CHAPTER 7

Second Order Ordinary Differential Equation

A second-order equation, quite a mess,
With derivatives causing plenty of distress.
But Euler, Cauchy, and Lagrange helped evolve,
Elegant methods to solve and resolve!

As noted in the previous chapter, the order of a differential equation is defined as the order of the highest derivative of the unknown function that appears in the equation. Therefore, a second-order differential equation is one in which the second derivative of the unknown function occurs, and no derivative of higher order is present.

Second order differential equations have a variety of applications in science and engineering such as vibrations and electric circuits. There are a host of multi dimensional engineering models that incorporate second order differential equations including wave motion, flow mechanics, Maxwell's electro-magnetic equations and Schroedinger equation in Nuclear Physics.

7.1 2nd Order Linear ODE

The standard form of a 2nd Order ODE is:

$$\boxed{y'' + p(x)y' + q(x)y = r(x)} \quad \text{It is linear in } y,\ y' \text{ and } y'' \tag{7.1.1}$$

If $r(x) = 0$, the ODE is homogeneous, else it is non-homogeneous. When the coefficients a and b are constant:

$$\boxed{y'' + ay' + by = 0} \tag{7.1.2}$$

Choose $e^{\lambda x}$ as a solution and substitute in the homogeneous ODE.

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0 \implies \lambda^2 + a\lambda + b = 0 \implies \lambda = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right)$$

$$\boxed{y_h = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}} \quad \text{(general solution to the homogeneous ODE)}$$

y_1, corresponding to λ_1, and y_2, corresponding to λ_2, are linearly independent and are called basis of solutions. The superposition principle also called the linearity principle, i.e., the homogeneous solution is a combination of y_1 and y_2 is true only for linear homogeneous ODE. The arbitrary constants c_1 and c_2 are determined from the initial conditions:

$$y(x_0) = k_0 \quad y'(x_0) = k_1$$

A particular solution is obtained if we assign specific values to c_1 and c_2.

7.2 Lagrange's Method of Reduction of Order

Consider a linear homogeneous 2nd Order ODE in its standard form:

$$\boxed{y'' + p(x)y' + q(x)y = 0} \tag{7.2.1}$$

If y_1 is a basis solution, we can find y_2 as follows:

$$y_1 = e^{\lambda x}$$

Let $y_2 = uy_1$

$$y_2' = u'y_1 + uy_1'$$

$$y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

Substituting,

$$(u''y_1 + 2u'y_1' + uy_1'') + p(u'y_1 + uy_1') + q(uy_1) = 0$$

$$y_1u'' + (2y_1' + py_1)u' + (y_1'' + py_1' + qy_1)u = 0$$

$$u'' + u'\frac{2y_1' + py_1}{y_1} = 0$$

Let $U = u'$

$$U' + U\left(\frac{2y_1'}{y_1} + p\right) = 0$$

$$\frac{U'}{U} = -\left(\frac{2y_1'}{y_1} + p\right)$$

$$\int \frac{U'}{U}dx + \int \left(\frac{2y_1'}{y_1}\right)dx = -pdx$$

$$\ln|U| + 2\ln|y_1| = -\int pdx$$

$$\ln|Uy_1^2| = -\int pdx$$

$$Uy_1^2 = e^{\int -pdx}$$

$$\boxed{U = \frac{1}{y_1^2}e^{\int -pdx}}$$

$$\boxed{u = \int Udx}$$

$$\boxed{y_2 = y_1\int Udx}$$

$$\boxed{y_2 = y_1\int \frac{1}{y_1^2}e^{\int -pdx}dx}$$

7.3 Homogeneous Linear ODE with Constant Coefficients

$$\begin{cases} \text{Case 1: 2 Real Roots when} & a^2-4b>0 \\ \text{Case 2: Double Root when} & a^2-4b=0 \\ \text{Case 3: Complex Conjugate Roots when} & a^2-4b<0 \end{cases}$$

Case 1: 2 Real Roots when $a^2-4b>0$. The general solution is given by:

$$y_1=e^{\lambda_1 x} \quad y_2=e^{\lambda_2 x}$$

$$\lambda_1=\frac{1}{2}\left(-a+\sqrt{a^2-4b}\right) \quad \lambda_2=\frac{1}{2}\left(-a-\sqrt{a^2-4b}\right)$$

$$\boxed{y_h=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}} \tag{7.3.1}$$

Case 2: $\boxed{\lambda_1=-\frac{a}{2}, y_1=e^{-\frac{ax}{2}}}$ Determine y_2 using the method of reduction of order.

