Chapter 27

Differential Geometry

Curves whisper truth where flat maps fail,
Angles bend as distances unveil;
Through metrics, forms, and spaces wide,
Geometry learns how worlds abide.

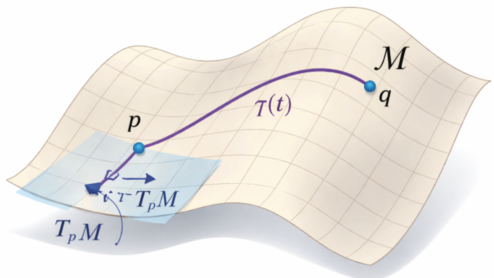
27.1 Manifolds

Differential geometry generalizes familiar geometric spaces. In elementary geometry, we work with lines, planes, and volumes embedded in Euclidean space. Differential geometry removes this restriction and studies spaces that may be curved, higher-dimensional, or not globally describable by a single coordinate system. Such spaces are called *manifolds*.

Informally, a manifold is a space that may be complicated when viewed as a whole, but which appears simple when examined locally. Just as the surface of the Earth looks flat over sufficiently small regions despite being globally curved, a manifold resembles ordinary Euclidean space in the neighborhood of every point. This local resemblance allows the tools of calculus—derivatives, integrals, and smooth functions—to be applied even when the global structure is non-Euclidean.

More precisely, while a manifold need not be a subset of $\mathbb{R}^n$, each small neighborhood of it can be described using coordinates drawn from $\mathbb{R}^n$. These local descriptions are called *charts*, and collections of compatible charts form an *atlas*. Smoothness refers to the requirement that transitions between overlapping charts are smooth functions.

In standard treatments, manifolds are assumed to satisfy mild topological conditions such as being Hausdorff and second countable. These assumptions ensure well-behaved global properties and will be taken for granted throughout.



27.1.1 Definition of a Manifold

With the intuition in place, we can state the formal definition.

A *smooth manifold* $\mathcal{M}$ of dimension n is a topological space such that each point $p \in \mathcal{M}$ has a neighborhood that is smoothly parameterized by coordinates

$$(x^1, x^2, \ldots, x^n)$$

drawn from $\mathbb{R}^n$, with smooth and invertible transition maps between overlapping coordinate charts.

27.1.2 Coordinate Transformations

On overlapping charts, the coordinates are related by smooth transformations

$$\tilde{x}^i = \tilde{x}^i(x^1, \ldots, x^n)$$

where the functions $\tilde{x}^i$ are smooth and invertible on their domains.

Differential geometry studies geometric and physical structures whose definitions and properties remain invariant under such coordinate transformations.

27.2 Curves and Tangent Vectors

27.2.1 Curves on a Manifold

A curve on a manifold $\mathcal{M}$ represents a smooth path through the space. Formally, a curve is a smooth mapping

$$\gamma : \mathbb{R} \to \mathcal{M}$$

that assigns to each real parameter value t a point $\gamma(t)$ on the manifold.

The parameter t merely orders points along the curve. It may represent time, arc length, or any other convenient parameter, but it has no intrinsic geometric meaning. As t varies smoothly, the image of $\gamma(t)$ traces out a path on $\mathcal{M}$.

Within a local chart, the curve is described by coordinate functions

$$x^i = x^i(t), \qquad i = 1, \ldots, n$$

Although these coordinate functions depend on the chosen chart, the curve itself is a geometric object independent of coordinates.

27.2.2 Tangent Vectors

Let $\gamma : \mathbb{R} \to \mathcal{M}$ be a smooth curve and let $p = \gamma(t_0)$ be a point on the curve. The tangent vector to the curve at p is defined by the derivatives of the coordinate functions at t_0:

$$v^i = \left.\frac{dx^i}{dt}\right|_{t=t_0}$$

These components define a tangent vector v at p. The collection of all tangent vectors at p forms a vector space called the *tangent space*, denoted $T_p\mathcal{M}$.

Equivalent, coordinate-independent definitions of tangent vectors exist (for example, as derivations acting on smooth functions), but the curve-based definition suffices for our purposes.

