Chapter 14
Complex Analysis

There is a number i *, quite imaginary,*
Yet it gets things done, quite extraordinary.
Who would imagine such a possibility?
But that is indeed the reality!

Complex analysis is the study of complex numbers together with their derivatives, manipulation, and other properties. Complex analysis is an extremely powerful tool with an unexpectedly large number of practical applications to the solution of physical problems. It is helpful in many areas such as hydrodynamics, thermodynamics, and particularly quantum mechanics. Complex analysis also has a wide range of applications in engineering fields such as nuclear, aerospace, mechanical and electrical engineering.

14.1 Complex Numbers

Complex numbers are the numbers that are expressed in the form of $x + i\,y$ where x, y are real numbers and i is the imaginary unit.

$$\boxed{z = x + i\,y \qquad \text{where } i = \sqrt{-1}}$$

Just as with real numbers, we can perform arithmetic operations on complex numbers. To add or subtract complex numbers, we combine the real parts and combine the imaginary parts. Addition, multiplication, and division of complex numbers are given below:

$$z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i\,(x_1 y_2 + x_2 y_1)$$

$$\frac{z_1}{z_2} = \frac{x_1 + i\,y_1}{x_2 + i\,y_2}, \quad z_2 \neq 0 = \frac{(x_1 + i\,y_1)(x_2 - i\,y_2)}{(x_2 + i\,y_2)(x_2 - i\,y_2)}$$

$$= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\,\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

14.1.1 Complex Conjugate

The **complex conjugate** of z is defined as:

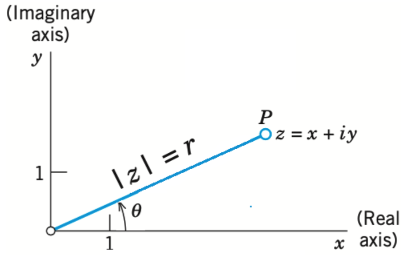
$$\boxed{\bar{z} = x - i\,y}$$

14.1.2 Polar Representation

$$\boxed{z = r\cos\theta + \boldsymbol{i}\, r\sin\theta = re^{\boldsymbol{i}\theta}, \quad r \geq 0} \qquad \text{(Polar representation)}$$

$$\boxed{z^n = r^n\left(\cos n\theta + \boldsymbol{i}\ \sin n\theta\right), \quad n \in \mathbb{Z}} \qquad \text{(De Moivre's theorem)}$$

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$



The complex plane

$$x = r\cos\theta, \quad y = r\sin\theta \quad \text{(radians, measured counterclockwise).}$$

θ is the **argument** of z, denoted by $\arg z$. Its **principal value** is:

$$\boxed{-\pi < \arg z \leq \pi, \quad z \neq 0}$$

The xy-plane is the complex plane, also known as the **Argand diagram**.

$$z_1 z_2 = r_1 r_2 e^{\boldsymbol{i}(\theta_1+\theta_2)} = r_1 r_2 \left[\cos(\theta_1+\theta_2) + \boldsymbol{i}\ \sin(\theta_1+\theta_2)\right]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{\boldsymbol{i}(\theta_1-\theta_2)}, \quad z_2 \neq 0 = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + \boldsymbol{i}\ \sin(\theta_1-\theta_2)\right]$$

$$\arg\left(\frac{z_1}{z_2}\right) \equiv \arg\ z_1 - \arg\ z_2 \pmod{2\pi}, \quad z_1 \neq 0,\ z_2 \neq 0$$

14.1.3 Properties

$z_1 z_2 = z_2 z_1$ (commutative)

$(z_1 z_2) z_3 = z_1 (z_2 z_3)$ (associative)

$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$(distributive)

$|z_1 z_2| = |z_1||z_2|$ and $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}, \quad z_2 \neq 0$

14.1.4 Roots

$$z = re^{\boldsymbol{i}\theta}$$

$$\boxed{z_k = r^{1/n} e^{\boldsymbol{i}\left(\frac{\theta+2k\pi}{n}\right)}, \quad k = 0, 1, \ldots, n-1}$$

