

SECOND ORDER ORDINARY DIFFERENTIAL EQUATION

*A second-order equation, quite a mess,
With derivatives causing plenty of distress.
But Euler, Cauchy, and Lagrange helped evolve,
Elegant methods to solve and resolve!*

As noted in the previous chapter, the order of a differential equation is defined as the order of the highest derivative of the unknown function that appears in the equation. Therefore, a second-order differential equation is one in which the second derivative of the unknown function occurs, and no derivative of higher order is present.

Second order differential equations have a variety of applications in science and engineering such as vibrations and electric circuits. There are a host of multi dimensional engineering models that incorporate second order differential equations including wave motion, flow mechanics, Maxwell's electro-magnetic equations and Schroedinger equation in Nuclear Physics.

7.1 2ND ORDER LINEAR ODE

The **standard form** of a **2nd Order ODE** is:

$$y'' + p(x)y' + q(x)y = r(x) \quad \text{It is linear in } y, y' \text{ and } y'' \quad (7.1.1)$$

If $r(x) = 0$, the ODE is homogeneous, else it is non-homogeneous. When the coefficients a and b are constant:

$$y'' + ay' + by = 0 \quad (7.1.2)$$

Choose $e^{\lambda x}$ as a solution and substitute in the homogeneous ODE.

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0 \implies \lambda^2 + a\lambda + b = 0 \implies \lambda = \frac{1}{2}(-a \pm \sqrt{a^2 - 4b})$$

$$y_h = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \quad (\text{general solution to the homogeneous ODE})$$

y_1 , corresponding to λ_1 , and y_2 , corresponding to λ_2 , are **linearly independent** and are called **basis of solutions**. The **superposition principle** also called the **linearity principle**, i.e., the homogeneous solution is a combination of y_1 and y_2 is true only for linear homogeneous ODE. The arbitrary constants c_1 and c_2 are determined from the **initial conditions**:

$$y(x_0) = k_0 \quad y'(x_0) = k_1$$

A **particular solution** is obtained if we assign specific values to c_1 and c_2 .

7.2 LAGRANGE'S METHOD OF REDUCTION OF ORDER

Consider a linear homogeneous 2nd Order ODE in its standard form :

$$y'' + p(x)y' + q(x)y = 0 \quad (7.2.1)$$

If y_1 is a basis solution, we can find y_2 as follows:

$$y_1 = e^{\lambda x}$$

$$\text{Let } y_2 = uy_1$$

$$y_2' = u' y_1 + u y_1'$$

$$y_2'' = u'' y_1 + 2u' y_1' + u y_1''$$

Substituting,

$$(u'' y_1 + 2u' y_1' + u y_1'') + p(u' y_1 + u y_1') + q(u y_1) = 0$$

$$y_1 u'' + (2y_1' + p y_1) u' + (y_1'' + p y_1' + q y_1) u = 0$$

$$u'' + u' \frac{2y_1' + p y_1}{y_1} = 0$$

$$\text{Let } U = u'$$

$$U' + U \left(\frac{2y_1'}{y_1} + p \right) = 0$$

$$\frac{U'}{U} = - \left(\frac{2y_1'}{y_1} + p \right)$$

$$\int \frac{U'}{U} dx + \int \left(\frac{2y_1'}{y_1} \right) dx = - \int p dx$$

$$\ln |U| + 2 \ln |y_1| = - \int p dx$$

$$\ln |U y_1^2| = - \int p dx$$

$$U y_1^2 = e^{\int -p dx}$$

$$U = \frac{1}{y_1^2} e^{\int -p dx}$$

$$u = \int U dx$$

$$y_2 = y_1 \int U dx$$

$$y_2 = y_1 \int \frac{1}{y_1^2} e^{\int -p dx} dx$$

7.3 HOMOGENEOUS LINEAR ODE WITH CONSTANT COEFFICIENTS

$$\begin{cases} \text{Case 1: 2 Real Roots when} & a^2 - 4b > 0 \\ \text{Case 2: Double Root when} & a^2 - 4b = 0 \\ \text{Case 3: Complex Conjugate Roots when} & a^2 - 4b < 0 \end{cases}$$

