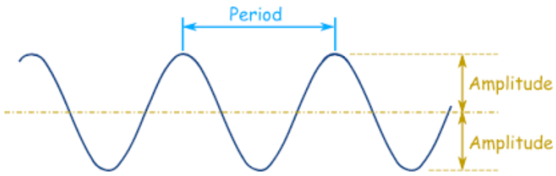
Chapter 13

Fourier Series

The Fourier transform is a mathematical tool that converts a signal from the time domain into the frequency domain, while its inverse transforms a signal from the frequency domain back into the time domain. It is named after **Joseph Fourier**. The Fourier transform has wide applications in engineering and physics, such as signal processing, vibration analysis, image processing, and communication systems.

13.1 Periodic Functions, Fourier Series

A function $f(x)$ is said to be **periodic** with period $T > 0$ if $f(x+T) = f(x), \quad \text{for all } x$



Let $f(x)$ be defined on the interval $[-L, L]$ and extended periodically outside this interval by

$$f(x + 2L) = f(x)$$

so that f has period $2L$. The **Fourier series** of $f(x)$ is

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)}$$

where the **Fourier coefficients** are:

$$\boxed{a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx}$$

$$\boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx}$$

$$\boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx}$$

13.2 Orthogonality of Trigonometric Systems

When $m \neq n$,

$$\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = 0,$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = 0,$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = 0.$$

Using the product-to-sum identities:

$$\cos nx\cos mx = \tfrac{1}{2}\big[\cos((n-m)x) + \cos((n+m)x)\big],$$

$$\sin nx\sin mx = \tfrac{1}{2}\big[\cos((n-m)x) - \cos((n+m)x)\big],$$

$$\sin nx\cos mx = \tfrac{1}{2}\big[\sin((n+m)x) + \sin((n-m)x)\big],$$

we have, for $m \neq n$,

$$\int_{-L}^{L} \cos nx\cos mx\,dx = \frac{1}{2}\int_{-L}^{L} \cos((n-m)x)\,dx + \frac{1}{2}\int_{-L}^{L} \cos((n+m)x)\,dx,$$

$$\int_{-L}^{L} \sin nx\sin mx\,dx = \frac{1}{2}\int_{-L}^{L} \cos((n-m)x)\,dx - \frac{1}{2}\int_{-L}^{L} \cos((n+m)x)\,dx,$$

$$\int_{-L}^{L} \sin nx\cos mx\,dx = \frac{1}{2}\int_{-L}^{L} \sin((n+m)x)\,dx + \frac{1}{2}\int_{-L}^{L} \sin((n-m)x)\,dx.$$

Each of these integrals evaluates to zero (for appropriate L), so the three orthogonality relations follow.

13.3 Fourier Coefficients

Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$$

Integrating both sides over $[-L, L]$,

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{L}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)\right]dx$$

Using orthogonality of sine and cosine,

$$\int_{-L}^{L} f(x)\,dx = a_0 L$$

hence

$$\boxed{a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx}$$

Now multiply the Fourier series by $\cos\frac{m\pi x}{L}$ and integrate:

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx = \int_{-L}^{L}\left[\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)\right]\cos\frac{m\pi x}{L}\,dx$$

By orthogonality,

$$\boxed{a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx.}$$

Similarly, multiplying by $\sin\frac{m\pi x}{L}$ and integrating,

$$\boxed{b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx.}$$

The first term on the right involving a_0 vanishes since

$$\int_{-L}^{L}\cos\frac{m\pi x}{L}\,dx = 0.$$

The integral of $a_n\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}$ is equal to $a_m L$ for $n = m$ and 0 for $n \neq m$. The integral of $b_n\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}$ is 0 for all m, n.

