

# CHAPTER 14

# COMPLEX ANALYSIS

*There is a number  $i$ , quite imaginary,  
Yet it gets things done, quite extraordinary.  
Who would imagine such a possibility?  
But that is indeed the reality!*

Complex analysis is the study of complex numbers together with their derivatives, manipulation, and other properties. Complex analysis is an extremely powerful tool with an unexpectedly large number of practical applications to the solution of physical problems. It is helpful in many areas such as hydrodynamics, thermodynamics, and particularly quantum mechanics. Complex analysis also has a wide range of applications in engineering fields such as nuclear, aerospace, mechanical and electrical engineering.

## 14.1 COMPLEX NUMBERS

Complex numbers are the numbers that are expressed in the form of  $x + iy$  where  $x, y$  are real numbers and  $i$  is the imaginary unit.

$$z = x + iy \quad \text{where } i = \sqrt{-1}$$

Just as with real numbers, we can perform arithmetic operations on complex numbers. To add or subtract complex numbers, we combine the real parts and combine the imaginary parts. Addition, multiplication, and division of complex numbers are given below:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{x_1 + iy_1}{x_2 + iy_2}, \quad z_2 \neq 0 = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} \\ &= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} \end{aligned}$$

### 14.1.1 COMPLEX CONJUGATE

The **complex conjugate** of  $z$  is defined as:

$$\bar{z} = x - iy$$

### 14.1.2 POLAR REPRESENTATION

$$z = r \cos \theta + i r \sin \theta = r e^{i\theta}, \quad r \geq 0 \quad (\text{Polar representation})$$

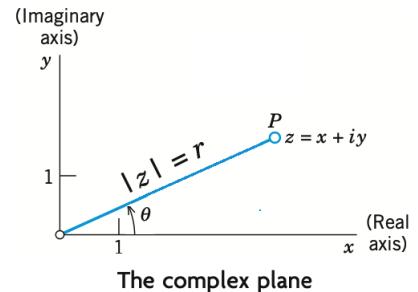
$$z^n = r^n (\cos n\theta + i \sin n\theta), \quad n \in \mathbb{Z} \quad (\text{De Moivre's theorem})$$

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

$x = r \cos \theta, \quad y = r \sin \theta$  (radians, measured counterclockwise).

$\theta$  is the **argument** of  $z$ , denoted by  $\arg z$ . Its **principal value** is:

$$-\pi < \arg z \leq \pi, \quad z \neq 0$$



The  $xy$ -plane is the complex plane, also known as the **Argand diagram**.

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}, \quad z_2 \neq 0 = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$$

$$\arg\left(\frac{z_1}{z_2}\right) \equiv \arg z_1 - \arg z_2 \pmod{2\pi}, \quad z_1 \neq 0, z_2 \neq 0$$

### 14.1.3 PROPERTIES

$$z_1 z_2 = z_2 z_1 \text{ (commutative)}$$

$$(z_1 z_2) z_3 = z_1 (z_2 z_3) \text{ (associative)}$$

$$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3 \text{ (distributive)}$$

$$|z_1 z_2| = |z_1| |z_2| \text{ and } \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, \quad z_2 \neq 0$$

### 14.1.4 ROOTS

$$z = r e^{i\theta}$$

$$z_k = r^{1/n} e^{i\left(\frac{\theta+2k\pi}{n}\right)}, \quad k = 0, 1, \dots, n-1$$

$$\text{Example: } 4i = 4e^{i\frac{\pi}{2}}$$

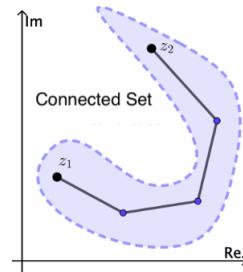
$$\sqrt{4i} = \sqrt{4e^{i\frac{\pi}{2}}} = 2e^{i\left(\frac{\pi}{4} + k\pi\right)}, \quad k = 0, 1$$

$$2e^{i\frac{\pi}{4}} = \sqrt{2}(1+i), \quad 2e^{i\frac{5\pi}{4}} = -\sqrt{2}(1+i)$$

$$\Rightarrow \sqrt{4i} = \sqrt{2}(1+i), -\sqrt{2}(1+i)$$

## 14.2 POINT SET & PATH

A **point set** is simply a collection of points in the complex plane. A set  $S$  is called **open** if it contains no boundary points. A set  $S$  is called **closed** if it contains all of its boundary points. A set  $S$  is called **connected** if it is all in one piece and cannot be broken into two separate open parts.



