Tensor Calculus and Differential Geometry

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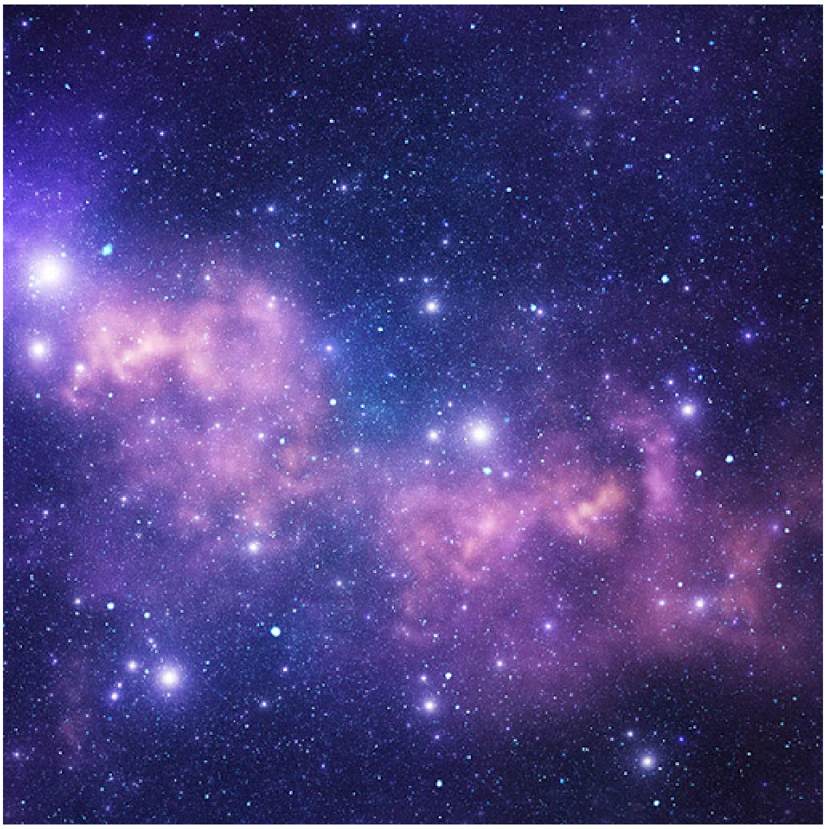

Contents

List of Figures

Preface

The objective in this book is to provide a compact explanation of the fundamental results in tensor theory and its application in differential geometry, engineering analysis and relativity.

Jaideep Ganguly received his degrees of Doctor of Science and Master of Science from the Massachusetts Institute of Technology. He had graduated from Indian Institute of Technology, Kharagpur. During his graduate studies at MIT, he was exposed to tensors. The motivation to write this book came from the desire to develop a comprehensive content that is necessary to provide a rigorous exposure to the theory of tensor calculus so that it can effectively applied in engineering analsysis and physics.

CHAPTER 1

INTRODUCTION

1.1 DEFINITION

Tensor calculus is concerned with a study of abstract quantities in geometry and physics called tensor, whose properties and relations are independent of the coordinate frame. A tensor is fully described if its components in any coordinate frame are given because, by definition, a specific kind of transformation law is used to represent the tensor in any other coordinate frame.

The beauty of tensor formulation of physical laws rests to a fair degree on two conventions. Consider the linear homogeneous function of the independent variables u_1, u_2 and u_3 which can be expressed as,

$$a_1 u^1 + a_2 u^2 + a_2 u^2 \equiv \sum_{m=1}^{3} a_m u^m$$

where a_i is a constant and the superscript in u^i are indices and not exponents and $\equiv$ means equivalent or identical to. We can get rid of the $\sum$ notation and write:

$$a_1 u^1 + a_2 u^2 + a_2 u^2 \equiv a_m u^m$$

adopting the notation that a repeated lower case index m is to be summed from 1 to 3. The repeated index is often called a dummy index as it is immaterial which letter of the alphabet is used. Two additional rules are:

1. The dummy index will almost always appear once as a subscript and once as a superscript
2. The same index cannot be used more than twice in the same term.

1.2 LEVI-CIVITA SYMBOL

The **Levi-Civita symbols**, named after the Italian mathematician Tullio Levi-Civita, make working with determinants compact and simpler and hence the motivation to study these symbols. In three dimensions, the Levi-Civita symbol, or the e symbol, is defined by:

$$e_{ijk} = \begin{cases} +1 & \text{if } i, j, k \text{ is an even permutation of } 1, 2, 3 \\ -1 & \text{if } i, j, k \text{ is an odd permutation of } 1, 2, 3 \\ 0 & \text{if i=j, j=k or k=i, i.e., any of the 2 labels is the same} \end{cases}$$

Each time two numbers are switched, it is a permutation. An easy way to remember is by referencing the following diagrams.

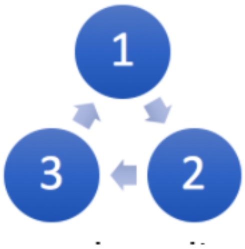

FIGURE 1.1 – e symbol sign is +ve when direction is clockwise.

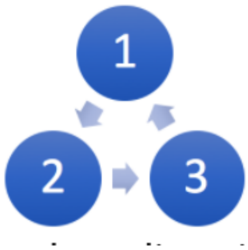

FIGURE 1.2 – e symbol sign is -ve when direction is anti-clockwise.

$$e_{ij} = \begin{cases} +1 & \text{if } i,j \text{ is clockwise} \\ -1 & \text{if } i,j \text{ is anticlockwise} \\ 0 & \text{if i=j} \end{cases}$$

So, $e_{123} = e_{231} = e_{312} = 1$ and $e_{132} = e_{321} = e_{213} = -1$. Also, e_{113}, e_{232}, etc., are 0.

In 2 dimensions, the Levi-Civita symbol is given as: This is a 2 x 2 anti-symmetric matrix.

$$\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

In three dimensions, the Levi-Civita symbols can be represented in a 3 x 3 x 3 matrix.

1.3 LINEAR TRANSFORMATION

Let us suppose that the variable u_r are transformed into a new set of variables $\tilde{u}$ according to the following linear transformation.

$$\tilde{u}^r = a^r_s u^s \tag{1.3.1}$$

where a^r_s are constants. Moreover, we assume that the determinant of the constants is not zero, i.e.,

$$a \equiv |a^r_s| \equiv \begin{vmatrix} a^1_1 & a^1_2 & a^1_3 \\ a^2_1 & a^2_2 & a^2_3 \\ a^3_1 & a^3_2 & a^3_3 \end{vmatrix}$$

1.3.1 DETERMINANT

In linear algebra, the determinant is a useful value that can be computed from the elements of a square matrix. The determinant of a 3x3 square matrix A with elements a_{ij}, given by a, is:

$$|A| = a = \begin{vmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & a_2^3 & a_3^3 \end{vmatrix} = a_1^1(a_2^2 a_3^3 - a_2^3 a_3^2) - a_2^1(a_1^2 a_3^3 - a_1^3 a_3^2) + a_3^1(a_1^2 a_2^3 - a_1^3 a_2^2)$$

Using the e symbol, we can express the right hand side in a compact format as follows.

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} e^{mnp} e_m^1 e_n^2 e_p^3$$

Dropping the summation notation and making it implicit we have, what is known as the Einstein notation,

$$a = \begin{vmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & a_2^3 & a_3^3 \end{vmatrix} = e^{mnp} e_m^1 e_n^2 e_p^3 \tag{1.3.2}$$

1.3.2 KRONECKER DELTA

The **Kronecker delta**, named after Leopold Kronecker, is defined as follows:

$$\delta_j^i = \begin{cases} +1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

The product of two Levi-Civita symbols can be expressed as a function of the Kronecker delta δ_{ij} as follows:

$$e^{mnp} e_{rst} = \begin{vmatrix} \delta_r^m & \delta_s^m & \delta_t^m \\ \delta_r^n & \delta_s^n & \delta_t^n \\ \delta_r^p & \delta_s^p & \delta_t^p \end{vmatrix}$$

For example, you can easily verify the following:

$$e^{123} e_{123} = \delta_1^1 \delta_2^2 \delta_3^3 = 1$$

Expanding the determinant, we have:

$$e^{mnp} e_{rst} = \delta_r^m \begin{vmatrix} \delta_s^n & \delta_t^n \\ \delta_s^p & \delta_t^p \end{vmatrix} - \delta_s^m \begin{vmatrix} \delta_r^n & \delta_t^n \\ \delta_r^p & \delta_t^p \end{vmatrix} + \delta_t^m \begin{vmatrix} \delta_r^n & \delta_s^n \\ \delta_r^p & \delta_s^p \end{vmatrix}$$

Contracting by setting $t = m$, we have:

$$\begin{aligned} e^{mnp} e_{rsm} &= \delta_r^m \begin{vmatrix} \delta_s^n & \delta_m^n \\ \delta_s^p & \delta_m^p \end{vmatrix} - \delta_s^m \begin{vmatrix} \delta_r^n & \delta_m^n \\ \delta_r^p & \delta_m^p \end{vmatrix} + \delta_m^m \begin{vmatrix} \delta_r^n & \delta_s^n \\ \delta_r^p & \delta_s^p \end{vmatrix} \\ & \begin{vmatrix} \delta_s^n & \delta_r^n \\ \delta_s^p & \delta_r^p \end{vmatrix} - \begin{vmatrix} \delta_r^n & \delta_s^n \\ \delta_r^p & \delta_s^p \end{vmatrix} + \delta_m^m \begin{vmatrix} \delta_r^n & \delta_s^n \\ \delta_r^p & \delta_s^p \end{vmatrix} \\ & - \begin{vmatrix} \delta_r^n & \delta_s^n \\ \delta_r^p & \delta_s^p \end{vmatrix} - \begin{vmatrix} \delta_r^n & \delta_s^n \\ \delta_r^p & \delta_s^p \end{vmatrix} + 3 \begin{vmatrix} \delta_r^n & \delta_s^n \\ \delta_r^p & \delta_s^p \end{vmatrix} = \begin{vmatrix} \delta_r^n & \delta_s^n \\ \delta_r^p & \delta_s^p \end{vmatrix} \end{aligned}$$

$$e^{mnp}e_{rsm} = \delta_r^n\delta_s^p - \delta_r^p\delta_s^n \tag{1.3.3}$$

Contracting again by setting $s = p$, we have:

$$e^{mnp}e_{rpm} = \delta_r^n\delta_p^p - \delta_r^p\delta_p^n = 3\delta_r^n - \delta_r^n = 2\delta_r^n$$

i.e.,

$$e^{mnp}e_{rpm} = 2\delta_r^n \tag{1.3.4}$$

1.3.3 COFACTOR

Now consider the following determinant:

$$|A| = \begin{vmatrix} a_1^1 & a_1^2 & a_1^3 \\ a_2^1 & a_2^2 & a_2^3 \\ a_3^1 & a_3^2 & a_3^3 \end{vmatrix} = e^{mnp}\,a_m^1\,a_n^2\,a_p^3$$

Define A_r^m as the cofactor of the element a_m^r in the determinant. Note carefully the reversed positions of m and r in a_m^r and A_r^m. The cofactor of a_m^r is obtained by deleting this element and we have:

$$A_r^m = e^{mnp}a_n^2a_p^3 \tag{1.3.5}$$

We can write the determinant as:

$$\begin{aligned} |A| &= a_1^1A_1^1 + a_1^2A_2^1 + a_1^3A_3^1 = a_1^rA_r^1 \\ |A| &= a_2^1A_1^2 + a_2^2A_2^2 + a_2^3A_3^2 = a_2^sA_s^2 \\ |A| &= a_3^1A_1^3 + a_3^2A_2^3 + a_3^3A_3^3 = a_3^tA_t^3 \end{aligned}$$

