ENGINEERING MATHEMATICS

A CRISP AND CONCISE REFERENCE USING SCIPY

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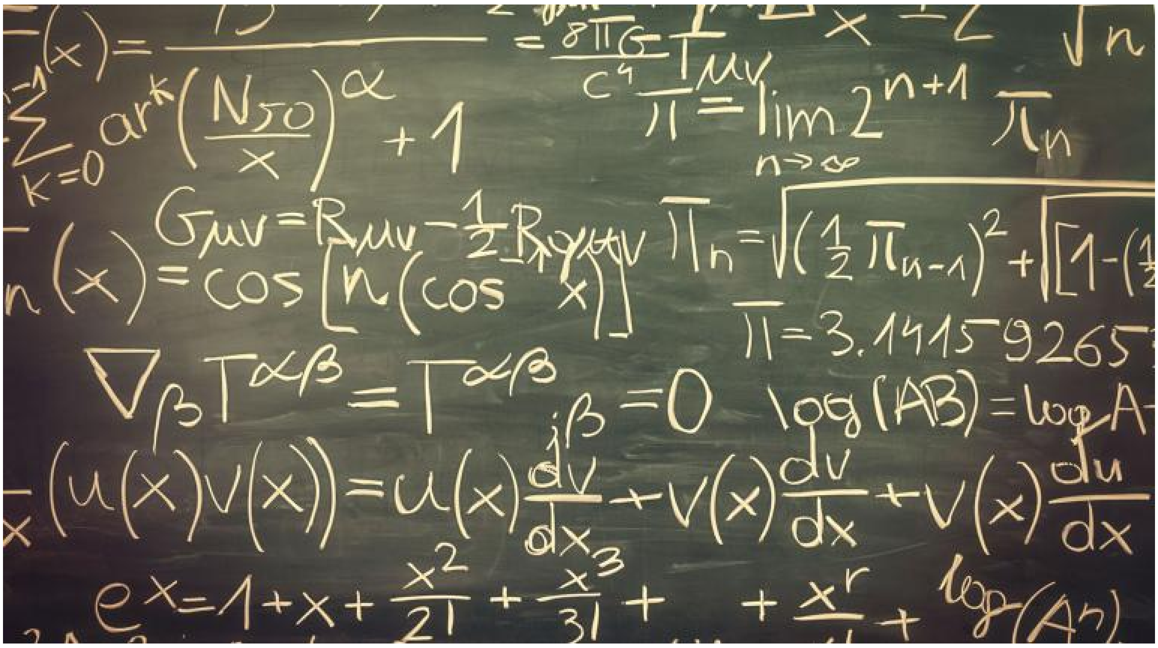


Contents

This book is dedictated to the memory of

Professor Anadi Sankar Gupta,

former Head of Department of Mathematics,

Indian Institute of Technology, Kharagpur,

who instilled in me a passion for mathematics.

Preface

"The language God talks" is a phrase attributed to Richard Feynman the distinguished and nobel prize winning physicist as an encouragement to learn calculus. Mathematics is indeed the language of science. Italian astronomer and physicist Galileo Galilei had stated *"Mathematics is the language in which God has written the universe."*

In 1865, James Clerk Maxwell, a Scottish mathematician at Cambridge University, came up with his seminal work where he presented mathematical equations that unified electricity, magnetism and optics that have contributed immensely to our civilisation. Albert Einstein had said, *"The work of James Clerk Maxwell changed the world forever."* Maxwell was a brilliant mathematician. The mathematical tools he had mastered had been developed a century before. While Michael Faraday and others had been investigating various electrical and magnetic phenonena, Maxwell looked for patterns and with his mathematics, he was able to synthesize all these phenomena.

Mathematics enables us to explore the connection between creativity and structure. Creativity flourishes in a formal mathematical structure, helps us sharpen our critical thinking and master the art of problem solving. One should study mathematics for the same reason that we study art, literature, history and science. The intellectual achievements of Newton, Gauss, Leibniz, Ramanujan in mathematics are at par with those of Tagore, Shakespeare and Leonardo da Vinci in art. Often the usefulness of mathematics is the only thing that is emphasized and mathematics is viewed as a toolbox, but it is much more than that. Understanding mathematics should be a desirable objective of everyone one and not limited to students of science and engineering. Mathematics enables us to be objective, quantitative and succinct in our communication. The process of problem-solving in mathematics helps us develop patience and resilience.

This book is intended to introduce students of engineering, physics, mathematics, computer science, and related fields to a comprehensive set of concepts in mathematics that are required for solving real world problems. Content is ever growing, the curriculum for many students is pretty much full and time is short and precious. This book provides a crisp and concise understanding of the fundamental concepts in engineering mathematics that are essential to comprehend natural and engineered phenomena. The book is described as a *handbook* as it is designed to be a ready reference to the fundamental concepts along with their proofs.

The most common misconception about mathematics is that it is a skill that comes naturally. Most of mathematics is not about natural talent, rather it is about one's approach to learning. Just as in any other skill, success comes with practice.

> With the advent of symbolic computing, the drudgery has been signficantly reduced. Throughout the book, examples are listed using *sympy*, *numpy*, *python* and *jupyterlab* to help visualize the solutions. The book is somewhat unique in that sense.

The author is a software executive by profession and has a deep interest in quantitative methods. He has received his degrees of Doctor of Science and Master of Science from the Massachusetts Institute of Technology after graduating from Indian Institute of Technology, Kharagpur.

I hope you will enjoy reading this book and develop a deep interest in mathematics.

Jaideep Ganguly

Hyderabad, India
Sunday 2^{nd} April, 2023

CHAPTER 1

Symbolic Computation

1.1 Introduction

Most people consider math and physics to be scary beasts from which it is best to keep one's distance. Computers, however, can help us tame the complexity and tedious arithmetic manipulations associated with these subjects. Indeed, math and physics are much more approachable once you have the power of computers on your side.

Symbolic computation or algebraic computation is a scientific area that refers to the study and development of algorithms and software for manipulating mathematical expressions and other mathematical objects. Macsyma is one of the oldest general-purpose computer algebra systems still in wide use. It was originally developed at MIT's Project MAC.

SymPy is a *Python* library for symbolic mathematics. Examples in the book make use of sympy and python for symbolic computation and visualization. *NumPy* is a numerical library for Python, Matplotlib is a plotting library for Python and Jupyterlab is a editor for Python that makes interactive computing very easy.

You will need *Python 3* installed in your computer. Thereafer, you will need to install *SymPy*, *NumPy*, *MatplotLib* and *JupyterLab*. In a mac, you can run the following commands from the shell to install these packages.

```
sudo pip3 install sympy
sudo pip3 install numpy
sudo pip3 install matplotlib
sudo pip3 install jupyterlab
```

```
import sympy as sp
import numpy as np
import matplotlib.pyplot as plt
from IPython.display import display, Math

%config Completer.use_jedi = False
```

CHAPTER 2

Function

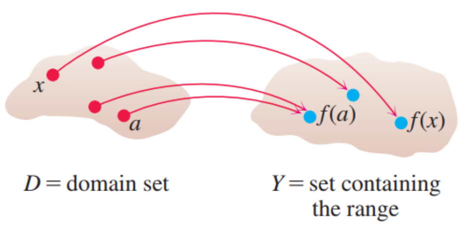
Function, in mathematics, is an expression, rule, or law that defines a relationship between one variable, the independent variable and another variable, which is the dependent variable. Functions are essential for formulating physical relationships in the sciences and are ubiquitous in mathematics.

2.1 Function, Domain & Range

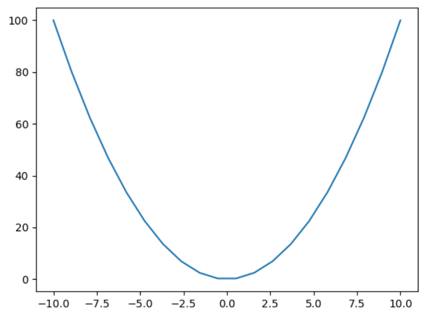
When a value of one variable x depends on another variable y, we say that *y is a function of x* and it is written symbolically as:

$\boxed{y = f(x)}$ and pronounced as "y equals f of x"

Formally, a function *f*, is a rule that assigns an *unique* value *f(x)* for each x in D where D is known as the *Domain* and the set of $y = f(x)$, or Y, is known as the *Range*.



Example:
$y = x^2$
Domain = $[-\infty, +\infty]$
Range = $[0, +\infty]$

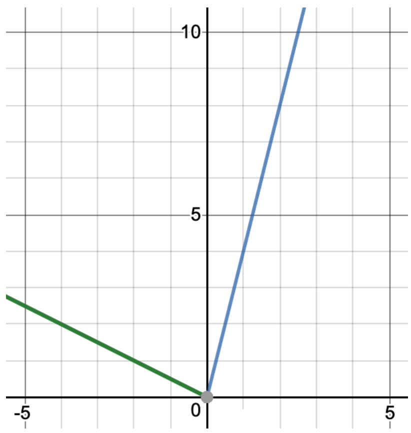


2.2 Piecewise Continuous & Discontinuous Functions

Sometimes a function is described in pieces by using different formulas on different parts of its domain.

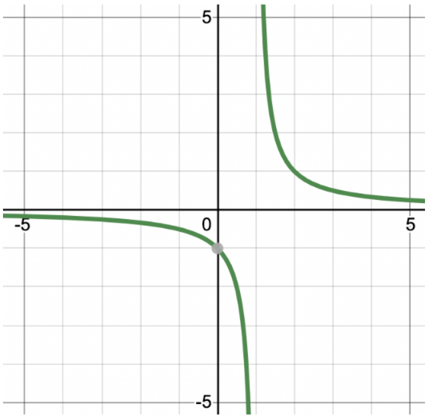
$$|y| = \begin{cases} 4x & \text{if } x \geq 0 \\ -0.5x & \text{if } x < 0 \end{cases}$$

y is *unique* for a given x. Such functions are *piecewise continous* as there are no "gaps".



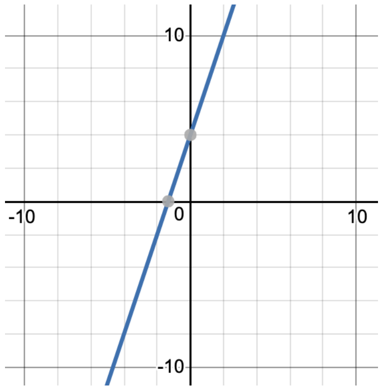
$$y = \frac{1}{x-1}$$

y does not exist for $x = 1$; The curve is not continuous at $x = 1$ and the function is *discontinous* .

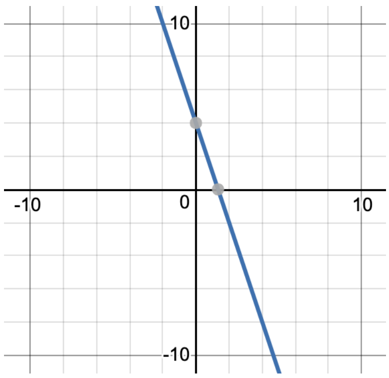


2.3 Increasing & Decreasing Functions

Increasing function:
$f(x_2) > f(x_1)$ when $x_2 > x_1$
Example: $y = 3x + 4$

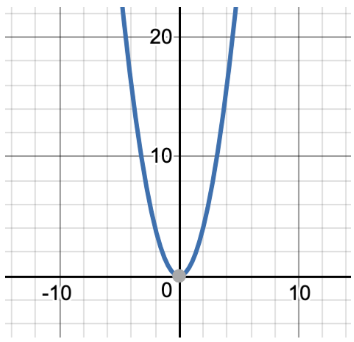


Decreasing function
$f(x_2) < f(x_1)$ when $x_2 > x_1$
Example: $y = -3x + 4$

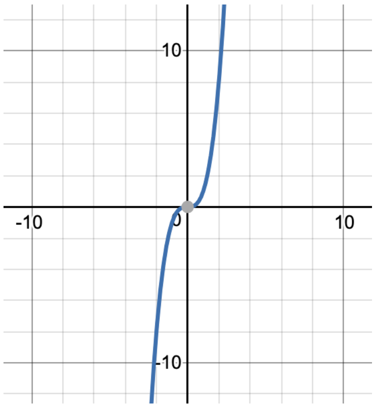


2.4 Even & Odd Functions

Even function $f(-x) = f(x)$
Example: $y = x^2$



Odd function $f(-x) = -f(x)$
Example: $y = x^3$



2.5 Types of Functons

Following are some types of functions.

1. **Linear** Functions $f(x) = mx + b$
2. **Polynomial** Functions $f(x) = a_0 + a_1x + a_2x^2 + \cdots a_{n-1}x^{n-1} + a_nx^n$
   $n = 2 \rightarrow$ Quadratic, $n = 3 \rightarrow$ Cubic.
3. **Rational** Functions $f(x) = p(x)/q(x)$
4. **Algebraic** Functions - constructed from polynomials using algebraic operations ($+$, $-$, $\times$, $\div$, and roots)
5. **Trigonometric** functions, e.g., $f(x) = sin(x)$
6. **Exponential** Functions, e.g., $y = 2^x$, Logarithmic Functions $y = log_5^x$
7. **Transcendental** Functions - functions that are not expressible as a finite combination of algebraic operations of addition, subtraction, multiplication, division, raising to a power, and extracting a root. E.g., $log\,x$, $sin\,x$, e^x and any functions containing them. Such functions are expressible in algebraic terms only as infinite series. In general, the term transcendental means nonalgebraic.

2.6 Sums, Differences, Products & Quotients of Functions

Much like numbers, functions can be added, subtracted, multiplied, and divided. By defnition:

$$(f + g)(x) = f(x) + g(x)$$
$$(f - g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$\frac{f}{g}(x) = \frac{f(x)}{g(x)} \text{ where } g(x) \neq 0$$

2.7 Function Composition

The output from is one function is the input to the second function.

$$\boxed{(f \circ g)(x) = f(g(x))}$$

2.8 Vertical & Horizontal Scaling, Reflecting a Function

Following are the transformation equations:

$y = cf(x)$ for c > 1, stretch vertically

$y = \frac{1}{c}f(x)$ for c > 1, compress vertically

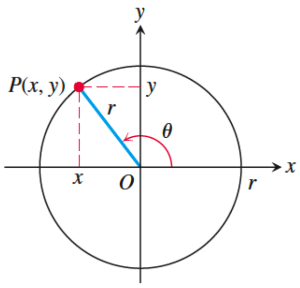
$y = f(cx)$ for c > 1, stretch horizontally

$y = f\left(\frac{x}{c}\right)$ for c > 1, compress horizontally

$y = -f(x)$ for c = -1, reflect across x axis

$y = f(-x)$ for c = -1, reflect across y axis

2.9 Basic Trigonometric Function Definitions



b = base
p = perpendicular
r = h (hypoetenuse)
$b^2 + p^2 = r^2$
(Pythagoras)

$sine\ \theta = \frac{p}{h}$ $\quad$ $cosecant\ \theta = \frac{1}{sin\ \theta}$

$cosine\ \theta = \frac{b}{h}$ $\quad$ $secant\ \theta = \frac{1}{cos\ \theta}$

$tangent\ \theta = \frac{p}{b} = \frac{sin\ \theta}{cos\ \theta}$ $\quad$ $cotangent\ \theta = \frac{1}{tan\ \theta}$

abbreviated as: sin, cos, tan, csc, sec, cot

2.10 Basic Trigonometric Identities

The following identities can be easily derived using the above definitions.

$$\boxed{sin^2\ \theta + \cos^2\ \theta = 1}$$

$$\boxed{sec^2\ \theta = 1 + tan^2\ \theta}$$

$$\boxed{csc^2\ \theta = 1 + cot^2\ \theta}$$

$$\boxed{sin\ (\theta_1 + \theta_2) = sin\ \theta_1\ cos\ \theta_2 + cos\ \theta_1\ sin\ \theta_2}$$

$$\boxed{cos\ (\theta_1 + \theta_2) = cos\ \theta_1\ cos\ \theta_2 - sin\ \theta_1\ sin\ \theta_2}$$

$$\boxed{c^2 = a^2 + b^2 - 2ab\ cos\ \theta \quad \text{(Law of Cosine)}}$$

$$\boxed{\left(\frac{sin\ A}{a}\right) = \left(\frac{sin\ B}{b}\right) = \left(\frac{sin\ C}{c}\right) \quad \text{(Law of Sine)}}$$

Where a,b,c are lengths, A,B,C are angles, we have:

[1]:
```
import sympy as sp
import numpy as np
import matplotlib.pyplot as plt
from IPython.display import display, Math
from sympy import sqrt, diff, integrate, oo
from sympy import sin, cos, tan, ln, exp, erf, trigsimp, expand_trig, simplify
from sympy import sinh, cosh

%config Completer.use_jedi = False

x = sp.symbols('x')

y = x**2
display(y)
```

x^2

[5]:
```
lamb_y = lamb_y = sp.lambdify(x,y)
x_num = np.linspace(start = -10, stop = 10, num = 20)
y_num = lamb_y(x_num)
display(x_num)
display(y_num)
```

```
array([-10.        ,  -8.94736842,  -7.89473684,  -6.84210526,
        -5.78947368,  -4.73684211,  -3.68421053,  -2.63157895,
        -1.57894737,  -0.52631579,   0.52631579,   1.57894737,
         2.63157895,   3.68421053,   4.73684211,   5.78947368,
         6.84210526,   7.89473684,   8.94736842,  10.        ])
```

```
array([100.        ,  80.05540166,  62.32686981,  46.81440443,
        33.51800554,  22.43767313,  13.5734072 ,   6.92520776,
         2.49307479,   0.27700831,   0.27700831,   2.49307479,
         6.92520776,  13.5734072 ,  22.43767313,  33.51800554,
        46.81440443,  62.32686981,  80.05540166, 100.        ])
```

[7]:
```
plt.plot(x_num, y_num)
plt.savefig("plot.png")
plt.show()
```

CHAPTER 3

Limit

In mathematics, a limit is the value that a function or a sequence approaches as the input approaches some value. Limits are essential to calculus and mathematical analysis, and are used to define continuity, derivatives, and integrals.

3.1 Definition of Limit

Consider a function $f(x)$ that is defined in a domain D which includes the point c. The function may or may not be defined at c. If, for all x that is *close* to c except for c, $f(x)$ is arbitrarily close to a number L (as close to L as we like), then it is said that f approaches the limit L as x approaches c and is written as:

$$\boxed{\lim_{x\to c} f(x) = L}$$

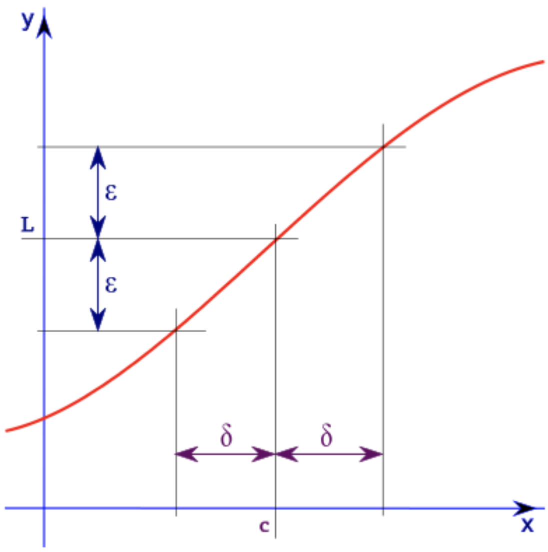
If the function can be evaluated at c, the limit L is simply $f(c)$. But, there can be situations where the function is not evaluable at c? For example, the following function cannot be evalauted at $x = 1$.

$$\lim_{x\to 1} \frac{x^2+x-2}{x-1}$$

But this function can be easily simplified to:

$$f(x) = \frac{(x-1)(x+2)}{x-1} = x+2$$
$$\to \lim_{x\to 1} f(x) = 3.$$

3.2 Formal Definition of Limit



Let $f(x)$ be a function that is defined on an interval that contains $x = c$, except possibly at c. Then, $\lim_{x\to c} f(x) = L$ if for every number $\epsilon > 0$, there is some number $\delta > 0$ such that, when $0 < |x-a| < \delta$, $|f(x) - L| < \epsilon$.

This means that for any number $\epsilon > 0$ that we pick, one can go to the graph and sketch two horizontal lines at $L+\epsilon$ and $L-\epsilon$. Then there must be another number $\delta > 0$ that can be determined to enable us to add in two vertical lines in the graph $a+\delta$ and $a-\delta$.

3.3 Laws of Limit

Given L, M, c, k are real numbers such that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. Then,

Sum Rule	$\lim_{x\to c}(f(x)+g(x)) = L+M$
Difference Rule	$\lim_{x\to c}(f(x)-g(x)) = L-M$
Constant Rule	$\lim_{x\to c}(kf(x)) = kL$
Product Rule	$\lim_{x\to c}(f(x)g(x)) = LM$
Quotient Rule	$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M}$
Power Rule	$\lim_{x\to c}[f(x)]^n = L^n \ (n>0)$
Root Rule	$\lim_{x\to c}\sqrt[n]{(f(x)} = \sqrt[n]{(L)} \ (n>0)$

Examples:

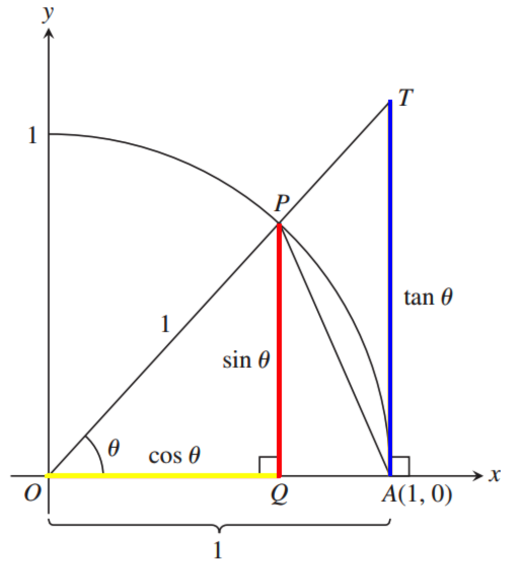
$$\lim_{x\to 3}\sqrt{(2x^3+10} = 8$$

$$\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}$$

The above function is not evaluable at $x=0$. The standard trick is to multiply both numerator and denominator by the conjugate radical expression.

$$\frac{\sqrt{x^2+9}-3}{x^2} = \frac{\sqrt{x^2+9}-3}{x^2}\cdot\frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3} = \frac{1}{\sqrt{x^2+9}+3} = \lim_{x\to 0}\frac{1}{\sqrt{x^2+9}+3} = \frac{1}{6}$$

3.4 An Important Limit



Consider the circle with a unit radius.

Area △ OAP < area sector OAP < area △ OAT

$\frac{1}{2}sin\ \theta \leq \pi 1^2\left(\frac{\theta}{2\pi}\right) \leq \frac{1}{2}\tan\ \theta$ (θ is in radians)

$1 \leq \frac{\theta}{sin\ \theta} \leq \frac{1}{cos\ \theta}$

$\rightarrow 1 \geq \frac{sin\ \theta}{\theta} \geq cos\ \theta$

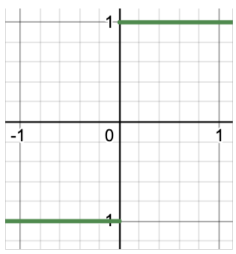
Hence,

$$\boxed{\lim_{\theta\to 0}\frac{Sin\ \theta}{\theta} = 1} \quad \text{where } \theta \text{ is in radians} \qquad (3.4.1)$$

Now consider the function $f(\theta) = \frac{1}{sin\ \theta}$. Does it have a limit as $t \to \theta$ from either side? As θ approaches 0, its reciprocal, $1/x$, grows without bound and the values of function cycle repeatedly from -1 to 1. There is no single number L that the function values stay increasingly close to as $\theta \to 0$. The function has neither a right-hand limit nor a lefthand limit at $\theta = 0$.

