# Chapter 8

# Higher Order Ordinary Differential Equation

*A higher order whispers through each change,*
*Derivatives weaving patterns wide and strange.*
*Roots shape motions—steady, wild, or deep—*
*In layered laws, the hidden forces sleep.*

## 8.1 Higher Order Homogeneous ODE

The concepts of second–order linear ODEs extend naturally to $n$th–order linear differential equations. A general $n$th–order linear ODE has the form

$$\boxed{y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x)}$$

When the coefficients are constant, a trial solution of the form $y = e^{\lambda x}$ leads to the **characteristic equation**:

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

If the characteristic equation has $n$ distinct roots $\lambda_1, \dots, \lambda_n$, then the general solution is

$$\boxed{y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_n e^{\lambda_n x}}$$

### The Wronskian

The Wronskian of the set $\{y_1, \dots, y_n\}$ is

$$W(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

If $y_k = e^{\lambda_k x}$, then

$$W(y_1, \dots, y_n) = E \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{vmatrix}$$

where

$$E = e^{(\lambda_1 + \cdots + \lambda_n)x}.$$

The remaining determinant is the **Vandermonde determinant**, which is nonzero exactly when all roots $\lambda_k$ are distinct.

Thus, $W \neq 0$ if and only if the exponential solutions $e^{\lambda_1 x}, \dots, e^{\lambda_n x}$ are linearly independent.

### Repeated Real Roots

If $\lambda$ is a real root of multiplicity $m$, the linearly independent solutions are

$$\boxed{e^{\lambda x},\ xe^{\lambda x},\ x^2 e^{\lambda x},\ \dots,\ x^{m-1} e^{\lambda x}}$$

### Complex Roots

If $\lambda = \gamma \pm i\omega$ is a complex conjugate pair, then the real solutions are

$$y_1 = e^{\gamma x} \cos(\omega x), \qquad y_2 = e^{\gamma x} \sin(\omega x)$$

If the root is repeated (multiplicity $m$), the corresponding solutions are

$$e^{\gamma x}(\cos \omega x,\ \sin \omega x,\ x \cos \omega x,\ x \sin \omega x,\ \dots)$$

For instance, a complex double root gives

$$y(x) = e^{\gamma x} \boxed{[(A_1 + A_2 x) \cos \omega x + (B_1 + B_2 x) \sin \omega x]}$$

## 8.2 Higher Order Non-Homogeneous ODE

### 8.2.1 Method of Undetermined Coefficients

For constant–coefficient ODEs of the form

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = r(x),$$

we choose a trial $y_p(x)$ that resembles $r(x)$.

If the trial function duplicates a homogeneous solution, multiply by $x^k$ where $k$ is the smallest integer making the result independent.

Examples of modified trial functions:

$$cxe^{\lambda x}, \quad cx^2e^{\lambda x}, \quad \dots$$

### 8.2.2 Method of Variation of Parameters

Consider

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = r(x)$$

Let $y_1, \dots, y_n$ be a fundamental set of solutions of the associated homogeneous equation. A particular solution is given by the general variation–of–parameters formula:

$$\boxed{y_p(x) = \sum_{k=1}^{n}(-1)^{k+1}y_k(x)\int \frac{W_k(x)}{W(x)}r(x)\,dx}$$

where $W(x)$ is the Wronskian and $W_k(x)$ is obtained from $W$ by replacing the $k$th column with $(0, \dots, 0, 1)^T$.

This is a direct generalization of the second–order formula.

## 8.3 Series Solutions of Homogeneous ODEs

ODEs with variable coefficients generally have nonelementary solutions such as Bessel and Legendre functions. These are obtained through power series.

A general power series is written as:

$$\boxed{y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots}$$

Differentiating, substituting into the ODE, and equating coefficients yields recurrence relations for $a_n$.

## 8.4 Existence of Power Series Solutions

Consider:

$$\boxed{y'' + p(x)y' + q(x)y = r(x)}$$

If $p, q, r$ are analytic at $x_0$, then the ODE admits a convergent power series solution around $x_0$.

## 8.5 Classical Differential Equations

Legendre: $\boxed{(1-x^2)y'' - 2xy' + k(k+1)y = 0}$

Chebyshev: $\boxed{(1-x^2)y'' - xy' + k^2y = 0}$

Hermite: $\boxed{y'' - 2xy' + 2ky = 0}$

Laguerre: $\boxed{xy'' + (1-x)y' + ky = 0}$

where $k$ is a constant

## 8.6 Legendre's Equation

$$\boxed{(1-x^2)y'' - 2xy' + k(k+1)y = 0}$$

Let

$$\boxed{y = \sum_{n=0}^{\infty} a_n x^n}$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting into the ODE and simplifying yields the recurrence:

$$\boxed{a_{m+2} = -\frac{(k-m)(k+m+1)}{(m+1)(m+2)} a_m}$$

This relation splits even and odd coefficients, producing two independent solutions $y_1$ and $y_2$, one even and one odd.

The general solution is

$$y(x) = a_0\, y_1(x) + a_1\, y_2(x)$$

### 8.6.1 Legendre Polynomials

When $k$ is a nonnegative integer, the series terminates because the factor $(k-m)$ eventually becomes zero. Thus, for integer $k$, the Legendre solution becomes a polynomial $P_k(x)$.