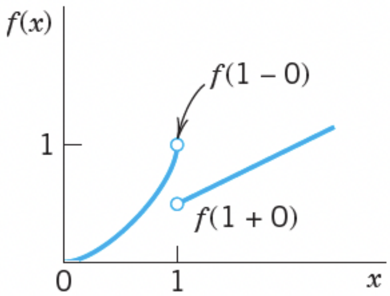
Hence,

$$\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx = a_m L \;\Rightarrow\; \boxed{a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx}$$

$$\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx = b_m L \;\Rightarrow\; \boxed{b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx}$$

13.4 Convergence

If $f(x)$ is **periodic** and piecewise continuous, and if the left-hand and right-hand derivatives exist at each point of the interval (except possibly at a finite number of points), then the Fourier series of $f(x)$ **converges**. At every point of continuity, the series converges to $f(x)$. At each point of discontinuity, the sum of the series converges to the **average of the left-hand and right-hand limits** of $f(x)$.



13.5 Fourier Series for an Arbitrary Period

Let $f(x)$ be a periodic function with arbitrary period p. Define

$$L = \frac{p}{2}$$

Then $f(x)$ has period $2L$ and its Fourier series representation is

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)}$$

where the Fourier coefficients are

$$\boxed{a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx}$$

$$\boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots}$$

$$\boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots}$$

13.6 Even & Odd Functions

Even Function:

$f(x) = f(-x)$

For an even function, all sine terms vanish, and the Fourier series reduces to a cosine series:

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}}$$

with

$$\boxed{a_0 = \frac{2}{L}\int_{0}^{L} f(x)\,dx}$$

$$\boxed{a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots}$$

Odd Function:

$f(x) = -f(-x)$

For an odd function, all cosine terms vanish (including a_0), and the Fourier series reduces to a sine series:

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}}$$

with

$$\boxed{b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots}$$

13.7 Approximation & Error Minimization

Let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

be the Fourier series of $f(x)$. The N-term approximation is:

$$F(x) = \frac{A_0}{2} + \sum_{n=1}^{N}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right)$$

which is chosen to best approximate $f(x)$ in the least–squares sense. Define the mean–square error

$$E = \int_{-L}^{L} [\, f(x) - F(x)\,]^2 \, dx$$

Using orthogonality of sine and cosine functions, we obtain

$$\int_{-L}^{L} F^2(x)\,dx = L\left(\frac{A_0^2}{2} + \sum_{n=1}^{N}(A_n^2 + B_n^2)\right),$$

$$\int_{-L}^{L} f(x)F(x)\,dx = L\left(\frac{A_0 a_0}{2} + \sum_{n=1}^{N}(A_n a_n + B_n b_n)\right).$$

Hence,

$$E = \int_{-L}^{L} f^2(x)\,dx - 2L\left(\frac{A_0 a_0}{2} + \sum_{n=1}^{N}(A_n a_n + B_n b_n)\right) + L\left(\frac{A_0^2}{2} + \sum_{n=1}^{N}(A_n^2 + B_n^2)\right).$$

Minimizing E with respect to A_n and B_n gives

$$\boxed{A_n = a_n}$$

$$\boxed{B_n = b_n}$$

so the best approximation is obtained by taking the Fourier coefficients themselves. Therefore, the minimum error is:

$$\boxed{E_{\min} = \int_{-L}^{L} f^2(x)\,dx - L\left(\frac{a_0^2}{2} + \sum_{n=1}^{N}(a_n^2 + b_n^2)\right).}$$

$$E - E_{\min} = L\left[\frac{(A_0 - a_0)^2}{2} + \sum_{n=1}^{N}\left\{(A_n - a_n)^2 + (B_n - b_n)^2\right\}\right] \geq 0$$

$$E = E_{\min} \quad \text{if and only if} \quad A_0 = a_0,\ A_n = a_n,\ B_n = b_n$$

The square error of F (with fixed N) relative to f on the interval $-L \leq x \leq L$ is minimum if and only if the coefficients of F are the Fourier coefficients of f. With increasing N, the partial sums of the Fourier series of f yield better and better approximations to f in the mean-square sense.

