Twelf Tutorial
Twelf Encoding of Minilang

John Altidor

- Proving language properties are important.
  - ▶ Rule out certain errors (e.g. assuming wrong number of bytes for an object).
  - ▶ Well-defined behavior throughout execution (e.g. no segmentation fault or accessing wrong parts of memory).
  - ▶ Publishing.

Motivation

- Proving language properties are important.
  - Rule out certain errors (e.g. assuming wrong number of bytes for an object).
  - Well-defined behavior throughout execution (e.g. no segmentation fault or accessing wrong parts of memory).
  - Publishing.
- But proofs are long, error prone, and difficult to validate.
  - **+20 pages** is common for a type safety proof.

Typical Proof Structure

- Example taken from type soundness proof of TameFJ calculus.

Lemma 33 (Inversion Lemma (method invocation)).

If:

   a. $\Delta; \Gamma \vdash e.\langle\overline{P}\rangle m(\overline{e}) : T \mid \Delta'$
   b. $\emptyset \vdash \Delta$ OK
   c. $\Delta \vdash \Delta'$ OK
   d. $\forall x \in dom(\Gamma) : \Delta \vdash \Gamma(x)$ OK

then:

   there exists Δ_n

where:

  $\Delta', \Delta_n = \Delta'', \overline{\Delta}$
  $\Delta \vdash \Delta', \Delta_n$ OK
  $\Delta; \Gamma \vdash e : \exists \Delta''.N \mid \emptyset$
  $mType(m, N) = \langle \overline{Y} \triangleleft \overline{B} \rangle \overline{U} \to U$
  $\Delta; \Gamma \vdash e : \exists \overline{\Delta}.\overline{R} \mid \overline{\emptyset}$
  $match(sift(\overline{R}, \overline{U}, \overline{Y}), \overline{P}, \overline{Y}, \overline{T})$
  $\Delta \vdash \overline{P}$ OK
  $\Delta, \Delta'', \overline{\Delta} \vdash \overline{T} <: \overline{[T/Y]B}$
  $\Delta, \Delta'', \overline{\Delta} \vdash \exists \emptyset.R <: \overline{[T/Y]U}$
  $\Delta, \Delta'', \Delta_n \vdash \overline{[T/Y]U} <: T$

Proof *by structural induction on the derivation of* $\Delta; \Gamma \vdash e.\langle\overline{P}\rangle m(\overline{e}) : T \mid \Delta'$ *with a case analysis on the last step:*

Case 1 (T-INVK)

1.	$\Delta' = \Delta'', \overline{\Delta}$	*by def* T-INVK
2.	$T = \overline{[T/Y]U}$	
3.	$\Delta; \Gamma \vdash e : \exists \Delta''.N \mid \emptyset$	
4.	$mType(m, N) = \langle \overline{Y} \triangleleft \overline{B} \rangle \overline{U} \to U$	
5.	$\Delta; \Gamma \vdash e : \exists \overline{\Delta}.\overline{R} \mid \overline{\emptyset}$	*by premises* T-INVK
6.	$match(sift(\overline{R}, \overline{U}, \overline{Y}), \overline{P}, \overline{Y}, \overline{T})$	
7.	$\Delta \vdash \overline{P}$ OK	
8.	$\Delta, \Delta'', \overline{\Delta} \vdash \overline{T} <: \overline{[T/Y]B}$	
9.	$\Delta, \Delta'', \overline{\Delta} \vdash \exists \emptyset.R <: \overline{[T/Y]U}$	
10.	*let* $\Delta_n = \emptyset$	
11.	$\Delta \vdash \exists \Delta''.N$ OK	*by 3, b, d, lemma 30*
12.	$\Delta \vdash \Delta''$ OK	*by 11, def* F-EXIST
13.	$\Delta \vdash \exists \overline{\Delta}.\overline{R}$ OK	*by 5, b, d, lemma 30*
14.	$\Delta \vdash \overline{\Delta}$ OK	*by 13, def* F-EXIST
15.	$dom(\Delta'') \cap dom(\overline{\Delta}) = \emptyset$	*by 3, 5, Barendregt*
16.	$\Delta \vdash \Delta'', \overline{\Delta}$ OK	*by 12, 14, 15, lemma 14*
17.	*done*	*by 10, 1, 16, 3, 4, 5,* *6, 7, 8, 9, 2, reflexivity*

- Lots of steps, lemmas, and opportunities for errors in proofs of language properties.

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  - ▸ Not all proofs can be derived automatically.
- **Proof checkers** simply verify the proofs they are given.
  - ▸ These proofs must be specified in an extremely detailed, low-level form.
- **Proof assistants** are a hybrid of both.
  - ▸ "Hard steps" of proofs (the ones requiring deep insight) are provided by human.
  - ▸ "Easy steps" of proofs can be filled in automatically.
- (above bullet points taken from UPenn's Software Foundations course slides)

Twelf Proof Assistant

- Automated support for **deriving proofs** and **checking proofs** of language properties.
- Implementation of the LF calculus (calculus for reasoning about deductive systems).
- Alternatives: Coq, Isabelle, Agda, etc.
- Presenting Example Twelf Encoding of Minilang.

