_n,T]$ where $\theta_1 \leq \cdots \leq \theta_n \in \mathcal{T}_{t,T}$. Thus, a natural thing to consider is the following: What happens as $n \to \infty$ and correspondingly $\theta_{i+1} - \theta_i \to 0$? What we obtain is the Hamilton-Jacobi-Bellman equation (HJB) which describes the dynamics of the value function over small increments of time. In this chapter and what follows, we will use the HJB equation as follows:

- Provide a formal derivation of the HJB equation.
- Obtain or try to show the existence of a smooth solution.
- Verification step: Show that the smooth solution is the value function.
- As a byproduct, we obtain an optimal feedback control.

Theorem 2.5.1 (Hamilton-Jacobi-Bellman Equation)**.** The dynamics of the value function $v(t,x)$ satisfy the following non-linear second-order partial differential equation:

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) + \sup_{a\in A}\left[\mathcal{L}^a v(t,x) + f(t,x,a)\right] = 0 \ \forall \ (t,x) \in [0,T) \times \mathbb{R} \\ v(T,x) = g(x) \ \forall \ x \in \mathbb{R}. \end{cases} \tag{2.19}$$

where $\mathcal{L}^a$ is the operator associated to the diffusion (2.1) and defined by (see Section 1.6)

$$\mathcal{L}^a v = b(t,x,a)v_x + \frac{1}{2}\sigma(t,x,a)^2 v_{xx}. \tag{2.20}$$

Proof. Let us consider time $\theta = t+h$ and a constant control $\alpha_s = a$ for some arbitrary $a \in A$, in our slightly stronger variant of the DPP (2.17):

$$v(t,x) \geq \mathbb{E}\left[\int_t^{t+h} f(s,X_s^{t,x},a)\mathrm{d}s + v(t+h,X_{t+h}^{t,x})\right]. \tag{2.21}$$

By assuming that v is smooth enough, we can apply Itô's formula between t and $t+h$:

$$v(t+h,X_{t+h}^{t,x}) = v(t,x) + \int_t^{t+h}\left(\frac{\partial v}{\partial t} + \mathcal{L}^a v\right)(s,X_s^{t,x})\mathrm{d}s + (\text{local})\ \text{martingale}. \tag{2.22}$$

We can then substitute back into (2.21) to obtain

$$0 \geq \mathbb{E}\left[\int_t^{t+h}\left(\frac{\partial v}{\partial t} + \mathcal{L}^a v\right)(s,X_s^{t,x}) + f(s,X_s^{t,x},a)\mathrm{d}s\right] \tag{2.23}$$

which if we divide by h and send $h \to 0$ we yield

$$0 \geq \frac{\partial v}{\partial t}(t,x) + \mathcal{L}^a v(t,x) + f(t,x,a) \tag{2.24}$$

by the mean-value theorem. Since this holds true for any $a \in A$, we obtain the inequality

$$-\frac{\partial v}{\partial t}(t,x) - \sup_{a \in A}[\mathcal{L}^a v(t,x) + f(t,x,a)] \geq 0. \tag{2.25}$$

On the other hand, suppose that α^* is an optimal control. Then in (2.16) we have

$$v(t,x) = \mathbb{E}\left[\int_t^{t+h} f(s, X_s^*, \alpha_s^*)\mathrm{d}s + v(t+h, X_{t+h}^*)\right], \tag{2.26}$$

where X^* is the solution to (2.1) starting from state x at time t with control α^*. Again by Itô's formula we have that

$$v(t+h, X_{t+h}^*) = v(t,x) + \int_t^{t+h}\left(\frac{\partial v}{\partial t} + \mathcal{L}^a v\right)(s, X_s^*)\mathrm{d}s + \text{(local) martingale} \tag{2.27}$$

which we can again substitute back into (2.26) to obtain

$$0 = \mathbb{E}\left[\int_t^{t+h}\left(\frac{\partial v}{\partial t} + \mathcal{L}^a v\right)(s, X_s^*) + f(s, X_s*, a)\mathrm{d}s\right] \tag{2.28}$$

and hence once again we divide by h and send $h \to 0$ yielding

$$-\frac{\partial v}{\partial t}(t,x) - \mathcal{L}^{\alpha_t^*} v(t,x) - f(t,x,\alpha_t^*) = 0. \tag{2.29}$$

Combining this with (2.25), v should satisfy

$$-\frac{\partial v}{\partial t}(t,x) - \sup_{a \in A}\left[\mathcal{L}^a v(t,x) + f(t,x,a)\right] = 0 \;\forall\; (t,x) \in [0,T) \times \mathbb{R}, \tag{2.30}$$

if the above supremum in a is finite. This may arise when the control space A is unbounded, and we will see how to deal with this later on. We can also obtain the terminal condition associated to this PDE:

$$v(T,x) = g(x) \;\forall\; x \in \mathbb{R} \tag{2.31}$$

which results immediately from the definition in (2.6) of the value function considered at the horizon time T. □

So the general process for solving a stochastic optimal control problem is the following:

1. Formulate the problem in terms of a controlled diffusion process
2. Define the value function
3. Formulate the Hamilton-Jacobi-Bellman equation that describes the dynamics of the value function
4. Solve the HJB equation to obtain an exact form or approximation for the value function
5. Utilise the resolved value function to find the optimal control process

There is an extra step that we have not covered here in chapter 2, namely the *verification theorem*. This theorem provides necessary and sufficient conditions for the candidate value function found in step 4. and candidate optimal control found in step 5. to constitute the full solution to the problem. For simplicity, we do not include it here as we will not be explicitly solving the HJB-equation associated to our value function for the market-making problem in chapter 3, however, the theorem and it's proof is included in appendix A.

To finish off chapter 2, we will work through a famous example, given by Merton 1969, of portfolio allocation over a finite time horizon in a Black-Scholes-Merton Model.

2.6 Finite-Horizon Merton Portfolio Allocation Problem

Consider an agent who, at time t, invests a proportion of her wealth α_t in a stock of price S governed by a geometric Brownian motion, and $1-\alpha_t$ in a riskless bond of price S^0 with interest rate r. We consider a finite horizon $[0,T]$, and at any time $t \in [0,T]$ the investor faces the constraint that α_t is valued in A, a closed convex subset of $\mathbb{R}$ (namely, $A=[0,1]$).

The wealth process evolves according to

$$\begin{aligned} \mathrm{d}X_t &= \frac{X_t\alpha_t}{S_t}\mathrm{d}S_t + \frac{X_t(1-\alpha_t)}{S_t^0}\mathrm{d}S_t^0 \\ &= X_t(\alpha_t\mu + (1-\alpha_t)r)\mathrm{d}t + X_t\alpha_t\sigma\mathrm{d}W_t. \end{aligned}$$

Denote by $\mathcal{A}$ the set of progressively measurable processes α valued in A and such that

$$\int_0^T \alpha_s^2 \mathrm{d}s < \infty \text{ a.s.}$$

This integrability condition ensures the existence and uniqueness of a strong solution to the SDE governing the wealth process controlled by α. Given a strategy α, we denote by $X^{t,x}$ the corresponding wealth process starting from initial capital $X_t = x > 0$ at time t. The agent wants to maximise the expected utility of terminal wealth, giving us the value function

$$v(t,x) = \sup_{\alpha\in\mathcal{A}} \mathbb{E}[U(X_T^{t,x})],\ (t,x) \in [0,T]\times\mathbb{R}^+. \tag{2.32}$$

The HJB equation for this stochastic control problem is given by

$$\frac{\partial v}{\partial t} + \sup_{a\in A}\{\mathcal{L}^a v(t,x)\} = 0, \tag{2.33}$$

together with the terminal condition

$$v(T,x) = U(x),\ x \in \mathbb{R}^+. \tag{2.34}$$

Here,

$$\mathcal{L}^a v(t,x) = x(a\mu + (1-a)r)\frac{\partial v}{\partial x} + \frac{1}{2}x^2a^2\sigma^2\frac{\partial^2 v}{\partial x^2}. \tag{2.35}$$

As originally considered by Merton, we will look at power utility functions of Constant Relative Risk Aversion (CRRA) type, taking the form

$$U(x) = \frac{x^p}{p},\ x \geq 0, p < 1, p \neq 0.$$

It turns out that explicit smooth solutions do exist for the problem given by (2.33)-(2.34). We are looking for a candidate solution of the form

$$v(t,x) = \phi(t)U(x)$$

for some positive function ϕ. By substituting into (2.33)-(2.34), we derive that ϕ should satisfy the ODE

$$\begin{cases} \phi'(t) + \rho\phi(t) = 0 \\ \phi(T) = 1 \end{cases}$$

where

$$\rho = p\sup_{a\in A}\{a(\mu-r) + r - \frac{1}{2}a^2(1-p)\sigma^2\}.$$

This ODE is solved by $\phi(t) = e^{\rho(T-t)}$. Hence we arrive at

$$v(t,x) = e^{\rho(T-t)}U(x),\ (t,x) \in [0,T] \times \mathbb{R}^+. \tag{2.36}$$

Moreover, $a(\mu-r)+r-\frac{1}{2}a^2(1-p)\sigma^2$ is a concave function of $a \in A$ and thus attains its maximum at some point $\hat{a}$. By construction, $\hat{a}$ also attains the supremum of $\sup_{a\in A}\{\mathcal{L}^a v(t,x)\}$. Moreover, the wealth process associated to the constant control $\hat{a}$

$$\mathrm{d}X_t = X_t(\hat{a}\mu + (1-\hat{a})r)\mathrm{d}t + X_t\hat{a}\sigma\mathrm{d}W_t$$

admits a unique solution given an initial condition. We can also use the verification theorem in appendix A to prove that the value function in problem (2.33) is equal to (2.35), and the optimal proportion of wealth to invest is given by $\hat{a}$. Finally, using the concavity of

$$a(\mu-r) + r - \frac{1}{2}a^2(1-p)\sigma^2$$

in a we obtain that

$$\hat{a} = \frac{\mu - r}{(1-p)\sigma^2} \tag{2.37}$$

and

$$\rho = \frac{(\mu-r)^2}{2\sigma^2}\frac{p}{1-p} + rp.$$

After walking through this short example, we hopefully have a little bit more understanding of how to apply stochastic control theory to problems in finance. In chapter 3, we will put this to the test by returning to the market-making problem introduced in section 1.2, and building up the model given by Avellaneda and Stoikov 2008.

Chapter 3

The Avellaneda-Stoikov Model

3.1 Introduction

In this chapter, we can finally return to the market-making problem introduced in Chapter 1, fully armed with the theory of stochastic optimal control that we have built up in Chapter 2. We will formulate the problem and our assumptions in the framework of Avellaneda and Stoikov 2008, and walk through their methodology and theoretical results.

We begin in section 3.2 by setting out our assumptions our assumptions about the dynamics of the market mid-point price and our agents utility function. In section 3.3, we introduce the concept of an indifference or reservation price in the context of a passive agent with constant inventory, and derive some expressions. In 3.4, we briefly analyse the infinite time horizon case, showing that analagous reservation prices exist, which may be of greater interest to dealers in markets that trade 24/7 such as FX and crypto. In section 3.5 we return to the finite horizon setting and define concepts such as market impact, arriving at the objective function of the agent who can set limit orders and thus influence the dynamics of their wealth over time.

Of crucial importance to this agent are the statistical properties of market orders: Their arrival frequency, the distribution of their size, and how they impact prices, which we discuss in section 3.6. Next we derive the Hamilton-Jacobi-Bellman equation in section 3.7, and introduce an ansatz which allows us to simplify our problem and derive some useful relations between the agents reservation price and optimal bid-ask spread. Finally, we introduce some analytical approximations in section 3.8 that enable us to derive an approximate solution in terms of our model parameters.