## 14.3 COMPLEX DIFFERENTIATION

Complex analysis is the study of complex-valued functions that are complex differentiable in a domain. The concepts of limits, derivatives, and integrals are similar in spirit to those in real calculus, but they possess much stronger consequences in the complex case.

A function  $f(z)$  of a complex variable  $z$  is called **analytic** in a domain  $D$  if it is **defined** and **complex differentiable** at every point of  $D$ .

### 14.3.1 CAUCHY-RIEMANN EQUATIONS

A necessary condition that  $f(z) = u(x, y) + \mathbf{i}v(x, y)$  be analytic in a region  $R$  is that  $u$  and  $v$  satisfy the Cauchy-Riemann equations as stated below

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{u(x + \Delta x, y + \Delta y) + \mathbf{i}v(x + \Delta x, y + \Delta y) - [u(x, y) + \mathbf{i}v(x, y)]}{\Delta x + \mathbf{i}\Delta y}$$

$$\text{with } \Delta y = 0 \quad f'(z) = \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + \frac{\mathbf{i}v(x + \Delta x, y) - \mathbf{i}v(x, y)}{\Delta x} = u_x + \mathbf{i}v_x$$

$$\text{with } \Delta x = 0 \quad f'(z) = \lim_{\Delta y \rightarrow 0} \frac{u(x, y + \Delta y) - u(x, y)}{\mathbf{i}\Delta y} + \frac{\mathbf{i}v(x, y + \Delta y) - \mathbf{i}v(x, y)}{\mathbf{i}\Delta y} = v_y - \mathbf{i}u_y$$

The Cauchy-Riemann equations are:

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad , \quad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

In polar coordinates:

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}, \quad r \neq 0}$$

#### Example

$$z = x + \mathbf{i}y \Rightarrow u = x, v = y, u_x = 1, v_y = 1, u_y = 0, v_x = 0$$

$$\text{Since } u_x = v_y \text{ and } u_y = -v_x, \quad z' = 1$$

$$\bar{z} = x - \mathbf{i}y \Rightarrow u = x, v = -y, u_x = 1, v_y = -1, u_y = 0, v_x = 0$$

$$\text{Since } u_x \neq v_y \text{ and } u_y \neq -v_x, \quad \bar{z} \text{ is not analytic}$$

### 14.3.2 LAPLACE'S EQUATION

Using the Cauchy–Riemann equations

$$u_x = v_y \quad u_y = -v_x$$

and assuming that  $u$  and  $v$  have continuous second partial derivatives, we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(u_x) = \frac{\partial}{\partial x}(v_y) = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(u_y) = \frac{\partial}{\partial y}(-v_x) = -\frac{\partial^2 v}{\partial y \partial x}$$

Since the mixed partial derivatives of  $v$  are equal,

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$$

it follows that

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.}$$

Similarly,

$$\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.}$$

Thus both  $u$  and  $v$  are harmonic in  $D$ . The function  $v$  is called the **harmonic conjugate** of  $u$  in  $D$  (not to be confused with  $\bar{z}$ ) when  $u$  and  $v$  satisfy the Cauchy–Riemann equations in  $D$ .

### 14.3.3 TRIGONOMETRIC & HYPERBOLIC FUNCTIONS

$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos x - i \sin x$$

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$$

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

$$\cosh z = \frac{1}{2}(e^z + e^{-z})$$

$$\sinh z = \frac{1}{2}(e^z - e^{-z})$$

$$(\cosh z)' = \sinh z$$

$$(\sinh z)' = \cosh z$$

## 14.4 COMPLEX INTEGRATION

Let  $f(z) = u(x, y) + \mathbf{i}v(x, y)$  be continuous on a piecewise smooth curve  $C$ . Then

$$\int_C f(z) dz = \int_C (u + \mathbf{i}v)(dx + \mathbf{i}dy) = \left[ \int_C u dx - \int_C v dy \right] + \mathbf{i} \left[ \int_C u dy + \int_C v dx \right]$$