$$y_1=e^{\frac{-ax}{2}}$$

$$y_2=y_1\int\frac{1}{y_1^2}e^{\int -pdx}dx=e^{\frac{-ax}{2}}\int\frac{1}{\left(e^{\frac{-ax}{2}}\right)^2}e^{\int -adx}dx=e^{-\frac{ax}{2}}\int e^{ax}e^{-ax}dx=xe^{-\frac{ax}{2}}$$

$$y_h=c_1e^{-ax/2}+c_2xe^{-\frac{a}{2}x}$$

$$\boxed{y_h=(c_1+c_2x)e^{-ax/2}}$$

Case 3: $\boxed{\lambda=-\frac{a}{2}\pm \mathbf{i}w, w=\sqrt{|a^2-4b|}}$

$$y_1=e^{\lambda_1 x}=e^{(-\frac{a}{2}+\mathbf{i}w)x}=e^{-\frac{ax}{2}}e^{\mathbf{i}wx}$$

$$y_2=e^{\lambda_2 x}=e^{(-\frac{a}{2}-\mathbf{i}w)x}=e^{-\frac{ax}{2}}e^{-\mathbf{i}wx}$$

$$e^{ix}=1+ix+\frac{(ix)^2}{2!}+\frac{(ix)^3}{3!}+\ldots=(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots)+i(x-\frac{x^3}{3!}-\frac{x^5}{5|}+\ldots)=cos\,x+\mathbf{i}sin\,x$$

$\boxed{e^{iwx}=cos\,wx+\mathbf{i}sin\,wx}$ (de Moivre's theorem) and $\boxed{e^{\mathbf{i}\pi}=-1}$ (Euler's Identity)

Any real solution is a linear combination of the real and imaginary parts:

$\boxed{y=e^{-\frac{ax}{2}}(c_1cos\,wx+c_2sin\,wx)}$ c_1, c_2 are constants

7.4 Euler-Cauchy Equations

The Euler-Cauchy equation is of the form:

$$\boxed{x^2y''+axy'+by=0} \text{ where } a,b \text{ are constants} \tag{7.4.1}$$

Let $y=x^m \implies y'=mx^{m-1} \implies y''=m(m-1)x^{m-2}$ and substituting

$x^2m(m-1)x^{m-2}+axmx^{m-1}+bx^m=0 \implies m^2+(a-1)m+b=0$

$$\boxed{m=\frac{1}{2}(1-a)\pm\sqrt{\frac{1}{4}(a-1)^2-b}}$$

Case 1: Roots are distinct. The basis solutions are :

$y_1(x) = x^{m_1} \quad y_2(x) = x^{m_2}$, the general solution is given by, $\boxed{y = c_1 x^{m_1} + c_2 x^{m_2}}$

Case 2: Double roots.

$$\boxed{y_1 = x^{\frac{1}{2}(1-a)}}$$

$$y'' + \frac{a}{x}y' + \frac{(1-a)^2}{4x^2}y = 0$$

Use method of reduction of order, $y_2 = uy_1$ and with $\boxed{p = \frac{a}{x}}$

$$\boxed{U = \frac{1}{y_1^2}e^{\int -pdx}} \quad \boxed{u = \int Udx} \quad \boxed{y_2 = y_1 \int Udx} \quad \boxed{y_2 = y_1 \int \frac{1}{y_1^2}e^{\int -pdx}dx}$$

$$\int pdx = \int \frac{a}{x}dx = alnx \implies e^{\int -pdx} = e^{-alnx} = e^{lnx^{-a}} = x^{-a} = \frac{1}{x^a}$$

$$U = \frac{1}{y_1^2}\frac{1}{x^a} = \frac{1}{x^{1-a}}\frac{1}{x^a} = \frac{1}{x} \implies u = \int Udx = \int \frac{1}{x}dx = lnx$$

$$y_2 = y_1 \int Udx = x^{\frac{1}{2}(1-a)}lnx$$

$\boxed{y_h = (c_1 + c_2 lnx)x^{\frac{1}{2}(1-a)}}$ c_1, c_2 are constants

7.5 The Wronskian

Two solutions y_1 and y_2 are linearly dependent if their Wronkskian W is 0.

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = 0$$

Because if the solutions are dependent, $y_1 = ky_2$, where k is a constant

$$\implies W(y_1, y_2) = y_1 y_2' - y_2 y_1' = ky_2 y_2' - y_2 k y_2' = 0$$

The Wronksian is expressed as a Wronkski Determinant:

$$\boxed{W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \tag{7.5.1}$$

7.6 Non-homogeneous ODE

Consider the following non-homogeneous ODE:

$$y'' + p(x)y' + q(x)y = r(x)$$

The complete solution is the sum of homogeneous (y_h) and particular (y_p)solutions.