The main result, which we summarise in section 3.9, is that optimal bid and ask quotes can be computed through an intuitive two-step procedure: First, the agent computes a personal reservation price for the asset, given her current inventory. Second, she calibrates her bid and ask quotes to the limit order book, by considering the probability with which her quotes will be executed as a function of their distance from the midpoint price.

3.2 Model assumptions

The paper of Avellaneda and Stoikov 2008 is closely related to that of Ho and Stoll 1981, with the crucial difference being that while Ho and Stoll consider a monopolistic dealer, Avellaneda and Stoikov consider a dealer who is potentially one of many dealers and many other market participants who may set limit orders.

In Ho and Stoll (1981), the authors specify a 'true' price for the asset, and then allow the dealer to set quotes around this price. This may be more applicable to OTC markets in illiquid products where there is no openly accessible limit orderbook, but Avellaneda and Stoikov consider a dealer operating in an openly accessible limit orderbook, and hence it makes sense to view the mid-point price in the orderbook as the true price of the security.

Another point to make here is that Avellaneda and Stoikov consider a dealer who is concerned only with inventory risk, not asymmetric information, and so assuming that other market participants are better informed and reacting to fundamental arbitrage opportunities, by the efficient market hypothesis we would have that the market mid price is the best available measure of the true price of the asset given all the information available up to a particular point in time.

We will assume that the market mid-point price evolves according to the SDE

$$\mathrm{d}S_u = \sigma \mathrm{d}W_u \tag{3.1}$$

with initial value $S_t = s$. W_t is a standard one-dimensional Brownian motion, and $\sigma > 0$ is constant. Underlying this model is an implicit assumption that the agent has no opinion on the drift or any autocorrelation or stochasticity of volatility for the stock.

We also assume for simplicity that the money market pays no interest. Moreover, the limit orders set by the agent can be continuously updated at no cost. In reality, the cost of trading will differ depending on the exchange in question, as most charge a small percentage fee of every executed trade and some only charge market orders, while providing rebates to dealers' trades for the liquidity they provide. Finally, we assume that the lot sizes our limit orders are constant at one share per order, and that the overall arrival frequency of market orders is constant.

We summarise our assumptions in the list below:

- The dealer being modelled is one of many players in the market
- The 'true' price is given by the market mid-price
- The mid-price evolves according to a brownian motion with constant volatility σ
- The agent has no opinion on drift or autocorrelation of the stock price
- The money-market pays no interest
- Limit orders can be continuously updated at no cost
- Limit orders are of fixed size 1
- The arrival frequency of market orders to the market is constant

3.3 Modelling an inactive trader

Our agents objective will be to maximise the expected utility of their wealth at a terminal time T. Avellaneda and Stoikov's choice of exponential utility is convenient since its convexity allows us to define reservation prices that are indepenent of the agents current wealth.

The utility function

Initially, we consider an inactive trader who holds a fixed inventory of q stocks until the terminal time T. The agent's value function is

$$v(x,s,q,t) = \mathbb{E}\left[-e^{-\gamma(x+qS_T)}|\mathcal{F}_t\right] \tag{3.2}$$

where x is the initial wealth in dollars, t is the present time and γ is a personal pre-defined risk-aversion parameter. By some simple manipulations, we can write this in a more convenient form as follows:

$$\begin{aligned} v(x,s,q,t) &= \mathbb{E}\left[-e^{-\gamma(x+qS_T)}|\mathcal{F}_t\right] \\ &= -e^{-\gamma x}\mathbb{E}\left[e^{-\gamma qS_T}|\mathcal{F}_t\right] \\ &= -e^{-\gamma x}e^{-\gamma qs+\frac{\gamma^2q^2\sigma^2(T-t)}{2}} \\ &= -e^{-\gamma x}e^{-\gamma qs}e^{\frac{\gamma^2q^2\sigma^2(T-t)}{2}} \end{aligned}$$

Reservation prices

Following Avellaneda and Stoikov 2008, we can now use our value function to define the agents reservation bid and ask prices. The reservation bid and reservation ask prices are simply the prices at which the agent is indifferent between buying/selling and doing nothing. In other words, the reservation bid (ask) is the price at which the agent is indifferent between her current portfolio and her current portfolio $\pm$ one stock and $\mp$ the cash price.

Definition 3.3.1 (Reservation bid price)**.** Let v be the value function of the agent. Its reservation bid price r^b is given implicitly by the relation

$$v(x-r^b(s,q,t),s,q+1,t) = v(x,s,q,t) \tag{3.3}$$

and the corresponding reservation ask price r^a is similarly implicit in the relation

$$v(x+r^a(s,q,t),s,q-1,t) = v(x,s,q,t). \tag{3.4}$$

We can determine an exact expression for $r^b(s,q,t)$ by plugging our prior definition for the value function, (3.2), in to our relation (3.3) as follows:

$$\begin{aligned} v(x-r^b(s,q,t),s,q+1,t) &= v(x,s,q,t) \\ -e^{-\gamma(x-r^b(s,q,t))}e^{-\gamma s(q+1)}e^{\frac{\gamma^2(q+1)^2\sigma^2(T-t)}{2}} &= -e^{-\gamma x}e^{-\gamma qs}e^{\frac{\gamma^2q^2\sigma^2(T-t)}{2}} \\ -\gamma(x-r^b(s,q,t))-\gamma s(q+1)+\frac{\gamma^2(q+1)^2\sigma^2(T-t)}{2} &= -\gamma x-\gamma qs+\frac{\gamma^2q^2\sigma^2(T-t)}{2} \\ \gamma r^b(s,q,t)-\gamma s+\frac{\gamma^2(1+2q)\sigma^2(T-t)}{2} &= 0, \end{aligned}$$

dividing by γ and rearranging to obtain

$$r^b(s,q,t) = s+(-1-2q)\frac{\gamma\sigma^2(T-t)}{2} \tag{3.5}$$

Similarly for $r^a(s,q,t)$:

$$\begin{aligned}
v(x+r^a(s,q,t),s,q-1,t) &= v(x,s,q,t)\\
-e^{-\gamma(x+r^a(s,q,t))}e^{-\gamma s(q-1)}e^{\frac{\gamma^2(q-1)^2\sigma^2(T-t)}{2}} &= -e^{-\gamma x}e^{-\gamma qs}e^{\frac{\gamma^2 q^2\sigma^2(T-t)}{2}}\\
-\gamma(x+r^a(s,q,t))-\gamma s(q-1)+\frac{\gamma^2(q-1)^2\sigma^2(T-t)}{2} &= -\gamma x-\gamma qs+\frac{\gamma^2q^2\sigma^2(T-t)}{2}\\
-\gamma r^a(s,q,t)+\gamma s+\frac{\gamma^2(1-2q)\sigma^2(T-t)}{2} &= 0,
\end{aligned}$$

again dividing by γ and rearranging to obtain

$$r^a(s,q,t) = s+(1-2q)\frac{\gamma\sigma^2(T-t)}{2} \tag{3.6}$$

We define the *reservation* or *indifference* price to be the average of these two *given* that the agent currently holds q stocks:

$$\begin{aligned}
r(s,q,t) &= \frac{r^a(s,q,t)+r^b(s,q,t)}{2}\\
&= \frac{s+(1-2q)\frac{\gamma\sigma^2(T-t)}{2}+s+(-1-2q)\frac{\gamma\sigma^2(T-t)}{2}}{2}\\
&= \frac{2s-2q\gamma\sigma^2(T-t)}{2}\\
&= s-q\gamma\sigma^2(T-t)
\end{aligned}$$

This price is nothing more than an adjustment to the mid-price which accounts for the effect of the inventory held by the agent on the agents preference to buy or sell. It is easy to see that if the agent is long stock ($q>0$), the reservation price will be lower than the mid-price, reflecting the agents willingness to sell at a discount in order to reduce its inventory. Conversely, if the agent is short stock ($q<0$), its reservation price will be greater than the mid-price, indicating the agents preference to buy at a premium to the market in order to return to a market-neutral position.

We note that the expressions derived above for r^a and r^b (and consequently r) exist in the setting where q is a fixed constant, and therefore it is not so simple to derive these expressions when our agent is permitted to set limit orders. However, they are important both as an illustrative example and because when we introduce our approximate solution in 3.8, we will arrive at a very similar reservation price.

3.4 The Optimising Agent with Infinite Horizon

We will now briefly analyse the infinite horizon variant of the dealer problem, showing that we can derive a stationary version of the reservation price through defining an infinite horizon variant of our value function including a discount factor. This is necessary since in our finite horizon case discussed above, our reservation price is dependent upon the time interval $T-t$. The intuition for this is that at or close to T, the agent may liquidate any remaining inventory for (or at least close to) S_T, hence the closer time is to T, the less risk there is in the dealer's position.

We consider an infinite-horizon value function of the form

$$\bar{v}(x,s,q) = \mathbb{E}\left[\int_0^\infty -e^{-\omega t}e^{-\gamma(x+qS_t)}\mathrm{d}t\right]$$

where ω is our discount factor. An interpretation of ω is that it represents an upper bound on the absolute inventory position that the agent is allowed to build up. A natural choice is to take $\omega = \frac{1}{2}\gamma^2\sigma^2(q_{\max}+1)^2$, this will be justified shortly.

Using the definition of reservation bid and ask prices given above in section 3.3, we can attain stationary versions of the reservation prices r^b and r^a with much the same method as before, only relying on slightly more advanced theory, appealing to Tonelli's theorem (1.13) which allows us to swap the expectation and integral in the value function. For r^b, we have the following:

$$
\begin{aligned}
\bar{v}(x-\bar{r}^b(s,q), s, q+1) &= \bar{v}(x,s,q) \\
\mathbb{E}\left[\int_0^\infty -e^{-\omega t}e^{-\gamma(x-\bar{r}^b(s,q)+(q+1)S_t)}\mathrm{d}t\right] &= \mathbb{E}\left[\int_0^\infty -e^{-\omega t}e^{-\gamma(x+qS_t)}\mathrm{d}t\right] \\
\int_0^\infty e^{-\omega t}e^{-\gamma(x-\bar{r}^b(s,q))}\mathbb{E}\left[e^{-\gamma(q+1)S_t}\right]\mathrm{d}t &= \int_0^\infty e^{-\omega t}e^{-\gamma x}\mathbb{E}\left[e^{-\gamma q S_t}\right]\mathrm{d}t \text{ (by Tonelli)} \\
e^{-\gamma(x-\bar{r}^b(s,q))}\int_0^\infty e^{-\omega t}e^{-\gamma(q+1)s+\frac{\gamma^2(q-1)^2\sigma^2 t}{2}}\mathrm{d}t &= e^{-\gamma x}\int_0^\infty e^{-\omega t}e^{-\gamma q s+\frac{\gamma^2 q^2\sigma^2 t}{2}}\mathrm{d}t \\
e^{-\gamma(x-\bar{r}^b(s,q))}e^{-\gamma(q+1)s}\int_0^\infty e^{-\omega t}e^{\frac{\gamma^2(q+1)^2\sigma^2 t}{2}}\mathrm{d}t &= e^{-\gamma x}e^{-\gamma q s}\int_0^\infty e^{-\omega t}e^{\frac{\gamma^2 q^2\sigma^2 t}{2}}\mathrm{d}t \\
e^{\gamma\bar{r}^b(s,q)}e^{-\gamma s}\int_0^\infty e^{\left(\frac{\gamma^2(q+1)^2\sigma^2-2\omega}{2}\right)t}\mathrm{d}t &= \int_0^\infty e^{\left(\frac{\gamma^2 q^2\sigma^2-2\omega}{2}\right)t}\mathrm{d}t \\
e^{\gamma\bar{r}^b(s,q)}e^{-\gamma s}\left(\frac{2}{2\omega-\gamma^2(q+1)^2\sigma^2}\right) &= \left(\frac{2}{2\omega-\gamma^2q^2\sigma^2}\right) \\
e^{\gamma(\bar{r}^b(s,q)-s)} &= \frac{2\omega-\gamma^2(q+1)^2\sigma^2}{2\omega-\gamma^2q^2\sigma^2} \\
e^{\gamma(\bar{r}^b(s,q)-s)} &= 1-\frac{(1+2q)\gamma^2\sigma^2}{2\omega-\gamma^2q^2\sigma^2} \\
\gamma\bar{r}^b(s,q)-\gamma s &= \log\left(1+\frac{(-1-2q)\gamma^2\sigma^2}{2\omega-\gamma^2q^2\sigma^2}\right) \\
\bar{r}^b(s,q) &= s+\frac{1}{\gamma}\log\left(1+\frac{(-1-2q)\gamma^2\sigma^2}{2\omega-\gamma^2q^2\sigma^2}\right)
\end{aligned}
$$

