CHAPTER 8

Higher Order Ordinary Differential Equation

A higher order whispers through each change,
Derivatives weaving patterns wide and strange.
Roots shape motions—steady, wild, or deep—
In layered laws, the hidden forces sleep.

8.1 Higher Order Homogeneous ODE

The concepts of the 2nd Order ODE can be extended to higher order ODE which has the form:

$$\boxed{y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = r(x)}$$

For constant coefficients, $y = e^{\lambda x}$ yields:
$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0 \quad$ (characteristic equation)

For n distinct roots, there are n distinct basis solutions:

$$\boxed{y = c_1 e^{\lambda^1 x} + c_2 e^{\lambda^2 x} + \ldots + c_n e^{\lambda^n x}}$$

The Wronkskian is given by:

$$W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ . & . & \ldots & . \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix} = E \begin{vmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_n^2 \\ . & . & \ldots & . \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \ldots & \lambda_n^{n-1} \end{vmatrix}$$

Where $E = e^{(\lambda_1 + \lambda_2 + \ldots + \lambda_n)x}$

The determinant is known as the Vandermonde or Cauchy determinant. $W \neq 0$, if and only if, all the n roots are different.

If λ is a real root of order m, i.e., a real root of multiplicity m, the corresponding solutions are:

$$\boxed{e^{\lambda x}, xe^{\lambda x}, x^2 e^{\lambda x}, \ldots, x^{m-1} e^{\lambda x}}$$

Complex roots occur in conjugate pairs $\lambda = \gamma \pm iw$ since the coefficients of the ODE are real.

$$y_1 = e^{\gamma x}\cos(wx), \qquad y_2 = e^{\gamma x}\sin(wx).$$

If $\lambda = \gamma + iw$ is a complex double root (and hence $\gamma - iw$ also), then the corresponding linearly independent solutions are: $e^{\gamma x}\cos(wx), e^{\gamma x}\sin(wx), xe^{\gamma x}\cos(wx), xe^{\gamma x}\sin(wx)$. The corresponding general solution is: $y = e^{\gamma x}\boxed{[(A_1 + A_2 x)\cos(wx) + (B_1 + B_2 x)\sin(wx)]}$

For complex triple roots, one would obtain two more solutions: $\boxed{x^2 e^{\gamma x} cos\, wx}$ $\boxed{x^2 e^{\gamma x} sin\, wx}$

8.2 Higher Order Non-Homogeneous ODE

8.2.1 Method of Undetermined Coefficients

Use the method of undetermined coefficients with a small adjustment. If a term you would normally choose for $y_p(x)$ is already a solution of the homogeneous equation, multiply it by x^k, where k is the smallest positive integer that makes the new term no longer a solution of the homogeneous equation.

In practice, try:
$cxe^{\lambda x}, \quad cx^2e^{\lambda x}, \quad \ldots, \quad cx^ke^{\lambda x},$
substitute into the ODE, and solve for c using the smallest k that works.

8.2.2 Method of Variation of Parameters

Consider the nth–order linear ODE in normalized form

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = r(x),$$

and let $y_1(x), \ldots, y_n(x)$ be a fundamental set of solutions of the corresponding homogeneous equation. Let $W(x)$ denote their Wronskian.

To find a particular solution of the nonhomogeneous equation, we replace the constants in the homogeneous solution by functions and obtain the general variation-of-parameters formula:

$$\boxed{y_p(x) = \sum_{k=1}^{n} (-1)^{k+1} y_k(x) \int \frac{W_k(x)}{W(x)} r(x)\, dx}$$

where $W_k(x)$ is the determinant obtained from the Wronskian $W(x)$ by replacing its kth column with the vector $(0, 0, \ldots, 0, 1)^T$.

In this construction,

- $W(x)$ ensures linear independence of the fundamental solutions;
- $W_k(x)$ comes from solving the system for the parameter derivatives using Cramer's rule;
- the alternating sign $(-1)^{k+1}$ reflects the cofactor expansion used in that determinant calculation.

Thus the formula generalizes the familiar second–order version to any order n, providing a systematic way to compute a particular solution once the homogeneous solutions are known.

8.3 Series Solutions of Homogeneous ODEs

Higher order linear ODEs with constant coefficients can be solved by algebraic methods as their solutions are often elementary functions which are known from calculus. For ODEs with variable coefficients the situation is complicated and their solutions are nonelementary special functions, e.g., Legendre and Bessel functions.