We can combine the two e symbols into a more compact form by defining a δ system as follows:

$$\delta_{rst}^{mnp} = e^{mnp}e_{rst} \tag{1.3.6}$$

The Kronecker delta is a member of this system. The value of the Kronecker delta such as $\delta_{rs}^{mn} = e^{mn}e_{rs}$ is:

$$\delta_{rs}^{mn} = \begin{cases} +1 & \text{if m,n and r,s are even permutations of the same numbers} \\ -1 & \text{if m,n and r,s are odd permutations of the same numbers} \\ 0 & \text{if m,n and r,s are not permutations} \end{cases}$$

For example,

$$\begin{aligned} \delta_{22}^{22} &= 0 && \text{since indices are not permutations} \\ \delta_{12}^{12} &= 1 && \text{since indices are permutations of the same numbers} \\ \delta_{21}^{12} &= -1 && \text{since indices are opposite permutations of the same numbers} \end{aligned}$$

Equation 1.3.5 can then be written as:

$$A_r^m = e^{mnp}a_n^2a_p^3 = \frac{1}{2!}\delta_{rst}^{mnp}\,a_n^s a_p^t \tag{1.3.7}$$

i.e.,

$$A_r^m = \frac{1}{2!}\delta_{rst}^{mnp}\,a_m^s a_n^t \tag{1.3.8}$$

If we solve the set of simultaneous equations in 1.3.1, we have the reverse transformation:

$$u^m = \tilde{a}_r^m\tilde{u}^r \tag{1.3.9}$$

CHAPTER 2

TENSORS

2.1 ORTHOGONAL CARTESIAN SYSTEM

The position of a point in three dimensional space can be determined by its coordinates referred to in an orthogonal Cartesian system of axes. The general functional transformation is given by:

$$x^m = f^m(y^1, y^2, y^3) \tag{2.1.1}$$

where:

1. y^m are orthogonal Cartesian coordinates
2. x^m are general curvilinear coordinates
3. f^m are single valued functions of y^1, y^2 and y^3

Henceforth, we will use y^m to denote Cartesian coordinates and x^m to denote curvilinear coordinates.

2.2 CURVILINEAR COORDINATE SYSTEM

In order for the transformation to be reversible, it is necessary for the determinant to be non-zero, i.e.:

$$\left|\frac{\partial x^r}{\partial y^s}\right| \equiv e^{mnp}\frac{\partial x^1}{\partial y^m}\frac{\partial x^2}{\partial y^n}\frac{\partial x^3}{\partial y^p} \neq 0 \tag{2.2.1}$$

The above equation can be solved to yield:

$$y^m = h^m(x^1, x^2, x^3) \tag{2.2.2}$$

In equation **??**, h^m represents single valued functions of x^1, x^2, x^3. The surfaces denoted by x^1 = constant, x^2 = constant and x^3 = constant are called coordinate surfaces. The intersection of these coordinate surfaces with one another are called coordinate curves. At each point in space, there are 3 intersecting curves. In general, these curves are not straight lines and hence they are called **curvilinear coordinates**. Since the similar relations must hold for another curvilinear system $\tilde{l}^r$, we can write:

$$\tilde{x}^r = \tilde{l}^r(x^1, x^2, x^3) \tag{2.2.3}$$

and,

$$x^r = l^r(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) \tag{2.2.4}$$

Substituting equation 2.2.3 into equation 2.2.4, we have:

$$x^r = l^r(\tilde{l}^1, \tilde{l}^2, \tilde{l}^3) \tag{2.2.5}$$

The partial derivative of the above equation w.r.t. x^s yields:

$$\frac{\partial x^r}{\partial x^s} = \frac{\partial l^r}{\partial \tilde{l}^m}\frac{\partial \tilde{l}^m}{\partial x^s} \equiv \frac{\partial x^r}{\partial \tilde{x}^m}\frac{\partial \tilde{x}^m}{\partial x^s} \tag{2.2.6}$$

Since the three curvilinear coordinates represented by x^r are independent, it means that when $r \neq s$, the above equation must equate to zero. For example, clearly:

$$\frac{\partial x^1}{\partial x^2} = 0$$

and,

$$\frac{\partial x^1}{\partial x^1} = 1$$

The above can be summarized as:

$$\frac{\partial x^r}{\partial \tilde{x}^m}\frac{\partial \tilde{x}^m}{\partial x^s} = \delta^r_s \tag{2.2.7}$$

where δ^r_s is the Kronecker delta.

Similarly, we can write:

$$\frac{\partial \tilde{x}^r}{\partial \tilde{x}^m}\frac{\partial x^m}{\partial \tilde{x}^s} = \delta^r_s \tag{2.2.8}$$

2.3 TENSOR AND TENSOR CALCULUS

A tensor is a mathematical object whose properties and relations are independent of the coordinate frame. A tensor is fully described if its components are given in a particular coordinate frame and specific transformation laws are defined to represent the tensor in any other coordinate frame.

Calculus deals with change, tensor calculus simply deals with how tensors change. Tensor and tensor calculus provide a natural and concise mathematical framework for formulating and solving problems in areas of physics and engineering such as elasticity, fluid mechanics and general relativity.

2.4 TENSOR OF ORDER ZERO

In this book, the words tensor and system are used interchangeably. A system of order zero, i.e., a single number is termed a **scalar** quantity. Such a quantity, by definition, has the same value regardless of the coordinate system used. This means:

$$f(x^1, x^2, x^3) \equiv g(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) \tag{2.4.1}$$

If the coordinate transformation law is represented by:

$$x^r = x^r(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) \tag{2.4.2}$$

then f is invariant if:

$$f[x^1(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3), x^2(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3), x^3(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)] = f(x^1, x^2, x^3) \tag{2.4.3}$$

Examples of scalar quantities are familiar entities such as energy, length of a line element and $\sigma^m_m = \delta^1_1 + \delta^2_2 + \delta^2_3 = 3$.

2.5 CONTRAVARIANT TENSOR OF ORDER ONE - CONTRAVARIANT VECTOR

Consider a system of order one such as differential elements of length dx_1, dx_2 and dx_3. In the new coordinate frame the differentials are related to the original frame by:

$$d\tilde{x}^m = \frac{\partial \tilde{x}^m}{\partial x^r} dx^r$$

Functions that transform in this manner are called contravariant tensors of order one or contravariant vectors. **Quantities which bear a single superscript are contravariant vectors and they obey the transformation law given by:**

$$\tilde{a}^m(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = \frac{\partial \tilde{x}^m}{\partial x^n} a^r(x^1, x^2, x^3) \tag{2.5.1}$$

The functions a^m are said to be the components of a contravariant tensor of order one if they transform according to equation 2.5.1.

From the above equation, we have:

$$\frac{\partial x^n}{\partial \tilde{x}^m} \tilde{a}^m = \frac{\partial x^n}{\partial \tilde{x}^m} \frac{\partial \tilde{x}^m}{\partial x^r} a^r = \delta^n_r a^r = a^n$$

We thus have the inverse transformation:

$$a^n(x^1, x^2, x^3) = \frac{\partial x^n}{\partial \tilde{x}^m} a^m(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) \tag{2.5.2}$$

2.6 COVARIANT TENSOR OF ORDER ONE - COVARIANT VECTOR

There is another way in which the components of a system of order one can transform. For example, consider the partial derivatives, $\frac{\partial f}{\partial x^n}$ of a scalar $f(x^m)$. Such a system arises in connection with the notion of a gradient of a potential function. The induced transformation law for these partial derivatives, subject to the coordinate transformation as defined in equation 2.4.3 is computed according to the rule for composite functions as:

$$\frac{\partial f}{\partial \tilde{x}^m} = \frac{\partial f}{\partial x^n} \frac{\partial x^r}{\partial \tilde{x}^m}$$

or,

$$\frac{\partial f}{\partial \tilde{x}^m} = \frac{\partial x^r}{\partial \tilde{x}^m} \frac{\partial f}{\partial x^n} \tag{2.6.1}$$

Functions which transform according to equation 2.6.1 are called covariant tensors of order one or covariant vectors. The components of a covariant vector are identified by a single subscript and the transformation law is written as:

$$\tilde{a}_m(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = \frac{\partial x^n}{\partial \tilde{x}^m} a_n(x^1, x^2, x^3) \tag{2.6.2}$$

The functions, a_m, are called the covariant components of a covariant tensor.

Note carefully the difference in the manner of transformation between the contravariant components and covariant components. An useful aid in remembering the two transformation laws is to keep in mind that differentials dx^m are contravariant in character and that the gradients $\frac{\partial f}{\partial x^m}$ are covariant.

2.7 TENSORS OF THE SECOND ORDER

A simple way to form a second order system is to multiply two tensors of first order together. Since we have defined both contravariant there are three kinds of products which can be obtained. First by if b^m and c^n are the components of two contravariant tensors, then the product $b^m c^n$ is a contravariant tensor of order two and such a tensor will be denoted by two superscripts.

$$a^{mn} = b^m c^n$$

The transformation law is given by:

$$\tilde{a}^{mn}(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = \frac{\partial \tilde{x}^m}{\partial x^n} b^n \frac{\partial \tilde{x}^n}{\partial x^s} c^s$$

i.e.,

$$\tilde{a}^{mn}(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = \frac{\partial \tilde{x}^m}{\partial x^r} \frac{\partial \tilde{x}^r}{\partial x^s} a^{rs}(x^1, x^2, x^3) \tag{2.7.1}$$

Systems which transform according to above equation known as contravariant tensors of the second order. A familiar example is the stress tensor. Secondly, if b_m and c_n are the components of two covariant tensors, then the product $b_m c_n$ is a covariant tensor of order two and will be denoted by two subscripts.

$$a_{mn} = b_m c_n$$

The transformation law for a_{mn} is given by:

$$\tilde{a}_{mn}(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = \frac{\partial x^r}{\partial \tilde{x}^m} \frac{\partial x^s}{\partial \tilde{x}^n} a_{rs}(x^1, x^2, x^3) \tag{2.7.2}$$

Systems which transform according to the above equation are known as covariant tensors of the second order. A familiar example is the strain tensor.

If b^m and c_m are contravariant and covariant, respectively, then the product b^m and c_n is a mixed tensor of order two and is denoted by one superscript and one subscript.

$$a^m_n = b^m c_n$$

a^m_n transform as below:

$$\tilde{a}^m_n(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = \frac{\partial \tilde{x}^m}{\partial x^r} \frac{\partial x^s}{\partial \tilde{x}^n} a^r_s(x^1, x^2, x^3) \tag{2.7.3}$$

Systems which transform according to the above equation are known as mixed tensors of the second order. Note that the superscript transforms like a contravariant vector and the subscript transforms like a covariant vector. An example of the mixed second order tensor is the Kronecker delta σ_n^m

$$\tilde{\delta}_n^m = \frac{\partial \tilde{x}^m}{\partial x^r} \frac{\partial x^s}{\partial \tilde{x}^n} \delta_s^r = \frac{\partial \tilde{x}^m}{\partial x^r} \frac{\partial x^r}{\partial \tilde{x}^n} = \delta_n^m \tag{2.7.4}$$

We will always use superscripts to denote contravariant behavior and subscripts to denote covariant behavior. In the above, the second-order tensors have been created by multiplication in order to demonstrate the transformation laws. However, second order tensors are not necessarily formed by multiplications since the only requirement is that the system obey the equations 2.7.1, 2.7.2 or 2.7.3.

2.8 HIGHER ORDER TENSORS

Tensors of higher order can be treated by extending the line of reasoning described in the previous section. The important feature is the form of the transformation law. For example, a third order mixed tensor is one which obeys the transformation law.

$$a_{sr}^r(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = \frac{\partial \tilde{x}^r}{\partial x^m} \frac{\partial x^n}{\partial \tilde{x}^s} \frac{\partial x^p}{\partial \tilde{x}^t} a_{np}^m(x^1, x^2, x^3)$$

Such a tensor is said to be contravariant to the first order and covariant to the second order. The transformation law for tensors of still higher order follow the same pattern.