which is valid for $\omega > \frac{1}{2}\gamma^2\sigma^2q^2$ and agrees exactly with the result presented in Avellaneda and Stoikov 2008. We can now perform the same procedure for the reservation ask price

r^a:

$$
\begin{aligned}
\bar{v}(x + r^a(s,q), s, q-1) &= \bar{v}(x, s, q) \\
\mathbb{E}\left[\int_0^\infty -e^{-\omega t} e^{-\gamma(x + r^a(s,q) + (q-1)S_t)} \mathrm{d}t\right] &= \mathbb{E}\left[\int_0^\infty -e^{-\omega t} e^{-\gamma(x+qS_t)} \mathrm{d}t\right] \\
\int_0^\infty e^{-\omega t} e^{-\gamma(x + r^a(s,q))} \mathbb{E}\left[e^{-\gamma(q-1)S_t}\right] \mathrm{d}t &= \int_0^\infty e^{-\omega t} e^{-\gamma x} \mathbb{E}\left[e^{-\gamma q S_t}\right] \mathrm{d}t \text{ (by Tonelli)} \\
e^{-\gamma(x + r^a(s,q))} \int_0^\infty e^{-\omega t} e^{-\gamma(q-1)s + \frac{\gamma^2 (q-1)^2 \sigma^2 t}{2}} \mathrm{d}t &= e^{-\gamma x} \int_0^\infty e^{-\omega t} e^{-\gamma q s + \frac{\gamma^2 q^2 \sigma^2 t}{2}} \mathrm{d}t \\
e^{-\gamma(x + r^a(s,q))} e^{-\gamma(q-1)s} \int_0^\infty e^{-\omega t} e^{\frac{\gamma^2 (q-1)^2 \sigma^2 t}{2}} \mathrm{d}t &= e^{-\gamma x} e^{-\gamma q s} \int_0^\infty e^{-\omega t} e^{\frac{\gamma^2 q^2 \sigma^2 t}{2}} \mathrm{d}t \\
e^{-\gamma r^a(s,q)} e^{\gamma s} \int_0^\infty e^{\left(\frac{\gamma^2 (q-1)^2 \sigma^2 - 2\omega}{2}\right) t} \mathrm{d}t &= \int_0^\infty e^{\left(\frac{\gamma^2 q^2 \sigma^2 - 2\omega}{2}\right) t} \mathrm{d}t \\
e^{-\gamma r^a(s,q)} e^{\gamma s} \left(\frac{2}{2\omega - \gamma^2 (q-1)^2 \sigma^2}\right) &= \left(\frac{2}{2\omega - \gamma^2 q^2 \sigma^2}\right) \\
e^{\gamma(s - r^a(s,q))} &= \frac{2\omega - \gamma^2 (q-1)^2 \sigma^2}{2\omega - \gamma^2 q^2 \sigma^2} \\
e^{\gamma(s - r^a(s,q))} &= 1 - \frac{(1-2q)\gamma^2 \sigma^2}{2\omega - \gamma^2 q^2 \sigma^2} \\
\gamma s - \gamma r^a(s,q) &= \log\left(1 - \frac{(1-2q)\gamma^2 \sigma^2}{2\omega - \gamma^2 q^2 \sigma^2}\right) \\
r^a(s,q) &= s - \frac{1}{\gamma} \log\left(1 - \frac{(1-2q)\gamma^2 \sigma^2}{2\omega - \gamma^2 q^2 \sigma^2}\right)
\end{aligned}
$$

which is again valid for $\omega > \frac{1}{2}\gamma^2\sigma^2 q^2$. However, this differs from the result presented in Avellaneda and Stoikov 2008, which they give to be:

$$
r^a(s,q) = s + \frac{1}{\gamma} \log\left(1 + \frac{(1-2q)\gamma^2 \sigma^2}{2\omega - \gamma^2 q^2 \sigma^2}\right)
$$

From our derivations, we can see that to ensure integrability in both cases, the user-specified parameter ω must satisfy

$$
\omega > \frac{1}{2}\gamma^2 \sigma^2 q^2,
$$

where the only variable quantity on the RHS is the inventory variable q. Therefore, if we want to ensure that these reserve prices always exist, we should bound the maximum inventory our agent can build up on either side. Hence, we set some $q_{\max} > 0$ and set

$$
\omega = \frac{1}{2}\gamma^2 \sigma^2 (q_{\max} + 1)^2
$$

to ensure integrability.

3.5 Modelling Limit Orders

Now that we have defined and discussed the idea of a reservation price for the dealer, we should move on to considering the case of the dealer who can indirectly influence both their inventory and cash flow through the setting of limit orders.

As mentioned in section 3.2, the agent quotes bid and ask limit orders in lot sizes of 1 only. We denote the agent's quotes p^a and p^b for the ask and bid respectively, and note that the agent is comitted to sell or buy 1 unit of stock respectively should these orders be "hit" or "lifted" by an incoming market order. These quotes can also be updated continuously at no cost. The distances

$$\delta^a := p^a - s \tag{3.7}$$

and

$$\delta^b := s - p^b \tag{3.8}$$

as well as the current shape of the orderbook determine the priority of execution when large market orders are placed.

For example, when a market order to buy Q shares arrives, the Q limit orders with the lowest ask prices will be lifted automatically by the exchanges matching engine. If Q is greater than the number of shares available at the lowest ask level in the orderbook, the order causes a temporary market impact since transactions will occur at a price not only higher than the mid-price, but higher than the best ask.

Definition 3.5.1 (Temporary market impact)**.** Let p^Q be the price of the highest (most expensive) limit order executed in this trade. Then

$$\Delta p := p^Q - s \tag{3.9}$$

is the temporary market impact of the trade of size Q.

Then we have that if our agent's $\delta^a < \Delta p$, our agents limit order will be executed. We will assume that market buy orders will lift our agent's sell limit orders with a Poisson intensity function denoted $\lambda^a(\delta^a)$ which is a decreasing function of δ^a. Likewise, we assume that market sell orders will hit our agent's bid limit orders with Poisson intensity $\lambda^b(\delta^b)$, decreasing in δ^b. Intuitively, this encapsulates the fact that further away from the mid-price the agent places her quotes, the less often she will receive market orders.

Now, our cash wealth and portfolio of stock is stochastic and depends on the incoming flow of market buy and sell orders. Naturally, both our cash flow and inventory jump every time a market order executes one of our agent's limit orders. Let N_t^a and N_t^b be Poisson point processes with intensities λ^a and λ^b, representing the amount of stocks sold or bought by the agent up to time t respectively. Our inventory at time t is thus

$$q_t := N_t^b - N_t^a \tag{3.10}$$

and our wealth process evolves according to

$$\mathrm{d}Xt = p^a \mathrm{d}N_t^a - p^b \mathrm{d}N_t^b. \tag{3.11}$$

Finally, we can reformulate our value function from section 3.3. The goal we set for our agent is still to maximise the expected expontential utility of terminal wealth, however now the cash and inventory components of our terminal portfolio are stochastic as well as the mid-price itself. Hence, our value function becomes the following:

Definition 3.5.2 (Value function of Market-Making Agent)**.**

$$u(s, x, q, t) := \max_{\delta^a, \delta^b} \mathbb{E}\left[-e^{-\gamma(X_T + q_T S_T)} | \mathcal{F}_t\right] \tag{3.12}$$

Notice that our agent choses its quote spreads δ^a and δ^b, and hence controls its quotes p^a and p^b. This means that the agent therefore indirectly influences the flow of orders she receives.

In the next section we will consider some realistic forms for the functions λ^a and λ^b based on results in the econophysics literature exploring the statistical properties of the limit orderbook, before turning to the application of Stochastic Control and solution to the above problem in section 3.7.

3.6 Modelling Trading Intensity

Here, we will focus on deriving a realistic form for the Poisson intensity λ with which a limit order will be executed as a function of its distance δ to the mid-price. In order to quantify this, we need to infer some statistics regarding

- The overall frequency of market orders
- The distribution of the size of market orders
- The temporary price impact of a large market order

For simplicity, we will assume a constant frequency Λ of market buy or sell orders. In practice, this could be estimated by simply dividing the total volume bought or sold in a given time interval by the average volume of market buy/sell orders in that interval.

Distribution of the size of market orders

The distribution of size of market orders has been found to obey a power law:

Theorem 3.6.1 (Density of Market Order Size)**.** The distribution of size of market orders has been found to obey a power law:

$$f^Q(x) \propto x^{-1-\alpha} \tag{3.13}$$

for large x, with $\alpha = 1.53$ in Gopikrishnan et al. 2000 for US stocks, $\alpha = 1.4$ in Maslov and Mills 2001 for shares traded on the NASDAQ and $\alpha = 1.5$ in Gabaix et al. 2006 for shares on the Paris Bourse.

Modelling market impact

Here there is much less consensus on market impact, due to lack of agreement on how to define it and how to measure it. Some papers find that the change in price Δp after a market order of size Q is described well by

$$\Delta p \propto Q^\beta \tag{3.14}$$

with $\beta = 0.5$ in Gabaix et al. 2006 and $\beta = 0.76$ in Weber and Rosenow 2005, while Potters and Bouchaud 2003 find a better fit to the relationship

$$\Delta p \propto \log(Q). \tag{3.15}$$

Using (3.13) and (3.15) we can derive the poisson intensity as follows:

$$
\begin{aligned}
\lambda(\delta) &= \Lambda\mathbb{P}(\delta < \Delta p) \\
&= \Lambda\mathbb{P}\left(\delta < \frac{\log Q}{K}\right) \\
&= \Lambda\mathbb{P}(K\delta < \log Q) \\
&= \Lambda\mathbb{P}\left(e^{K\delta} < Q\right) \\
&= \Lambda\int_{e^{K\delta}}^{\infty} x^{-1-\alpha}dx \\
&= \Lambda\left[\frac{-x^{-\alpha}}{\alpha}\right]_{e^{K\delta}}^{\infty} \\
&= \Lambda\left(\lim_{t\to\infty}\frac{-t^{-\alpha}}{\alpha} + \frac{e^{-K\delta\alpha}}{\alpha}\right) \\
&= \frac{\Lambda}{\alpha}\left(e^{-K\delta\alpha} - \lim_{t\to\infty}\frac{1}{t^{\alpha}}\right) \\
&= \frac{\Lambda}{\alpha}e^{-\alpha K\delta} \\
&= Ae^{-k\delta}
\end{aligned}
$$

where $A = \frac{\Lambda}{\alpha}$ and $k = \alpha K$. On the other hand, (3.13) and (3.14) yield:

$$
\begin{aligned}
\lambda(\delta) &= \Lambda\mathbb{P}(\delta < \Delta p) \\
&= \Lambda\mathbb{P}(\delta < kQ^{\beta}) \\
&= \Lambda\mathbb{P}\left(Q > \left(\frac{\delta}{k}\right)^{-\beta}\right) \\
&= \Lambda\int_{\left(\frac{\delta}{k}\right)^{-\beta}}^{\infty} x^{-1-\alpha}dx \\
&= \Lambda\left[\lim_{t\to\infty}\frac{-t^{-\alpha}}{\alpha} + \frac{\left(\frac{\delta}{k}\right)^{-\frac{\alpha}{\beta}}}{\alpha}\right] \\
&= \frac{\Lambda\left(\frac{\delta}{k}\right)^{-\frac{\alpha}{\beta}}}{\alpha} \\
&= B\delta^{-\frac{\alpha}{\beta}}
\end{aligned}
$$

where $B = \frac{\Lambda}{k\alpha}$. Alternatively, we could derive the price impact function Δp directly by integrating the density of the orderbook as described in Weber and Rosenow 2005 and Smith et al. 2003. However, this leads to a function which is highly dependent on the orderbook in question, so in the interest of obtaining a more general result in the following sections, we will not cover this method here.