3.5 One Sided Limits

$$\lim_{x\to 0^+} f(x) = 1$$
$$\lim_{x\to 0^-} f(x) = -1$$



3.6 Continuous Function

Function is right-continous at c (continuous from right) if $\lim_{x \to c^+} f(x) = f(c)$
Function is left-continous at c (continuous from left) if $\lim_{x \to c^-} f(x) = f(c)$
A function is continous at c if $\lim_{x \to c} f(x) = f(c)$

If a function is discontinuous at one or more points of its domain, it is called a discontinuous function.

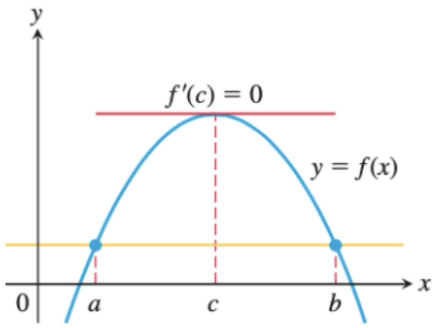
3.7 Infinite Limits

$\lim_{x \to 0^+} \frac{1}{x} = \infty$, $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

Note that this does not mean that the limit exists as there is no real number such as ∞. It is simply a concise way of saying that the limit does not exist.

3.8 Rolle's Theorem

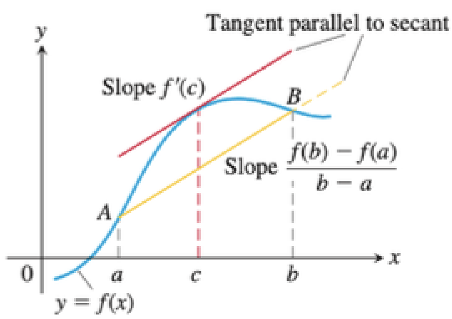
If f is a continuous function on a closed interval $[a, b]$ and If $f(a) = f(b)$, then there is at least one point c in (a, b) where $f'(c) = 0$.



3.9 Mean Value Theorem

There is at least one number c in the interval (a, b) such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



[2]:
```
import sympy as sp
import numpy as np
import matplotlib.pyplot as plt
from IPython.display import display, Math
from sympy import sin, cos, tan, trigsimp, expand_trig
from sympy import oo
from sympy import limit
```

[3]:
```
%config Completer.use_jedi = False
```

[5]:
```
x = sp.symbols('x')
```

[6]:
```
y = (x**2 + x - 2) / (x - 1)
lim = limit(y, x, 1)
display(lim)
```

3

[7]:
```
y = ( (x**2 + 9)**0.5 - 3 ) / x**2
display(y)
lim = limit(y, x, 0)
display(lim)
```

$$\frac{\left(x^2+9\right)^{0.5}-3}{x^2}$$

$$\frac{1}{6}$$

[8]:
```
y = sin(x)/x
lim = limit(y, x, 0)
display(lim)
```

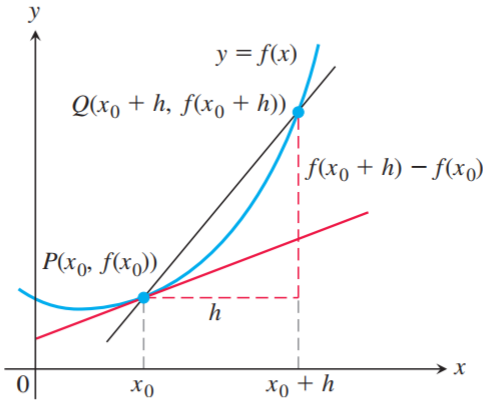
1

CHAPTER 4

Derivative

In mathematics, the derivative of a function of a real variable measures the sensitivity to change of the function value, i.e., the output value with respect to a change in its argument, i.e., the input value. Derivatives are a fundamental to calculus.

4.1 Definition of a Derivative



Consider the limit: $\lim_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}$

This limit is called the *derivative* and is written as:

$$\boxed{\frac{df}{dx} = \frac{dy}{dx} = f'(x)}$$

Its value at a is represented as: $\boxed{f'(a) = \left.\dfrac{dy}{dx}\right|_{x=a}}$

A derivative is rate of change, it is the *tangent* at the point .

A function $f(x)$ is differentiable at $x = a$ if $f'(a)$ exists and $f(x)$ is called differentiable on an interval if the derivative exists for each point in that interval. If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$. $\boxed{\dfrac{d}{dx}}$ is known as the Differential Operator .

4.2 Derivative of a Polynomial Term

$$
\begin{aligned}
f(x) &= x^n \\
f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^n - a^n}{x - a} \\
x^n - a^n &= (x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}) \\
f'(a) &= \lim_{x \to a} (x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots a^{n-2}x + a^{n-1}) = na^{n-1}
\end{aligned}
$$

$\boxed{\dfrac{dx^n}{dx} = nx^{n-1}}$ and obviously $\dfrac{d}{dx}(constant) = 0$

4.3 Derivatives of a Trigonometric Function

$$\begin{aligned}
\frac{d}{dx}(sin(x)) &= \lim_{h\to 0}\frac{sin(x+h)-sin(x)}{h}\\
&= \lim_{h\to 0}\frac{sin(x)\,cos(h)+cos(x)\,sin(h)-sin(x)}{h}\\
&= sin(x)\lim_{h\to 0}\frac{cos(h)-1}{h}+cos(x)\lim_{h\to 0}\frac{sin(h)}{h}
\end{aligned}$$

From **??**, **??** & 3.4.1 we have,

$$1-cos(h)=2sin^2\left(\frac{h}{2}\right)$$

$$\frac{d}{dx}(sin(x))=sinx(x).0+cos(x).1$$

$$\boxed{\frac{d}{dx}(sin(x))=cos(x)}$$

4.4 Derivative of a log function

Compute $\frac{d}{dx}(ln\;x)$

$$\frac{d}{dx}ln\;x=\lim_{h\to 0}\frac{ln(x+h)-ln(x)}{h}=\lim_{h\to 0}\frac{ln\frac{(x+h)}{x}}{h}=\lim_{h\to 0}\frac{1}{h}ln\left(1+\frac{h}{x}\right)=\lim_{h\to 0}ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}}$$

Let $h=nx\to\frac{1}{h}=\frac{1}{n}.\frac{1}{x}\to\lim_{h\to 0}ln(1+n)^{\frac{1}{n}\frac{1}{x}}=\lim_{h\to 0}ln\left((1+n)^{\frac{1}{n}}\right)^{\frac{1}{x}}=\frac{1}{x}ln\left(\lim_{h\to 0}(1+n)^{\frac{1}{n}}\right)$

$$\boxed{\frac{d}{dx}ln\;x=\frac{1}{x}}$$

4.5 Chain Rule

Compute $\frac{d}{dx}(v(u(x)))$

$$\frac{dv}{dx}=\lim_{x\to 0}\frac{\Delta v}{\Delta x}=\lim_{x\to 0}\left(\frac{\Delta v}{\Delta u}\times\frac{\Delta u}{\Delta x}\right)=\lim_{x\to 0}\left(\frac{\Delta v}{\Delta u}\right)\times\lim_{x\to 0}\left(\frac{\Delta u}{\Delta x}\right)$$

$$\boxed{\frac{dv}{dx}=\frac{dv}{du}\times\frac{du}{dx}}$$

4.6 Derivative of an exponential function

Compute $\frac{d}{dx}(a^x)$

Let $y=a^x$

$$\begin{aligned}
ln\,y &= ln\,a\\
\frac{1}{y}\frac{dy}{dx} &= ln\,a
\end{aligned}$$

$$\boxed{\frac{d}{dx}a^x=a^x\,ln\,a}$$

4.7 Implicit Differentiation

In implicit differentiation, we differentiate each side of an equation with two variables (usually x and y) by treating one of the variables as a function of the other. This calls for using the chain rule. Example:

$$x^2+y^2=1 \to \frac{d}{dx}(x^2+y^2)=\frac{d}{dx}x^2+\frac{d}{dx}y^2=2x+2y\frac{dy}{dx}=0 \to \frac{dy}{dx}=-\frac{x}{y}$$

4.8 Product Rule

$$\begin{aligned}(fg)' &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}\\ &= \lim_{h\to 0} f(x+h)\frac{(g(x+h)-g(x))}{h}+\lim_{h\to 0} g(x)\frac{f(x+h)-f(x)}{h}\end{aligned}$$

$$\boxed{(fg)'=f(x)g'(x)+g(x)f'(x)} \tag{4.8.1}$$

4.9 Quotient Rule

$$\begin{aligned}\left(\frac{f}{g}\right)' &= \lim_{h\to 0}\frac{f'g-fg'}{g^2}=\lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h}=\lim_{h\to 0}\frac{1}{h}\frac{f(x+h)g(x)-f(x)g(x+h)}{g(x+h)g(x)}\\ &= \lim_{h\to 0}\frac{1}{g(x+h)g(x)}\frac{f(x+h)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(x+h)}{h}\\ &= \lim_{h\to 0}\left(g(x)\frac{f(x+h)-f(x)}{h}-f(x)\frac{g(x+h)-g(x)}{h}\right)\end{aligned}$$

$$\boxed{\left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}} \tag{4.9.1}$$

4.10 L'Hôpital's rule

First, need to do mathematical manipulations to get the limit into a l'Hôpital form, i.e., $0/0$ or ∞/∞ form. Let $f(x)$ and $g(x)$ be continuous functions on an interval containing $x=a$, with $f(a)=g(a)=0$. Suppose that f and g are differentiable, and that f' and g' are continuous. and, suppose that $g'(a)\neq 0$. Then,

$$\begin{aligned}\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}\\ &= \lim_{x\to a}\frac{(f(x)-f(a))/(x-a)}{(g(x)-f(a))/(x-a)}\\ &= \frac{\lim_{x\to a}(f(x)-f(a))/(x-a)}{\lim_{x\to a}(g(x)-f(a))/(x-a)}\end{aligned}$$

$$\boxed{\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{f'(x)}{g'(x)}} \tag{4.10.1}$$

4.11 Concave Up (Convex) & Concave Down

Let $y=f(x)$ be twice-differentiable on an interval I. If f″ > 0 on I, the graph of f over I is concave up (also called convex). If f″ < 0 on I, the graph of f over I is concave down.

4.12 Euler's number

Euler's number is written as e.

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to 0}(1+n)^{\frac{1}{n}} = \sum_{n=0}^{\infty}\frac{1}{n!} \text{ such that, } \lim_{h\to 0}\frac{e^h-1}{h} = 1$$

$$e = 2.718281\ldots$$

Note: $\boxed{\frac{d}{dx}e^x = e^x}$

4.13 Hyperbolic Functions

$$\boxed{sinh\, x = \frac{e^x - e^{-x}}{2}} \tag{4.13.1}$$

$$\boxed{cosh\, x = \frac{e^x + e^{-x}}{2}} \tag{4.13.2}$$

$$\boxed{tanh\, x = \frac{e^x - e^{-x}}{e^x + e^{-x}}} \tag{4.13.3}$$

$$\boxed{cosh^2 x - sinh^2 x = 1} \tag{4.13.4}$$

4.14 Partial Derivatives

$$\begin{aligned}
f_x(x,y) &= \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} \\
f_y(x,y) &= \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h} \\
f_x = f_x(x,y) &= \frac{\partial}{\partial x}f(x,y) \\
f_y = f_x(x,y) &= \frac{\partial}{\partial y}f(x,y)
\end{aligned}$$

Example,

$$f(x,y) = x^2y - 10y^2z^3 + 43x - 7tan(4y)$$

$$\begin{aligned}
\frac{\partial}{\partial x}f(x,y,z) &= 2xy + 43 \\
\frac{\partial}{\partial y}f(x,y,z) &= x^2 - 20yz^3 - 28sec^2(4y) \\
\frac{\partial}{\partial z}f(x,y,z) &= -30y^2z^2
\end{aligned}$$

[1]:
```
import sympy as sp
import numpy as np
import matplotlib.pyplot as plt
from IPython.display import display, Math
from sympy import sqrt, diff
from sympy import sin, cos, tan, ln, trigsimp, expand_trig, simplify
from sympy import sinh, cosh
```

[2]:
```
%config Completer.use_jedi = False
```

[3]:
```
x = sp.symbols('x')
y = x**2
der = diff(y, (x, 2))     # 2nd derivative of y wrt x
display(der)
```

2

[4]:
```
y = sin(x)
der = diff(y, x)          # derivative of y wrt x
display(der)
```

$\cos(x)$

[5]:
```
y = ln(x)
der = diff(y, x)          # derivative of y wrt x
display(der)
```

$\frac{1}{x}$

[6]:
```
a = sp.symbols('a')
y = a**x
der = diff(y, x)          # derivative of y wrt x
display(der)
```

$a^x \log(a)$

[7]:
```
y = sinh(x)
der = diff(y, x)          # derivative of y wrt x
display(der)
```

$\cosh(x)$

[8]:
```
y = cosh(x)
der = diff(y, x)          # derivative of y wrt x
display(der)
```

$\sinh(x)$

[9]:
```
x, y, z = sp.symbols('x y z')
f = (x**2) * y - 10 * (y**2)*(z**3) + 43*x - 7*tan(4*y)
pdx = diff(f,x)
display(pdx)
pdy = simplify(diff(f,y))
display(pdy)
pdz = diff(f,z)
display(pdz)
```

$2xy + 43$

$x^2 - 20yz^3 - \frac{28}{\cos^2(4y)}$

$-30y^2z^2$

CHAPTER 5

Integral

Integration, in mathematics, is the technique of finding a function $g(x)$ the derivative of which is equal to a given function $f(x)$. This is indicated by the integral sign $\int$ as in $\int f(x)dx$ and is called the indefinite integral of the function. The symbol dx represents an infinitesimal displacement along x. Hence, $\int f(x)dx$ is the summation of the product of $f(x)$ and dx. The definite integral, written as $\int_a^b f(x)dx$ where a and b are called the limits of integration, is equal to $g(b) - g(a)$, where $\frac{d}{dx}g(x) = f(x)$.

5.1 Integral

Given a function $f(x)$, an *anti-derivative* of $f(x)$ is any function $g(x)$ such that $g'(x) = f(x)$. The most general anti-derivative is called the *indefinite integral* .

$\int f(x)dx = g(x) + c$ where c is a constant of integration

Note the following inequalities.

$$\int f(x)g(x)dx \neq \int f(x)dx \int g(x)dx$$

$$\int \frac{f(x)}{g(x)}dx \neq \frac{\int f(x)dx}{\int g(x)dx}$$

5.2 Common Integrals

$$\int x^n dx = \frac{x^{(n+1)}}{(n+1)} + c \tag{5.2.1}$$

$$\int e^x dx = e^x + c \tag{5.2.2}$$

$$\int a^x dx = \frac{a^x}{ln\ a} + c \tag{5.2.3}$$

$$\int \frac{1}{x} dx = ln|x| + c \tag{5.2.4}$$

$$\int cos(x)dx = sin(x) + c \tag{5.2.5}$$

5.3 Substitution Technique

$$\int 18x^2 \sqrt[4]{(6x^3+5)}dx$$

Let $u = 6x^3 + 5$

$$\rightarrow du = 18x^2 dx$$

$$\rightarrow \int \sqrt[4]{u}\, du = \frac{u^{(\frac{1}{4}+1)}}{\frac{1}{4}+1} = \frac{4}{5}u^{\frac{5}{4}} = \frac{4}{5}\left(6x^3+5\right)^{\frac{5}{4}}$$

5.4 Integration by Parts

$$[f(x)g(x)]' = f(x)g'(x) + f'(x)g(x)$$

$$f(x)g'(x) = [f(x)g(x)]' - f'(x)g(x)$$

$$\int f(x)g'(x)dx = \int [f(x)g(x)]'dx - \int f'(x)g(x)dx$$

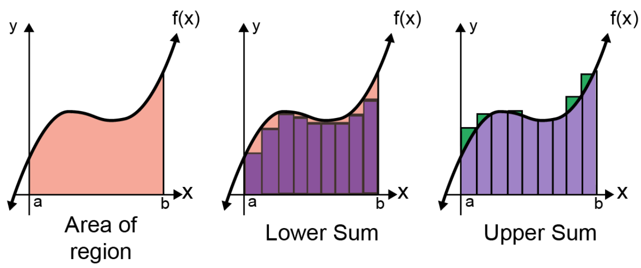
$$a\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

$$\boxed{\int u(x)v(x)dx = u(x)\int v(x) - \int [u'(x)\int v(x)]dx} \tag{5.4.1}$$

Hence, integral of two functions = first function × integral of second function − integral of (differntiation of the first function × integral of the second function).

5.5 Definite Integral

A definite integral is a the area under its curve .



$$\boxed{\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = g(x)\Big|_a^b = g(b) - g(a)}$$

where $f(x_i^*)$ is the value at the middle of the strip Δx.

$$\boxed{f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx}$$

$$\boxed{\int_a^b f(x)dx = f(c)(b-a) \text{ where c is in [a,b]}}$$

5.6 Some Integration Strategies

- Simplify the integrand. E.g., $cos^2(\theta) = \frac{1}{2}(1 + cos(2\theta))$
- Check if simple substitution will work
- If integrand is a rational expression, partial functions may work
- If integrand is polynomial x, $trig$, exp, ln function, integration by parts may work
- If integrand involves $\sqrt{b^2x^2 + a^2}$, trigonometric substitution may work
- If integrand has a quadratic in it, completing the square may work.

[1]:
```
import sympy as sp
import numpy as np
import matplotlib.pyplot as plt
from IPython.display import display, Math
from sympy import sqrt, diff, integrate, oo
from sympy import sin, cos, tan, ln, exp, erf, trigsimp, expand_trig, simplify
from sympy import sinh, cosh
```

[2]:
```
%config Completer.use_jedi = False

x = sp.symbols('x')
```

[3]:
```
y = x**2 + x + 1
int = integrate(y,x)              # integrate y wrt x
display(int)
```

$$\frac{x^3}{3} + \frac{x^2}{2} + x$$

[4]:
```
y = exp(-x**2)*erf(x)
int = integrate(y,x)              # integrate y wrt x
display(int)
```

$$\frac{\sqrt{\pi}\operatorname{erf}^2(x)}{4}$$

[5]:
```
y = exp(-x)
int = integrate(y, (x, 0, oo))    # definite integral, limits 0 & infinity
display(int)
```

1

CHAPTER 6

First Order Ordinary Differential Equation

Differential equation is a mathematical statement containing one or more derivatives, i.e., terms representing the rates of change of continuously varying quantities. Differential equations are very common in fields of quantitative study such as science and engineering. Generally, the solution of a differential equation is an equation expressing the functional dependence of one variable upon one or more variables. It ordinarily contains constant terms that are not present in the original differential equation. In other words, the solution of a differential equation produces a function that can be used to predict the behaviour of the original system within certain constraints.

6.1 Differential Equation

A differential equation (DE) is an equation involving an unknown function and its derivatives. A DE is an ordinary differential equation (ODE) if the unknown function depends on only one variable. If the unknown function depends on 2 or more independent variables, the DE is a partial differential equation.

A DE along with subsidiary conditions on the unknown function and its derivatives, all given at the same value of the independent variable, constitutes an initial-value problem. The subsidiary conditions are initial conditions.

If the subsidiary conditions are given at more than one value of the independent variable, the problem is a boundary-value problem and the conditions are the boundary conditions.

6.2 Standard & Differential forms of an ODE

The Standard form for first order DE is:

$$\boxed{\frac{dy}{dx} = f(x,y)}$$

and the differential form is:

$$\boxed{M(x,y)dx + N(x,y)dy = 0}$$

6.3 Order & Degree of a Differential Equation

The order of a differential equation is the order of the highest derivative which is also known as the differential coefficient. E.g.,

$$\frac{d^3x}{dx} + 3x\frac{dy}{dx} = e^y$$

the order of the above differential equation is 3. A first order differential equation is of the form:

$$\boxed{\frac{dy}{dx} + Py = Q} \tag{6.3.1}$$

where P & Q are constants or functions of independent variables. E.g.,

$$\frac{dy}{dx} + (x^2+5)y = \frac{x}{5}$$

The degree of the differential equation is represented by the power of the highest order derivative in the given differential equation.

$$\left[\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right]^4 = k^2\left(\frac{d^3y}{dx^3}\right)^2$$ the degree of the above differential equation is 2.

For the equation:

$$tan\left(\frac{dy}{dx}\right) = x + y$$ the degree is undefined.

