Chapter 7

Second Order Ordinary Differential Equation

A second-order equation, quite a mess,
With derivatives causing plenty of distress.
But Euler, Cauchy, and Lagrange helped evolve,
Elegant methods to solve and resolve!

7.1 Introduction

As discussed in the previous chapter, the *order* of a differential equation is defined by the highest derivative of the unknown function that appears in the equation. Accordingly, a *second-order differential equation* is one in which the second derivative of the unknown function occurs, and no derivative of higher order is present. Such equations naturally arise when the evolution of a system depends not only on the rate of change of a quantity, but also on how that rate itself varies.

Second-order differential equations occupy a position of particular importance in science and engineering. They form the mathematical foundation of many fundamental models, including mechanical vibrations, oscillatory motion, and electrical circuits containing inductance and capacitance. In these systems, the balance between inertia, restoring forces, and damping effects leads directly to second-order equations.

Beyond ordinary differential equations, second-order formulations also appear prominently in multidimensional engineering and physical models. Wave motion in solids and fluids, flow mechanics, and electromagnetic phenomena are governed by equations that are second order in space and time. Notable examples include Maxwell's equations in electromagnetism and the Schrödinger equation in quantum and nuclear physics, both of which encapsulate essential physical principles through second-order differential relationships.

This chapter focuses on second-order ordinary differential equations and the methods used to analyze and solve them. Emphasis is placed on equations with constant coefficients, classification of solution behavior, and the interpretation of solutions in physical contexts. Understanding second-order differential equations is essential for modeling dynamic systems and serves as a gateway to more advanced topics in applied mathematics, physics, and engineering analysis.

7.2 2nd Order Linear ODE

The **standard form** of a **2nd Order ODE** is:

$$\boxed{y^{''} + p(x)y^{'} + q(x)y = r(x)} \text{ It is linear in } y, y' \text{ and } y'' \tag{7.1}$$

If $r(x) = 0$, the ODE is homogeneous, else it is non-homogeneous. When the coefficients a and b are constant:

$$\boxed{y^{''} + ay^{'} + by = 0} \tag{7.2}$$

Choose $e^{\lambda x}$ as a solution and substitute in the homogeneous ODE.

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0 \implies \lambda^2 + a\lambda + b = 0 \implies \lambda = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right)$$

$$\boxed{y_h = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}} \text{ (general solution to the homogeneous ODE)}$$

y_1, corresponding to λ_1, and y_2, corresponding to λ_2, are **linearly independent** and are called **basis** of solutions. The **superposition principle** also called the **linearity principle,** i.e., the homogeneous solution is a combination of y_1 and y_2 **is true only for linear homogeneous ODE.** The arbitrary constants c_1 and c_2 are determined from the **initial conditions:**

$$y(x_0) = k_0 \quad y^{'}(x_0) = k_1$$

A **particular solution** is obtained if we assign specific values to c_1 and c_2.

7.3 Lagrange's Method of Reduction of Order

Consider a linear homogeneous 2nd Order ODE in its **standard form**:

$$\boxed{y^{''} + p(x)y^{'} + q(x)y = 0} \tag{7.3}$$

If y_1 is a **basis solution**, we can find y_2 as follows:

$$y_1 = e^{\lambda x}$$

Let $y_2 = u\, y_1$

$$y_2' = u' y_1 + u y_1'$$

$$y_2'' = u'' y_1 + 2u' y_1' + u y_1''$$

Substituting into $y'' + py' + qy = 0$:

$$(u'' y_1 + 2u' y_1' + u y_1'') + p(u' y_1 + u y_1') + q(u y_1) = 0$$

$$u''y_1 + (2y_1' + py_1)u' + (y_1'' + py_1' + qy_1)u = 0$$

$$u'' + u'\frac{2y_1' + py_1}{y_1} = 0$$

Let $U = u'$

$$U' + U\left(\frac{2y_1'}{y_1} + p\right) = 0$$

$$\frac{U'}{U} = -\left(\frac{2y_1'}{y_1} + p\right)$$

$$\int \frac{U'}{U}\,dx = -\int\left(\frac{2y_1'}{y_1} + p\right)dx$$

$$\ln|U| = -2\ln|y_1| - \int p\,dx$$

$$\ln|Uy_1^2| = -\int p\,dx$$

$$Uy_1^2 = e^{-\int p\,dx}$$

$$\boxed{U = \frac{1}{y_1^2}e^{-\int p\,dx}}$$

$$\boxed{u = \int U\,dx}$$

$$\boxed{y_2 = y_1\int U\,dx}$$

$$\boxed{y_2 = y_1\int \frac{1}{y_1^2}e^{-\int p\,dx}\,dx}$$

7.4 Homogeneous Linear ODE with Constant Coefficients

$$\begin{cases} \text{Case 1: 2 Real Roots when} & a^2 - 4b > 0 \\ \text{Case 2: Double Root when} & a^2 - 4b = 0 \\ \text{Case 3: Complex Conjugate Roots when} & a^2 - 4b < 0 \end{cases}$$