Example: $4\boldsymbol{i} = 4e^{\boldsymbol{i}\frac{\pi}{2}}$

$$\sqrt{4\boldsymbol{i}} = \sqrt{4e^{\boldsymbol{i}\frac{\pi}{2}}} = 2e^{\boldsymbol{i}\left(\frac{\pi}{4}+k\pi\right)}, \quad k = 0, 1$$

$$2e^{\boldsymbol{i}\frac{\pi}{4}} = \sqrt{2}(1+\boldsymbol{i}), \qquad 2e^{\boldsymbol{i}\frac{5\pi}{4}} = -\sqrt{2}(1+\boldsymbol{i})$$

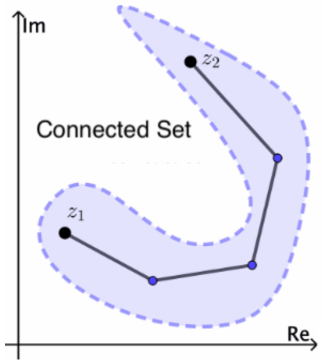
$$\Rightarrow \quad \sqrt{4\boldsymbol{i}} = \sqrt{2}(1+\boldsymbol{i}),\ -\sqrt{2}(1+\boldsymbol{i})$$

14.2 Point Set & Path

A **point set** is simply a collection of points in the complex plane. A set is called **open** if, around every point in the set, we can draw a small open circle that lies completely inside the set. An open circle consists of all the points inside the circle but does not include the circle itself, which is why it is called "open."

A set is called **closed** if it contains all the points that can be approached from within the set. In other words, a set is closed if everything outside it forms an open set.

A set S is called **connected** if it is all in one piece and cannot be broken into two separate open parts.



14.3 Complex Differentiation

Complex analysis is the study of complex-valued functions that are complex differentiable in a domain. The concepts of limits, derivatives, and integrals are similar in spirit to those in real calculus, but they possess much stronger consequences in the complex case. A function $f(z)$ of a complex variable z is called **analytic** in a domain D if it is **defined and complex differentiable** at every point of D.

14.3.1 Cauchy–Riemann Equations

A necessary condition that $f(z) = u(x, y) + i\,v(x, y)$ be analytic in a region R is that u and v satisfy the Cauchy–Riemann equations as stated below

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x + \Delta x, y + \Delta y) + i\,v(x + \Delta x, y + \Delta y) - [u(x, y) + i\,v(x, y)]}{\Delta x + i\,\Delta y}$$

$$\text{with } \Delta y = 0 \quad f'(z) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + \frac{i\,v(x + \Delta x, y) - i\,v(x, y)}{\Delta x} = u_x + i\,v_x$$

$$\text{with } \Delta x = 0 \quad f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\,\Delta y} + \frac{i\,v(x, y + \Delta y) - i\,v(x, y)}{i\,\Delta y} = v_y - i\,u_y$$

The Cauchy-Riemann equations are:

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

In polar coordinates:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}, \quad r \neq 0$$

Example

$$z = x + i\,y \quad \Rightarrow \quad u = x,\ v = y, \quad u_x = 1,\ v_y = 1,\ u_y = 0,\ v_x = 0$$

Since $u_x = v_y$ and $u_y = -v_x$, $\quad z' = 1$

$$\bar{z} = x - i\,y \quad \Rightarrow \quad u = x,\ v = -y, \quad u_x = 1,\ v_y = -1,\ u_y = 0,\ v_x = 0$$

Since $u_x \neq v_y$ and $u_y \neq -v_x$, $\quad \bar{z}$ is not analytic

14.3.2 Laplace's Equation

Using the Cauchy–Riemann equations

$$u_x = v_y, \qquad u_y = -v_x$$

and assuming that u and v have continuous second partial derivatives, we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(u_x) = \frac{\partial}{\partial x}(v_y) = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(u_y) = \frac{\partial}{\partial y}(-v_x) = -\frac{\partial^2 v}{\partial y \partial x}$$

Since the mixed partial derivatives of v are equal,

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$$

it follows that

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

and similarly

$$\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$$

Thus both u and v are harmonic in D. The function v is called the **harmonic conjugate** of u in D (not to be confused with $\bar{z}$) when u and v satisfy the Cauchy–Riemann equations in D.

