

# HIGHER ORDER ORDINARY DIFFERENTIAL EQUATION

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*A function bends, its path unclear,  
Higher-order terms draw fate near.  
Roots decide if rise or fall,  
In chaos or order, it governs all.*

## 8.1 HIGHER ORDER HOMOGENEOUS ODE

The concepts of the 2nd Order ODE can be extended to higher order ODE which has the form:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_1(x)y' + p_0(x)y = r(x)$$

For constant coefficients,  $y = e^{\lambda x}$  yields  $\lambda^{(n)} + a_{n-1}\lambda^{(n-1)} + \dots + a_1\lambda + a_0 = 0$  (characteristic equation). For  $n$  distinct roots, there are  $n$  distinct basis solutions:

$$y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x} + \dots + c_n e^{\lambda_n x}$$

The Wronskian is given by:

$$W(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \cdot & \cdot & \dots & \cdot \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix} = E \begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_n^2 \\ \cdot & \cdot & \dots & \cdot \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{vmatrix}$$

Where  $E = e^{(\lambda_1 + \lambda_2 + \dots + \lambda_n)x}$ .  $W = 0$  if and only if the determinant, known as the Vandermonde or Cauchy determinant, is zero.  $W \neq 0$ , if and only if, all the  $n$  roots are different.

If a real double root occurs, say,  $\lambda_1 = \lambda_2$ , then we take  $y_1$  and  $xy_1$  as corresponding linearly independent solutions. If  $\lambda$  is a real root of order  $m$ , then the corresponding basis solutions are:

$$e^{\lambda x}, xe^{\lambda x}, x^2e^{\lambda x}, \dots, x^{m-1}e^{\lambda x}$$

Complex roots occur in conjugate pairs  $\lambda = \gamma \pm iw$  since the coefficients of the ODE are real.

$$y_1 = e^{\gamma x} \cos wx \quad y_2 = e^{\gamma x} \sin wx$$

If  $\lambda = \gamma + iw$  is a complex double root, so is the conjugate  $\lambda = \gamma - iw$  and the corresponding linearly independent solutions are:

$$e^{\gamma x} \cos wx \quad e^{\gamma x} \sin wx \quad x e^{\gamma x} \cos wx \quad x e^{\gamma x} \sin wx$$

The first two of these result from  $e^{\lambda x}$  and  $e^{\bar{\lambda}x}$  as before, and the second two from  $xe^{\lambda x}$  and  $xe^{\bar{\lambda}x}$  in the same fashion. The corresponding general solution is:

$$y = e^{\gamma x}[(A_1 + A_2 x)\cos wx + (B_1 + B_2 x)\sin wx]$$

For complex triple roots, which is quite rare, one would obtain two more solutions:

$$x^2 e^{\gamma x} \cos wx \quad x^2 e^{\gamma x} \sin wx$$

## 8.2 HIGHER ORDER NON-HOMOGENEOUS ODE

### 8.2.1 METHOD OF UNDETERMINED COEFFICIENTS

Apply the method of undetermined coefficients for solving 2nd order ODE with a modification. If a term in the choice for  $y_p(x)$  is a solution of the homogeneous equation, then multiply this term by  $x^k$ , where  $k$  is the smallest positive integer and satisfies the condition that this term  $\times x^k$  is not a solution of the homogeneous equation. So, we try  $cxe^{\lambda x}, cx^2e^{\lambda x}, \dots, cx^ke^{\lambda x}$  as a solution, plug into the ODE, and solve for  $c$  for the minimum  $k$ .

### 8.2.2 METHOD OF VARIATION OF PARAMETERS

Extending the concept that we used for 2nd order ODE to arbitrary order  $n$  we have:

$$y_p(x) = \sum_{k=1}^n y_k(x) \int \frac{W_k(x)}{W(x)} r(x) dx$$

## 8.3 SERIES SOLUTIONS OF HOMOGENEOUS ODES

Higher order linear ODEs with constant coefficients can be solved by algebraic methods as their solutions are often elementary functions which are known from calculus. For ODEs with variable coefficients the situation is complicated and their solutions are nonelementary special functions, e.g., Legendre and Bessel functions.