2.9 PROPERTIES OF TENSORS

The components of a tensor in the new variables, $\tilde{x}^r$ are linear combinations of the variables x^r. Consequently, if all the components a tensor vanish in any particular coordinate system, then they vanish in all coordinate systems. Or, if the components are zero at some particular point, then they are zero at this same point for all coordinate systems. This is an extremely important property of tensors.

2.10 SYMMETRY AND SKEW-SYMMETRY

The positional order of the indices in a tensor is important. Thus the component a^{mn} is not necessarily the same as component a^{nm}. This is clearly seen if the tensor system a^{mn} is written out in matrix form and we will see that a^{nm} is the transpose of a^{mn}. If it turns out that they are the same, then it means that the tensor is symmetric. Also, it is trivial to prove by a simple application of the summation notation that the symmetrical properties of a tensor are unchanged if new coordinates are used. Furthermore, it is generally not meaningful to define symmetry with respect to two indices one of which is contravariant and the other is covariant. A completely symmetric third order tensor must satisfy the relations:

$$a_{mnp} = a_{nmp} = a_{pmn} = a_{pnm} = a_{npm} = a_{mpn}$$

A tensor a_{mn} is said to be skew-symmetric if the interchange of indices alters the sign of the component but not its magnitude. A completely skew-symmetric system of the third order must satisfy the relations:

$$a_{mnp} = -a_{nmp} = a_{npm} = -a_{mpn} = a_{pmn} = -a_{pnm}$$

2.11 ADDITION, SUBTRACTION AND MULTIPLICATION OF TENSORS

The algebraic operations of addition and subtraction applied to tensors can have meaning if and only if the tensors which are added or subtracted are of the same order and type. Thus, it is meaningless to consider the addition of a_m and b_mn or a_m and b^n. If we have two triple systems a^r_{st} and b^r_{st} then we can define a new system c^r_{st} as:

$$c^r_{st} = a^r_{st} + b^r_{st}$$

or

$$b^r_{st} = c^r_{st} - a^r_{st}$$

We have already used the operation of multiplication on two tensors earlier. Thus if a^r_s is a second order system and b^r_{mn} is a third order system, then a fifth order system can be formed by multiplying each component or either system by every component of the other system to yield:

$$c^{rt}_{smn} = a^r_s b^t_{mn}$$

This type of multiplication is called outer multiplication and the resulting tensor is called the outer product of a^r_s and b^t_{mn}. The familiar vector or cross product of vector analysis is outer multiplication applied to two first order tensors. An important point to remember is that this process of the multiplication of two tensors produces another tensor.

2.12 INNER MULTIPLICATION - CONTRACTION OF TENSORS

The process of multiplication and contraction can be combined in an algebraic operation called inner multiplication. From the tensors a^r_s and b^t_{nm} we can obtain the following tensor:

$$f^t_{sn} = a^m_s b^t_{nm}$$

which is called the inner product. If the contraction process is applied until there is no longer any free indices, then the resulting tensor is an invariant or tensor of zero order.

2.13 QUOTIENT LAW

It is of great importance to be able to recognize a tensor without having to show directly that the tensor transformation law is satisfied since a direct verification may be inconvenient or very difficult. The quotient law is the means by which this can be accomplished. The quotient law is stated as follows. Given the following relation:

$$a(r, s, t) b^i_{rs} = c^i_t$$

then $a(r, s, t)$ has to be represented by the tensor a^{rs}_t. The proof is easy to demonstrate by showing that a^{rs}_t follows the transformation law.

CHAPTER 3

GEOMETRICAL REPRESENTATION OF TENSORS

3.1 FUNDAMENTAL OR THE METRIC TENSOR

In a three dimensional Euclidean space, the element of length or the line element will be denoted by ds and is the distance between neighboring points x and $x + dx$.

$$(ds)^2 = (dy^1)^2 + (dy^2)^2 + (dy^3)^2 = \sum_{r=1}^{3} dy^r dy^r$$

Note that the summation convention does not apply to repeated superscripts or subscripts and and hence summing on r must be specified by the summation sign. The differentials are contravariant vectors and thus:

$$dy^r = \frac{\partial y^r}{\partial x^m} dx^m$$

where x^m are general curvilinear coordinates. Hence, we obtain for the line element:

$$(ds)^2 = \sum_{r=1}^{3} \frac{\partial y^r}{\partial x^m} \frac{\partial y^r}{\partial x^n} dx^m dx^n$$

We now define the **fundamental tensor** or the **metric tensor** as:

$$g_{mn} = \sum_{r=1}^{3} \frac{\partial y^r}{\partial x^m} \frac{\partial y^r}{\partial x^n} \tag{3.1.1}$$

Hence,

$$(ds)^2 = g_{mn} dx^m dx^n$$

Since the line element, i.e., the distance between two neighboring points is invariant, using the Quotient Law we can deduce that g_{mn} is a covariant tensor of the second order. Further, equation 3.1.1 indicates that g_{mn} is symmetric. Note, also, in a Euclidean space that $(ds)^2$ is positive-definite, i.e., it is zero only if $dx_1 = dx_2 = 0$; otherwise it is always positive.

Let $\hat{G}^{mn}$ be the cofactor of the element g_{mn} in the determinant formed by the components of the metric tensor. From the theory of determinants we have the following results:

$$g = |g_{mn}| = \frac{1}{3!} e^{rst} e^{mnp} g_{rm} g_{sn} g_{rp}$$

$$\hat{G}^{mn} = \frac{1}{2!} e^{mpq} e^{nrs} g_{pr} g_{qs}$$

$$g_{mn}\hat{G}^{mp} = g_{nm}\hat{G}^{pm} = g\delta^p_n$$

Setting:

$$g^{mp} = \frac{\hat{G}^{mp}}{g} \tag{3.1.2}$$

we have:

$$g_{mn}g^{mp} = \delta^p_n \tag{3.1.3}$$

The contravariant system denoted by g^{mp} is called the contravariant metric tensor a second order contravariant tensor. The relation between g_{mn} and g^{mn} as shown in equation 3.1.3 is said to be reciprocal and tensors which obey such a relation are called **conjugate tensors**.

3.2 MAGNITUDE OF A FIRST ORDER TENSOR

The contracted tensor product denoted by $g_{mn}a^m a^n$ is a scalar quantity where a^m is a contravariant vector in the coordinate system x^r and g_{mn} is a covariant vector in the same coordinate system. Using the appropriate transformation law we can write:

$$\begin{aligned} \tilde{g}_{mn}\tilde{a}^m\tilde{a}^n &= \frac{\partial x^p}{\partial \tilde{x}^m}\frac{\partial x^q}{\partial \tilde{x}^n}\frac{\partial \tilde{x}^m}{\partial x^r}\frac{\partial \tilde{x}^n}{\partial x^s} \\ &= \delta^p_r\delta^q_s g_{pq}a^r a^s \end{aligned}$$

Hence,

$$\tilde{g}_{mn}\tilde{a}^m\tilde{a}^n = g_{pq}a^p a^q \tag{3.2.1}$$

This demonstrates that the quantity $g_{pq}a^p a^q$ is indeed a scalar or invariant. **The square root of this quantity is called the magnitude or length of the tensor.**

$$|a| = (g_{mn}a^m a^n)^{\frac{1}{2}} \tag{3.2.2}$$

In a rectangular coordinate system, the components of the fundamental tensor are all equal to unity and the length of a tensor in rectangular Cartesian coordinates is given by:

$$|a| = \sum_{r=1}^{3} a^m(y^1, y^2, y^3)a^m(y^1, y^2, y^3)$$

For this special case, the $a^m(y^1, y^2, y^3)$ are the rectangular components of the tensor. If the a^m are such that the magnitude is unity, i.e., it is a unit vector, then:

$$g_{mn}a^m a^n = 1$$

3.3 ASSOCIATED TENSORS - RAISING AND LOWERING OF INDICES

The existence of the fundamental metric g^{mn} and g_{mn} permits us to generate new tensors by the process of in a modification. For example the inner multiplication of a^{rs}_{tp} with either the contravariant or covariant metric tensor yields a tensor which is said to be associated with a^{rs}_{tp}. Thus we can have:

$$g^{mt}a^{rs}_{tp} = a^{mrs}_{.p}$$

This process of inner multiplication is more descriptively known as the process of lowering or raising the indices. In order to indicate clearly which index has been removed, a dot is placed in the space which has been vacated. A dot is also necessary as in general $a^r_{.s}$ is not the same as $a_r^{.s}$ as can be seen from the example below:

$$a^r_{.s} = g^{rm} a_{ms} a^r_{.r} = g^{sm} a_{rm}$$

However, it is evident that if a_{ms} is symmetric, then $a^r_{.s}$is equal to to $a_r^{.s}$. For example:

$$a^{rs} = a^{rm} g^{sn} a_{mn}$$

All of the tensors constructed by inner multiplication with the fundamental metric tensors are said to be associated to the given tensor. Geometrically, as associated tensors can be interpreted as representing the same tensor in different coordinates reference frames. There are two special cases of interest:

1. A dummy index can be raised from its lower position and lowered from its upper position without altering the value of the term. For example: $a_{ms} b^s = a^s_{m.} b_s$
2. A free index in a tensor equation can be raised or lowered wherever it occurs to yield an equivalent equation. For example:

$$a_{rst} = b^p_{rs} c_{tp}$$

can be operated as follows:

$$g^{rm} a_{rst} = g^{rm} b^p_{rs} c_{tp}$$

to yield:

$$a^m_{.st} = b^{mp}_{.s} c_{tp}$$

3.4 BASE VECTORS

In this section we will interpret some of the results obtained thus far in terms of ordinary vector analysis. This will enable us to easily draw pictures of some of our results. It turns out that a judicious combination of vector analysis and tensor analysis is optimal to get the desired results.

Let us recall that y^i denotes orthogonal Cartesian coordinates and x^i denotes general curvilinear coordinates. A bar will be used to denote vectors in the ordinary sense of the word. Let $\bar{r}(x^1, x^2, x^3)$ denote the position vector to a point p from point 0 which is the origin of the y^i coordinate frame and let $\bar{r} + \overline{dr}$ denote the position vector to a neighboring point q. The rectangular Cartesian components of $\bar{r}$ are y^1, y^2, y^3 and this is written as:

$$\bar{r} = y^n \bar{i}_n$$

where i_n are the unit vectors associated with the y^n coordinate system. The transformation equations from the rectangular cartesian coordinates to the general curvilinear coordinates are given by:

$$x^m = x^m(y^1, y^2, y^3)$$

The differential of the position vector, i.e., the vector connecting p to q is:

$$d\bar{r} = \frac{\partial \bar{r}}{\partial x^m} dx^m$$

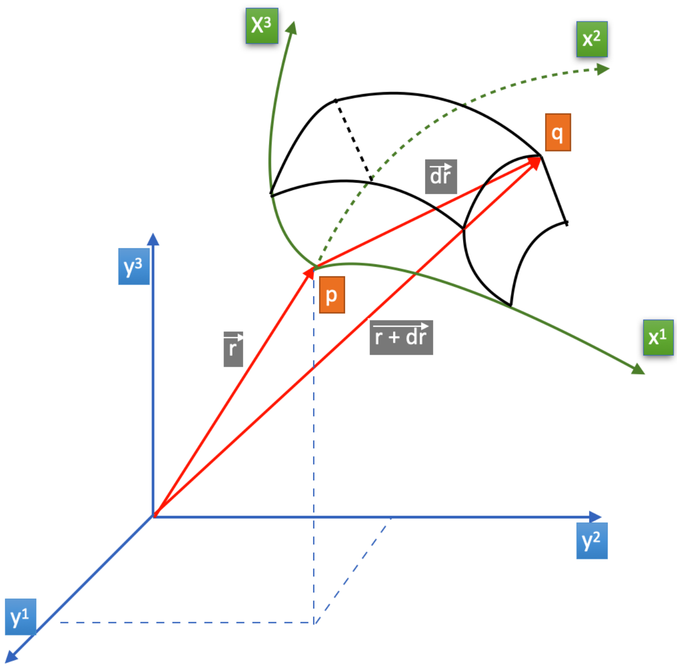

FIGURE 3.1 – Euclidean and Curvilinear Space.

and the square of the line element s:

$$(ds)^2 = d\overline{r}.d\overline{r} = \frac{\partial \overline{r}}{x^m}\frac{\partial \overline{r}}{\partial x^n} dx^m dx^n$$

where ds is the length of the vector joining p to q. The dot, as in common usage, between two vectors signifies the scalar product.