Constructive Logic

- Twelf is a **constructive** (not classical) proof assistant.
- Proposition is true **iff** there exists a proof of it.
- Law of excluded middle not assumed: $P \vee \neg P$.
  - Proving $P \vee \neg P$ **requires** either:
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- No choice operator ($\epsilon x.P(x)$ proposed by David Hilbert).
  - $ln(x) = u$ such that $x = e^u$.
  - Definition in Isabelle/HOL:
    ```
    definition ln :: real => real where
    ln x = THE u. exp u = x.
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- In Twelf: Writing a proof $=$ Writing a program.
  - Proofs are programs.

- Lecture will involve in-class exercises.
- Can try Twelf without installation.
- Twelf Live Server:

<div style="text-align: center;">

`http://twelf.org/live/`

</div>

- Links to starter code of examples will be provided.

Kinds: Category of Types

- Three **levels** of objects in Twelf:
  - ▶ **Kinds** are at highest level.
  - ▶ **Types** are at second level.
  - ▶ **Terms** are at lowest level.

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- Each type is of a certain kind.
  (Twelf syntax: "someType : someKind")
- Each term is of a certain type.
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- Twelf overloads languages constructs with same syntax.
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- Contrived examples:
    - Term [1, 2, 3] is of type ArrayInt.
    - Type ArrayInt is of kind Array.

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- Contrived examples:
  - Term [1, 2, 3] is of type ArrayInt.
  - Type ArrayInt is of kind Array.
- The kind type is a pre-defined kind in Twelf.

Functions

- Twelf supports defining functions:

  ```
  int :  type.   one :  int.
  plusOne :  int -> int.
  ```

  - plusOne is a **function term**.
  - plusOne takes in a term of type int and returns a term of type int.
  - The type of function term plusOne is int -> int.
  - (plusOne one) has type int.

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- Functions taking in **multiple** arguments are represented using their **curried form**:

- plus: int -> int -> int.

  - int -> int -> int is curried form of (int, int) -> int.
  - int -> int -> int = int -> (int -> int).
  - (plus one) has type int -> int.
  - (plus one one) has type int.

Functions Returning Types

- Recall that type is a **kind** (type of types).
- Functions can also return **types**:
- equalsOne : int -> type.
  - equalsOne is a function **term**.
  - equalsOne takes in a term of type int and returns a **type** of kind type.
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- `oneIsOne : (equalsOne one).`
  - Defines a new **term** `oneIsOne` of **type** `(equalsOne one)`.

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- `oneIsOne : (equalsOne one).`
  - Defines a new **term** `oneIsOne` of **type** `(equalsOne one)`.
- A function type is a kind if its return type is also a kind.
  - `int -> type` is a kind.
  - `int -> (int -> type)` is a kind.
  - `int -> int -> type = int -> (int -> type)` is a kind.
- `type` is **not** allowed on the left-hand side of arrow (`->`).

Minilang Syntax in Twelf

- The object language is Minilang (the object of study).
- Syntactic categories encoded w/ object types (defined types).
  - `exp : type.`
  - Defines type `exp` of kind `type`.
  - `exp` represents syntactic category e.
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  - `exp` represents syntactic category e.
  - Terms in the grammar of e have type `exp`.
- Grammar productions encoded w/ **functions** between syntactic categories.
  - `add : exp -> exp -> exp.`
  - *Expression e* $::= +(e_1; e_2)$
  - `add` takes in **two** arguments.
  - `exp -> exp -> exp` is curried form of `(exp, exp) -> exp`.

Terms w/ variables using Higher-Order Abstract Syntax (HOAS)

- Abstract syntax from earlier slides is **first-order abstract syntax (FOAS)**.
    - Each AST has form $o(t_1, t_2, \ldots, t_n)$, where o is operator and $t_1, \ldots, t_n$ are ASTs. Example:

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    - `+(num[3]; num[4])`
    - `add (enat 3) (enat 4)`
- ASTs in **Higher-Order Abstract Syntax (HOAS)**:
- Each t_i in $o(t_1, t_2, \ldots, t_n)$ has form:

$$x_1, x_2, \ldots x_k.t$$

- t is a FO-AST.
- Each x_j is a variable bound in t.
- $k \geq 0$; if $k = 0$, then no variable is declared.

HOAS encoding of `let` expression

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- "$x.e_2$" captures that x is bound in e_2.
- HOAS lets us know where variables are being bound.

$$\mathtt{let}(3;\ x.+(x;\ 4)) \equiv \mathtt{let}(3;\ y.+(y;\ 4))$$

- Two preceding terms above are **alpha-equivalent**.