Using a parametric representation,

$$z(t) = x(t) + \mathbf{i}y(t), \quad a \leq t \leq b$$

$$\dot{z}(t) = \frac{dz}{dt}$$

we obtain

$$\boxed{\int_C f(z) dz = \int_a^b f[z(t)] \dot{z}(t) dt}$$

### Example 1

$$\begin{aligned} z &= 3t - \mathbf{i}t^2 \\ \frac{dz}{dt} &= 3 - \mathbf{i}2t \\ \int f(z) dz &= \int (3t - \mathbf{i}t^2)(3 - \mathbf{i}2t) dt \\ &= \int (9t - 2t^3 - \mathbf{i}9t^2) dt \\ &= \left( -\frac{t^4}{2} + \frac{9t^2}{2} \right) - \mathbf{i}3t^3 \end{aligned}$$

### Example 2

$$\begin{aligned} \oint_C \frac{dz}{z} \\ z &= re^{\mathbf{i}\theta}, \quad dz = \mathbf{i}re^{\mathbf{i}\theta}d\theta \\ \oint_C \frac{dz}{z} &= \int_0^{2\pi} \frac{\mathbf{i}re^{\mathbf{i}\theta}}{re^{\mathbf{i}\theta}} d\theta \\ &= \mathbf{i} \int_0^{2\pi} d\theta \\ &= 2\pi\mathbf{i} \end{aligned}$$

### Example 3

$$\begin{aligned} \oint_C (z - z_0)^m dz \\ z(t) &= z_0 + re^{\mathbf{i}t}, \quad dz = \mathbf{i}re^{\mathbf{i}t}dt \\ \oint_C (z - z_0)^m dz &= \int_0^{2\pi} r^m e^{\mathbf{i}mt} \mathbf{i}re^{\mathbf{i}t} dt \\ &= \mathbf{i}r^{m+1} \int_0^{2\pi} e^{\mathbf{i}(m+1)t} dt \\ &= \begin{cases} 2\pi\mathbf{i}, & m = -1 \\ 0, & m \neq -1 \end{cases} \end{aligned}$$

#### 14.4.1 PATH DEPENDENCE

If we integrate a given function  $f(z)$  from a point  $z_1$  to a point  $z_2$  along different paths, the integrals will in general have different values. **A complex line integral depends not only on the end points of the path but also, in general, on the path itself.**

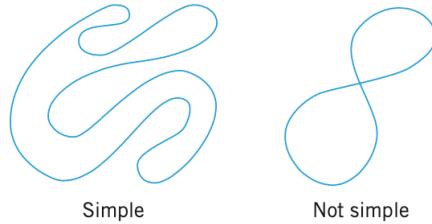
#### 14.4.2 ML-INEQUALITY

$$\left| \oint_C f(z) dz \right| \leq ML, \quad \text{where } |f(z)| \leq M \text{ on } C$$

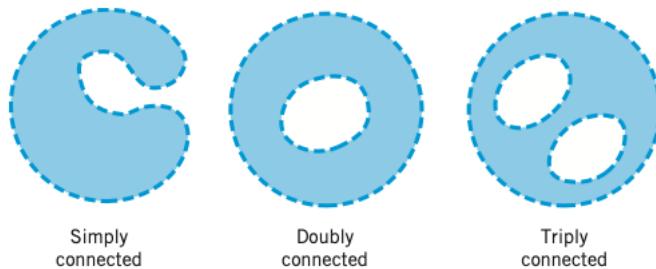
Here  $L$  is the length of the contour  $C$  and  $|f(z)| \leq M$ , where  $M$  is a constant. This follows from the fact that  $|f(z)|$  is bounded on the contour  $C$ , and its maximum value on  $C$  is denoted by  $M$ .