3.5 FUNDAMENTAL METRIC TENSOR

Comparing equation 3.1.1 with equation 3.4 yields the following vector formula for the **fundamental metric tensor**.

$$g_{mn} = \frac{\partial \overline{r}}{\partial x^m}\frac{\partial \overline{r}}{\partial x^n}$$

Geometrically, the vector $\frac{\partial \overline{r}}{\partial x^m}$ is a vector which is directed tangentially to the x^m coordinate curve. Such a vector is called a **base vector** and will be denoted by $\overline{g}_m$:

$$\overline{g}_m = \frac{\partial \overline{r}}{\partial x^m}$$

These are three independent base vectors in **Euclidean** space and all other vectors in this space can be expressed as a linear combination of three independent base vectors. The components of the fundamental metric tensor are now seen to be the scalar product of the base vectors.

$$g_{mn} = \overline{g}_m \cdot \overline{g}_n \tag{3.5.1}$$

Also, equation 3.5.1 can be written as:

$$d\overline{r} = dx^m \overline{g}_m \tag{3.5.2}$$

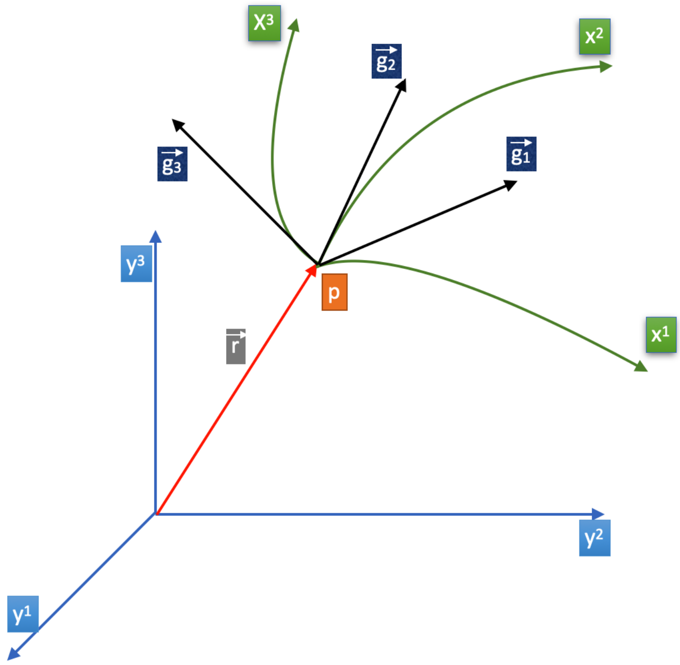

FIGURE 3.2 – Orthogonality of Vectors.

In orthogonal Cartesian coordinates, this becomes:

$$d\bar{r} = dy^m \bar{i}_m \tag{3.5.3}$$

where the $\bar{i}_m$ are the base vectors associated with orthogonal Cartesian coordinates. We recall that the $\bar{i}_m$ are unit vectors which are constant throughout Euclidean space. On substitution of the transformation equation for dy^m in equation 3.5.3 we have:

$$d\bar{r} = \frac{\partial y^m}{\partial x^n} dx^n \bar{i}_m \tag{3.5.4}$$

Comparing equation 3.5.2 with equation 3.5.4 yields, since dx^m is arbitrary, the following transformation law for base vectors:

$$\boxed{\bar{g}_m(x^1, x^2, x^3) = \frac{\partial y^n}{\partial x^m} \bar{i}_n} \tag{3.5.5}$$

The relation given by equation 3.5.5 is of the same form as the covariant transformation law which governs the first order covariant tensors. In order to ascribe a geometrical meaning to the contravariant fundamental metric tensor, g^{mn}, we proceed by first defining a set of contravariant base vectors $\bar{g}^n$as below:

$$\boxed{\bar{g}_m \cdot \bar{g}^n = \delta^n_m} \tag{3.5.6}$$

So,

$$\bar{g}_1 \cdot \bar{g}^1 = 1$$
$$\bar{g}_2 \cdot \bar{g}^1 = 0$$
$$\bar{g}_3 \cdot \bar{g}^1 = 0$$

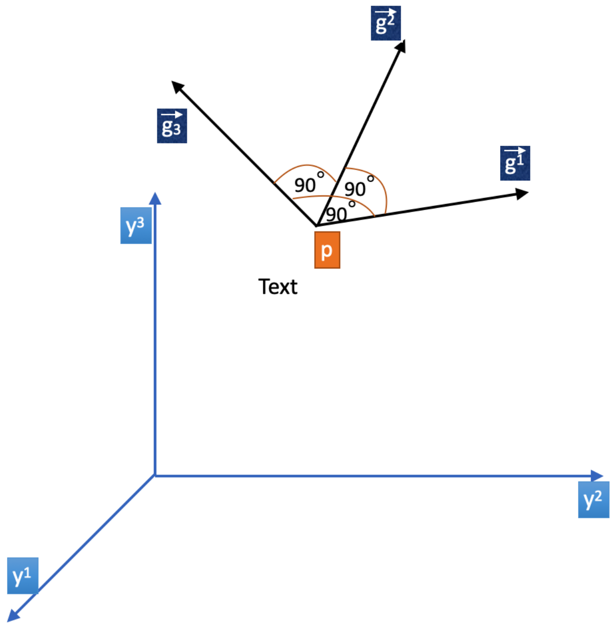

FIGURE 3.3 – Euclidean and Curvilinear Space.

Since $\overline{g}_1$ is perpendicular to both $\overline{g}_2$ and $\overline{g}_3$, we can write:

$$\overline{g}^1 = \lambda \overline{g}_2 \times \overline{g}_3 \tag{3.5.7}$$

where x denotes the operation of vector multiplication and λ is a scalar which is determined by taking the scalar product of equation 3.5.7 with $\overline{g}_1$.

$$\lambda = \frac{1}{\overline{g}_1 \cdot \overline{g}_2 \times \overline{g}_3}$$

We can then write:

$$\overline{g}^1 = \frac{\overline{g}_2 \times \overline{g}_3}{\sqrt{g}}$$
$$\overline{g}^2 = \frac{\overline{g}_3 \times \overline{g}_1}{\sqrt{g}}$$
$$\overline{g}^3 = \frac{\overline{g}_1 \times \overline{g}_2}{\sqrt{g}}$$

where $\sqrt{g}$ is the triple scalar product given by:

$$\sqrt{g} \equiv \overline{g}_1 \cdot \overline{g}_2 \times \overline{g}_3$$

Similarly, the covariant base vectors can be expressed in terms of the contravariant base vectors as follows:

$$\overline{g}_1 = \sqrt{g}(\overline{g}^2 \times \overline{g}^3)$$
$$\overline{g}_2 = \sqrt{g}(\overline{g}^3 \times \overline{g}^1)$$
$$\overline{g}_3 = \sqrt{g}(\overline{g}^1 \times \overline{g}^2)$$

where,

$$\frac{1}{\sqrt{g}} = \overline{g}^1 \cdot \overline{g}^2 \times \overline{g}^3$$

We can then easily show:

$$(\overline{g}_2 \times \overline{g}_3) \cdot (\overline{g}^2 \times \overline{g}^3) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

because:

$$\overline{g}_1 \cdot \overline{g}^1 = 1$$

The differential of the position vector $d\overline{r}$ can be expressed in terms of the contravariant base vectors as:

$$d\overline{r} = dx_m \overline{g}^m$$

where dx_m are the appropriate covariant components of $d\overline{r}$. The line element can also be written in terms of the covariant metric tensor as follows:

$$(ds)^2 = d\overline{r} \cdot d\overline{r} = \overline{g}^m \cdot \overline{g}^n dx_m dx_n = g^{mn} dx_m dx_n$$

Hence, the components of the contravariant metric tensor are seen to be the scalar product of the contravariant base vectors.

$$\boxed{g^{mn} = \overline{g}^m \cdot \overline{g}^n} \tag{3.5.8}$$

If the curvilinear coordinates have the very desirable feature of being orthogonal, then the results of this section become greatly simplified. The orthogonality feature means that the base vectors $\overline{g}^m$ are mutually orthogonal and hence the fundamental metric tensor becomes:

$$g_{mn} = \begin{vmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{vmatrix} = 1$$

and the determinant g has the value:

$$g = g_{11} g_{22} g_{33}$$

The vector product $\overline{g_2} \times \overline{g_3}$ in these special circumstances will result in a vector parallel to $\overline{g_1}$ and hence the contravariant base vectors become:

$$\overline{g^1} = \frac{\overline{g}_1}{g_{11}}$$

$$\overline{g^2} = \frac{\overline{g}_2}{g_{22}}$$

$$\overline{g^3} = \frac{\overline{g}_3}{g_{33}}$$

The contravariant metric tensor will then assume the form:

$$g^{mn} = \begin{vmatrix} \frac{1}{g_{11}} & 0 & 0 \\ 0 & \frac{1}{g_{22}} & 0 \\ 0 & 0 & \frac{1}{g_{33}} \end{vmatrix}$$

Each set of the three contravariant and covariant base vectors are linearly independent and hence non co-planar.

3.6 GEOMETRICAL REPRESENTATION OF FIRST ORDER TENSORS

It can be shown that any vector $\bar{a}$ can be resolved into three linearly independent components which are directed along the covariant base vectors, i.e.,

$$\bar{a} = a^m \bar{g}_m \tag{3.6.1}$$

The familiar scheme of resolving a vector into rectangular Cartesian components is a special case of equation 3.6.1 wherein the base vectors are the unit vectors i_m associated with the y^m coordinate system. The a_m quantities defined in equation 3.6.1 can be

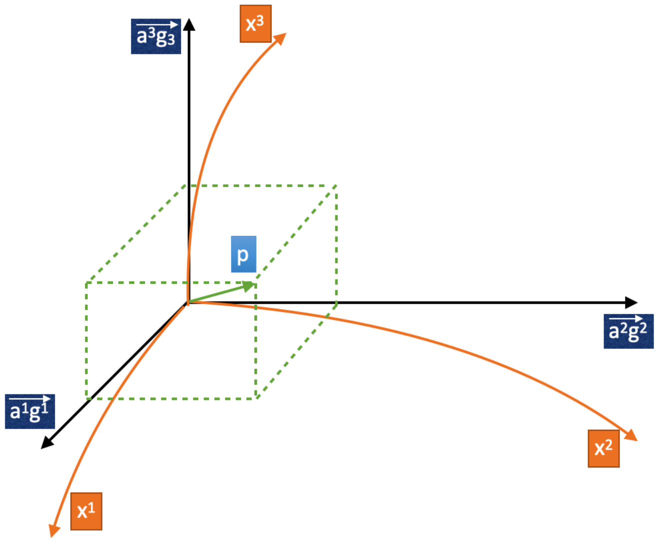

FIGURE 3.4 – Components of contravariant tensor.

considered as the components of the contravariant tensor and hence can be called the contravariant components of the vector $\bar{a}$. The magnitude of $\bar{a}$ is given by the scalar product, i.e.,

$$|\bar{a}| = [a^m g_m \cdot a^n g_n]^{\frac{1}{2}} = [a^m a^n \, g_m \cdot g_n]^{\frac{1}{2}} = [a^m a^n \, g_{mn}]^{\frac{1}{2}} \tag{3.6.2}$$

The above equation is the same as 3.2.2. In other words, the contravariant tensor of the first order denoted by the tensor symbol, a^m, designates exactly the same entity denoted by the symbol, $\bar{a}$, which represents the vector in the usage of ordinary vector analysis. Thus, a one to one correspondence between much of the vector analysis and tensor analysis has been and can be established.