3.7 The Hamilton-Jacobi-Bellman Equation

Now that we have formulated our agent's value function, and discussed some empirical results on the form of the Poisson intensity λ, we turn to the solution of the problem at hand. Following on from our discussion of the theory of stochastic control in Chapter 2, our first goal will be to formulate the Hamilton-Jacobi-Bellman PDE associated to our value function which we defined in (3.12). Recall that this is given by

$$v(s,x,q,t) = \max_{\delta^a,\delta^b} \mathbb{E}\left[-e^{-\gamma(X_T+q_TS_T)}|\mathcal{F}_t\right] \tag{3.16}$$

where our optimal control processes δ^a and δ^b will turn out to be time and state dependent. This type of optimal dealer problem was first studied by Ho and Stoll 1981, who use the Dynamic Programming Principle to show that v satisfies the following HJB:

Theorem 3.7.1 (Ho and Stoll (1981) HJB)**.**

$$\begin{cases} v_t + \dfrac{1}{2}\sigma^2 v_{ss} + \max\limits_{\delta^b} \lambda^b(\delta^b)[v(s,x-s+\delta^b,q+1,t) - v(s,x,q,t)] \\ \qquad\qquad + \max\limits_{\delta^a} \lambda^a(\delta^a)[v(s,x+s+\delta^a,q-1,t) - v(s,x,q,t)] = 0, \\ \qquad\qquad v(s,x,q,T) = -e^{-\gamma(x+qs)}. \end{cases} \tag{3.17}$$

Proof. Recall from chapter 2 that the value function in general follows the form given in (2.19) and shown below:

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) + \sup\limits_{a\in A}\left[\mathcal{L}^a v(t,x) + f(t,x,a)\right] = 0 \ \forall\ (t,x) \in [0,T) \times \mathbb{R} \\ v(T,x) = g(x)\ \forall\ x \in \mathbb{R}. \end{cases}$$

First, we will check our boundary condition. At time T, the trading period has ended, and so in (3.12) we can ignore the maximisation over controles δ^a and δ^b when evaluating v at T. Moreover, X_T, q_T and S_T are all adapted and hence measurable with respect to $\mathcal{F}_T$, and we can remove the expectation. Hence we find that

$$v(s,x,q,T) = -e^{-\gamma(X_T+q_TS_T)} \tag{3.18}$$

as expected. Next, for the dynamics of the system, note that the operator $\mathcal{L}^\alpha v$ defined as

$$\begin{aligned} \mathcal{L}^\alpha v(t,x) &= b(t,x,\alpha)v_x + \frac{1}{2}\sigma(t,x,\alpha)^2 v_{xx} \\ &= \frac{1}{2}\sigma(t,x,\alpha)^2 v_{xx} \text{ since we assumed a brownian motion without drift} \\ &= \frac{1}{2}\sigma^2 v_{xx} \text{ since we assumed constant volatility } \sigma \end{aligned}$$

and we also notice that this has no dependence on our control process $\alpha = \begin{pmatrix}\delta^a \\ \delta^b\end{pmatrix}$.

For the incremental gains encapuslated by the function f, we notice that by the properties of the Poisson process, the density of ask or bid orders arriving at time t is $\lambda^a(\delta^a)$ or $\lambda^b(\delta^b)$ respectively. Hence the density of changes to the value function over infinitessimal units of time is given by

$\lambda^b(\delta^b)\times$(increment to v caused by 1 bid order) $= \lambda^b(\delta^b)[v(s,x-s+\delta^b,q+1,t)-v(s,x,q,t)]$

for market sell orders hitting our bid orders, and the corresponding expression for market buy orders lifting our ask orders is

$$\lambda^a(\delta^a)[v(s, x + s + \delta^a, q - 1, t) - v(s, x, q, t)].$$

Putting this all together, we have that

$$\frac{\partial v}{\partial t}(t, x) + \sup_{a \in A} [\mathcal{L}^a v(t, x) + f(t, x, a)] = 0$$

equates to

$$\begin{cases} v_t + \frac{1}{2}\sigma^2 v_{ss} + \max_{\delta^b} \lambda^b(\delta^b)[v(s, x - s + \delta^b, q + 1, t) - v(s, x, q, t)] \\ \qquad\qquad + \max_{\delta^a} \lambda^a(\delta^a)[v(s, x + s + \delta^a, q - 1, t) - v(s, x, q, t)] = 0 \end{cases}$$

for our particular value function v. □

Remark. Avellaneda and Stoikov 2008 throughout their paper refer to maxima, rather than suprema, over the value functions. This is justified because although the optimal quotes are real-valued, modern electronic markets typically trade in integer amounts of fractions of pennies, restricting dealers to a countable set of possible quotes. Optimal quotes can simply be rounded to the nearest attainable quote in order to be used for trading.

In order to simplify the problem before turning to look at its solution, Avellaneda and Stoikov 2008 argue that due to our choice of exponential utility, we can introduce the following ansatz:

$$v(s, x, q, t) = -e^{-\gamma x} e^{-\gamma \theta(s,q,t)} \tag{3.19}$$

The intuition behind this is that considering our value function at time t, our current wealth X_t is a predetermined constant and thus measurable w.r.t $\mathcal{F}_t$. Hence we can take $-e^{-\gamma x}$ out from the expectation. The remainder, being our future cash flow, future inventory and terminal portfolio value are all time and state dependent and hence encapsulated by some function θ of s, q and t. Moreover, thanks to the properties of the exponential function, the expectation of the utility of our future wealth can also be written in an exponential form. Finally, we also assume that the function θ factors in our optimal control $\alpha^* = \begin{pmatrix} \delta^{a*} \\ \delta^{b*} \end{pmatrix}$

By substitution of Avellaneda and Stoikov's ansatz (3.19) into Ho and Stoll's HJB (3.17) we obtain the following HJB equation for θ:

Theorem 3.7.2 (Avellaneda and Stoikov (2008) HJB)**.**

$$\begin{cases} \theta_t + \frac{1}{2}\sigma^2\theta_{ss} - \frac{1}{2}\sigma^2\gamma\theta_s^2 + \max_{\delta^b}\left[\frac{\lambda^b(\delta^b)}{\gamma}(1 - e^{\gamma(s-\delta^b-r^b)})\right] \\ \qquad\qquad + \max_{\delta^a}\left[\frac{\lambda^a(\delta^a)}{\gamma}(1 - e^{-\gamma(s+\delta^a-r^a)})\right] = 0, \\ \qquad\qquad \theta(s, q, T) = qs. \end{cases} \tag{3.20}$$

Relations for the reserve prices

Before we can prove that this subsitution provides an equivalent formulation of our Hamilton-Jacobi-Bellman equation, we need a lemma relating the definitions we gave of the dealer's reservation bid and ask prices in section 3.3 to our new function θ. We find that we can express r^b and r^a directly in terms of θ as follows:

Lemma 3.7.3. We have using the ansatz 3.19 that the reservation bid and ask prices defined in definition 3.3.1 are given by

$$r^b(s, q, t) = \theta(s, q + 1, t) - \theta(s, q, t) \tag{3.21}$$

and

$$r^a(s, q, t) = \theta(s, q, t) - \theta(s, q - 1, t). \tag{3.22}$$

Proof. We prove the above directly from the definition of the reserve bid and ask respectively:

$$\begin{aligned}
v(s, x - r^b(s, q, t), q + 1, t) &= v(s, x, q, t) \\
-e^{-\gamma(x - r^b(s,q,t))}e^{-\gamma\theta(s,q+1,t)} &= -e^{-\gamma x}e^{-\gamma\theta(s,q,t)} \\
-\gamma(x - r^b(s, q, t)) - \gamma\theta(s, q + 1, t) &= -\gamma x - \gamma\theta(s, q, t) \\
x - r^b(s, q, t) + \theta(s, q + 1, t) &= x + \theta(s, q, t) \\
r^b(s, q, t) &= \theta(s, q + 1, t) - \theta(s, q, t)
\end{aligned}$$

and

$$\begin{aligned}
v(s, x + r^a(s, q, t), q - 1, t) &= v(s, x, q, t) \\
-e^{-\gamma(x + r^a(s,q,t))}e^{-\gamma\theta(s,q-1,t)} &= -e^{-\gamma x}e^{-\gamma\theta(s,q,t)} \\
-\gamma(x + r^a(s, q, t)) - \gamma\theta(s, q - 1, t) &= -\gamma x - \gamma\theta(s, q, t) \\
x + r^a(s, q, t) + \theta(s, q - 1, t) &= x + \gamma\theta(s, q, t) \\
r^a(s, q, t) &= \theta(s, q, t) - \theta(s, q - 1, t).
\end{aligned}$$

□

Using this result, we can check that the ansatz 3.19 allows us to derive the HJB equation given in theorem 3.7.2.

Proof of Theorem 3.7.2. First we check the terminal condition:

$$\begin{aligned}
v(s, x, q, T) &= -e^{-\gamma(x+qs)} \text{ from (3.17)} \\
&= -e^{-\gamma x}e^{-\gamma\theta(s,q,T)} \text{ from (3.19)} \\
&= -e^{-\gamma(x+\theta(s,q,T))} \\
\implies \theta(s, q, T) &= qs.
\end{aligned}$$

which is what we expected. Next, by direct substitution, note that

$$v_t = -\frac{\partial}{\partial t}e^{-\gamma x}e^{-\gamma\theta} = -e^{-\gamma x} \times -\gamma\theta_t e^{-\gamma\theta} = \gamma e^{-\gamma x}\theta_t e^{-\gamma\theta}$$

and

$$\begin{aligned}
\frac{1}{2}\sigma^2 v_{ss} &= -e^{-\gamma x}\frac{1}{2}\sigma^2\frac{\partial^2}{\partial s^2}e^{-\gamma\theta} \\
&= \gamma e^{-\gamma x}\frac{1}{2}\sigma^2\frac{\partial}{\partial s}\theta_s e^{-\gamma\theta} \\
&= \gamma e^{-\gamma x}\frac{1}{2}\sigma^2\left(\theta_{ss}e^{-\gamma\theta} - \gamma\theta_s^2 e^{-\gamma\theta}\right).
\end{aligned}$$