6.4 Solving ODE- Method of Separation of Variables

Through algebraic manipulations, some ODEs can be reduced to:

$$\boxed{g(y)\frac{dy}{dx} = f(x)} \tag{6.4.1}$$

By integrating both sides:

$$\int g(y)dy = \int f(x)dx + c$$

Example:

$$\frac{dy}{dx} = 1 + y^2 \implies \frac{dy}{1+y^2} = dx$$

Let $y = tan\,\theta \implies \frac{dy}{d\theta} = sec^2\theta \implies \frac{sec^2\theta}{1+tan^2\theta}d\theta = dx \implies x = \theta + c \implies x = \tan^{-1}y + c$

6.5 Solving ODE - Reduction to Separable Form

Consider the ODE of the form:

$$\boxed{\frac{dy}{dx} = f\left(\frac{y}{x}\right)} \tag{6.5.1}$$

Let $y = ux$

$$\frac{dy}{dx} = x\frac{du}{dx} + u$$

$$f(u) = x\frac{du}{dx} + u$$

$$\frac{du}{f(u)-u} = \frac{dx}{x}$$

$$\int \frac{du}{f(u)-u} = \int \frac{dx}{x} + c$$

$$\boxed{\int \frac{du}{f(u)-u} = \ln|x| + c} \tag{6.5.2}$$

6.6 Solving ODE - Exact ODE & Integrating Factor

If an ODE has an implicit solution :

$$\boxed{u(x,y) = c = constant}$$

then,

$$\boxed{du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0}$$

$$\boxed{M(x,y)dx + N(x,y)dy = 0}$$

$$M = \frac{\partial u}{\partial x}$$
$$N = \frac{\partial u}{\partial y}$$

A 1st order ODE is an exact DE if:

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}}$$

$$\boxed{u = \int M dx + k(y) = \int N dx + l(x)}$$

Example:

$$cos(x+y)dx + (3y^2 + 2y + cos(x+y))dy = 0$$
$$M = \frac{\partial u}{\partial x} = cos(x+y)$$
$$u = sin(x+y) + k(y) \implies \frac{\partial u}{\partial y} = cos(x+y) + \frac{dk}{dy}$$
$$N = \frac{\partial u}{\partial y} = 3y^2 + 2y + cos(x+y) = cos(x+y) + \frac{dk}{dy}$$
$$k = y^3 + y^2 + c^*$$
$$u = sin(x+y) + y^3 + y^2 + c$$

6.7 Inexact ODE

Consider the ODE:

$$-ydx + xdy = 0$$

Here the above approach will not work, because:

$$M = \frac{\partial u}{\partial x} = -y \quad N = \frac{\partial u}{\partial y} = x \quad \frac{\partial M}{\partial y} = \frac{\partial^2 M}{\partial x \partial y} = -1 \quad \frac{\partial N}{\partial x} = \frac{\partial^2 N}{\partial x \partial y} = 1 \quad \frac{\partial^2 M}{\partial x \partial y} \neq \frac{\partial^2 N}{\partial x \partial y} \text{ (inexact)}$$
$$u = -y \int dx + k(y) = -xy + k(y)$$
$$\frac{\partial u}{\partial y} = -x + \frac{dk}{dy}$$

But $N = \dfrac{\partial u}{\partial y} = x$ which contradicts the above equation

6.8 Integrating Factor to transform to an exact ODE

Multiply the equation by a factor $F(x,y)$ to make it exact.

$$FMdx + FNdy = 0$$

and impose the conditions:

$$\frac{\partial}{\partial y}(FM) = \frac{\partial}{\partial x}(FN) \rightarrow F_y M + FM_y = F_x N + FN_x$$

Let F depend only on x,

$$FM_y = F^{'}N + FN_x$$

$$\frac{M_y}{N} = \frac{F^{'}}{F} + \frac{N_x}{N}$$

$$\int \frac{df}{F}dx = \int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx$$

Let $\boxed{R = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}$

$$ln(F) = \int Rdx \implies \boxed{F(x) = e^{\int R(x)dx}}$$

Similarly,

$$R^* = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \implies \boxed{F(y) = e^{\int R^*(y)dy}}$$

E.g., solve:

$$(e^{x+y} + ye^y)dx + (xe^y - 1)dy = 0$$

$$M = \frac{\partial u}{\partial x} = e^{x+y} + ye^y \quad N = \frac{\partial u}{\partial y} = xe^y - 1$$

$$\frac{\partial M}{\partial y} = e^{x+y} + ye^y + e^y \quad \frac{\partial N}{\partial x} = e^y \quad \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = e^{x+y} + ye^y$$

$$R = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xe^y - 1}\left(e^{x+y} + ye^y\right)$$

R does not work as it is a function of both x and y. So we try with R^*

$$R^* = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{-1}{e^{x+y} + ye^y}\left(e^{x+y} + ye^y\right) = -1$$

$e^{\int R^* dy} = e^{-y}$ this works as it is a function y only

Multiplying the ODE by $e^{R^*} = e^{-y}$

$$(e^x + y)dx + (x - e^{-y})dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = 1 \text{ (exact ODE!)}$$

$$M = \frac{\partial u}{\partial x} = e^x + y \implies u = e^x + xy + k(y) \implies \frac{\partial u}{\partial y} = x + \frac{dk}{dy} = x - e^{-y} \implies k = e^y + c^*$$

$$u = e^x + xy + e^y + c$$

6.9 1st Order Linear ODE - Homogeneous

A first order ODE is linear if it is of the following form:

$$\boxed{\frac{dy}{dx} + p(x)y = r(x)}$$

and is non-linear if it cannot be brought to the above form. The above ODE is linear in both y and y' where p and q are any function of x. When $r(x) = 0$, the ODE is called homogeneous.

$$\boxed{\frac{dy}{dx} + p(x)y = 0}$$

By the method of separation of variables we have,

$$\int \frac{dy}{y} = -\int p(x)dx \implies \ln|y| = -\int p(x)dx + c^*$$

$$\boxed{y = ce^{-\int p(x)dx} \text{ (homogeneous solution } y_h)}$$

6.10 1st Order ODE - Non Homogeneous

When $r(x) \neq 0$, the ODE is called non homogeneous. We multiply the ODE by a function $F(x)$.

$$F y' + F p(x)y = F r(x)$$

$$\text{Let } F p(x) = F' \implies \frac{F'}{F} = p(x) \implies \ln|F| = \int p(x)dx \quad \text{Let } h = \int p(x)dx \implies F = e^h$$

$$\text{Now } (F y)' = F r(x) \implies (e^h y)' = r(x)e^h \implies e^h y = \int e^h r(x)dx + c$$

$$\boxed{y_p = e^{-h}\int e^h r dx + c} \quad \boxed{y = y_h + y_p = ce^{-h} + e^{-h}\int e^h r dx + c}$$

6.11 Reduction to Linear Form - Bernoulli Equation

The Bernoulli equation, a non-linear ODE is given by:

$$\boxed{y' + p(x)y = r(x)y^n} \tag{6.11.1}$$

where n is any real number.

$$\text{Let } u = y^{1-n}$$
$$u' = (1-n)y^{-n}y'$$
$$u' = (1-n)y^{-n}(ry^n - py)$$
$$u' = (1-n)(r - py^{1-n})$$
$$u' = (1-n)(r - pu)$$
$$u' + (1-n)pu = (1-n)r \text{ (Linear ODE)}$$

[1]:
```
from sympy import Function, dsolve, Eq, diff, Derivative, sin, cos, symbols, pprint
```

[13]:
```
x = Function('x')
t = symbols('t')
deq = Eq(diff(x(t),t), x(t)) # Eq(LHS, RHS)
display(deq)
xsoln = dsolve(deq, x(t))     # dsolve wrt x(t)
display(xsoln)
```

$$\frac{d}{dt}x(t) = x(t)$$

$$x(t) = C_1 e^t$$

[6]:
```
x = Function('x')
t = symbols('t')
deq = Eq(diff(x(t),t), (x(t) - 900) / 2)
display(deq)
xsoln = dsolve(deq, x(t))
display(xsoln)
```

$$\frac{d}{dt}x(t) = \frac{x(t)}{2} - 450$$

$$x(t) = C_1 e^{\frac{t}{2}} + 900$$

[5]:
```
y = Function('y')
x = symbols('x')
deq = Eq(diff(y(x),x), (1 + y(x)**2))
display(deq)
xsoln = dsolve(deq, y(x))
display(xsoln)
```

$$\frac{d}{dx}y(x) = y^2(x) + 1$$

$$y(x) = -\tan(C_1 - x)$$

CHAPTER 7

Second Order Ordinary Differential Equation

Second order differential equations have a variety of applications in science and engineering such as vibrations and electric circuits. There are a host of multi dimensional engineering models that incorporate second order differential equations including wave motion, flow mechanics, Maxwell's electro-magnetic equations and Schroedinger equation in Nuclear Physics.

7.1 Power Series

7.1.1 Power Series, Taylor Series & Maclaurin Series

Consider the following function that is represented as a power series.

$$
\begin{aligned}
&f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n \\
&f(a) = c_0 \\
&f^{'}(a) = c_1 \\
&f^{''}(a) = 2c_2 \rightarrow c_2 = \frac{1}{2}f^{''}(a) \\
&f^{'''}(a) = 3\times 2c_3 \rightarrow c_3 = \frac{1}{3!}f^{'''}(a) \\
&\vdots \\
&f^n(a) = n(n-1)(n-2)...1 \rightarrow c_n = \frac{1}{n!}f^n(a)
\end{aligned}
$$

If $f^n(x)$ exists at $x = a$, the Taylor series for $f(x)$ at a is given by:

$$
\boxed{\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a) = f(a) + f^{'}(a)(x-a) + \frac{f^{''}(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n + \ldots} \tag{7.1.1}
$$

A Maclaurin series is a Taylor series expansion about 0.

$$
\boxed{f(x) = f(0) + f^{'}(0)x + \frac{f^{''}(0)}{2!}x^2 + \ldots \frac{f^n(0)}{n!}x^n + \ldots} \tag{7.1.2}
$$

The series solution may or may not converge at $x = x_p$. To converge, for any ϵ, there is exists an N that satisfies:

$$
|R_n(x_p)| = |s(x_p) - s_n(x_p)| < \epsilon\, \forall n > N \quad \text{(for all } n > N)
$$

where $s_n(x)$ is the nth partial sum:

$$s_n(x) = a_0 + a_a(x - x_0) + \cdot + a_n(x - a_0)^n$$

and $R_n(x_p)$ is the remainder.

$$R_n(x) = a_{n+1}(x - x_0)^{n+1} + a_{n+2}(x - x_0)^{n+2} + \dots$$

The convergence interval is $|x - x_0| < R$ (radius of convergence). This means that in the case of convergence, we can approximate the sum $s(x_1)$ by $s_n(x_1)$ as accurately as we want by taking a large enough n. $f(x)$ is called analytic at a point $x = x_0$ if it can be represented by a power series in powers of $x - x_0$ with a positive radius of convergence R. This means that a real analytic function has to be an infinitely differentiable function.

7.2 2nd Order Linear ODE

The standard form of a 2nd Order ODE is:

$$\boxed{y'' + p(x)y' + q(x)y = r(x)} \quad \text{It is linear in } y, y' \text{ and } y'' \tag{7.2.1}$$

If $r(x) = 0$, the ODE is homogeneous, else it is non-homogeneous. When the coefficients a and b are constant:

$$\boxed{y'' + ay' + by = 0} \tag{7.2.2}$$

Choose $e^{\lambda x}$ as a solution and substitute in the homogeneous ODE.

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0 \implies \lambda^2 + a\lambda + b = 0 \implies \lambda = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right)$$

$$\boxed{y_h = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}} \quad \text{(general solution to the homogeneous ODE)}$$

y_1, corresponding to λ_1, and y_2, corresponding to λ_2, are linearly independent and are called basis of solutions. The superposition principle also called the linearity principle, i.e., the homogeneous solution is a combination of y_1 and y_2 is true only for linear homogeneous ODE. The arbitrary constants c_1 and c_2 are determined from the initial conditions:

$$y(x_0) = k_0 \quad y'(x_0) = k_1$$

A particular solution is obtained if we assign specific values to c_1 and c_2.

7.3 Lagrange's Method of Reduction of Order

Consider a linear homogeneous 1st Order ODE in its standard form:

$$\boxed{y'' + p(x)y' + q(x)y = 0} \tag{7.3.1}$$

If y_1 is a basis solution, we can find y_2 as follows:

Let $y = y_2 = uy_1 \implies y_2' = u'y_1 + uy_1' \implies y_2'' = u''y_1 + 2u'y_1' + uy_1''$

Substituting, $(u''y_1 + 2u'y_1' + uy_1'') + p(u'y_1 + uy_1') + q(uy_1) = 0$

$y_1 u'' + (2y_1' + py_1)u' + (y_1'' + py_1' + qy_1)u = 0$

$$u'' + u' \frac{2y_1' + py_1}{y_1} = 0$$

Let $U = u^{'} \implies U^{'} + U\left(\frac{2y_1^{'}}{y_1} + p\right) = 0 \implies \frac{U'}{U} = -\left(\frac{2y_1^{'}}{y_1} + p\right)$

$$\int \frac{U'}{U}dx + \int \left(\frac{2y_1^{'}}{y_1}\right)dx = -pdx \implies \ln|U| + 2\ln|y_1| = -\int pdx \implies \ln|Uy_1^2| = -\int pdx$$

$$Uy_1^2 = e^{\int -pdx}$$

$$\boxed{U = \frac{1}{y_1^2}e^{\int -pdx}} \quad \boxed{u = \int Udx} \quad \boxed{y_2 = y_1 \int Udx}$$

$$\boxed{y_2 = y_1 \int \frac{1}{y_1^2}e^{\int -pdx}dx}$$

7.4 Homogeneous Linear ODE with Constant Coefficients

$$\begin{cases} \text{Case 1: 2 Real Roots when} & a^2 - 4b > 0 \\ \text{Case 2: Double Root when} & a^2 - 4b = 0 \\ \text{Case 3: Complex Conjugate Roots when} & a^2 - 4b < 0 \end{cases}$$

Case 1: 2 Real Roots when $a^2 - 4b > 0$. The general solution is given by:

$$y_1 = e^{\lambda_1 x} \quad y_2 = e^{\lambda_2 x}$$

$$\lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right) \quad \lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right)$$

$$\boxed{y_h = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}} \tag{7.4.1}$$

Case 2: $\boxed{\lambda_1 = -\frac{a}{2}, y_1 = e^{-\frac{ax}{2}}}$ Determine y_2 using the method of reduction of order.

$$y_1 = e^{\frac{-ax}{2}}$$

$$y_2 = y_1 \int \frac{1}{y_1^2}e^{\int -pdx}dx = e^{\frac{-ax}{2}} \int \frac{1}{\left(e^{\frac{-ax}{2}}\right)^2}e^{\int -adx}dx = e^{-\frac{ax}{2}} \int e^{ax}e^{-ax}dx = xe^{-\frac{ax}{2}}$$

$$y_h = c_1 e^{-ax/2} + c_2 x e^{-\frac{a}{2}x} \implies \boxed{y_h = (c_1 + c_2 x)e^{-ax/2}}$$

Case 3: $\boxed{\lambda = -\frac{a}{2} \pm \mathbf{i}w, w = \sqrt{|a^2 - 4b|}}$

$$y_1 = e^{\lambda_1 x} = e^{(-\frac{a}{2} + \mathbf{i}w)x} = e^{-\frac{ax}{2}}e^{\mathbf{i}wx} \qquad y_2 = e^{\lambda_2 x} = e^{(-\frac{a}{2} - \mathbf{i}w)x} = e^{-\frac{ax}{2}}e^{-\mathbf{i}wx}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots = cos\,x$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \ldots = (1 - \frac{x^2}{2!} + \ldots) + i(x - \frac{x^3}{3!} + \ldots) = sin\,x$$

$\implies \boxed{e^{iwx} = cos\,wx + \mathbf{i}sin\,wx}$ (de Moivre's theorem) and $\boxed{e^{\mathbf{i}\pi} = -1}$ (Euler's Identity)

The general solution is given by, $\boxed{y = e^{-\frac{ax}{2}}(c_1 cos\,wx + c_2 sin\,wx)}$ c_1, c_2 are constants

7.5 Euler-Cauchy Equations

The Euler-Cauchy equation is of the form:

$$\boxed{x^2y'' + axy' + by = 0} \quad \text{where } a, b \text{ are constants} \tag{7.5.1}$$

Let $y = x^m \implies y' = mx^{m-1} \implies y'' = m(m-1)x^{m-2}$
Substituting, $x^2m(m-1)x^{m-2} + axmx^{m-1} + bx^m = 0 \implies m^2 + (a-1)m + b = 0$

$$\boxed{m = \frac{1}{2}(1-a) \pm \sqrt{\frac{1}{4}(a-1)^2 - b}}$$

Case 1: Roots are distinct. The basis solutions are :

$y_1(x) = x^{m_1} \quad y_2(x) = x^{m_2}$, the general solution is given by, $\boxed{y = c_1x^{m_1} + c_2x^{m_2}}$

Case 2: Double roots.

$$b = \frac{1}{4}(1-a)^2 \quad m = \frac{1}{2}(1-a) \quad \boxed{y_1 = x^{\frac{1}{2}(1-a)}}$$

$$y'' + \frac{a}{x}y' + \frac{(1-a)^2}{4x^2}y = 0$$

Use method of reduction of order, $y_2 = uy_1$ and with $\boxed{p = \frac{a}{x}}$

$$\boxed{U = \frac{1}{y_1^2}e^{\int -pdx}} \quad \boxed{u = \int Udx} \quad \boxed{y_2 = y_1\int Udx} \quad \boxed{y_2 = y_1\int \frac{1}{y_1^2}e^{\int -pdx}dx}$$

$$\int pdx = \int \frac{a}{x}dx = alnx \implies e^{\int -pdx} = e^{-alnx} = e^{lnx^{-a}} = x^{-a} = \frac{1}{x^a}$$

$$U = \frac{1}{y_1^2}\frac{1}{x^a} = \frac{1}{x^{1-a}}\frac{1}{x^a} = \frac{1}{x} \implies u = \int Udx = \int \frac{1}{x}dx = lnx$$

$$y_2 = y_1\int Udx = x^{\frac{1}{2}(1-a)}lnx$$

$$\boxed{y_h = (c_1 + c_2lnx)x^{\frac{1}{2}(1-a)}} \quad c_1, c_2 \text{ are constants}$$

7.6 The Wronskian

Two solutions y_1 and y_2 are linearly dependent if their Wronkskian W is 0.

$W(y_1, y_2) = y_1y_2' - y_2y_1' = 0$
Because if the solutions are dependent, $\quad y_1 = ky_2$, where k is a constant
$\implies W(y_1, y_2) = y_1y_2' - y_2y_1' = ky_2y_2' - y_2ky_2' = 0$

The Wronksian is expressed as a Wronski Determinant:

$$\boxed{W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \tag{7.6.1}$$

7.7 Non-homogeneous ODE

Consider the following non-homogeneous ODE:

$$y'' + p(x)y' + q(x)y = r(x)$$

The complete solution is the sum of homogeneous (y_h) and particular (y_p)solutions.

$$y(x) = y_h(x) + y_p(x) \text{ where } y_h = c_1y_1 + c_2y_2 \quad \text{(general solution)}$$

y_p is a solution of the non-homogeneous equation without any constants. A particular solution is obtained by assigning specific values to the constants. The Method of Undetermined Coeffcients is an approach to finding a particular solution to nonhomogeneous ODEs. If the term in $r(x)$ contains the following term, the choice for $y_p(x)$ is given by:

Term in r(x)	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$Kx^n (n = 0, 1, \ldots)$	$K_nx^n + K_{n-1}x^{n-1} + \ldots + K_1x + K_0$
$k cos wx$ or $k sin wx$	$K cos wx + M sin wx$
$ke^{\alpha x} cos wx$ or $ke^{\alpha x} sin wx$	$e^{\alpha x}(K cos wx + M sin wx)$

7.8 Particular Solution by Variation of Parameters (Lagrange)

The particular solution for the standard form ODE is derived as follows:

$$y'' + p(x)y' + q(x)y = r(x)$$

Find a pair of functions $u_1(x)$ and $u_2(x)$ such that:

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \implies y_p'(x) = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$$

Set constraint, $\boxed{u_1'y_1 + u_2'y_2 = 0}$

$$y_p'(x) = u_1y_1' + u_2y_2'$$
$$y_p''(x) = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$$

Substituting,

$$(u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'') + p(u_1y_1' + u_2y_2') + q(u_1y_1 + u_2y_2) = r$$
$$(y_1'' + py_1' + qy_1)u_1 + (y_2'' + py_2' + qy_2)u_2 + (u_1'y_1' + u_2'y_2') = r$$

Since y_1 and y_2 are solutions to the homogeneous ODE,

$$u_1'y_1' + u_2'y_2' = r$$

We now have the following simultaneous equations:

$$u_1'y_1 + u_2'y_2 = 0$$
$$u_1'y_1' + u_2'y_2' = r$$

Solving,

$$u_1' = -\frac{y_2r}{y_1y_2' - y_1'y_2} = -\frac{y_2r}{W}$$
$$u_2' = -\frac{y_1r}{y_1y_2' - y_1'y_2} = -\frac{y_1r}{W}$$

$$\boxed{y_p(x) = -y_1\int\frac{y_2r}{W}dx + y_2\int\frac{y_1r}{W}dx}$$

[148]:
```
import sympy as sp
from sympy import sin, cos, tan, exp, E, I, simplify
from sympy.abc import x, y, z, t, w
```

[172]:
```
fun = E**(x)
display(fun.series(x,n=10))

fun = E**(I*w*x)
display(fun.series(x,n=10))

fun = cos(w*x)
s1 = fun.series(x,n=10)
display(s1)

fun = I*sin(w*x)
s2 = fun.series(x,n=10)
display(s2)

display(s1+s2)
```

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362880} + O\left(x^{10}\right)$$

$$1 + iwx - \frac{w^2x^2}{2} - \frac{iw^3x^3}{6} + \frac{w^4x^4}{24} + \frac{iw^5x^5}{120} - \frac{w^6x^6}{720} - \frac{iw^7x^7}{5040} + \frac{w^8x^8}{40320} + \frac{iw^9x^9}{362880} + O\left(x^{10}\right)$$

$$1 - \frac{w^2x^2}{2} + \frac{w^4x^4}{24} - \frac{w^6x^6}{720} + \frac{w^8x^8}{40320} + O\left(x^{10}\right)$$

$$iwx - \frac{iw^3x^3}{6} + \frac{iw^5x^5}{120} - \frac{iw^7x^7}{5040} + \frac{iw^9x^9}{362880} + O\left(x^{10}\right)$$

$$1 + iwx - \frac{w^2x^2}{2} - \frac{iw^3x^3}{6} + \frac{w^4x^4}{24} + \frac{iw^5x^5}{120} - \frac{w^6x^6}{720} - \frac{iw^7x^7}{5040} + \frac{w^8x^8}{40320} + \frac{iw^9x^9}{362880} + O\left(x^{10}\right)$$

CHAPTER 8

Higher Order ODE

8.1 Higher Order Homogeneous ODE

The concepts of the 2nd Order ODE can be extended to higher order ODE which has the form:

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots+p_1(x)y'+p_0(x)y=r(x)$$

For constant coefficients, $y=e^{\lambda x}$ yields $\lambda^{(n)}+a_{n-1}\lambda^{(n-1)}+\ldots+a_1\lambda+a_0=0$ (characteristic equation). For n distinct roots, there are n distinct basis solutions:

$$y=c_1e^{\lambda^1 x}+c_2e^{\lambda^2 x}+\ldots+c_ne^{\lambda^n x}$$

The Wronkskian is given by:

$$W(y_1,y_2,\ldots,y_n)=\begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ . & . & \ldots & . \\ y_1^{(n-1)} & y_2^{(n-1)} & \ldots & y_n^{(n-1)} \end{vmatrix}=E\begin{vmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_n^2 \\ . & . & \ldots & . \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \ldots & \lambda_n^{n-1} \end{vmatrix}$$

Where $E=e^{(\lambda_1+\lambda_2+\ldots+\lambda_n)x}$. $W=0$ if and only if the determinant, known as the Vandermonde or Cauchy determinant, is zero. $W\neq 0$, if and only if, all the n roots are different.

If a real double root occurs, say, $\lambda_1=\lambda_2$, then we take y_1 and xy_1 as corresponding linearly independent solutions. If λ is a real root of order m, then the corresponding basis solutions are:

$$e^{\lambda x}, xe^{\lambda x}, x^2e^{\lambda x}, \ldots, x^{m-1}e^{\lambda x}$$

Complex roots occur in conjugate pairs $\lambda=\gamma\pm \mathbf{i}w$ since the coefficients of the ODE are real.

$$y_1=e^{\gamma x}cos\ wx \qquad y_2=e^{\gamma x}sin\ wx$$

If $\lambda=\gamma+\mathbf{i}w$ is a complex double root, so is the conjugate $\lambda=\gamma-\mathbf{i}w$ and the corresponding linearly independent solutions are:

$$e^{\gamma x}cos\,wx \qquad e^{\gamma x}sin\,wx \qquad xe^{\gamma x}cos\,wx \qquad xe^{\gamma x}sin\,wx$$

The first two of these result from $e^{\lambda x}$ and $e^{\bar{\lambda}x}$ as before, and the second two from $xe^{\lambda x}$ and $xe^{\bar{\lambda}x}$ in the same fashion. The corresponding general solution is:

$$y=e^{\gamma x}[(A_1+A_2x)cos\,wx+(B_1+B_2x)sin\,wx]$$

For complex triple roots, which is quite rate, one would obtain two more solutions:

$$x^2e^{\gamma x}cos\,wx \qquad x^2e^{\gamma x}sin\,wx$$

8.2 Higher Order Non-Homogeneous ODE

8.2.1 Method of Undetermined Coefficients

Apply the method of undetermined coefficients for solving 2nd order ODE with a modification. If a term in the choice for $y_p(x)$ is a solution of the homogeneous equation, then multiply this term by x^k, where k is the smallest positive integer and satisfies the condition that this term $\times x^k$ is not a solution of the homogeneous equation. So, we try $cxe^{\lambda x}, cx^2e^{\lambda x}, \ldots, cx^ke^{\lambda x}$ as a solution, plug into the ODE, and solve for c for the minimum k.