14.3.3 Trigonometric & Hyperbolic Functions

$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos x - i \sin x$$

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$$

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

$$\cosh z = \frac{1}{2}(e^{z} + e^{-z})$$

$$\sinh z = \frac{1}{2}(e^{z} - e^{-z})$$

$$(\cosh z)' = \sinh z$$

$$(\sinh z)' = \cosh z$$

14.4 Complex Integration

Let $f(z) = u(x, y) + i\, v(x, y)$ be continuous on a piecewise smooth curve C. Then

$$\int_C f(z)\, dz = \int_C (u + i\, v)(dx + i\, dy) = \left[\int_C u\, dx - \int_C v dy\right] + i \left[\int_C u\, dy + \int_C v dx\right].$$

Using parametric representation,

$$z(t) = x(t) + i\, y(t)$$

$$\dot{z}(t) = \frac{dz}{dt}$$

$$\boxed{\int_C f(z)\, dz = \int_a^b f[z(t)]\, \dot{z}(t)\, dt}$$

Example

$$z = 3t - i\, t^2$$

$$\frac{dz}{dt} = 3 - i\, 2t$$

$$\int f(z)\, dz = \int (3t - i\, t^2)(3 - i\, 2t)\, dt$$

$$= \int (9t - 2t^3 - i\, 9t^2)\, dt$$

$$= \left(-\frac{t^4}{2} + \frac{9t^2}{2}\right) - i\, 3t^3$$

Example

$$\oint_C \frac{dz}{z}$$

$$z = re^{i\theta}, \quad dz = i\, re^{i\theta} d\theta$$

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} \frac{i\, re^{i\theta}}{re^{i\theta}}\, d\theta$$

$$= i \int_0^{2\pi} d\theta$$

$$= 2\pi i$$

Example

$$\oint_C (z - z_0)^m\, dz$$

$$z(t) = z_0 + re^{i\, t}, \quad dz = i\, re^{i\, t} dt$$

$$\oint_C (z - z_0)^m\, dz = \int_0^{2\pi} r^m e^{i\, mt}\, i\, re^{i\, t}\, dt$$

$$= i\, r^{m+1} \int_0^{2\pi} e^{i\, (m+1)t}\, dt$$

$$= \begin{cases} 2\pi i, & m = -1 \\ 0, & m \neq -1 \end{cases}$$

14.4.1 Path Dependence

If we integrate a given function $f(z)$ from a point z_1 to a point z_2 along different paths, the integrals will in general have different values. A complex line integral depends not only on the end points of the path but also, in general, on the path itself.

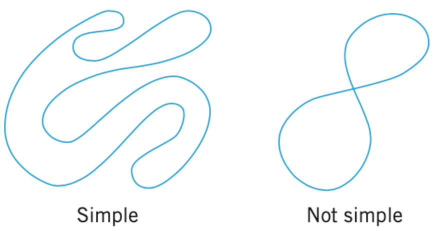
14.4.2 ML-Inequality

$$\boxed{\left|\oint_C f(z)\,dz\right| \le ML, \quad \text{where } |f(z)| \le M \text{ on } C}$$

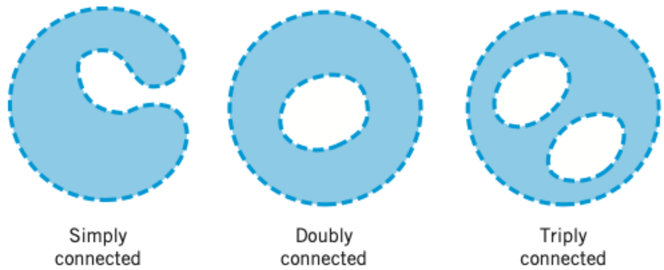
Here L is the length of the contour C and $|f(z)| \le M$, where M is a constant. This follows from the fact that $|f(z)|$ is bounded on the contour C, and its maximum value on C is denoted by M.

