Chapter 3
Limit

A limit's a dream in sight,
We chase it with all our might!
But just as we draw near,
It grins, and disappears.

3.1 Introduction

In mathematics, a **limit** is the value that a function or a sequence approaches as the input approaches some value. Limits are essential to calculus and mathematical analysis, as they provide the foundation for defining continuity, derivatives, and integrals.

Informally, a limit describes what value a function is getting close to when the input gets close to a certain point. Importantly, the function does not need to be defined at that point; what matters is how it behaves near it. When we write

$$\lim_{x \to c} f(x) = L$$

we mean that as the variable x moves closer and closer to c, the values of $f(x)$ move closer and closer to the number L. The value L is the number the function is approaching, not necessarily the value of the function at $x = c$.

Sometimes a function cannot be evaluated at a particular point because of division by zero or some other issue. Even in such cases, the limit may still exist if the function values approach a definite number as x gets close to that point. Simplifying algebraic expressions often helps reveal this behavior.

To make the idea of a limit precise, mathematicians use the ϵ–δ definition. This definition says that a limit exists if we can make the function values as close as we like to L by restricting x to be sufficiently close to c. In simple terms, no matter how small an error we allow in the output, we can always find a range around c that keeps the function within that error.

The chapter also introduces several *laws of limits*. These laws explain how limits behave under common operations such as addition, subtraction, multiplication, division, powers, and roots. They allow us to compute limits efficiently without returning to the formal definition each time, provided the individual limits exist.

Some limits are especially important and appear frequently in calculus. One such example is

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

when angles are measured in radians. This limit plays a key role in defining derivatives of trigonometric functions.

The notion of a limit can also be restricted to one side. A *right-hand limit* considers values of x greater than the point, while a *left-hand limit* considers values less than the point. A two-sided limit exists only if both one-sided limits exist and are equal.

Limits are closely connected to the idea of *continuity*. A function is continuous at a point if the limit of the function at that point exists and equals the function's value there. If this condition fails at any point, the function is said to be discontinuous at that point.

The chapter also discusses *infinite limits*, where function values grow without bound as x approaches a point. Writing such limits as ∞ or $-\infty$ does not mean the limit exists in the usual sense; it simply describes the behavior of the function.

Finally, the chapter introduces two fundamental results in calculus: *Rolle's Theorem* and the *Mean Value Theorem*. These theorems connect limits, continuity, and derivatives, and they formalize the idea that a smooth function must have points where its rate of change reflects its overall behavior on an interval.

Overall, this chapter lays the conceptual foundation for calculus by explaining how limits describe the local behavior of functions and how this behavior leads naturally to continuity and differentiation.

3.2 Definition of Limit

Consider a function $f(x)$ that is defined in a domain D which includes the point c. The function may or may not be defined at c. If, for all x that is *close* to c except for c, $f(x)$ is arbitrarily close to a number L (as close to L as we like), then it is said that f approaches the limit L as x approaches c and is written as:

$$\boxed{\lim_{x \to c} f(x) = L}$$

If the function can be evaluated at c, the limit L is simply $f(c)$. But there can be situations where the function is not evaluable at c. E.g., the following function cannot be evaluated at $x = 1$.

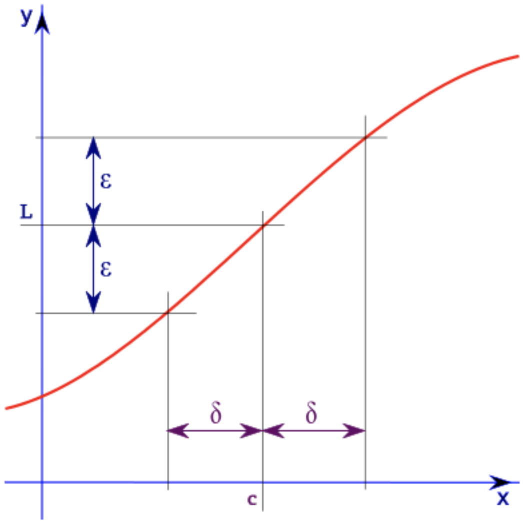
$$\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1}$$

