Chapter 13
Fourier Series

From complex waves that rise and fall,
Fourier answers nature's call.
Through sines and cosines, pure and true,
The whole emerges from parts we knew.

13.1 Introduction

Fourier series provide a powerful method for representing periodic functions as infinite sums of sine and cosine functions. This idea, introduced by Joseph Fourier, reveals that even complex and irregular periodic phenomena can be decomposed into simple harmonic components. Such representations form the mathematical foundation for the analysis of waves, vibrations, heat flow, and signal processing.

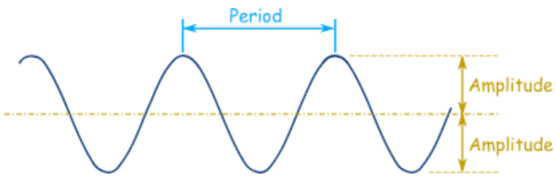
This chapter begins with the definition of periodic functions and the formulation of the Fourier series on a finite interval. The orthogonality of trigonometric functions is established and used to derive explicit expressions for the Fourier coefficients. Conditions for convergence of the Fourier series are discussed, clarifying how and where the series represents the original function.

The theory is then extended to functions with arbitrary periods and to special cases involving even and odd functions, leading to simplified sine and cosine series. The role of Fourier series in function approximation is examined through least-squares error minimization and Bessel's inequality, highlighting their optimality in the mean-square sense.

Finally, the chapter connects Fourier series to broader mathematical frameworks through Sturm–Liouville problems, eigenvalues, eigenfunctions, and generalized Fourier series. Computational aspects are illustrated using symbolic and numerical tools, demonstrating both the theoretical depth and practical utility of Fourier series in engineering and applied mathematics.

13.2 Periodic Functions, Fourier Series

A function $f(x)$ is said to be **periodic** with period $T > 0$ if $f(x+T) = f(x), \quad$ for all x



Let $f(x)$ be defined on the interval $[-L, L]$ and extended periodically outside this interval by

$$f(x + 2L) = f(x)$$

so that f has period $2L$. The **Fourier series** of $f(x)$ is

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)}$$

where the **Fourier coefficients** are:

$$\boxed{a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx}$$

$$\boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx}$$

$$\boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx}$$

13.3 Orthogonality of Trigonometric Systems

When $m \neq n$,

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = 0,$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = 0,$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = 0.$$

Using the product-to-sum identities:

$$\cos nx\cos mx = \tfrac{1}{2}\big[\cos((n-m)x) + \cos((n+m)x)\big],$$

$$\sin nx\sin mx = \tfrac{1}{2}\big[\cos((n-m)x) - \cos((n+m)x)\big],$$

$$\sin nx\cos mx = \tfrac{1}{2}\big[\sin((n+m)x) + \sin((n-m)x)\big],$$

we have, for $m \neq n$,

$$\int_{-L}^{L} \cos nx\cos mx\,dx = \frac{1}{2}\int_{-L}^{L}\cos((n-m)x)\,dx + \frac{1}{2}\int_{-L}^{L}\cos((n+m)x)\,dx,$$

$$\int_{-L}^{L} \sin nx\sin mx\,dx = \frac{1}{2}\int_{-L}^{L}\cos((n-m)x)\,dx - \frac{1}{2}\int_{-L}^{L}\cos((n+m)x)\,dx,$$

$$\int_{-L}^{L} \sin nx\cos mx\,dx = \frac{1}{2}\int_{-L}^{L}\sin((n+m)x)\,dx + \frac{1}{2}\int_{-L}^{L}\sin((n-m)x)\,dx.$$

Each of these integrals evaluates to zero (for appropriate L), so the three orthogonality relations follow.