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- Holes filled in by **applying** (terms w/ holes)/functions.
- Holes **abstract** details.

Higher-Order Terms are Functions

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- Holes are represented by **variables**.
- Holes filled in by **applying** (terms w/ holes)/functions.
- Holes **abstract** details.
- "$x.e$" represented by **lambda abstraction** "$\lambda x : \tau.e$".
- Twelf's syntax of "$\lambda x : \tau.e$": "$[x : \tau]\ e$"

- Twelf type signature of `let`:

$$\text{let} : \quad \text{exp} \to \underbrace{(\text{exp} \to \text{exp})}_{x.e_2} \to \text{exp}.$$

let expression in Twelf HOAS

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- Example HOAS term in Twelf:

Concrete Syntax	Twelf HOAS
`let x = ` $\underbrace{\texttt{1 + 2}}_{e_1}$ ` in ` $\underbrace{\texttt{x + 3}}_{e_2}$	`let ` $\underbrace{\texttt{(add 1 2)}}_{e_1}$ $\underbrace{\texttt{([x:exp] add x 3)}}_{x.e_2}$

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- No need to define object (Minilang) variables.
- **LF variables** remove need for object variables.
- No need to define substitution (nor requisite theorems) as well.

Predicates in Twelf

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  Functions that return types (not terms).
- Typing Predicate: $e : \tau$
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  - Let `Vec` be a function that given **nat** n, `Vec(`n`)` returns the type of n-dimensional vectors.
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  - `[4, 1, 3]` is a **term** of **type** `Vec(3)`.

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  - Vec(3) is a **dependent type** representing 3-dimensional vectors.
  - [4, 1, 3] is a **term** of **type** Vec(3).
  - Vec is a **type family** because it is a function that returns dependent types.

Judgments are Dependent Types

- Judgments/Propositions (instantiations of predicates) represented by dependent types.
- Judgment z : num represented by type `(of (enat z) num)`.
- Dependent type (`of` e τ) represents judgment "e : τ".

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- Curry-Howard Correspondence:
  Proofs are terms.
  Propositions/Judgments are types.

Pi-Abstractions

- Function/**Lambda Abstraction** "$\lambda x : S.e$" of type $S \to T$:
  - Takes in a term s of type S.
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  {x:S} T

Inference Rules are Functions

$$\frac{}{\texttt{num}[n] \ : \ \texttt{num}} \ \text{T.1}$$

- of/nat : {N:nat} of (enat N) num.
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    - Constants start with lower-case letters.
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- (of/nat z) ≠ (of (enat z) num).
- of/nat **returns terms** not types.
- (of/nat z) = **term** of type (of (enat z) num).
  - ▶ Example legal assignment:
    y : (of (enat z) num) = (of/nat z).

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- `(of/nat z)` $\neq$ `(of (enat z) num)`.
- `of/nat` **returns terms** not types.
- `(of/nat z) =` **term** of type `(of (enat z) num)`.
  - ▶ Example legal assignment:
    `y : (of (enat z) num) = (of/nat z).`
- `(of/nat z)` is a **derivation/term** of judgment z : `num` represented by type of `(enat z) num`.

Premises are Inputs

$$\frac{e_1: \quad \text{num} \quad e_2: \quad \text{num}}{+(e_1; \; e_2): \quad \text{num}} \; \text{T.4}$$

- Twelf Encoding:
  ```
  of/add :  of (add E1 E2) num
      <- of E1 num
      <- of E2 num.
  ```
- Given a proof of (of E2 num) **and**
- Given a proof of (of E1 num)
- of/add returns proof of (of (add E1 E2) num)

Implicit and explicit parameters

- `of/nat:` `{N:nat}` `of (enat N) num.`
- Parameter `N` is **explicit** in the above signature.
- Explicit parameters **must be specified** in function applications.
- `D : of (enat z) num = of/nat z.`

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- `of/nat:` `of (enat N) num.`
- Parameter `N` is **implicit** in the above signature.
- Implicit parameters **cannot be specified** by programmer in function applications.
- `D : of (enat z) num = of/nat.`
- Twelf figures out from the context that `z` is the implicit parameter that `of/nat` should be applied to.

First-Order Quantification Only

- Can only quantify over first-order terms.
- **Allowed**:
  - add : exp -> exp -> exp.
  - let : exp -> (exp -> exp) -> exp.
    - A **higher-order term** is a function, where one of its inputs is also a function.

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- **Not allowed**:
  - ▶ quantifyTypes : exp -> type -> exp.
  - ▶ allIsTrue : {Prop:type} Prop.
- The kind type categorizes Twelf types.

