CHAPTER 12

LAPLACE TRANSFORM

The Laplace transform is a special type of integral transform introduced by the French mathematician Pierre-Simon Laplace and later developed extensively by the British physicist Oliver Heaviside. It is used to simplify the solution of differential equations that arise in physical processes. For example, it is widely used by electrical engineers to analyze and solve circuit problems. By applying the Laplace transform, the task of solving an ordinary differential equation is reduced to an algebraic problem.

12.1 DEFINITION

The Laplace transform of a function $f(t)$ is defined as:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt,$$

where e^{-st} is called the kernel of the transform. The inverse transform $\mathcal{L}^{-1}\{F\}$ will yield $f(t)$.

12.2 BASIC TRANSFORMS

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{(a-s)t} dt = \frac{1}{(a-s)} e^{(a-s)t}\Big|_0^\infty = 0 - \frac{1}{(a-s)}$$

$$\implies \boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}} \qquad \text{where } s - a > 0$$

The Laplace transform of trigonometric functions are as follows:

$$\mathcal{L}(\cos wt) = \int_0^\infty e^{-st}\cos wt\, dt = \frac{e^{-st}}{-s}\cos wt\Big|_0^\infty - \frac{w}{s}\int_0^\infty e^{-st}\sin wt\, dt = \frac{1}{s} - \frac{w}{s}\mathcal{L}\{\sin wt\}$$

$$\mathcal{L}\{\sin wt\} = \int_0^\infty e^{-st}\sin wt\, dt = \frac{e^{-st}}{-s}\sin wt\Big|_0^\infty + \frac{w}{s}\int_0^\infty e^{-st}\cos wt\, dt = \frac{w}{s}\mathcal{L}\{\cos wt\}$$

Solving the simultaneous equations we get,

$$\boxed{\mathcal{L}\{\cos wt\} = \frac{s}{s^2 + w^2}}$$

$$\boxed{\mathcal{L}\{\sin wt\} = \frac{w}{s^2 + w^2}}$$

The hyperbolic functions are given by:

$$cosh\ wt = \frac{1}{2}(e^{wt} + e^{-wt})$$

$$sinh\ wt = \frac{1}{2}(e^{wt} - e^{-wt})$$

The Laplace transform of hyperbolic functions are:

$$\mathscr{L}\{cosh\ wt\} = \frac{1}{2}\mathscr{L}\{e^{wt}\} + \frac{1}{2}\mathscr{L}\{e^{-wt}\} = \frac{1}{2}\left(\frac{1}{s-w} + \frac{1}{s+w}\right) = \frac{s}{s^2 - w^2}$$

$$\mathscr{L}\{sinh\ wt\} = \frac{1}{2}\mathscr{L}\{e^{wt}\} - \frac{1}{2}\mathscr{L}\{e^{-wt}\} = \frac{1}{2}\left(\frac{1}{s-w} - \frac{1}{s+w}\right) = \frac{a}{s^2 - w^2}$$

$$\boxed{\mathscr{L}\{cosh\ wt\} = \frac{s}{s^2 - w^2}} \qquad \boxed{\mathscr{L}\{sinh\ wt\} = \frac{w}{s^2 - w^2}}$$

The Laplace transform of a polynomial is computed below.

$$\mathscr{L}\{t^{n+1}\} = \int_0^\infty e^{-st}t^{n+1}dt = -\frac{1}{s}e^{-st}t^{n+1}\Big|_0^\infty + \frac{n+1}{s}\int_0^\infty e^{-st}t^n dt$$

In the first term, the numerator grows like a power, the denominator grows like an exponential. Since exponentials grow faster than any power, the value is 0.

$$\implies \mathscr{L}\{t^{n+1}\} = \frac{n+1}{s}\mathscr{L}\{t^n\} \text{ for } s > 0$$

$$\implies \mathscr{L}\{t^n\} = \frac{n}{s}\mathscr{L}\{t^{n-1}\}$$

$$\implies \boxed{\mathscr{L}\{t^{n+1}\} = \frac{(n+1)!}{s^{n+2}}} \text{ (by induction, where } n = 0, 1, \cdots)$$

