Automated Market-Making

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22nd March 2024

Introduction

- How do financial markets work? (at the level of individual participants)
- How can we use stochastic control theory to model the behaviour of market participants?
- The Avellaneda-Stoikov Model
- Results on the statistical properties of the limit orderbook

What is a (financial) market?

- Stocks, bonds, commodities, all manner of derivatives
- Multiple buyers, multiple sellers
- Prices determined by trading: literally supply & demand
- So how does trading actually occur inside the exchange?

The limit orderbook

Side	Price /£	Volume
A	1.02	50
A	1.01	30
_	1.00	0
B	0.99	25
B	0.98	45

- Can either place *limit* orders or *market* orders
- *Limit* orders guaruntee price but not execution
  - Limit orders can also be ammended/updated for as long as they exist
- *Market* orders guaruntee execution but not price
- *Spread* between best bid and best ask → £0.02
- Can view the mid-price (here £1.00) as the "true" price

The limit orderbook

After market order for 20 shares:

Side	Price /£	Volume
A	1.02	50
A	1.01	10
_	1.00	0
B	0.99	25
B	0.98	45

After another market order for 30 shares:

Side	Price /£	Volume
A	1.02	30
A	1.01	0
_	1.00	0
B	0.99	25
B	0.98	45

Dealer considerations

Problem:

- What if no one wants to sell? (resp. buy?)
- What if buyers and sellers have wildly different price expectations?

Idea:

- Simultaneously place bid and ask limit orders $\rightarrow$ simultaneously buying and selling
- Enables other market participants to always have someone to trade against $\rightarrow$ "providing liquidity"
- Narrows the spread between bid and ask prices, decreasing implied cost of trading
- Dealer profits the (small) spread between buying and selling (multiplied across large trading volume)

Risks:

- Informed traders
- Inventory

Modelling dealer behavior

If dealer accrues positive inventory:

- Want to sell more than buy
- Set lower ask price

If dealer accrues negative inventory:

- Want to buy more than sell
- Set a higher bid price

Now the maths...

Stochastic Optimal Control

Following Pham (2009):

System:

$$\mathrm{d}X_t = b(X_t, \alpha_t)\mathrm{d}t + \sigma(X_t, \alpha_t)\mathrm{d}W_t \tag{1}$$

Objective (Finite time horizon):

$$v(t, x) := \sup_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_t^T f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + g(X_T^{t,x})\right] \tag{2}$$

Dynamic Programming Principle:

$$v(t, x) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_t^\theta f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + v(\theta, X_\theta^{t,x})\right] \tag{3}$$

Hamilton-Jacobi-Bellman Equation:

$$\frac{\partial v}{\partial t}(t, x) + \sup_{\alpha \in \mathcal{A}} \left[\mathcal{L}^\alpha v(t, x) + f(t, x, \alpha)\right] = 0 \tag{4}$$

The Avellaneda-Stoikov model

Following Avellaneda & Stoikov (2008):
Model market mid-price as Brownian motion with variance σ^2 (no drift)

$$\mathrm{d}S_t = \sigma \mathrm{d}W_t, \; t \in [0, T]$$

Dealer's value function: Expected exponential utility of terminal wealth

$$v(x, s, q, t) = \mathbb{E}\left[-e^{-\gamma(x+qS_T)} | \mathcal{F}_t\right] \quad (5)$$

- x = dealer's initial wealth (cash)
- q = dealer's inventory (assume fixed for now)
- γ = dealer's risk aversion

The Avellaneda-Stoikov model

Reservation bid price: The price at which the dealer is indifferent between buying an extra share and doing nothing: Set

$$v(x - r^b(s,q,t), s, q+1, t) = v(x,s,q,t) \tag{6}$$

and by substitution of (5) into (6) we obtain

$$r^b(s,q,t) = s - (1+2q)\frac{\gamma\sigma^2(T-t)}{2} \tag{7}$$

An analagous expression exists for r^a. We define the dealer's reservation price $r(s,q,t) := \frac{r^a(s,q,t)+r^b(s,q,t)}{2}$ and arrive at

$$r(s,q,t) = s - q\gamma\sigma^2(T-t) \tag{8}$$

The Avellaneda-Stoikov model

Now consider a dealer who sets limit orders. The dealer quotes bid p^b and ask p^a with spreads $\delta^b = s - p^b$ and $\delta^a = p^a - s$ respectively. We assume that

- Market buy orders will 'lift' the dealer's sell orders at Poisson rate $\lambda^a(\delta^a)$ (a decreasing function of δ^a)
- Market sell orders will 'hit' the dealer's bid orders at rate $\lambda^b(\delta^b)$ (decreasing in δ^b).