First-Order Quantification Only

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    - A **higher-order term** is a function, where one of its inputs is also a function.
- **Not allowed**:
  - `quantifyTypes : exp -> type -> exp.`
  - `allIsTrue : {Prop:type} Prop.`
- The kind `type` categorizes Twelf types.
- No type polymorphism implies no **general** logical connectives.
- **Not allowed**:
  ```
  conjunction :
  {P:type} {Q:type} P -> Q -> (and P Q).
  ```

Predicativity

- Different term levels used to restrict quantification.
    - Twelf terms are first-order terms; e.g., (s z).
    - Twelf types are second-order terms; e.g., nat.
    - Twelf kinds are third-order terms; e.g., type.

Predicativity

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  - Twelf terms are first-order terms; e.g., `(s z)`.
  - Twelf types are second-order terms; e.g., `nat`.
  - Twelf kinds are third-order terms; e.g., `type`.
- Twelf only allows **predicative** definitions:
  - Cannot apply term to itself. (Cannot quantify over oneself.)
  - No term has itself as type. (Not allowed: `typ : typ`.)
  - Disallows Russell's paradox:
    Let $H = \{x \mid x \notin x\}$. Then $\underbrace{H \in H \iff H \notin H}_{\textit{False}}$.
- Helps Twelf avoid logical inconsistency
  (i.e. proving false/uninhabited type).
- False implies any proposition (including false ones).
- False/uninhabited types used for constructive proofs by
  contradiction.

- Create language of numbers with subtyping in Twelf.

Category	Item		Abstract	Concrete
Terms	*e*	$: : =$	zero	0
		|	pi	π
		|	img	$\sqrt{-1}$
Types	*t*	$: : =$	number	*num*
		|	real	*real*
		|	complex	*complex*
		|	int	*int*

More Exercises

- **Subtyping** Rules (not all):

$$\overline{complex <: num} \qquad \overline{real <: num} \qquad \overline{int <: real}$$

- **Typing** Rules (not all):

$$\overline{0 : int} \qquad \overline{\pi : real} \qquad \overline{\sqrt{-1} : complex}$$

- Define **reflexive** and **transitive** rules for subtyping.
- Define **subsumption** rule for typing judgment.
- **Prove** 0 : num.
  - Fill in the blank below:
  - D : (of zero number) = •

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- **Hypothetical Judgments**:
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Hypothetical Judgments in Twelf

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- Encoded w/ **higher-order types**:
  Function types where one of the inputs is also a function type.

Hypothetical Judgments in Twelf

- What happened to typing context Γ?
- **Hypothetical Judgments**:
  Judgments made under the assumption of other judgments.
- Encoded w/ **higher-order types**:
  Function types where one of the inputs is also a function type.
- Input function types represent hypothetical assumptions.
- Similar to higher-order terms.
  (Another application of HOAS)
- Γ does not need to be defined.

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, x : \tau_1 \vdash e_2 : \tau_2}{\Gamma \vdash \texttt{let}(x;\ e_1;\ e_2) : \ \tau_2} \text{ T.6}$$

- Twelf Encoding:

```
of/let :  ({x:  exp} of x T1 -> of (E2 x) T2) ->
    of E1 T1 ->
    of (let E1 ([x] E2 x)) T2.
```

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- Twelf Encoding:
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  of/let :  ({x:  exp} of x T1 -> of (E2 x) T2) ->
       of E1 T1 ->
       of (let E1 ([x] E2 x)) T2.
  ```
- First, a Twelf **coding convention**:
  Return type (of (let E1 ([x] E2 x)) T2) could be
  replaced with (of (let E1 E2) T2).
- E2 in both cases is of type (exp -> exp).
- ([x] E2 x) used for readability: # of inputs explicit.
- ([x] E2 x) is called the **eta-expansion** of E2.

- Let f be a function of of/let's first input type:
  ({x: exp} of x T1 -> of (E2 x) T2).
- Reminder of hypothetical judgment: $\Gamma, x : \tau_1 \vdash e_2 : \tau_2$.

of/let's first input type

- Let f be a function of `of/let`'s first input type:
  (`{x:  exp} of x T1 -> of (E2 x) T2`).
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- f's first input is an `exp` term bound to LF variable `x`.
- f's second input is a term `dx` of type (`of x T1`).
- f's output is a term of type (`of (E2 x) T2`).

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- f's second input is a term `dx` of type (`of x T1`).
- f's output is a term of type (`of (E2 x) T2`).
- `dx` of type (`of x T1`) can be **used in the definition of** f to return a proof/term of type (`of (E2 x) T2`).
- The ability to use a proof (`dx`) of type (`of x T1`) to derive a proof of type (`of (E2 x) T2`) simulates the ability to use an assumption $x : \tau_1$ to prove $e_2 : \tau_2$.

Exercise Applying Hypothetical Judgment

- Derive the judgment $\vdash$ `let x be 1 in x + 0 : num` in Twelf.
- Twelf encoding of above judgment:
  `of (let (enat (s z)) ([x:exp] add x (enat z))) num.`
- Recall important signatures (displaying implicit parameters):