8.3.1 Power Series Method

$$\boxed{y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3^3 + \ldots}$$

Compute $y^{'}, y^{''}, \ldots, y^{(n)}$, substitute in the ODE and compute the coefficients of the powers of $x, x^2, x^3, \ldots, x^n$. Equate each of the coefficients to 0 to determine $a_0, a_1, a_2, \ldots, a_n$.

8.4 Existence of Power Series Solutions

Consider the following ODE:

$$\boxed{y'' + p(x)y' + q(x)y = r(x)}$$

If p, q, r have Taylor series representations (analytic) then every solution of the ODE can be represented by a power series in powers of $x - x_0$ with a positive radius of convergence R. A power series can be added, multiplied and differentiated term by term.

8.5 Classical Differential Equations

Legendre: $\boxed{(1-x^2)y^{''} - 2xy^{'} + k(k+1)y = 0}$

Chebyshev: $\boxed{(1-x^2)y'' - xy' + k^2 y = 0}$

Herimite: $\boxed{y'' - 2xy' + 2ky = 0}$

Laguerre: $\boxed{xy'' + (1-x)y' + ky = 0}$

where k is a constant

8.6 Legendre's Equation

$\boxed{(1-x^2)y^{''} - 2xy^{'} + k(k+1)y = 0}$ k is a constant

Let $\boxed{y = \sum_{n=0}^{\infty} a_n x^n}$ and compute $y, y^{'}, y^{''}$ to substitute in the above equation.

$$y^{'} = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y^{''} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + k(k+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

Since $n(n-1)$ is 0 for $n = 0$ and $n = 1$, the lower indices start from 2 and 1.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{\infty} n a_n x^n + k(k+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

Let $n - 2 = m$ and use m as the index in the remaining terms as it is a dummy index:

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m - \sum_{m=2}^{\infty} m(m-1)a_m x^m - 2\sum_{m=1}^{\infty} m a_m x^m + k(k+1)\sum_{m=0}^{\infty} a_m x^m = 0$$

a_0 and a_1 are arbitrary constants, the remaining constants are expressed in terms of these.

For $m = 0$,

$$2a_2 + k(k+1)a_0 = 0$$

$$\boxed{a_2 = -\frac{k(k+1)}{2!}a_0}$$

For $m = 1$,

$$6a_3 + [-2 + k(k+1)]a_1 = 0$$

$$\boxed{a_3 = -\frac{(k-1)(k+2)}{3!}a_1}$$

For $m \geq 2$,

$$(m+2)(m+1)a_{m+2} = [m(m-1) + 2m - k(k+1)]a_m = (m^2 + m - k^2 - k)a_m$$

$$\boxed{a_{m+2} = -\frac{(k-m)(k+m+1)}{(m+1)(m+2)}a_m} \quad \text{for } m = 0, 1, 2, \ldots$$

Notice that the recurrence relation separates the coefficients into two independent groups: all *even* coefficients depend only on even ones, and all *odd* coefficients depend only on odd ones. Thus the full power series naturally splits into two independent series.

Independence of the solutions. A second-order linear ODE admits exactly two linearly independent solutions. Setting the initial data $(a_0 = 1, a_1 = 0)$ produces the even solution $y_1(x)$, while $(a_0 = 0, a_1 = 1)$ produces the odd solution $y_2(x)$. Even and odd functions cannot be constant multiples of one another, so these two solutions are necessarily independent.

Even-power series. Starting with a_0, the recurrence generates only even coefficients:

$$y_1(x) = 1 + a_2x^2 + a_4x^4 + \cdots$$

This series contains exclusively even powers of x and forms one solution of Legendre's equation.

Odd-power series. Starting with a_1, the recurrence generates only odd coefficients:

$$y_2(x) = x + a_3x^3 + a_5x^5 + \cdots$$

This series contains exclusively odd powers of x and forms the second, linearly independent solution.

General solution. Any solution of the Legendre equation can therefore be expressed as a linear combination of these two fundamental series:

$$y(x) = a_0\, y_1(x) + a_1\, y_2(x)$$

where a_0 and a_1 are arbitrary constants determined by boundary conditions.