8.2.2 Method of Variation of Parameters

Extending the concept that we used for 2nd order ODE to arbitrary order n we have:

$$\boxed{y_p(x) = \sum_{k=1}^{n} y_k(x) \int \frac{W_k(x)}{W(x)} r(x)dx}$$

8.3 Series Solutions of Homogeneous ODEs

Higher order linear ODEs with constant coefficients can be solved by algebraic methods as their solutions are often elementary functions which are known from calculus. For ODEs with variable coefficients the situation is complicated and their solutions are nonelementary special functions, e.g., Legendre and Bessel functions.

8.3.1 Power Series Method

$$\boxed{y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3^3 + \ldots}$$

Compute $y', y'', \ldots, y^{(n)}$, substitute in the ODE and compute the coefficients of the powers of $x, x^2, x^3, \ldots, x^n$. Equate each of the coefficients to 0 to determine $a_0, a_1, a_2, \ldots, a_n$.

8.4 Existence of Power Series Solutions

Consider the following ODE:

$$\boxed{y'' + p(x)y' + q(x)y = r(x)}$$

If p, q, r have Taylor series representations (analytic) then every solution of the ODE can be represented by a power series in powers of $x - x_0$ with a positive radius of convergence R. A power series can be added, multiplied and differentiated term by term.

8.5 Classical Differential Equations

Legendre: $\boxed{(1-x^2)y'' - 2xy' + k(k+1)y = 0}$

Chebyshev: $\boxed{(1-x^2)y'' - xy' + k^2 y = 0}$

Herimite: $\boxed{y'' - 2xy' + 2ky = 0}$

Laguerre: $\boxed{xy'' + (1-x)y' + ky = 0}$

where k is a constant

8.6 Legendre's Equation

$\boxed{(1-x^2)y'' - 2xy' + k(k+1)y = 0}$ k is a constant

Let $\boxed{y = a_n \sum_{n=0}^{\infty} x^n}$

Compute y, y', y'' and substitute in the above equation.

$$y' = na_n \sum_{n=0}^{\infty} x^{n-1} \qquad y'' = n(n-1)a_n \sum_{n=0}^{\infty} x^{n-2}$$

$$(1-x^2)\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=0}^{\infty} na_n x^{n-1} + k(k+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

Since $n(n-1)$ is 0 for $n=0$ and $n=1$, the lower indices start from 2 and 1.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{\infty} na_n x^n + k(k+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

Let $n-2=m$ and use m as the index in the remaining terms as it is a dummy index:

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m - \sum_{m=2}^{\infty} m(m-1)a_m x^m - 2\sum_{m=1}^{\infty} ma_m x^m + k(k+1)\sum_{m=0}^{\infty} a_m x^m = 0$$

a_0 and a_1 are abitrary constants, the remaining constants are expressed in terms of these.

$$m=0 \implies 2a_2 + k(k+1)a_0 = 0 \qquad \implies \boxed{a_2 = -\frac{-k(k+1)}{2!}a_0}$$

$$m=1 \implies 6a_3 + [-2+k(k+1)]a_1 = 0 \implies \boxed{a_3 = -\frac{(k-1)(k+2)}{3!}a_1}$$

$$m \geq 2 \implies (m+2)(m+1)a_{m+2} = [m(m-1)+2m-k(k+1)]a_m = (m^2+m-k^2-k)a_m$$

$$\boxed{a_{m+2} = -\frac{(k-m)(k+m+1)}{(m+1)(m+2)}a_m}$$

$$\boxed{a_4 = \frac{(k-2)k(k+1)(k+3)}{4!}a_0}$$

$$\boxed{a_5 = \frac{(k-3)(k-1)(k+2)(k+4)}{5!}a_1}$$

$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \ldots + a_nx^n + \ldots$

$y = a_0y_1(x) + a_1y_2(x)$ a_0, a_1 are arbitrary constants, y_1 is the even series & y_2 is the odd)

$y_1 = 1 + a_2x^2 + a_4x^4 + \cdots$

$y_2 = x + a_3x^3 + a_5x^5 + \cdots$

8.6.1 Legendre Polynomials

When:

$$\boxed{m = k,\ a_{m+2} = a_{m+4} = a_{m+6} \cdots = 0}$$

If k is even, $y_1(x)$ reduces to a polynomial of degree k.
If k is odd, $y_2(x)$ reduces to a polynomial of degree k.

The reduction of power series to polynomials is a great advantage because then we have solutions for all x without convergence restrictions. These polynomials, multiplied by some constants, are called Legendre polynomials and are denoted by $P_n(x)$.

The standard choice of such constants is to choose the coefficient an of the highest power x^n as:

$$\boxed{a_k = \frac{(2k)!}{2^k(k!)^2}}$$

We then calculate the other coefficients as follows:

$$\boxed{a_m = -\frac{(m+1)(m+2)}{(k-m)(k+m+1)}a_{m+2}}$$

With $m = k - 2$

$$\begin{aligned}
a_{k-2} &= -\frac{k(k-1)}{2(2k-1)}a_k \\
&= -\frac{k(k-1)}{2(2k-1)}\frac{2k!}{2^k(k!)^2} \\
&= -\frac{k(k-1)}{2(2k-1)}\frac{2k(2k-1)(2k-2)!}{2^k k(k-1)!k(k-1)(k-2)!} \\
&= \frac{(2k-2)!}{2^k(k-1)!(k-2)!}
\end{aligned}$$

With $m = k - 4$

$$\begin{aligned}
a_{k-4} &= \frac{(k-2)(k-3)}{4(2k-3)}a_{k-2} \\
&= \frac{(k-2)(k-3)}{4(2k-3)}\frac{(2k-2)!}{2^k(k-1)!(k-2)!} \\
&= \frac{(2k-4)!}{2^k 2!(k-2)!(k-4)!}
\end{aligned}$$

In general,

$$\boxed{a_{k-2m} = (-1)^m\frac{(2k-2m)!}{2^k m!(k-m)!(k-2m)!}}$$

8.7 Frobenius Method

Several important 2nd order ODEs have coefficients that are not analytic. Yet these ODEs can be solved through an extension of the power series method that is credited to Frobenius. Consider the ODE:

$$y'' + \frac{b(x)}{x}y' + \frac{c(x)}{x^2}y = 0 \quad \text{Note: } b(x), c(x) \text{ are analytic at } x = 0$$

This ODE has at least one solution of the form:

$$\boxed{y(x) = x^r \sum_{m=0}^{\infty} a_m x^m}$$

Where r is real or complex and $a_0 \neq 0$.

Multiply the ODE by x^2 and expand b(x) and c(x) in Taylor series.

$$x^2y'' + xb(x)y' + c(x)y = 0$$

$$b(x) = \sum_{m=0}^{\infty} b_m x^m \quad c(x) = \sum_{m=0}^{\infty} c_m x^m$$

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m \quad y'(x) = \sum_{m=0}^{\infty} (m+r)a_m x^{m+r-1} \quad y''(x) = \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r-2}$$

Substituting in the ODE,

$$x^r[r(r-1)a_0 + \cdots] + (b_0 + b_1x + \cdots)x^r(ra_0 + \cdots) + (c_0 + c_1x + \cdots)x^r(a_0 + a_1x + \cdots) = 0$$

Equate coefficients of x^r, x^{r+1}, x^{r+2} to 0.

$[r(r-a) + b_0r + c_0]a_0 = 0 \implies [r(r-a) + b_0r + c_0] = 0$ (indicial equation)

The Frobenius method yields a basis of solutions.

Distinct roots not differing by an integer

$y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \ldots)$

$y_2(x) = x^{r_2}(A_0 + A_1x + A_2x^2 + \ldots)$

Double root $r_1 = r_2 = r = \frac{1}{2}(1 - b_0)$

$y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \ldots)$

$y_2(x) = y_1(x)ln x + x^{r_1}(A_0 + A_1x + A_2x^2 + \ldots)$

Roots differing by an integer

$y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \ldots)$

$y_2(x) = ky_1(x)ln x + x^{r_2}(A_0 + A_1x + A_2x^2 + \ldots)$

$r_1 > r_2, k$ can be 0

For cases 2 and 3, the second independent solution can be obtained by reduction of order.

8.8 Bessel's Equation

$\boxed{x^2y'' + xy' + (x^2 - \nu^2)y = 0}$ (ν is a real number ≥ 0)

Applying Frobenius technique, the solution is of the form

$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}$

$y'(x) = \sum_{m=0}^{\infty} (m+r)a_m x^{m+r-1} = x^{r-1}[ra_0 + (r+1)a_1x + (r+2)a_2x^2 + \ldots]$

$y''(x) = \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r-2} = x^{r-2}[r(r-1)a_0 + (r+1)ra_1x + (r+2)(r+1)a_2x^2 + \ldots]$

substituting in the ODE

$\sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r} + \sum_{m=0}^{\infty} (m+r)a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} - \nu^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0$

$r(r-1)a_0 + ra_0 - \nu^2 a_0 = 0$ $(m = 0)$

$(r+\nu)(r-\nu) = 0, \implies \boxed{r = \pm\nu}$

$(r+1)ra_1 + (r+1)a_1 - \nu^2 a_1 = 0$ $(m = 1)$

$((\nu+1)\nu + (\nu+1) - \nu^2)a_1 = 0 \implies (2\nu+1)a_1 = 0 \implies \boxed{a_1 = 0}$

$(m+r)(m+r-1)a_m + (m+r)a_m + a_{m-2} - \nu^2 a_m = 0$ $(m = 2, 3, \ldots)$

$(m+\nu)[(m+\nu-1+(m+\nu) - \nu^2]a_m + a_{m-2} = 0 \implies \boxed{m(m+2\nu)a_m + a_{m-2} = 0}$

since $a_1 = 0 \implies \boxed{a_3 = a_5 = \ldots = 0}$

$2m(2m+2\nu)a_{2m} + a_{2m-2} = 0$ (ensure even numbers only, $m = 1, 2, \ldots$)

$a_{2m} = -\frac{a_{2m-2}}{2^2 m(m+\nu)}$ $(m = 1, 2, \ldots)$

$a_2 = -\frac{a_0}{2^2(\nu+1)}$

$a_4 = -\frac{a_2}{2^2 2(\nu+2)} = \frac{a_0}{2^4 2!(\nu+1)(\nu+2)}$

When ν is an integer, denote it as by n

$a_{2m} = -\frac{(-1)^n a_0}{2^{2m} m!(n+1)(n+2)\ldots(n+m)}$ $(m = 1, 2, \ldots)$

choose, $a_0 = \frac{1}{2^n n!}$

$$\boxed{a_{2m} = \frac{(-1)^m}{2^{2m+n} m!(n+m)!}} \quad (m = 1, 2, \ldots)$$

A particular solution to Bessel's equation is then given by,

$$\boxed{J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!}} \quad (m = 1, 2, \ldots, \text{ and } n \geq 0)$$

$J_n(x)$ is called the Bessel function of the first kind of order n and converges $\forall x$.

For $n = 0$, $J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m!^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} + \ldots$ (Bessel function of order 0, similar to cosine)

For $n = 1$, $J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m} m!(m+1)!} = \frac{x}{2} - \frac{x^3}{2^3 1! 2!} + \frac{x^5}{2^5 2! 3!} + \ldots$ (Bessel function of order 1, similar to sine)

8.8.1 Bessel functions for real number

Choose $a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}$ where the Gamma function is defined as:

$\boxed{\Gamma(\nu+1) = \int_0^\infty e^{-t} t^\nu dt}$ $\quad (\nu > -1)$

$\Gamma(\nu+1) = -e^{-t} t^\nu \Big|_0^\infty + \nu \int_0^\infty e^{-t} t^{\nu-1} dt = 0 + \nu\Gamma(\nu)$

$\boxed{\Gamma(\nu+1) = \nu\Gamma(\nu)}$ for $n = 0, 1, \ldots$ $\boxed{\Gamma(n+1) = n!}$ (The Gamma function is a generalised factorial)

$a_{2m} = -\frac{(-1)^m a_0}{2^{2m} m!(\nu+1)(\nu+2)...(\nu+m) 2^\nu \Gamma(\nu+1)}$
$a_{2m} = -\frac{(-1)^m a_0}{2^{2m+\nu} m! \Gamma(\nu+m+1)}$

$$\boxed{J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m! \Gamma(\nu+m+1)!}}$$

$J_\nu(x)$ is called the Bessel function of the first kind of order ν

Bessel functions satisfy many relationships such as the following:

$[x^\nu J_\nu(x)]' = x^\nu J_{\nu-1}(x)$	$[x^{-\nu} J_\nu(x)]' = -x^{-\nu} J_{\nu+1}(x)$
$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$	$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_\nu(x)$
$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} sinx$	$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} cosx$

8.8.2 General Solution

For a general solution of Bessel's equation in addition to J_ν we need a second linearly independent solution. If ν is not an integer, the general solution can be obtained by replacing ν with $-\nu$. The general solution is then given by:

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$$

This cannot be the general solution for an integer $\nu = n$ because that will lead to linear dependence.

8.8.3 Bessel functions of the second kind, $Y_\nu(x)$

For $n = 0$, the Bessel function can be written as:

$$xy'' + y' + xy = 0$$

The indical equation has a double root and the desired solution must be of the form:

$$\boxed{y_2(x) = J_0 ln\, x + \sum_{m=1}^{\infty} A_m x^m}$$

$$y_2' = J_0' ln\, x + \frac{J_0}{x} + \sum_{m=1}^{\infty} m A_m x^{m-1}$$

$$y_2'' = J_0'' ln\, x + \frac{2J_0'}{x} - \frac{J_0}{x^2} + \sum_{m=1}^{\infty} m(m-1) A_m x^{m-2}$$

Substituting y_2'', y_2', y in the equation we have:

$$(xJ_0''lnx + 2J_0' - \cancel{\frac{J_0}{x}} + \sum_{m=1}^{\infty} m(m-1)A_m x^{m-1}) + (J_0' lnx + \cancel{\frac{J_0}{x}} + \sum_{m=1}^{\infty} mA_m x^{m-1}) +$$
$$(xJ_0 lnx + \sum_{m=1}^{\infty} A_m x^{m+1}) = 0$$

$$\underbrace{(xJ_0'' + J_0' + xJ_0)}_{0} lnx + 2J_0' + \sum_{m=1}^{\infty} m(m-1)A_m x^{m-1} + \sum_{m=1}^{\infty} mA_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0$$

$$2J_0' + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0$$

Now, $J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m!^2}$

$$J_0'(x) = \sum_{m=1}^{\infty} \frac{(-1)^m 2m x^{2m-1}}{2^{2m} m!^2} = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-1} m!(m-1)!}$$

$$\sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-2} m!(m-1)!} + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0$$

The power of x_0 occurs only in the 2nd series, hence $\boxed{A_1 = 0}$.

Comparing coeffient of even powers of x in 2nd & 3rd series (1st series has none), we have:

$(2s+1)^2 A_{2s+1} + A_{2s-1} = 0$ (where $s = 0, 1, 2, \cdots$)

Since $A_1 = 0 \implies A_3 = A_5 = \cdots = 0$

$$-1 + 4A_2 = 0 \implies \boxed{A_2 = \frac{1}{4}}$$

Matching the odd power of x in all 3 series, we have:

$$\frac{(-1)^{s+1}}{2^{2s}(s+1)!s!} + (2s+2)^2 A_{2s+2} + A_{2s} = 0$$

$$\boxed{A_{2m} = \frac{(-1)^{m-1}}{2^{2m}(m!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right) \quad (m = 0, 1, 2, \cdots)}$$

$$\boxed{y_2(x) = J_0(x) lnx + \frac{(-1)^{m-1} h_m}{2^{2m}(m!)^2} x^{2m} \quad \text{where } h_m = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right)}$$

J_0, y_2 are linearly independent functions (basis for $x > 0$), express y_2 as particular solution:x

$\boxed{Y_0(x) = a(y_2 + bJ_0)}$ and choose $\boxed{a = \pi/2}$ and $\boxed{b = \gamma - ln2}$

Let, $\boxed{\gamma = \lim_{s\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{s}\right) - ln\, s = 0.57721566490}$ (Euler constant)

The standard particular solution thus obtained is called the Bessel function of the second kind of order zero or Neumann's function of order zero and is denoted by $Y_0(x)$.

$$Y_0(x) = \frac{2}{\pi}\left(J_0(x) lnx + \frac{(-1)^{m-1} h_m}{2^{2m}(m!)^2} x^{2m} + (\gamma - ln2) J_0\right)$$

$$\boxed{Y_0(x) = \frac{2}{\pi}\left[J_0(x)\left(ln\frac{x}{2} + \gamma\right) + \frac{(-1)^{m-1} h_m}{2^{2m}(m!)^2} x^{2m}\right]}$$

8.8.4 Bessel functions of the second kind, $Y_n(x)$

For $n = 1, 2, \cdots$ a second solution can be obtained by manipulations similar to those for $n = 0$. It turns out that in these cases the solution also contains a logarithmic term.

A standard second solution known as the Bessel function of the 2nd kind of order ν or Neumann's function of order ν is defined as:

$$Y_\nu(x) = \frac{1}{sin\,\nu\pi}[J_\nu(x) cos\,\nu\pi - J_{-\nu}(x)]$$
$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x)$$

$$y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x)$$

CHAPTER 9

Laplace Transform

Laplace transform is a particular integral transform invented by the French mathematician Pierre-Simon Laplace and subsequently developed by British physicist Oliver Heaviside to simplify the solution of differential equations that describe physical processes. As an example. it is widely used by electrical engineers to solve circuit problems. With Laplace transforms the process of solving an ODE is simplified to an algebraic problem.

9.1 Definition

The Laplace transform is an integral transform defined as:

$$\boxed{F(s) = \mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)dt} \quad \text{where } k(s,t) \text{ is the kernel function } e^{-st}$$

The inverse transform $\mathcal{L}^{-1}\{F\}$ will yield $f(t)$.

9.2 Basic Transforms

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt = \frac{1}{-(s-a)} e^{-(s-a)t}\Big|_0^\infty = 0 - \frac{1}{-(s-a)}$$

$$\implies \boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}} \quad \text{where } s-a>0$$

The Laplace transform of trigonometric functions are as follows:

$$\mathcal{L}(cos\ wt) = \int_0^\infty e^{-st} cos\ wt\ dt = \frac{e^{-st}}{-s} cos\ wt\Big|_0^\infty - \frac{w}{s}\int_0^\infty e^{-st} sin\ wt\ dt = \frac{1}{s} - \frac{w}{s}\mathcal{L}\{sin\ wt\}$$

$$\mathcal{L}\{sin\ wt\} = \int_0^\infty e^{-st} sin\ wt\ dt = \frac{e^{-st}}{-s} sin\ wt\Big|_0^\infty + \frac{w}{s}\int_0^\infty e^{-st} cos\ wt\ dt = \frac{w}{s}\mathcal{L}\{cos\ wt\}$$

Solving the simulatneous equations we get,

$$\boxed{\mathcal{L}\{cos\ wt\} = \frac{s}{s^2+w^2}} \quad \boxed{\mathcal{L}\{sin\ wt\} = \frac{w}{s^2+w^2}}$$

The hyperbolic functions are given by:

$$cosh\, wt = \frac{1}{2}(e^{wt} + e^{-wt})$$

$$sinh\, wt = \frac{1}{2}(e^{wt} - e^{-wt})$$

The Laplace transform of hyperbolic functions are:

$$\mathcal{L}\{cosh\ wt\} = \frac{1}{2}\mathcal{L}\{e^{wt}\} + \frac{1}{2}\mathcal{L}\{e^{-wt}\} = \frac{1}{2}\left(\frac{1}{s-w} + \frac{1}{s+w}\right) = \frac{s}{s^2-w^2}$$
$$\mathcal{L}\{sinh\ wt\} = \frac{1}{2}\mathcal{L}\{e^{wt}\} - \frac{1}{2}\mathcal{L}\{e^{-wt}\} = \frac{1}{2}\left(\frac{1}{s-w} - \frac{1}{s+w}\right) = \frac{a}{s^2-w^2}$$
$$\boxed{\mathcal{L}\{cosh\ wt\} = \frac{s}{s^2-w^2}} \quad \boxed{\mathcal{L}\{sinh\ wt\} = \frac{w}{s^2-w^2}}$$

The Laplace transform of a polynomial is computed below.

$$\mathcal{L}\{t^{n+1}\} = \int_0^\infty e^{-st}t^{n+1}dt = -\frac{1}{s}e^{-st}t^{n+1}\Big|_0^\infty + \frac{n+1}{s}\int_0^\infty e^{-st}t^n dt$$
$$\implies \mathcal{L}\{t^{n+1}\} = \frac{n+1}{s}\mathcal{L}\{t^n\} \text{ for } s > 0$$
$$\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\}$$
$$\implies \boxed{\mathcal{L}\{t^{n+1}\} = \frac{(n+1)!}{s^{n+2}}} \text{ (by induction, where } n = 0, 1, \cdots)$$

Now consider a to be real positive. Laplace transform of a polynomial can be expressed in terms of the *Gamma function.*

$$\mathcal{L}\{t^a\} = \int_0^\infty e^{-st}t^a dt$$

Let $st = x$,

$$\mathcal{L}\{t^a\} = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^a \frac{dx}{s} = \frac{1}{s^{a+1}}\int_0^\infty e^{-x}x^a dx$$

Gamma function is defined as $\boxed{\Gamma(a) = \int_0^\infty e^{-x}x^{a-1}}$

$$\implies \boxed{\mathcal{L}\{t^a\} = \frac{\Gamma(a+1)}{s^{a+1}}}$$

9.3 Linearity

Obviously, Laplace transform is a linear operation. i.e.,

$$\boxed{\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}} \quad \text{where } a \text{ and } b \text{ are constants}$$

9.4 s - Shifting

If Laplace transform of $f(t)$ is given by $F(a)$, we can get the Laplace transform $e^{at}f(t)$ as $F(s-a)$.

$$F(s-a) = \int_0^\infty e^{-(s-a)}f(t)dt = \int_0^\infty e^{st}[e^{at}f(t)]dt = \mathcal{L}\{e^{at}f(t)\}$$
$$\boxed{F(s-a) = \mathcal{L}\{e^{at}f(t)\}}$$

9.5 Existance & Uniqueness

For Laplace transform $\mathcal{L}\{f\}$ to exist, the following condition must be satisfied:

$$\boxed{|f(t)| \le Me^{kt}} \quad \text{where } M, k \text{ are constants}$$

$$|\mathcal{L}\{f\}| = \left| \int_0^\infty e^{-st} f(t) dt \right| \le \int_0^\infty Me^{kt} e^{-st} dt$$

$$M \int_0^\infty e^{-(s-k)t} dt = \frac{M}{-(s-k)} e^{-(s-k)t} \Big|_0^\infty = 0 + \frac{M}{s-k} = \frac{M}{s-k} \quad \text{where } s > k$$

$$\boxed{|\mathcal{L}\{f\}| \le \frac{M}{s-k} \quad \text{where } s > k}$$

If the Laplace transform of a given function exists, it is uniquely determined. If two continuous functions have the same transform, they are identical.