```
of/let :  {T1:typ} {E2:exp -> exp} {T2:typ} {E1:exp}
  ({x:exp} of x T1 -> of (E2 x) T2) -> of E1 T1
    -> of (let E1 ([x:exp] E2 x)) T2.

of/nat : {N:nat} of (enat N) num.

of/add :  {E2:exp} {E1:exp}
  of E2 num -> of E1 num -> of (add E1 E2) num.
```

Relations w/ Inputs and Outputs (Modes)

- Inputs/Outputs defined with `%mode` declaration.
  ```
  of :  exp -> typ -> type.
  %mode of +E -T.
  ```
- Inputs marked with +.
- Outputs marked with −.

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- Modes are necessary for **specifying theorems**.
- Modes used also for **checking proofs of theorems**.

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- Not all relations required modes.
- Modes are necessary for **specifying theorems**.
- Modes used also for **checking proofs of theorems**.
- Only **ground** terms may be applied to relations w/ modes in rules (details later).

Backward Arrow vs. Forward Arrow

- Output terms must be **ground** given ground input terms.
  - ▸ Ground terms do not contain **free** variables.
  - ▸ Output terms are fixed (ground) wrt (ground) inputs.
- Forward "->" reflects **order that premises are passed to rules/functions** and makes proofs more natural.
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- Order of args that **causes error**:

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  ```
- Error message:
  ```
  Occurrence of variable T1 in output (-) argument
  not necessarily ground
  ```

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- `right1 :  of E1 num -> of (add E1 (enat z)) num.`

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- Terms in input position of return type: `(add E1 (enat z))`.

Universally-Quantified Inputs

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- Tokens starting with capital letters are assumed by Twelf to be variables in type: `E1`.

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- Terms in input position of return type: `(add E1 (enat z))`.
- Tokens starting with capital letters are assumed by Twelf to be variables in type: `E1`.
- Free variables in input position of return type, `E1`, are inferred by Twelf to be **universally**-quantified inputs to function `right1`.
  - Only these terms are allowed to be universal inputs to function `right1`.

Resolving Ground Terms

- All terms must be **ground** terms:
  constants or terms **without** free variables
  assuming that input terms (**from return type**) are also
  ground (do not contain free variables).

- Next Step:
  Check that input terms in type preceding return type are
  ground:

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  ground:

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  `num.`

- `E1` in premise type (`of E1 num`) is ground wrt `E1` in return
  type because they are the same.

- `num` in premise type (`of E1 num`) is ground wrt return type
  because `num` is a constant.

- `wrong1 : of E2 num -> of (add E1 (enat z)) num.`
- E2 term not coming from conclusion (return type).

- Output terms resulting from grounded input terms are also ground.
- Second argument of the of relation is an **output** argument.

```
right2 :  of E T -> of (add E (enat z)) T.
```

Ground Term From Output

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```
right2 :  of E T -> of (add E (enat z)) T.
```

- Term T is **computed**/result of premise/**recursive call** (of E T).

- Output term in conclusion not grounded:

  ```
  wrong2 :  of E T1 -> of (add E (enat z)) T2.
  ```

- Output term T2 is universally quantified instead of a grounded result of the input term.
  This violates the %mode declaration of the of relation.

- `right1 :  of E1 num -> of (add E1 (enat z)) num.`
- `wrong1 :  of E2 num -> of (add E1 (enat z)) num.`
- `right2 :  of E T -> of (add E (enat z)) T.`
- `wrong2 :  of E T1 -> of (add E (enat z)) T2.`

Decidable Predicate Definitions

- Decidable Predicate Definition or Algorithmic Definition:
  Definition of predicate that gives an **algorithm** for deciding
  predicate thats halts on all inputs within a **finite** number of
  steps.

- Constructive Logic Requirement:
  Proposition is true **iff** there exists a proof of it.

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  Definition of predicate that gives an **algorithm** for deciding
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  steps.
- Constructive Logic Requirement:
  Proposition is true **iff** there exists a proof of it.
- For every true proposition/instance of predicate, algorithm
  finds a proof of proposition.
- For every false proposition of predicate, algorithm determines
  no proof exists.

Termination

- %terminates checks a program succeeds or fails in a finite number of steps given ground inputs.
- Modes with termination ensure decidable definitions.
- Termination not guaranteed with **transitive** rule.

```
subtype : typ -> typ -> type.
%mode subtype +T1 -T2.
subtype/int/rea : subtype int real.
subtype/rea/num : subtype real number.
subtype/num/num : subtype number number.
subtype/trans:
  subtype T1 T3 <- subtype T1 T2 <- subtype T2 T3.
%terminates T (subtype T _).
```

- **Error**: Termination violation: ---> (T1) < (T1)
- First input to subtype not **smaller** in premise/recursive call.