The coefficients satisfy

$$\boxed{a_{k-2m} = (-1)^m \frac{(2k-2m)!}{2^k\, m!\,(k-m)!\,(k-2m)!}}$$

Thus

$$\boxed{P_k(x) = \sum_{m=0}^{\lfloor k/2 \rfloor} (-1)^m \frac{(2k-2m)!}{2^k m!(k-m)!(k-2m)!} x^{k-2m}}$$

## 8.7 Frobenius Method

The Frobenius method applies to ODEs with a regular singular point, such as

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

Seek a solution of the form

$$\boxed{y = x^r \sum_{m=0}^{\infty} a_m x^m}$$

The indicial equation becomes

$$\boxed{r^2 + (b_0 - 1)r + c_0 = 0}$$

Let the roots be $r_1$ and $r_2$.

Depending on the relationship between $r_1$ and $r_2$, the Frobenius method gives solutions involving power series or logarithmic terms.

## 8.8 Bessel's Equation

$$\boxed{x^2 y'' + x y' + (x^2 - \nu^2) y = 0}$$

Seeking a Frobenius solution

$$\boxed{y = \sum_{m=0}^{\infty} a_m x^{m+r}}$$

yields an indicial equation

$$r = \pm\nu.$$

### 8.8.1 Solution for $r = \nu$

Seek a Frobenius solution of the form

$$\boxed{y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}}, \qquad a_0 \neq 0,$$

and substitute into Bessel's equation

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0.$$

Differentiating termwise gives

$$y' = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1}, \qquad y'' = \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-2}.$$

Substituting and collecting like powers of $x$ yields

$$\sum_{m=0}^{\infty} \Big( \big[ (m+r)^2 - \nu^2 \big] a_m + a_{m-2} \Big) x^{m+r} = 0,$$

with the convention $a_{-1} = a_{-2} = \cdots = 0$. The lowest power ($m = 0$) gives the indicial equation

$$\big[ r^2 - \nu^2 \big] a_0 = 0,$$

so

$$\boxed{r = \pm \nu}.$$

Now take $r = \nu$. The general recurrence (valid for $m \geq 2$) is

$$\boxed{a_m = -\frac{a_{m-2}}{(m+\nu)^2 - \nu^2}}$$

which simplifies to

$$a_m = -\frac{a_{m-2}}{m(m+2\nu)}, \qquad m \geq 2.$$

Examine the $m = 1$ equation (using $a_{-1} = 0$):

$$\big( (1+\nu)^2 - \nu^2 \big) a_1 = (2\nu + 1) a_1 = 0,$$

hence $a_1 = 0$ (since $2\nu + 1 \neq 0$ for all $\nu$). By the recurrence, all odd coefficients vanish:

$$a_1 = a_3 = a_5 = \cdots = 0.$$

Set $m = 2k$. Then for $k \geq 1$,

$$a_{2k} = -\frac{a_{2k-2}}{2k(2k+2\nu)}.$$

Iterating and simplifying yields the closed form (for $k \geq 0$)

$$\boxed{a_{2k} = \frac{(-1)^k a_0\,\Gamma(\nu+1)}{2^{2k} k!\,\Gamma(\nu+k+1)}}$$

Choosing the normalization

$$a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}$$

gives the standard series for the Bessel function of the first kind:

$$\boxed{J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k+\nu}}$$

This derivation explains why, for $r = \nu$, the Frobenius solution produces the even-power series (in $x^\nu$) that defines $J_\nu(x)$.

### 8.8.2 Solution for $r = -\nu$

Similarly one obtains the second Frobenius solution $J_{-\nu}(x)$.

Together they yield the general solution for noninteger $\nu$:

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$$

For integer order, the second solution requires a logarithmic term and produces the Neumann function $Y_\nu(x)$.

## 8.9 Bessel Functions of the Second Kind

For $\nu = 0$,

$$Y_0(x) = \frac{2}{\pi}\left[J_0(x)\left(\ln\frac{x}{2} + \gamma\right) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} h_m}{2^{2m}(m!)^2} x^{2m}\right]$$

Here $\gamma$ is the Euler–Mascheroni constant.

## 8.10 General Solution

For noninteger $\nu$, the general solution of Bessel's equation is

$$y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x)$$

For integer $\nu = n$,

$$y(x) = C_1 J_n(x) + C_2 Y_n(x)$$

## 8.11 SymPy

```python
import sympy as sp

x, nu = sp.symbols('x nu')
y = sp.Function('y')

ode_general = sp.Eq(x**2*sp.diff(y(x), x, 2)
                + x*sp.diff(y(x), x)
                + (x**2 - nu**2)*y(x), 0)

sol_general = sp.dsolve(ode_general)
sol_general
```

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$$

$$\frac{\left(Y_{2}\left(1\right) - Y_{0}\left(1\right)\right) J_{1}\left(x\right)}{J_{1}\left(1\right) Y_{2}\left(1\right) + J_{0}\left(1\right) Y_{1}\left(1\right) - J_{1}\left(1\right) Y_{0}\left(1\right) - J_{2}\left(1\right) Y_{1}\left(1\right)} + \frac{\left(- J_{2}\left(1\right) + J_{0}\left(1\right)\right) Y_{1}\left(x\right)}{J_{1}\left(1\right) Y_{2}\left(1\right) + J_{0}\left(1\right) Y_{1}\left(1\right) - J_{1}\left(1\right) Y_{0}\left(1\right) - J_{2}\left(1\right) Y_{1}\left(1\right)}$$