### 8.3.1 POWER SERIES METHOD

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

Compute  $y'$ ,  $y''$ , ...,  $y^{(n)}$ , substitute in the ODE and compute the coefficients of the powers of  $x^2, x^3, \dots, x^n$ . Equate each of the coefficients to 0 to determine  $a_0, a_1, a_2, \dots, a_n$ .

## 8.4 EXISTENCE OF POWER SERIES SOLUTIONS

Consider the following ODE:

$$y'' + p(x)y' + q(x)y = r(x)$$

If  $p, q, r$  have Taylor series representations (analytic) then every solution of the ODE can be represented by a power series in powers of  $x - x_0$  with a positive radius of convergence  $R$ . A power series can be added, multiplied and differentiated term by term.

## 8.5 CLASSICAL DIFFERENTIAL EQUATIONS

Legendre:  $(1 - x^2)y'' - 2xy' + k(k + 1)y = 0$

Chebyshev:  $(1 - x^2)y'' - xy' + k^2y = 0$

Hermitte:  $y'' - 2xy' + 2ky = 0$

Laguerre:  $xy'' + (1 - x)y' + ky = 0$

where  $k$  is a constant

## 8.6 LEGENDRE'S EQUATION

$(1 - x^2)y'' - 2xy' + k(k + 1)y = 0$   $k$  is a constant

Let  $y = a_n \sum_{n=0}^{\infty} x^n$

Compute  $y, y', y''$  and substitute in the above equation.

$$\begin{aligned} y' &= na_n \sum_{n=0}^{\infty} x^{n-1} & y'' &= n(n-1)a_n \sum_{n=0}^{\infty} x^{n-2} \\ (1 - x^2) \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 2x \sum_{n=0}^{\infty} na_n x^{n-1} + k(k+1) \sum_{n=0}^{\infty} a_n x^n &= 0 \end{aligned}$$

Since  $n(n-1)$  is 0 for  $n = 0$  and  $n = 1$ , the lower indices start from 2 and 1.

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - 2 \sum_{n=1}^{\infty} na_n x^n + k(k+1) \sum_{n=0}^{\infty} a_n x^n = 0$$

Let  $n - 2 = m$  and use  $m$  as the index in the remaining terms as it is a dummy index:

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m - \sum_{m=2}^{\infty} m(m-1)a_m x^m - 2 \sum_{m=1}^{\infty} ma_m x^m + k(k+1) \sum_{m=0}^{\infty} a_m x^m = 0$$

$a_0$  and  $a_1$  are arbitrary constants, the remaining constants are expressed in terms of these.

$$m = 0 \implies 2a_2 + k(k+1)a_0 = 0 \implies a_2 = -\frac{-k(k+1)}{2!}a_0$$

$$m = 1 \implies 6a_3 + [-2 + k(k+1)]a_1 = 0 \implies a_3 = -\frac{(k-1)(k+2)}{3!}a_1$$

$$m \geq 2 \implies (m+2)(m+1)a_{m+2} = [m(m-1) + 2m - k(k+1)]a_m = (m^2 + m - k^2 - k)a_m$$

$$a_{m+2} = -\frac{(k-m)(k+m+1)}{(m+1)(m+2)}a_m$$

$$a_4 = \frac{(k-2)k(k+1)(k+3)}{4!}a_0$$

$$a_5 = \frac{(k-3)(k-1)(k+2)(k+4)}{5!}a_1$$

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots + a_nx^n + \dots$$

$y = a_0y_1(x) + a_1y_2(x)$   $a_0, a_1$  are arbitrary constants,  $y_1$  is the even series &  $y_2$  is the odd)

$$y_1 = 1 + a_2x^2 + a_4x^4 + \dots$$

$$y_2 = x + a_3x^3 + a_5x^5 + \dots$$

### 8.6.1 LEGENDRE POLYNOMIALS

When:

$$m = k, a_{m+2} = a_{m+4} = a_{m+6} \dots = 0$$

If  $k$  is even,  $y_1(x)$  reduces to a polynomial of degree  $k$ .

If  $k$  is odd,  $y_2(x)$  reduces to a polynomial of degree  $k$ .

The reduction of power series to polynomials is a great advantage because then we have solutions for all  $x$  without convergence restrictions. These polynomials, multiplied by some constants, are called Legendre polynomials and are denoted by  $P_n(x)$ .

