Chapter 26
Tensors

Tensors form a rigorous mathematical framework for describing physical quantities in a way that does not depend on the choice of coordinate system. By extending the familiar notions of scalars and vectors, tensors provide a unified language for expressing laws of physics, and are central to continuum mechanics, electromagnetism, relativity, and much of modern physics.

26.1 Scalars and Vectors

26.1.1 Scalars

A scalar is a quantity that is fully described by a single number and remains unchanged under coordinate transformations.

Examples include mass, temperature, and energy.

26.1.2 Vectors

A vector is a physical quantity characterized by both **magnitude** (how large it is) and **direction** (which way it points). Unlike scalars, which have only magnitude (e.g., temperature or mass), vectors describe quantities like displacement, velocity, or force.

In a 3D Cartesian coordinate system, we represent a vector **A** using its components along the basis vectors $\mathbf{e}_1 = \hat{i}$, $\mathbf{e}_2 = \hat{j}$, $\mathbf{e}_3 = \hat{k}$:

$$\mathbf{A} = A^i \mathbf{e}_i = A^1 \mathbf{e}_1 + A^2 \mathbf{e}_2 + A^3 \mathbf{e}_3 = (A^1, A^2, A^3)$$

Here, A^i are the *contravariant components* (with the Einstein summation convention implying summation over $i = 1, 2, 3$), and the parentheses denote the standard component tuple.

Example

The vector pointing 3 units east, 4 units north, and 0 units up is $\mathbf{A} = (3, 4, 0)$, with magnitude

$$\|\mathbf{A}\| = \sqrt{3^2 + 4^2 + 0^2} = 5.$$

Under a change of coordinates, the components transform linearly.

26.2 Coordinate Transformations

Consider two coordinate systems related by a linear transformation

$$\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j} x^j$$

The transformation matrix is defined as

$$\tilde{\mathbf{x}} = \Lambda \mathbf{x}$$

where:

- $\mathbf{x} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$ (original coordinates)
- $\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x}^1 \\ \tilde{x}^2 \\ \tilde{x}^3 \end{pmatrix}$ (new coordinates)
- $\Lambda = \left(\frac{\partial \tilde{x}^i}{\partial x^j} \right)$ is the Jacobian matrix

$$\Lambda^i_j = \frac{\partial \tilde{x}^i}{\partial x^j}$$

Explicit matrix form

$$\begin{pmatrix} \tilde{x}^1 \\ \tilde{x}^2 \\ \tilde{x}^3 \end{pmatrix} = \begin{pmatrix} \frac{\partial \tilde{x}^1}{\partial x^1} & \frac{\partial \tilde{x}^1}{\partial x^2} & \frac{\partial \tilde{x}^1}{\partial x^3} \\ \frac{\partial \tilde{x}^2}{\partial x^1} & \frac{\partial \tilde{x}^2}{\partial x^2} & \frac{\partial \tilde{x}^2}{\partial x^3} \\ \frac{\partial \tilde{x}^3}{\partial x^1} & \frac{\partial \tilde{x}^3}{\partial x^2} & \frac{\partial \tilde{x}^3}{\partial x^3} \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

26.3 Definition of a Tensor

A tensor is a mathematical object defined by how its components transform under coordinate changes. Specifically, a tensor of type (k, l) transforms as:

$$T'^{i_1 \ldots i_k}_{j_1 \cdots j_l} = \frac{\partial x'^{i_1}}{\partial x^{m_1}} \cdots \frac{\partial x'^{i_k}}{\partial x^{m_k}} \frac{\partial x^{n_1}}{\partial x'^{j_1}} \cdots \frac{\partial x^{n_l}}{\partial x'^{j_l}} T^{m_1 \ldots m_k}_{n_1 \ldots n_l}$$

Vectors as special cases

- Contravariant vector $(1, 0)$:

$$V'^i = \frac{\partial x'^i}{\partial x^j} V^j$$

- Covariant vector $(0, 1)$:

$$W'_i = \frac{\partial x^j}{\partial x'^i} W_j$$

Key insight: Tensors are the objects whose transformation rules are consistent across coordinate systems, distinguishing them from general functions.

26.3.1 Contravariant Vectors

A contravariant vector V^i transforms according to the chain rule:

$$\tilde{V}^i = \frac{\partial \tilde{x}^i}{\partial x^j} V^j$$

The components "stretch" with the new coordinate basis.

26.3.2 Covariant Vectors (Dual Vectors)

A covariant vector W_i (one-form) transforms inversely:

$$\tilde{W}_i = \frac{\partial x^j}{\partial \tilde{x}^i} W_j$$

This ensures $W_i V^i$ remains invariant under coordinate changes.

Note the pattern reciprocity: Contravariant uses $\frac{\partial \text{new}}{\partial \text{old}}$, covariant uses $\frac{\partial \text{old}}{\partial \text{new}}$.

Why Covariant Vectors Are Called Dual Vectors

A covariant vector is called a **dual vector** because it belongs to the **dual space** V^* of the original vector space V.

