Chapter 29
Space Time Geometry

Spacetime geometry provides the geometric framework of special and general relativity. Space and time are unified into a four-dimensional manifold equipped with a metric that encodes causal structure, intervals, and the motion of matter and fields.

29.1 Spacetime as a Manifold

Spacetime is modeled as a four-dimensional smooth manifold $\mathscr{M}$.

Each event is specified by coordinates

$$x^\mu = (ct, x, y, z)$$

where $\mu = 0, 1, 2, 3$.

Coordinate transformations between inertial frames preserve the geometric structure of spacetime.

29.2 Minkowski Metric

29.2.1 Metric Tensor

In special relativity, spacetime is flat and equipped with the Minkowski metric

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The spacetime interval between two events is

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

This interval is invariant under Lorentz transformations.

29.2.2 Timelike, Spacelike, and Null Intervals

Intervals are classified according to the sign of ds^2.

For timelike intervals

$$ds^2 > 0$$

For spacelike intervals

$$ds^2 < 0$$

For null intervals

$$ds^2 = 0$$

This classification determines causal relationships between events.

29.3 Four-Vectors

Physical quantities are expressed as four-vectors to ensure Lorentz invariance.

29.3.1 Position and Velocity Four-Vectors

The position four-vector is

$$x^\mu = (ct, x, y, z)$$

The four-velocity is defined as

$$u^\mu = \frac{dx^\mu}{d\tau}$$

where τ is the proper time.

The normalization condition is

$$\eta_{\mu\nu} u^\mu u^\nu = c^2$$

29.3.2 Energy–Momentum Four-Vector

The energy–momentum four-vector is

$$p^\mu = (E/c, \mathbf{p})$$

It satisfies the invariant relation

$$\eta_{\mu\nu} p^\mu p^\nu = m^2 c^2$$

29.4 Proper Time and Worldlines

The proper time interval along a worldline is defined by

$$d\tau = \frac{1}{c}\sqrt{ds^2}$$

A particle's motion through spacetime is represented by a worldline

$$x^\mu = x^\mu(\tau)$$

Free particles follow straight worldlines in flat spacetime.

29.5 Lorentz Transformations as Isometries

Lorentz transformations preserve the Minkowski metric

$$\eta_{\alpha\beta}\Lambda^\alpha{}_\mu\Lambda^\beta{}_\nu = \eta_{\mu\nu}$$

They are therefore isometries of spacetime geometry.

29.6 Curved Spacetime

In the presence of gravitation, spacetime becomes curved.

The metric is generalized to

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$$

where $g_{\mu\nu}$ varies with position.

29.7 Geodesics in Spacetime

The motion of free particles in curved spacetime follows geodesics.

The geodesic equation is

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

where $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols.

29.8 Curvature and Gravitation

Spacetime curvature is described by the Riemann curvature tensor

$$R^\mu{}_{\nu\alpha\beta}$$

Curvature determines the relative acceleration of nearby worldlines and encodes the gravitational interaction.

29.9 Spacetime Geometry and Physics

Spacetime geometry unifies kinematics and gravitation:

- Lorentz invariance arises from flat spacetime geometry
- Gravitation is interpreted as spacetime curvature
- Physical laws are expressed in covariant tensor form

29.10 Closing Remarks

Spacetime geometry replaces the notion of absolute space and time with an intrinsic geometric structure. By describing physics in terms of invariant intervals, worldlines, and curvature, it provides the foundation for modern relativistic theories.