It is important to observe that the a^m quantities, which we have called the components of the contravariant tensor, are not the usual components of the ordinary vector analysis. This means that the tensor components, a^m, will generally not possess the proper dimensions of the quantity being represented, and further, the different components will have different dimensions. The components which are dimensionally correct are called the **physical components** in contradistinction to the tensor components. Thus the physical components of $\bar{a}$ are the lengths of the parallelepiped which encloses $\bar{a}$. For example, the physical component of $\bar{a}$ which is directly along the g_1 direction is:

$$|a^1 \bar{g}_1| = [a^1 g_1 \cdot a^1 g_1]^{\frac{1}{2}} = a^1 \sqrt{g_{11}}$$

whereas the tensor component is merely a^1. In general, the physical component of the r_{th} component of a contravariant tensor of the first order is given by the following relation.

$$|a^r \bar{g}_r| = a^r \sqrt{g_{rr}} \quad \text{Note: no sum over } r$$

where g_{rr} (no sum) is a component of the fundamental metric tensor.

$$g_{rr} = \bar{g}_r \cdot \bar{g}_r \quad \text{Note: no sum over } r$$

A more meaningful representation of the connection between the physical and tensor components is arrived at by recalling that in elementary vector analysis, a vector is resolved into components parallel to unit tangent vectors. The unit tangent vectors are simply obtained by dividing the base vector $\bar{g}_m$ by its length g_{mm} (no sum), i.e.,

$$t_m = \frac{\bar{g}_m}{\sqrt{g_{mm}}} \quad \text{(no sum)}$$

Hence, equation 3.6.1 can be written as:

$$\bar{a} = \sqrt{g_{11}}a^1\left(\frac{\bar{g}_1}{\sqrt{g_{11}}}\right) + \sqrt{g_{22}}a^2\left(\frac{\bar{g}_2}{\sqrt{g_{22}}}\right) + \sqrt{g_{33}}a^3\left(\frac{\bar{g}_3}{\sqrt{g_{33}}}\right)$$

The three terms enclosed in parenthesis are unit tangent vectors t_1, t_2 and t_3.

The vector $\bar{a}$, which has been expressed as a linear combination of the covariant base vectors in equation 3.6.1 can also be expressed as a linear combination of the contravariant base vectors as:

$$\bar{a} = a_m \bar{g}^m \tag{3.6.3}$$

where a_1, a_2 and a_3 are the covariant components of the vector $\bar{a}$. These covariant components can be interpreted as the edges of a parallelepiped, as in the case of contravariant components. There is another useful interpretation of the which can be given to the covariant components of a vector. If we form the scalar product of equation 3.6.3 with the covariant base vector $\bar{g}_1$, we have:

$$\bar{g}_1 \cdot \bar{a} = a_m \bar{g}_1 \bar{g}^m = a_m \delta_1^m = a_1 \tag{3.6.4}$$

The scalar product of $\bar{g}_1 \cdot \bar{a}$ is interpreted in vector analysis as the product of the orthogonal projection of $\bar{a}$ onto the direction of $\bar{g}_1$ and the length $|\bar{g}_1|$. By definition, the scalar product is:

$$\bar{g}_1 \cdot \bar{a}^1 = |\bar{g}_1||\bar{a}|cos\theta \tag{3.6.5}$$

where θ is the angle between $\bar{g}_1$ and $\bar{a}$. By combining equations 3.6.4 and 3.6.5, we can see that $\frac{a_1}{\sqrt{g_{11}}}$ is the orthogonal projection of the vector $\bar{a}$ onto the direction of $\bar{g}_1$. This result, when generalized, reveals another geometrical representation of the covariant components of a first order tensor, namely that $\frac{a_m}{g_{mm}}$ (no sum) of a vector $\bar{a}$ is the length of the orthogonal projection of the vector $\bar{a}$ onto the tangent to the x^m coordinate curve at the point p (see figure 3.5). The vector $\bar{a}$ has been represented in two different coordinate frames, refer equations 3.6.1 and 3.6.3. Hence,

$$\bar{a} = a^m \bar{g}_m = a_m \bar{g}^m \tag{3.6.6}$$

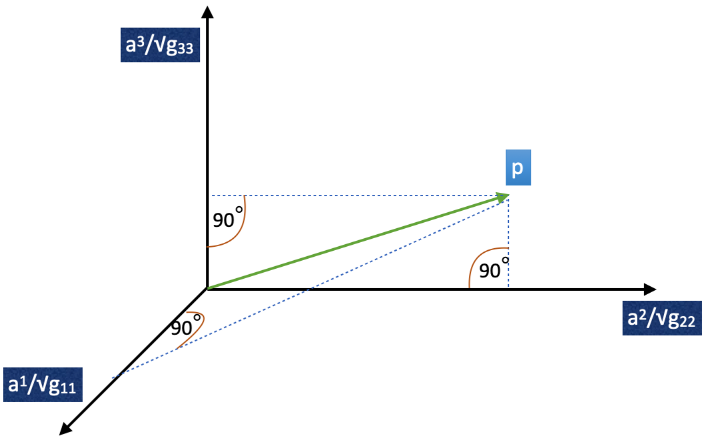

FIGURE 3.5 – Components of covariant tensor.

The scalar product of equation 3.6.6 with $\bar{g}_n$ yields,

$$\bar{g}_n \cdot a^m \bar{g}_m = \bar{g}_n \cdot a_m \bar{g}^m$$

i.e.,

$$g_{nm} a^m = \delta^n_m a_m = a_n$$

Thus, the tensor process of lowering an index by contraction with the fundamental metric tensor has an equivalent operation in ordinary vector analysis. This equivalent operation is the formation of the scalar product with the covariant base vector. Similarly, the scalar product of equation 3.6.6 with g^n yields,

$$\bar{g}^n \cdot a^m \bar{g}_m = \bar{g}^n \cdot a_m \bar{g}^m$$

$$g^{nm} a_m = a^n$$

The operation is equivalent to raising an index is the formation of the scalar product with the contravariant base vector.

It should be observed that the quantities obtained by lowering the index in a^m are precisely the covariant components a_m. Geometrically, a^m and a_m are seen to represent the same vector in two base systems which are different but nonetheless related. If the base vectors g_m are orthogonal and of unit length, i.e., the base vectors of a rectangular Cartesian frame.

It has been shown in equation 3.6.2 that the scalar quantity represented in tensor analysis by $g_{mn} a^m a^n$ is proportional to the length of the vector $\bar{a}$. The scalar product of two vectors has also its counterpart in tensor analysis. Let $\bar{b}$ and $\bar{c}$ be two vectors with tensor components given by,

$$\bar{b} = b^m \bar{g}_m$$

and

$$\bar{c} = c^n \bar{g}_n$$

Then, the scalar product of these two vectors yields,

$$\bar{b} \cdot \bar{c} = g_{mn} b^m c^n \tag{3.6.7}$$

The quantity $g_m n b^m c^n$, which is a scalar because there are no free indices, is therefore precisely the scalar product of ordinary vector analysis.

Finally, let us consider the angle between the vectors $\bar{b}$ and $\bar{c}$. This angle can be calculated by the formula:

$$cos\theta = \frac{\bar{b} \cdot \bar{c}}{|b|\,|c|} \tag{3.6.8}$$

By substituting equation 3.6.7 into the numerator of equation 3.6.8 and leveraging equation 3.6.2 into the denominator, the tensor formula for θ is obtained.

$$cos\theta = \frac{g_{mn} b^m c^n}{g_{mn} b^m b^n g_{pr} c^p c^r} \tag{3.6.9}$$

Equation 3.6.9 demonstrates that the scalar quantity $g_{mn} b^m c^n$ is proportional to the angle between the contravariant tensors of the first order b^m and c^n.

CHAPTER 4

TENSOR CALCULUS

4.1 CHRISTOFFEL SYMBOLS

We will find that the equations involving the first derivatives of the fundamental metric tensor become simpler in form with the introduction of the following two symbols.

$$[mn, p] = \frac{1}{2}\left(\frac{\partial g_{np}}{\partial x^m} + \frac{\partial g_{mp}}{\partial x^n} - \frac{\partial g_{mn}}{\partial x^p}\right) \tag{4.1.1}$$

$$\Gamma^k{}_{mn} = g^{kp}[mn, p] \tag{4.1.2}$$

The symbols described by equation 4.1.1 and 4.1.13 are called **Christoffel symbols of the first and second kind**. Although, these two symbols are not tensors, the summation convention applied to a superscript which is repeated as a subscript has been retained. This means that the indices in the Christoffel symbol of the first kind are to be regarded as subscripts whereas there are one superscript and two subscripts in Christoffel symbols of the second kind. We can write:

$$g_{lk}\Gamma^k{}_{mn} = g_{lk}g^{kr}[m, r] = \delta^r_l[mn, r] = [mn, l] \tag{4.1.3}$$

Also, it is easily verified that,

$$\frac{\partial g_{mp}}{\partial x^n} = [mn, p] + [pn, m] \tag{4.1.4}$$

which can also be put in the form:

$$\frac{\partial g_{mp}}{\partial x^n} = -g_{rp}\Gamma^r{}_{mn} - g_{rm}\Gamma^r{}_{pn} \tag{4.1.5}$$

Differentiating equation **??** with respect to x^l, there results,

$$g_{mn}\frac{\partial g_{mp}}{\partial x^l} + g^{mp}\frac{\partial g_{mm}}{\partial x^l} = 0 \tag{4.1.6}$$

The inner product of equation 4.1.6 with g^{nr} yields,

$$\frac{\partial g^{rn}}{\partial x^l} = -g^{pm}\Gamma^r{}_{ml} - g^{nr}\Gamma^p{}_{kl} \tag{4.1.7}$$

Next the transformation laws for the Christoffel symbols will be determined. We proceed by differentiating with respect to a second coordinate frame, $\tilde{x}^r$, the equation,

$$\tilde{g}_{pq} = \frac{\partial x^m}{\partial \tilde{x}^p}\frac{\partial x^n}{\partial \tilde{x}^q}g_{mn} \tag{4.1.8}$$

to obtain,

$$\frac{\partial \tilde{g}_{pq}}{\partial \tilde{x}^r} = \frac{\partial g_{mn}}{\partial x^k}\frac{\partial x^m}{\partial \tilde{x}^p}\frac{\partial x^n}{\partial \tilde{x}^q}\frac{\partial x^k}{\partial \tilde{x}^r} + g_{mn}\left(\frac{\partial x^m}{\partial \tilde{x}^p}\frac{\partial^2 x^n}{\partial \tilde{x}^q \tilde{x}^r} + \frac{\partial x^n}{\partial \tilde{x}^q}\frac{\partial^2 x^m}{\partial \tilde{x}^p \tilde{x}^r}\right) \tag{4.1.9}$$

where the tilde over the metric tensor, $\tilde{g}_{pq}$, indicates the independent variables are $\tilde{x}^1$, $\tilde{x}^2$, $\tilde{x}^3$. By suitably interchanging the free and dummy indices in equation 4.1.9, there will appear two additional equations of the same type.