The standard choice of such constants is to choose the coefficient  $a_n$  of the highest power  $x^n$  as:

$$a_k = \frac{(2k)!}{2^k(k!)^2}$$

We then calculate the other coefficients as follows:

$$a_m = -\frac{(m+1)(m+2)}{(k-m)(k+m+1)}a_{m+2}$$

With  $m = k - 2$

$$\begin{aligned} a_{k-2} &= -\frac{k(k-1)}{2(2k-1)}a_k \\ &= -\frac{k(k-1)}{2(2k-1)} \frac{2k!}{2^k(k!)^2} \\ &= -\frac{k(k-1)}{2(2k-1)} \frac{2k(2k-1)(2k-2)!}{2^k k(k-1)! k(k-1)(k-2)!} \\ &= \frac{(2k-2)!}{2^k (k-1)!(k-2)!} \end{aligned}$$

With  $m = k - 4$

$$\begin{aligned}
a_{k-4} &= \frac{(k-2)(k-3)}{4(2k-3)} a_{k-2} \\
&= \frac{(k-2)(k-3)}{4(2k-3)} \frac{(2k-2)!}{2^k(k-1)!(k-2)!} \\
&= \frac{(2k-4)!}{2^k 2!(k-2)!(k-4)!}
\end{aligned}$$

In general,

$$a_{k-2m} = (-1)^m \frac{(2k-2m)!}{2^k m! (k-m)! (k-2m)!}$$

## 8.7 FROBENIUS METHOD

Several important 2nd order ODEs have coefficients that are not analytic. Yet these ODEs can be solved through an extension of the power series method that is credited to Frobenius. Consider the ODE:

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0 \quad \text{Note: } b(x), c(x) \text{ are analytic at } x = 0$$

This ODE has at least one solution of the form:

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m$$

Where  $r$  is real or complex and  $a_0 \neq 0$ .

Multiply the ODE by  $x^2$  and expand  $b(x)$  and  $c(x)$  in Taylor series.

$$\begin{aligned}
&x^2 y'' + x b(x) y' + c(x) y = 0 \\
&b(x) = \sum_{m=0}^{\infty} b_m x^m \quad c(x) = \sum_{m=0}^{\infty} c_m x^m \\
&y(x) = x^r \sum_{m=0}^{\infty} a_m x^m \quad y'(x) = \sum_{m=0}^{\infty} (m+r)a_m x^{m+r-1} \quad y''(x) = \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r-2}
\end{aligned}$$

Substituting in the ODE,

$$x^r [r(r-1)a_0 + \dots] + (b_0 + b_1 x + \dots) x^r (ra_0 + \dots) + (c_0 + c_1 x + \dots) x^r (a_0 + a_1 x + \dots) = 0$$

Equate coefficients of  $x^r, x^{r+1}, x^{r+2}$  to 0.

$$[r(r-1) + b_0 r + c_0] a_0 = 0$$

$$[r^2 + (b_0 - 1)r + c_0] = 0 \quad (\text{indicial equation})$$

The Frobenius method yields a basis of solutions.

Distinct roots not differing by an integer

$$y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \dots)$$

$$y_2(x) = x^{r_2}(A_0 + A_1x + A_2x^2 + \dots)$$

Double root  $r_1 = r_2 = r = \frac{1}{2}(1 - b_0)$

$$y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \dots)$$

$$y_2(x) = y_1(x)\ln x + x^{r_1}(A_0 + A_1x + A_2x^2 + \dots)$$

Roots differing by an integer

$$y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \dots)$$

$$y_2(x) = ky_1(x)\ln x + x^{r_2}(A_0 + A_1x + A_2x^2 + \dots)$$

$r_1 > r_2, k$  can be 0

For cases 2 and 3, the second independent solution can be obtained by reduction of order.