Next we consider the maximised terms:

$$\begin{aligned}\lambda^b(\delta^b)[v(s,x-s+\delta^b,q+1,t)-v(s,x,q,t)] &= \lambda^b(\delta^b)[-e^{-\gamma(x-s+\delta^b)}e^{-\gamma\theta(s,q+1,t)}+e^{-\gamma x}e^{-\gamma\theta(s,q,t)}] \\ &= \lambda^b(\delta^b)[e^{-\gamma x}e^{-\gamma\theta(s,q,t)}-e^{-\gamma x}e^{\gamma s}e^{-\gamma\delta^b}e^{-\gamma\theta(s,q+1,t)}]\end{aligned}$$

and

$$\begin{aligned}\lambda^a(\delta^a)[v(s,x+s+\delta^a,q-1,t)-v(s,x,q,t)] &= \lambda^a(\delta^a)[-e^{-\gamma(x+s+\delta^a)}e^{-\gamma\theta(s,q-1,t)}+e^{-\gamma x}e^{-\gamma\theta(s,q,t)}] \\ &= \lambda^a(\delta^a)[e^{-\gamma x}e^{-\gamma\theta(s,q,t)}-e^{-\gamma x}e^{-\gamma s}e^{-\gamma\delta^a}e^{-\gamma\theta(s,q-1,t)}].\end{aligned}$$

We note that since the R.H.S. of our equation is 0 and all of our expressions contain $e^{-\gamma x}$, we can mutliply by this term. We can remove all $e^{-\gamma\theta}$ terms similarly. Dividing all expressions by γ and substituting into (3.17) yields a L.H.S. of

$$\begin{aligned}\theta_t+\frac{1}{2}\sigma^2\theta_{ss}-\frac{1}{2}\sigma^2\gamma\theta_s^2+\max_{\delta^b}\frac{\lambda^b(\delta^b)}{\gamma}[1-e^{\gamma s}e^{-\gamma\delta^b}e^{-\gamma(\theta(s,q+1,t)-\theta(s,q,t))}] \\ +\max_{\delta^a}\frac{\lambda^a(\delta^a)}{\gamma}[1-e^{-\gamma s}e^{-\gamma\delta^a}e^{\gamma(\theta(s,q,t)-\theta(s,q-1,t))}]\end{aligned}$$

which by lemma 3.7.3 simplifies to

$$\begin{aligned}&\theta_t+\frac{1}{2}\sigma^2\theta_{ss}-\frac{1}{2}\sigma^2\gamma\theta_s^2+\max_{\delta^b}\frac{\lambda^b(\delta^b)}{\gamma}[1-e^{\gamma s}e^{-\gamma\delta^b}e^{-\gamma r^b(s,q,t)}] \\ &\qquad+\max_{\delta^a}\frac{\lambda^a(\delta^a)}{\gamma}[1-e^{-\gamma s}e^{-\gamma\delta^a}e^{\gamma r^a(s,q,t)}] \\ &=\theta_t+\frac{1}{2}\sigma^2\theta_{ss}-\frac{1}{2}\sigma^2\gamma\theta_s^2+\max_{\delta^b}\frac{\lambda^b(\delta^b)}{\gamma}[1-e^{\gamma(s-\delta^b-r^b(s,q,t))}] \\ &\qquad+\max_{\delta^a}\frac{\lambda^a(\delta^a)}{\gamma}[1-e^{-\gamma(s+\delta^a-r^a(s,q,t))}]\end{aligned}$$

□

Implicit relations for the optimal spreads δ^a and δ^b

We can derive some relations for the optimal distances δ^a and δ^b that are implicit in our slightly simplified HJB equation (3.20). Inspecting the maximised terms in the HJB, we can invoke a first-order optimality condition to find an expression involving the optimal spreads, the reservation prices, and our Poisson intensity λ.

Theorem 3.7.4 (Implicit relations for the optimal spreads δ^a and δ^b)**.** For δ^b, we obtain the following:

$$s-r^b(s,q,t)=\delta^b-\frac{1}{\gamma}\log\left(1-\gamma\frac{\lambda^b(\delta^b)}{\frac{\partial\lambda^b}{\partial\delta}(\delta^b)}\right) \tag{3.23}$$

while for δ^a we have

$$r^a(s,q,t)-s=\delta^a-\frac{1}{\gamma}\log\left(1-\gamma\frac{\lambda^a(\delta^a)}{\frac{\partial\lambda^a}{\partial\delta}(\delta^a)}\right) \tag{3.24}$$

Proof. Taking the derivative of the term in 3.20 that we maximise w.r.t. δ^b and setting it equal to 0, we find:

$$\begin{aligned}
\frac{\partial}{\partial \delta}\left[\frac{\lambda^b(\delta)}{\gamma}(1-e^{\gamma(s-\delta-r^b(s,q,t))})\right](\delta^b) &= 0\\
\frac{1}{\gamma}\left[\frac{\partial \lambda^b}{\partial \delta}(\delta^b) - \frac{\partial}{\partial \delta}\lambda^b(\delta^b)e^{\gamma(s-\delta^b-r^b(s,q,t))}\right] &= 0\\
\frac{\partial \lambda^b}{\partial \delta}(\delta^b) - \frac{\partial \lambda^b}{\partial \delta}(\delta^b)e^{\gamma(s-\delta^b-r^b(s,q,t))} + \gamma\lambda^b(\delta^b)e^{\gamma(s-\delta^b-r^b(s,q,t))} &= 0\\
\left(\gamma\lambda^b(\delta^b) - \frac{\partial \lambda^b}{\partial \delta}(\delta^b)\right)e^{\gamma(s-\delta^b-r^b(s,q,t))} &= -\frac{\partial \lambda^b}{\partial \delta}(\delta^b)\\
-\left(\frac{\partial \lambda^b}{\partial \delta}(\delta^b)\right)e^{-\gamma(s-\delta^b-r^b(s,q,t))} &= \gamma\lambda^b(\delta^b) - \frac{\partial \lambda^b}{\partial \delta}(\delta^b)\\
e^{-\gamma(s-\delta^b-r^b(s,q,t))} &= 1-\gamma\frac{\lambda^b(\delta^b)}{\frac{\partial \lambda^b}{\partial \delta}(\delta^b)}\\
-\gamma(s-\delta^b-r^b(s,q,t)) &= \log\left(1-\gamma\frac{\lambda^b(\delta^b)}{\frac{\partial \lambda^b}{\partial \delta}(\delta^b)}\right)\\
s-\delta^b-r^b(s,q,t) &= -\frac{1}{\gamma}\log\left(1-\gamma\frac{\lambda^b(\delta^b)}{\frac{\partial \lambda^b}{\partial \delta}(\delta^b)}\right)\\
s-r^b(s,q,t) &= \delta^b-\frac{1}{\gamma}\log\left(1-\gamma\frac{\lambda^b(\delta^b)}{\frac{\partial \lambda^b}{\partial \delta}(\delta^b)}\right)
\end{aligned}$$

while a similar procedure for δ^a yields:

$$\begin{aligned}
\frac{\partial}{\partial \delta}\left[\frac{\lambda^a(\delta)}{\gamma}(1-e^{-\gamma(s+\delta-r^a(s,q,t))})\right](\delta^a) &= 0\\
\frac{1}{\gamma}\left[\frac{\partial \lambda^a}{\partial \delta}(\delta^a) - \frac{\partial}{\partial \delta}\lambda^a(\delta^a)e^{-\gamma(s+\delta^a-r^a(s,q,t))}\right] &= 0\\
\frac{\partial \lambda^a}{\partial \delta}(\delta^a) - \frac{\partial \lambda^a}{\partial \delta}(\delta^a)e^{-\gamma(s+\delta^a-r^a(s,q,t))} + \gamma\lambda^a(\delta^a)e^{-\gamma(s+\delta^a-r^a(s,q,t))} &= 0\\
\left(\gamma\lambda^a(\delta^a) - \frac{\partial \lambda^a}{\partial \delta}(\delta^a)\right)e^{-\gamma(s+\delta^a-r^a(s,q,t))} &= -\frac{\partial \lambda^a}{\partial \delta}(\delta^a)\\
-\left(\frac{\partial \lambda^a}{\partial \delta}(\delta^a)\right)e^{\gamma(s+\delta^a-r^a(s,q,t))} &= \gamma\lambda^a(\delta^a) - \frac{\partial \lambda^a}{\partial \delta}(\delta^a)\\
e^{\gamma(s+\delta^a-r^a(s,q,t))} &= 1-\gamma\frac{\lambda^a(\delta^a)}{\frac{\partial \lambda^a}{\partial \delta}(\delta^a)}\\
\gamma(s+\delta^a-r^a(s,q,t)) &= \log\left(1-\gamma\frac{\lambda^a(\delta^a)}{\frac{\partial \lambda^a}{\partial \delta}(\delta^a)}\right)\\
s+\delta^a-r^a(s,q,t) &= \frac{1}{\gamma}\log\left(1-\gamma\frac{\lambda^a(\delta^a)}{\frac{\partial \lambda^a}{\partial \delta}(\delta^a)}\right)\\
r^a(s,q,t)-s &= \delta^a-\frac{1}{\gamma}\log\left(1-\gamma\frac{\lambda^a(\delta^a)}{\frac{\partial \lambda^a}{\partial \delta}(\delta^a)}\right).
\end{aligned}$$

□

To briefly summarise, the optimal bid and ask spreads δ^a and δ^a are derived in an intuitive two-step procedure. First, we solve the HJB PDE (3.20) to obtain the reservation bid and ask prices. Then, we use the relations (3.23) and (3.24) to find the optimal bid and ask spreads $\delta^b(s,q,t)$ and $\delta^a(s,q,t)$ between the mid-price and obtimal bid and ask quotes respectively.

3.8 Asymptotic Expansion in q

The main computational difficulty in this advance is solving equation 3.20 due to the order-arrival terms (the terms to be maximised) being highly nonlinear and dependent on the inventory variable q. To get around this, Avellaneda and Stoikov 2008 suggest an asymptotic expansion of θ in the inventory variable q, and a linear approximation of the order arrival terms.

Following on from our work in section 3.6, we will assume that our arrival rates are symmetric and exponential given by

$$\lambda^a(\delta) + \lambda^b(\delta) = Ae^{-k\delta} \tag{3.25}$$

in which case, our agents indifference prices $r^a(s,q,t)$ and $r^b(s,q,t)$ coincide with their frozen inventory values given in section 3.3.

With some elementary calculus we can see that our exponential arrival rates satisfy the following property:

$$\frac{\lambda(\delta)}{\frac{\partial\lambda}{\partial\delta}(\delta)} = \frac{Ae^{-k\delta}}{-kAe^{-k\delta}} = -\frac{1}{k}.$$

Hence by plugging in the relations (3.23) and (3.24) into the maximised terms in the HJB equation (3.20) under the assumption of symmetric exponential arrival rates (3.24), we see that

$$\begin{aligned}
&\max_{\delta^b}\left[\frac{\lambda^b(\delta^b)}{\gamma}(1-e^{\gamma(s-\delta^b-r^b)})\right] + \max_{\delta^a}\left[\frac{\lambda^a(\delta^a)}{\gamma}(1-e^{-\gamma(s+\delta^a-r^a)})\right] \\
&= \frac{Ae^{-k\delta^b}}{\gamma}\left(1-e^{\gamma\left(-\frac{1}{\gamma}\log\left(1+\frac{\gamma}{k}\right)\right)}\right) + \frac{Ae^{-k\delta^a}}{\gamma}\left(1-e^{-\gamma\left(\frac{1}{\gamma}\log\left(1+\frac{\gamma}{k}\right)\right)}\right) \\
&= \left[\frac{A}{\gamma}\left(1-e^{-\log\left(1+\frac{\gamma}{k}\right)}\right)\right](e^{-k\delta^b}+e^{-k\delta^a}) \\
&= \left[\frac{A}{\gamma}\left(1-\frac{1}{1+\frac{\gamma}{k}}\right)\right](e^{-k\delta^b}+e^{-k\delta^a}) \\
&= \left(\frac{A}{\gamma}-\frac{A}{\gamma+\frac{\gamma^2}{k}}\right)(e^{-k\delta^b}+e^{-k\delta^a}) \\
&= \left(\frac{A\left(1+\frac{\gamma}{k}\right)-A}{\gamma+\frac{\gamma^2}{k}}\right)(e^{-k\delta^b}+e^{-k\delta^a}) \\
&= \left(\frac{A\frac{\gamma}{k}}{\gamma+\frac{\gamma^2}{k}}\right)(e^{-k\delta^b}+e^{-k\delta^a}) \\
&= \frac{A}{k+\gamma}(e^{-k\delta^b}+e^{-k\delta^a})
\end{aligned}$$

which results in the simplified HJB equation below:

$$\begin{cases}\theta_t + \frac{1}{2}\sigma^2\theta_{ss} - \frac{1}{2}\sigma^2\gamma\theta_s^2 + \frac{A}{k+\gamma}(e^{-k\delta^a}+e^{-k\delta^b}) = 0 \\ \theta(s,q,T) = qs.\end{cases} \tag{3.26}$$

Next, we consider an asymptotic expansion of θ in the inventory variable q:

$$\theta(q,s,t) = \theta^0(s,t) + q\theta^1(s,t) + \frac{1}{2}q^2\theta^2(s,t) + \ldots \tag{3.27}$$

where the superscripts denote different functions, not powers - we do not use subscripts to avoid conflicts with our notation for partial derivatives later on.