9.6 Laplace Transforms of Derivatives

$$\mathcal{L}\{f'\} = \int_0^\infty e^{-st} f'(t) dt = e^{-st} f(t) \Big|_0^\infty + s \int_0^\infty e^{-st} f(t) dt$$

Since $|f(t)| \le Me^{kt}$, the upper limit of $e^{-st} f(t) \Big|_0^\infty$ is 0

$$\Longrightarrow \boxed{\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)}$$

$$\mathcal{L}\{f''\} = s\mathcal{L}\{f'\} - f'(0) = s\left[s\mathcal{L}\{f\} - f(0)\right] - f'(0)$$

$$\boxed{\mathcal{L}\{f''\} = s^2\mathcal{L}\{f\} - sf(0) - f'(0)}$$

Similarily,

$$\boxed{\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) \cdots - f^{n-1}(0)}$$

9.7 Laplace Transforms of Integrals

Let $g(t) = \int_0^t f(u) du$

$\Longrightarrow g'(t) = f(t)$

$$\mathcal{L}\{g'(t)\} = \mathcal{L}\{f(t)\}$$

$$s\mathcal{L}\{g(t)\} - g(0) = \mathcal{L}\{f(t)\}$$

Now $g(0) = \int_0^0 f(u) du = 0$

$$\Longrightarrow \boxed{\mathcal{L}\{g(t)\} = \frac{1}{s}\mathcal{L}\{f(t)\}}$$

9.8 Laplace Transforms for solving ODEs

Consider the ODE,

$$\boxed{y'' + ay' + by = r(t)}$$

where $y(0) = K_0,\ y'(0) = K_1,$ and a, b are constants

$$\left[s^2Y - s(y(0)) - y'(0)\right] + a\left[sY - (y(0)\right] + bY = R(s)$$

where $Y = \mathcal{L}(y)$ and $R = \mathcal{L}(r)$

$$(s^2 + as + b)Y = (s+a)y(0) + y'(0) + R(s)$$

The Transfer Function is defined as: $Q(s) = \dfrac{1}{s^2 + as + b} = \dfrac{1}{(s+\frac{1}{2}a)^2 + (b - \frac{1}{4}a^2)}$

$$Y(s) = [(s+a)y(0) + y'(0)]\,Q(s) + R(s)Q(s)$$

$$\text{if } y(0) = y'(0) = 0, \quad Q(s) = \frac{Y}{R} = \frac{\mathcal{L}\{output\}}{\mathcal{L}\{(input\}}$$

The ODE is transformed into an algebraic equation which is also known as the *subsidiary equation.* If the initial conditions are at some t_0 and not at 0, set $t = \tau + t_0$. Solve the subsidiary equation. Finally, compute the inverse transform to get the solution to the ODE. This technique can be used to solve *systems of ODEs* .

9.9 Unit Step Function (Heaviside Function)

Unit Step Function or Heaviside Function is defined as:

$$u(t-a) = \begin{cases} 0 \text{ for } t < a \\ 1 \text{ for } t > a \end{cases}$$

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)dt = \int_a^\infty e^{-st} dt = -\frac{e^{-st}}{s}\bigg|_a^\infty$$

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

9.10 Time Shifting (t-Shifting)

Consider a function $f(t)$ that has its Laplace transform $F(s)$. The *shifted function* is given by:

$$\tilde{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a \end{cases}$$

$$e^{-as}F(s) = e^{-as}\int_a^\infty e^{-s\tau} f(\tau)d\tau = \int_a^\infty e^{-s(\tau+a)} f(\tau)d\tau$$

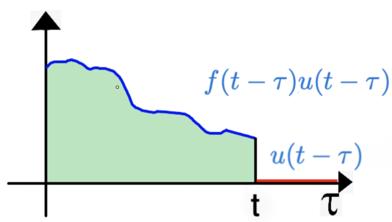
Let $t = a + \tau$

$$e^{-as}F(s) = \int_a^\infty e^{-st} f(t-a)dt$$

Introduce $u(t-a)$ to change the lower limit

$$e^{-as}F(s) = \int_0^\infty e^{-st} f(t-a)u(t-a)dt$$

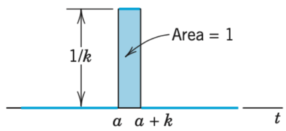
$$\mathcal{L}\{\tilde{f}(t)\} = \int_0^\infty e^{-st} f(t-a)u(t-a)dt = e^{-as}F(s)$$



9.11 Dirac Delta Function

Consider the following function and the integral:

$$f_k(t-a) = \begin{cases} \frac{1}{k} & \text{if } a \le t \le a+k \\ 0 & \text{otherwise} \end{cases}$$ $$I_k = \int_0^\infty f_k(t-a)dt = \int_a^{a+k} \frac{1}{k} dt = 1$$



We take the limit of f_k as $k \to 0 \quad (k > 0)$, the *Dirac Delta* function, $(\delta - a)$, is then defined as:

$$\delta(t-a) = \lim_{k \to 0} f_k(t-a)$$ $$\delta(t-a) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{otherwise} \end{cases} \implies \int_0^\infty \delta(t-a)dt = 1$$

$$\mathcal{L}\{\delta(t-a)\} = \int_0^\infty e^{-st} f(t)\delta(t-a)dt = \int_0^\infty e^{-st} f(a)\delta(t-a)dt = f(a)e^{-as} \int_0^\infty \delta(t-a)dt$$

$$\implies \int_0^\infty f(t)\delta(t-a)dt = f(a) \text{ (Sifting)}$$

The impulse function $\delta(t-a)$ sifts through the function $f(t)$ and pulls out the value $f(a)$

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}$$

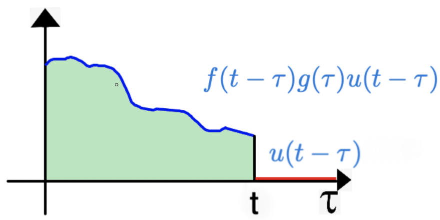
The use of the unit step function (Heaviside function) and the Dirac delta function make the method particularly powerful for problems with inputs, i.e., driving forces, that have discontinuities or represent short impulses or complicated periodic functions.

9.12 Convolution

Convolution of two functions $f(t)$ and $g(t)$ is defined by the following integral:

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

$$\begin{aligned} \mathcal{L}\{(f*g)(t)\} &= \int_0^\infty e^{-st} \int_0^t f(\tau)g(t-\tau)d\tau\, dt \\ &= \int_0^\infty e^{-st} \int_0^\infty f(\tau)g(t-\tau)u(t-\tau)d\tau\, dt \\ &= \int_0^\infty e^{-st} f(\tau)d\tau \int_0^\infty g(t-\tau)u(t-\tau)dt \end{aligned}$$



$$\mathcal{L}\{(f*g)(t)\} = F(s)G(s)$$

The transform of a product is generally different from the product of the transforms of the factors.

9.13 Differentiation of Transforms

$$F(s) = \int_0^\infty e^{-st} f(t)dt$$

$$\frac{d}{ds}F(s) = \int_0^\infty \frac{d}{ds} e^{-st} f(t)dt = \int_0^\infty -te^{-st} f(t) = -\int_0^\infty e^{-st} t f(t)$$

$$F'(s) = -\mathcal{L}(tf(t))$$

9.14 Integration of Transforms

$$F(s) = \int_0^\infty e^{-st} f(t) dt$$

$$\int_s^\infty F(\tilde{s}) d\tilde{s} = \int_s^\infty \left[\int_0^\infty e^{-\tilde{s}t} f(t) dt \right] d\tilde{s} = \int_0^\infty \left[\int_s^\infty e^{-\tilde{s}t} f(t) d\tilde{s} \right] dt = \int_0^\infty f(t) \left[\int_s^\infty e^{-\tilde{s}t} d\tilde{s} \right] dt$$

$$\int_s^\infty F(\tilde{s}) d\tilde{s} = \int_0^\infty e^{-st} \frac{f(t)}{t}$$

$$\boxed{\int_0^\infty F(s) ds = \mathcal{L}\left(\frac{f(t)}{t} \right)}$$

CHAPTER 10

Complex Analysis

Complex analysis is the study of complex numbers together with their derivatives, manipulation, and other properties. Complex analysis is an extremely powerful tool with an unexpectedly large number of practical applications to the solution of physical problems. It is helpful in many areas such as hydrodynamics, thermodynamics, and particularly quantum mechanics. Complex analysis also has a wide range of applications in engineering fields such as nuclear, aerospace, mechanical and electrical engineering.

10.1 Complex Number

Complex numbers are the numbers that are expressed in the form of $x+\mathbf{i}y$ where, x,y are real numbers and $\mathbf{i}$is an imaginary number called "iota" defined as follows:

$$\boxed{z = x+\mathbf{i}y} \qquad \mathbf{i} = \sqrt{-1}$$

Just as with real numbers, we can perform arithmetic operations on complex numbers. To add or subtract complex numbers, we combine the real parts and combine the imaginary parts. Addition, multiplication and division of complex numbers are given below.

$$z_1+z_2 = (x_1+x_2)+\mathbf{i}(y_1+y_2)$$
$$z_1z_2 = (x_1x_2-y_1y_2)+\mathbf{i}(x_1y_2+x_2y_1)$$
$$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} = \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)} = \frac{(x_1x_2+y_1y_2)}{x_2^2+y_2^2}+\mathbf{i}\frac{(x_2y_1-x_1y_2)}{x_2^2+y_2^2}$$

10.1.1 Complex Conjugate

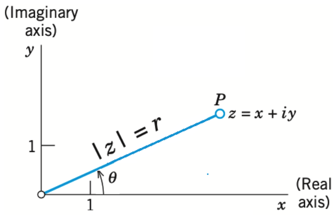
The *complex conjugate* of z is defined as:

$$\boxed{\bar{z} = x-\mathbf{i}y}$$

10.1.2 Polar Representation

$$\boxed{z = r\cos\theta+\mathbf{i}r\sin\theta = re^{\theta}}$$ (Polar representation)

$$\boxed{z^n = r^n(\cos n\theta+\sin n\theta)}$$ (De Moivre's theorem)



The complex plane

$$|z| = r = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$$

$\tan\theta = \dfrac{y}{x}$ (radians, counterclockwise). θ is argument of z

denoted by $\arg z$,its Principal value is $\boxed{i \leq \arg z \leq i}$

XY-plane is complex plane, also known as the Argand diagram

$$
\begin{aligned}
z_1 z_2 &= r_1 r_2 e^{i(\theta_1+\theta_2)} = r_1 r_2 \left[cos(\theta_1+\theta_1) + \mathbf{i} sin(\theta_1+\theta_2)\right] \\
\frac{z_1}{z_2} &= \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} = \frac{r_1}{r_2}\left[cos(\theta_1-\theta_1) + \mathbf{i} sin(\theta_1-\theta_2)\right] \\
arg\,(z_1 z_2) &= arg\, z_1 + arg\, z_2 \\
arg\left(\frac{z_1}{z_2}\right) &= arg\, z_1 - arg\, z_2
\end{aligned}
$$

10.1.3 Properties

$$
\begin{aligned}
&z_1 z_2 = z_2 z_1 \text{ (commutative)} \\
&(z_1 z_2) z_3 = z_1 (z_2 z_3) \text{ (associative)} \\
&z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3 \text{(distributive)} \\
&|z_1 z_2| = |z_1||z_2| \text{ and } \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}
\end{aligned}
$$

10.1.4 Roots

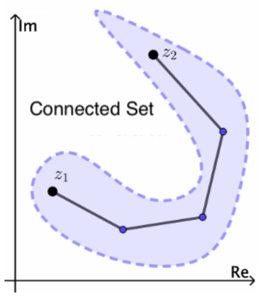
$$z = re^{i\theta}$$

$$\boxed{\sqrt[n]{z} = r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n}+\frac{2ki}{n}\right)}} \qquad k = 1, 2, \cdots$$

Example, $\sqrt{4i} = \sqrt{4e^{\frac{i}{2}}} = 2e^{i\frac{i}{4}}, 2e^{i\left(\frac{i}{4}+i\right)} = m\sqrt{2}(1+i)$

10.2 Point Set & Path

A *Point Set* is a collection of a finite or infinite points in the complex plane. A set is *open* if every point in is an interior point. A set is *closed* if it contains all of its boundary points. A set S is called *connected* if any two of its points can be joined by a chain of finitely many straight-line segments all of whose points belong to S.



10.3 Complex Differentiation

Complex analysis is about complex functions that are differentiable in a domain. The concepts of limits, derivatives, integrals are similar to those in calculus with real numbers. A function $f(z)$ of a complex variable z is called analytic in a domain D if it is defined and differentiable at all points of D.

10.3.1 Cauchy Reimann Equations

A necessary condition that $f(z) = u(x,y) + \mathbf{i}v(x,y)$ be analytic in a region R is that u and v satisfy the Cauchy-Riemann equations as stated below:

$$
\begin{aligned}
f'(z) &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \\
f'(z) &= \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y+\Delta y) + \mathbf{i}v(x+\Delta x, y+\Delta y) - [u(x,y) + \mathbf{i}v(x,y)]}{\Delta x + \mathbf{i}\Delta y} \\
\text{with } \Delta y = 0,\ f'(z) &= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + \frac{\mathbf{i}v(x+\Delta x, y) - \mathbf{i}v(x,y)}{\Delta x} = u_x + \mathbf{i}v_x \\
\text{with } \Delta x = 0,\ f'(z) &= \lim_{\Delta x \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\mathbf{i}\Delta y} + \frac{\mathbf{i}v(x, y+\Delta y) - \mathbf{i}v(x,y)}{\mathbf{i}\Delta y} = v_y - \mathbf{i}u_y
\end{aligned}
$$

The Cauchy-Reimann equations are then given by:

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \text{ and } \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Similarly, for polar coordinates we have,

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}} \quad \boxed{\frac{\partial v}{\partial r} = -\frac{1}{\frac{\partial u}{\partial \theta}}}$$

Example,

$z = x + \mathbf{i}y$ is differentiable because $u_x = v_y = 1$ and $u_y = -v_x = 0$, $z' = 1 + \mathbf{i}$

$\bar{z} = z = x - \mathbf{i}y$ is not differentiable because $u_x = 1, v_y = -1, u_x \neq v_y$ although $u_y = -v_x = 0$.

10.3.2 Laplace's Equation

Using Cauchy-Reimann equations we arrive at the Laplace's equation,

$$\frac{\partial^2 u}{\partial^2 x} = \frac{\partial^2 v}{\partial x \partial y} \qquad \frac{\partial^2 u}{\partial^2 y} = -\frac{\partial^2 v}{\partial x \partial y} \qquad \Longrightarrow \boxed{\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} = 0} \boxed{\frac{\partial^2 v}{\partial^2 x} + \frac{\partial^2 v}{\partial^2 y} = 0}$$

v is called the harmonic conjugate function of u in D (not to be confused with $\bar{z}$).

10.3.3 Trigonometric & Hyperbolic Functions

$$e^{\mathbf{i}x} = \cos x + \mathbf{i}\sin x \qquad e^{-\mathbf{i}x} = \cos x - \mathbf{i}\sin x$$

$$\cos x = \frac{1}{2}(e^{\mathbf{i}x} + e^{-\mathbf{i}x}) \qquad \sin x = \frac{1}{2}(e^{\mathbf{i}x} - e^{-\mathbf{i}x})$$

$$\cosh z = \frac{1}{2}(e^{z} + e^{-z}) \qquad \sinh z = \frac{1}{2}(e^{z} - e^{-z})$$

$$(\cosh z)' = \sinh z \qquad (\sinh z)' = \cosh z$$

10.4 Complex Integration

$$\int_C f(z)dz = \int_C (u + \mathbf{i}v)(dx + \mathbf{i}dy) = \left[\int_C udx - \int_C vdy\right] + \mathbf{i}\left[\int_C udy + \int_C vdx\right]$$

Using parametric representation,

$$z(t) = x(t) + \mathbf{i}y(t)$$

$$\dot{z} = \frac{dz}{dt}$$

$$\boxed{\int_C f(z)dz = \int_a^b f[z(t)]\dot{z}(t)dt}$$

Examples,

$$z = 3t - \mathbf{i}t^2$$

$$\frac{dz}{dt} = 3 - \mathbf{i}2t$$

$$\int f(z)dz = \int (3t - \mathbf{i}t^2)(3 - \mathbf{i}2t)dt = \int (9t - 2t^3 - \mathbf{i}9t^2)dt = \left(-\frac{t^4}{2} + \frac{9t^2}{2}\right) - \mathbf{i}3t^3$$

Evaluate $\oint_C \frac{dz}{z} = \frac{r\mathbf{i}e^{\mathbf{i}\theta}}{re^{\mathbf{i}\theta}} d\theta \implies \oint_C \frac{dz}{z} = 2\pi\mathbf{i}$

$\oint_C (z - z_0)^m dz$

Let $z(t) = z_0 + re^{\mathbf{i}t} \implies dz = \mathbf{i}re^{\mathbf{i}t} dt$

$$\oint_C (z - z_0)^m dz = \int_0^{2\pi} r^m e^{\mathbf{i}mt} \mathbf{i}re^{\mathbf{i}t} dt = \mathbf{i}r^{m+1} \int_0^{2i} e^{\mathbf{i}(m+1)t} dt = \begin{cases} 2\pi\mathbf{i} & (m = -1) \\ 0 & (m \neq -1) \end{cases}$$

10.4.1 Path Dependence

If we integrate a given function f(z) from a point z_1 to a point z_2 along different paths, the integrals will in general have different values. A complex line integral depends not only on the endpoints of the path but in general also on the path itself.

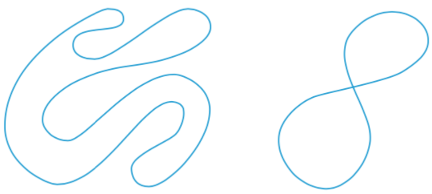
10.4.2 ML-Inequality

$\boxed{\left| \oint_C f(z) dz \right| \leq ML}$ (L is length of C, $|f(z)| \leq M$, where M is a constant)

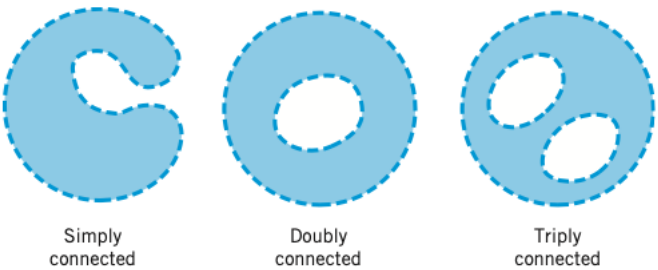
This is evident from the fact that $f(z) = re^{\mathbf{i}\theta}$ has to be bounded in a given domain whose upper limit is represented as M.

10.5 Cauchy's Integral Theorem

A *simple closed path* is a closed path that does not intersect or touch itself.



An *open and connected* set is called a *domain*. In a simply connected domain D, any simple closed curve C is the boundary of some region E which is contained in D. In simple words, a region is simply connected if every closed curve within it can be shrunk continuously to a point that is within the region. That means, a simply connected region is one that has no holes.



If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D,

$$\boxed{\oint_C f(z) dz = 0}$$

Since $f(z)$ is analytic in D, $f'(z)$ exists in D. Assume $f'(z)$ to be continuous, i.e., u & v have continuous ∂ derivatives in D [1].

$$\int_C f(z)dz = \int_C (u+iv)(dx+idy) = \left[\int_C udx - \int_C vdy\right] + i\left[\int_C udy + \int_C vdx\right]$$

(Replacing v with $-v$) in Green's Theorem

$$\boxed{\oint_c u(x,y)dx - \oint_c v(x,y)dy = \int_R\int \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy}$$

and using Cauchy-Reimann equations we have,

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \qquad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

$$\left[\int_C udx - \int_C vdy\right] = \int_R\int \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy = 0$$

$$\left[\int_C udy + \int_C vdx\right] = \int_R\int \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy = 0$$

$$\implies \boxed{\oint_c f(z)dz = 0}$$

10.5.1 Path Independence

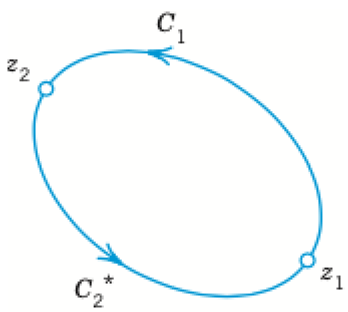
If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of the path in D. This follows from Cauchy's Integral Theorem.

$$\oint_c f(z)dz = 0$$

$$\int_{c1} f(z)dz + \int_{c_2^*} f(z)dz = 0$$

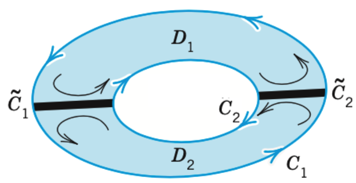
$$\int_{c1} f(z)dz = -\int_{c_2^*} f(z)dz$$

$$\boxed{\int_{c1} f(z)dz = \int_{c2} f(z)dz}$$



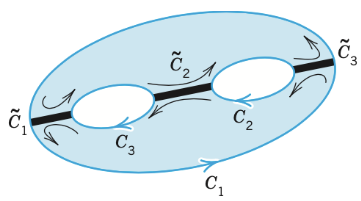
10.5.2 Cauchy's Integral Theorem for Multiply Connected Domains

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$



Doubly connected domain

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz$$



Triply connected domain

[1]Goursat proved without the condition that $f'(z)$ is continuous but the proof is complex.