$$\frac{\partial \tilde{g}_{rq}}{\partial \tilde{x}^q} = \frac{\partial g_{kj}}{\partial x^i}\frac{\partial x^i}{\partial \tilde{x}^p}\frac{\partial x^j}{\partial \tilde{x}^q}\frac{\partial x^k}{\partial \tilde{x}^r} + g_{ij}\left(\frac{\partial x^i}{\partial \tilde{x}^r}\frac{\partial^2 x^j}{\partial \tilde{x}^q \tilde{x}^p} + \frac{\partial x^j}{\partial \tilde{x}^q}\frac{\partial^2 x^i}{\partial \tilde{x}^p \tilde{x}^r}\right) \tag{4.1.10}$$

$$\frac{\partial \tilde{g}_{pr}}{\partial \tilde{x}^q} = \frac{\partial g_{ik}}{\partial x^j}\frac{\partial x^i}{\partial \tilde{x}^p}\frac{\partial x^j}{\partial \tilde{x}^q}\frac{\partial x^k}{\partial \tilde{x}^r} + g_{ij}\left(\frac{\partial x^i}{\partial \tilde{x}^r}\frac{\partial^2 x^j}{\partial \tilde{x}^q \tilde{x}^r} + \frac{\partial x^j}{\partial \tilde{x}^r}\frac{\partial^2 x^i}{\partial \tilde{x}^p \tilde{x}^q}\right) \tag{4.1.11}$$

Now if equation 4.1.9 is subtracted from the sum of equations 4.1.10 and 4.1.11, and the Christoffel symbol of the first kind is introduced, there results,

$$[\widetilde{pq}, r] = [ij,k]\frac{\partial x^i}{\partial \tilde{x}^p}\frac{\partial x^j}{\partial \tilde{x}^q}\frac{\partial x^k}{\partial \tilde{x}^r} + g_{ij}\frac{\partial x^i}{\partial \tilde{x}^r}\frac{\partial^2 x^j}{\partial \tilde{x}^p \tilde{x}^q} \tag{4.1.12}$$

where the tilde over the Christoffel symbol indicates that the independent variables are $\tilde{x}^1$, $\tilde{x}^2$, $\tilde{x}^3$. This is the transformation law for Christoffel symbol of the first kind. It is evident that the system represented by [mn,p] is not a tensor unless the coordinate transformation is such as to make the second term of equation 4.1.12 vanish.

To obtain the transformation law for the Christoffel symbol of the second kind we first write the transformation law for the contravariant tensor.

$$\tilde{g}^{rm} = g^{hl}\frac{\partial \tilde{x}^r}{\partial x^h}\frac{\partial \tilde{x}^m}{\partial x^l} \tag{4.1.13}$$

Next, the left and right hand sides of equation 4.1.12 by the left and right sides of equation 4.1.13 respectively, and summed on r.
introduction of the Christoffel symbol of the second kind.

$$\Gamma^{\tilde{m}}{}_{pq} = g^{hl}[ij,k]\frac{\partial x^i}{\partial x^p}\frac{\partial x^j}{\partial x^q}\delta^k_l\frac{\partial \tilde{x}^m}{\partial x^h} + g^{hl}g_{ij}\delta^i_l\frac{\partial \tilde{x}^m}{\partial x^n}\frac{\partial^2 \tilde{x}^j}{\partial \tilde{x}^p \partial \tilde{x}^q} \tag{4.1.14}$$

which becomes

$$\Gamma^{\tilde{m}}{}_{pq} = \Gamma^h{}_{ij} = \frac{\partial x^i}{\partial \tilde{x}^p}\frac{\partial x^j}{\partial \tilde{x}^q}\frac{\partial \tilde{x}^m}{\partial x^h} + \frac{\partial \tilde{x}^m}{\partial x^h}\frac{\partial^2 \tilde{x}^h}{\partial \tilde{x}^p \partial \tilde{x}^q} \tag{4.1.15}$$

This indicates that the Christoffel symbol of the second kind is not a tensor unless the coordinate transformation is such that the second term of equation 4.1.15 vanishes.

A formula for the transformation law of the mixed second derivatives can be obtained by multiplying equation 4.1.15 by $\frac{\partial x^s}{\partial \tilde{x}^m}$ and summing on m. The formula is:

$$\frac{\partial^2 x^s}{\partial \tilde{x}^p \partial \tilde{x}^q} = \Gamma^{\tilde{m}}{}_{pq}\frac{\partial x^s}{\partial \tilde{x}^m} - \Gamma^{\tilde{s}}{}_{ij}\frac{\partial x^i}{\partial \tilde{x}^p}\frac{\partial x^j}{\partial \tilde{x}^q} \tag{4.1.16}$$

It will be informative and useful to derive the vector formulae for Christoffel symbols. To this end we differentiate equation **??** to obtain

$$\frac{\partial g_{mn}}{\partial x^p} = \frac{\partial \bar{g}_m}{\partial \tilde{x}^p}\cdot \bar{g}_n + \frac{\partial \bar{g}_n}{\partial \tilde{x}^p}\cdot \bar{g}_m \tag{4.1.17}$$

Since the order of differentiation is unimportant the indices $\frac{\partial \bar{g}_m}{\partial x^p}$ can be interchanged. This can be shown by the following bit of algebraic manipulation:

$$\frac{\partial \bar{g}_m}{\partial x^n} = \frac{\partial}{\partial x^n}\left(\frac{\partial \bar{r}}{\partial x^m}\right) = \frac{\partial}{\partial x^m}\left(\frac{\partial \bar{r}}{\partial x^n}\right) = \frac{\partial \bar{g}_n}{\partial x^m} \tag{4.1.18}$$

The right hand side of the equation 4.1.1 can be obtained by adding together the three equations of the form given by 4.1.17 in which m,n and p have been properly permuted. We will obtain the following vector analysis definition of the Christoffel symbol of the first kind:

$$[mn,p] = \frac{\partial \bar{g}_m}{\partial x^n} \cdot \bar{g}_p \tag{4.1.19}$$

In arriving at this formula, equation 4.1.18 has been used.

It follows from the reciprocal relation equations 3.5.6 and 4.1.19 that

$$[mn,k]\bar{g}^k = \frac{\bar{g}_m}{x^n} \tag{4.1.20}$$

and hence

$$[mn,k]\bar{g}^k \cdot \bar{g}^r = [mn,k]g^{kr} = \frac{\partial \bar{g}_m}{\partial x^n} \cdot \bar{g}^r \tag{4.1.21}$$

Thus, upon comparison of equation 4.1.21 with equation 4.1.13 we have the following vector analysis definition for the Christoffel symbol of the second kind:

$$\Gamma^r{}_{mn} = \frac{\partial \bar{g}_m}{\partial x^n} \cdot \bar{g}^r = \frac{\partial \bar{g}_n}{\partial x^m} \cdot \bar{g}^r \tag{4.1.22}$$

Again, by making use of the reciprocal relation in equation 3.5.6 and the above, it follows that.

$$\bar{g}_p \Gamma^p{}_{mn} = \frac{\partial \bar{g}_m}{\partial x^n} \tag{4.1.23}$$

If we differentiate the reciprocal relation, equation 3.5.6 with respect to x^m, we obtain.

$$\frac{\partial \bar{g}_r}{\partial x^m} \cdot \bar{g}_n + \bar{g}^r \cdot \frac{\partial \bar{g}_n}{\partial x^m} = 0 \tag{4.1.24}$$

Therefore, the Christoffel symbol of the second kind is also given by:

$$\Gamma^r{}_{mn} = -\frac{\partial \bar{g}_r}{\partial x^m} \cdot \bar{g}_n \tag{4.1.25}$$

and the companion to equation **??** is

$$\bar{g}^p \Gamma^r{}_{mp} = -\frac{\partial \bar{g}^r}{\partial x^m} \tag{4.1.26}$$

There are some additional useful formulae which will be introduced at this point. By contracting the Christoffel symbol of the second kind, there is obtained

$$\Gamma^k{}_{kn} = \frac{1}{2} g^{kp}\left(\frac{\partial g_{np}}{\partial x^k} + \frac{\partial g_{kp}}{\partial x^n} - \frac{\partial g_{kn}}{\partial x^p}\right) = \frac{1}{2} g^{kp} \frac{\partial g_{np}}{\partial x^k} \tag{4.1.27}$$

If $\hat{G}^{rs}$ is the cofactor of g_{rs} in the determinant formed by the components of the fundamental metric tensor, then the expansion of the determinant in terms of the r_{th} row is.

$$g = g_{rs}\hat{G}^{rs}\text{no sum on r, sum on s only} \tag{4.1.28}$$

and hence,

$$\frac{\partial g_{rn}}{\partial x^p} = g_{rs}\frac{\partial \hat{G}^{rs}}{\partial x^{rn}} + \frac{\partial g^{rs}}{\partial g^{rn}}\text{sum on s only} \tag{4.1.29}$$

Since the cofactors G^{rs} (r is specified, s = 1,2,3) will not contain the specific component g_{rn}, the first term of equation 4.1.29 is zero. Additionally, the $g'_{rn}s$ are independent and hence equation 4.1.29 becomes

$$\frac{\partial g}{\partial g_{rn}} = \frac{\partial g_{rs}}{\partial g_{rn}}\hat{G}^{rs} = \delta^n_s G^{rn} \tag{4.1.30}$$

There is also the relation

$$\frac{\partial g}{\partial g^m} = \frac{\partial g}{\partial g_{rs}}\frac{\partial g_{rs}}{\partial x^m} = \hat{G}^{rs}\frac{\partial g_{rs}}{\partial x^m} \tag{4.1.31}$$

and if it is remembered that

$$g^{rs} = \frac{\hat{G}^{rs}}{g} \tag{4.1.32}$$

then

$$\frac{\partial g}{\partial x^m} = gg^{rs}\frac{\partial g_{rs}}{\partial x^m} \tag{4.1.33}$$

By combining equation 4.1.33 with equation 4.1.27, we obtain the very useful result that

$$\frac{\partial g}{\partial x^m} = 2g\Gamma^r{}_{rm} \tag{4.1.34}$$

or in another form

$$\frac{\partial \sqrt{g}}{\partial x^m} = \sqrt{g}\Gamma^r{}_{rm} \tag{4.1.35}$$

4.2 COVARIANT DIFFERENTIATION

We have observed in section ? that the derivatives of tensor of zero order, i.e., a scalar, are the components of a tensor of the first order. It will be shown in this section that this is the only case in which the derivatives of a tensor yields another tensor. However, in this section there will be an operation defined akin to ordinary differentiation which, when performed on a tensor, yields another tensor of one higher covariant order. This tensor operation is known as **covariant differentiation**

Let us consider the transformation law for a contravariant tensor

$$v^m = \frac{\partial x^m}{\partial \tilde{x}^p}\tilde{v}^p \tag{4.2.1}$$

and differentiate it with respect to x^n.

$$\frac{\partial v^m}{\partial x^n} = \frac{\partial x^m}{\partial \tilde{x}^p}\frac{\partial \tilde{x}^q}{\partial x^n}\frac{\partial \tilde{v}^p}{\partial \tilde{x}^q} + \frac{\partial^2 x^m}{\partial \tilde{x}^p \partial \tilde{x}^q}\frac{\partial \tilde{x}^q}{\partial x^n}\tilde{v}^p \tag{4.2.2}$$

Clearly, equation 4.2.2 does not conform to the tensor transformation law due to the presence of the second term. The second derivative in equation 4.2.2 can be replaced by its equivalent which is given in equation 4.1.16. The result is

$$\frac{\partial x^m}{\partial x^n} = \frac{\partial x^m}{\partial \tilde{x}^q}\frac{\partial \tilde{x}^q}{\partial x^n}\frac{\partial \tilde{v}^p}{\partial \tilde{x}^q} + \left[\frac{\partial x^m}{\partial \tilde{x}^m}\Gamma^r{}_{pq} - \frac{\partial x^i}{\partial \tilde{x}^p}\frac{\partial x^j}{\partial \tilde{x}^q}\Gamma^m{}_{ij}\right]\frac{\partial \tilde{x}^q}{\partial x^n}\tilde{v}^p \tag{4.2.3}$$

By taking cognizance of the following formulae (see equation 2.2.7)

$$\frac{\partial x^j}{\partial \tilde{x}^q}\frac{\partial \tilde{x}^q}{\partial \tilde{x}^n} = \delta^i_n \tag{4.2.4}$$

$$\frac{\partial x^i}{\partial \tilde{x}^p}\tilde{v}^p = v^i \tag{4.2.5}$$

equation 4.2.3 becomes, with a slight rearrangement,

$$\left[\frac{\partial v^m}{\partial \tilde{x}^n} + \Gamma^m{}_{in}v^i\right] = \left[\frac{\partial \tilde{v}^p}{\partial \tilde{x}^q}\Gamma^{\tilde{p}}{}_{rq}\tilde{v}^r\right]\frac{\partial x^m}{\partial \tilde{x}^q}\frac{\partial \tilde{x}^q}{\partial \tilde{x}^n} \tag{4.2.6}$$

It is evident from an examination of 4.2.6 that the quantities inside the square brackets transform according to the law for a mixed tensor of the second order. We are permitted, therefore, to introduce the following as a mixed tensor.

$$v^m_{,n} = \frac{\partial v^m}{\partial x^n} + \Gamma^m{}_{in}v^i \tag{4.2.7}$$

where the squiggle from the n is a comma and is used to symbolize the process of covariant differentiation of the tensor a^m with respect to x^n.

