

CHAPTER 3

LIMIT

*A limit's a dream in sight,
We chase it with all our might!
But just as we draw near,
It grins, and disappears.*

In mathematics, a limit is the value that a function or a sequence approaches as the input approaches some value. Limits are essential to calculus and mathematical analysis, and are used to define continuity, derivatives, and integrals.

3.1 DEFINITION OF LIMIT

Consider a function $f(x)$ that is defined in a domain D which includes the point c . The function may or may not be defined at c . If, for all x that is close to c except for c , $f(x)$ is arbitrarily close to a number L (as close to L as we like), then it is said that f approaches the limit L as x approaches c and is written as:

$$\lim_{x \rightarrow c} f(x) = L$$

If the function can be evaluated at c , the limit L is simply $f(c)$. But there can be situations where the function is not evaluable at c . E.g., the following function cannot be evaluated at $x = 1$.

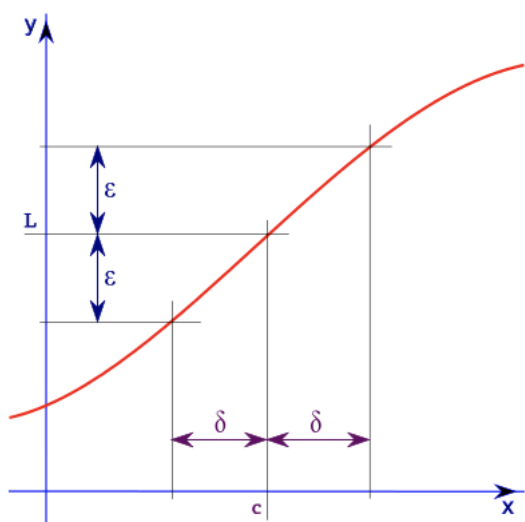
$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x - 1}$$

But this function can be easily simplified to:

$$f(x) = \frac{(x - 1)(x + 2)}{x - 1} = x + 2$$

$$\rightarrow \lim_{x \rightarrow 1} f(x) = 3.$$

3.2 FORMAL DEFINITION OF LIMIT



Let $f(x)$ be a function that is defined on an interval that contains $x = c$, except possibly at c . Then, $\lim_{x \rightarrow c} f(x) = L$ if for every number $\epsilon > 0$, there is some number $\delta > 0$ such that, when $0 < |x - a| < \delta$, $|f(x) - L| < \epsilon$.

This means that for any number $\epsilon > 0$ that we pick, one can go to the graph and sketch two horizontal lines at $L + \epsilon$ and $L - \epsilon$. Then there must be another number $\delta > 0$ that can be determined to enable us to add in two vertical lines in the graph $a + \delta$ and $a - \delta$.

3.3 LAWS OF LIMIT

Given L, M, c, k are real numbers such that $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} g(x) = M$. Then,

Sum Rule	$\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$
Difference Rule	$\lim_{x \rightarrow c} (f(x) - g(x)) = L - M$
Constant Rule	$\lim_{x \rightarrow c} (kf(x)) = kL$
Product Rule	$\lim_{x \rightarrow c} (f(x)g(x)) = LM$
Quotient Rule	$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M}$
Power Rule	$\lim_{x \rightarrow c} [f(x)]^n = L^n \ (n > 0)$
Root Rule	$\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{L} \ (n > 0)$

Examples:

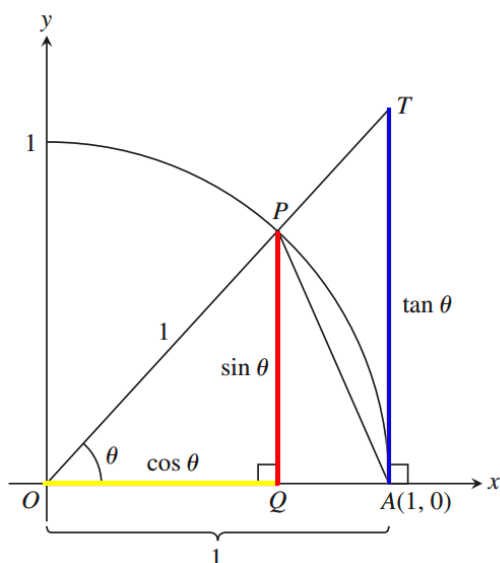
$$\lim_{x \rightarrow 3} \sqrt{2x^3 + 10} = 8$$

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 9} - 3}{x^2}$$

The above function is not evaluable at $x = 0$. The standard trick is to multiply both numerator and denominator by the conjugate radical expression.

$$\frac{\sqrt{x^2 + 9} - 3}{x^2} = \frac{\sqrt{x^2 + 9} - 3}{x^2} \cdot \frac{\sqrt{x^2 + 9} + 3}{\sqrt{x^2 + 9} + 3} = \frac{1}{\sqrt{x^2 + 9} + 3} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x^2 + 9} + 3} = \frac{1}{6}$$

3.4 AN IMPORTANT LIMIT



Consider the circle with a unit radius.