10.5.3 Existence of Indefinite Integral

If $f(z)$ is analytic in a simply connected domain D, then there exists $F(z) = \int_{z_1}^{z_2} f(z)dz$ which is analytic in D and hence $F'(z) = f(z)$. The integral can be evaluated as:

$$\boxed{F(z) = \int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)}$$

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z^*)dz^* = \frac{f(z)}{\Delta z}\int_{z}^{z+\Delta z} dz^*$$

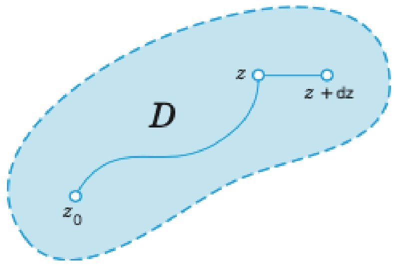
$$f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)dz^* = \frac{f(z)}{\Delta z}\int_{z}^{z+\Delta z} dz^* \text{ } (f(z) \text{ is constant})$$

$$\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| = \frac{1}{|\Delta z|}\left|\int_{z}^{z+\Delta z} [f(z)^* - f(z)]\, dz^*\right|$$

$$\leq \frac{1}{|\Delta z|}\epsilon|\Delta z| = \epsilon$$

$$\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = F'(z) = f(z)$$

$$F(z) = \int f(z)dz$$



10.6 Cauchy's Integral Formula

If $f(z)$ is analytic in a simply connected domain D we have,

$$\boxed{\oint_C \frac{f(z)}{z - z_0}dz = 2\pi \mathbf{i} f(z_0)}$$

$$f(z) = f(z_0) + [f(z) - f(z_0)]$$

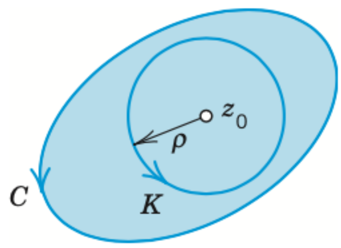
$$\oint_C \frac{f(z)}{z - z_0}dz = f(z_0)\oint_C \frac{1}{z - z_0}dz + \oint_C \frac{f(z) - f(z_0)}{z - z_0}dz = 2\pi \mathbf{i} f(z_0) + \cancelto{0}{\oint_C \frac{f(z) - f(z_0)}{z - z_0}dz}$$

The second term $\to 0$ because, given $\epsilon > 0$, it is possible to find $\delta > 0$ such that $f(z) - f(z_0) < \epsilon$ for all z in the disk $|z - z_0| < \delta$

Choosing the radius ρ of K smaller. we have.

$$\left|\frac{f(z) - f(z_0)}{z - z_0}\right| < \frac{\epsilon}{\rho}$$

$$\left|\oint \frac{f(z) - f(z_0)}{z - z_0}dz\right| < \frac{\epsilon}{\rho}2\pi\rho = 2\pi\epsilon = 0$$



10.6.1 Multiply Connected Domain

By extension, Cauchy's theorem for multiply connected domain is given by:

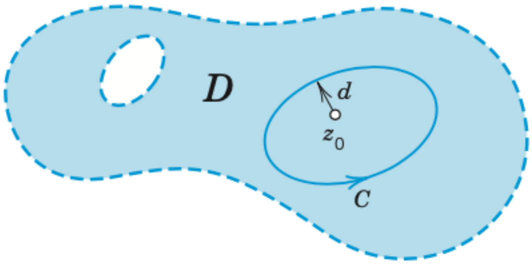
$$\boxed{\oint_{C_1} \frac{f(z)}{z - z_0}dz + \oint_{C_2} \frac{f(z)}{z - z_0}dz + \cdots + \oint_{C_n} \frac{f(z)}{z - z_0}dz = 2\pi \mathbf{i} f(z_0)}$$

10.7 Derivatives of Analytic Functions

$$\boxed{f'(z_0) = \frac{1}{2\pi \mathbf{i}} \oint_C \frac{f(z)}{(z-z_0)^2} dz}$$

$$\boxed{f''(z_0) = \frac{2!}{2\pi \mathbf{i}} \oint_C \frac{f(z)}{(z-z_0)^2} dz}$$

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi \mathbf{i}} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz}$$



$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z)}{\Delta_z}$$

Let's represent $f(z_0 + \Delta z)$ and $f(z_0)$ by Cauchy's integral formula:

$$f'(z_0) = \frac{1}{2\pi \mathbf{i} \Delta z} \oint_C \frac{f(z)}{z-(z_0+\Delta z)} dz - \frac{1}{2\pi \mathbf{i} \Delta z} \oint_C \frac{f(z)}{z-z_0} dz = \frac{1}{2\pi \mathbf{i}} \oint_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)} dz$$

Now, $\displaystyle \oint_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)} dz - \oint_C \frac{f(z)}{(z-z_0)^2} dz = \oint_C \frac{f(z)\Delta z}{(z-z_0-\Delta z)(z-z_0)^2} dz \rightarrow 0$

The integral on the right $\to 0$ as $\Delta z \to 0$ as is evident from the following

Let $|z-z_0|^2 \geq d^2 \implies \dfrac{1}{|z-z_0|^2} \leq \dfrac{1}{d^2}$

$d \leq |z-z_0| = |z-z_0-\Delta z + \Delta z| \leq |z-z_0-\Delta z| + |\Delta z|$

Let $|\Delta z| \leq d/2 \implies \dfrac{1}{|z-z_0-\Delta z|} \leq \dfrac{2}{d}$

$\displaystyle \left| \oint_C \frac{f(z)\Delta z}{(z-z_0-\Delta z)(z-z_0)^2)} dz \right| \leq ML|\Delta z| \frac{2}{d}\frac{1}{d^2}$ (refer ML inequality)

As $\displaystyle\lim_{\Delta z \to 0}$ the above integral $\to 0$

Hence, $\boxed{f'(z_0) = \dfrac{1}{2\pi \mathbf{i}} \displaystyle\oint_C \frac{f(z)}{(z-z_0)^2} dz}$

10.7.1 Cauchy's Inequality

$$\boxed{|f^{(n)}(z_0)| = \frac{n!}{2\pi} \left| \oint_C \frac{f(z)}{(z-z_0)^{(n+1)}} dz \right| \leq \frac{n!}{2\pi} M \frac{1}{r^{(n+1)}} 2\pi r = \frac{n! M}{r^n}}$$

10.7.2 Liouville's Theorem

If an entire function is bounded in absolute value in the whole complex plane, then this function must be a constant. This is because if $|f(z)| < M$ for all z, then by Cauchy's inequality theorem $|f'(z) < M/r$. We can choose r to be arbitrarily large and hence $f'(z) = 0$ and $f(z)$ is constant.

10.7.3 Morera's Theorem (Converse of Cauchy's Integral Theorem)

If $f(z)$ is continuous in a simply connected domain D and if $\oint_C f(z) = 0$ for every closed path in D, then $f(z)$ is analytic in D.

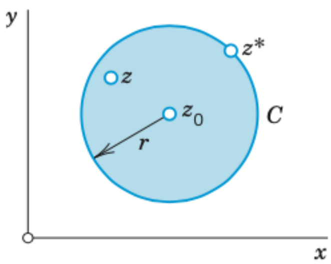
10.8 Power Series

Complex power series are analogs of real power in calculus. Complex power series represent analog functions and conversely, every analytic function can be represented as a power series.

10.8.1 Taylor Series

The Taylor series is given by:

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + R_n(z)$$
$$a_n = \frac{1}{n!} f^{(n)}(z_0) = \frac{1}{2\pi \mathbf{i}} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$
$$R_n(z) = \frac{(z-z_0)^{n+1}}{2\pi \mathbf{i}} \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}(z^*-z)} dz^*$$



A Maclaurin series is a Taylor series with center $z_0 = 0$.

$$\left| \frac{z-z_0}{z^*-z_0} \right| < 1$$
$$\frac{1}{z^*-z} = \frac{1}{z^*-z_0-(z-z_0)} = \frac{1}{(z^*-z_0)} \times \frac{1}{\left(1-\frac{z-z_0}{z^*-z_0}\right)}$$

Let $q = \dfrac{z-z_0}{z^*-z_0}$

$$1-q = z^*-z$$
$$1+q+q^2+\cdots+q^n = \frac{1-q^{n+1}}{1-q} = \frac{1}{1-q} - \frac{q^{n+1}}{1-q}$$
$$\frac{1}{1-q} = 1+q+q^2+\cdots+q^n + \frac{q^{n+1}}{1-q}$$
$$\frac{1}{z^*-z} = \frac{1}{z^*-z_0}\left[1+\frac{z-z_0}{z^*-z_0} + \left(\frac{z-z_0}{z^*-z_0}\right)^2 + \cdots + \left(\frac{z-z_0}{z^*-z_0}\right)^n\right] + \frac{1}{z^*-z}\left(\frac{z-z_0}{z^*-z_0}\right)^{n+1}$$

In Cauchy's integral formula, use z instead of z_0 and z^* instead of z,

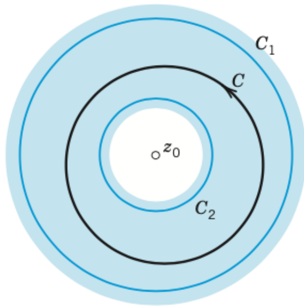
z^* is the variable of integration

$$\oint_C \frac{f(z^*)}{z^*-z} dz^* = 2\pi \mathbf{i} f(z)$$

$$\boxed{f(z) = \frac{1}{2\pi \mathbf{i}} \oint_C \frac{f(z^*)}{(z^*-z_0)} dz^* + \frac{1}{2\pi \mathbf{i}} \oint_C \frac{f(z^*)}{(z^*-z_0)^2} dz^* + \cdots \frac{1}{2\pi \mathbf{i}} \oint_C \frac{f(z^*)}{(z^*-z_0)^n} dz^* + R_n(z)}$$

10.8.2 Laurent's Series

Laurent series generalize Taylor series. Indeed, whereas a Taylor series has positive integer powers (and a constant term) and converges in a disk, a Laurent series is a series of positive and negative integer powers of $z-z_0$ and converges in an annulus (a circular ring) with center z_0. It converges for $0 < |z-z_0| < R$, that is, everywhere except at z_0 which is a singular point of $f(z)$.



The nonnegative powers are those of a Taylor series. The series, i.e. the finite sum of the negative powers of the Laurent series is called the *principal part* of the singularity of $f(z)$ at z_0 and is used to classify this singularity. The coefficient of the power $\frac{1}{z-z_0}$ of this series is called the *residue* of $f(z)$ at z_0. Residues are used in a technique called residue integration for complex contour integrals and for certain complicated real integrals.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=0}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$a_n = \frac{1}{2\pi \mathbf{i}} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

$$b_n = \frac{1}{2\pi \mathbf{i}} \oint_C (z-z_0)^{n-1} f(z) dz$$

$$f(z) = f(z) + g(z) = \frac{1}{2\pi \mathbf{i}} \oint_{C_1} \frac{f(z^*)}{z^* - z} dz^* \frac{1}{2\pi \mathbf{i}} \oint_{C_2} \frac{f(z^*)}{z^* - z} dz^*$$

The first integral is the Taylor series of $g(z)$.

$$\left| \frac{z^* - z_0}{z - z_0} \right| < 1$$

$$\frac{1}{z^* - z} = \frac{1}{z^* - z_0 - (z - z_0)} = \frac{-1}{(z-z_0)} \times \frac{1}{\left(1 - \frac{z^* - z_0}{z - z_0}\right)}$$

$$q = \frac{z^* - z_0}{z - z_0}$$

$$1 - q = z - z^*$$

$$\frac{1}{1-q} = 1 + q + q^2 + \cdots + q^n + \frac{q^{n+1}}{1-q}$$

$$\frac{1}{z^* - z} = -\frac{1}{z - z_0} \left[1 + \left(\frac{z^* - z_0}{z - z_0}\right) + \left(\frac{z^* - z_0}{z - z_0}\right)^2 + \cdots + \left(\frac{z^* - z_0}{z - z_0}\right)^n \right] - \frac{1}{z - z^*} \left(\frac{z^* - z_0}{z - z_0}\right)^{n+1}$$

In Cauchy's integral formula, use z instead of z_0 and z^* instead of z, z^* is the variable of integration

$$h(z) = -\frac{1}{2\pi \mathbf{i}} \oint_{C_2} \frac{f(z^*)}{(z^* - z)} dz^*$$

$$= \frac{1}{2\pi \mathbf{i}(z - z_0})} \oint_{C_2} f(z^*) dz^* + \frac{1}{2\pi \mathbf{i}(z - z_0)^2}) \oint_{C_2} f(z^*) dz^* + \cdots$$

$$+ \frac{1}{2\pi \mathbf{i}(z - z_0)^{n+1}} \oint_{C_2} (z^* - z_0)^n f(z^*) dz^* + \frac{1}{2\pi \mathbf{i}(z - z_0)^{n+1}} \oint_{C_2} \frac{(z^* - z_0)^{n+1}}{z - z^*} f(z^*) dz^*$$

10.9 Zero, Singularity, Infinity

A *zero* is a z at which $f(z) = 0$.

A function $f(z)$ is *singular* or has a singularity at a point $z = z_0$ if $f(z)$ is not analytic and may not be even defined at $z = z_0$. If the principal part has finite terms such as:

$$\frac{b_1}{z - z_0} + \cdots + \frac{b_1}{(z - z_0)^m} \text{ where } b_m \neq 0$$

The singularity of $f(z)$ at $z = z_0$ is called a *pole*, and m is called its *order*. Poles of the first order are also known as *simple* poles.

If $f(z)$ is analytic and has a pole at $z = z_0$, then $|f(z)| \to \infty$ as $z \to z_0$.

10.10 Residue Integration Method

The coefficient b_1 of the first negative power $\frac{1}{z-z_0}$ of this Laurent series is given by the following integral formula with $n = 1$:

$$b_1 = \frac{1}{2\pi \mathbf{i}} \oint_C f(z)dz$$

The coefficient b_1 is called the residue of $f(z)$ at $z = z_0$ and we denote it by:

$$b_1 = \operatorname*{Res}_{z=z_0} f(z)$$

For a simple pole at $z = z_0$

$$f(z) = \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \qquad (0 < |z-z_0| < R)$$

$$\implies \operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z\to z_0}(z-z_0)f(z)$$

The Taylor series of $q(z)$ at a simple zero z_0 is

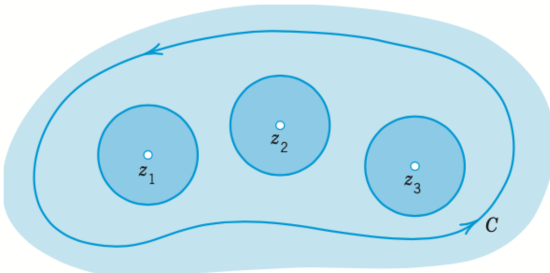
$$q(z) = (z-z_0)q' + \frac{(z-z_0)^2}{2!}q''(z_0) + \cdots$$

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z)}{q'(z_0)} = \lim_{z\to z_0} \frac{(z-z_0)p(z)}{(z-z_0)[q'(z_0) + (z-z_0)q''(z_0)/2 + \cdots]} = \frac{p(z_0)}{q'(z_0)}$$

In general,

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \left[\frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z)\right]$$

$$\boxed{\oint_C f(z)dz = 2\pi \mathbf{i} \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)}$$



CHAPTER 11

Calculus of Variations

A functional is a function that accepts one or more functions as inputs and produces a real valued number as an output. The calculus of variations or variational calculus is a field of mathematical analysis that uses variations, which are small changes in functions and functionals, to find maxima and minima of functionals.

One of the main problems of the calculus of variations is to determine that curve con-necting two given points which either minimizes or maximizes some given integral. Consider a curve connecting two points. Its length, S, is given by:

$$\boxed{S = \int_{x_1}^{x_2} \sqrt{(dx)^2 + (dy)^2} dx}$$

$$S = \int_{x_1}^{x_2} \sqrt{(1 + \left(\frac{dy}{dx}\right)^2} dx$$

The problem of determining that curve connecting two points (x_1, y_1) and (x_2, y_2) whose length is a minimum is the same as that of finding the curve $Y = y(x)$ where $y(x_1) = y_1$, $y(x_2) = y_2$ such that:

$$\int_{x_1}^{x_2} \sqrt{(1 + \left(\frac{dy}{dx}\right)^2} dx \quad \text{is a minimum}$$

In general, we want to find the curve $Y = y(x)$ where $y(x_1) = y_1$, $y(x_2) = y_2$ such that for some given function $F(x, y, y^{'})$:

$$\boxed{\int_{x_1}^{x_2} F(x, y, y^{'}) dx} \tag{11.0.1}$$

is either a *minimum* or a *maximum,* otherwise also referred to as an *extremum* or *stationary* value. The function which satisfies this property is called an *extremal*. The above integral assumes a numerical value for some class of functions $y(x)$ is a *functional.*

11.1 Euler-Lagrange Equation

Let $y = f(x)$ be the function be the curve joining (x_1, y_1), (x_2, y_2) which makes $\int_{x_1}^{x_2} F(x, y, y^{'}) dx$ an extremum.

Let,

$$\boxed{\bar{y}(x) = y(x) + \epsilon \eta(x)} \quad \epsilon \ \text{ is a constant}$$

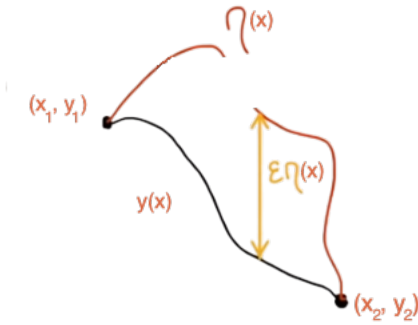
$$\eta(x_1) = \eta(x_2) = 0$$

$$\bar{y}^{'}(x) = \bar{y}^{'}(x) + \epsilon \eta^{'}(x)$$

$$I = \int_{x_1}^{x_2} F(x, y, y^{'}) dx$$

be a neighbouring curve connecting these points. To satisfy the boundary conditions, we have:

$$
\begin{aligned}
\left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} &= \left.\frac{dI}{d\epsilon}\right|_{\epsilon=0} \int_{x_1}^{x_2} F(x,y,y^{'})dx \\
&= \int_{x_1}^{x_2} \frac{dI}{d\epsilon}\left.\left(F(x,y,y^{'})\right)\right|_{\epsilon=0} dx = 0 \quad \text{(Leibnitz's rule)} \\
&= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial x}\cancelto{0}{\frac{\partial x}{\partial \epsilon}} + \frac{\partial F}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial \epsilon} + \left.\frac{\partial F}{\partial \bar{y}'}\frac{\partial \bar{y}^{'}}{\partial \epsilon}\right|_{\epsilon=0}\right) dx = 0 \\
&= \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \bar{y}}\eta + \left.\frac{\partial F}{\partial \bar{y}'}\eta^{'}\right|_{\epsilon=0}\right) dx = 0
\end{aligned}
$$



When $\epsilon = 0 \implies \bar{y} = y, \bar{y}^{'} = y^{'}$

$$\boxed{\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\eta + \left.\frac{\partial F}{\partial y'}\eta^{'}\right|_{\epsilon=0}\right) dx = 0}$$ 1st Variation, Weak Form

$$\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\right)\eta dx + \cancelto{0}{\left.\left(\frac{\partial F}{\partial y'}\right)\eta\right|_{x_1}^{x_2}} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\eta dx = 0$$

$$\boxed{\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)\eta dx = 0}$$ 1st Variation, Strong Form

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0}$$ Euler-Lagrange equation

CHAPTER 12

Matrices

Matrices are a rectangular arrangement of numbers, expressions, symbols which are arranged as rows and columns. The numbers represented in the matrix are called as entries. Matrices find many applications in solving practical real life problems making it an indispensable concept. Matrices have wide applications in engineering analysis and design, physics, economics, and statistics. Matrices also have important applications in computer graphics for image transformations. More recently, matrices have found wide use in the field of Machine Learning (ML). Modern computers are equipped with specially designed hardware called a Graphics Processing Unit or a GPU that is used for parallel processing of matrix operations for much quicker results than ordinary sequential processing.

12.1 Definition of a Matrix

A matrix of order $m \times n$, or m by n matrix, is a rectangular array of numbers having m rows and n columns. It is represented as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m21} & \cdots & a_{mn} \end{bmatrix}$$

12.2 DEFINITIONS & OPERATIONS INVOLVING MATRICS

12.2.1 Equality

Two matrices A and B are equal, i.e., $A = B$, if and only if they are of the same size and their corresponding entries are equal, i.e., $a_{ij} = b_{ij}$.

12.2.2 Addition (or subtraction):

If two matrices A and B have the same size, then $A + B$ has the entries $[a_{ij} \pm b_{ij}]$. Example,

$$\begin{bmatrix} 3 & 2 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 6 & 6 \end{bmatrix}$$

12.2.3 Scalar Multiplication

$cA = [ca_{ij}]$ where c is a number. Example,

$$2 \times \begin{bmatrix} 3 & 2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 8 & 4 \end{bmatrix}$$

12.2.4 Matrix Multiplication

$AB = C$, the entries of C are given by:

$$\boxed{c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}}$$

If A is matrix of size $m \times n$, B is a matrix of size $n \times p$, then the resulting matrix C from their multiplication is of size $m \times p$. Example,

$$\begin{bmatrix} 3 & 2 \\ 4 & 2 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 13 & 11 \\ 16 & 12 \end{bmatrix}$$

Matrix addition is commutative and associative. Matrix Multiplication is not commutative.

12.2.5 Transpose of a Matrix

The transpose of matrix a_{ij} is a matrix with its elements as a_{ji}. The rows of A become the columns of A^T, i.e., the entries of $A^T = [a_{ji}]$. Example,

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \\ 3 & 2 & 1 \end{bmatrix} \quad A^T = \begin{bmatrix} 3 & 4 & 3 \\ 2 & 5 & 2 \\ 1 & 6 & 1 \end{bmatrix}$$

12.2.6 Principal Diagonal

If A is a square matrix, then the diagonal which contains all elements a_{jk} for which $j = k$ is called the *principal* or *main diagonal*. Example: *Principal Diagonal of A is* $[3 \quad 5 \quad 1]$.

12.2.7 Trace of a Matrix

The sum of elements of the principal diagonal of a matrix is called the $trace$ of A.

12.3 Types of Matrices

12.3.1 Diagonal Matrix

A $Diagonal$ matrix is a square matrix that has non-zero entries on its diagonal while all other entries above and below the the diagonal are 0. Example,

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

12.3.2 Zero or Null Matrix

A matrix whose elements are all equal to zero is called the null or zero matrix and is often denoted by O or simply 0. Example,

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

12.3.3 Unit or Identity Matrix

All entries in the diagonal matrix are 1 and all other elements are 0. This implies $AI = IA$, where I is the *Identity Matrix*. Example,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

12.3.4 Symmetric Matrix & Skew Symmetric Matrix

Symmetric matrices are square matrices whose transpose equals the matrix itself, i.e., $A^T = A$. Skew-symmetric matrices are square matrices whose transpose equals the negative of the matrix, i.e., $A^T = -A$. Example,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 3 \end{bmatrix} \text{(Symmetric)} \qquad \begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & -5 \\ -3 & 5 & 3 \end{bmatrix} \text{(Skew Symmetric)}$$

12.3.5 Orthogonal Matrix

A square matrix A is called an *orthogonal matrix* if its transpose is the same as its inverse, i.e., $A^T = A^{-1}$ or $A^T A = I$. Example,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad A \cdot A^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

12.3.6 Complex Conjugate of a Matrix

A complex conjugate is formed by changing the sign between two terms in a complex number. If all elements a_{jk} of a matrix A are replaced by their complex conjugates $\bar{a}_{jk}$, the matrix obtained is called the complex conjugate of A and is denoted by $\bar{A}$. Example,

$$A = \begin{bmatrix} 1+5i & 3-2i \\ 2-6i & 4+4i \end{bmatrix} \quad \bar{A} = \begin{bmatrix} 1-5i & 3+2i \\ 2+6i & 4-4i \end{bmatrix}$$

12.3.7 Hermitian & Skew-Hermitian Matrices

A square matrix A, which is the same as the complex conjugate of its transpose, i.e. if $A = \bar{A}^T$, is called *Hermitian* matrix. If $A = -\bar{A}^T$, then A is called *skew-Hermitian* matrix. If A is real, these reduce to symmetric and skew-symmetric matrices respectively. Example,

$$A = \begin{bmatrix} 3 & 1-i \\ 1+i & -2 \end{bmatrix} \quad \bar{A} = \begin{bmatrix} 3 & 1+i \\ 1-i & -2 \end{bmatrix} \quad \bar{A}^T = \begin{bmatrix} 3 & 1-i \\ 1+i & -2 \end{bmatrix} = A \text{ (Hermitian)}$$

$$A = \begin{bmatrix} 3i & 1+i \\ -1+i & -i \end{bmatrix} \quad \bar{A} = \begin{bmatrix} -3i & 1-i \\ -1-i & i \end{bmatrix} \quad \bar{A}^T = \begin{bmatrix} -3i & -1-i \\ 1-i & i \end{bmatrix} = -A \text{ (Skew Hermitian)}$$

12.3.8 Unitary Matrix

A complex square matrix A is called a *unitary matrix* if its complex conjugate transpose is the same as its inverse, i.e., $\bar{A}^T = A^{-1}$ or $\bar{A}^T A = I$. Example,

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}i & -\frac{1}{\sqrt{2}}i \end{bmatrix} \quad \bar{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}}i \end{bmatrix} \quad \bar{A}^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}i \end{bmatrix} \quad A \cdot \bar{A}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A \text{ (Unitary)}$$

The real analogue of a unitary matrix is an orthogonal matrix, i.e., if all the entries of a unitary matrix are real (i.e., their complex parts are all zero), then the matrix is orthogonal.