Case 1: 2 Real Roots when $a^2 - 4b > 0$. The general solution is given by:

$$y_1 = e^{\lambda_1 x} \quad y_2 = e^{\lambda_2 x} \qquad \lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right) \quad \lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right)$$

$$\boxed{y_h = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$$

Case 2: $\boxed{\lambda_1 = -\frac{a}{2},\ y_1 = e^{-\frac{ax}{2}}}$ Determine y_2 using the method of reduction of order.

$$y_1 = e^{\frac{-ax}{2}}$$
$$y_2 = y_1 \int \frac{1}{y_1^2} e^{\int -pdx} dx = e^{\frac{-ax}{2}} \int \frac{1}{\left(e^{\frac{-ax}{2}}\right)^2} e^{\int -adx} dx = e^{-\frac{ax}{2}} \int e^{ax} e^{-ax} dx = xe^{-\frac{ax}{2}}$$
$$y_h = c_1 e^{-ax/2} + c_2 x e^{-\frac{a}{2}x}$$
$$\boxed{y_h = (c_1 + c_2 x) e^{-ax/2}}$$

Case 3: $\boxed{\lambda = -\frac{a}{2} \pm \boldsymbol{i} w, w = \sqrt{|a^2 - 4b|}}$

$$y_1 = e^{\lambda_1 x} = e^{(-\frac{a}{2} + \boldsymbol{i} w)x} = e^{-\frac{ax}{2}} e^{\boldsymbol{i} wx}$$
$$y_2 = e^{\lambda_2 x} = e^{(-\frac{a}{2} - \boldsymbol{i} w)x} = e^{-\frac{ax}{2}} e^{-\boldsymbol{i} wx}$$
$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \ldots = (1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots) + i(x - \frac{x^3}{3!} - \frac{x^5}{5|} + \ldots) = cos\, x + \boldsymbol{i}\, sin\, x$$

$\boxed{e^{iwx} = cos\, wx + \boldsymbol{i}\, sin\, wx}$ (de Moivre's theorem) and $\boxed{e^{\boldsymbol{i}\pi} = -1}$ (Euler's Identity)

Any real solution is a linear combination of the real and imaginary parts:

$\boxed{y = e^{-\frac{ax}{2}} (c_1 cos\, wx + c_2 sin\, wx)}$ c_1, c_2 are constants

7.5 Euler-Cauchy Equations

The **Euler-Cauchy** equation is of the form:

$$\boxed{x^2 y'' + axy' + by = 0} \text{ where } a, b \text{ are constants} \tag{7.4}$$

Let $y = x^m \implies y' = mx^{m-1} \implies y'' = m(m-1)x^{m-2}$ and substituting

$x^2 m(m-1)x^{m-2} + axmx^{m-1} + bx^m = 0 \implies m^2 + (a-1)m + b = 0$

$$\boxed{m = \frac{1}{2}(1-a) \pm \sqrt{\frac{1}{4}(a-1)^2 - b}}$$

Case 1: Roots are distinct. The basis solutions are :

$y_1(x) = x^{m_1}$ $y_2(x) = x^{m_2}$, the general solution is given by, $\boxed{y = c_1 x^{m_1} + c_2 x^{m_2}}$

Case 2: Double roots.

$$\boxed{y_1 = x^{\frac{1}{2}(1-a)}}$$

$$y'' + \frac{a}{x}y' + \frac{(1-a)^2}{4x^2}y = 0$$

Use method of reduction of order, $y_2 = uy_1$ and with $\boxed{p = \frac{a}{x}}$

$$\boxed{U = \frac{1}{y_1^2}e^{\int -pdx}} \quad \boxed{u = \int Udx} \quad \boxed{y_2 = y_1 \int Udx} \quad \boxed{y_2 = y_1 \int \frac{1}{y_1^2}e^{\int -pdx}dx}$$

$$\int pdx = \int \frac{a}{x}dx = alnx \implies e^{\int -pdx} = e^{-alnx} = e^{lnx^{-a}} = x^{-a} = \frac{1}{x^a}$$

$$U = \frac{1}{y_1^2}\frac{1}{x^a} = \frac{1}{x^{1-a}}\frac{1}{x^a} = \frac{1}{x} \implies u = \int Udx = \int \frac{1}{x}dx = lnx$$

$$y_2 = y_1 \int Udx = x^{\frac{1}{2}(1-a)}lnx$$

$\boxed{y_h = (c_1 + c_2 lnx)x^{\frac{1}{2}(1-a)}}$ c_1, c_2 are constants

7.6 The Wronskian

Two solutions y_1 and y_2 are **linearly dependent** if their **Wronkskian** W is 0.