Now consider a to be real positive. Laplace transform of a polynomial can be expressed in terms of the **Gamma function** which is defined as:

$$\boxed{\Gamma(a) = \int_0^\infty e^{-x}x^{a-1}dx, \quad a > 0}$$

$$\mathscr{L}\{t^a\} = \int_0^\infty e^{-st}t^a dt$$

Let $st = x$,

$$\mathscr{L}\{t^a\} = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^a \frac{dx}{s} = \frac{1}{s^{a+1}}\int_0^\infty e^{-x}x^a dx$$

$$\implies \boxed{\mathscr{L}\{t^a\} = \frac{\Gamma(a+1)}{s^{a+1}}}$$

12.3   LINEARITY

Obviously, Laplace transform is a linear operation. i.e.,

$$\boxed{\mathscr{L}\{af(t) + bg(t)\} = a\mathscr{L}\{f(t)\} + b\mathscr{L}\{g(t)\}}$$ where a and b are constants

12.4   S-SHIFTING

If the Laplace transform of $f(t)$ is $F(s)$, then

$$\mathscr{L}\{e^{at}f(t)\} = F(s-a)$$

$$\mathscr{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a)$$

$$\boxed{\mathscr{L}\{e^{at}f(t)\} = F(s-a)}$$

12.5   EXISTANCE & UNIQUENESS

For Laplace transform $\mathscr{L}\{f\}$ to exist, the following condition must be satisfied:

$$\boxed{|f(t)| \le Me^{kt}}$$ where M, k are constants

$$|\mathscr{L}\{f\}| = \left|\int_0^\infty e^{-st}f(t)dt\right| \le \int_0^\infty Me^{kt}e^{-st}dt$$

$$M\int_0^\infty e^{-(s-k)t}dt = \frac{M}{-(s-k)}e^{-(s-k)t}\bigg|_0^\infty = 0 + \frac{M}{s-k} = \frac{M}{s-k}$$ where $s > k$

$$\boxed{|\mathscr{L}\{f\}| \le \frac{M}{s-k} \text{ where } s > k}$$

If the Laplace transform of a given function exists, it is uniquely determined. If two continuous functions have the same transform, they are identical.

12.6   LAPLACE TRANSFORMS OF DERIVATIVES

$$\mathscr{L}\{f'\} = \int_0^\infty e^{-st}f'(t)dt = e^{-st}f(t)\bigg|_0^\infty + s\int_0^\infty e^{-st}f(t)dt$$

Since $|f(t)| \le Me^{kt}$, the upper limit of $e^{-st}f(t)\bigg|_0^\infty$ is 0

$$\implies \boxed{\mathscr{L}\{f'\} = s\mathscr{L}\{f\} - f(0)}$$

$$\mathscr{L}\{f''\} = s\mathscr{L}\{f'\} - f'(0) = s[s\mathscr{L}\{f\} - f(0)] - f'(0)$$

$$\boxed{\mathscr{L}\{f''\} = s^2\mathscr{L}\{f\} - sf(0) - f'(0)}$$

Similarily,

$$\boxed{\mathscr{L}\{f^{(n)}\} = s^n\mathscr{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0)\cdots - f^{n-1}(0)}$$

12.7   Laplace Transforms of Integrals

Let $g(t) = \int_0^t f(u)du \implies g'(t) = f(t)$

$\mathscr{L}\{g'(t)\} = \mathscr{L}\{f(t)\}$

$s\mathscr{L}\{g(t)\} - g(0) = \mathscr{L}\{f(t)\}$

Now $g(0) = \int_0^0 f(u)du = 0$

$\implies$ $\boxed{\mathscr{L}\{g(t)\} = \dfrac{1}{s}\mathscr{L}\{f(t)\}}$

12.8   Laplace Transforms for solving ODEs

Consider the ODE,

$$\boxed{y'' + ay' + by = r(t)}$$

where $y(0) = K_0$, $y'(0) = K_1$, and a, b are constants

$$\left[s^2Y - sy(0) - y'(0)\right] + a\left[sY - (y(0)\right] + bY = R(s)$$

where $Y = \mathscr{L}(y)$ and $R = \mathscr{L}(r)$

$$(s^2 + as + b)Y = (s + a)y(0) + y'(0) + R(s)$$

The Transfer Function is defined as:

$$\boxed{Q(s) = \frac{1}{s^2 + as + b} = \frac{1}{(s + \frac{1}{2}a)^2 + (b - \frac{1}{4}a^2)}}$$

$$\boxed{Y(s) = \left[(s + a)y(0) + y'(0)\right]Q(s) + R(s)Q(s)}$$

If $y(0) = y'(0) = 0$,

$$\boxed{Q(s) = \frac{Y(s)}{R(s)} = \frac{\mathscr{L}\{output\}}{\mathscr{L}\{input\}}}$$

The ODE is transformed into an algebraic equation, which is also known as the **subsidiary equation**. If the initial conditions are specified at some $t_0 \neq 0$, we first shift the origin by setting $t = \tau + t_0$. The subsidiary equation is then solved in the s-domain, and finally the inverse Laplace transform is computed to obtain the solution of the original ODE. This technique can also be extended to solve **systems of ODEs**.

12.9   UNIT STEP FUNCTION (HEAVISIDE FUNCTION)

The Unit Step Function or Heaviside Function is defined as:

$$u(t-a) = \begin{cases} 0 \text{ for } t < a \\ 1 \text{ for } t > a \end{cases}$$

$$\mathscr{L}\{u(t-a)\} = \int_0^\infty e^{-st}u(t-a)dt = \int_a^\infty e^{-st}dt = -\frac{e^{-st}}{s}\Big|_a^\infty$$

$$\mathscr{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

12.10   TIME SHIFTING (t-SHIFTING)

Consider a function $f(t)$ that has its Laplace transform $F(s)$. The *shifted function* is given by:

$$\tilde{f}(t) = f(t-a)u(t-a) = \begin{cases} 0, & t < a, \\ f(t-a), & t \geq a. \end{cases}$$

Let

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt.$$

Then,

$$e^{-as}F(s) = e^{-as}\int_0^\infty e^{-s\tau}f(\tau)\,d\tau.$$

Now let

$$t = \tau + a \quad \Rightarrow \quad \tau = t - a, \quad d\tau = dt.$$

Then the limits change as:

$$\tau = 0 \Rightarrow t = a, \qquad \tau \to \infty \Rightarrow t \to \infty.$$

Hence,
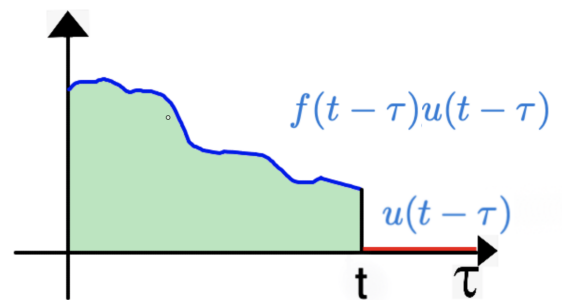
$$e^{-as}F(s) = \int_a^\infty e^{-st}f(t-a)\,dt.$$

Introduce the unit step function to shift the lower limit to zero:

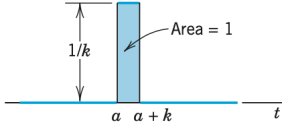$$e^{-as}F(s) = \int_0^\infty e^{-st}f(t-a)u(t-a)\,dt.$$

Therefore,

$$\mathscr{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$$

12.11 DIRAC DELTA FUNCTION

Consider the following function and the integral:

$$f_k(t - a) = \begin{cases} \dfrac{1}{k} & \text{if } a \le t \le a + k \\ 0 & \text{otherwise} \end{cases} \qquad I_k = \int_0^\infty f_k(t - a)\,dt = \int_a^{a+k} \frac{1}{k}\,dt = 1$$