12.4 Linear System of Equations

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ &\vdots \\ &a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{aligned}$$

The matrix form is: $Ax = b$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_1 \\ . \\ x_m \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_1 \\ . \\ b_m \end{bmatrix}$$

Augmented matrix is given by:

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ . & . & \cdots & . & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

12.4.1 Gaussian Elimination

Consider a system of 3 equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

Eliminating x_1 using the 2nd and 3rd equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a'_{22}x_2 + a'_{23}x_3 &= b'_2 \\ a'_{32}x_2 + a'_{33}x_3 &= b'_3 \end{aligned}$$

Eliminating x_2 using the 2nd and 3rd equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a'_{22}x_2 + a'_{23}x_3 &= b'_2 \\ a''_{33}x_3 &= b''_3 \end{aligned}$$

We can then solve for x_3, then x_2 and then x_1 from the 3rd, 2nd and 1st equations in that order.

$$x_3 = b_3''/a_{33}''$$
$$x_2 = (b_2' - a_{23}'x_3)/a_{22}'$$
$$x_1 = (b_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$$

At the end of the Gauss elimination the form of the coefficient matrix and the augmented matrix is called the row echelon form. For the above system of 3 equations, the augmented matrix is:

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}' & a_{23}' & b_2' \\ 0 & 0 & a_{33}'' & b_3'' \end{bmatrix}$$

12.4.2 Jacobi's Iterative Method

Consider the linear system of equations $AX = B$ where,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n21} & \cdots & a_{nn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Then the solution can be obtained iteratively from:

$$\boxed{x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j \neq i} a_{ij}x_j^{(k)}\right)} \quad i = 1, 2, \cdots n, \quad x^{(k)} \text{ \& } x^{(k+1)} \text{ are } k^{th} \text{ \& } (k+1)^{th} \text{ iteration of } x$$

12.4.3 Gauss - Seidel Method

The Guass-Seidel method is a modification of the Jacobi method that results in higher degree of accuracy within fewer iterations. In Jacobi method the value of the variables is not modified until next iteration. In Gauss-Seidel method the value of the variables are modified as soon as new value is evaluated, i.e., in iteration $(k+1)$, use previously computed value $x_i^{(k+1)}$ if available, otherwise use $x_i^{(k)}$.

12.5 Rank of a Matrix, Linear Independence

12.5.1 Rank

Rank of a matrix A, denoted as rank (A), is the maximum number of linearly independent row vectors of A. It is the number of non-zero rows in its row echelon form.

12.5.2 Existence & Uniqueness of Solutions in Linear Systems

A consistent system of equations has at least one solution. A linear system of n equations with n unknowns has an unique solution. This holds true when the *rank of coefficient matrix* A, r, is the same as *rank of augmented matrix* $\tilde{A}$. An inconsistent system has no solution. If $r < n$, then the number of solutions is ∞.

12.5.3 Null Space and Nullity

The null space of any matrix A consists of all the vectors B such that $AB = 0$ and B is not zero. It can also be thought as the solution obtained from $AB = 0$ where A is a known matrix of size $m \times n$ and B is a matrix to be found of size $n \times k$. The size of the null space of the matrix provides us with the number of linear relations among attributes. $AB = 0$ implies every row of A when multiplied by B goes to zero. This establishes the linear relationships between the variables. Every null space vector corresponds to one linear relationship. Nullity is number of vectors in the null space of matrix A.

12.5.4 Rank Nullity Theorem

Rank of A + Nullity of A = Total number of columns of A

Example,

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix} \implies \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 3 & 6 & 1 \end{bmatrix}$$

The rank of the matrix A which is the number of non-zero rows in its echelon form is 2.

With $AB = 0$,

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = 0 \implies b_1 + 2b_2 = 0, b_3 = 0 \implies B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \implies b_1 \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

Thus nullity, i.e., the dimension of the null space is 1. Thus, the sum of the rank and the nullity of A is $2 + 1 = 3$ which is equal to the number of columns of A.

12.6 Determinant

A determinant of order n is a scalar of an $n \times n$ (square) matrix $A[ij]$ is given by:

$$D = det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}$$

$$D = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \quad (j = 1, 2, \cdots, n) \text{ where } M_{ij} \text{ is a determinant of order } n-1$$

The determinant M_{ij} is obtained by removing the row and column in A corresponding to the element a_{ij}. M_{ij} is called the minor of a_{ij}. C_{ij}, called the cofactor of a_{ij}, is defined as $(-1)^{i+j} M_{ij}$. Hence, $D = \sum_{i=1}^{n} a_{ij} C_{ij} \quad (j = 1, 2, \cdots, n)$ where C_{ij} is a determinant of order $n-1$.

Adjoint of a matrix, written as $adj(A)$, is defined as the transpose of the cofactor matrix of A. Example,

$$det \begin{vmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{vmatrix} = 2\,det \begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix} - (-3)\,det \begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix} + 1\,det \begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix}$$

$$= 2(0+4) + 3(10+1) + 1(8-0) = 49$$

12.6.1 Properties of Determinants

1. The value of the determinant is unchanged if the rows and columns are interchanged.
2. Addition of a multiple of a row to another row does not alter the value of the determinant.

3. A zero row or column renders the value of a determinant zero.
4. A determinant with two identical rows or columns has the value zero. Proportional rows or columns render the value of a determinant zero.
5. Interchange of two rows multiplies the value of the determinant by -1.
6. Multiplication of a row by a non zero constant c multiplies the value of the determinant by c. $det(cA) = c\,det(A)$.
7. A $m \times n$ matrix A has rank $r \geq 1$ *iff* A has a $r \times r$ submatrix whose determinant $\neq 0$.
8. An $n \times n$ square matrix A has rank n *iff* $detA \neq 0$.
9. $det(AB) = det(BA) = det(A)det(B)$

12.6.2 Cramer's Rule

$$\begin{aligned}
&a_{11}x_1 + a_{11}x_1 + \ldots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{21}x_1 + \ldots + a_{2n}x_n = b_2 \\
&\vdots \\
&a_{n1}x_1 + a_{n1}x_1 + \ldots + a_{nn}x_n = b_n
\end{aligned}$$

$$\boxed{x_1 = \frac{D_1}{D}, x_2 = \frac{D_2}{D}, \cdots, x_n = \frac{D_n}{D}} \quad \text{(Cramer's Rule)}$$

where D_k is the determinant obtained by replacing the k^{th} column by the entries $b_1, b_2, \cdots, b_n$.

The proof is simple:

Let $A = [a_1 \quad a_2 \quad \cdots a_n]$

where a_i is a column vector.

$$\text{Let } I_i(X) = \begin{bmatrix} 1 & 0 & \cdots & x_1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & x_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & x_n & 0 & \cdots & 1 \end{bmatrix} = [e_1 \quad e_2 \quad \cdots \quad x_i \quad e_{i+1} \quad e_n]$$

$$\begin{aligned}
AI_i(X) &= [Ae_1 \quad Ae_2 \quad \cdots \quad Ax_i \quad Ae_{i+1} \quad Ae_n] \\
&= [a_1 \quad a_2 \cdots \quad a_{i-1} \quad b \quad a_{i+1} \quad \cdots \quad a_n] = A_i(b) \text{ (replace } i^{th} \text{ column of } A \text{ with } b)
\end{aligned}$$

$$det(A_i(b)) = det(A)\,I_i(X) = det(A)\,det(I_i(X)) = det(A)\,x_i$$

$$\implies \boxed{x_i = \frac{det(A_i(b))}{det(A)}}$$

Example,

$$\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

$$det \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix} = -5, \quad x = -\frac{1}{5} det \begin{bmatrix} 5 & 1 \\ -4 & -3 \end{bmatrix} = 11/5, \quad y = -\frac{1}{5} det \begin{bmatrix} 1 & 5 \\ 2 & -4 \end{bmatrix} = 14/5$$

12.7 Inverse of a Matrix

The inverse of a square matrix A, denoted by A^{-1} is a $n \times n$ matrix that satisfies the following:

$AA^{-1} = A^{-1}A = I$ (I is an $n \times n$ unit matrix)

If A^{-1} exists,$A$is called a non-singular matrix, else it is called a singular matrix. If the inverse exists, it is always unique. A has an inverse *iff* rank $A = r$.

12.7.1 Inverse by Gauss Jordan method

To determine A^{-1},

1. Create augmented matrix $\tilde{A} = [A\ I]$ of size $n \times 2n$.
2. Apply Gauss elimination to $\tilde{A}$ to reduce to upper triangular form $[UH]$.
3. Eliminate the entries of U above the diagonal and make the diagonal entries 1 to get to arrive at the form $[I\ K]$.
4. Then, $A^{-1} = K$

12.7.2 Inverse by Cofactors

$$A\,adj(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \begin{vmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} det(A) & 0 & \cdots & 0 \\ 0 & det(A) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & det(A) \end{vmatrix} = det(A)\,I$$

$$\implies A\left[\frac{adj(A)}{det(A)}\right] = I$$

$$\implies \boxed{A^{-1} = \frac{adj(A)}{det(A)}}$$

Example,

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

$$det(A) = 10, \quad cof(A) = \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}, \quad adj(A) = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}, \quad A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$$

12.7.3 Property of Matrix Inverse

$(AB)^{-1} = B^{-1}A^{-1}$ because $(AB)(AB)^{-1} = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$ or $I = I$

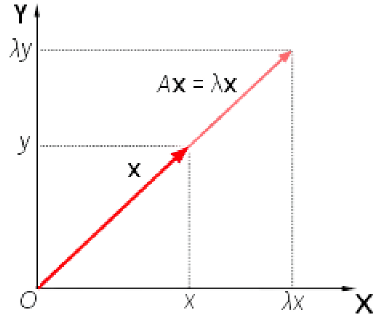
Generalizing, $\boxed{(ABC \cdots PQR)^{-1} = R^{-1}Q^{-1}P^{-1} \cdots C^{-1}B^{-1}A^{-1}}$

12.8 Eigenvalue & Eigenvector

Consider the following system of equations in matrix form.

$AX = \lambda X$ (where A is a $n \times n$ matrix and λ is a scalar)
$(A - \lambda I)X = 0$

The number, i.e., the scalar value λ is an eigenvalue of A and X, a non zero vector, is called an eigenvector of A. Geometrically, an eigenvector, corresponding to a real nonzero eigenvalue, points in a direction in which it is stretched by the transformation and the eigenvalue is the factor by which it is stretched. If the eigenvalue is negative, the direction is reversed. A^T has the same eigenvalues as A.



Using Cramer's rule:

$$\boxed{det(A - \lambda I) = 0}$$

Solve for λ, substitute in equation, and determine x. $det(\lambda)$ is called the characteristic determinant and the polynomial is called the characteristic polynomial. A $n \times n$ matrix as at least 1 eigenvalue, at most n different eigenvalues.

12.8.1 Algebraic Multiplicity

The algebraic multiplicity of an eigenvalue, μ, is the number of times it appears, i.e., repeated, as a root of the characteristic polynomial. Example,

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 2 \end{bmatrix} \quad det(A - \lambda I) = \begin{bmatrix} 4-\lambda & 2 \\ 1 & 2-\lambda \end{bmatrix} = \lambda^2 - 6\lambda + 6 = 0, \quad \lambda_1 = 3 + \sqrt{3}, \quad \lambda_1 = 3 - \sqrt{3}$$

$$\mu(\lambda_1) = 1, \quad \mu(\lambda_2) = 1$$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad det(A - \lambda I) = \begin{bmatrix} 1-\lambda & 0 \\ 2 & 1-\lambda \end{bmatrix} = (\lambda - 1)^2 = 0, \quad \lambda_1 = 1, \quad \lambda_2 = 1$$

$$\mu(\lambda_1) = 2, \quad \mu(\lambda_2) = 2$$

12.8.2 Geometric Multiplicity

Eigenspace is the collection of eigenvectors associated with each eigenvalue for the linear transformation applied to the eigenvector. The geometric multiplicity of an eigenvalue is the dimension of the linear space of its associated eigenvectors (i.e., its eigenspace). Example,

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \quad det(A - \lambda I) = \begin{bmatrix} 2-\lambda & 0 \\ 1 & 1-\lambda \end{bmatrix} = (\lambda - 2)(\lambda - 1) = 0, \quad \lambda_1 = 2, \quad \lambda_2 = 1$$

$$\begin{bmatrix} 2-\lambda_1 & 0 \\ 1 & 1-\lambda_1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This yields $x_{11} = x_{21}$ and is non-zero. Hence, the eigenspace of λ_1 is the linear space that contains all vectors of the form $X_2 = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where α is any non-zero scalar. Thus, the eigenspace of λ_1 is generated by the single vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and so has dimension 1 and the geometric multiplicity of λ_1 is 1.

Note that the second set of equations, corresponding to $\lambda_2 = 1$, yields $x_{12} = x_{22} = 0$ and hence the vector X_2 is not non-zero and is of no use.

Now consider,

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad det(A - \lambda I) = \begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} = (\lambda - 2)(\lambda - 2) = 0, \quad \lambda_1 = 2, \quad \lambda_2 = 2$$

$$\begin{bmatrix} 2-\lambda_1 & 0 \\ 1 & 2-\lambda_1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This system of equations is satisfied for any value of x_{11} and x_{21}.

Hence, the eigenspace of λ_1 is the linear space that contains all vectors x_1 are:

$$X_1 = x_{11} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_{21} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where x_{11} and x_{21} are scalars that can be arbitrarily chosen. Thus, the eigenspace of λ_1 is generated by the two linearly independent vectors $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Hence, it has dimension 2. As a consequence, the geometric multiplicity of λ_1 is 2, equal to its algebraic multiplicity.

12.8.3 Defective eigenvalues

The algebraic and geometric multiplicity of an eigenvalue do not necessarily coincide. When the geometric multiplicity of a repeated eigenvalue is strictly less than its algebraic multiplicity, then that eigenvalue is said to be defective.

An eigenvalue that is not repeated has an associated eigenvector which is different from zero. Therefore, the dimension of its eigenspace is equal to 1, its geometric multiplicity is equal to 1 and equals its algebraic multiplicity. Thus, an eigenvalue that is not repeated is also non-defective.

12.8.4 Real Eigenvalues

Let A be a real symmetric matrix and let λ be a complex eigenvalue of A.

$Ax = \lambda x, x \neq 0$

Taking complex conjugates of both sides, and since A is real we have,

$A\bar{x} = \bar{\lambda}\bar{x}$

Taking transpose and with A as symmetric we have,

$\bar{x}^T A = \bar{\lambda}\bar{x}^T$

$\bar{x}^T Ax = \bar{\lambda}\bar{x}^T x$

$\bar{x}^T \lambda x = \bar{\lambda}\bar{x}^T x$

$\lambda = \bar{\lambda}$

The eigenvalues of a symmetric matrix are real. Similarly, we can establish that the eigenvalues of a skew-symmetric matrix are pure imaginary or zero.

12.8.5 Matrix Diagonalization

Two square matrices A and B are said to be similar if there exists an invertible P such that,

$$B = P^{-1}AP$$

If two matrices are similar, then they have the same rank, trace, determinant and eigenvalues. Not only two similar matrices have the same eigenvalues, but their eigenvalues have the same algebraic and geometric multiplicities. When A is diagonalizable, then there exists an invertible matrix P such that,

$$D = P^{-1}AP$$

where D is a diagonal matrix.

This is because multiplying the above with P we have,

$$AP = PD$$

Since D is diagonal, P_k is an eigenvector associated with D_{kk}. The matrix P used in the diagonalization must be invertible. Therefore, its columns must be linearly independent. Stated differently, there must be k linearly independent eigenvectors of A.

For some matrices, called defective matrices, it is not possible to find k linearly independent eigenvectors. A matrix is defective when it has at least one repeated eigenvalue whose geometric multiplicity is strictly less than its algebraic multiplicity (called a defective eigenvalue). Therefore, defective matrices cannot be diagonalized.

Matrix A is diagonalizable if and only if it does not have any defective eigenvalue. If all the eigenvalues of A are distinct, then A does not have any defective eigenvalue. Therefore, possessing distinct eigenvalues is a sufficient condition for diagonalizability.

12.8.6 Positive Definite Matrix

A square matrix A is positive definite if pre-multiplying and post-multiplying it by the same vector x always gives a positive number as a result, independently of how we choose the vector, i.e., $x^T A\, x > 0$. Positive definite symmetric matrices have the property that all their eigenvalues are positive.

12.8.7 Quadratic Form & Positive Definiteness

A quadratic form in A is a transformation $x^T Ax$ and is a scalar. When A is symmetric, we can also write the transformation as $x^T(\frac{A}{2} + \frac{A^T}{2})x$. A is said to be positive definite *iff* $x^T Ax > 0$ for any non-zero x. It is said to be semi positive definite iff $x^T Ax \geq 0$ for any non-zero x.

If A is positive definite, then it is full-rank. A matrix is said to have full rank if its rank equals the largest possible for a matrix of the same dimensions, which is the lesser of the number of rows and columns.

CHAPTER 13

Vector

Vectors were first used to express the laws of electromagnetism. Since that time, vectors have become essential in physics, mechanics, electrical engineering, and other sciences to describe forces mathematically.

Some quantities in physics are characterized by both magnitude and direction, such as displacement, velocity, force and acceleration. To describe such quantities, we introduce the concept of a *vector* as a directed line segment. There are other quantities in physics that are characterized by magnitude only, such as mass, length and temperature. Such a quantity is called a *scalars.*

For example, speed, say 10 KM/Hr is a scalar whereas velocity, say 10 KM/Hr towards north-east is a vector and is denoted as:

$$\vec{\mathbf{v}} = 10 \cdot \frac{1}{\sqrt{2}}\hat{\mathbf{i}} + 10 \cdot \frac{1}{\sqrt{2}}\hat{\mathbf{j}}$$

where $\hat{\mathbf{i}}$ is an unit vector along the x direction and $\hat{\mathbf{j}}$ is an unit vector along the y direction.

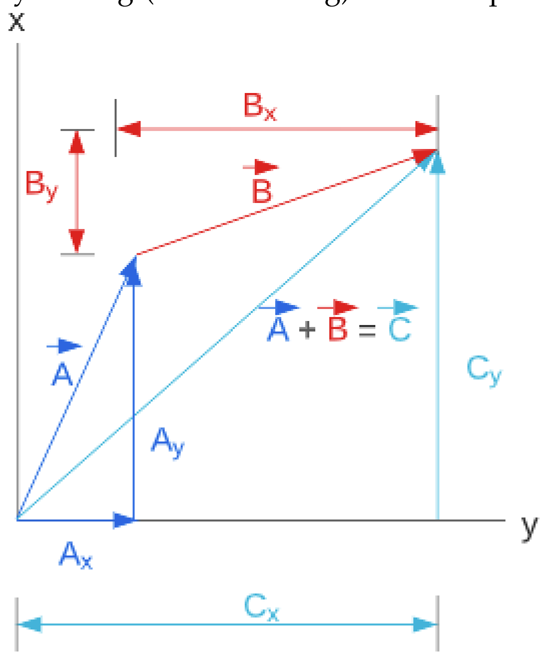
13.1 Vector Algebra

13.1.1 Vector Addition & Subtraction

Vector addtion (or subtraction) is performed by adding (or subtracting) their components.

$$\vec{\mathbf{A}} + \vec{\mathbf{B}} = \vec{\mathbf{C}}$$

$$A_x + B_x = C_x$$
$$A_y + B_y = C_y$$



13.1.2 Scalar Multiplication

Multiplication of a vector $\vec{\mathbf{A}}$ by a scalar m produces a vector $m\vec{\mathbf{A}}$ with magnitude $m \times \|A\|$ where $\|A\|$ is the magnitude of $\vec{\mathbf{A}}$.

13.1.3 Unit Vector

Unit vectors are vectors having unit length.

$$A = A_1\hat{\mathbf{i}} + A_2\hat{\mathbf{j}} + A_3\hat{\mathbf{k}}$$

$$\|A\| = \sqrt{A_1^2 + A_2^2 + A_3^2} = 1$$

13.1.4 Linear Independence & Dependence

Vectors $\vec{\mathbf{A_1}}$, $\vec{\mathbf{A_2}}$, $\cdots$, $\vec{\mathbf{A_n}}$ are linearly dependent if there exist scalars $\vec{\mathbf{a_1}}$, $\vec{\mathbf{a_2}}$, $\cdots$, $\vec{\mathbf{a_n}}$, not all zero, such that:

$$a_1\vec{\mathbf{A_1}} + a_2\vec{\mathbf{A_2}} + \cdots + a_n\vec{\mathbf{A_n}} = 0$$

Otherwise, the vectors are linearly independent.

13.1.5 Scalar & Vector Fields

For each point (x,y,z) of a region D in space, if there corresponds a number (scalar) $\phi(x,y,z)$, then ϕ is called a scalar function of position and we say that a scalar field f has been defined on D. A scalar field ϕ, which is independent of time, is called a stationary or steady-state scalar field.

For each point (x,y,z) of a region D in space, if there corresponds a vector $V(x,y,z)$, then $\vec{\mathbf{V}}$ is called a vector function of position, and we say that a vector field $\vec{\mathbf{V}}$ has been defined on D. A vector field $\vec{\mathbf{V}}$ which is independent of time is called a stationary or steady-state vector field.

13.1.6 Vector Space R^n

Let $V = R^n$ where R^n consists of all n-element sequences $u = (a_1, a_2, \cdots, a_n)$ of real numbers called the components of u. The term vector is used for the elements of V and we denote them using the letters u, v, and w, with or without a subscript. The real numbers arel scalars and we denote them using letters other than u, v, or w.

We define two operations on $V = R^n$:

$$\vec{\mathbf{u}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad \vec{\mathbf{v}} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad \vec{\mathbf{u}} + \vec{\mathbf{v}} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

where a_i, b_i are the components of vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ and,

$$k\vec{\mathbf{u}} = \begin{bmatrix} ku_1 \\ ku_2 \\ \vdots \\ ku_n \end{bmatrix}$$

13.2 Vector Spaces

Vectors with n real numbers as components are elements of real n dimensional vector space R^n. Each vector in R^n is an ordered n-tuple of real numbers. Instead of real numbers, we can have complex numbers to obtain the complex vector space.

13.2.1 Dimension

For a non-empty set $\vec{\mathbf{V}}$ of vectors where each vector has the same number of components. If, for any two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ in $\vec{\mathbf{V}}$, all linear combinations $\alpha a + \beta b$ where α, β are real numbers, are also elements of $\vec{\mathbf{V}}$.

The maximum number of linearly independent vectors in $\vec{\mathbf{V}}$ is called the dimension of $\vec{\mathbf{V}}$ and is denoted as $\mathbf{dim}\vec{\mathbf{V}}$. Hence, a vector space having vectors with **n** components has the dimension **n**.

13.2.2 Basis

A linearly independent set in $\vec{\mathbf{V}}$ consisting of a maximum possible number of vectors in $\vec{\mathbf{V}}$ is called the **basis** for $\vec{\mathbf{V}}$.

13.2.3 Span

Span (a vector space) is the set of all linear combinations of the vectors.

13.2.4 Subspace

Subspace of A is a non-empty subset of **V** including **V** itself.