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = 0$$

Because if the solutions are dependent, $y_1 = ky_2,$ where k is a constant

$$\implies W(y_1, y_2) = y_1 y_2' - y_2 y_1' = ky_2 y_2' - y_2 k y_2' = 0$$

The Wronksian is expressed as a **Wronkski Determinant:**

$$\boxed{W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \tag{7.5}$$

7.7 Non-homogeneous ODE

Consider the following non-homogeneous ODE:

$$y'' + p(x)y' + q(x)y = r(x)$$

The complete solution is the sum of homogeneous (y_h) and particular (y_p)solutions.

$y(x) = y_h(x) + y_p(x)$ where $y_h = c_1 y_1 + c_2 y_2$ (general solution)

y_p is a solution of the non-homogeneous equation without any constants. A particular solution is obtained by assigning specific values to the constants. The **Method of Undetermined Coeffcients** is an approach to finding a particular solution to nonhomogeneous ODEs. If the term in $r(x)$ contains the following term, the choice for $y_p(x)$ is given by:

Term in r(x)	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$Kx^n (n = 0, 1, \dots)$	$K_n x^n + K_{n-1}x^{n-1} + \dots + K_1 x + K_0$
$k\cos wx$ or $k\sin wx$	$K\cos wx + M\sin wx$
$ke^{\alpha x}\cos wx$ or $ke^{\alpha x}\sin wx$	$e^{\alpha x}(K\cos wx + M\sin wx)$

7.8 Particular Solution by Variation of Parameters (Lagrange)

The particular solution for the standard form ODE is derived as follows:

$$y'' + p(x)y' + q(x)y = r(x)$$

Find a pair of functions $u_1(x)$ and $u_2(x)$ such that:

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \implies y_p'(x) = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$$

Set constraint, $\boxed{u_1'y_1 + u_2'y_2 = 0}$

$$y_p'(x) = u_1y_1' + u_2y_2'$$

$$y_p''(x) = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$$

Substituting,

$$(u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'') + p(u_1y_1' + u_2y_2') + q(u_1y_1 + u_2y_2) = r$$

$$(y_1'' + py_1' + qy_1)u_1 + (y_2'' + py_2' + qy_2)u_2 + (u_1'y_1' + u_2'y_2') = r$$

Since y_1 and y_2 are solutions to the homogeneous ODE,

$$u_1'y_1' + u_2'y_2' = r$$

We now have the following simultaneous equations:

$$u_1'y_1 + u_2'y_2 = 0$$

$$u_1'y_1' + u_2'y_2' = r$$

Solving,

$$u_1' = -\frac{y_2 r}{y_1y_2' - y_1'y_2} = -\frac{y_2 r}{W}$$

$$u_2' = -\frac{y_1 r}{y_1y_2' - y_1'y_2} = -\frac{y_1 r}{W}$$

$$\boxed{y_p(x) = -y_1\int \frac{y_2 r}{W}\,dx + y_2\int \frac{y_1 r}{W}\,dx}$$

7.9 SymPy

Solve the following non-homogeneous ordinary differential equation:

$$2y(x) - 3\frac{d}{dx}y(x) + \frac{d^2}{dx^2}y(x) = x^2$$

```
import sympy as sp
from IPython.display import display, Math

# Variables and function
x = sp.symbols('x')
y = sp.Function('y')

# Nonhomogeneous ODE: y'' - 3 y' + 2 y = x^2
ode = sp.Eq(sp.diff(y(x), x, 2) - 3*sp.diff(y(x), x) + 2*y(x), x**2)

# Solve full ODE
soln_full = sp.dsolve(ode)

# Extract complementary (homogeneous) and particular parts
# dsolve returns: Eq(y(x), C1*exp…()+C2*exp…() + particular)
rhs = soln_full.rhs
C1, C2 = sp.symbols('C1 C2')

# The homogeneous part is the expression containing C1, C2
soln_homo = rhs.expand().coeff(C1)*C1 + rhs.expand().coeff(C2)*C2

# The particular solution is the rest (remove terms with C1, C2)
soln_part = sp.simplify(rhs - soln_homo)

# Display results
print(sp.latex(ode))
print(sp.latex(soln_homo))
print(sp.latex(soln_part))
print(sp.latex(soln_full))

display(Math(r"\textbf{Homogeneous solution: } y_h = " + sp.latex(soln_homo)))
display(Math(r"\textbf{Particular solution: } y_p = " + sp.latex(soln_part)))
display(Math(r"\textbf{General solution: } y = " + sp.latex(rhs)))

print(sp.latex(soln_full))
soln_full
```

General solution:

$$y = C_1e^x + C_2e^{2x} + \frac{x^2}{2} + \frac{3x}{2} + \frac{7}{4}$$

Homogeneous solution:

$$y_h = C_1 e^x + C_2 e^{2x}$$

Particular solution:

$$y_p = \frac{x^2}{2} + \frac{3x}{2} + \frac{7}{4}$$

General (or Full) solution:

$$y(x) = C_1 e^x + C_2 e^{2x} + \frac{x^2}{2} + \frac{3x}{2} + \frac{7}{4}$$