13.4 Fourier Coefficients

Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

Integrating both sides over $[-L, L]$,

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L} \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \right] dx$$

Using orthogonality of sine and cosine,

$$\int_{-L}^{L} f(x)\,dx = a_0 L$$

hence

$$\boxed{a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\,dx}$$

Now multiply the Fourier series by $\cos \frac{m\pi x}{L}$ and integrate:

$$\int_{-L}^{L} f(x) \cos \frac{m\pi x}{L}\,dx = \int_{-L}^{L} \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \right] \cos \frac{m\pi x}{L}\,dx$$

By orthogonality,

$$\boxed{a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{m\pi x}{L}\,dx.}$$

Similarly, multiplying by $\sin \frac{m\pi x}{L}$ and integrating,

$$\boxed{b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L}\,dx.}$$

The first term on the right involving a_0 vanishes since

$$\int_{-L}^{L} \cos \frac{m\pi x}{L}\,dx = 0.$$

The integral of $a_n \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L}$ is equal to $a_m L$ for $n = m$ and 0 for $n \neq m$. The integral of $b_n \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L}$ is 0 for all m, n.

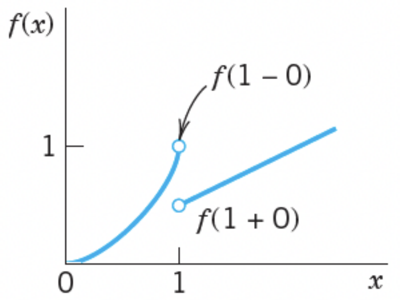
Hence,

$$\int_{-L}^{L} f(x) \cos \frac{m\pi x}{L}\,dx = a_m L \;\Rightarrow\; \boxed{a_m = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{m\pi x}{L}\,dx}$$

$$\int_{-L}^{L} f(x) \sin \frac{m\pi x}{L}\,dx = b_m L \;\Rightarrow\; \boxed{b_m = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L}\,dx}$$

13.5 Convergence

If $f(x)$ is **periodic** and piecewise continuous, and if the left-hand and right-hand derivatives exist at each point of the interval (except possibly at a finite number of points), then the Fourier series of $f(x)$ **converges**. At every point of continuity, the series converges to $f(x)$. At each point of discontinuity, the sum of the series converges to the **average of the left-hand and right-hand limits** of $f(x)$.



13.6 Fourier Series for an Arbitrary Period

Let $f(x)$ be a periodic function with arbitrary period p. Define

$$L = \frac{p}{2}$$

Then $f(x)$ has period $2L$ and its Fourier series representation is

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)}$$

where the Fourier coefficients are

$$\boxed{a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx}$$

$$\boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots}$$

$$\boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots}$$

13.7 Even & Odd Functions

Even Function:

$f(x) = f(-x)$

For an even function, all sine terms vanish, and the Fourier series reduces to a cosine series:

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}}$$

with

$$\boxed{a_0 = \frac{2}{L}\int_{0}^{L} f(x)\,dx}$$

$$\boxed{a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots}$$

Odd Function:
$f(x) = -f(-x)$

For an odd function, all cosine terms vanish (including a_0), and the Fourier series reduces to a sine series:

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}}$$

with

$$\boxed{b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots}$$

13.8 Approximation & Error Minimization

Let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$$

be the Fourier series of $f(x)$. The N-term approximation is:

$$F(x) = \frac{A_0}{2} + \sum_{n=1}^{N}\left(A_n\cos\frac{n\pi x}{L} + B_n\sin\frac{n\pi x}{L}\right)$$

which is chosen to best approximate $f(x)$ in the least–squares sense. Define the mean–square error

$$E = \int_{-L}^{L}[\,f(x) - F(x)\,]^2\,dx$$

Using orthogonality of sine and cosine functions, we obtain

$$\int_{-L}^{L} F^2(x)\,dx = L\left(\frac{A_0^2}{2} + \sum_{n=1}^{N}(A_n^2 + B_n^2)\right),$$

$$\int_{-L}^{L} f(x)F(x)\,dx = L\left(\frac{A_0a_0}{2} + \sum_{n=1}^{N}(A_na_n + B_nb_n)\right).$$

Hence,

$$E = \int_{-L}^{L} f^2(x)\,dx - 2L\left(\frac{A_0a_0}{2} + \sum_{n=1}^{N}(A_na_n + B_nb_n)\right) + L\left(\frac{A_0^2}{2} + \sum_{n=1}^{N}(A_n^2 + B_n^2)\right).$$

