

CHAPTER 8

HIGHER ORDER ORDINARY DIFFERENTIAL EQUATION

*A higher order whispers through each change,
Derivatives weaving patterns wide and strange.
Roots shape motions—steady, wild, or deep—
In layered laws, the hidden forces sleep.*

8.1 INTRODUCTION

In earlier chapters, we studied first- and second-order ordinary differential equations and the methods used to analyze and solve them. Many physical and engineering systems, however, cannot be adequately described by equations of such low order. More complex models often involve higher derivatives of the unknown function, leading to what are known as *higher-order ordinary differential equations*.

A higher-order ordinary differential equation is one in which the highest derivative of the unknown function is of order greater than two. Such equations arise naturally in systems where multiple interacting effects influence the evolution of a quantity. Examples include the motion of elastic beams and plates, advanced vibration models, control systems, electrical networks with multiple energy-storage elements, and higher-order approximations in mechanics and field theories.

Although higher-order differential equations may appear more complicated, many of them can be analyzed systematically by extending ideas developed for lower-order equations. In particular, linear higher-order equations with constant coefficients admit general solution techniques based on characteristic equations, superposition, and the structure of homogeneous and non-homogeneous solutions.

This chapter focuses on the formulation and solution of higher-order ordinary differential equations, with emphasis on linear equations that commonly arise in applications. Methods for reducing the order, constructing general solutions, and applying initial or boundary conditions are presented. The goal is to develop both technical proficiency and physical insight, enabling the reader to recognize and solve higher-order models encountered in physics, engineering, and applied mathematics.

8.2 HIGHER ORDER HOMOGENEOUS ODE

A *homogeneous ordinary differential equation* is said to be homogeneous if it can be written in the form:

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0$$

where the right-hand side is zero.

The concepts of second-order linear ODEs extend naturally to n th-order linear differential equations. A general n th-order linear ODE has the form

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x)$$

When the coefficients are constant, a trial solution of the form $y = e^{\lambda x}$ leads to the **characteristic equation**:

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

If the characteristic equation has n distinct roots $\lambda_1, \dots, \lambda_n$, then the general solution is

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_n e^{\lambda_n x}$$

THE WRONSKIAN

The Wronskian of the set $\{y_1, \dots, y_n\}$ is

$$W(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

If $y_k = e^{\lambda_k x}$, then

$$W(y_1, \dots, y_n) = E \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{vmatrix}$$

where

$$E = e^{(\lambda_1 + \cdots + \lambda_n)x}$$

The remaining determinant is the **Vandermonde determinant**, which is nonzero exactly when all roots λ_k are distinct.

Thus, $W \neq 0$ if and only if the exponential solutions $e^{\lambda_1 x}, \dots, e^{\lambda_n x}$ are linearly independent.

REPEATED REAL ROOTS

If λ is a real root of multiplicity m , the linearly independent solutions are

$$e^{\lambda x}, xe^{\lambda x}, x^2 e^{\lambda x}, \dots, x^{m-1} e^{\lambda x}$$

COMPLEX ROOTS

If $\lambda = \gamma \pm i\omega$ is a complex conjugate pair, then the real solutions are

$$y_1 = e^{\gamma x} \cos(\omega x), \quad y_2 = e^{\gamma x} \sin(\omega x)$$

If the root is repeated (multiplicity m), the corresponding solutions are

$$e^{\gamma x} (\cos \omega x, \sin \omega x, x \cos \omega x, x \sin \omega x, \dots)$$

For instance, a complex double root gives

$$y(x) = e^{\gamma x} [(A_1 + A_2 x) \cos \omega x + (B_1 + B_2 x) \sin \omega x]$$

8.3 HIGHER ORDER Non-HOMOGENEOUS ODE

8.3.1 METHOD OF UNDETERMINED COEFFICIENTS

For constant-coefficient ODEs of the form

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = r(x),$$

we choose a trial $y_p(x)$ that resembles $r(x)$.

