

# INTEGRATION

*Integration seems fancy and grand,  
But it's just adding, hand over hand.  
Dress it in symbols, make it look prime,  
Yet it's summing, just one at a time!*

Integration, in mathematics, is the technique of finding a function  $g(x)$  the derivative of which is equal to a given function  $f(x)$ . This is indicated by the integral sign  $\int$  as in  $\int f(x)dx$  and is called the indefinite integral of the function. The symbol  $dx$  represents an infinitesimal displacement along  $x$ . Hence,  $\int f(x)dx$  is the summation of the product of  $f(x)$  and  $dx$ . The definite integral, written as  $\int_a^b f(x)dx$  where  $a$  and  $b$  are called the limits of integration, is equal to  $g(b) - g(a)$ , where  $\frac{d}{dx}g(x) = f(x)$ .

## 1.1 INTEGRAL

Given a function  $f(x)$ , an **anti-derivative** of  $f(x)$  is any function  $g(x)$  such that  $g'(x) = f(x)$ . The most general anti-derivative is called the **indefinite integral**.

$$\int f(x)dx = g(x) + c \text{ where } c \text{ is a constant of integration}$$

Note the following inequalities.

$$\int f(x)g(x)dx \neq \int f(x)dx \int g(x)dx$$

$$\int \frac{f(x)}{g(x)}dx \neq \frac{\int f(x)dx}{\int g(x)dx}$$

## 1.2 COMMON INTEGRALS

$$\int x^n dx = \frac{x^{(n+1)}}{(n+1)} + c \quad (1.2.1)$$

$$\int e^x dx = e^x + c \quad (1.2.2)$$

$$\int a^x dx = \frac{a^x}{\ln a} + c \quad (1.2.3)$$

$$\int \frac{1}{x} dx = \ln|x| + c \quad (1.2.4)$$

$$\int \cos(x)dx = \sin(x) + c \quad (1.2.5)$$

### 1.3 SUBSTITUTION TECHNIQUE

$$\int 18x^2 \sqrt[4]{(6x^3 + 5)} dx$$

$$\text{Let } u = 6x^3 + 5$$

$$\rightarrow du = 18x^2 dx$$

$$\rightarrow \int \sqrt[4]{u} du = \frac{u^{(\frac{1}{4}+1)}}{\frac{1}{4}+1} = \frac{4}{5} u^{\frac{5}{4}} = \frac{4}{5} (6x^3 + 5)^{\frac{5}{4}}$$

### 1.4 INTEGRATION BY PARTS

$$[f(x)g(x)]' = f(x)g'(x) + f'(x)g(x)$$

$$f(x)g'(x) = [f(x)g(x)]' - f'(x)g(x)$$

$$\int f(x)g'(x)dx = \int [f(x)g(x)]' dx - \int f'(x)g(x)dx$$

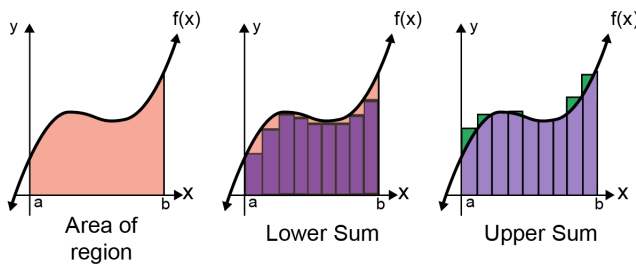
$$a \int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

$$\int u(x)v(x)dx = u(x) \int v(x) - \int [u'(x) \int v(x)]dx \quad (1.4.1)$$

Hence, integral of two functions = first function  $\times$  integral of second function – integral of ( differentiation of the first function  $\times$  integral of the second function ).

### 1.5 DEFINITE INTEGRAL

A definite integral is a the area under its curve .



$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x = g(x) \Big|_a^b = g(b) - g(a)$$

where  $f(x_i^*)$  is the value at the middle of the strip  $\Delta x$ .

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x)dx$$

$$\int_a^b f(x)dx = f(c)(b-a) \text{ where } c \text{ is in } [a,b]$$

## 1.6 SOME INTEGRATION STRATEGIES

- Simplify the integrand. E.g.,  $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$
- Check if simple substitution will work
- If integrand is a rational expression, partial functions may work
- If integrand is polynomial  $x$ ,  $\text{trig}$ ,  $\text{exp}$ ,  $\ln$  function, integration by parts may work
- If integrand involves  $\sqrt{b^2x^2 + a^2}$ , trigonometric substitution may work
- If integrand has a quadratic in it, completing the square may work.

## 1.7 SYMPY

Integration the following functions:

$$\int (x^2 + x + 1) dx$$

Import Python packages

```
1 import sympy as sp
2 import numpy as np
3 import matplotlib.pyplot as plt
4 from IPython.display import display, Math
5 from sympy import sqrt, diff, integrate, oo
6 from sympy import sin, cos, tan, ln, exp, erf, trigsimp, expand_trig, simplify
7 from sympy import sinh, cosh
```

```
1 y = x**2 + x + 1
2 int = integrate(y,x) # integrate y wrt x
3 display(int)
```

$$\frac{x^3}{3} + \frac{x^2}{2} + x + C$$

The **error function** is defined as:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$\int e^{-x^2} \text{erf}(x) dx$$

```
1 y = exp(-x**2)*erf(x)
2 int = integrate(y,x) # integrate y wrt x
3 display(int)
```

$$\frac{\sqrt{\pi} \text{erf}^2(x)}{4}$$

$$\int_0^\infty e^{-x} dx$$

```
1 y = exp(-x)
2 int = integrate(y, (x, 0, oo)) # definite integral, limits 0 & infinity
```

```
3 display(int)
```

```
1
```