Minimizing E with respect to A_n and B_n gives

$$\boxed{A_n = a_n}$$

$$\boxed{B_n = b_n}$$

so the best approximation is obtained by taking the Fourier coefficients themselves. Therefore, the minimum error is:

$$\boxed{E_{\min} = \int_{-L}^{L} f^2(x)\,dx - L\left(\frac{a_0^2}{2} + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right).}$$

$$E - E_{\min} = L\left[\frac{(A_0 - a_0)^2}{2} + \sum_{n=1}^{N}\left\{(A_n - a_n)^2 + (B_n - b_n)^2\right\}\right] \geq 0$$

$$E = E_{\min} \quad \text{if and only if} \quad A_0 = a_0,\ A_n = a_n,\ B_n = b_n$$

The square error of F (with fixed N) relative to f on the interval $-L \leq x \leq L$ is minimum if and only if the coefficients of F are the Fourier coefficients of f. With increasing N, the partial sums of the Fourier series of f yield better and better approximations to f in the mean-square sense.

$$\boxed{\frac{a_0^2}{2} + \sum_{n=1}^{N}(a_n^2 + b_n^2) \leq \frac{1}{L}\int_{-L}^{L} f(x)^2\,dx \quad \text{(Bessel's Inequality)}}$$

13.9 Sturm–Liouville Problems & Orthogonality

The idea of the Fourier series is to represent general periodic functions in terms of cosines and sines. These functions form a trigonometric system which possesses the important property of **orthogonality**. This orthogonality allows us to compute the coefficients of the Fourier series using Euler's formulas. A natural question then arises: can we replace the trigonometric system by other orthogonal systems (sets of orthogonal functions)? The answer is *yes*, and this leads to **generalized Fourier series**, including the Fourier–Legendre series and the Fourier–Bessel series.

A **Sturm–Liouville problem** consists of the differential equation

$$\boxed{\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + \left[q(x) + \lambda r(x)\right]y = 0, \quad a \leq x \leq b,}$$

together with the boundary conditions

$$k_1 y(a) + k_2 y'(a) = 0,$$

$$l_1 y(b) + l_2 y'(b) = 0.$$

If $y_m(x)$ and $y_n(x)$ are eigenfunctions corresponding to distinct eigenvalues $\lambda_m \neq \lambda_n$, then they satisfy the **orthogonality condition**

$$\boxed{(y_m, y_n) = \int_a^b r(x)\, y_m(x)\, y_n(x)\,dx = 0, \quad m \neq n.}$$

The **norm** of an eigenfunction is defined by

$$\boxed{\|y_n\| = \sqrt{(y_n, y_n)} = \sqrt{\int_a^b r(x)\, y_n^2(x)\,dx}.}$$

13.10 Eigenvalues & Eigenfunctions

The eigenfunctions $y_n(x)$ of a Sturm–Liouville problem satisfy the weighted inner product

$$(y_m, y_n) = \int_a^b r(x)\, y_m(x)\, y_n(x)\, dx$$

If the eigenfunctions are normalized, then

$$(y_m, y_n) = \delta_{mn} = \begin{cases} 1, & m = n, \\ 0, & m \neq n \end{cases}$$

If $r(x) = 1$, then

$$(y_m, y_n) = \int_a^b y_m(x)\, y_n(x)\, dx = 0, \quad m \neq n.$$

The norm of an eigenfunction is defined as

$$\boxed{\|y_m\| = \sqrt{(y_m, y_m)} = \sqrt{\int_a^b r(x)\, y_m^2(x)\, dx}}$$

The eigenfunctions satisfy

$$(p y_m')' + (q + \lambda_m r) y_m = 0$$

$$(p y_n')' + (q + \lambda_n r) y_n = 0$$

Multiplying the first equation by y_n, the second by y_m, and subtracting, we obtain

$$(\lambda_m - \lambda_n) r(x) y_m y_n = y_n (p y_m')' - y_m (p y_n')'$$

Integrating from a to b gives

$$(\lambda_m - \lambda_n) \int_a^b r(x)\, y_m y_n\, dx = \left[p(x)\big(y_n y_m' - y_m y_n'\big) \right]_a^b$$