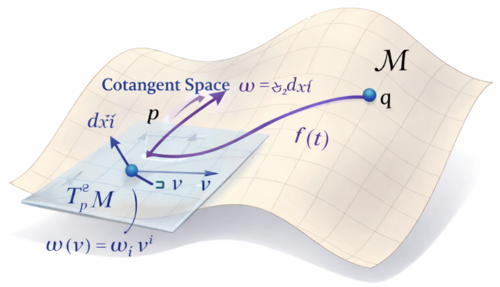
27.3 Cotangent Space

At each point $p \in \mathscr{M}$, the tangent space $T_p\mathscr{M}$ describes possible directions of motion through the point.

The *dual space* of the tangent space, denoted $T_p^*\mathscr{M}$, consists of all linear maps from the tangent space to the real numbers:

$$T_p^*\mathscr{M} = \left\{\omega : T_p\mathscr{M} \to \mathbb{R} \;\middle|\; \omega \text{ is linear}\right\}$$

This space is called the *cotangent space*. Its elements, known as *covectors* or *one-forms*, act on tangent vectors to produce real numbers. While tangent vectors encode directions of motion, cotangent vectors measure rates of change of scalar quantities along those directions.



27.3.1 Differential Forms

Differential forms generalize covectors to higher dimensions and provide a coordinate-independent framework for integration and differentiation on manifolds.

At each point $p \in \mathscr{M}$, a k-*form* is a totally antisymmetric, multilinear map

$$\omega_p : \underbrace{T_p\mathscr{M} \times \cdots \times T_p\mathscr{M}}_{k \text{ times}} \to \mathbb{R}$$

A *one-form* acts on a single tangent vector. In local coordinates, it can be written as

$$\omega = \omega_i \, dx^i$$

where $\{dx^i\}$ form a basis of the cotangent space. Gradients of scalar functions provide natural examples.

Higher-order forms are constructed using the *wedge product*, which enforces antisymmetry:

$$dx^i \wedge dx^j = -\, dx^j \wedge dx^i, \qquad dx^i \wedge dx^i = 0$$

A two-form has the local expression

$$\omega = \frac{1}{2}\, \omega_{ij} \, dx^i \wedge dx^j$$

with $\omega_{ij} = -\omega_{ji}$.

Differential forms are naturally suited for integration. A k-form can be integrated over a k-dimensional oriented manifold. Differentiation of forms is performed using the *exterior derivative*, and the *generalized Stokes' theorem* unifies the fundamental theorems of calculus into a single geometric statement.

27.4 Metric Tensor

A *metric tensor* assigns an inner product to tangent vectors at each point of a manifold. Given two tangent vectors V^i and W^j, their inner product is defined as $g_{ij}V^iW^j$.

The components of the metric are

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

and the squared line element is

$$ds^2 = g_{ij}\, dx^i dx^j$$

The metric determines distances, angles, and volumes on the manifold.

27.5 Connection and Parallel Transport

Tangent vectors at different points belong to different vector spaces and cannot be compared directly. A *connection* provides a rule for relating tangent spaces at nearby points. In Euclidean space, you can slide a vector along a curve without changing it. On a curved manifold, "keeping the vector the same" is ambiguous. Parallel transport defines what "the same" means geometrically.

Parallel transport describes how a vector is moved along a curve while changing as little as possible according to the geometry of the manifold.

Definition 27.1. (Parallel Transport) *Let $(\mathcal{M}, \nabla)$ be a smooth manifold equipped with a connection ∇, and let*

$$\gamma : [a, b] \to \mathcal{M}$$

be a smooth curve. A vector field $V(t)$ along γ is said to be parallel transported *along γ if it satisfies*

$$\nabla_{\dot{\gamma}(t)} V(t) = 0 \quad \textit{for all } t \in [a, b]$$

27.6 Christoffel Symbols

27.6.1 Conceptual Meaning

On a smooth manifold $\mathcal{M}$, tangent vectors at different points belong to different tangent spaces and cannot be compared directly. A *connection* provides a rule for comparing vectors at nearby points and for differentiating vector fields along directions on the manifold.

The fundamental object is the connection ∇. The Christoffel symbols are not fundamental geometric objects themselves; rather, they are the *coordinate components* of a connection with respect to a chosen coordinate basis.

