

INTEGRATION

*Integration seems fancy and grand,
But it's just adding, hand over hand.
Dress it in symbols, make it look prime,
Yet it's summing, just one at a time!*

Integration, in mathematics, is the technique of finding a function $g(x)$ the derivative of which is equal to a given function $f(x)$. This is indicated by the integral sign \int as in $\int f(x)dx$ and is called the indefinite integral of the function. The symbol dx represents an infinitesimal displacement along x . Hence, $\int f(x)dx$ is the summation of the product of $f(x)$ and dx . The definite integral, written as $\int_a^b f(x)dx$ where a and b are called the limits of integration, is equal to $g(b) - g(a)$, where $\frac{d}{dx}g(x) = f(x)$.

5.1 INTEGRAL

Given a function $f(x)$, an *anti-derivative* of $f(x)$ is any function $g(x)$ such that $g'(x) = f(x)$. The most general anti-derivative is called the *indefinite integral*.

$$\int f(x)dx = g(x) + c \text{ where } c \text{ is a constant of integration}$$

Note the following inequalities.

$$\int f(x)g(x)dx \neq \int f(x)dx \int g(x)dx$$

$$\int \frac{f(x)}{g(x)}dx \neq \frac{\int f(x)dx}{\int g(x)dx}$$

5.2 COMMON INTEGRALS

$$\int x^n dx = \frac{x^{(n+1)}}{(n+1)} + c \quad (5.2.1)$$

$$\int e^x dx = e^x + c \quad (5.2.2)$$

$$\int a^x dx = \frac{a^x}{\ln a} + c \quad (5.2.3)$$

$$\int \frac{1}{x} dx = \ln|x| + c \quad (5.2.4)$$

$$\int \cos(x) dx = \sin(x) + c \quad (5.2.5)$$

5.3 SUBSTITUTION TECHNIQUE

$$\int 18x^2 \sqrt[4]{(6x^3 + 5)} dx$$

Let $u = 6x^3 + 5$

$$\rightarrow du = 18x^2 dx$$

$$\rightarrow \int \sqrt[4]{u} du = \frac{u^{(\frac{1}{4}+1)}}{\frac{1}{4}+1} = \frac{4}{5} u^{\frac{5}{4}} = \frac{4}{5} (6x^3 + 5)^{\frac{5}{4}}$$

5.4 INTEGRATION BY PARTS

$$[f(x)g(x)]' = f(x)g'(x) + f'(x)g(x)$$

$$f(x)g'(x) = [f(x)g(x)]' - f'(x)g(x)$$

$$\int f(x)g'(x) dx = \int [f(x)g(x)]' dx - \int f'(x)g(x) dx$$

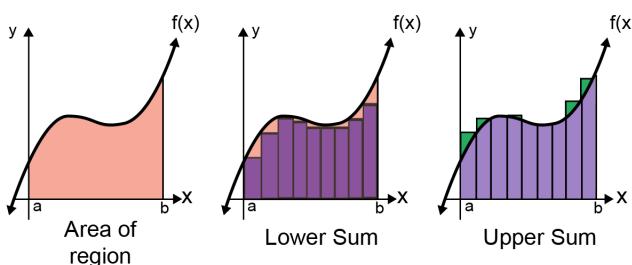
$$a \int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$$

$$\boxed{\int u(x)v(x) dx = u(x) \int v(x) dx - \int [u'(x) \int v(x) dx] dx} \quad (5.4.1)$$

Hence, integral of two functions = first function \times integral of second function – integral of (differentiation of the first function \times integral of the second function).

5.5 DEFINITE INTEGRAL

A definite integral is the area under its curve .



$$\boxed{\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x = g(x) \Big|_a^b = g(b) - g(a)}$$

where $f(x_i^*)$ is the value at the middle of the strip Δx .

$$\boxed{f_{avg} = \frac{1}{b-a} \int_a^b f(x) dx}$$

$$\boxed{\int_a^b f(x) dx = f(c)(b-a) \text{ where } c \text{ is in } [a,b]}$$

5.6 SOME INTEGRATION STRATEGIES

- Simplify the integrand. E.g., $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$
- Check if simple substitution will work
- If integrand is a rational expression, partial functions may work
- If integrand is polynomial x , trig, exp, ln function, integration by parts may work
- If integrand involves $\sqrt{b^2x^2 + a^2}$, trigonometric substitution may work
- If integrand has a quadratic in it, completing the square may work.

5.7 SYMPY

Integration the following functions:

$$\int (x^2 + x + 1) dx$$

Import Python packages

```

1 import sympy as sp
2 import numpy as np
3 import matplotlib.pyplot as plt
4 from IPython.display import display, Math
5 from sympy import sqrt, diff, integrate, oo
6 from sympy import sin, cos, tan, ln, exp, erf, trigsimp, expand_trig, simplify
7 from sympy import sinh, cosh

```



```

1 y = x**2 + x + 1
2 int = integrate(y,x) # integrate y wrt x
3 display(int)

```

$$\frac{x^3}{3} + \frac{x^2}{2} + x + C$$

The **error function** is defined as:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$\int e^{-x^2} \operatorname{erf}(x) dx$$

```

1 y = exp(-x**2)*erf(x)
2 int = integrate(y,x)           # integrate y wrt x
3 display(int)

```

$$\frac{\sqrt{\pi} \operatorname{erf}^2(x)}{4}$$

$$\int_0^\infty e^{-x} dx$$

```

1 y = exp(-x)
2 int = integrate(y, (x, 0, oo))    # definite integral, limits 0 & infinity

```

```
3 display(int)
```

```
1
```