$$\boxed{\frac{a_0^2}{2} + \sum_{n=1}^{N}(a_n^2 + b_n^2) \leq \frac{1}{L}\int_{-L}^{L} f(x)^2\,dx \quad \text{(Bessel's Inequality)}}$$

13.8 Sturm–Liouville Problems & Orthogonality

The idea of the Fourier series is to represent general periodic functions in terms of cosines and sines. These functions form a trigonometric system which possesses the important property of **orthogonality**. This orthogonality allows us to compute the coefficients of the Fourier series using Euler's formulas. A natural question then arises: can we replace the trigonometric system by other orthogonal systems (sets of orthogonal functions)? The answer is *yes*, and this leads to **generalized Fourier series**, including the Fourier–Legendre series and the Fourier–Bessel series.

A **Sturm–Liouville problem** consists of the differential equation

$$\boxed{\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + \left[q(x) + \lambda r(x)\right]y = 0, \quad a \le x \le b,}$$

together with the boundary conditions

$$k_1 y(a) + k_2 y'(a) = 0,$$

$$l_1 y(b) + l_2 y'(b) = 0.$$

If $y_m(x)$ and $y_n(x)$ are eigenfunctions corresponding to distinct eigenvalues $\lambda_m \neq \lambda_n$, then they satisfy the **orthogonality condition**

$$\boxed{(y_m, y_n) = \int_a^b r(x)\, y_m(x)\, y_n(x)\, dx = 0, \quad m \neq n.}$$

The **norm** of an eigenfunction is defined by

$$\boxed{\|y_n\| = \sqrt{(y_n, y_n)} = \sqrt{\int_a^b r(x)\, y_n^2(x)\, dx}.}$$

13.9 Eigenvalues & Eigenfunctions

The eigenfunctions $y_n(x)$ of a Sturm–Liouville problem satisfy the weighted inner product

$$(y_m, y_n) = \int_a^b r(x)\, y_m(x)\, y_n(x)\, dx$$

If the eigenfunctions are normalized, then

$$(y_m, y_n) = \delta_{mn} = \begin{cases} 1, & m = n, \\ 0, & m \neq n \end{cases}$$

If $r(x) = 1$, then

$$(y_m, y_n) = \int_a^b y_m(x)\, y_n(x)\, dx = 0, \quad m \neq n.$$

The norm of an eigenfunction is defined as

$$\boxed{\|y_m\| = \sqrt{(y_m, y_m)} = \sqrt{\int_a^b r(x)\, y_m^2(x)\, dx}}$$

The eigenfunctions satisfy

$$(py_m')' + (q + \lambda_m r)y_m = 0$$

$$(py_n')' + (q + \lambda_n r)y_n = 0$$

Multiplying the first equation by y_n, the second by y_m, and subtracting, we obtain

$$(\lambda_m - \lambda_n)r(x)y_m y_n = y_n(py_m')' - y_m(py_n')'$$

Integrating from a to b gives

$$(\lambda_m - \lambda_n)\int_a^b r(x)\, y_m y_n\, dx = \left[p(x)\big(y_n y_m' - y_m y_n'\big)\right]_a^b$$

The boundary term on the right-hand side vanishes due to the boundary conditions (separated or periodic). Hence,

$$(\lambda_m - \lambda_n)\int_a^b r(x)\, y_m y_n\, dx = 0$$

For $\lambda_m \neq \lambda_n$, this implies the orthogonality relation

$$\boxed{\int_a^b r(x)\, y_m(x) y_n(x)\, dx = 0, \qquad m \neq n}$$

13.10 Generalized Fourier Series

Let $\{y_m(x)\}_{m=0}^{\infty}$ be an orthogonal set of functions on $[a, b]$ with respect to the weight function $r(x)$. Then a function $f(x)$ may be expanded as

$$\boxed{f(x) = \sum_{m=0}^{\infty} a_m y_m(x)}$$

This is called an **orthogonal expansion** or a **generalized Fourier series**. If the functions $y_m(x)$ are eigenfunctions of a Sturm–Liouville problem, the expansion is called an **eigenfunction expansion**. Taking the inner product of both sides with $y_n(x)$, we obtain

$$(f, y_n) = \int_a^b r(x) f(x) y_n(x)\, dx = \int_a^b r(x)\left(\sum_{m=0}^{\infty} a_m y_m(x)\right) y_n(x)\, dx = \sum_{m=0}^{\infty} a_m (y_m, y_n)$$