8.6.1 Legendre Polynomials

When k is a nonnegative integer, the recurrence relation

$$a_{m+2} = -\frac{(k-m)(k+m+1)}{(m+1)(m+2)}a_m$$

eventually produces a factor $(k-m)$ in the numerator. Once $m = k$, this factor becomes zero, so:

$$a_{k+2} = a_{k+4} = a_{k+6} = \cdots = 0$$

This means the power series stops after finitely many terms — it becomes a polynomial. If k is even, the even series $y_1(x)$ terminates and becomes a polynomial of degree k. If k is odd, the odd series $y_2(x)$ terminates and becomes a polynomial of degree k.

These finite series are the *Legendre polynomials*, denoted by $P_k(x)$. Because they are polynomials, they are valid for all x (no convergence issues).

A common normalization is to choose the leading coefficient (the coefficient of x^k) as

$$a_k = \frac{(2k)!}{2^k (k!)^2}$$

To find the remaining coefficients, we use the recurrence in reverse:

$$a_m = -\frac{(m+1)(m+2)}{(k-m)(k+m+1)} a_{m+2} \qquad m < k$$

Example 1: $m = k-2$

$$a_{k-2} = -\frac{k(k-1)}{2(2k-1)} a_k = \frac{(2k-2)!}{2^k (k-1)!(k-2)!}$$

Example 2: $m = k-4$

$$a_{k-4} = \frac{(k-2)(k-3)}{4(2k-3)} a_{k-2} = \frac{(2k-4)!}{2^k\, 2!\,(k-2)!(k-4)!}$$

In general, the coefficients are

$$a_{k-2m} = (-1)^m \frac{(2k-2m)!}{2^k\, m!\,(k-m)!\,(k-2m)!} \qquad m = 0, 1, 2, \ldots, \left\lfloor \frac{k}{2} \right\rfloor .$$

Thus the Legendre polynomial can be written as

$$\boxed{P_k(x) = \sum_{m=0}^{\lfloor k/2 \rfloor} (-1)^m \frac{(2k-2m)!}{2^k\, m!\,(k-m)!\,(k-2m)!} x^{k-2m}} \quad \lfloor k/2 \rfloor \text{ is floor of } k/2$$

8.7 Frobenius Method

Several important 2nd order ODEs have coefficients that are not analytic. Yet these ODEs can be solved through an extension of the power series method that is credited to Frobenius. Consider the ODE:

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0 \quad \text{Note: } b(x), c(x) \text{ are analytic at } x = 0$$

This ODE has at least one solution of the form:

$$\boxed{y(x) = x^r \sum_{m=0}^{\infty} a_m x^m}$$

Where r is real or complex and $a_0 \neq 0$.

Multiply the ODE by x^2 and expand b(x) and c(x) in Taylor series.

$x^2y'' + xb(x)y' + c(x)y = 0$

$b(x) = \sum_{m=0}^{\infty} b_m x^m \quad c(x) = \sum_{m=0}^{\infty} c_m x^m$

$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m \quad y'(x) = \sum_{m=0}^{\infty} (m+r)a_m x^{m+r-1} \quad y''(x) = \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r-2}$

Substituting in the ODE,

$$x^r[r(r-1)a_0 + \cdots] + (b_0 + b_1x + \cdots)x^r(ra_0 + \cdots) + (c_0 + c_1x + \cdots)x^r(a_0 + a_1x + \cdots) = 0$$

Equate coefficients of x^r, x^{r+1}, x^{r+2} to 0.

$[r(r-1) + b_0r + c_0]a_0 = 0$

$\boxed{[r^2 + (b_0-1)r + c_0] = 0}$ (indicial equation)

The Frobenius method yields a basis of solutions.

Distinct roots not differing by an integer

$y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \ldots)$

$y_2(x) = x^{r_2}(A_0 + A_1x + A_2x^2 + \ldots)$

Double root $r_1 = r_2 = r = \frac{1}{2}(1-b_0)$

$y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \ldots)$

$y_2(x) = y_1(x)ln\,x + x^{r_1}(A_0 + A_1x + A_2x^2 + \ldots)$

Roots differing by an integer

$y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \ldots)$

$y_2(x) = ky_1(x)ln\,x + x^{r_2}(A_0 + A_1x + A_2x^2 + \ldots)$

$r_1 > r_2, k$ can be 0

For cases 2 and 3, the second independent solution can be obtained by reduction of order.