$y(x) = y_h(x) + y_p(x)$ where $y_h = c_1 y_1 + c_2 y_2$ (general solution)

y_p is a solution of the non-homogeneous equation without any constants. A particular solution is obtained by assigning specific values to the constants. The Method of Undetermined Coeffcients is an approach to finding a particular solution to nonhomogeneous ODEs. If the term in $r(x)$ contains the following term, the choice for $y_p(x)$ is given by:

Term in r(x)	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$Kx^n (n = 0, 1, \ldots)$	$K_n x^n + K_{n-1}x^{n-1} + \ldots + K_1 x + K_0$
$kcos\, wx$ or $ksin\, wx$	$Kcos\, wx + Msin\, wx$
$ke^{\alpha x} cos\, wx$ or $ke^{\alpha x} sin\, wx$	$e^{\alpha x}(Kcos\, wx + Msin\, wx)$

7.7 Particular Solution by Variation of Parameters (Lagrange)

The particular solution for the standard form ODE is derived as follows:

$$y'' + p(x)y' + q(x)y = r(x)$$

Find a pair of functions $u_1(x)$ and $u_2(x)$ such that:

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \implies y_p^{'}(x) = u_1^{'}y_1 + u_1y_1^{'} + u_2^{'}y_2 + u_2y_2^{'}$$

Set constraint, $\boxed{u_1^{'}y_1 + u_2^{'}y_2 = 0}$

$$y_p^{'}(x) = u_1y_1^{'} + u_2y_2^{'}$$
$$y_p^{''}(x) = u_1^{'}y_1^{'} + u_1y_1^{''} + u_2^{'}y_2^{'} + u_2y_2^{''}$$

Substituting,

$$(u_1^{'}y_1^{'} + u_1y_1^{''} + u_2^{'}y_2^{'} + u_2y_2^{''}) + p(u_1y_1^{'} + u_2y_2^{'}) + q(u_1y_1 + u_2y_2) = r$$
$$(y_1'' + py_1' + qy_1)u_1 + (y_2'' + py_2' + qy_2)u_2 + (u_1'y_1' + u_2'y_2') = r$$

Since y_1 and y_2 are solutions to the homogeneous ODE,

$$u_1'y_1' + u_2'y_2' = r$$

We now have the following simultaneous equations:

$$u_1^{'}y_1 + u_2^{'}y_2 = 0$$
$$u_1'y_1' + u_2'y_2' = r$$

Solving,

$$u_1^{'} = -\frac{y_2 r}{y_1y_2^{'} - y_1^{'}y_2} = -\frac{y_2 r}{W}$$
$$u_2^{'} = -\frac{y_1 r}{y_1y_2^{'} - y_1^{'}y_2} = -\frac{y_1 r}{W}$$

$$\boxed{y_p(x) = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx}$$

7.8 SymPy

```python
import sympy as sp
from IPython.display import display, Math

# Variables and function
x = sp.symbols('x')
y = sp.Function('y')

# Nonhomogeneous ODE: y'' - 3 y' + 2 y = x^2
ode_nonhom = sp.Eq(sp.diff(y(x), x, 2) - 3*sp.diff(y(x), x) + 2*y(x), x**2)

# Solve full ODE
solution_full = sp.dsolve(ode_nonhom)

```

```
# Extract complementary (homogeneous) and particular parts
# dsolve returns: Eq(y(x), C1*exp()+C2*exp() + particular)
rhs = solution_full.rhs
C1, C2 = sp.symbols('C1 C2')

# The homogeneous part is the expression containing C1, C2
homogeneous_part = rhs.expand().coeff(C1)*C1 + rhs.expand().coeff(C2)*C2

# The particular solution is the rest (remove terms with C1, C2)
particular_part = sp.simplify(rhs - homogeneous_part)

# Display results
print(sp.latex(ode_nonhom))
display(Math(r"\textbf{Non-homogeneous ODE: }" + sp.latex(ode_nonhom)))
display(Math(r"\textbf{General solution: } y = " + sp.latex(rhs)))
display(Math(r"\textbf{Homogeneous solution: } y_h = " +
    ↪ sp.latex(homogeneous_part)))
display(Math(r"\textbf{Particular solution: } y_p = " + sp.latex(particular_part)))

print(sp.latex(solution_full))
solution_full
```

$$2y(x) - 3\frac{d}{dx}y(x) + \frac{d^2}{dx^2}y(x) = x^2$$

$$C_1e^x + C_2e^{2x} + \frac{x^2}{2} + \frac{3x}{2} + \frac{7}{4}$$

$$C_1e^x + C_2e^{2x}$$

$$\frac{x^2}{2} + \frac{3x}{2} + \frac{7}{4}$$

$$y(x) = C_1e^x + C_2e^{2x} + \frac{x^2}{2} + \frac{3x}{2} + \frac{7}{4}$$