## 8.8 BESSEL'S EQUATION

$$\boxed{x^2y'' + xy' + (x^2 - \nu^2)y = 0} \quad (\nu \text{ is a real number } \geq 0)$$

Applying Frobenius technique, the solution is of the form

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}$$

$$y'(x) = \sum_{m=0}^{\infty} (m+r)a_m x^{m+r-1} = x^{r-1}[ra_0 + (r+1)a_1x + (r+2)a_2x^2 + \dots]$$

$$y''(x) = \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r-2} = x^{r-2}[r(r-1)a_0 + (r+1)ra_1x + (r+2)(r+1)a_2x^2 + \dots]$$

substituting in the ODE

$$\sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r} + \sum_{m=0}^{\infty} (m+r)a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} - \nu^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$$r(r-1)a_0 + ra_0 - \nu^2 a_0 = 0 \quad (m=0)$$

$$(r+\nu)(r-\nu) = 0, \implies r = \pm\nu$$

$$(r+1)ra_1 + (r+1)a_1 - \nu^2 a_1 = 0 \quad (m=1)$$

$$((\nu+1)\nu + (\nu+1) - \nu^2)a_1 = 0 \implies (2\nu+1)a_1 = 0 \implies a_1 = 0$$

$$(m+r)(m+r-1)a_m + (m+r)a_m + a_{m-2} - \nu^2 a_m = 0 \quad (m=2,3,\dots)$$

$$(m+\nu)[(m+\nu-1 + (m+\nu) - \nu^2)a_m + a_{m-2}] = 0 \implies m(m+2\nu)a_m + a_{m-2} = 0$$

$$\text{since } a_1 = 0 \implies a_3 = a_5 = \dots = 0$$

$$2m(2m+2\nu)a_{2m} + a_{2m-2} = 0 \quad (\text{ensure even numbers only, } m=1,2,\dots)$$

$$a_{2m} = -\frac{a_{2m-2}}{2^2 m(m+\nu)} \quad (m=1,2,\dots)$$

$$a_2 = -\frac{a_0}{2^2(\nu+1)}$$

$$a_4 = -\frac{a_2}{2^2 2(\nu+2)} = \frac{a_0}{2^4 2!(\nu+1)(\nu+2)}$$

When  $\nu$  is an integer, denote it as by  $n$

$$a_{2m} = -\frac{(-1)^n a_0}{2^{2m} m!(n+1)(n+2)\dots(n+m)} \quad (m=1,2,\dots)$$

$$\text{choose, } a_0 = \frac{1}{2^n n!}$$

$$a_{2m} = \frac{(-1)^m}{2^{2m+n} m!(n+m)!} \quad (m=1,2,\dots)$$

A particular solution to Bessel's equation is then given by,

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!(n+m)!} \quad (m=1,2,\dots, \text{and } n \geq 0)$$

$J_n(x)$  is called the Bessel function of the first kind of order  $n$  and converges  $\forall x$ .

For  $n=0$ ,  $J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m!^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} + \dots$  (Bessel function of order 0, similar to cosine)

For  $n=1$ ,  $J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m} m!(m+1)!} = \frac{x}{2} - \frac{x^3}{2^3 1!2!} + \frac{x^5}{2^5 2!3!} + \dots$  (Bessel function of order 1, similar to sine)

### 8.8.1 BESSEL FUNCTIONS FOR REAL NUMBER

Choose  $a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}$  where the Gamma function is defined as:

$$\Gamma(\nu+1) = \int_0^\infty e^{-t} t^\nu dt \quad (\nu > -1)$$

$$\Gamma(\nu+1) = -e^{-t} t^\nu \Big|_0^\infty + \nu \int_0^\infty e^{-t} t^{\nu-1} dt = 0 + \nu \Gamma(\nu)$$

$$\boxed{\Gamma(\nu+1) = \nu \Gamma(\nu)} \quad \text{for } n = 0, 1, \dots \quad \boxed{\Gamma(n+1) = n!} \quad (\text{The Gamma function is a generalised factorial})$$

$$a_{2m} = -\frac{(-1)^m a_0}{2^{2m} m!(\nu+1)(\nu+2)\dots(\nu+m) 2^\nu \Gamma(\nu+1)}$$

$$a_{2m} = -\frac{(-1)^m a_0}{2^{2m+\nu} m! \Gamma(\nu+m+1)}$$

$$\boxed{J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m! \Gamma(\nu+m+1)}}$$

$J_\nu(x)$  is called the Bessel function of the first kind of order  $\nu$

Bessel functions satisfy many relationships such as the following:

$[x^\nu J_\nu(x)]' = x^\nu J_{\nu-1}(x)$	$[x^{-\nu} J_\nu(x)]' = -x^{-\nu} J_{\nu+1}(x)$
$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$	$J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_\nu(x)$
$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$	$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

### 8.8.2 GENERAL SOLUTION

For a general solution of Bessel's equation in addition to  $J_\nu$  we need a second linearly independent solution. If  $\nu$  is not an integer, the general solution can be obtained by replacing  $\nu$  with  $-\nu$ . The general solution is then given by:

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$$

This cannot be the general solution for an integer  $\nu = n$  because that will lead to linear dependence.