Now have stochastic wealth and inventory:

$$\mathrm{d}X_t = p^a \mathrm{d}N_t^a - p^b \mathrm{d}N_t^b \tag{9}$$

$$q_t = N_t^b - N_t^a \tag{10}$$

The Avellaneda-Stoikov model

Need to adapt our objective function: Maximise terminal wealth over possible bid/ask spreads δ^a, δ^b

$$u(s, x, q, t) = \max_{\delta^a, \delta^b} \mathbb{E}\left[-e^{-\gamma(X_T + q_T S_T)} | \mathcal{F}_t\right] \tag{11}$$

- Very difficult (maybe impossible) to solve analytically
- Through some asymptotic approximations we can work out some simple expressions for an approximate solution in terms of our model paramaters

The Avellaneda-Stoikov model

We obtain

$$r(s, q, t) = s - q\gamma\sigma^2(T - t) \tag{12}$$

which coincides with our indifference price for the dealer with static inventory, and

$$\delta^a + \delta^b = \gamma\sigma^2(T - t) + \frac{2}{\gamma}\log\left(1 + \frac{\gamma}{k}\right) \tag{13}$$

with γ and σ as before, k is a parameter from the orderbook describing how market order size impacts prices.

Statistical properties of the limit order book

Poisson intensity $\lambda(\delta)$ describes how likely a limit order is to be executed as a function of it's distance to the mid-price. Need some statistics regarding:

- Overall frequency of market orders
  - For simplicity, assume constant Λ
- Size distribution of market orders
  - "Econophysics" $\implies$ power law $f^Q(x) \propto x^{-1-\alpha}$
- Price impact of large market orders
  - "Econophysics" $\implies$ either $\Delta p \propto Q^\beta$ or $\Delta p \propto \log Q$

Using the second result for price impact we obtain:

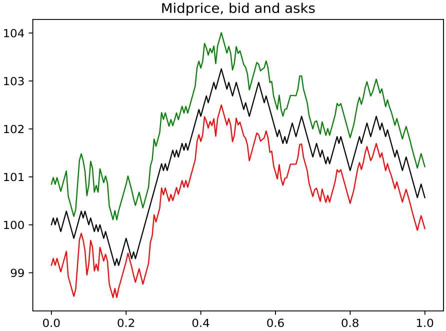
$$\begin{aligned}
\lambda(\delta) = \Lambda\mathbb{P}(\Delta p > \delta) &= \Lambda\mathbb{P}(\log Q > K\delta) \\
&= \Lambda\mathbb{P}\left(Q > e^{K\delta}\right) \\
&= \Lambda\int_{e^{K\delta}}^{\infty} x^{-1-\alpha} dx \\
&= Ae^{-k\alpha\delta}
\end{aligned}$$

The Avellaneda-Stoikov model - Summary

- Estimate parameters from market data

At each timestep, given current inventory and estimated parameters:

- Compute reservation price $r(s, q, t)$
- Compute spread $\delta^a + \delta^b$
- Set quotes $p^a = s + \frac{\delta^a+\delta^b}{2}$, $p^b = s - \frac{\delta^a+\delta^b}{2}$



Conclusion

- Hopefully you know a little bit more about how financial markets work!
- Stochastic control provides a useful framework through which to analyse the behaviour of market participants
- We can apply this framework to model the optimal behaviour of a dealer in financial markets
- This behaviour depends on the statistical properties of the particular market, which we can infer from market data

Thank you for your attention!

Questions?

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