27.6.2 Definition

Let $\{\partial/\partial x^i\}$ be the coordinate basis of tangent vectors. The Christoffel symbols Γ^i_{jk} are defined by the action of the connection on the basis vectors:

$$\boxed{\nabla_j\left(\frac{\partial}{\partial x^k}\right)=\Gamma^i_{jk}\frac{\partial}{\partial x^i}}$$

This equation is the *definition* of the Christoffel symbols. They represent how the coordinate basis vectors vary from point to point on the manifold.

27.6.3 Geometric Interpretation

Geometrically, Γ^i_{jk} measure how the direction $\partial/\partial x^k$ changes when one moves in the direction $\partial/\partial x^j$. They encode the local geometric structure of the manifold and determine how vectors are parallel transported, how derivatives are taken, and how geodesics curve. Because they depend on the choice of coordinates, Christoffel symbols are *not tensors*.

27.6.4 Connection Versus Symbols

The connection ∇ is a geometric object. The Christoffel symbols Γ^i_{jk} are merely its coordinate representation. Different coordinate systems produce different Christoffel symbols for the same connection.

27.6.5 Metric Compatibility

Let g_{ij} be a metric tensor on $\mathcal{M}$. A connection is said to be *metric-compatible* if the covariant derivative of the metric vanishes:

$$\nabla_k g_{ij}=0$$

This condition means that lengths and angles are preserved under parallel transport.

Expanding this condition gives

$$\partial_k g_{ij}-\Gamma^l_{ki}g_{lj}-\Gamma^l_{kj}g_{il}=0$$

27.6.6 Torsion

The torsion tensor T of a connection is defined by

$$T(X,Y)=\nabla_X Y-\nabla_Y X-[X,Y]$$

A connection is *torsion-free* if

$$T(X,Y)=0$$

In coordinates, this implies symmetry of the Christoffel symbols in the lower indices

$$\Gamma^i_{jk}=\Gamma^i_{kj}$$

27.6.7 Uniqueness of the Levi–Civita Connection

A fundamental theorem of differential geometry states: On a smooth manifold with metric g, there exists a unique connection that is both metric-compatible and torsion-free. This unique connection is called the *Levi–Civita connection.*

27.6.8 Derivation of the Metric Formula

Imposing metric compatibility gives the equations

$$\nabla_k g_{ij} = 0, \quad \nabla_i g_{jk} = 0, \quad \nabla_j g_{ki} = 0$$

Expanding each using the definition of the covariant derivative produces a system of linear equations in the unknowns Γ^i_{jk}. Using the torsion-free symmetry condition $\Gamma^i_{jk} = \Gamma^i_{kj}$ and solving this system uniquely determines the connection coefficients.

27.6.9 Levi–Civita Formula

The Christoffel symbols of the Levi–Civita connection are

$$\boxed{\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk} \right)}$$

This expression is *not a definition*. It is the unique solution imposed by:

- Metric compatibility
- Vanishing torsion
- Linearity of the connection

27.6.10 Interpretation

The Christoffel symbols encode the intrinsic geometry of the manifold. They determine:

- Parallel transport
- Covariant differentiation
- Geodesic motion
- Curvature

Although they depend on coordinates, the geometric objects they define are coordinate-independent.

27.6.11 Summary

- Christoffel symbols are the coordinate components of a connection
- They are defined by the action of the connection on basis vectors
- They are not tensors
- The metric formula is derived, not postulated
- The Levi–Civita connection is uniquely fixed by geometry

27.7 Covariant Derivative

The need for the covariant derivative arises from the fact that, on a curved manifold, basis vectors vary from point to point. As a result, ordinary partial derivatives of vector components do not transform tensorially.

27.7.1 Contravariant Vector Fields

Let V be a contravariant vector field on a manifold $\mathscr{M}$, written in local coordinates as

$$V = V^i \frac{\partial}{\partial x^i}$$

Taking the ordinary partial derivative with respect to x^j gives

$$\partial_j V = \partial_j \left(V^i \frac{\partial}{\partial x^i} \right) = (\partial_j V^i) \frac{\partial}{\partial x^i} + V^i \, \partial_j \left(\frac{\partial}{\partial x^i} \right)$$

The first term differentiates the components of the vector field, while the second term accounts for the variation of the coordinate basis vectors themselves.

By definition, the change of the coordinate basis vectors is expressed as

$$\partial_j \left(\frac{\partial}{\partial x^k} \right) = \Gamma^i_{jk} \frac{\partial}{\partial x^i}$$

where the coefficients Γ^i_{jk} are called the Christoffel symbols.