We take the limit of f_k as $k \to 0$ $(k > 0)$, the **Dirac Delta** function, $(\delta - a)$, is then defined as:

$$\delta(t - a) = \lim_{k \to 0} f_k(t - a) \qquad \delta(t - a) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{otherwise} \end{cases} \implies \int_0^\infty \delta(t - a)\,dt = 1$$

$$\mathcal{L}\{\delta(t - a)\} = \int_0^\infty e^{-st}\,\delta(t - a)\,dt = e^{-as} \int_0^\infty \delta(t - a)\,dt = e^{-as}$$

$$\implies \int_0^\infty f(t)\delta(t - a)\,dt = f(a),\ a > 0 \text{ (Sifting)}$$

The impulse function $\delta(t - a)$ sifts through the function $f(t)$ and pulls out the value $f(a)$

$$\mathcal{L}\{\delta(t - a)\} = e^{-as}$$

The use of the unit step function (Heaviside function) and the Dirac delta function make the method particularly powerful for problems with inputs, i.e., driving forces, that have discontinuities or represent short impulses or complicated periodic functions.

12.12 CONVOLUTION

Convolution of two functions $f(t)$ and $g(t)$ is defined by

$$(f * g)(t) = \int_0^t f(\tau)\,g(t - \tau)\,d\tau$$

Let $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$. Then

$$\mathcal{L}\{(f * g)(t)\} = \int_0^\infty e^{-st}\left[\int_0^t f(\tau)\,g(t - \tau)\,d\tau\right] dt = \int_0^\infty \int_0^t e^{-st} f(\tau)\,g(t - \tau)\,d\tau\,dt$$

Interchange the order of integration:

$$\int_0^\infty \int_0^t \cdots d\tau\,dt = \int_0^\infty \int_\tau^\infty \cdots dt\,d\tau$$

$$\mathcal{L}\{(f * g)(t)\} = \int_0^\infty f(\tau)\left[\int_\tau^\infty e^{-st} g(t - \tau)\,dt\right] d\tau$$

Now set $u = t - \tau$ so that $t = u + \tau$, $dt = du$, and when t goes from τ to ∞, u goes from 0 to ∞:

$$\mathscr{L}\{(f * g)(t)\} = \int_0^\infty f(\tau)\left[\int_0^\infty e^{-s(u+\tau)}g(u)\,du\right]d\tau$$

$$= \int_0^\infty f(\tau)e^{-s\tau}d\tau \int_0^\infty e^{-su}g(u)\,du$$

$$= F(s)\,G(s)$$

$$\boxed{\mathscr{L}\{(f * g)(t)\} = F(s)G(s)}$$

12.13 DIFFERENTIATION OF TRANSFORMS

$$F(s) = \int_0^\infty e^{-st}f(t)dt$$

$$\frac{d}{ds}F(s) = \int_0^\infty \frac{d}{ds}e^{-st}f(t)dt = \int_0^\infty -te^{-st}f(t) = -\int_0^\infty e^{-st}tf(t)$$

$$\boxed{F'(s) = -\mathscr{L}(tf(t))}$$

12.14 INTEGRATION OF TRANSFORMS

$$F(s) = \int_0^\infty e^{-st}f(t)dt$$

$$\int_s^\infty F(\tilde{s})d\tilde{s} = \int_s^\infty \left[\int_0^\infty e^{-\tilde{s}t}f(t)dt\right]d\tilde{s}$$

$$\int_0^\infty \left[\int_s^\infty e^{-\tilde{s}t}f(t)d\tilde{s}\right]dt$$

$$\int_0^\infty f(t)\left[\int_s^\infty e^{-\tilde{s}t}d\tilde{s}\right]dt$$

$$\int_s^\infty F(\tilde{s})d\tilde{s} = \int_0^\infty e^{-st}\frac{f(t)}{t}dt$$

$$\boxed{\int_s^\infty F(\tilde{s})\,d\tilde{s} = \mathscr{L}\left\{\frac{f(t)}{t}\right\}(s)}$$