The transformation law of this mixed tensor is

$$v,^m_n = \frac{\partial x^m}{\partial \tilde{x}^p}\frac{\partial \tilde{x}^q}{\partial x^n}\tilde{v}^p_{,q} \tag{4.2.8}$$

as given by equation 4.2.6. The system $v,^m_n$ obtained in the manner indicated by equation 4.2.8 is said to be the result of the covariant differentiation of the tensor v^m with respect to x^n. We will use a comma placed before the n to indicate this process of covariant differentiation.

A similar process can be defined for a covariant tensor. By differentiating with respect to x^n, the equation

$$\tilde{u}_m = \frac{\partial x^p}{\tilde{x}^m}u_p \tag{4.2.9}$$

which governs the transformation of a covariant tensor fo the first order, and again making use of equation 4.1.16, there is obtained

$$\tilde{u}_{m,n} = \frac{\partial x^p}{\partial \tilde{x}^m}\frac{\partial x^q p}{\partial \tilde{x}^n}u_{p,q} \tag{4.2.10}$$

where $u_{p,q}$ is defined by the relation

$$u_{p,q} \equiv \frac{\partial u_p}{\partial x^q} - \Gamma^i{}_{pq} \tag{4.2.11}$$

The quantities $u_{p,q}$ defined in equation **??** are the components of a covariant tensor of the second order and are the result of the covariant differentiation of the covariant tensor u_p with respect to x^q.

It can be verified by differentiation of the appropriate transformation law that the covariant derivatives of higher order tensors are also tensors. Thus, the covariant derivatives of the second order tensors are defined as

$$v^{mn},k \equiv \frac{\partial v^{mn}}{\partial x^k} + \Gamma^m{}_{hk} v^{hn} + \Gamma^h{}_{nk} v^{mh} \tag{4.2.12}$$

$$v^m_{n,k} \equiv \frac{\partial v^m_n}{\partial x^k} - \Gamma^m{}_{hk} w^h_n - \Gamma^h{}_{nk} w^m_h \tag{4.2.13}$$

$$v^m_{n,k} \equiv \frac{\partial v^m_n}{\partial x^k} - \Gamma^m{}_{hk} w^h_n - \Gamma^h{}_{nk} w^m_h \tag{4.2.14}$$

An examination of equations 4.2.7, 4.2.11, 4.2.12, 4.2.13, 4.2.14 disclosed the following facts.

1. Covariant differentiation produces a tensor of one covariant higher order.
2. The first term in each definition consists of the ordinary partial derivative of the original tensor.
3. There is **added** a term involving a Christoffel symbol of the second kind for each **contravariant** index in the original tensor.
4. There is **subtracted** a term involving a Christoffel symbol of the second kind for each **covariant** index in the original tensor.

Observe also that in each term there is the same up and down arrangement of free indices; once these are arranged, the dummy indices go into the remaining slots.

It is a simple task to demonstrate that the covariant differentiation of the sum, difference, outer and inner product of tensors obey the same rules as ordinary differentiation. For example,

$$(u_{mn} w^k l), p = u_{mn,p} w^{kl} + u_{mn} w,^{kl}_p \tag{4.2.15}$$

Once again, we can add to our understanding of the covariant derivative by appealing to ordinary vector analysis. In the language of section ?, a vector $\bar{v}$ can be expressed in component form in terms of the covariant vectors $\bar{g}_m$:

$$\bar{v} = v^m \bar{g}_m \tag{4.2.16}$$

If equation 4.2.16 is differentiated with respect to x^n, we obtain

$$\frac{\partial v}{\partial x_n} = \frac{\partial \bar{g}^m}{\partial x^n} \bar{g}_m + v^m \frac{\partial \bar{g}^m}{\partial x^n} \tag{4.2.17}$$

The partial derivatives of the base vector, $\frac{\partial \bar{g}_m}{\partial x^n}$, can be replaced by means of equation 4.1.23. With a change in dummy indices, equation 4.2.17 becomes

$$\frac{\partial \bar{v}}{\partial x^n} = \left[\frac{\partial v^m}{\partial x^n} + \Gamma^m{}_{rn} v^r \right] \bar{g}_m \tag{4.2.18}$$

The expression in the square brackets is precisely the covariant derivative (see equation 4.2.7 of v^m. Therefore

$$\frac{\partial \bar{v}}{\partial x^n} = v,_n^m \bar{g}_m \tag{4.2.19}$$

Thus, we are able to interpret the covariant derivative of the contravariant tensor v^m as the components $\frac{\partial \bar{v}}{\partial x^n}$ referred to the covariant base vectors $\bar{g}_m$.

In an analogous manner, we can represent the vector $\bar{v}$ in terms of its covariant components (see equation **??**) and prove that

$$\frac{\partial \bar{v}}{\partial x^m} = u_{v,m} \bar{g}^r \tag{4.2.20}$$

In this case, the covariant derivative of the covariant tensor is seen to be the components of $\frac{\partial v}{\partial x^m}$ referred to the contravariant base vectors $\bar{g}^r$

4.3 INTRINSIC DIFFERENTIATION

In the previous section the covariant derivative has been shown to be the tensor equivalent of the partial derivative, There will be developed in the section the tensor equivalent of the total derivative. Let us consider a contravariant tensor defined along a curve C which is specified in terms of a parameter t i.e.,

$$C : x^i = x^i(t) \tag{4.3.1}$$

Frequently we are interested in the rate of change of v^m along the curve C. This is accomplished by forming the derivative with respect to the parameter t of the invariant $g_{mn} v^m v^n L$:

$$\frac{d}{dt}(g_{mn} v^m v^n) = \frac{\partial g_{mn}}{\partial x^p}^m v^n \cdot \frac{dx^r}{dt} + 2 g_{mn} v^n \frac{\partial v^n}{\partial dt} \tag{4.3.2}$$

By introducing equation 4.1.5, the above becomes

$$\frac{d}{dt}(g_{mn} v^m v^n) = v^m v^n \frac{dx^p}{dt} \left[g_r \Gamma^r{}_{mp} + g_r \Gamma^r{}_{np} \right] + 2 g_{mn} v^m \frac{dv^n}{dt} \tag{4.3.3}$$

Next, with a suitable rearrangement of the dummy indices there is obtained

$$\frac{d}{dt}(g_{mn} v^m v^n) = 2 g_{mn} v^m \left[\frac{dv^n}{dt} + v^s \Gamma{}_{sp} + \frac{dx^p}{dt} \right] \tag{4.3.4}$$

The quantity inside the square brackets is called the intrinsic derivative (after McConnell) of v^m with respect to t and will be shown to be a tensor. We represent it as

$$\frac{\delta v^m}{\partial dt} \equiv \frac{\partial d v^m}{\partial dt} + v^s \Gamma^m{}_{sp} \frac{\delta x^p}{\partial dt} \tag{4.3.5}$$

We differentiate with respect to t, the transformation law

$$v^m = \frac{\partial x^m}{\partial \bar{x}^p} \bar{v}^p \tag{4.3.6}$$

with the result

$$\frac{dv^m}{dt} = \frac{\partial \tilde{x}^m}{\partial \tilde{x}^p}\frac{d\tilde{v}^p}{dt} + \tilde{v}^p \frac{\partial^2 x^m}{\partial \tilde{x}^p \partial \tilde{x}^r}\frac{d\tilde{x}^r}{dt} \tag{4.3.7}$$

and introduce the equation 4.1.16 as well as

$$\tilde{v}^p = \frac{\partial x^p}{\partial x^n} v^n \tag{4.3.8}$$

and

$$\frac{d\tilde{x}^r}{dt} = \frac{\partial x^r}{\partial x^k}\frac{dx^k}{dt} \tag{4.3.9}$$

in the proper places with the result

$$\frac{dv^m}{dt} + v^n \Gamma^m{}_{nk}\frac{dx^k}{dt} = \frac{\partial x^m}{\partial \tilde{x}^p}\left[\frac{d\tilde{v}^p}{dt} + \tilde{v}^l \frac{d\tilde{x}^r}{dt}\right] \tag{4.3.10}$$

This is recognized as the proper transformation law for a contravariant tensor of the first order and justifies the use of the one free contravariant index for the intrinsic derivative. In a like manner it can be proved that the intrinsic derivative of a covariant vector, defined as

$$\frac{\delta u_m}{\delta t} \equiv \frac{dU_m}{dt} - u_p \frac{dx^n}{dt} \tag{4.3.11}$$

is a covariant vector

The extension of the process of intrinsic differentiation to tensors of higher order is straight forward. Thus, we write

$$\frac{\delta a^i_{jk}}{\delta t} \equiv \frac{da^i_{jk}}{\delta t}dt + a^m_{jk}\Gamma^i{}_{mn}\frac{dx^n}{dt} - a^i_{mk}\Gamma^m{}_{jn}\frac{dx^n}{dt} - a^i_{jm}\Gamma^m{}_{kn}\frac{dx^n}{dt} \tag{4.3.12}$$

It should be carefully noted that the intrinsic differentiation of a tensor leads to a tensor of the same order and type. The rules of ordinary calculus regarding differentiation of the sum, difference, and product of quantities apply also to intrinsic differentiation.

As in the case of covariant differentiation, there is an operation in vector analysis which is equivalent to tensor differentiation. By differentiating the vector $\bar{v}$ with respect to the parameter t we obtain

$$\frac{d\bar{v}}{dt} = \frac{\bar{v}}{x^n}\frac{dx^n}{dt} \tag{4.3.13}$$

The partial derivatives $\frac{\partial \bar{v}}{\partial x^n}$ is related to the covariant derivative by equation 4.2.19 and this, when substitute into equation 4.3.13 yields.

$$\frac{barv}{dt} = v,^m_n = g_m \frac{dx^n}{dt} = \left[\frac{\partial v^m}{\partial x^n} + v^r \Gamma^m{}_{rn}\right]\bar{g}_m \frac{dx^n}{dt} \tag{4.3.14}$$

We recognize that

$$\frac{dv^m}{dt} = \frac{\partial v^m}{\partial x^n}\frac{dx^n}{dt} \tag{4.3.15}$$

and hence equation 4.3.14 can be rearranged to read

$$\frac{dv^m}{dt} = \left[\frac{dv^m}{dt} + v^r \Gamma^m{}_{rn}\frac{dx^n}{dt}\right]\bar{g}_m \tag{4.3.16}$$

The quantity inside the square bracket is precisely what has been defined as the tensor derivative of v^m (c.f. equation 4.3.5. Thus equation 4.3.16 can be written as

$$\frac{d\bar{v}}{dt} = \frac{\delta v^m}{\delta t}\bar{g}_m \tag{4.3.17}$$

We are able, therefore, to interpret the tensor derivative of v^m as the components with respect to the covariant base vectors of the vector $\frac{d\bar{v}}{dt}$.