Dual Space Concept

For any vector space V with basis $\{\mathbf{e}_i\}$:

- **Vectors** in V: $V = v^i \mathbf{e}_i$ (contravariant components)
- **Dual vectors** in V^*: $\omega = \omega_i \mathbf{e}^i$ where $\{\mathbf{e}^i\}$ is the **dual basis**

Dual basis definition: $\mathbf{e}^i(\mathbf{e}_j) = \delta^i_j$ (Kronecker delta)

Perfect Pairing

The scalar product is **invariant**:

$$\omega(\mathbf{v}) = \omega_i v^i$$

When coordinates change, contravariant components v^i "stretch" one way, covariant ω_i stretch oppositely—**duality ensures the product stays constant**.

Transformation Duality

$$\tilde{v}^i = \frac{\partial \tilde{x}^i}{\partial x^j} v^j \qquad \text{(stretches with new basis)}$$
$$\tilde{\omega}_i = \frac{\partial x^j}{\partial \tilde{x}^i} \omega_j \qquad \text{(stretches inversely)}$$

Analogy: Vectors point **with** the basis arrows →, dual vectors measure **across** them ↔.

26.4 Metric Tensor

The metric tensor g_{ij} defines geometry—distances, angles, and volumes in curved spaces.

The squared infinitesimal distance (line element) is:

$$ds^2 = g_{ij}\,dx^i dx^j$$

g_{ij} is symmetric ($g_{ij} = g_{ji}$) and defines the *inner product*.

Example: In flat 3D Cartesian: $g_{ij} = \delta_{ij}$. In polar: $g_{rr} = 1$, $g_{\theta\theta} = r^2$.

26.5 Raising and Lowering Indices

The metric tensor allows conversion between covariant and contravariant components.

Lowering an index is performed by

$$V_i = g_{ij} V^j$$

Raising an index is performed using the inverse metric g^{ij}

$$V^i = g^{ij} V_j$$

26.6 General Tensors

A tensor of type (p, q) has p contravariant indices and q covariant indices.

Its components transform as

$$\tilde{T}^{i_1 \dots i_p}_{\ \ j_1 \dots j_q} = \frac{\partial \tilde{x}^{i_1}}{\partial x^{k_1}} \cdots \frac{\partial \tilde{x}^{i_p}}{\partial x^{k_p}} \frac{\partial x^{l_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{l_q}}{\partial \tilde{x}^{j_q}} T^{k_1 \dots k_p}_{\ \ l_1 \dots l_q}$$

26.7 Tensor Operations

26.7.1 Addition and Scalar Multiplication

Tensors of the same type may be added componentwise.

26.7.2 Tensor Product

The tensor product of two tensors A and B produces a tensor of higher rank

$$(A \otimes B)^{ij} = A^i B^j$$

Example,

$$A = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \qquad A \otimes B = \begin{pmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \end{pmatrix}$$

26.7.3 Contraction

Contraction reduces the rank of a tensor by summing over a repeated upper and lower index (Einstein summation convention). For example,

$$T^i{}_i$$

is a scalar obtained by contracting the indices of the rank-2 tensor $T^i{}_j$.

26.8 Important Physical Tensors

26.8.1 Stress Tensor

In continuum mechanics, the stress tensor σ_{ij} relates force to area

$$F_i = \sigma_{ij} n^j$$

where n^j is the normal vector.

26.8.2 Moment of Inertia Tensor

The moment of inertia tensor is defined as

$$I_{ij} = \sum_k m_k (\delta_{ij} r_k^2 - x_{k,i} x_{k,j})$$

26.8.3 Electromagnetic Field Tensor

In relativistic electrodynamics, the electromagnetic field is represented by a rank-2 tensor.

26.9 Tensor Calculus

26.9.1 Partial Derivatives

The partial derivative of a tensor is generally not a tensor, because partial derivatives do not transform covariantly under general coordinate transformations.

To preserve tensorial character, the covariant derivative of a contravariant vector field V^i is introduced:

$$\nabla_k V^i = \partial_k V^i + \Gamma^i_{kj} V^j.$$

where Γ^i_{kj} are the Christoffel symbols, which encode the effects of curvature and the coordinate system.

26.9.2 Christoffel Symbols

The Christoffel symbols describe how coordinate basis vectors vary from point to point and are used to define covariant derivatives in curved or curvilinear coordinate systems. They are not tensors, but are constructed from the metric tensor and its derivatives.

Christoffel Symbols of the Second Kind

The Christoffel symbols of the second kind are defined by

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{i\ell} \left(\partial_j g_{k\ell} + \partial_k g_{j\ell} - \partial_\ell g_{jk} \right)$$

Christoffel Symbols of the First Kind

Lowering the upper index with the metric yields the Christoffel symbols of the first kind:

$$\Gamma_{ijk} = g_{i\ell} \Gamma^\ell{}_{jk} = \frac{1}{2} \left(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk} \right)$$

Role in Covariant Differentiation

For a contravariant vector field V^i, the covariant derivative is given by

$$\nabla_k V^i = \partial_k V^i + \Gamma^i{}_{kj} V^j$$

which ensures that $\nabla_k V^i$ transforms as a tensor under general coordinate transformations.

26.10 Closing Remarks

Tensors provide a coordinate-independent language for physical laws. Their systematic use, together with covariant differentiation, unifies diverse areas of physics and enables the formulation of laws valid in arbitrary coordinate systems.