Syntax-Directed Definition

- **Syntax-Directed Definition**: For each syntactic form of input, there is at most one applicable rule.
- Syntax of input term tells us which rule to use. (or if no rule applies)
- Each true proposition of a syntax-directed predicate has exactly one unique derivation.
- Only one way to derive +(5; 3) : num.

$$\dfrac{\dfrac{}{5 \ : \ \ num}\text{ of/num} \quad \dfrac{}{3 \ : \ \ num}\text{ of/num}}{+(5; \ 3): \ \ num}\text{ of/add}$$

- No need for exhaustive proof search with syntax-directed predicates.

Checking Syntax-Directed

- `%unique` checks if outputs are uniquely determined by inputs.
- `%unique` check can also ensure rules are syntax-directed.

```
subtype : typ -> typ -> type.
%mode subtype +T1 -T2.
subtype/int/rea : subtype int real.
subtype/rea/num : subtype real number.
subtype/num/num : subtype number number.
subtype/trans:
  subtype T1 T3 <- subtype T1 T2 <- subtype T2 T3.
%worlds () (subtype _ _).
%unique subtype +T1 +T2.
```

- **Error**: subtype/rea/num and subtype/trans overlap
- Both rules could be used to derive subtype real number.

- Twelf can derive (search) for proofs:
- `%solve D1 :`
  `of (estr (a , b , c , a , eps)) string.`
- Twelf will save proof term in D1.

Printing Proof Terms

- To print all (implicit) terms in proofs:
- From Twelf Server:
  "set Print.implicit true"
- From ML (SML) Prompt:
  "Twelf.Print.implicit := true"
- Then just execute "Check File":
  Emacs Key Sequence: ^C ^S

Proof Term in Sample Output

```
loadFile test_typing.elf
[Opening file test_typing.elf]
%solve
of (estr (, a (, b (, c (, a eps))))) string.
 OK
D1 : of (estr (, a (, b (, c (, a eps))))) string
   = of/str (, a (, b (, c (, a eps)))).
```

Twelf Theorems

- **Preservation Theorem**:
  If (of E T) and (step E E'), then (of E' T).

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- Twelf allows expressing $\forall\exists$-type properties.
- **Preservation, re-formulated**:
    - **For every** derivation of (of E T) and (step E E'),
    - **there exists** at least one derivation of (of E' T).

Twelf Theorems

- **Preservation Theorem**:
  If (of E T) and (step E E'), then (of E' T).
- Twelf allows expressing ∀∃-type properties.
- **Preservation, re-formulated**:
  - ▶ **For every** derivation of (of E T) and (step E E'),
  - ▶ **there exists** at least one derivation of (of E' T).
- %theorem
  preservation :
       forall* {E} {E'} {T}
       forall {O:of E T} {S:step E E'}
       exists {O':of E' T}
       true.
- Verbose syntax above.
  Desugared, concise alternative on next slide.

Theorems are Function Types w/ Specified Inputs/Outputs

- Preservation theorem is a function returning types (type family):

  ```
  preservation:
      of E T -> step E E' -> of E' T -> type.
  ```

- Premises are **inputs**. Conclusions are **outputs**.

  ```
  %mode preservation +O +S -O'.
  ```

- To prove preservation theorem, need to show `preservation` is a **total relation** on all **possible inputs**.
  - For each possible derivation of premises (inputs), need at least one derivation of conclusion (output).

Proofs of Theorems

- Proofs of theorems are **total** relations over inputs.
- Proving theorem
  = Constructing functions **for each case**:
  - For each **constructor** of term to perform structural induction on.

Proofs of Theorems

- Proofs of theorems are **total** relations over inputs.
- Proving theorem
  = Constructing functions **for each case**:
    - For each **constructor** of term to perform structural induction on.
- Note:
  No case-split or **pattern match** construct in Twelf.
    - This is the reason why **multiple functions** are required to prove theorem for multiple cases.
    - Results in smaller proof terms but more of them.

Preservation Proof - Addition Case 2 - Informal

Case: (T.4, D.1)

$$\frac{e_1: \text{ num } \quad e_2: \text{ num}}{+(e_1; \ e_2): \text{ num}} \text{ T.4} \qquad \frac{e_1 \mapsto e_1'}{+(e_1; \ e_2) \mapsto +(e_1'; \ e_2)} \text{ D.1}$$

We assume preservation holds for subexpressions. Hence, by the **inductive hypothesis**, $e_1: \text{ num}$ and $e_1 \mapsto e_1'$ implies $e_1': \text{ num}$. Rule T.4 gives us:

$$\frac{e_1': \text{ num } \quad e_2: \text{ num}}{+(e_1'; \ e_2): \text{ num}} \text{ T.4} \quad \square$$

```
of/add :
    of (add E1 E2) num <- of E1 num <- of E2 num.

- :
{E1-num  : of E1 num }
{E2-num  : of E2 num }
{E1=>E1' : step E1 E1' }
{E1'-num : of E1' num }
preservation E1-num E1=>E1' E1'-num ->
 preservation
  ((of/add E2-num E1-num) : (of (add E1 E2) num))
  ((step/add1 E1=>E1') :
      (step (add E1 E2) (add E1' E2)))
  ((of/add E2-num E1'-num) : (of (add E1' E2) num)).
```