The boundary term on the right-hand side vanishes due to the boundary conditions (separated or periodic). Hence,

$$(\lambda_m - \lambda_n) \int_a^b r(x)\, y_m y_n\, dx = 0$$

For $\lambda_m \neq \lambda_n$, this implies the orthogonality relation

$$\boxed{\int_a^b r(x)\, y_m(x) y_n(x)\, dx = 0, \quad m \neq n}$$

13.11 Generalized Fourier Series

Let $\{y_m(x)\}_{m=0}^{\infty}$ be an orthogonal set of functions on $[a, b]$ with respect to the weight function $r(x)$. Then a function $f(x)$ may be expanded as

$$\boxed{f(x) = \sum_{m=0}^{\infty} a_m y_m(x)}$$

This is called an **orthogonal expansion** or a **generalized Fourier series**. If the functions $y_m(x)$ are eigenfunctions of a Sturm–Liouville problem, the expansion is called an **eigenfunction expansion**. Taking the inner product of both sides with $y_n(x)$, we obtain

$$(f, y_n) = \int_a^b r(x) f(x) y_n(x)\, dx = \int_a^b r(x) \left(\sum_{m=0}^{\infty} a_m y_m(x) \right) y_n(x)\, dx = \sum_{m=0}^{\infty} a_m (y_m, y_n)$$

By orthogonality, all terms vanish except when $m = n$, hence

$$(f, y_n) = a_n (y_n, y_n) = a_n \|y_n\|^2$$

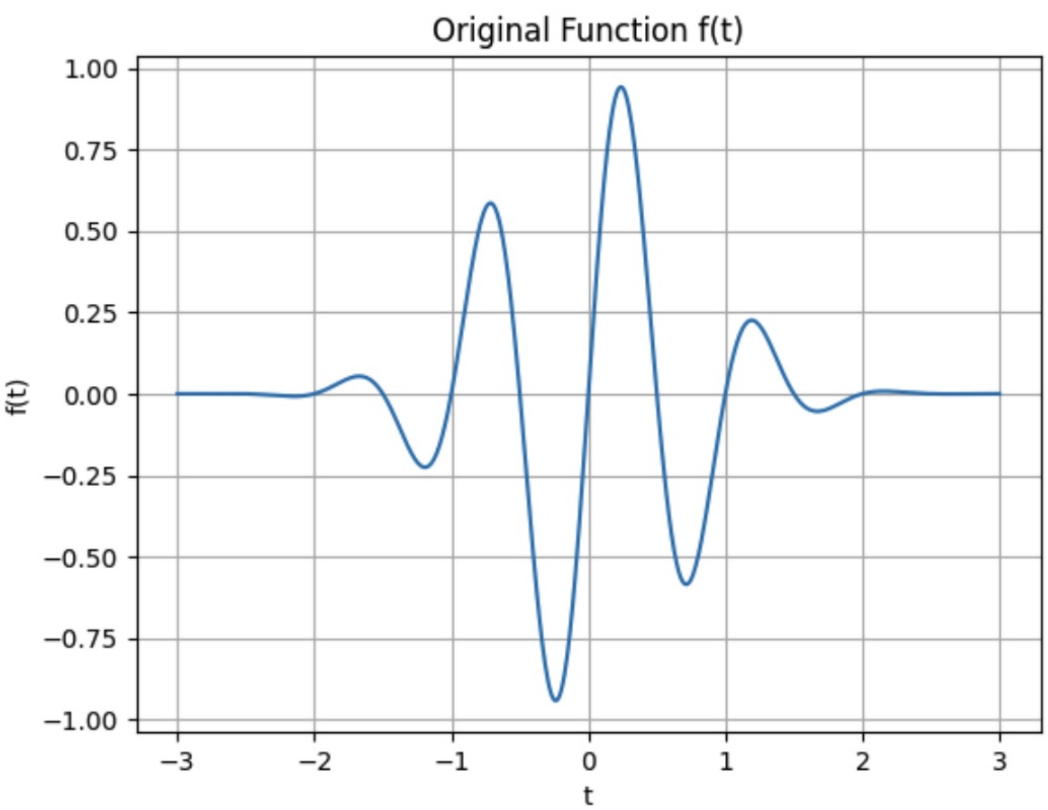
Therefore, the coefficients are given by

$$\boxed{a_n = \frac{(f, y_n)}{\|y_n\|^2} = \frac{1}{\|y_n\|^2} \int_a^b r(x)\, f(x)\, y_n(x)\, dx}$$