14.5 Cauchy's Integral Theorem

A **simple closed path** is a closed path that does not intersect or touch itself



An **open and connected** set is called a **domain**. In a **simply connected domain** D, any simple closed curve C is the boundary of some region R which is contained in D. In simple words, a region is simply connected if every closed curve within it can be shrunk continuously to a point that is within the region. That means, a simply connected region is one that has no holes



If **$f(z)$ is analytic** in a simply connected domain D, then for every simple closed path C in D,

$$\boxed{\oint_C f(z)\,dz = 0}$$

Since $f(z)$ is analytic in D, $f'(z)$ exists in D. Assume $f'(z)$ to be continuous, i.e., u and v have continuous partial derivatives in D[1]

$$\int_C f(z)\,dz = \int_C (u + iv)(dx + i\,dy) = \left[\int_C u\,dx - \int_C v\,dy\right] + i\left[\int_C u\,dy + \int_C v\,dx\right]$$

(Replacing v with $-v$) in Green's Theorem

[1]Goursat proved this without the condition that $f'(z)$ is continuous, but the proof is more involved

$$\oint_C u(x,y)\,dx - \oint_C v(x,y)\,dy = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

and using the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\left[\int_C u\,dx - \int_C v\,dy\right] = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = 0$$

$$\left[\int_C u\,dy + \int_C v\,dx\right] = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy = 0$$

$$\oint_C f(z)\,dz = 0$$

14.5.1 Path Independence

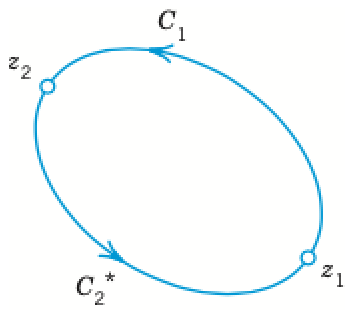
If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of the path in D. This follows from Cauchy's Integral Theorem.

$$\oint_C f(z)\,dz = 0$$

$$\int_{C_1} f(z)\,dz + \int_{C_2^*} f(z)\,dz = 0$$

$$\int_{C_1} f(z)\,dz = -\int_{C_2^*} f(z)\,dz$$

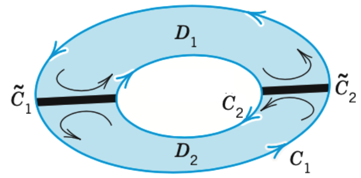
$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$



14.5.2 Cauchy's Integral Theorem for Multiply Connected Domains

Suppose $f(z)$ is analytic in the region between the curves (and on the curves themselves). Then:

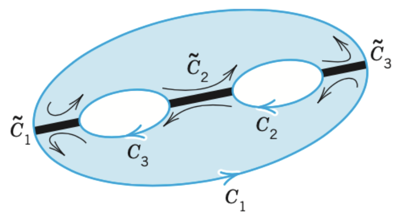
$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$



Doubly connected domain

and, in the triply connected case,

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz$$



Triply connected domain

Here C_1 is the outer boundary oriented counterclockwise, and C_2 (and C_3) are the inner boundaries oriented clockwise, so that $C_1 + C_2(+C_3)$ is the positively oriented boundary of the region.

14.5.3 Existence of Indefinite Integral

If $f(z)$ is analytic in a simply connected domain D, then the integral

$$F(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta$$

is independent of the path in D and hence defines a single-valued function $F(z)$. It is analytic in D and hence $F'(z) = f(z)$. The definite integral can be evaluated as

$$\boxed{F(z_2) - F(z_1) = \int_{z_1}^{z_2} f(\zeta)\, d\zeta}$$

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(\zeta)\, d\zeta$$

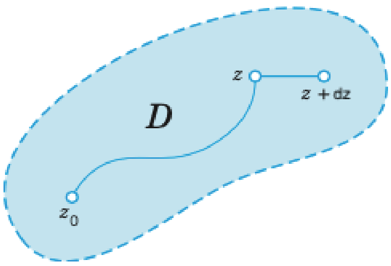
$$= \frac{1}{\Delta z}\int_{z}^{z+\Delta z} \left[f(\zeta) - f(z) + f(z)\right] d\zeta$$

$$= \frac{f(z)}{\Delta z}\int_{z}^{z+\Delta z} d\zeta + \frac{1}{\Delta z}\int_{z}^{z+\Delta z} \left[f(\zeta) - f(z)\right] d\zeta$$

$$= f(z) + \frac{1}{\Delta z}\int_{z}^{z+\Delta z} \left[f(\zeta) - f(z)\right] d\zeta$$