### 14.5 CAUCHY'S INTEGRAL THEOREM

A **simple closed path** is a closed path that does not intersect or touch itself



An **open and connected** set is called a **domain**. In a **simply connected domain**  $D$ , any simple closed curve  $C$  is the boundary of some region  $R$  which is contained in  $D$ . In simple words, a region is simply connected if every closed curve within it can be shrunk continuously to a point that is within the region. That means, a simply connected region is one that has no holes



If  $f(z)$  is analytic in a simply connected domain  $D$ , then for every simple closed path  $C$  in  $D$ ,

$$\oint_C f(z) dz = 0$$

Since  $f(z)$  is analytic in  $D$ ,  $f'(z)$  exists in  $D$ . Assume  $f'(z)$  to be continuous, i.e.,  $u$  and  $v$  have continuous partial derivatives in  $D$ <sup>1</sup>

$$\int_C f(z) dz = \int_C (u + iv)(dx + idy) = \left[ \int_C u dx - \int_C v dy \right] + i \left[ \int_C u dy + \int_C v dx \right]$$

<sup>1</sup>Goursat proved this without the condition that  $f'(z)$  is continuous, but the proof is more involved.

Applying Green's theorem to the vector field  $(P, Q) = (u, -v)$ ,

$$\oint_C u(x, y) dx - \oint_C v(x, y) dy = \iint_R \left( -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy$$

and using the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\left[ \int_C u dx - \int_C v dy \right] = \iint_R \left( -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy = 0$$

$$\left[ \int_C u dy + \int_C v dx \right] = \iint_R \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy = 0$$

$$\oint_C f(z) dz = 0$$

#### 14.5.1 PATH INDEPENDENCE

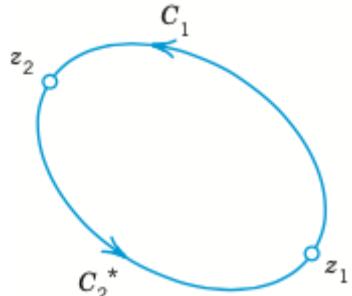
If  $f(z)$  is analytic in a simply connected domain  $D$ , then the integral of  $f(z)$  is independent of the path in  $D$ . This follows from Cauchy's Integral Theorem.

$$\oint_C f(z) dz = 0$$

$$\int_{C_1} f(z) dz + \int_{C_2^*} f(z) dz = 0$$

$$\int_{C_1} f(z) dz = - \int_{C_2^*} f(z) dz$$

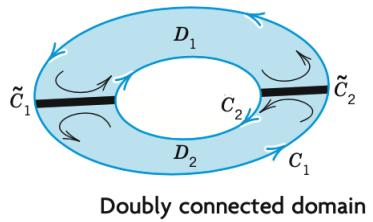
$$\int_{C_1} f(z) dz = \int_{C_2} f(z) dz$$



#### 14.5.2 CAUCHY'S INTEGRAL THEOREM FOR MULTIPLY CONNECTED DOMAINS

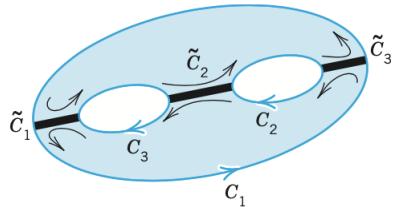
Suppose  $f(z)$  is analytic in the region between the curves (and on the curves themselves). Then:

$$\oint_{C_1} f(z) dz = \oint_{C_2} f(z) dz$$



and, in the triply connected case,

$$\oint_{C_1} f(z) dz = \oint_{C_2} f(z) dz + \oint_{C_3} f(z) dz$$



Here  $C_1$  is the outer boundary oriented counterclockwise, and  $C_2$  (and  $C_3$ ) are the inner boundaries oriented clockwise, so that  $C_1 + C_2 (+C_3)$  is the positively oriented boundary of the region.