The exact relations for the reserve bid and ask prices obtained in lemma 3.7.3 yield

$$r^b(s,q,t) = \theta^1(s,t) + (1+2q)\theta^2(s,t) + \ldots \tag{3.28}$$

$$r^a(s,q,t) = \theta^1(s,t) + (-1-2q)\theta^2(s,t) + \ldots \, . \tag{3.29}$$

Then our reservation price

$$r(s,q,t) = \frac{r^a(s,q,t) + r^b(s,q,t)}{2} = \theta^1(s,t) + 2q\theta^2(s,t) \tag{3.30}$$

follows immediately. We can interpret this expression nicely: θ^1 is the reserve price when the inventory is 0, and θ^2 is our agent's sensitivity to changes in inventory. We might then expect that $\theta^2 < 0 \; \forall (s,t)$, since then a long position will result in lower quotes (more willing to sell) and vice-versa. We also have that

$$\begin{aligned}
\delta^a + \delta^b &= \frac{1}{\gamma}\log\left(1+\frac{\gamma}{k}\right) + r^a(s,q,t) - s + \frac{1}{\gamma}\log\left(1+\frac{\gamma}{k}\right) + s - r^b(s,q,t) \\
&= r^a(s,q,t) - r^b(s,q,t) + \frac{2}{\gamma}\log\left(1+\frac{\gamma}{k}\right) \\
&= -2\theta^2(s,t) + \frac{2}{\gamma}\log\left(1+\frac{\gamma}{k}\right)
\end{aligned} \tag{3.31}$$

through our approximation and the relations (3.23) and (3.24). Now consider a first-order approximation of the order arrival term:

$$\frac{A}{k+\gamma}(e^{-\gamma\delta^a} + e^{-\gamma\delta^b}) = \frac{A}{k+\gamma}(2 - k(\delta^a+\delta^b) + \ldots) \tag{3.32}$$

where we notice that the linear term does not depend on the inventory q. Therefore, by substituting (3.27) and (3.32) into (3.26) and grouping terms of order q we obtain

$$\begin{cases} \theta^1_t + \frac{1}{2}\sigma^2\theta^1_{ss} = 0 \\ \theta^1(s,T) = s. \end{cases} \tag{3.33}$$

which admits the solution $\theta^1(s,t) = s$ by inspection.

Grouping terms of order q^2 yields

$$\begin{cases} \theta^2_t + \frac{1}{2}\sigma^2\theta^2_{ss} - \frac{1}{2}\sigma^2\gamma(\theta^1_s)^2 = 0 \\ \theta^2(s,T) = 0 \end{cases} \tag{3.34}$$

which simplifies by our previous solution to (3.33) to

$$\begin{cases} \theta^2_t + \frac{1}{2}\sigma^2\theta^2_{ss} - \frac{1}{2}\sigma^2\gamma = 0 \\ \theta^2(s,T) = 0 \end{cases}$$

with solution $\theta^2(s,t) = -\frac{1}{2}\sigma^2\gamma(T-t)$ by inspection.

Thus for this linear approximation of the order arrival term, we can substitute our solutions back into (3.30) to obtain the same indifference price

$$r(s,t) = s - q\gamma\sigma^2(T-t) \tag{3.35}$$

as in the case where no trading is allowed. We quote a bid-ask spread that is symmetric about this reservation price and is given by the below expression, which is again acquired through subsituting our solutions for θ^1 and θ^2 back into (3.31).

$$\delta^a + \delta^b = \gamma\sigma^2(T-t) + \frac{2}{\gamma}\log\left(1+\frac{\gamma}{k}\right) \tag{3.36}$$

3.9 Summary

In the above section, we have seen how the problem of finding the optimal behaviour of a dealer in a limit orderbook, which we introduced in chapter 1, can be formulated as a stochastic control problem using the theoretical framework we built up in chapter 2. We have also noted a possible miscalculation or typographical error in the short section of the original paper of Avellaneda and Stoikov 2008 on the infinite-horizon agent, which we address in section 3.4.

We then provide full derivations of all of the results presented in Avellaneda and Stoikov 2008, explain the use of the ansatz (3.19) as a key simplifying assumption, and walk through the asymptotic approximation used to yield the final results that Avellaneda and Stoikov present.

In the next chapter, we will provide code and results for the numerical simulation results that Avellaneda and Stoikov describe, attaining results that are very close to those presented in the original paper, and demonstrating some of the concepts that we have worked with mathematically through visualisations of our simulations.

Chapter 4

Numerical Analysis and Simulations

4.1 Introduction

With our simple expressions for the reservation price and quote spread derived from our approximations, and the intuitive procedure described at the end of section 3.8, we can easily test the performance of our strategy using numerical simulation. In the first section, we will replicate the results obtained by Avellaneda and Stoikov 2008 and give the full python implementation.

4.2 Numerical Simulations - Avellaneda & Stoikov

First, we need to import a couple of libraries and set up some auxiliary functions. The only libraries required are numpy for easy sampling and dealing with vectors, and matplotlib for producing the plots and charts later on.

Listing 4.1: Auxiliary Functions

```
import numpy as np
import matplotlib as plt

def computeReservePrice(s,q,gamma,sigma,t,T):
    return s-q*gamma*(sigma**2)*(T-t)

def computeSpread(gamma,sigma,t,T,k):
    return (gamma * (sigma ** 2) * (T-t)) + ((2 / gamma) *
        np.log(1 + (gamma / k)))

def computeRate(A,k,delta):
    return A*np.exp(-k*delta)

def computeSamplePath(S0, sigma, dt, T):
    return np.insert(S0 + np.cumsum(sigma * np.sqrt(dt) *
        np.random.choice([1,-1], int(T/dt))), 0, S0)
```

`computeReservePrice` takes the current stock price, current inventory, risk aversion, volatility, current time and end time as arguments and returns the current reservation price as given in (3.35).

`computeSpread` takes the risk aversion, volatility, current time, end time and orderbook parameter k as arguments and computes the current spread as given by (3.36).

`computeRate` computes the function λ in the form assumed by (3.25), taking the orderbook parameters A and k and the current spread δ as arguments.

`computeSamplePath` generates a sample path from a brownian motion from t to T with stepsize dt, volatility σ and starting value S_0.

Next, we define a function which we can call as many times as we would like to compare the performance of our "inventory" strategy derived in chapter 3 to that of a benchmark "symmetric" strategy, where we compute the average spread over time of the inventory strategy and constantly maintain this spread symmetric about the midprice, regardless of our inventory. Comments are included throughout the code. We use the same parameters as Avellaneda and Stoikov 2008, namely $S_0 = 100$, $T = 1$, $\sigma = 2$, d$t = 0.005$, $k = 1.5$, $A = 140$, and $q_0 = 0$.

Listing 4.2: Avellaneda-Stoikov Model

```
def simulateBothStrategies(gamma, plots=False):
    # Initialise model parameters
    S0 = 100
    T = 1
    sigma = 2
    dt = 0.005
    k = 1.5
    A = 140

    # Initialise variables to keep track of inventory strategy
    inv_q = 0
    inv_X = 0
    inv_bids = []
    inv_asks = []
    inv_wealth = []
    inv_adj_wealth = []
    inv_inventory = []

    # Initialise variables to keep track of symmetric strategy
    sym_q = 0
    sym_X = 0
    sym_bids=[]
    sym_asks=[]
    sym_wealth = []
    sym_adj_wealth = []
    sym_inventory = []

    # generate sample path for midprice
    price_process = computeSamplePath(S0, sigma, dt, T)

    # compute average inventory strat spread over sample path
    sym_spread = 0
    for i in np.arange(0, T, dt):
        sym_spread += computeSpread(gamma, sigma, i, T, k)
```

```
        av_sym_spread = (sym_spread / (T / dt))
        sym_prob = min(A*np.exp(-k*av_sym_spread / 2) * dt, 1)
        sym_bids = price_process - av_sym_spread/2
        sym_asks = price_process + av_sym_spread/2

    # iterate through price process
    for step, s in enumerate(price_process):
        # compute reserve price and spread
        r = computeReservePrice(s,inv_q,gamma,sigma,step*dt,T)
        spread = computeSpread(gamma, sigma, step*dt, T, k) / 2
        delta_a = (spread + r) - s
        delta_b = s - (r - spread)

        # keep track of any updated variables
        inv_asks.append(s + delta_a)
        inv_bids.append(s - delta_b)
        inv_wealth.append(inv_X)
        inv_adj_wealth.append(inv_X + inv_q * s)
        inv_inventory.append(inv_q)

        # sample possible incoming market orders
        prob_a = min(computeRate(A,k,delta_a) * dt, 1)
        prob_b = min(computeRate(A,k,delta_b) * dt, 1)
        p = np.random.default_rng().uniform(0,1,None)
        if p <= prob_a:
            inv_q -= 1
            inv_X += (s+delta_a)
        p = np.random.default_rng().uniform(0,1,None)
        if p <= prob_b:
            inv_q += 1
            inv_X -= (s-delta_b)

        # keep track of symmetric strategy
        sym_wealth.append(sym_X)
        sym_adj_wealth.append(sym_X + sym_q * s)
        sym_inventory.append(sym_q)

        # sample incoming market orders for symmetric strat
        p = np.random.default_rng().uniform(0,1,None)
        if p <= sym_prob:
            sym_q -= 1
            sym_X += (s + av_sym_spread / 2)
        p = np.random.default_rng().uniform(0,1,None)
        if p <= sym_prob:
            sym_q += 1
            sym_X -= (s-av_sym_spread/2)

    if plots==True:
        t = np.arange(0, T+dt, dt)
        plt.plot(t, price_process, 'black', linewidth = 1.0,
            label = "S")
```

```
        plt.plot(t, asks, 'green', linewidth=1.0,label="p_a")
        plt.plot(t, bids, 'red',linewidth=1.0,label="p_b")
        ax = plt.gca()
        ax.set_facecolor((0.9,0.9,0.9,1))
        plt.xlabel("t")
        plt.ylabel("S")
        plt.legend()
        plt.show()

        plt.plot(t, inventory, 'purple', linewidth = 1.0,
            label = "q")
        ax = plt.gca()
        ax.set_facecolor((0.9,0.9,0.9,1))
        plt.xlabel("t")
        plt.ylabel("q")
        plt.legend()
        plt.show()

        plt.plot(t, wealth, "red", linewidth=1.0,label="X")
        plt.plot(t, adj_wealth, "blue", linewidth = 1.0,
            label = "X+q*S")
        ax = plt.gca()
        ax.set_facecolor((0.9,0.9,0.9,1))
        plt.xlabel("t")
        plt.ylabel("X")
        plt.legend()
        plt.show()

    # Return final performance of both strategies
    return((inv_wealth[-1], inv_inventory[-1],
        price_process[-1], sym_wealth[-1],
        sym_inventory[-1], av_sym_spread))
```

Using the function above, we run 10000 simulations of both the inventory and symmetric strategies and report means and standard deviations of profit ($X_T + q_T S_T$) and final inventory q_T. We also report the mean spread for the inventory strategy, which is also the spread employed by the symmetric strategy throughout. We are also interested in the effect of varying the risk-aversion parameter γ on the final inventory and shape of the Profit and Loss (PnL) profile.