Hence

$$\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| = \frac{1}{|\Delta z|}\left|\int_{z}^{z+\Delta z} \left[f(\zeta) - f(z)\right] d\zeta\right|.$$



Since f is continuous at z, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(\zeta) - f(z)| < \epsilon \quad \text{whenever } |\zeta - z| < \delta.$$

For $|\Delta z| < \delta$, and taking the straight-line path from z to $z + \Delta z$, the path length is $|\Delta z|$, so

$$\left|\int_{z}^{z+\Delta z} \left[f(\zeta) - f(z)\right] d\zeta\right| \le \epsilon |\Delta z|.$$

Therefore

$$\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| \le \frac{1}{|\Delta z|}\epsilon|\Delta z| = \epsilon.$$

Since ϵ is arbitrary,

$$\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = F'(z) = f(z).$$

14.6 Cauchy's Integral Formula

If $f(z)$ is analytic in a simply connected domain D, and C is a positively oriented simple closed curve in D with z_0 inside C, then:

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0).$$

$$f(z) = f(z_0) + [f(z) - f(z_0)]$$

$$\oint_C \frac{f(z)}{z-z_0}\,dz = f(z_0)\oint_C \frac{1}{z-z_0}\,dz + \oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

$$= 2\pi i\, f(z_0) + \cancelto{0}{\oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz}$$

The second term tends to zero because given $\epsilon > 0$ it is possible to find $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ for all z in the disk $|z - z_0| < \delta$

14.6.1 Multiply Connected Domain

By extension, Cauchy's theorem for a multiply connected domain is given by

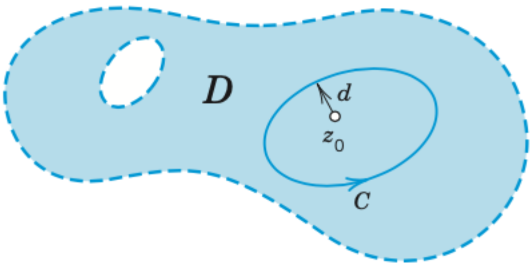
$$\boxed{\oint_{C_1} \frac{f(z)}{z-z_0}\,dz + \oint_{C_2} \frac{f(z)}{z-z_0}\,dz + \cdots + \oint_{C_n} \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)}$$

14.7 Derivatives of Analytic Functions

$$\boxed{f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz}$$

$$\boxed{f''(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}\,dz}$$

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz}$$



$$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

Let us represent $f(z_0 + \Delta z)$ and $f(z_0)$ by Cauchy's integral formula

$$f'(z_0) = \frac{1}{2\pi i\,\Delta z}\oint_C \frac{f(z)}{z-(z_0+\Delta z)}\,dz - \frac{1}{2\pi i\,\Delta z}\oint_C \frac{f(z)}{z-z_0}\,dz$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}\,dz$$

$$\oint_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}\,dz - \oint_C \frac{f(z)}{(z-z_0)^2}\,dz = \cancelto{0}{\oint_C \frac{f(z)\Delta z}{(z-z_0-\Delta z)(z-z_0)^2}\,dz}$$

$$|z-z_0|^2 \geq d^2 \implies \frac{1}{|z-z_0|^2} \leq \frac{1}{d^2}$$

$$d \leq |z-z_0| \leq |z-z_0-\Delta z| + |\Delta z|$$

$$|\Delta z| \leq \frac{d}{2} \implies \frac{1}{|z-z_0-\Delta z|} \leq \frac{2}{d}$$

$$\left|\oint_C \frac{f(z)\Delta z}{(z-z_0-\Delta z)(z-z_0)^2}\,dz\right| \leq ML|\Delta z|\frac{2}{d}\frac{1}{d^2}$$

$$\lim_{\Delta z \to 0} \oint_C \frac{f(z)\Delta z}{(z-z_0-\Delta z)(z-z_0)^2}\,dz = 0$$

$$\boxed{f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz}$$

14.7.1 Cauchy's Inequality

$$\boxed{|f^{(n)}(z_0)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right| \leq \frac{n!}{2\pi}M\frac{1}{r^{n+1}}2\pi r = \frac{n!M}{r^n}}$$