### 14.5.3 EXISTENCE OF INDEFINITE INTEGRAL

If  $f(z)$  is analytic in a simply connected domain  $D$ , then its complex integral is **path independent**. This means that the value of the integral from a fixed point  $z_0$  to any point  $z$  is the same for all paths lying in  $D$ . Hence the function

$$F(z) = \int_{z_0}^z f(\zeta) d\zeta$$

is a **single-valued** function in  $D$ . Moreover, this function  $F(z)$  is itself analytic and satisfies

$$F'(z) = f(z)$$

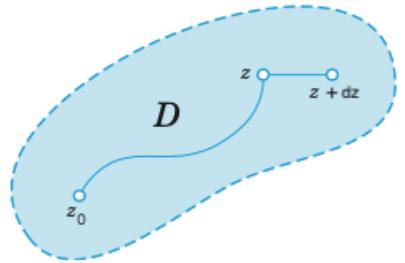
Thus, every analytic function possesses an **antiderivative** in a simply connected domain. Consequently, complex integration is the inverse process of complex differentiation. If  $f(z)$  is analytic in a simply connected domain  $D$ , then the integral

$$F(z) = \int_{z_0}^z f(\zeta) d\zeta$$

is independent of the path in  $D$  and hence defines a single-valued function  $F(z)$ . It is analytic in  $D$  and hence  $F'(z) = f(z)$ . The definite integral can be evaluated as

$$F(z_2) - F(z_1) = \int_{z_1}^{z_2} f(\zeta) d\zeta$$

$$\begin{aligned} \frac{F(z + \Delta z) - F(z)}{\Delta z} &= \frac{1}{\Delta z} \int_z^{z + \Delta z} f(\zeta) d\zeta \\ &= \frac{1}{\Delta z} \int_z^{z + \Delta z} [f(\zeta) - f(z) + f(z)] d\zeta \\ &= \frac{f(z)}{\Delta z} \int_z^{z + \Delta z} d\zeta + \frac{1}{\Delta z} \int_z^{z + \Delta z} [f(\zeta) - f(z)] d\zeta = f(z) + \frac{1}{\Delta z} \int_z^{z + \Delta z} [f(\zeta) - f(z)] d\zeta \end{aligned}$$



Hence

$$\left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) \right| = \frac{1}{|\Delta z|} \left| \int_z^{z + \Delta z} [f(\zeta) - f(z)] d\zeta \right|$$

Since  $f$  is continuous at  $z$ , for every  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$|f(\zeta) - f(z)| < \epsilon \quad \text{whenever } |\zeta - z| < \delta$$

For  $|\Delta z| < \delta$ , and taking the straight-line path from  $z$  to  $z + \Delta z$ , the path length is  $|\Delta z|$ , so

$$\left| \int_z^{z + \Delta z} [f(\zeta) - f(z)] d\zeta \right| \leq \epsilon |\Delta z|.$$

Therefore

$$\left| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) \right| \leq \frac{1}{|\Delta z|} \epsilon |\Delta z| = \epsilon.$$

Since  $\epsilon$  is arbitrary,

$$\lim_{\Delta z \rightarrow 0} \frac{F(z + \Delta z) - F(z)}{\Delta z} = F'(z) = f(z).$$

## 14.6 CAUCHY'S INTEGRAL FORMULA

**THEOREM 14.1. (CAUCHY'S INTEGRAL FORMULA)** *Let  $f$  be analytic in a simply connected domain  $D$ , and let  $C$  be a positively oriented simple closed curve in  $D$  such that  $z_0$  lies in the interior of  $C$  and the interior of  $C$  is contained in  $D$ . Then*

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz.$$

Equivalently,

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0).$$

*Proof.* First, we prove the formula for a circle centered at  $z_0$ .

Let  $C_r$  be the circle  $|z - z_0| = r$  contained in  $D$ . For  $z$  on  $C_r$ ,

$$f(z) = f(z_0) + [f(z) - f(z_0)],$$

hence

$$\oint_{C_r} \frac{f(z)}{z - z_0} dz = f(z_0) \oint_{C_r} \frac{1}{z - z_0} dz + \oint_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz.$$

We compute the first integral by the parametrization  $z = z_0 + re^{it}$ ,  $0 \leq t \leq 2\pi$ :

$$\oint_{C_r} \frac{1}{z - z_0} dz = \int_0^{2\pi} \frac{1}{re^{it}} \cdot ire^{it} dt = i \int_0^{2\pi} dt = 2\pi i.$$

Therefore,

$$f(z_0) \oint_{C_r} \frac{1}{z - z_0} dz = 2\pi i f(z_0).$$

We now show that the second integral is zero. Since  $f$  is continuous at  $z_0$ , for every  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$|f(z) - f(z_0)| < \epsilon \quad \text{whenever } |z - z_0| < \delta.$$