Proof Case without Explicit Types

```
-  : preservation
      (of/add E2-num E1-num)
      (step/add1 E1=>E1')
      (of/add E2-num E1'-num)
        <- preservation E1-num E1=>E1' E1'-num.
```

- Types of terms in proofs: usually not required to specify.
- Allowed to be manually specified.
- Output from Twelf server contains (some) inferred types.

```
- : preservation
      (of/add E2-num E1-num)
      (step/add1 E1=>E1')
      (of/add E2-num E1'-num)
        <- preservation E1-num E1=>E1' E1'-num.
```

- Applying inductive hypothesis = recursive call.

Checking Proof Totality

- After proving all cases, ask Twelf to check we covered all cases.

```
%worlds () (preservation _ _ _).
%total E-T (preservation E-T _ _).
```

- `%total E-T` tells Twelf to check proof of totality by structural induction on typing derivation E–T.
- Details of `%world` declaration later.

Missing Case

- If we forget to prove a case, `%total` command will fail.
- Twelf prints error message to help user "debug" proof:

```
preservation.elf:69.8-69.11 Error:
Coverage error --- missing cases:
{E1:exp} {E2:exp} {E3:exp}
{O1:of (add E1 E2) num} {S1:step E1 E3}
{O2:of (add E3 E2) num}
   |- preservation O1 (step/add1 S1) O2.
```

- Forgot the case where we could derive:
  - ▸ (of (add E1 E2) num)
  - ▸ (step (add E1 E2) (add E3 E2)))
- Need to construct proof of (of (add E3 E2) num).

- Cannot prove theorem by just assuming conclusion of theorem holds.

Assuming What Needs To Be Proven

- Cannot prove theorem by just assuming conclusion of theorem holds.
- Also, cannot assume propositions not derived from premises of theorem.

Assuming What Needs To Be Proven

- Cannot prove theorem by just assuming conclusion of theorem holds.
- Also, cannot assume propositions not derived from premises of theorem.
- Such a proof will contain a **non-ground** term.
  - ▸ `%mode` declarations used to check proofs.

Recall Valid Proof of Case

```
- : {E1-num   : of E1 num}
    {E2-num   : of E2 num}
    {E1=>E1' : step E1 E1'}
    {E1'-num : of E1' num}
    preservation E1-num E1=>E1' E1'-num
      -> preservation (of/add E2-num E1-num)
                      (step/add1 E1=>E1')
                      (of/add E2-num E1'-num).
```

Invalid Proof of Case

```
- : {E1-num  : of E1 num }
    {E2-num  : of E2 num }
    {E1=>E1' : step E1 E1' }
    {E1'-num : of E1' num }
    preservation (of/add E2-num E1-num)
                 (step/add1 E1=>E1')
                 (of/add E2-num E1'-num).
```

- Proof above just assumes of E1' num, which is not one of the assumptions for the case.
- E1'-num is not an input term in the conclusion (third) argument of preservation.
- E1'-num is not an output term derived from ground terms.
- Twelf reports error for function above.

Checking Entire Proofs of Theorems

- Twelf checks proofs of theorems by verifying three key aspects:
  - Type checking – Proof of correct proposition
  - Grounds checking – Valid assumptions
  - Coverage checking – Proved all cases of theorem
- Next few slides describes Twelf's coverage checking of proofs

Specifying Worlds – Possible Inputs

- To check totality of function/theorem, need to define all possible inputs or **worlds**.
  - World = Set of terms of a type (inhabitants of a type)
- Example world of natural numbers:

```
nat :  type.
z  :  nat.
s  :  nat -> nat.
%worlds () (nat).
```

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    - World = Set of terms of a type (inhabitants of a type)
- Example world of natural numbers:
  ```
  nat :  type.
  z :  nat.
  s :  nat -> nat.
  %worlds () (nat).
  ```
- No term of type nat containing LF variables.
- No such nat of form (s x), where x of variable of type nat.

- Let expression contains binders.

```
add :  exp -> exp -> exp.
let :  exp -> (exp -> exp) -> exp.
%worlds () (exp).
```

Terms Containing Binders

- Let expression contains binders.
  ```
  add :  exp -> exp -> exp.
  let :  exp -> (exp -> exp) -> exp.
  %worlds () (exp).
  ```
- Error message:
  ```
  syntax.elf:38.15-38.25 Error:
  While checking constant let:
  World violation for family exp:  {_:exp} </:  1
  ```

Terms Containing Binders

- Let expression contains binders.
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  ```
- Error message:
  ```
  syntax.elf:38.15-38.25 Error:
  While checking constant let:
  World violation for family exp:  {_:exp} </:  1
  ```
- Need to tell Twelf about possible variables that can arise from rules.