Area $\triangle OAP < \text{area sector OAP} < \text{area } \triangle OAT$

$$\frac{1}{2} \sin \theta \leq \pi 1^2 \left(\frac{\theta}{2\pi} \right) \leq \frac{1}{2} \tan \theta \quad (\theta \text{ is in radians})$$

$$1 \leq \frac{\theta}{\sin \theta} \leq \frac{1}{\cos \theta}$$

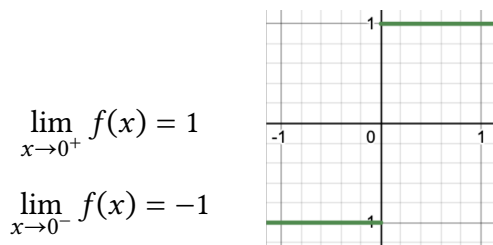
$$\rightarrow 1 \geq \frac{\sin \theta}{\theta} \geq \cos \theta$$

Hence,

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \quad \text{where } \theta \text{ is in radians} \quad (3.1)$$

Now consider the function $f(\theta) = \frac{1}{\sin \theta}$. Does it have a limit as $t \rightarrow \theta$ from either side? As θ approaches 0, its reciprocal, $1/x$, grows without bound and the values of function cycle repeatedly from -1 to 1. There is no single number L that the function values stay increasingly close to as $\theta \rightarrow 0$. The function has neither a right-hand limit nor a lefthand limit at $\theta = 0$.

3.5 ONE SIDED LIMITS



$$\lim_{x \rightarrow 0^+} f(x) = 1$$

$$\lim_{x \rightarrow 0^-} f(x) = -1$$

3.6 CONTINUOUS FUNCTION

Function is right-continuous at c (continuous from right) if $\lim_{x \rightarrow c^+} f(x) = f(c)$

Function is left-continuous at c (continuous from left) if $\lim_{x \rightarrow c^-} f(x) = f(c)$

A function is continuous at c if $\lim_{x \rightarrow c} f(x) = f(c)$

If a function is discontinuous at one or more points of its domain, it is called a discontinuous function.

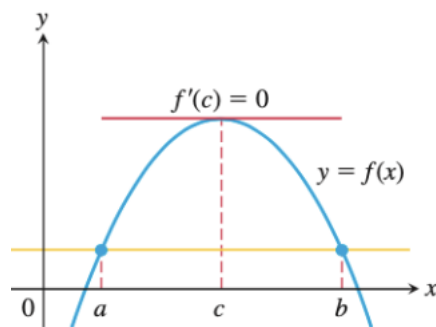
3.7 INFINITE LIMITS

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty, \quad \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$

Note that this does not mean that the limit exists as there is no real number such as ∞ . It is simply a concise way of saying that the limit does not exist.

3.8 ROLLE'S THEOREM

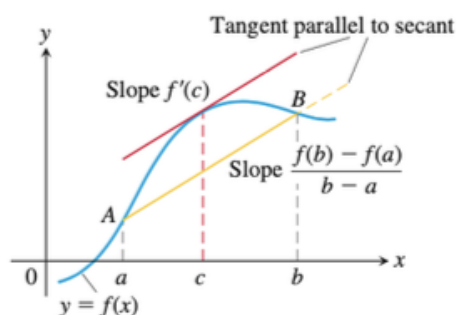
If f is a continuous function on a closed interval $[a, b]$ and If $f(a) = f(b)$, then there is at least one point c in (a, b) where $f'(c) = 0$.



3.9 MEAN VALUE THEOREM

There is at least one number c in the interval (a, b) such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



3.10 SYMPY CODE

Determine the limits of:

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{(x - 1)}$$

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 9} - 3}{x^2}$$

$$\lim_{x \rightarrow 0} \sin(x)$$

```
1 import sympy as sp
2 import numpy as np
3 import matplotlib.pyplot as plt
4 from IPython.display import display, Math
5 from sympy import sin, cos, tan, trigsimp, expand_trig
6 from sympy import oo
7 from sympy import limit
8
```

```

9 x = sp.symbols('x')
10 y = (x**2 + x - 2) / (x - 1)
11 lim = limit(y, x, 1)
12 display(lim)
13
14 y = ( (x**2 + 9)**0.5 - 3 ) / x**2
15 lim = limit(y, x, 0)
16 display(lim)
17
18 y = sin(x)/x
19 lim = limit(y, x, 0)
20 display(lim)

```

3

$\frac{1}{6}$

$\cos(x)$