Case 1: 2 Real Roots when $a^2 - 4b > 0$. The general solution is given by:

$$\begin{aligned} y_1 &= e^{\lambda_1 x} \quad y_2 = e^{\lambda_2 x} \\ \lambda_1 &= \frac{1}{2} \left(-a + \sqrt{a^2 - 4b} \right) \quad \lambda_2 = \frac{1}{2} \left(-a - \sqrt{a^2 - 4b} \right) \\ y_h &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \end{aligned} \tag{7.3.1}$$

Case 2: $\lambda_1 = -\frac{a}{2}, y_1 = e^{-\frac{ax}{2}}$ Determine y_2 using the method of reduction of order.

$$\begin{aligned} y_1 &= e^{-\frac{ax}{2}} \\ y_2 &= y_1 \int \frac{1}{y_1^2} e^{\int -p dx} dx = e^{-\frac{ax}{2}} \int \frac{1}{\left(e^{-\frac{ax}{2}}\right)^2} e^{\int -a dx} dx = e^{-\frac{ax}{2}} \int e^{ax} e^{-ax} dx = x e^{-\frac{ax}{2}} \\ y_h &= c_1 e^{-ax/2} + c_2 x e^{-\frac{ax}{2}} \implies y_h = (c_1 + c_2 x) e^{-ax/2} \end{aligned}$$

Case 3: $\lambda = -\frac{a}{2} \pm iw, w = \sqrt{|a^2 - 4b|}$

$$\begin{aligned} y_1 &= e^{\lambda_1 x} = e^{(-\frac{a}{2} + iw)x} = e^{-\frac{ax}{2}} e^{iwx} \\ y_2 &= e^{\lambda_2 x} = e^{(-\frac{a}{2} - iw)x} = e^{-\frac{ax}{2}} e^{-iwx} \\ e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \dots = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right) = \cos x + i \sin x \\ e^{iwx} &= \cos wx + i \sin wx \quad (\text{de Moivre's theorem}) \quad \text{and} \quad e^{i\pi} = -1 \quad (\text{Euler's Identity}) \end{aligned}$$

Any real solution is a linear combination of the real and imaginary parts:

$$y = e^{-\frac{ax}{2}} (c_1 \cos wx + c_2 \sin wx) \quad c_1, c_2 \text{ are constants}$$

7.4 EULER-CAUCHY EQUATIONS

The Euler-Cauchy equation is of the form:

$$x^2 y'' + ax y' + by = 0 \quad \text{where } a, b \text{ are constants} \tag{7.4.1}$$

$$\text{Let } y = x^m \implies y' = m x^{m-1} \implies y'' = m(m-1) x^{m-2}$$

$$\text{Substituting, } x^2 m(m-1) x^{m-2} + ax m x^{m-1} + b x^m = 0 \implies m^2 + (a-1)m + b = 0$$

$$m = \frac{1}{2} (1-a) \pm \sqrt{\frac{1}{4} (a-1)^2 - b}$$

Case 1: Roots are distinct. The basis solutions are :

$y_1(x) = x^{m_1}$ $y_2(x) = x^{m_2}$, the general solution is given by, $y = c_1 x^{m_1} + c_2 x^{m_2}$

Case 2: Double roots.

$$b = \frac{1}{4}(1-a)^2 \quad m = \frac{1}{2}(1-a) \quad y_1 = x^{\frac{1}{2}(1-a)}$$

$$y'' + \frac{a}{x}y' + \frac{(1-a)^2}{4x^2}y = 0$$

Use method of reduction of order, $y_2 = uy_1$ and with $p = \frac{a}{x}$

$$U = \frac{1}{y_1^2} e^{\int -p dx}$$

$$u = \int U dx$$

$$y_2 = y_1 \int U dx$$

$$y_2 = y_1 \int \frac{1}{y_1^2} e^{\int -p dx} dx$$

$$\int p dx = \int \frac{a}{x} dx = a \ln x \implies e^{\int -p dx} = e^{-a \ln x} = e^{\ln x^{-a}} = x^{-a} = \frac{1}{x^a}$$

$$U = \frac{1}{y_1^2} \frac{1}{x^a} = \frac{1}{x^{1-a}} \frac{1}{x^a} = \frac{1}{x} \implies u = \int U dx = \int \frac{1}{x} dx = \ln x$$

$$y_2 = y_1 \int U dx = x^{\frac{1}{2}(1-a)} \ln x$$

$$y_h = (c_1 + c_2 \ln x) x^{\frac{1}{2}(1-a)} \quad c_1, c_2 \text{ are constants}$$

7.5 THE WRONSKIAN

Two solutions y_1 and y_2 are linearly dependent if their Wronskian W is 0.