8.8 Bessel's Equation

$\boxed{x^2y'' + xy' + (x^2 - \nu^2)y = 0}$ (ν is a real number ≥ 0)

Applying Frobenius technique, the solution is of the form

$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}$

$y'(x) = \sum_{m=0}^{\infty} (m+r)a_m x^{m+r-1} = x^{r-1}[ra_0 + (r+1)a_1x + (r+2)a_2x^2 + \ldots]$

$y''(x) = \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r-2} = x^{r-2}[r(r-1)a_0 + (r+1)ra_1x + (r+2)(r+1)a_2x^2 + \ldots]$

substituting in the ODE

$\sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r} + \sum_{m=0}^{\infty} (m+r)a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} - \nu^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0$

$r(r-1)a_0 + ra_0 - \nu^2 a_0 = 0 \ (m=0)$
$(r+\nu)(r-\nu) = 0, \implies \boxed{r = \pm\nu}$

$(r+1)ra_1 + (r+1)a_1 - \nu^2 a_1 = 0 \ (m=1)$
$((\nu+1)\nu + (\nu+1) - \nu^2)a_1 = 0 \implies (2\nu+1)a_1 = 0 \implies \boxed{a_1 = 0}$

$(m+r)(m+r-1)a_m + (m+r)a_m + a_{m-2} - \nu^2 a_m = 0 \ (m = 2,3,\ldots)$
$(m+\nu)[(m+\nu-1+(m+\nu)-\nu^2]a_m + a_{m-2} = 0 \implies \boxed{m(m+2\nu)a_m + a_{m-2} = 0}$

since $a_1 = 0 \implies \boxed{a_3 = a_5 = \ldots = 0}$
$2m(2m+2\nu)a_{2m} + a_{2m-2} = 0$ (ensure even numbers only, $m = 1,2,\ldots$)

$a_{2m} = -\frac{a_{2m-2}}{2^2 m(m+\nu)} \ (m = 1,2,\ldots)$
$a_2 = -\frac{a_0}{2^2(\nu+1)}$
$a_4 = -\frac{a_2}{2^2 2(\nu+2)} = \frac{a_0}{2^4 2!(\nu+1)(\nu+2)}$

When ν is an integer, denote it as by n
$a_{2m} = -\frac{(-1)^n a_0}{2^{2m} m!(n+1)(n+2)\ldots(n+m)} \ (m = 1,2,\ldots)$
choose, $a_0 = \frac{1}{2^n n!}$

$$\boxed{a_{2m} = \frac{(-1)^m}{2^{2m+n} m!(n+m)!}} \quad (m = 1,2,\ldots)$$

A particular solution to Bessel's equation is then given by,

$$\boxed{J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}} \quad (m = 1,2,\ldots, \text{and } n \geq 0)$$

$J_n(x)$ is called the Bessel function of the first kind of order n and converges $\forall x$.

For $n = 0$, $J_0(x) = \sum\limits_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m!^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} + \ldots$ (Bessel function of order 0, similar to cosine)
For $n = 1$, $J_1(x) = \sum\limits_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m} m!(m+1)!} = \frac{x}{2} - \frac{x^3}{2^3 1!2!} + \frac{x^5}{2^5 2!3!} + \ldots$ (Bessel function of order 1, similar to sine)

8.8.1 Bessel functions for real number

Choose $a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}$ where the Gamma function is defined as:

$$\boxed{\Gamma(\nu+1) = \int_0^\infty e^{-t} t^\nu dt} \quad (\nu > -1)$$

$$\Gamma(\nu+1) = -e^{-t}t^\nu \Big|_0^\infty + \nu \int_0^\infty e^{-t}t^{\nu-1}dt = 0 + \nu\Gamma(\nu)$$

$\boxed{\Gamma(\nu+1) = \nu\Gamma(\nu)}$ for $n = 0, 1, \ldots$ $\boxed{\Gamma(n+1) = n!}$ (The Gamma function is a generalised factorial)

$$a_{2m} = -\frac{(-1)^m a_0}{2^{2m} m!(\nu+1)(\nu+2)\ldots(\nu+m)2^\nu\Gamma(\nu+1)}$$

$$a_{2m} = -\frac{(-1)^m a_0}{2^{2m+\nu} m!\Gamma(\nu+m+1)}$$

$$\boxed{J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m!\Gamma(\nu+m+1)!}}$$