By orthogonality, all terms vanish except when $m = n$, hence

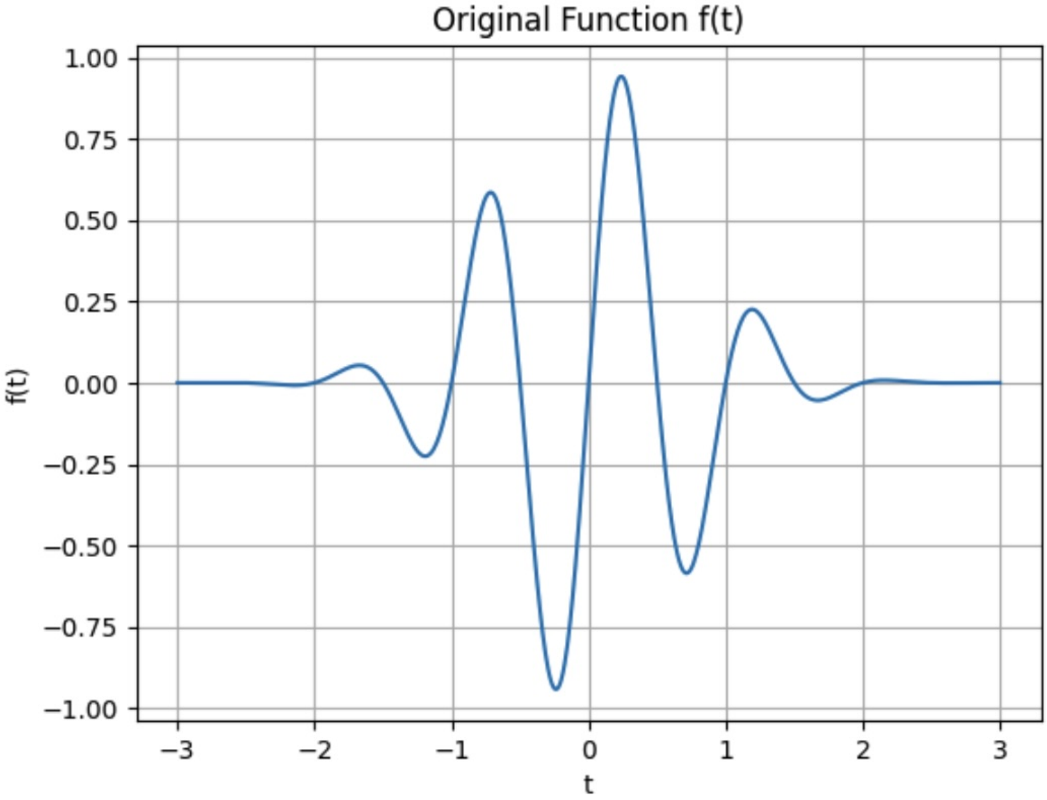
$$(f, y_n) = a_n (y_n, y_n) = a_n \|y_n\|^2$$

Therefore, the coefficients are given by

$$\boxed{a_n = \frac{(f, y_n)}{\|y_n\|^2} = \frac{1}{\|y_n\|^2}\int_a^b r(x)\, f(x)\, y_n(x)\, dx}$$