Blocks

- **Blocks**: Patterns describing fragment of contexts.
- Update addressing previous error:
  ```
  add :  exp -> exp -> exp.
  let :  exp -> (exp -> exp) -> exp.
  %block exp-block :  block {x:exp}.
  %worlds (exp-block) (exp).
  ```
- Informs Twelf that terms of type exp can contain binders of type exp.

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- Update addressing previous error:
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  let :  exp -> (exp -> exp) -> exp.
  %block exp-block :  block {x:exp}.
  %worlds (exp-block) (exp).
  ```
- Informs Twelf that terms of type exp can contain binders of type exp.
- Worlds can take in multiple blocks. Syntax:
  ```
  %worlds (block1 | block2 | ... | blockN) (exp).
  ```

- Specifying world requires specifying how variables are quantified (universal inputs or ground outputs).

```
of : exp -> typ -> type.
%mode of +E -T.
...
of/let : of (let E1 ([x] E2 x)) T2
  <- of E1 T1
  <- ({x: exp} of x T1 -> of (E2 x) T2).
%block of-block :
  some {T:typ} block {x: exp}{_: of x T}.
%worlds (of-block) (of _ _).
```

- Number of args specified by pattern in %worlds declaration:
  (of _ _)

Checking Proof Totality

- After defining the worlds of all inputs to a theorem/function type, we can ask Twelf to check that the proof/function is total: **defined over the world**.

```
%worlds () (preservation _ _ _).
%total E-T (preservation E-T _ _).
```

- `%total E-T` tells Twelf to check proof of totality by structural induction on typing derivation E-T.

Checking Proof Totality

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```
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%total E-T (preservation E-T _ _).
```

- `%total E-T` tells Twelf to check proof of totality by structural induction on typing derivation `E-T`.
- Twelf checks proofs of theorems by:
  - Type checking – Proof of correct proposition
  - Grounds checking – Valid assumptions
  - Coverage checking – Proved all cases of theorem

- Ask Twelf to **derive** proof for all cases of theorem:
  `%prove 3 E-T (preservation E-T _ _).`
  - by structural induction on typing derivation E-T
  - 3 is bound on the size of proof terms.

Totality Proof Automation

- Ask Twelf to **derive** proof for all cases of theorem:
  `%prove 3 E-T (preservation E-T _ _).`
  - by structural induction on typing derivation E–T
  - 3 is bound on the size of proof terms.
- Twelf fails to find proof of `progress` theorem because it requires nested case analysis.
  - Need extra theorems for sub-cases (no case-split construct).
  - See Twelf page on **Output Factoring** for more details:
    `http://twelf.org/wiki/Output_factoring`

My Review of Twelf: The Good

- **Language Simplicity**: Fewer language constructs
  - Functions encode many language elements
    (e.g., grammar, judgments, theorems, etc.)
- Good tool to **start with** for learning about proof assistants
  because of language simplicity and less syntactic sugar
  (my opinion).

My Review of Twelf: The Good

- **Language Simplicity**: Fewer language constructs
  - Functions encode many language elements (e.g., grammar, judgments, theorems, etc.)
- Good tool to **start with** for learning about proof assistants because of language simplicity and less syntactic sugar (my opinion).
- Language support (HOAS) for **variable binding**
  - Do not need to define substitution and prove substitution lemmas (sometimes).
- Language support for **context-sensitive propositions** (hypothetical judgments).
  - Do not need to define context of judgments and related lemmas (e.g. weakening) (sometimes).

My Review of Twelf: The Bad

- Language sometimes too simple
  - Missing support for frequent use-cases
    (e.g., nested case analysis, no case-split construct).

- Less verbosity can lead to cryptic code:
  Intent and meaning of code not clear without significant
  background:
  - No text suggesting this is a proof case:

    ```
    - : preservation (of/len _) (step/lenV _) of/nat.
    ```

  - No text suggesting this checks a proof of a theorem:

    ```
    %worlds () (preservation _ _ _).
    %total E-T (preservation E-T _ _).
    ```

- Error messages could be improved
  (e.g. missing cases messages).
  - Type annotations of function applications and defined names
    desired.

My Review of Twelf: The Bad (cont.)

- No support for stepping through proof instead of just reading proof trees.
- Lack of automation: Many proofs require manual specification (e.g. proofs requiring nested case analysis).
- No libraries.
  - No standard library
  - No import statements – All code must be included (repeat definition of `nat` for every project using them)
- No polymorphism
  - Separate definitions for (`int_list`), (`str_list`), etc.
  - Each type needs its own definition of equality.
- Many contexts require explicit definition (HOAS not always sufficient).

Summary

- Twelf is a proof assistant tool for checking and deriving proofs of properties of languages and deductive logics.
- A tool for language design and implementation.
- Imposes healthy reality and sanity check on language designs.
- Exposes, and helps correct, subtle design errors early in the process.