In a similar manner it can be verified that the intrinsic derivatives of a covariant tensor are the components with respect to the contravariant base vectors of the vector $\frac{d\bar{u}}{dt}$:

$$\frac{d\bar{u}}{dt} = \frac{\delta u_m}{\delta t}\bar{g}^m \tag{4.3.18}$$

4.4 RICCI'S THEOREM

Ricci's theorem states that the covariant derivatives of the fundamental tensors g_{mn} and g^{mn} are zero. The proof is quite simple. First, the tensor derivative of g_{mn}, according to the prescription given in equation 4.2.13, is

$$g_{mn.k} = \frac{\partial g_{mn}}{\partial x^k} - g_{mr}\Gamma^m{}_{nk} - g_{nr}\Gamma^n{}_{mk} \tag{4.4.1}$$

Second, the partial derivative $\frac{\partial g_{mn}}{\partial x^k}$ as given by the equation 4.1.5 is substituted into equation 4.4.1, leading to the result

$$g_{mn,k} = 0 \tag{4.4.2}$$

Similarly, it can be demonstrated that

$$g^{mn}_{,k} = 0 \tag{4.4.3}$$

which is the other part of Ricci's Theorem.

It can be demonstrated that the covariant and intrinsic derivatives of the generalized Kronecker deltas and the e-systems are also zero. A direct consequence which results in great simplification is that in calculating the covariant or intrinsic derivative of any product, the fundamental tensors, the e-systems and the Kronecker deltas can be regarded as constants wherever they occur.

4.5 NOTE ON THE COVARIANT AND INTRINSIC DERIVATIVE

We observe that if the coordinate system is rectilinear, then all the components of the fundamental metric are constants and consequently all of the Christoffel symbols are zero. An examination, therefore, of the definitions (e.g., 4.2.7, 4.2.11, 4.3.5, 4.2.11) discloses that in rectilinear coordinates the covariant derivative of a tensor is identical with its partial derivative and the intrinsic derivative is identical with the ordinary derivative. are led the fascinating and noteworthy result that any relationship in rectilinear coordinates containing the ordinary or partial derivatives of tensors can be converted to a corresponding relation which is true in all coordinate systems by merely replacing the ordinary derivative by the intrinsic derivative and the partial derivative by the covariant derivative. For example, it is a relatively simple matter to derive the equations of equilibrium in the theory of elasticity for rectangular Cartesian coordinate system. These are, in the absence of body and inertia forces.

$$\frac{\partial \sigma^{ij}}{\partial y^i} = 0 \tag{4.5.1}$$

where σ^{ij} are the usual physical components of stress in rectangular Cartesian coordinates. It is a much more difficult task to derive the same equations in spherical, or cylindrical or toroidal coordinates. However, once it has been established that the components of stress constitute a tensor, then we can immediately write the equations of equilibrium which are valid in any coordinate system by merely substituting the covariant derivative for the partial derivative. Thus, equation 4.5.1 in general curvilinear coordinates is written as

$$\tau,^{ij}_{j} = 0 \tag{4.5.2}$$

where τ^{ij} are the tensor components of stress. A different symbol has been used to emphasize the fact that these are tensor components and not physical components (see section ?). We observe that equation **??** ahs the same outward simplicity exhibited by equation 4.5.1. Nevertheless, the specialization to any coordinate system is given by the prescriptions to be found in equations 4.2.12, 4.1.1, 4.1.13 and 3.1.1.

4.6 REIMANN-CHRISTOFFEL TENSOR

The process of covariant differentiation in a manner has been defined such that the result is also a tensor. This process can be continued to obtain covariant derivatives of higher order. For example, let us examine the second covariant derivative of the covariant tensor v_j. The first covariant derivative is exhibited again for convenience:

$$v_{j,n} = \frac{\partial v}{\partial x^n} - v_l \Gamma^l{}_{jn} \tag{4.6.1}$$

In taking the covariant derivative of $v_{j,n}$, we keep in mind that it is a covariant tensor of the second order and hence equation 4.2.13 is to be followed. There results

$$v_{j,np} = \frac{\partial}{\partial x^p}(v_{j,n}) - v_{l,n}\Gamma^l{}_{jp} - v_{j,l}\Gamma^l{}_{np} \tag{4.6.2}$$

All three terms on the right-hand side of equation 4.6.2 can be expanded by introducing equation 4.6.1 to yield

$$v_{j,np} = \frac{\partial^2 v_j}{\partial x^n \partial x^p} - v_l \frac{\partial}{\partial x^p}\Gamma^l{}_{jn} - \frac{\partial}{\partial}\Gamma^l{}_{jn} - \frac{\partial v_l}{\partial x^n}\Gamma^l{}_{jp} + v_k \Gamma^l{}_{jp}\Gamma^k{}_{ln} - \frac{\partial v_j}{\partial x^l}\Gamma^l{}_{np} + \tag{4.6.3}$$

We will now investigate the question of the commutability of covariant differentiation. In other words, is the order of covariant differentiation important, i.e., does $v_{j,np} = v_{j,pn}$

? Let us interchange n and p in equation **??** and then subtract the result from equation **??** :

$$v_{j,np} - v_{j,pn} = v_l Rl_{.jnp} \tag{4.6.4}$$

The system represented by $R^l_{.jnp}$ is known as **Reimann-Christoffel** tensor and is a function of the Christoffel symbols.

$$R^l_{.jnp} = \frac{\partial}{\partial x^n}\Gamma^l_{\ jp} - \frac{\partial}{\partial x^n}\Gamma^l_{\ jn} - \Gamma^l_{\ ns}\Gamma^s_{\ jp} - \Gamma^l_{\ ps}\Gamma^l_{\ ps} \tag{4.6.5}$$

It is clear from equation 4.6.4 that the necessary and sufficient condition that the covariant differentiation be commutative is the vanishing of the Reimann-Christoffel tensors.

We note that $R^l_{.jnp}$ is indeed a tensor of the order and type indicated by the position of the indices because the quotient law can be applied to equation 4.6.4. It should be noted that it is formed exclusively from the fundamental tensor and its derivatives up to the second order since only Christoffel symbols appear in defining equation. An examination of equation 4.6.5 reveals that $R^l_{.jnp}$ is skew-symmetric with respect to the last two subscripts.

$$R^l_{.jnp} = -R^l_{.jpn} \tag{4.6.6}$$

It is clear from equation 4.6.4 that covariant differentiation is commutative if and only if the Reimann-Christoffel tensor is zero. Actually, all of our developments thus far have been assumed to occur in an Euclidean space which means that there exists a transformation of coordinates which will reduce the line element from its general form (equation **??**) to the form given by by equation 3.1. In an Euclidean space and in an orthogonal Cartesian coordinate system, therefore, the left-hand side of equation 1.25.4 becomes.

$$V_{j,np}(y^1,y^2,y^3) - V_{j,pn}(y^1,y^2,y^3) = \frac{\partial^2 v_j}{\partial y^n \partial y^p} - \frac{\partial^2 v_j}{\partial y^p \partial y^n} - \tag{4.6.7}$$

But this is identically zero since the order of partial differentiation is immaterial. Consequently, the Reimann-Christoffel tensor is identically zero in three-dimensional Euclidean space. We recall that Euclidean space is one in which a transformation can be found such that the components of the metric tensor are constants. Clearly this can be always be accomplished in the three dimensional space we live.

An associated tensor can be obtained by lowering the contravariant index:

$$R_{rjnp} = g_{rl} R^l_{.jnp} \tag{4.6.8}$$

The substitution of equation 4.6.5 into 4.6.8 yields

$$R_{rjnp} = \frac{\partial}{\partial x^n}[jp,r] - \frac{\partial}{\partial x^p}[jn,r] + \Gamma^l_{\ jn}[rp,l] - \Gamma^l_{\ jp}[rn,l] \tag{4.6.9}$$

This can be further reduced to the following:

$$R_{rjnp} = \frac{1}{2}\left(\frac{\partial^2 g_{rp}}{\partial x^j \partial x^n} + \frac{\partial^2 g_{jn}}{\partial x^r \partial x^p} + \frac{\partial^2 g_{rn}}{\partial x^j \partial x^p} - \frac{\partial^2 g_{jp}}{\partial x^r \partial x^n}\right) + \tag{4.6.10}$$

$$g^{ts}([jn,s][rp,t] - [jp,s][rn,t]) \tag{4.6.11}$$

It is observed upon examination of equation 4.6.11 that R_{rjnp} has the following special properties:

$$R_{rjnp} = -R_{jrnp} \tag{4.6.12}$$
$$R_{rjnp} = -R_{rjpn} \tag{4.6.13}$$
$$R_{rjnp} = -R_{nprj} \tag{4.6.14}$$
$$R_{rjnp} + R_{rnpj} + R_{rpjn} = 0 \tag{4.6.15}$$

Thus, the Reimann-Christoffel tensor R_{rjnp} is skew-symmetric in its first two and last two indices, and is symmetric if the first pair of indices is interchanged with the second pair. As a result, there are only six independent component of R_{rjnp} in a three dimensional Euclidean space; namely,

$$R_{3131}\ R_{3232}\ R_{1212}\ R_{3132}\ R_{3212}\ R_{3112} \tag{4.6.16}$$

It is convenient to introduce at this point the tensor S^{kl} defined by

$$S^{kl} \equiv \frac{1}{4}\epsilon^{krj}\epsilon^{lnp}R_{rjnp} \tag{4.6.17}$$

which is seen to be symmetric. The systems denoted by ϵ^{mnp} and ϵ_{mnp} are permutation tensors defined as follows:

$$\epsilon^{mnp} = \frac{1}{\sqrt{g}}e^{mnp} \tag{4.6.18}$$

$$\epsilon_{mnp} = \sqrt{g}e_{mnp} \tag{4.6.19}$$

Upon contraction with $\epsilon_{kim}\epsilon_{lqs}$ there results

$$\epsilon_{kim}\epsilon_{lqs}S^{kl} \quad = \tfrac{1}{4}\delta^{rj}_{im}\delta^{np}_{qs}R_{rjnp} \tag{4.6.20}$$
$$\tfrac{1}{2}\delta^{np}_{qs}R_{impn} = R_{imqs} \tag{4.6.21}$$

and hence

$$R_{imqs} = \epsilon_{kim}\epsilon_{lqs}S^{kl} \tag{4.6.22}$$

Therefore, in a space of three dimensions, the Reimann-Christoffel tensor can be expressed in terms of the symmetric double tensor S^{kl}.

It is easy to see on the basis of S^{kl} that R_{rjnp} has only six distinct components as stated above. Also it is clear that if $S^{kl} = 0$ then $R^{l}_{.jnp}$ is also zero.

$$a_{ij,rs} - a_{ij,sr} = a_{mj}R^{m}_{.irs} + a_{im}R^{m}_{.jrs} \tag{4.6.23}$$

$$a,^{ij}_{rs} - a,^{ij}_{sr} = -a^{mj}R^{i}_{.mrs} - a^{im}R^{j}_{.mrs} \tag{4.6.24}$$
$$a,^{m}_{nr} - a,^{m}_{rn} = a^{p}R^{m}_{.prn} \tag{4.6.25}$$