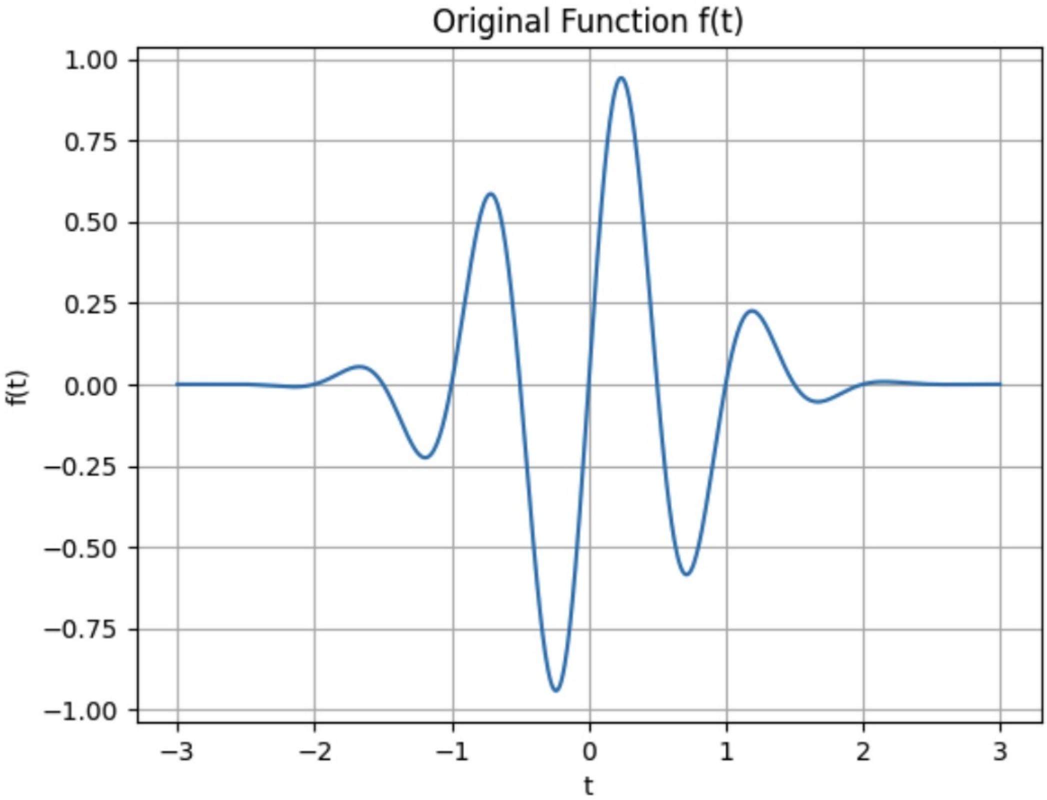
13.11 SymPy

```python
import numpy as np
import matplotlib.pyplot as plt
from sympy import symbols, exp, sin, pi, fourier_transform, lambdify

# Define symbols
t, w = symbols('t w', real=True)

# Define the function
f = exp(-t**2) * sin(2*pi*t)

# Compute Fourier Transform
F = fourier_transform(f, t, w)

# Convert to numerical functions for plotting
f_num = lambdify(t, f, "numpy")
F_num = lambdify(w, F, "numpy")

# Generate numerical data
t_vals = np.linspace(-3, 3, 600)
w_vals = np.linspace(-10, 10, 600)

f_vals = f_num(t_vals)
F_vals = F_num(w_vals)

# Plot f(t)
plt.figure()
plt.plot(t_vals, f_vals)
plt.xlabel("t")
plt.ylabel("f(t)")
plt.title("Original Function f(t)")
plt.grid(True)
plt.show()
display("Original function f(t):")
display(f)

# Plot Fourier Transform magnitude |F(w)|
plt.figure()
plt.plot(w_vals, np.abs(F_vals))
plt.xlabel("ω")
plt.ylabel("|Fω()|")
plt.title("Magnitude of Fourier Transform |Fω()|")
plt.grid(True)
plt.show()
display("Fourier Transform Fω():")
display(F)
```



'Original function f(t):'

$e^{-t^2} \sin(2\pi t)$



'Original function f(t):'

$e^{-t^2} \sin(2\pi t)$