Listing 4.3: run-simulations

```
series = []
for i in range(10000):
    series.append(simulateBothStrategies(0.1))

series = np.array(series)
inv_final_inv = series[:,1]
inv_adj_wealth = series[:,0]+inv_final_inv*series[:,2]
print("Inventory strat mean PNL: {}".format(
    np.mean(inv_adj_wealth)))
print("Inventory strat PNL stdev: {}".format(
```

```
    np.std(inv_adj_wealth)))
print("Inventory strat final q mean: {}".format(
    np.mean(inv_final_inv)))
print("Inventory strat final q stdev: {}".format(
    np.std(inv_final_inv)))

sym_final_inv = series[:,4]
sym_adj_wealth = series[:,3]+sym_final_inv*series[:,2]
print("Symmetric strat mean PNL: {}".format(
    np.mean(sym_adj_wealth)))
print("Symmetric strat PNL stdev: {}".format(
    np.std(sym_adj_wealth)))
print("Symmetric strat final q mean: {}".format(
    np.mean(sym_final_inv)))
print("Symmetric strat final q stdev: {}".format(
    np.std(sym_final_inv)))

print("Average spread: {}".format(np.mean(series[:,5])))

bins=np.histogram(np.hstack((inv_adj_wealth,sym_adj_wealth)),
     bins=100)[1]
plt.hist(inv_adj_wealth, bins, alpha = 1,
     label = "Inventory strategy", edgecolor = "white",
     color = "red")
plt.hist(sym_adj_wealth, bins, label = "Symmetric strategy",
     edgecolor = "black", color = "white", fc = (1,0,1,0))
plt.legend()
plt.ylim((0,1200))
plt.xlabel("PnL")
plt.ylabel("# of runs")
ax = plt.gca()
ax.set_facecolor((0.9,0.9,0.9,1))
plt.show()
```

In figure 4.1 we have the results and PnL distributions for 10000 simulations with $\gamma = 0.1$. We obtain an identical mean spread to Avellaneda and Stoikov 2008, which is to be expected as the calculation for the mean spread does not depend on either the sampling of the mid-price nor the sampling of market orders. All of our results for the mean and standard deviation of the profit and final inventory are within 0.1 of those presented in the original paper, with the exception of the standard deviation of final inventory where my figure of 13.1 is 0.4 above the 12.7 reported by Avellaneda and Stoikov.

In figure 4.2 we present the results for 10000 simulations with $\gamma = 0.01$. Again, all of our figures closely match those presented by Avellaneda and Stoikov. In their paper, they run 1000 simulations for each level of γ, whereas here we run 10000. Moreover, the reported results are statistical properties and hence it is very unlikely that we should obtain exactly the same figures.

Finally, we plot an example simulation of the model, including the midprice, bid and ask prices, cash flow, wealth and inventory. We can identify the effect of varying levels of inventory on our bid and ask quotes: at $t \approx 0.3$ for example, the agent was short stock and hence set her quotes around a high reservation price, resulting in a bid price that

Strategy	μ (Spread)	μ (Profit)	σ (Profit)	μ (Final q)	σ (Final q)
Inventory	1.49	64.9	6.7	0.03	2.9
Symmetric	1.49	68.2	13.1	-0.1	8.3

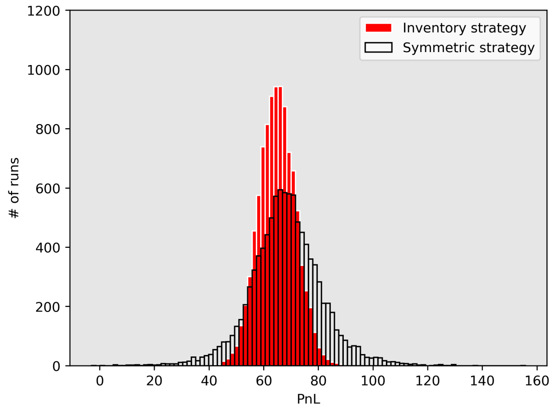

Figure 4.1: Results for $\gamma = 0.1$

almost coincided with the midprice. We can also see the effect of time - as we reach T, the quotes become more and more symmetric as we become more and more certain about the terminal price S_T.

Strategy	μ (Spread)	μ (Profit)	σ (Profit)	μ (Final q)	σ (Final q)
Inventory	1.35	68.5	9.1	0.04	5.3
Symmetric	1.35	68.6	13.8	0.04	8.6

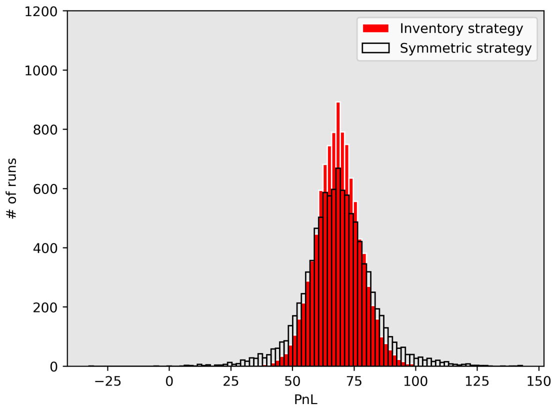

Figure 4.2: Results for $\gamma = 0.01$

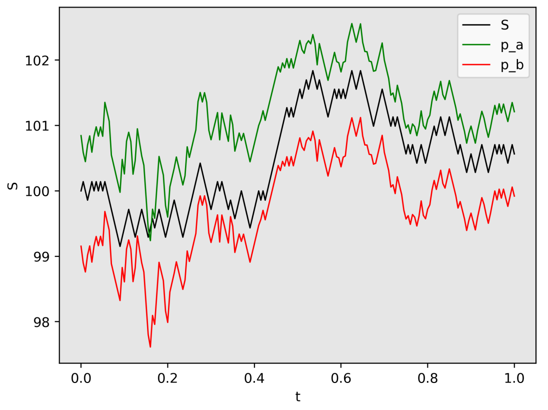

Figure 4.3: Sample path for $\gamma = 0.1$

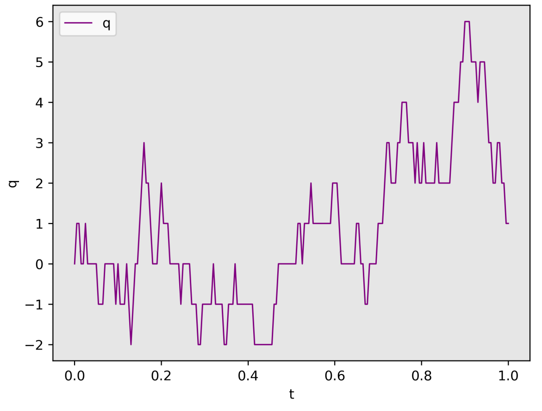

Figure 4.4: Sample inventory for $\gamma = 0.1$

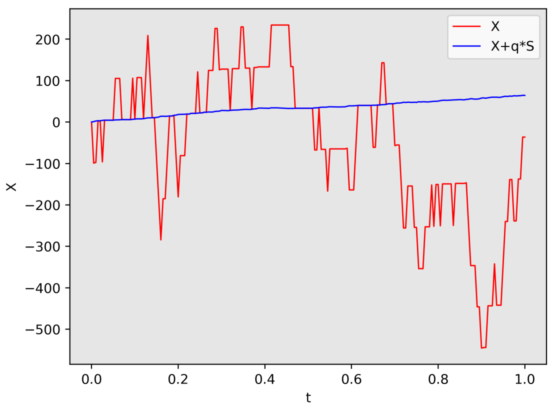

Figure 4.5: Sample profit for $\gamma = 0.1$

Chapter 5

Conclusion

The main goal of this project was to clearly formulate and solve the problem of market-making under inventory risk in the framework of stochastic optimal control. With the background material in stochastic calculus and stochastic differential equations from chapter 1, and our development of the theory of stochastic optimal control in chapter 2, we were then able to formulate and prove the results of Avellaneda and Stoikov 2008 concerning market-making under inventory risk in chapter 3. Indeed, we may even have corrected an error in their calculation of reservation prices in the infinite-horizon case, although this does not impact the main result of their paper. We go on to perform the same numerical simulations as in Avellaneda and Stoikov 2008, with a larger sample size, and replicate their results.

Our conclusion is thus that under some potentially quite restrictive assumptions, we can formulate a general problem statement for the dealer operating in a limit orderbook in terms of a value function to be optimised. We can then derive the Hamilton-Jacobi-Bellman equation, which we may solve explicitly (if a solution exists), numerically (if this is computationally tractable), or in our case, approximately, using an important ansatz and analytical approximations.

As mentioned above, the assumptions under which we can formulate the general problem of a dealer may not be the most realistic. One particular example of this is that we assume the arrival frequency of market orders to be constant. Trading volume tends to peak around the opening and closing bell of the trading session (Sampath and Gopalaswamy 2020) as new information that may have revealed pre or post-market is traded on, before settling down for the rest of the day. Corporate events such as earnings releases also cause short-term spikes in trading volume (Lamont and Frazzini 2007).

To incorporate these exogenous events into our current model would require our Poisson intensity λ to be a function of both distance to the midprice and time, which would introduce extra complexity and potentially make our problem harder to solve.

However, a possible alternative model for incoming market orders would be the Hawkes process (Hawkes 2018), a kind of self-exciting Poisson process where arrivals increase the probability of more arrivals in the near future. This may be a more natural model for market-order arrivals as trades beget more trades in reaction. "Hawkes Process-Driven Models for Limit Order Book Dynamics" 2020 presents an application of the Hawkes process to model order arrivals.

Therefore, we recommend that future research in this area attempts to incorporate such self-exciting processes into the current stochastic control framework.

Bibliography

Daley, D. J. and D. Vere-Jones (2008). *An Introduction to the Theory of Point Processes: Volume II: General Theory and Structure*. Springer New York, NY.

Karatzas, Ioannis and Steven E. Shreve (1998). *Brownian Motion and Stochastic Calculus*. Springer.

Pham, Huyen (2009). *Continuous-time Stochastic Control and Optimization with Financial Applications*. Ed. by B. Rozovskii. Springer. ISBN: 978-3-540-89499-5. DOI: 10.1007/978-3-540-89500-8.

Shreve, Steven (2008). *Stochastic Calculus for Finance II*. Ed. by Marco Avellaneda. Springer.

Sarkka, Simo and Arno Solin (2019). *Applied Stochastic Differential Equations*. Ed. by Nancy Reid et al. Cambridge University Press.

Krylov, N. V. (1980). *Controlled Diffusion Processes*. Ed. by A. V. Balakrishnan. Springer-Verlag.

Bertsekas, Dimitri and Steven Shreve (1978). *Stochastic Optimal Control: The Discrete Time Case*. Academic Press Inc.

Merton, Robert C. (1969). "Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case". In: *The Review of Economics and Statistics* 51.3, pp. 247–257. ISSN: 00346535, 15309142. URL: http://www.jstor.org/stable/1926560 (visited on 04/16/2024).

Avellaneda, Marco and Sasha Stoikov (2008). "High-frequency trading in a limit order book". In: *Quantitative Finance* 8.3, pp. 217–224. DOI: 10.1080/14697680701381228. eprint: https://doi.org/10.1080/14697680701381228.