14.7.2 Liouville's Theorem

If an entire function is bounded in absolute value in the whole complex plane, then this function must be a constant. This is because if $|f(z)| < M$ for all z, then by Cauchy's inequality

$$|f'(z)| < \frac{M}{r}$$

We may choose r arbitrarily large, hence $f'(z) = 0$ and therefore $f(z)$ is constant

14.7.3 Morera's Theorem (Converse of Cauchy's Integral Theorem)

If $f(z)$ is continuous in a simply connected domain D and

$$\oint_C f(z)\,dz = 0$$

for every closed path C in D, then $f(z)$ is analytic in D

14.8 Power Series

Complex power series are analogs of real power series in calculus. Every analytic function can be represented as a power series

14.8.1 Taylor Series

The Taylor series is given by

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + R_n(z)$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

$$R_n(z) = \frac{(z-z_0)^{n+1}}{2\pi i} \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}(z^*-z)}\, dz^*$$

A Maclaurin series is a Taylor series with center $z_0 = 0$

$$\left|\frac{z-z_0}{z^*-z_0}\right| < 1$$

$$\frac{1}{z^*-z} = \frac{1}{z^*-z_0-(z-z_0)} = \frac{1}{z^*-z_0} \cdot \frac{1}{1-\frac{z-z_0}{z^*-z_0}}$$

$$q = \frac{z-z_0}{z^*-z_0}$$

$$\frac{1}{1-q} = 1 + q + q^2 + \cdots + q^n + \frac{q^{n+1}}{1-q}$$

$$\frac{1}{z^*-z} = \frac{1}{z^*-z_0}\left[1 + \frac{z-z_0}{z^*-z_0} + \left(\frac{z-z_0}{z^*-z_0}\right)^2 + \cdots + \left(\frac{z-z_0}{z^*-z_0}\right)^n\right] + \frac{1}{z^*-z}\left(\frac{z-z_0}{z^*-z_0}\right)^{n+1}$$

$$\oint_C \frac{f(z^*)}{z^*-z}\, dz^* = 2\pi i\, f(z)$$

$$\boxed{f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^*-z_0}\, dz^* + \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^2}\, dz^* + \cdots + \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^n}\, dz^* + R_n(z)}$$

14.8.2 Laurent's Series

A Laurent series generalizes a Taylor series by allowing both positive and negative integer powers of $(z-z_0)$. It converges in an annulus

$$0 < |z-z_0| < R$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

$$b_n = \frac{1}{2\pi i} \oint_C (z - z_0)^{n-1} f(z)\, dz$$

14.9 Zero, Singularity, Infinity

A **zero** is a value z at which

$$f(z) = 0$$

If the principal part has finitely many terms

$$\frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m} \qquad (b_m \neq 0)$$

then z_0 is a **pole** of order m. A first-order pole is called a **simple pole**

If $f(z)$ has a pole at z_0, then

$$|f(z)| \to \infty \qquad \text{as } z \to z_0$$

14.10 Residue Integration Method

$$b_1 = \frac{1}{2\pi i} \oint_C f(z)\, dz$$

$$b_1 = \operatorname{Res}_{z=z_0} f(z)$$

For a simple pole

$$f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

$$\operatorname{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z)$$

If

$$f(z) = \frac{p(z)}{q(z)}$$

then

$$\operatorname{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)}$$

For a pole of order m

$$\operatorname{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z)\right]$$

$$\boxed{\oint_C f(z)\, dz = 2\pi i \sum_{j=1}^{k} \operatorname{Res}_{z=z_j} f(z)}$$