CHAPTER 5

Integration

Integration sounds lofty and grand,
Yet it's just adding, hand over hand.
Cloaked in fine symbols, solemn, sublime -
Still it's just summing, one step at a time!

Integration, in mathematics, is the technique of finding a function $g(x)$ the derivative of which is equal to a given function $f(x)$. This is indicated by the integral sign $\int$ as in $\int f(x)dx$ and is called the indefinite integral of the function. The symbol dx represents an infinitesimal displacement along x. Hence, $\int f(x)dx$ is the summation of the product of $f(x)$ and dx. The definite integral, written as $\int_a^b f(x)dx$ where a and b are called the limits of integration, is equal to $g(b) - g(a)$, where $\frac{d}{dx}g(x) = f(x)$.

1 Integral

Given a function $f(x)$, an ***ANTI-DERIVATIVE*** of $f(x)$ is any function $g(x)$ such that $g'(x) = f(x)$. The most general anti-derivative is called the ***INDEFINITE INTEGRAL***.

$$\boxed{\int f(x)dx = g(x) + c \text{ where c is a constant of integration}}$$

Note the following inequalities.

$$\int f(x)g(x)dx \neq \int f(x)dx \int g(x)dx$$

$$\int \frac{f(x)}{g(x)}dx \neq \frac{\int f(x)dx}{\int g(x)dx}$$

2 Common Integrals

$$\int x^n dx = \frac{x^{(n+1)}}{(n+1)} + c \tag{2.1}$$

$$\int e^x dx = e^x + c \tag{2.2}$$

$$\int a^x dx = \frac{a^x}{ln\ a} + c \tag{2.3}$$

$$\int \frac{1}{x} dx = ln|x| + c \tag{2.4}$$

$$\int cos(x)dx = sin(x) + c \tag{2.5}$$

3 Substitution Technique

$$\int 18x^2 \sqrt[4]{(6x^3+5)}dx$$

Let $u = 6x^3 + 5$

$$\rightarrow du = 18x^2 dx$$

$$\rightarrow \int \sqrt[4]{u}\, du = \frac{u^{\left(\frac{1}{4}+1\right)}}{\frac{1}{4}+1} = \frac{4}{5}u^{\frac{5}{4}} = \frac{4}{5}\left(6x^3+5\right)^{\frac{5}{4}}$$

4 Integration by Parts

$$[f(x)g(x)]' = f(x)g'(x) + f'(x)g(x)$$

$$f(x)g'(x) = [f(x)g(x)]' - f'(x)g(x)$$

$$\int f(x)g'(x)\,dx = \int [f(x)g(x)]'\,dx - \int f'(x)g(x)\,dx$$

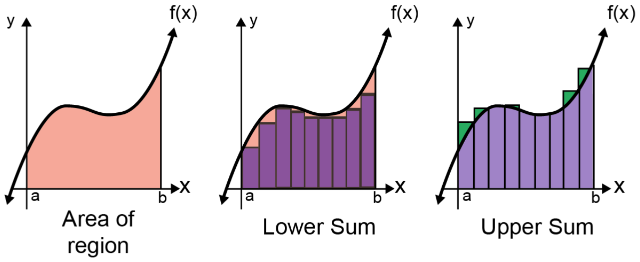
$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$$\boxed{\int u\,dv = uv - \int v\,du} \tag{4.1}$$

Hence, integral of two functions = first function × integral of second function − integral of (differntiation of the first function × integral of the second function).

5 Definite Integral

A definite integral is a the area under its curve.



$$\boxed{\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = g(x)\Big|_a^b = g(b) - g(a)}$$

where $f(x_i^*)$ is the value at the middle of the strip Δx.

$$\boxed{f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx}$$

$$\boxed{\int_a^b f(x)dx = f(c)(b-a) \text{ where c is in [a,b]}}$$

6 Some Integration Strategies

1. Simplify the integrand. E.g., $cos^2(\theta) = \frac{1}{2}(1 + cos(2\theta))$
2. Check if simple substitution will work
3. If integrand is a rational expression, partial functions may work
4. If integrand is polynomial x, $trig$, exp, ln function, integration by parts may work
5. If integrand involves $\sqrt{b^2x^2 + a^2}$, trigonometric substitution may work
6. If integrand has a quadratic in it, completing the square may work.

7 SymPy

Integrate the following functions:

$$\int (x^2 + x + 1)\, dx$$

$$\int e^{-x^2}\,\mathrm{erf}(x) dx \text{where} \,\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

$$\int y = e^x\, dx$$

```python
import sympy as sp
import numpy as np
import matplotlib.pyplot as plt
from IPython.display import display, Math
from sympy import sqrt, diff, integrate, oo
from sympy import sin, cos, tan, ln, exp, erf, trigsimp, expand_trig, simplify
from sympy import sinh, cosh

x = sp.symbols('x')

y = x**2 + x + 1
int_expr = integrate(y,x) # integrate y wrt x
print(sp.latex(int_expr))
display(int_expr)

y = exp(-x**2)*erf(x)
int_expr = integrate(y,x)           # integrate y wrt x
print(sp.latex(int_expr))
display(int_expr)

y = exp(-x)
int_expr = integrate(y, (x, 0, oo))     # definite integral, limits 0 & infinity
print(sp.latex(int_expr))
display(int_expr)
```

$\frac{x^3}{3} + \frac{x^2}{2} + x$

$\frac{\sqrt{\pi}\,\mathrm{erf}^2(x)}{4}$

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