13.3 Vector Products

13.3.1 Dot Product

The dot or scalar product of two vectors A and B, denoted by $A \cdot B$, is defined as the product of the magnitudes of A and B and the cosine of the angle θ between them.

$$\boxed{A\cdot = |A||B|cos\theta, \quad 0 \leq \theta \leq \pi} \tag{13.3.1}$$

13.3.2 Inner Product

An inner product is a generalization of the dot product. In a vector space, it is a way to multiply vectors together, with the result of this multiplication being a scalar. an inner product $<\mathring{u},\mathring{u}>$ satisfies the following four properties. Let u, v, and w be vectors and α be a scalar, then:

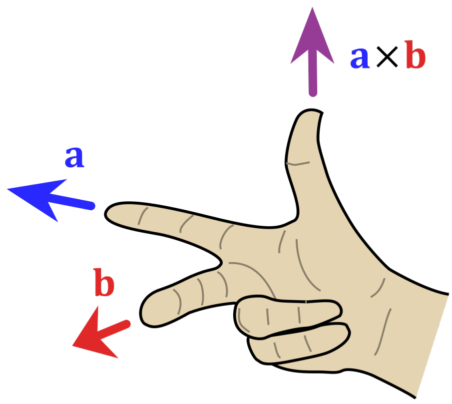
1. $<u+v,w>=<u,w>+<v,w>$
2. $<\alpha v,w>=\alpha<v,w>$
3. $<v,w>=<w,v>$
4. $<v,v>\geq 0$ and equal if and only if $v=0$.

13.3.3 Cross Product

The cross product of vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is a vector $\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}}$ (read as $\vec{\mathbf{A}}$ *cross* $\vec{\mathbf{B}}$) defined as follows.

$$\boxed{\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}} = |A||B|sin\theta\,\hat{u}, \quad 0 \leq \theta \leq \pi}$$

The magnitude of $\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}}$ is equal to the product of the magnitudes of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ and the *sine* of the angle θ between them. The direction of $\vec{\mathbf{C}}$ is perpendicular to the plane of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ so that $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$, and $\vec{\mathbf{C}}$ form a right-handed system. where $\hat{u}$ is a unit vector indicating the direction of $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$.



The cross product of two vectors can be expressed in terms of *determinant* as follows:

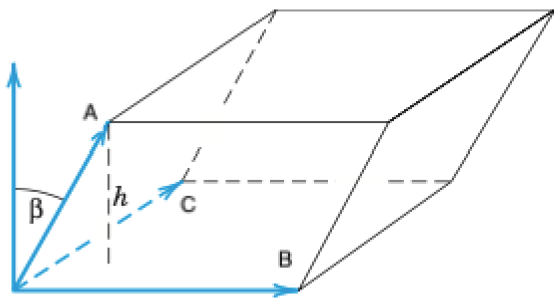
$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \begin{vmatrix} A_2 & A_3 \\ B_2 & B_3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} A_1 & A_3 \\ B_1 & B_3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix} \hat{\mathbf{k}}$$

13.3.4 Scalar Triple Product

The scalar triple product of three vectors $\vec{\mathbf{A}}, \vec{\mathbf{B}}, \vec{\mathbf{C}}$ is defined as:

$$(\vec{\mathbf{A}}\,\vec{\mathbf{B}}\,\vec{\mathbf{C}}) = \vec{\mathbf{A}} \cdot (\vec{\mathbf{B}} \times \vec{\mathbf{C}}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = det[\vec{\mathbf{A}}\,\vec{\mathbf{B}}\,\vec{\mathbf{C}}]$$

Geometrically, the absolute value of $\|(\vec{\mathbf{A}}\ \vec{\mathbf{B}}\ \vec{\mathbf{C}})\|$ is the volume of the parallelepiped with $\vec{\mathbf{A}}, \vec{\mathbf{B}}, \vec{\mathbf{C}}$ as edge vectors. The three vectors in R^3 are linearly independent if and only if their scalar triple product is not zero.



Properties of scalar triple product are as follows:

$$(\vec{\mathbf{A}}\,\vec{\mathbf{B}}\,\vec{\mathbf{C}}) = \vec{\mathbf{A}} \cdot (\vec{\mathbf{B}} \times \vec{\mathbf{C}}) = \vec{\mathbf{A}} \cdot (\vec{\mathbf{B}} \times \vec{\mathbf{C}}) = (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) \cdot \vec{\mathbf{C}}$$
$$\vec{\mathbf{A}} \cdot (\vec{\mathbf{B}} \times \vec{\mathbf{C}}) = (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) \cdot \vec{\mathbf{C}}$$

13.3.5 Reciprocal Set

A reciprocal set $a^{'}$ satisfies the following:

$$a \cdot a^{'} = 1$$

13.3.6 Vector Properties

Given three vectors $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$ and $\vec{\mathbf{C}}$; they satisfy the following properties:

1. Commutative: $\vec{\mathbf{A}} + \vec{\mathbf{B}} = \vec{\mathbf{B}} + \vec{\mathbf{A}}$
2. Associative: $(\vec{\mathbf{A}} + \vec{\mathbf{B}}) + \vec{\mathbf{C}} = \vec{\mathbf{A}} + (\vec{\mathbf{B}} + \vec{\mathbf{C}})$

3. Distributative: $\vec{\mathbf{A}} \cdot (\vec{\mathbf{B}} + \vec{\mathbf{C}}) = \vec{\mathbf{A}} \cdot \vec{\mathbf{B}} + \vec{\mathbf{A}} \cdot \vec{\mathbf{C}}$
4. Distributative: $\vec{\mathbf{A}} \times (\vec{\mathbf{B}} + \vec{\mathbf{C}}) = (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) + (\vec{\mathbf{A}} \times \vec{\mathbf{C}})$
5. Distributive: $(\vec{\mathbf{A}} + \vec{\mathbf{B}}) \times \vec{\mathbf{C}} = (\vec{\mathbf{A}} \times \vec{\mathbf{C}}) + (\vec{\mathbf{B}} \times \vec{\mathbf{C}})$
6. Zero Vector: $\vec{\mathbf{A}} + 0 = \vec{\mathbf{A}}$
7. Scalar Multiplication by m: $m(\vec{\mathbf{A}} + \vec{\mathbf{B}}) = m\vec{\mathbf{A}} + m\vec{\mathbf{B}}$
8. Inner Product: $(\vec{\mathbf{A}}, \vec{\mathbf{B}}) = \vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \vec{\mathbf{A}}^T \vec{\mathbf{B}}$ (n-Dimensional Euclidean Space)
9. Linear Transformation of Space R^n into Space R^m:
   $X = R^n, Y = R^m, Y = AX$ where A is an $m \times n$ matrix.

13.3.7 Gram–Schmidt orthonormalization

The Gram–Schmidt orthonormalization process is a procedure for orthonormalizing a set of vectors in an inner product space. Let $\{v_1, v_2, \cdots, v_k\}$ to be a non-orthonormal basis for V. Then, we need to determine $\{u_1, u_2, \cdots, u_k\}$ an orthonormal basis for the span of $\{v_1, v_2, \cdots v_p\}$. We define the projection operator by:

$$proj_u(v) = \frac{< u, v >}{< u, u >} u$$

where $< u, v >$ denotes the inner product of the vectors u and v. This operator projects the vector v orthogonally onto the line spanned by vector u. The Gram–Schmidt process is given by:

$$\begin{aligned}
u_1 &= v_1 & e_1 &= \frac{u_1}{||u_1||} \\
u_2 &= v_2 - proj_{u_1}(v_2) & e_2 &= \frac{u_2}{||u_2||} \\
u_3 &= v_3 - proj_{u_1}(v_3) - proj_{u_2}(v_3) & e_3 &= \frac{u_3}{||u_3||} \\
&\vdots \\
u_k &= v_k - \sum_{j=1}^{k-1} proj_{u_j}(v_k) & e_k &= \frac{u_k}{||u_k||}
\end{aligned}$$

The sequence $u_1, u_2, \cdots, u_k$ is the required system of orthogonal vectors, and the normalized vectors $e_1, e_2, \cdots, e_k$ form an orthonormal set. The calculation of the sequence $u_1, u_2, \cdots, u_k$ is known as Gram–Schmidt orthogonalization, while the calculation of the sequence $e_1, e_2, \cdots, e_k$ is known as Gram–Schmidt orthonormalization as the vectors are normalized.

13.4 Vector Differentiation

$$\frac{\Delta\vec{\mathbf{R}}}{\Delta\vec{\mathbf{u}}} = \frac{\vec{\mathbf{R}}(\vec{\mathbf{u}} + \Delta\vec{\mathbf{u}}) - \vec{\mathbf{R}}(\vec{\mathbf{u}})}{\Delta\vec{\mathbf{u}}}$$

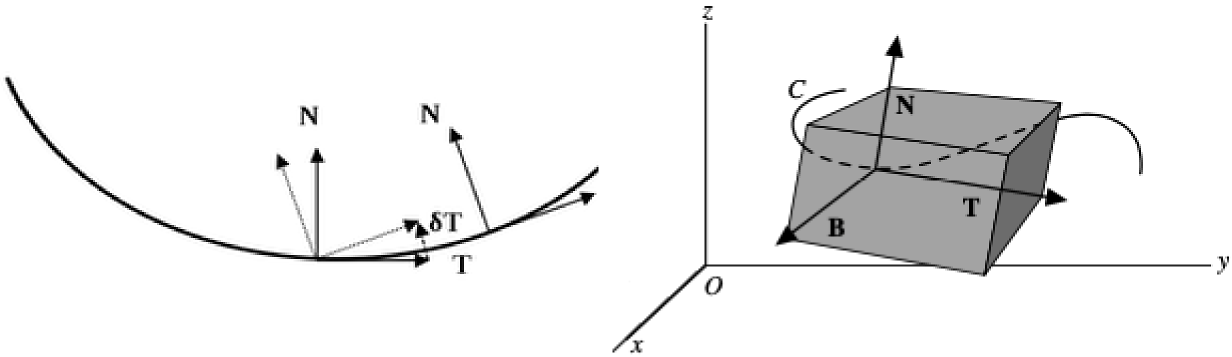
$$\frac{d\vec{\mathbf{R}}}{d\vec{\mathbf{U}}} = \lim_{\Delta\vec{\mathbf{u}} \to 0} \frac{\Delta\vec{\mathbf{R}}}{\Delta\vec{\mathbf{u}}} = \lim_{\Delta\vec{\mathbf{u}} \to 0} \frac{\vec{\mathbf{R}}(\vec{\mathbf{u}} + \Delta\vec{\mathbf{u}}) - \vec{\mathbf{R}}(\vec{\mathbf{u}})}{\Delta\vec{\mathbf{u}}}$$

If, $r(u) = x(u)\hat{\mathbf{i}} + y(u)\hat{\mathbf{j}} + z(u)\hat{\mathbf{k}}$

$$\frac{dr}{du} = \frac{dx}{du}\hat{\mathbf{i}} + \frac{dy}{du}\hat{\mathbf{j}} + \frac{dz}{du}\hat{\mathbf{k}}$$

13.5 Differential Geometry

13.5.1 Frenet–Serret Formulae



Consider C to be a space curve defined by the function $r(u)$. Then, dr/du is a vector in the direction of the tangent to C. If the scalar u is taken as the arc length s measured from some fixed point on C, then dr/ds is a unit tangent vector to C and is denoted by T. The rate at which T changes with respect to s is a measure of the curvature of C and is given by dT/ds. The direction of dT/ds at any given point on C is normal to the curve at that point. If N is a unit vector in this normal direction, it is called the *principal normal* to the curve. Then $dT/ds = kN$, where k is called the curvature of C at the specified point. The quantity $\rho = 1/k$ is called the *radius of curvature* .

A unit vector B perpendicular to the plane of T and N and such that $B = T \times N$, is called the *binormal* to the curve. It follows that directions T, N, B form a localized right-handed rectangular coordinate system at any specified point of C. This coordinate system is called the trihedral or triad at the point. As s changes, the coordinate system moves and is known as the moving trihedral. The *Frenet–Serret* formulae are given by:

$$\boxed{\frac{dT}{ds} = \kappa N} \quad \boxed{\frac{dN}{ds} = \tau B - \kappa T} \quad \boxed{\frac{dB}{ds} = \tau N} \tag{13.5.1}$$

where τ is a scalar called the *torsion.* The quantity $s = 1/\tau$ is called the *radius of torsion* . The osculating plane to a curve at a point P is the plane containing the tangent and principal normal at P. The normal plane is the plane through P perpendicular to the tangent. The rectifying plane is the plane through P, which is perpendicular to the principal normal.

13.5.2 Gradient

The differential operator *del* , written as ∇ is defined as:

$$\boxed{\nabla = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial y}\hat{\mathbf{k}}}$$

if $\phi(x, y, z)$ be a scalar function defined and differentiable at each point (x, y, z) in a certain region of space, then the gradient of ϕ, written $\nabla\phi$ or $grad\,\phi$ is defined as follows:

$$\nabla\phi = \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial y}\hat{\mathbf{k}}\right)\phi = \frac{\partial\phi}{\partial x}\hat{\mathbf{i}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{j}} + \frac{\partial\phi}{\partial z}\hat{\mathbf{k}}$$

13.5.3 Divergence

If $V(x, y, z) = V_1\hat{\mathbf{i}} + V_2\hat{\mathbf{j}} + V_3\hat{\mathbf{k}}$ is defined and differentiable at each point (x, y, z) in a region of space, then the divergence of V, a scalar, is defined as follows:

$$\boxed{\nabla \cdot V = \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial y}\hat{\mathbf{k}}\right) \cdot (V_i\hat{\mathbf{i}} + V_j\hat{\mathbf{j}} + V_k\hat{\mathbf{k}}) = \frac{\partial V_i}{\partial x}\hat{\mathbf{i}} + \frac{\partial V_j}{\partial y}\hat{\mathbf{j}} + \frac{\partial V_k}{\partial z}\hat{\mathbf{k}} \quad \text{(scalar)}}$$

13.5.4 Curl

The *curl* or rotation of V, a vector, is defined as:

$$\boxed{curl\ V = \nabla \times V = \left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial y}\hat{\mathbf{k}}\right) \times (V_1\hat{\mathbf{i}} + V_2\hat{\mathbf{j}} + V_3\hat{\mathbf{k}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}}$$

13.6 Vector Integration

13.6.1 Line Integration

Let $A(x, y, z) = A_1\hat{\mathbf{i}} + A_2\hat{\mathbf{j}} + A_3\hat{\mathbf{k}}$ be a vector function of position defined and continuous along C. Then the integral of the tangential component of $\vec{\mathbf{A}}$ along C from P_1 to P_2, written as:

$$\int_{P_1}^{P_2} \vec{\mathbf{A}} \cdot \vec{\mathbf{dr}} = \int_C \vec{\mathbf{A}} \cdot \vec{\mathbf{dr}} = \int_C (A_1 dx + A_2 dy + A_3 dz)$$

If C is a closed curve (which we shall suppose is a simple closed curve, that is, a curve that does not intersect itself anywhere), the integral around C is often denoted by:

$$\boxed{\oint_C \vec{\mathbf{A}} \cdot \vec{\mathbf{dr}} = \oint_C (A_1 dx + A_2 dy + A_3 dz)}$$

13.6.2 Surface Integration

Consider a differential of surface area dS a vector $\mathbf{dS}$ whose magnitude is dS and whose direction is that of $\hat{n}$. Then $\vec{\mathbf{dS}} = \hat{n} dS$. The integral is given by:

$$\boxed{\iint_S \vec{\mathbf{A}} \cdot \vec{\mathbf{dS}} = \iint_S \vec{\mathbf{A}} \cdot \hat{n}\, dS}$$

13.6.3 Volume Integration

Consider a closed surface in space enclosing a volume V. The volume integral is given by:

$$\boxed{\iiint_V A\, dV}$$

13.6.4 Gauss' Divergence Theorem

Suppose V is the volume bounded by a closed surface S and $\vec{\mathbf{F}}$ is a vector function of position with continuous derivatives. Then:

$$\iiint_V \nabla\cdot\vec{\mathbf{F}}\,dV = \iiint_V \left(\frac{\partial}{\partial x}\hat{\mathbf{i}}+\frac{\partial}{\partial y}\hat{\mathbf{j}}+\frac{\partial}{\partial y}\hat{\mathbf{k}}\right)\cdot(F_x\hat{\mathbf{i}}+F_y\hat{\mathbf{j}}+F_z\hat{\mathbf{k}})\,dV$$

$$= \iiint_V \frac{\partial F_x}{\partial x}\,dx\,dy\,dz + \iiint_V \frac{\partial F_y}{\partial y}\,dx\,dy\,dz + \iiint_V \frac{\partial F_z}{\partial z}\,dx\,dy\,dz$$

$$\text{Now, } \iiint_V \frac{\partial F_z}{\partial z}\,dx\,dy\,dz = \int_{x_1}^{x_2}\int_{y_1}^{y_2}[F_z(x,y,z_2)-F_z(x,y,z_1)]\,dx\,dy = \int_{S_1}\int_{S_2} F_z\cdot ds$$

$$\implies \boxed{\iiint_V \nabla\cdot\vec{\mathbf{F}}\,dV = \iint_S F_x\cdot ds + \iint_S F_y\cdot ds + \iint_S F_z\cdot ds = \oiint \vec{\mathbf{F}}\cdot dS}$$

13.6.5 Stoke's Theorem

The line integral of a vector field over a loop is equal to the flux of its curl through the enclosed surface. Suppose S is an open, two sided surface bounded by a closed, non intersecting curve C (simple closed curve), and suppose $\vec{\mathbf{F}}$ is a vector function of position with continuous derivatives. Then,

$$\oint_C \vec{\mathbf{F}}\cdot dl = \oiint_S (\nabla\times\vec{\mathbf{F}})\cdot\vec{\mathbf{n}}\,dS$$

$$curl\,\vec{\mathbf{F}} = \nabla\times\vec{\mathbf{F}} = \left(\frac{\partial}{\partial x}\hat{\mathbf{i}}+\frac{\partial}{\partial y}\hat{\mathbf{j}}+\frac{\partial}{\partial y}\hat{\mathbf{k}}\right)\times(F_x\hat{\mathbf{i}}+F_y\hat{\mathbf{j}}+F_z\hat{\mathbf{k}}) = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_x & F_y & F_z\end{vmatrix}$$

$$= \begin{vmatrix}\frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_y & F_z\end{vmatrix}\hat{\mathbf{i}} - \begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial z}\\ F_x & F_z\end{vmatrix}\hat{\mathbf{j}} + \begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ F_x & F_y\end{vmatrix}\hat{\mathbf{k}}$$

$$= \left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\hat{\mathbf{i}} - \left(\frac{\partial F_z}{\partial x}-\frac{\partial F_x}{\partial z}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\hat{\mathbf{k}}$$

$$= \left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\hat{\mathbf{k}}$$

$$\oiint_S (\nabla\times\vec{\mathbf{F}})\cdot\vec{\mathbf{n}}\,dS = \oiint_S \left[\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\hat{\mathbf{k}}\right]\cdot\hat{\mathbf{n}}\,dS$$

Using a parametric representation of the surface we have,

$$r(u,v) = [x(u,v), y(u,v), z(u,v)] = x(u,v)\hat{\mathbf{i}} + y(u,v)\hat{\mathbf{j}} + z(u,v)\hat{\mathbf{k}}$$

$$\text{Normal Vector, } \vec{\mathbf{N}} = \vec{\mathbf{r_u}}\times\vec{\mathbf{r_v}}, \quad \text{Unit Normal Vecrtor } \hat{\mathbf{n}} = \frac{1}{\|N\|}\vec{\mathbf{N}}$$

$$\|\vec{\mathbf{r}}_u\times\vec{\mathbf{r}}_v\| = \|N\| \quad \text{(area of the parallelogram with sides } \vec{\mathbf{r}}_u \text{ and } \vec{\mathbf{r}}_v\text{)}$$

$$\implies \boxed{\vec{\mathbf{n}}\,dS = \hat{\mathbf{n}}\,\|N\|\vec{\mathbf{du}}\cdot\vec{\mathbf{dv}} = \vec{\mathbf{N}}\,\vec{\mathbf{du}}\cdot\vec{\mathbf{dv}}}$$

$$\implies \oiint_S (\nabla\times\vec{\mathbf{F}})\cdot\vec{\mathbf{n}}\,dS = \oiint_S \left[\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\hat{\mathbf{k}}\right]\cdot\vec{\mathbf{N}}\,du\,dv$$

Setting, $u=x,\ v=y,\ r(u,v)=r(x,y)=x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+f\hat{\mathbf{k}}$

$$N = |r_u\times r_v| = |r_x\times r_y| = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 0 & f_x\\ 0 & 1 & f_y\end{vmatrix} = -f_x\hat{\mathbf{i}} - f_y\hat{\mathbf{j}} + \hat{\mathbf{k}} \text{ where } f_x = \frac{\partial r(x,y))}{\partial x} \text{ and } f_y = \frac{\partial r(x,y))}{\partial y}$$

$$\oiint_S \left[\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\hat{\mathbf{k}}\right]\cdot[-f_x\hat{\mathbf{i}} - f_y\hat{\mathbf{j}} + \hat{\mathbf{k}}]\,dx\,dy$$

$$= \oiint_S \left[\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right)(-f_x) + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right)(-f_y) + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \right] dx\, dy$$

From Chain Rule, $-\frac{\partial}{\partial y} F_x(x,y,f(x,y)) = -\frac{\partial}{\partial y} F_x(x,y,z) - \frac{\partial}{\partial z} F_x(x,y,z) \frac{\partial}{\partial y} f(x,y)$

$$\Longrightarrow \oiint_S \left[\left(-\frac{\partial F_x}{\partial y} - \frac{\partial F_x}{\partial z} f_y \right) \right] dx\, dy = \oiint_S -\frac{\partial}{\partial y} F_x(x,y,f(x,y)) dx\, dy = \oint_C F_x dx$$

Similarly use, $y = g(x,z)$, $\quad z = h(x,y)$ to arrive at $\oint_C F_y dy$, $\oint_C F_z dz$

$$\Longrightarrow \boxed{\oiint_S (\nabla \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\, dS = \oint_C \vec{\mathbf{F}} \cdot dl = \oint_C F_x\, dx + \oint_C F_y\, dy + \oint_C F_z\, dz}$$

13.6.6 Green's Theorem

Suppose R is a closed region in the xy plane bounded by a simple closed curve C, and suppose M and N are continuous functions of x and y having continuous derivatives in R. Then,

$$\boxed{\oiint_R \left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} \right) dx\, dy = \oint_C (M\, dx + N\, dy)}$$

This can be proven from the following:

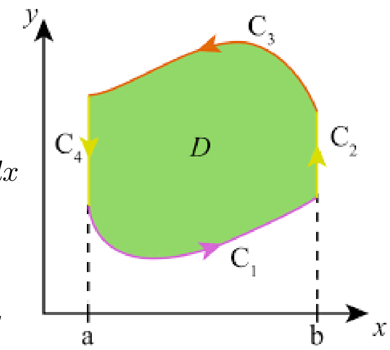
$$\oiint_R \frac{\partial N}{\partial y} dx dy = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial N}{\partial y} dy dx$$

$$= \int_a^b N(x, g_2(x)) dx - \int_b^a N(x, g_1(x)) dx$$

$$= -\int_{C_3} N(x, g_2(x)) dx - \int_{C_1} N(x, g_1(x)) dx$$

$$\int_{C_2} N(x,y) dx = \int_{C_4} N(x,y) dx = 0$$

Similarly, $\int_C M(x,y) dy = \int_{C_2} M(y, h_1(y)) dy + \int_{C_4} M(x, h_2(y)) dy$



where C is traversed in the positive (counter clockwise) direction. Green's theorem is a planar case of Stoke's theorem.