Substituting this into the previous expression yields

$$\partial_j V = (\partial_j V^i) \frac{\partial}{\partial x^i} + V^k \Gamma^i_{jk} \frac{\partial}{\partial x^i}$$

Factoring out the basis vectors, we obtain

$$\partial_j V = \left(\partial_j V^i + \Gamma^i_{jk} V^k \right) \frac{\partial}{\partial x^i}$$

This motivates the definition of the **covariant derivative of a contravariant vector field**

$$\boxed{\nabla_j V^i = \partial_j V^i + \Gamma^i_{jk} V^k}$$

The additional term compensates for the variation of the basis vectors and ensures that the result transforms as a tensor.

27.7.2 Covariant Vector Fields

Now consider a covariant vector field (one-form)

$$\omega = V_i \, dx^i$$

Taking the partial derivative gives

$$\partial_j \omega = \partial_j (V_i dx^i) = (\partial_j V_i) \, dx^i + V_i \, \partial_j (dx^i)$$

The dual basis $\{dx^i\}$ is defined by the pairing

$$dx^i \left(\frac{\partial}{\partial x^j} \right) = \delta^i{}_j$$

which must remain constant under differentiation. Differentiating this identity and using the definition of the Christoffel symbols implies

$$\partial_j(dx^i) = -\Gamma^i_{jk}\, dx^k$$

Substituting this relation into the expression for $\partial_j\omega$ yields

$$\partial_j\omega = (\partial_j V_i)\, dx^i - \Gamma^k_{ji} V_k\, dx^i$$

Factoring out the basis covectors, we obtain

$$\partial_j\omega = \left(\partial_j V_i - \Gamma^k_{ji} V_k\right) dx^i$$

This leads to the definition of the **covariant derivative of a covariant vector field**

$$\boxed{\nabla_j V_i = \partial_j V_i - \Gamma^k_{ji} V_k}$$

27.7.2.1 Interpretation

The difference in sign between the contravariant and covariant cases reflects their distinct transformation properties under coordinate changes. Upper indices acquire correction terms with a plus sign, while lower indices acquire correction terms with a minus sign. In both cases, the covariant derivative ensures that the resulting object transforms as a tensor.

27.8 Geodesics

Geodesics generalize straight lines to curved spaces. They extremize the length functional

$$s = \int \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}}\, dt$$

When the curve is parameterized by an affine parameter (such as arc length), the resulting Euler–Lagrange equations yield the geodesic equation.

Equivalently, geodesics are curves whose tangent vectors are parallel transported along themselves:

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0$$

27.9 Curvature

27.9.1 Riemann Curvature Tensor

Curvature measures the failure of covariant derivatives to commute when acting on vector fields. This failure reflects the intrinsic geometry of the manifold and is independent of any embedding.

The Riemann curvature tensor is defined by

$$\boxed{R^i{}_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{km} \Gamma^m_{jl} - \Gamma^i_{lm} \Gamma^m_{jk}}$$

Its geometric meaning is revealed through the commutator of covariant derivatives

$$[\nabla_k, \nabla_l]V^i = R^i{}_{jkl}V^j$$

which measures the change of a vector under parallel transport around an infinitesimal closed loop.

27.9.2 Ricci Tensor and Scalar Curvature

Contracting the Riemann tensor yields the Ricci tensor

$$R_{ij} = R^k{}_{ikj}$$

which captures the trace part of curvature relevant for volume deformation and gravitational dynamics.

Further contraction with the inverse metric defines the scalar curvature

$$\boxed{R = g^{ij}R_{ij}}$$

27.10 Integration on Manifolds

The presence of a metric endows the manifold with a natural notion of volume, allowing integration of scalar functions in a coordinate-independent manner.

On an oriented manifold equipped with a metric, the invariant volume element is

$$\boxed{dV = \sqrt{|g|}\, d^n x}$$

27.11 Differential Geometry in Physics

Differential geometry underlies modern physical theories:

- Geodesics describe free particle motion
- Curvature encodes gravitation
- Differential forms unify electromagnetic laws

27.12 Closing Remarks

Differential geometry replaces flat intuition with intrinsic structure. By expressing physical laws in coordinate-independent language, it provides the mathematical foundation for modern theoretical physics.