Chapter 19
Mechanics

Motion begins where silence yields to force
Time traces change along a measured course
Stillness holds when balances agree
Law gives form to how the world must be

Mechanics is the branch of classical physics concerned with the equilibrium and motion of material bodies. It is divided into statics and dynamics. Dynamics is further subdivided into kinematics and dynamics proper.

19.1 Statics

Statics deals with mechanical systems in equilibrium. In this regime, all accelerations vanish and the net force acting on each body is zero.

19.1.1 Forces and Equilibrium

Let a particle be acted upon by a finite set of forces $\{\mathbf{F}_i\}$. The condition for translational equilibrium is

$$\sum_i \mathbf{F}_i = \mathbf{0}$$

For rigid bodies, rotational equilibrium must also be satisfied. Let $\mathbf{r}_i$ denote the position vector of the point of application of $\mathbf{F}_i$ relative to a chosen origin. The total moment (torque) is

$$\tau = \sum_i \mathbf{r}_i \times \mathbf{F}_i$$

The conditions for static equilibrium of a rigid body are therefore

$$\sum_i \mathbf{F}_i = \mathbf{0} \tag{19.1}$$

$$\sum_i \mathbf{r}_i \times \mathbf{F}_i = \mathbf{0} \tag{19.2}$$

19.1.2 Constraints and Reaction Forces

Mechanical constraints restrict the possible configurations of a system. In statics, constraints give rise to reaction forces which enforce equilibrium. These forces do no virtual work

$$\sum_i \mathbf{R}_i \cdot \delta\mathbf{r}_i = 0$$

where $\delta \mathbf{r}_i$ are admissible virtual displacements.

19.2 Dynamics

Dynamics concerns systems in which motion is present. It is divided into kinematics and dynamics proper.

19.2.1 Kinematics

Kinematics describes motion without reference to the forces causing it.

19.2.1.1 Position, Velocity, and Acceleration

The motion of a particle is described by a trajectory

$$\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$$

The velocity and acceleration are defined as successive time derivatives

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$$

$$\mathbf{a}(t) = \frac{d^2\mathbf{r}}{dt^2}$$

For one-dimensional motion along a straight line

$$v = \frac{dx}{dt}$$

$$a = \frac{dv}{dt}$$

19.2.1.2 Uniform and Uniformly Accelerated Motion

If the acceleration vanishes

$$a = 0$$

the velocity is constant and the position is given by

$$x(t) = x_0 + vt$$

If the acceleration is constant, $a(t) = a_0$, integration yields

$$v(t) = v_0 + a_0 t \tag{19.3}$$

$$x(t) = x_0 + v_0 t + \frac{1}{2} a_0 t^2 \tag{19.4}$$

$$v^2 = v_0^2 + 2a_0(x - x_0) \tag{19.5}$$

19.2.1.3 Relative Motion

For two particles with position vectors $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$, the relative position, velocity, and acceleration are

$$\mathbf{r}_{21} = \mathbf{r}_2 - \mathbf{r}_1 \tag{19.6}$$
$$\mathbf{v}_{21} = \mathbf{v}_2 - \mathbf{v}_1 \tag{19.7}$$
$$\mathbf{a}_{21} = \mathbf{a}_2 - \mathbf{a}_1 \tag{19.8}$$

19.2.2 Dynamics Proper

Dynamics proper relates motion to the forces producing it.

19.2.2.1 Newton's Laws of Motion

First Law In an inertial frame, a particle not acted upon by external forces moves with constant velocity

$$\mathbf{F} = \mathbf{0} \Rightarrow \mathbf{a} = \mathbf{0}$$

Second Law The net force acting on a particle of mass m is equal to the time rate of change of its momentum

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

For constant mass

$$\mathbf{F} = m\mathbf{a}$$

Third Law For every action there is an equal and opposite reaction

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

19.2.2.2 Equations of Motion

The fundamental equation of dynamics for a particle is the second-order differential equation

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}(\mathbf{r}, \mathbf{v}, t)$$

Given initial conditions

$$\mathbf{r}(t_0), \quad \mathbf{v}(t_0)$$

the motion is determined uniquely under suitable regularity conditions on $\mathbf{F}$.

19.2.2.3 Work and Energy

The work done by a force along a trajectory is

$$W = \int \mathbf{F} \cdot d\mathbf{r}$$

The kinetic energy is defined as

$$T = \frac{1}{2} m v^2$$

Using Newton's second law

$$\frac{dT}{dt} = \mathbf{F} \cdot \mathbf{v}$$

which leads to the work–energy theorem

$$\Delta T = W$$

19.2.2.4 Conservative Forces

If the force derives from a potential $V(\mathbf{r})$ such that

$$\mathbf{F} = -\nabla V$$

then the total mechanical energy

$$E = T + V$$

is conserved.

19.3 Closing Remarks

Statics characterizes equilibrium through balance of forces and moments. Kinematics describes motion as a geometric function of time. Dynamics proper relates acceleration to force through Newton's laws.

Together they constitute the classical structure of mechanics.