$J_\nu(x)$ is called the Bessel function of the first kind of order ν

Bessel functions satisfy many relationships such as the following:

$[x^\nu J_\nu(x)]' = x^\nu J_{\nu-1}(x)$	$[x^{-\nu} J_\nu(x)]' = -x^{-\nu} J_{\nu+1}(x)$
$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$	$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_\nu(x)$
$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} sinx$	$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} cosx$

8.8.2 General Solution

For a general solution of Bessel's equation in addition to J_ν we need a second linearly independent solution. If ν is not an integer, the general solution can be obtained by replacing ν with $-\nu$. The general solution is then given by:

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$$

This cannot be the general solution for an integer $\nu = n$ because that will lead to linear dependence.

8.8.3 Bessel functions of the second kind, $Y_\nu(x)$

For $n = 0$, the Bessel function can be written as:

$$xy'' + y' + xy = 0$$

The indical equation has a double root and the desired solution must be of the form:

$$\boxed{y_2(x) = J_0 ln\, x + \sum_{m=1}^{\infty} A_m x^m}$$

$$y_2' = J_0' ln\, x + \frac{J_0}{x} + \sum_{m=1}^{\infty} m A_m x^{m-1}$$

$$y_2'' = J_0'' ln\, x + \frac{2J_0'}{x} - \frac{J_0}{x^2} + \sum_{m=1}^{\infty} m(m-1) A_m x^{m-2}$$

Substituting y_2'', y_2', y in the equation we have:

$$(xJ_0''lnx+2J_0'-\cancel{\frac{J_0}{x}}+\sum_{m=1}^{\infty}m(m-1)A_mx^{m-1})+(J_0'lnx+\cancel{\frac{J_0}{x}}+\sum_{m=1}^{\infty}mA_mx^{m-1})+$$
$$(xJ_0lnx+\sum_{m=1}^{\infty}A_mx^{m+1})=0$$

$$\underbrace{\cancel{(xJ_0''+J_0'+xJ_0)}}_{0}lnx+2J_0'+\sum_{m=1}^{\infty}m(m-1)A_mx^{m-1}+\sum_{m=1}^{\infty}mA_mx^{m-1}+\sum_{m=1}^{\infty}A_mx^{m+1}=0$$

$$2J_0'+\sum_{m=1}^{\infty}m^2A_mx^{m-1}+\sum_{m=1}^{\infty}A_mx^{m+1}=0$$

Now, $J_0(x)=\sum_{m=0}^{\infty}\frac{(-1)^mx^{2m}}{2^{2m}m!^2}$

$$J_0'(x)=\sum_{m=1}^{\infty}\frac{(-1)^m2mx^{2m-1}}{2^{2m}m!^2}=\sum_{m=1}^{\infty}\frac{(-1)^mx^{2m-1}}{2^{2m-1}m!(m-1)!}$$

$$\sum_{m=1}^{\infty}\frac{(-1)^mx^{2m-1}}{2^{2m-2}m!(m-1)!}+\sum_{m=1}^{\infty}m^2A_mx^{m-1}+\sum_{m=1}^{\infty}A_mx^{m+1}=0$$

The power of x_0 occurs only in the 2nd series, hence $\boxed{A_1=0}$.

Comparing coeffient of even powers of x in 2nd & 3rd series (1st series has none), we have:

$(2s+1)^2A_{2s+1}+A_{2s-1}=0$ (where $s=0,1,2,\cdots$)

Since $A_1=0 \implies A_3=A_5=\cdots=0$

$$-1+4A_2=0 \implies \boxed{A_2=\frac{1}{4}}$$

Matching the odd power of x in all 3 series, we have:

$$\frac{(-1)^{s+1}}{2^{2s}(s+1)!s!}+(2s+2)^2A_{2s+2}+A_{2s}=0$$

$$\boxed{A_{2m}=\frac{(-1)^{m-1}}{2^{2m}(m!)^2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right)\quad(m=0,1,2,\cdots)}$$

$$\boxed{y_2(x)=J_0(x)lnx+\frac{(-1)^{m-1}h_m}{2^{2m}(m!)^2}x^{2m}\quad\text{where } h_m=\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}\right)}$$

J_0, y_2 are linearly independent functions (basis for $x>0$), express y_2 as particular solution:x