But this function can be easily simplified to:

$$f(x) = \frac{(x-1)(x+2)}{x-1} = x + 2$$

$$\to \lim_{x \to 1} f(x) = 3.$$

3.3 Formal Definition of Limit



Let $f(x)$ be a function that is defined on an interval that contains $x = c$, except possibly at c. Then, $\lim_{x \to c} f(x) = L$ if for every number $\epsilon > 0$, there is some number $\delta > 0$ such that, when $0 < |x - a| < \delta, \quad |f(x) - L| < \epsilon$.

This means that for any number $\epsilon > 0$ that we pick, one can go to the graph and sketch two horizontal lines at $L + \epsilon$ and $L - \epsilon$. Then there must be another number $\delta > 0$ that can be determined to enable us to add in two vertical lines in the graph $a + \delta$ and $a - \delta$.

3.4 Laws of Limit

Given L, M, c, k are real numbers such that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. Then,

Sum Rule	$\lim_{x \to c}(f(x) + g(x)) = L + M$
Difference Rule	$\lim_{x \to c}(f(x) - g(x)) = L - M$
Constant Rule	$\lim_{x \to c}(kf(x)) = kL$
Product Rule	$\lim_{x \to c}(f(x)g(x)) = LM$
Quotient Rule	$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$
Power Rule	$\lim_{x \to c}[f(x)]^n = L^n \; (n > 0)$
Root Rule	$\lim_{x \to c} \sqrt[n]{(f(x)} = \sqrt[n]{(L)} \; (n > 0)$

Examples:

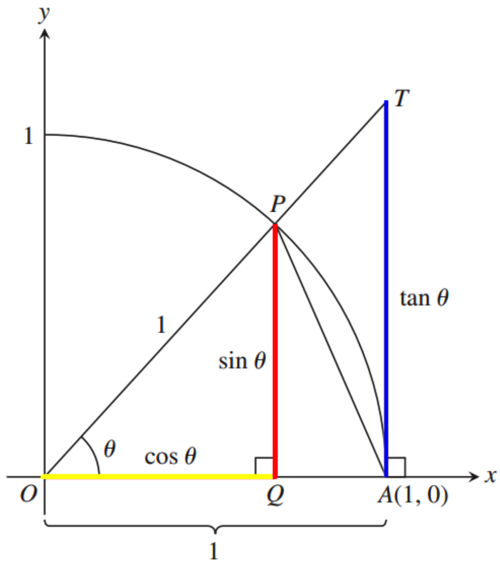
$$\lim_{x \to 3} \sqrt{(2x^3 + 10} = 8$$

$$\lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x^2}$$

The above function is not evaluable at $x = 0$. The standard trick is to multiply both numerator and denominator by the conjugate radical expression.

$$\frac{\sqrt{x^2 + 9} - 3}{x^2} = \frac{\sqrt{x^2 + 9} - 3}{x^2} \cdot \frac{\sqrt{x^2 + 9} + 3}{\sqrt{x^2 + 9} + 3} = \frac{1}{\sqrt{x^2 + 9} + 3} = \lim_{x \to 0} \frac{1}{\sqrt{x^2 + 9} + 3} = \frac{1}{6}$$

3.5 An Important Limit



Consider the circle with a unit radius.