If the trial function duplicates a homogeneous solution, multiply by x^k where k is the smallest integer making the result independent.

Examples of modified trial functions:

$$cx e^{\lambda x}, \quad cx^2 e^{\lambda x}, \quad \dots$$

8.3.2 METHOD OF VARIATION OF PARAMETERS

Consider

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = r(x)$$

Let y_1, \dots, y_n be a fundamental set of solutions of the associated homogeneous equation. A particular solution is given by the general variation-of-parameters formula:

$$y_p(x) = \sum_{k=1}^n (-1)^{k+1} y_k(x) \int \frac{W_k(x)}{W(x)} r(x) dx$$

where $W(x)$ is the Wronskian and $W_k(x)$ is obtained from W by replacing the k th column with $(0, \dots, 0, 1)^T$.

This is a direct generalization of the second-order formula.

8.4 SERIES SOLUTIONS OF HOMOGENEOUS ODEs

ODEs with variable coefficients generally have nonelementary solutions such as Bessel and Legendre functions. These are obtained through power series.

A general power series is written as:

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Differentiating, substituting into the ODE, and equating coefficients yields recurrence relations for a_n .

8.5 EXISTENCE OF POWER SERIES SOLUTIONS

Consider:

$$y'' + p(x)y' + q(x)y = r(x)$$

If p, q, r are analytic at x_0 , then the ODE admits a convergent power series solution around x_0 .

8.6 CLASSICAL DIFFERENTIAL EQUATIONS

Legendre: $(1 - x^2)y'' - 2xy' + k(k + 1)y = 0$

Chebyshev: $(1 - x^2)y'' - xy' + k^2y = 0$

Hermite: $y'' - 2xy' + 2ky = 0$

Laguerre: $xy'' + (1 - x)y' + ky = 0$

where k is a constant

8.7 LEGENDRE'S EQUATION

$$(1 - x^2)y'' - 2xy' + k(k + 1)y = 0$$

Let

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting into the ODE and simplifying yields the recurrence:

$$a_{m+2} = -\frac{(k-m)(k+m+1)}{(m+1)(m+2)} a_m$$

This relation splits even and odd coefficients, producing two independent solutions y_1 and y_2 , one even and one odd.

The general solution is

$$y(x) = a_0 y_1(x) + a_1 y_2(x)$$

8.7.1 LEGENDRE POLYNOMIALS

When k is a nonnegative integer, the series terminates because the factor $(k - m)$ eventually becomes zero. Thus, for integer k , the Legendre solution becomes a polynomial $P_k(x)$.

The coefficients satisfy

$$a_{k-2m} = (-1)^m \frac{(2k-2m)!}{2^k m! (k-m)! (k-2m)!}$$

Thus

$$P_k(x) = \sum_{m=0}^{\lfloor k/2 \rfloor} (-1)^m \frac{(2k-2m)!}{2^k m! (k-m)! (k-2m)!} x^{k-2m}$$

8.8 FROBENIUS METHOD

The Frobenius method applies to ODEs with a regular singular point, such as

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

Seek a solution of the form

$$y = x^r \sum_{m=0}^{\infty} a_m x^m$$

The indicial equation becomes

$$r^2 + (b_0 - 1)r + c_0 = 0$$

Let the roots be r_1 and r_2 .

Depending on the relationship between r_1 and r_2 , the Frobenius method gives solutions involving power series or logarithmic terms.