$\boxed{Y_0(x)=a(y_2+bJ_0)}$ and choose $\boxed{a=\pi/2}$ and $\boxed{b=\gamma-ln2}$

Let, $\boxed{\gamma=\lim_{s\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{s}\right)-ln\,s=0.57721566490}$ (Euler constant)

The standard particular solution thus obtained is called the Bessel function of the second kind of order zero or Neumann's function of order zero and is denoted by $Y_0(x)$.

$$Y_0(x)=\frac{2}{\pi}\left(J_0(x)lnx+\frac{(-1)^{m-1}h_m}{2^{2m}(m!)^2}x^{2m}+(\gamma-ln2)J_0\right)$$

$$\boxed{Y_0(x)=\frac{2}{\pi}\left[J_0(x)\left(ln\frac{x}{2}+\gamma\right)+\frac{(-1)^{m-1}h_m}{2^{2m}(m!)^2}x^{2m}\right]}$$

8.8.4 Bessel functions of the second kind, $Y_n(x)$

For $n = 1, 2, \cdots$ a second solution can be obtained by manipulations similar to those for $n = 0$. It turns out that in these cases the solution also contains a logarithmic term.

Depneding on whether ν is an integer or not, the standard second solution known as the Bessel function of the 2nd kind of order ν or Neumann's function of order ν is given by:

$$\boxed{Y_\nu(x) = \frac{1}{sin\,\nu\pi}\left[J_\nu(x)cos\,\nu\pi - J_{-\nu}(x)\right]}$$

$$\boxed{Y_n(x) = \lim_{\nu\to n} Y_\nu(x)}$$

The general solution of Bessel's equation $\forall x \wedge x > 0$ is given by:

$$\boxed{y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x)}$$

8.9 SymPy

```python
# Solving Bessel's equation with SymPy
# Demonstrates symbolic solution for general order nu, and a concrete example for nu=1.
import sympy as sp

# symbols and function
x, nu = sp.symbols('x nu')
y = sp.Function('y')

# General Bessel's equation: x^2 y'' + x y' + (x^2 - nu^2) y = 0
ode_general = sp.Eq(x**2*sp.diff(y(x), x, 2) + x*sp.diff(y(x), x) + (x**2 - nu**2)*y(x), 0)
print(sp.latex(ode_general))

# Solve symbolically (returns solution in terms of BesselJ and BesselY)
sol_general = sp.dsolve(ode_general)
sol_general_simpl = sp.simplify(sol_general.rhs)  # RHS is the general solution expression

# Concrete example: nu = 1 (order 1 Bessel equation)
ode_nu1 = sp.Eq(x**2*sp.diff(y(x), x, 2) + x*sp.diff(y(x), x) + (x**2 - 1)*y(x), 0)
sol_nu1 = sp.dsolve(ode_nu1)

# Example with initial conditions: y(1)=1, y'(1)=0 for nu=1
ics = {y(1): 1, sp.diff(y(x), x).subs(x, 1): 0}
sol_nu1_ics = sp.dsolve(ode_nu1, ics=ics)

# Show results
print("General solution (order 'nu'):\n", sol_general_simpl, "\n")
print("Solution for nu = 1:\n", sol_nu1.rhs, "\n")
print(sp.latex(sol_nu1.rhs))
print("Solution for nu = 1 with y(1)=1, y'(1)=0:\n", sol_nu1_ics.rhs, "\n")

# Also explicitly show the independent basis functions
C1, C2 = sp.symbols('C1 C2')
basis = sp.Matrix([sp.besselj(nu, x), sp.bessely(nu, x)])
print("Fundamental solutions (Bessel J and Y):\n", basis)

# Return objects for inspection if desired
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37 sol_general, sol_nu1, sol_nu1_ics, basis
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$$x^2 \frac{d^2}{dx^2} y(x) + x \frac{d}{dx} y(x) + \left(-\nu^2 + x^2\right) y(x) = 0$$

$$\frac{\left(Y_2(1) - Y_0(1)\right) J_1(x)}{J_1(1) Y_2(1) + J_0(1) Y_1(1) - J_1(1) Y_0(1) - J_2(1) Y_1(1)} + \frac{\left(-J_2(1) + J_0(1)\right) Y_1(x)}{J_1(1) Y_2(1) + J_0(1) Y_1(1) - J_1(1) Y_0(1) - J_2(1) Y_1(1)}$$