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = 0$$

Because if the solutions are dependent, $y_1 = k y_2$, where k is a constant

$$\implies W(y_1, y_2) = y_1 y_2' - y_2 y_1' = k y_2 y_2' - y_2 k y_2' = 0$$

The Wronskian is expressed as a Wronski Determinant:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \quad (7.5.1)$$

7.6 NON-HOMOGENEOUS ODE

Consider the following non-homogeneous ODE:

$$y'' + p(x)y' + q(x)y = r(x)$$

The complete solution is the sum of homogeneous (y_h) and particular (y_p) solutions.

$$y(x) = y_h(x) + y_p(x) \text{ where } y_h = c_1 y_1 + c_2 y_2 \text{ (general solution)}$$

y_p is a solution of the non-homogeneous equation without any constants. A particular solution is obtained by assigning specific values to the constants. The Method of Undetermined Coefficients is an approach to finding a particular solution to nonhomogeneous ODEs. If the term in $r(x)$ contains the following term, the choice for $y_p(x)$ is given by:

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$Kx^n (n = 0, 1, \dots)$	$K_n x^n + K_{n-1} x^{n-1} + \dots + K_1 x + K_0$
$k \cos wx$ or $k \sin wx$	$K \cos wx + M \sin wx$
$ke^{\alpha x} \cos wx$ or $ke^{\alpha x} \sin wx$	$e^{\alpha x} (K \cos wx + M \sin wx)$

7.7 PARTICULAR SOLUTION BY VARIATION OF PARAMETERS (LAGRANGE)

The particular solution for the standard form ODE is derived as follows:

$$y'' + p(x)y' + q(x)y = r(x)$$

Find a pair of functions $u_1(x)$ and $u_2(x)$ such that:

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \implies y'_p(x) = u'_1y_1 + u_1y'_1 + u'_2y_2 + u_2y'_2$$

Set constraint, $u'_1y_1 + u'_2y_2 = 0$

$$y'_p(x) = u_1y'_1 + u_2y'_2$$

$$y''_p(x) = u'_1y'_1 + u_1y''_1 + u'_2y'_2 + u_2y''_2$$

Substituting,

$$(u'_1y'_1 + u_1y''_1 + u'_2y'_2 + u_2y''_2) + p(u_1y'_1 + u_2y'_2) + q(u_1y_1 + u_2y_2) = r$$

$$(y''_1 + py'_1 + qy_1)u_1 + (y''_2 + py'_2 + qy_2)u_2 + (u'_1y'_1 + u'_2y'_2) = r$$

Since y_1 and y_2 are solutions to the homogeneous ODE,

$$u'_1y'_1 + u'_2y'_2 = r$$

We now have the following simultaneous equations:

$$u'_1y_1 + u'_2y_2 = 0$$

$$u'_1y'_1 + u'_2y'_2 = r$$

Solving,

$$u'_1 = -\frac{y_2r}{y_1y'_2 - y'_1y_2} = -\frac{y_2r}{W}$$

$$u'_2 = -\frac{y_1r}{y_1y'_2 - y'_1y_2} = -\frac{y_1r}{W}$$

$$y_p(x) = -y_1 \int \frac{y_2r}{W} dx + y_2 \int \frac{y_1r}{W} dx$$

7.8 SYMPY

```

1 import sympy as sp
2
3 # Define the variable and the unknown function
4 x = sp.symbols('x')
5 y = sp.Function('y')
6
7 # Define the differential equation
8 ode = sp.Eq(sp.diff(y(x), x, 2) - 3*sp.diff(y(x), x) + 2*y(x), 0)
9
10 # Solve the ODE
11 solution = sp.dsolve(ode)
12 print(sp.latex(ode))
13 display(ode)

```

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14  
15 print(sp.latex(solution))  
16 display(solution)
```

$$2y(x) - 3\frac{d}{dx}y(x) + \frac{d^2}{dx^2}y(x) = 0$$

$$y(x) = (C_1 + C_2e^x)e^x$$