### 8.8.3 BESSEL FUNCTIONS OF THE SECOND KIND, $Y_\nu(x)$

For  $n = 0$ , the Bessel function can be written as:

$$xy'' + y' + xy = 0$$

The indicial equation has a double root and the desired solution must be of the form:

$$\boxed{y_2(x) = J_0 \ln x + \sum_{m=1}^{\infty} A_m x^m}$$

$$y_2' = J_0' \ln x + \frac{J_0}{x} + \sum_{m=1}^{\infty} m A_m x^{m-1}$$

$$y_2'' = J_0'' \ln x + \frac{2J_0'}{x} - \frac{J_0}{x^2} + \sum_{m=1}^{\infty} m(m-1) A_m x^{m-2}$$

Substituting  $y_2'', y_2', y$  in the equation we have:

$$\begin{aligned}
& (xJ_0'' \ln x + 2J_0' - \frac{J_0}{x} + \sum_{m=1}^{\infty} m(m-1)A_m x^{m-1}) + (J_0' \ln x + \frac{J_0}{x} + \sum_{m=1}^{\infty} mA_m x^{m-1}) + \\
& (xJ_0 \ln x + \sum_{m=1}^{\infty} A_m x^{m+1}) = 0 \\
& (\cancel{xJ_0''} + \cancel{J_0'} + xJ_0) \ln x + 2J_0' + \sum_{m=1}^{\infty} m(m-1)A_m x^{m-1} + \sum_{m=1}^{\infty} mA_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0 \\
& 2J_0' + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0 \\
& \text{Now, } J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m!^2} \\
& J_0'(x) = \sum_{m=1}^{\infty} \frac{(-1)^m 2m x^{2m-1}}{2^{2m} m!^2} = \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-1} m! (m-1)!} \\
& \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m-1}}{2^{2m-2} m! (m-1)!} + \sum_{m=1}^{\infty} m^2 A_m x^{m-1} + \sum_{m=1}^{\infty} A_m x^{m+1} = 0
\end{aligned}$$

The power of  $x_0$  occurs only in the 2nd series, hence  $A_1 = 0$ .

Comparing coefficient of even powers of  $x$  in 2nd & 3rd series (1st series has none), we have:

$$(2s+1)^2 A_{2s+1} + A_{2s-1} = 0 \quad (\text{where } s = 0, 1, 2, \dots)$$

Since  $A_1 = 0 \implies A_3 = A_5 = \dots = 0$

$$-1 + 4A_2 = 0 \implies A_2 = \frac{1}{4}$$

Matching the odd power of  $x$  in all 3 series, we have:

$$\frac{(-1)^{s+1}}{2^{2s}(s+1)!s!} + (2s+2)^2 A_{2s+2} + A_{2s} = 0$$

$$A_{2m} = \frac{(-1)^{m-1}}{2^{2m}(m!)^2} \left( 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m} \right) \quad (m = 0, 1, 2, \dots)$$

$$y_2(x) = J_0(x) \ln x + \frac{(-1)^{m-1} h_m}{2^{2m} (m!)^2} x^{2m} \quad \text{where } h_m = \left( 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m} \right)$$

$J_0, y_2$  are linearly independent functions (basis for  $x > 0$ ), express  $y_2$  as particular solution:

$$Y_0(x) = a(y_2 + bJ_0) \quad \text{and choose } a = \pi/2 \quad \text{and } b = \gamma - \ln 2$$

$$\text{Let, } \gamma = \lim_{s \rightarrow \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{s} \right) - \ln s = 0.57721566490 \quad (\text{Euler constant})$$

The standard particular solution thus obtained is called the Bessel function of the second kind of order zero or Neumann's function of order zero and is denoted by  $Y_0(x)$ .