Area △ OAP < area sector OAP < area △ OAT

$\frac{1}{2}\sin\ \theta \leq \pi 1^2 \left(\frac{\theta}{2\pi}\right) \leq \frac{1}{2}\tan\ \theta$ (θ is in radians)

$1 \leq \frac{\theta}{\sin\ \theta} \leq \frac{1}{\cos\ \theta}$

$\rightarrow 1 \geq \frac{\sin\ \theta}{\theta} \geq \cos\ \theta$

Hence,

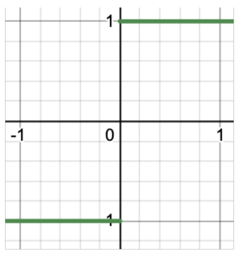
$$\boxed{\lim_{\theta\to 0}\frac{\sin\ \theta}{\theta} = 1} \quad \text{where } \theta \text{ is in radians} \tag{3.1}$$

Now consider the function $f(\theta) = \frac{1}{\sin}\ \theta$. Does it have a limit as $t \rightarrow \theta$ from either side? As θ approaches 0, its reciprocal, $1/x$, grows without bound and the values of function cycle repeatedly from -1 to 1. There is no single number L that the function values stay increasingly close to as $\theta \rightarrow 0$. The function has neither a right-hand limit nor a lefthand limit at $\theta = 0$.

3.6 One Sided Limits

$$\lim_{x\to 0^+} f(x) = 1$$

$$\lim_{x\to 0^-} f(x) = -1$$



3.7 Continuous Function

Function is right-continous at c (continuous from right) if $\lim_{x\to c^+} f(x) = f(c)$
Function is left-continous at c (continuous from left) if $\lim_{x\to c^-} f(x) = f(c)$
A function is continous at c if $\lim_{x\to c} f(x) = f(c)$

If a function is discontinuous at one or more points of its domain, it is called a discontinuous function.

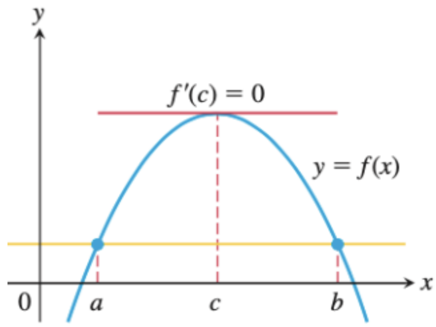
3.8 Infinite Limits

$$\lim_{x\to 0^+} \frac{1}{x} = \infty, \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

Note that this does not mean that the limit exists as there is no real number such as ∞. It is simply a concise way of saying that the limit does not exist.

3.9 Rolle's Theorem

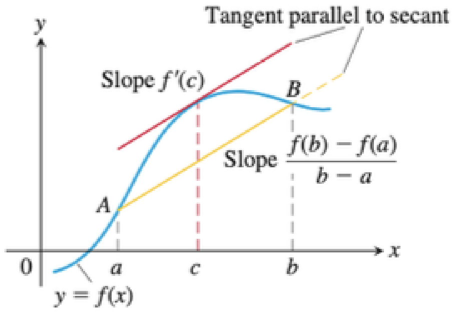
If f is a continuous function on a closed interval $[a, b]$ and If $f(a) = f(b)$, then there is at least one point c in (a, b) where $f'(c) = 0$.



3.10 Mean Value Theorem

There is at least one number c in the interval (a, b) such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



3.11 SymPy Code

Determine the limits of:

$$\lim_{x \to 1} \frac{x^2 + x - 2}{(x - 1)}$$

$$\lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x^2}$$

$$\lim_{x \to 0} \sin(x)$$

```python
import sympy as sp
import numpy as np
import matplotlib.pyplot as plt
from IPython.display import display, Math
from sympy import sin, cos, tan, trigsimp, expand_trig
from sympy import oo
from sympy import limit

x = sp.symbols('x')
y = (x**2 + x - 2) / (x - 1)
lim = limit(y, x, 1)
display(lim)
```

```

y = ( (x**2 + 9)**0.5 - 3 ) / x**2
lim = limit(y, x, 0)
display(lim)

y = sin(x)/x
lim = limit(y, x, 0)
display(lim)
```

3

$\frac{1}{6}$

$cos(x)$