Choose  $r$  with  $0 < r < \delta$ , so that on  $C_r$  we have  $|z - z_0| = r$  and hence  $|f(z) - f(z_0)| < \epsilon$ . Then

$$\left| \oint_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leq \max_{z \in C_r} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| \cdot \text{length}(C_r) \leq \frac{\epsilon}{r} \cdot 2\pi r = 2\pi\epsilon.$$

Since  $\epsilon > 0$  is arbitrary, this implies

$$\oint_{C_r} \frac{f(z) - f(z_0)}{z - z_0} dz = 0.$$

Combining the two parts, we obtain

$$\oint_{C_r} \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0).$$

Now let  $C$  be any positively oriented simple closed curve in  $D$  with  $z_0$  in its interior, and let  $C_r$  be a small circle around  $z_0$  contained entirely in the interior of  $C$ . The function

$$g(z) = \frac{f(z)}{z - z_0}$$

is analytic in the region between  $C$  and  $C_r$ , since  $z_0$  is outside that annular region. By Cauchy's theorem, the integrals of  $g$  over  $C$  and  $C_r$  are equal:

$$\oint_C \frac{f(z)}{z - z_0} dz = \oint_{C_r} \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0).$$

This proves the formula.  $\square$

#### 14.6.1 MULTIPLY CONNECTED DOMAIN

Let  $D$  be a multiply connected domain whose boundary consists of the outer positively oriented simple closed curve  $C_0$  and the inner negatively oriented simple closed curves  $C_1, C_2, \dots, C_n$ . If  $f(z)$  is analytic in  $D$  and  $z_0 \in D$ , then

$$\oint_{C_0} \frac{f(z)}{z - z_0} dz - \sum_{k=1}^n \oint_{C_k} \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0).$$

Equivalently,

$$\oint_{\partial D} \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0),$$

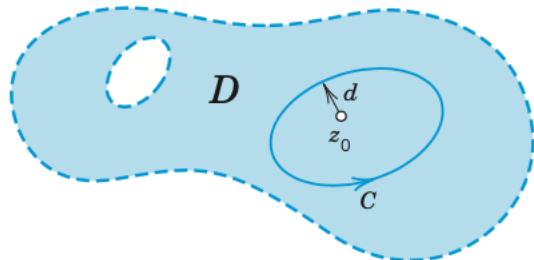
where  $\partial D = C_0 - C_1 - \dots - C_n$ .

### 14.7 DERIVATIVES OF ANALYTIC FUNCTIONS

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} dz$$

$$f''(z_0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^3} dz$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$



We derive the formula for  $f'(z_0)$ ; the higher derivatives follow similarly.

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

Let  $C$  be a simple closed curve contained in the domain of analyticity of  $f$  and enclosing both  $z_0$  and  $z_0 + \Delta z$  for all sufficiently small  $\Delta z$ . By Cauchy's integral formula,

$$f(z_0 + \Delta z) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - (z_0 + \Delta z)} dz, \quad f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz.$$

Hence

$$\begin{aligned}\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{\Delta z} \left[ \frac{1}{z - z_0 - \Delta z} - \frac{1}{z - z_0} \right] dz \\ &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)} dz.\end{aligned}$$

We compare this with

$$\frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} dz.$$

Their difference is

$$\frac{1}{2\pi i} \oint_C \left[ \frac{f(z)}{(z - z_0 - \Delta z)(z - z_0)} - \frac{f(z)}{(z - z_0)^2} \right] dz = \frac{1}{2\pi i} \oint_C \frac{f(z) \Delta z}{(z - z_0 - \Delta z)(z - z_0)^2} dz.$$

Let

$$M = \max_{z \in C} |f(z)|, \quad L = \text{length}(C), \quad d = \min_{z \in C} |z - z_0| > 0.$$

Then on  $C$  we have  $|z - z_0| \geq d$ , so

$$\frac{1}{|z - z_0|^2} \leq \frac{1}{d^2}.$$

Moreover,

$$|z - z_0| \leq |z - z_0 - \Delta z| + |\Delta z| \implies |z - z_0 - \Delta z| \geq d - |\Delta z|.$$