8.9 BESSEL'S EQUATION

$$x^2 y'' + xy' + (x^2 - v^2)y = 0$$

Seeking a Frobenius solution

$$y = \sum_{m=0}^{\infty} a_m x^{m+r}$$

yields an indicial equation

$$r = \pm v.$$

8.9.1 SOLUTION FOR $r = \nu$

Seek a Frobenius solution of the form

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}, \quad a_0 \neq 0,$$

and substitute into Bessel's equation

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0.$$

Differentiating termwise gives

$$y' = \sum_{m=0}^{\infty} (m+r)a_m x^{m+r-1}, \quad y'' = \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r-2}.$$

Substituting and collecting like powers of x yields

$$\sum_{m=0}^{\infty} \left([(m+r)^2 - \nu^2]a_m + a_{m-2} \right) x^{m+r} = 0,$$

with the convention $a_{-1} = a_{-2} = \dots = 0$. The lowest power ($m = 0$) gives the indicial equation

$$[r^2 - \nu^2]a_0 = 0,$$

so

$$r = \pm\nu.$$

Now take $r = \nu$. The general recurrence (valid for $m \geq 2$) is

$$a_m = -\frac{a_{m-2}}{(m+\nu)^2 - \nu^2}$$

which simplifies to

$$a_m = -\frac{a_{m-2}}{m(m+2\nu)}, \quad m \geq 2.$$

Examine the $m = 1$ equation (using $a_{-1} = 0$):

$$((1+\nu)^2 - \nu^2)a_1 = (2\nu+1)a_1 = 0,$$

hence $a_1 = 0$ (since $2\nu+1 \neq 0$ for all ν). By the recurrence, all odd coefficients vanish:

$$a_1 = a_3 = a_5 = \dots = 0.$$

Set $m = 2k$. Then for $k \geq 1$,

$$a_{2k} = -\frac{a_{2k-2}}{2k(2k+2\nu)}.$$

Iterating and simplifying yields the closed form (for $k \geq 0$)

$$a_{2k} = \frac{(-1)^k a_0 \Gamma(v+1)}{2^{2k} k! \Gamma(v+k+1)}$$

Choosing the normalization

$$a_0 = \frac{1}{2^v \Gamma(v+1)}$$

gives the standard series for the Bessel function of the first kind:

$$J_v(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(v+k+1)} \left(\frac{x}{2}\right)^{2k+v}$$

This derivation explains why, for $r = v$, the Frobenius solution produces the even-power series (in x^v) that defines $J_v(x)$.

8.9.2 SOLUTION FOR $r = -v$

Similarly one obtains the second Frobenius solution $J_{-v}(x)$.

Together they yield the general solution for noninteger v :

$$y(x) = c_1 J_v(x) + c_2 J_{-v}(x)$$

For integer order, the second solution requires a logarithmic term and produces the Neumann function $Y_v(x)$.

8.10 BESSEL FUNCTIONS OF THE SECOND KIND

For $v = 0$,

$$Y_0(x) = \frac{2}{\pi} \left[J_0(x) \left(\ln \frac{x}{2} + \gamma \right) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} h_m}{2^{2m} (m!)^2} x^{2m} \right]$$

Here γ is the Euler–Mascheroni constant.

8.11 GENERAL SOLUTION

For noninteger v , the general solution of Bessel's equation is

$$y(x) = C_1 J_v(x) + C_2 Y_v(x)$$

For integer $v = n$,

$$y(x) = C_1 J_n(x) + C_2 Y_n(x)$$

8.12 SYMPY

```
1 import sympy as sp
2
3 x, nu = sp.symbols('x nu')
4 y = sp.Function('y')
5
6 ode_general = sp.Eq(x**2*sp.diff(y(x), x, 2)
7                      + x*sp.diff(y(x), x)
8                      + (x**2 - nu**2)*y(x), 0)
9
10 sol_general = sp.dsolve(ode_general)
11 sol_general
```

$$x^2 y'' + x y' + (x^2 - v^2)y = 0$$

$$\frac{(Y_2(1) - Y_0(1)) J_1(x)}{J_1(1) Y_2(1) + J_0(1) Y_1(1) - J_1(1) Y_0(1) - J_2(1) Y_1(1)} + \frac{(-J_2(1) + J_0(1)) Y_1(x)}{J_1(1) Y_2(1) + J_0(1) Y_1(1) - J_1(1) Y_0(1) - J_2(1) Y_1(1)}$$