Ho, Thomas and Hans R. Stoll (1981). "Optimal dealer pricing under transactions and return uncertainty". In: *Journal of Financial Economics* 9.1, pp. 47–73. ISSN: 0304-405X. DOI: https://doi.org/10.1016/0304-405X(81)90020-9. URL: https://www.sciencedirect.com/science/article/pii/0304405X81900209.

Gopikrishnan, Parameswaran et al. (Oct. 2000). "Statistical properties of share volume traded in financial markets". In: *Phys. Rev. E* 62 (4), R4493–R4496. DOI: 10.1103/PhysRevE.62.R4493. URL: https://link.aps.org/doi/10.1103/PhysRevE.62.R4493.

Maslov, Sergei and Mark Mills (2001). "Price fluctuations from the order book perspective—empirical facts and a simple model". In: *Physica A: Statistical Mechanics and its Applications* 299.1. Application of Physics in Economic Modelling, pp. 234–246. ISSN: 0378-4371. DOI: https://doi.org/10.1016/S0378-4371(01)00301-6. URL: https://www.sciencedirect.com/science/article/pii/S0378437101003016.

Gabaix, Xavier et al. (May 2006). "Institutional Investors and Stock Market Volatility*". In: *The Quarterly Journal of Economics* 121.2, pp. 461–504. ISSN: 0033-5533. DOI: 10.1162/qjec.2006.121.2.461. eprint: https://academic.oup.com/qje/article-pdf/121/2/461/5324363/121-2-461.pdf. URL: https://doi.org/10.1162/qjec.2006.121.2.461.

Weber, P. and B. Rosenow (2005). “Order book approach to price impact”. In: *Quantitative Finance* 5.4, pp. 357–364. DOI: 10.1080/14697680500244411. eprint: https://doi.org/10.1080/14697680500244411. URL: https://doi.org/10.1080/14697680500244411.

Potters, Marc and Jean-Philippe Bouchaud (2003). “More statistical properties of order books and price impact”. In: *Physica A: Statistical Mechanics and its Applications* 324.1. Proceedings of the International Econophysics Conference, pp. 133–140. ISSN: 0378-4371. DOI: https://doi.org/10.1016/S0378-4371(02)01896-4. URL: https://www.sciencedirect.com/science/article/pii/S0378437102018964.

Smith, Eric et al. (Sept. 2003). “Statistical theory of the continuous double auction”. In: *Quantitative Finance* 3.6, p. 481. DOI: 10.1088/1469-7688/3/6/307. URL: https://dx.doi.org/10.1088/1469-7688/3/6/307.

Sampath, Aravind and Arun Kumar Gopalaswamy (2020). “Intraday Variability and Trading Volume: Evidence from National Stock Exchange”. In: *Journal of Emerging Market Finance* 19.3, pp. 271–295. DOI: 10.1177/0972652720930586. eprint: https://doi.org/10.1177/0972652720930586. URL: https://doi.org/10.1177/0972652720930586.

Lamont, Owen and Andrea Frazzini (2007). *The Earnings Announcement Premium and Trading Volume*. NBER Working Papers 13090. National Bureau of Economic Research, Inc. URL: https://EconPapers.repec.org/RePEc:nbr:nberwo:13090.

Hawkes, Alan G. (2018). “Hawkes processes and their applications to finance: a review”. In: *Quantitative Finance* 18.2, pp. 193–198. DOI: 10.1080/14697688.2017.1403131. eprint: https://doi.org/10.1080/14697688.2017.1403131. URL: https://doi.org/10.1080/14697688.2017.1403131.

“Hawkes Process-Driven Models for Limit Order Book Dynamics” (2020). MA thesis. University of Oxford. URL: https://www.maths.ox.ac.uk/system/files/attachments/Hawkes%20Process-Driven%20Models%20for%20Limit%20Order%20Book%20Dynamics_0.pdf.

Appendix A

The Verification Theorem

Here we state and prove the verification theorem mentioned in chapter 2, as given by Pham 2009.

Theorem A.0.1 (Verification Theorem)**.** Let $w : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function which is continuously differentiable at least once in its first argument and twice in its second. Let w also satisfy a quadratic growth condition, i.e. there exists a constant C such that

$$|w(t,x)| \leq C(1+x^2) \ \forall \ (t,x) \in [0,T] \times \mathbb{R}.$$

(i) Suppose that

$$\frac{\partial w}{\partial t}(t,x) + \sup_{a \in A}[\mathcal{L}^a w(t,x) + f(t,x,a)] \geq 0, \ (t,x) \in [0,T] \times \mathbb{R} \tag{A.1}$$

$$w(T,x) \geq g(x), \ x \in \mathbb{R}. \tag{A.2}$$

Then $w \geq v$ on $[0,T] \times \mathbb{R}$.
(ii) Suppose further that $w(T,.) = g$ and that there exists a measurable function $\hat{\alpha} : [0,T] \times \mathbb{R} \to A$ such that

$$\frac{\partial w}{\partial t}(t,x) + \sup_{a \in A}[\mathcal{L}^a w(t,x) + f(t,x,a)] = \frac{\partial w}{\partial t}(t,x) + \mathcal{L}^{\hat{\alpha}(t,x)} w(t,x) + f(t,x,\hat{\alpha}(t,x)) = 0,$$

the SDE

$$\mathrm{d}X_t = b(s,X_s,\hat{\alpha}(s,X_s))\mathrm{d}s + \sigma(s,X_s,\hat{\alpha}(s,X_s))\mathrm{d}W_s$$

admits a unique solution denoted by $\hat{X}_s^{t,x}$ given an initial condition $X_t = x$, and the process $\{\hat{\alpha}(s,\hat{X}_s^{t,x}) : t \leq s \leq T\}$ lies in $\mathcal{A}(t,x)$. Then

$$w = v \text{ on } [0,T] \times \mathbb{R} \tag{A.3}$$

and $\hat{\alpha}$ is an optimal Markovian control.

Proof. (i) Since $w \in C^{1,2}([0,T) \times \mathbb{R}^n)$, we have that for all $(t,x) \in [0,T) \times \mathbb{R}^n, \alpha \in \mathcal{A}(t,x), s \in [t,T)$, and any stopping time τ valued in $[t,\infty)$, by Itô's formula

$$\begin{aligned} w(s \wedge \tau, X_{s\wedge\tau}^{t,x}) = w(t,x) + \int_t^{s\wedge\tau} \frac{\partial w}{\partial t}(u, X_u^{t,x}) + \mathcal{L}^{\alpha_u} w(u, X_u^{t,x})\mathrm{d}u \\ + \int_t^{s\wedge\tau} D_x w(u, X_u^{t,x})^\top \sigma(X_u^{t,x}, \alpha_u)\mathrm{d}W_u. \end{aligned}$$

We choose $\tau = \tau_n = \inf\{s \geq t : \int_t^s |D_x w(u, X_u^{t,x})^\top \sigma(X_u^{t,x}, \alpha_u)|^2 \mathrm{d}u \geq n\}$, and we notice that $\tau_n \nearrow \infty$ as $n \to \infty$. The stopped process $\{\int_t^{s\wedge\tau_n} D_x w(u, X_u^{t,x})^\top \sigma(X_u^{t,x}) \mathrm{d}W_u, t \leq s \leq T\}$ is then a martingale, and by taking the expectation, we get

$$\mathbb{E}[w(s \wedge \tau_n, X_{s\wedge\tau_n}^{t,x})] = w(t,x) + \mathbb{E}\left[\int_t^{s\wedge\tau_n} \frac{\partial w}{\partial t}(u, X_u^{t,x}) + \mathcal{L}^{\alpha_u} w(u, X_u^{t,x}) \mathrm{d}u\right].$$

Since w satisfies (A.1), we have

$$\frac{\partial w}{\partial t}(u, X_u^{t,x}) + \mathcal{L}^{\alpha_u} w(u, X_u^{t,x}) + f(X_u^{t,x}, \alpha_u) \leq 0, \ \forall\, \alpha \in \mathcal{A}(t,x)$$

and so

$$\mathbb{E}[w(s \wedge \tau_n, X_{s\wedge\tau_n}^{t,x})] \leq w(t,x) - \mathbb{E}\left[\int_t^{s\wedge\tau_n} f(X_u^{t,x}, \alpha_u) \mathrm{d}u\right], \ \forall\, \alpha \in \mathcal{A}(t,x). \tag{A.4}$$

We have

$$\left|\int_t^{s\wedge\tau_n} f(X_u^{t,x}, \alpha_u) \mathrm{d}u\right| \leq \int_t^T |f(X_u^{t,x}, \alpha_u)| \mathrm{d}u,$$

and the RHS is integrable by the integrability condition on $\mathcal{A}(t,x)$. Since w satisfies a quadratic growth condition, we have

$$|w(s \wedge \tau_n, X_{s\wedge\tau_n}^{t,x})| \leq C(1 + \sup_{s\in[t,T]} |X_s^{t,x}|^2),$$

and the RHS term is integrable from the fact that

$$\mathbb{E}\left[\sup_{t\leq s\leq T} |X_s^{t,x}|^2\right] < \infty.$$

We can then apply the dominated convergence theorem, and send n to infinity in (A.4):

$$\mathbb{E}[w(s, X_s^{t,x})] \leq w(t,x) - \mathbb{E}\left[\int_s^t f(X_u^{t,x}, \alpha_u) \mathrm{d}u\right], \ \forall\, \alpha \in \mathcal{A}(t,x).$$

Since w is continuous on $[0,T] \times \mathbb{R}^n$, by taking the limit as $s \to T$, we obtain by the dominated convergence theorem and by (A.2)

$$\mathbb{E}[g(X_T^{t,x})] \leq w(t,x) - \mathbb{E}\left[\int_t^T f(X_u^{t,x}, \alpha_u) \mathrm{d}u\right], \ \forall\, \alpha \in \mathcal{A}(t,x).$$

From the arbitrariness of $\alpha \in \mathcal{A}(t,x)$, we deduce that $w(t,x) \leq v(t,x)$ for all $(t,x) \in [0,T] \times \mathbb{R}^n$.

(ii) We apply Itô's formula to $w(u, \hat{X}_u^{t,x})$ between $t \in [0,T]$ and $s \in [t,T)$, obtaining (after an eventual localisation to remove the stochastic integral term in the localisation):

$$\mathbb{E}[w(s, \hat{X}_s^{t,x})] = w(t,x) + \mathbb{E}\left[\int_t^s \frac{\partial w}{\partial t}(u, \hat{X}_u^{t,x}) + \mathcal{L}^{\hat{\alpha}(u, \hat{X}_u^{t,x})} w(u, \hat{X}_u^{t,x}) \mathrm{d}u\right].$$

Now, by the definition of $\hat{\alpha}(t,x)$, we have

$$\frac{\partial w}{\partial t} + \mathcal{L}^{\hat{\alpha}(t,x)} w(t,x) - f(t,x,\hat{\alpha}(t,x)) = 0,$$

and so

$$\mathbb{E}[w(s, \hat{X}_s^{t,x})] = w(t,x) - \mathbb{E}\left[\int_t^s f(\hat{X}_u^{t,x}, \hat{\alpha}(u, \hat{X}_u^{t,x})) \mathrm{d}u\right].$$

By taking the limit as s approaches T, we obtain

$$w(t,x) = \mathbb{E}\left[\int_t^T f(\hat{X}_u^{t,x}, \hat{\alpha}(u, \hat{X}_u^{t,x})) \mathrm{d}u + g(\hat{X}_T^{t,x})\right] = J(t,x,\hat{\alpha}).$$

This shows that $w(t,x) = J(t,x,\hat{\alpha}) \leq v(t,x)$, and finally that $w = v$ with $\hat{\alpha}$ as an optimal Markovian control. $\square$