If  $|\Delta z| \leq d/2$ , then  $|z - z_0 - \Delta z| \geq d/2$  and hence

$$\frac{1}{|z - z_0 - \Delta z|} \leq \frac{2}{d}.$$

Therefore,

$$\left| \oint_C \frac{f(z) \Delta z}{(z - z_0 - \Delta z)(z - z_0)^2} dz \right| \leq M |\Delta z| \cdot \frac{2}{d} \cdot \frac{1}{d^2} \cdot L = \frac{2ML}{d^3} |\Delta z| \xrightarrow[\Delta z \rightarrow 0]{} 0.$$

Thus

$$\lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} dz,$$

that is,

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} dz.$$

Repeated differentiation of Cauchy's integral formula yields the general result

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

### 14.7.1 CAUCHY'S INEQUALITY

Let  $f$  be analytic inside and on the circle  $|z - z_0| = r$ , and let

$$M = \max_{|z-z_0|=r} |f(z)|.$$

Then for all  $n \geq 0$ ,

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi} \left| \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz \right| \leq \frac{n!}{2\pi} M \frac{1}{r^{n+1}} 2\pi r = \frac{n!M}{r^n}.$$

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi} \left| \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz \right| \leq \frac{n!}{2\pi} M \frac{1}{r^{n+1}} 2\pi r = \frac{n!M}{r^n}$$

### 14.7.2 LIOUVILLE'S THEOREM

If an entire function is bounded in the whole complex plane, then it must be constant.

*Proof.* Suppose  $f$  is entire and  $|f(z)| \leq M$  for all  $z \in \mathbb{C}$ . Fix any  $z_0 \in \mathbb{C}$  and apply Cauchy's inequality to the circle  $|z - z_0| = r$ . Then

$$|f'(z_0)| \leq \frac{M}{r}.$$

Since  $r$  can be chosen arbitrarily large, we obtain

$$f'(z_0) = 0.$$

Because  $z_0$  is arbitrary,  $f'(z) = 0$  for all  $z$ , hence  $f(z)$  is constant.  $\square$

If an entire function is bounded in absolute value in the whole complex plane, then this function must be a constant. This is because if  $|f(z)| < M$  for all  $z$ , then by Cauchy's inequality

$$|f'(z)| < \frac{M}{r}$$

We may choose  $r$  arbitrarily large, hence  $f'(z) = 0$  and therefore  $f(z)$  is constant

### 14.7.3 MORERA'S THEOREM (CONVERSE OF CAUCHY'S INTEGRAL THEOREM)

If  $f(z)$  is continuous in a simply connected domain  $D$  and

$$\oint_C f(z) dz = 0$$

for every closed path  $C$  in  $D$ , then  $f(z)$  is analytic in  $D$ .

## 14.8 POWER SERIES

Complex power series are the natural analogs of real power series in calculus. Every analytic function can be represented locally by a power series.

### 14.8.1 TAYLOR SERIES

Let  $f$  be analytic inside and on a simple closed curve  $C$  enclosing the point  $z_0$ . Then  $f$  admits a Taylor expansion about  $z_0$  of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

The remainder after  $n$  terms is given by

$$R_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1} (\zeta - z)} d\zeta.$$

A Maclaurin series is a Taylor series with center  $z_0 = 0$ .

Assume that  $|z - z_0| < r$ , where  $r = |z^* - z_0|$  for all  $z^* \in C$ . Then

$$\left| \frac{z - z_0}{z^* - z_0} \right| < 1.$$

We write

$$\frac{1}{z^* - z} = \frac{1}{z^* - z_0 - (z - z_0)} = \frac{1}{z^* - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{z^* - z_0}},$$

and define

$$q = \frac{z - z_0}{z^* - z_0}.$$

Since  $|q| < 1$ , we use the geometric series expansion

$$\frac{1}{1 - q} = 1 + q + q^2 + \dots + q^n + \frac{q^{n+1}}{1 - q}.$$

Therefore,

$$\frac{1}{z^* - z} = \frac{1}{z^* - z_0} \left[ 1 + \frac{z - z_0}{z^* - z_0} + \left( \frac{z - z_0}{z^* - z_0} \right)^2 + \dots + \left( \frac{z - z_0}{z^* - z_0} \right)^n \right] + \frac{1}{z^* - z} \left( \frac{z - z_0}{z^* - z_0} \right)^{n+1}.$$