$$Y_0(x) = \frac{2}{\pi} \left( J_0(x) \ln x + \frac{(-1)^{m-1} h_m}{2^{2m} (m!)^2} x^{2m} + (\gamma - \ln 2) J_0 \right)$$

$$Y_0(x) = \frac{2}{\pi} \left[ J_0(x) \left( \ln \frac{x}{2} + \gamma \right) + \frac{(-1)^{m-1} h_m}{2^{2m} (m!)^2} x^{2m} \right]$$

### 8.8.4 BESSEL FUNCTIONS OF THE SECOND KIND, $Y_n(x)$

For  $n = 1, 2, \dots$  a second solution can be obtained by manipulations similar to those for  $n = 0$ . It turns out that in these cases the solution also contains a logarithmic term.

Depending on whether  $\nu$  is an integer or not, the standard second solution known as the Bessel function of the 2nd kind of order  $\nu$  or Neumann's function of order  $\nu$  is given by:

$$Y_\nu(x) = \frac{1}{\sin \nu \pi} [J_\nu(x) \cos \nu \pi - J_{-\nu}(x)]$$

$$Y_n(x) = \lim_{\nu \rightarrow n} Y_\nu(x)$$

The general solution of Bessel's equation  $\forall x \wedge x > 0$  is given by:

$$y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x)$$

## 8.9 SymPy

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1 # Solving Bessel's equation with SymPy
2 # Demonstrates symbolic solution for general order nu, and a concrete example for
2   ↪ nu=1.
3 import sympy as sp
4
5 # symbols and function
6 x, nu = sp.symbols('x nu')
7 y = sp.Function('y')
8
9 # General Bessel's equation: x^2 y'' + x y' + (x^2 - nu^2) y = 0
10 ode_general = sp.Eq(x**2*sp.diff(y(x), x, 2) + x*sp.diff(y(x), x) + (x**2 -
11   ↪ nu**2)*y(x), 0)
11 print(sp.latex(ode_general))
12
13 # Solve symbolically (returns solution in terms of BesselJ and BesselY)
14 sol_general = sp.dsolve(ode_general)
15 sol_general_simpl = sp.simplify(sol_general.rhs) # RHS is the general solution
16   ↪ expression
16
17 # Concrete example: nu = 1 (order 1 Bessel equation)
18 ode_nu1 = sp.Eq(x**2*sp.diff(y(x), x, 2) + x*sp.diff(y(x), x) + (x**2 - 1)*y(x), 0)
19 sol_nu1 = sp.dsolve(ode_nu1)
20
21 # Example with initial conditions: y(1)=1, y'(1)=0 for nu=1
22 ics = {y(1): 1, sp.diff(y(x), x).subs(x, 1): 0}
23 sol_nu1_ics = sp.dsolve(ode_nu1, ics=ics)
24
25 # Show results
26 print("General solution (order 'nu'):\n", sol_general_simpl, "\n")
27 print("Solution for nu = 1:\n", sol_nu1.rhs, "\n")
28 print(sp.latex(sol_nu1.rhs))
29 print("Solution for nu = 1 with y(1)=1, y'(1)=0:\n", sol_nu1_ics.rhs, "\n")
30
31 # Also explicitly show the independent basis functions
32 C1, C2 = sp.symbols('C1 C2')
33 basis = sp.Matrix([sp.besselj(nu, x), sp.bessely(nu, x)])
34 print("Fundamental solutions (Bessel J and Y):\n", basis)
35
36 # Return objects for inspection if desired

```

37 sol\_general, sol\_nu1, sol\_nu1\_ics, basis

$$x^2 \frac{d^2}{dx^2}y(x) + x \frac{d}{dx}y(x) + (-\nu^2 + x^2)y(x) = 0$$
$$\frac{(Y_2(1) - Y_0(1))J_1(x)}{J_1(1)Y_2(1) + J_0(1)Y_1(1) - J_1(1)Y_0(1) - J_2(1)Y_1(1)} + \frac{(-J_2(1) + J_0(1))Y_1(x)}{J_1(1)Y_2(1) + J_0(1)Y_1(1) - J_1(1)Y_0(1) - J_2(1)Y_1(1)}$$