13.12 SymPy

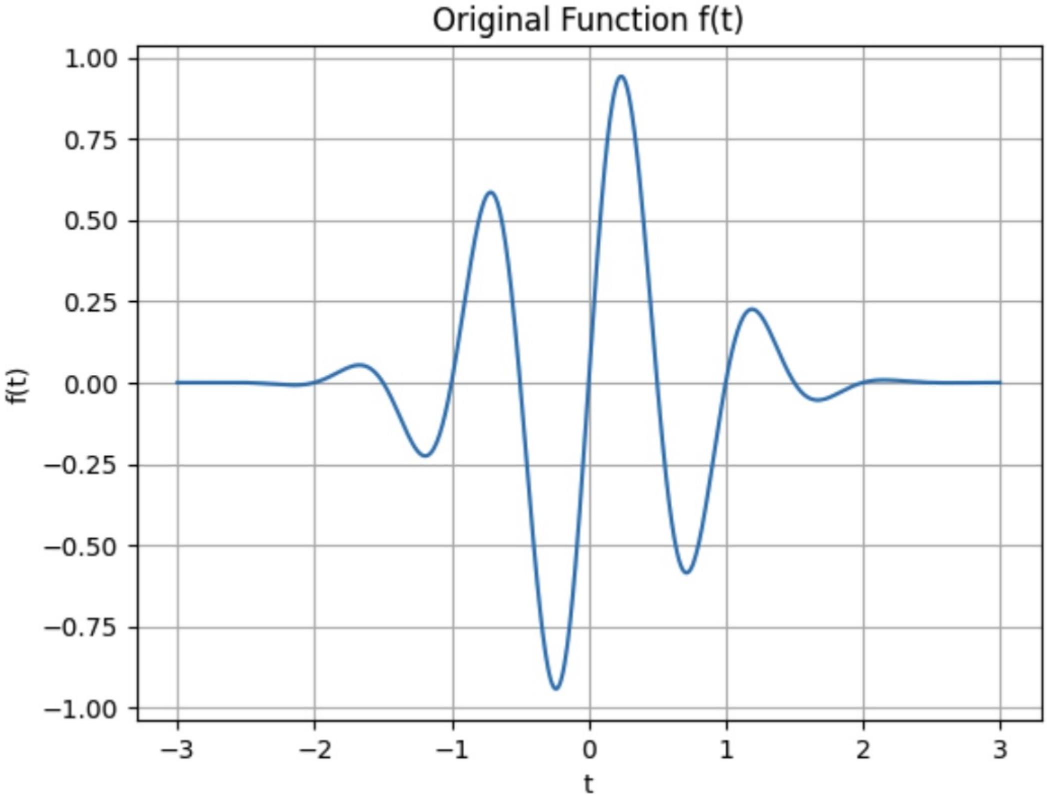
```python
import numpy as np
import matplotlib.pyplot as plt
from sympy import symbols, exp, sin, pi, fourier_transform, lambdify

# Define symbols
t, w = symbols('t w', real=True)

# Define the function
f = exp(-t**2) * sin(2*pi*t)

# Compute Fourier Transform
F = fourier_transform(f, t, w)

# Convert to numerical functions for plotting
f_num = lambdify(t, f, "numpy")
F_num = lambdify(w, F, "numpy")

# Generate numerical data
t_vals = np.linspace(-3, 3, 600)
w_vals = np.linspace(-10, 10, 600)

f_vals = f_num(t_vals)
```

```python
F_vals = F_num(w_vals)

# Plot f(t)
plt.figure()
plt.plot(t_vals, f_vals)
plt.xlabel("t")
plt.ylabel("f(t)")
plt.title("Original Function f(t)")
plt.grid(True)
plt.show()
display("Original function f(t):")
display(f)

# Plot Fourier Transform magnitude |F(w)|
plt.figure()
plt.plot(w_vals, np.abs(F_vals))
plt.xlabel("ω")
plt.ylabel("|Fω()|")
plt.title("Magnitude of Fourier Transform |Fω()|")
plt.grid(True)
plt.show()
display("Fourier Transform Fω():")
display(F)
```




```
'Original function f(t):'
```

$$e^{-t^2}\sin(2\pi t)$$




```
'Original function f(t):'
```

$e^{-t^2} \sin(2\pi t)$