Since

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{z^* - z} dz^*,$$

substitution gives

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{z^* - z_0} dz^* + \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^2} dz^* + \dots + \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^* + R_n(z)$$

### 14.8.2 LAURENT'S SERIES

A Laurent series generalizes a Taylor series by allowing both positive and negative integer powers of  $(z - z_0)$ . It converges in an annulus

$$r < |z - z_0| < R,$$

where  $0 \leq r < R \leq \infty$ .

If  $f$  is analytic in the annulus  $r < |z - z_0| < R$ , then  $f$  can be represented as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

for all  $z$  in the annulus.

The coefficients are given by

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz, \quad n \geq 0,$$

$$b_n = \frac{1}{2\pi i} \oint_C (z - z_0)^{n-1} f(z) dz, \quad n \geq 1,$$

where  $C$  is any positively oriented simple closed curve in the annulus  $r < |z - z_0| < R$ .

## 14.9 ZEROS AND SINGULARITIES

### 14.9.1 ZEROS

A **zero** of an analytic function  $f$  is a point  $z_0$  such that

$$f(z_0) = 0.$$

If

$$f(z) = (z - z_0)^m g(z),$$

where  $g(z)$  is analytic and  $g(z_0) \neq 0$ , then  $z_0$  is called a **zero of order  $m$** . A zero of order 1 is called a **simple zero**.

### 14.9.2 SINGULARITIES

Let  $f$  be analytic in a punctured neighborhood of  $z_0$ . Then  $z_0$  is called an **isolated singularity** of  $f$  if  $f$  is not analytic at  $z_0$ .

Let the Laurent expansion of  $f$  about  $z_0$  be

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}.$$

### 14.9.3 CLASSIFICATION OF ISOLATED SINGULARITIES

- ▷ If all  $b_n = 0$ , then  $z_0$  is a **removable singularity**.
- ▷ If the principal part has finitely many terms,

$$\frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m}, \quad b_m \neq 0,$$

then  $z_0$  is a **pole of order  $m$** . A pole of order 1 is called a **simple pole**.

- ▷ If the principal part has infinitely many terms, then  $z_0$  is called an **essential singularity**.

If  $f(z)$  has a pole at  $z_0$ , then

$$|f(z)| \rightarrow \infty \quad \text{as } z \rightarrow z_0.$$

## 14.10 RESIDUE INTEGRATION METHOD

Let  $f$  be analytic in a punctured neighborhood of  $z_0$  and suppose its Laurent expansion about  $z_0$  is

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}.$$

Then the coefficient  $b_1$  is called the **residue** of  $f$  at  $z_0$  and is denoted by

$$\text{Res}_{z=z_0} f(z) = b_1.$$

By Cauchy's coefficient formula,

$$b_1 = \frac{1}{2\pi i} \oint_C f(z) dz,$$

where  $C$  is any positively oriented simple closed curve enclosing  $z_0$ .

### RESIDUE AT A SIMPLE POLE

If  $z_0$  is a simple pole, then

$$f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots,$$

and therefore

$$\text{Res}_{z=z_0} f(z) = \lim_{z \rightarrow z_0} (z - z_0) f(z).$$

If

$$f(z) = \frac{p(z)}{q(z)},$$

where  $p$  and  $q$  are analytic and  $z_0$  is a simple zero of  $q$ , then

$$\text{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)}.$$

## RESIDUE AT A POLE OF ORDER $m$

If  $z_0$  is a pole of order  $m$ , then

$$\text{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)].$$

## RESIDUE THEOREM

Let  $f$  be analytic inside and on a simple closed positively oriented contour  $C$ , except for finitely many isolated singularities  $z_1, z_2, \dots, z_k$  inside  $C$ . Then

$$\oint_C f(z) dz = 2\pi i \sum_{j=1}^k \text{Res}_{z=z_j} f(z)$$