Chapter 29

Spacetime Geometry

Space bends where time learns how to flow,
Events whisper distances we cannot know.
Light draws the rules by which causes agree,
And geometry remembers what motion must be.

Spacetime geometry provides the geometric framework of special and general relativity. Space and time are unified into a four-dimensional manifold equipped with a metric that encodes causal structure, invariant intervals, and the motion of matter and fields. This unification is not merely a change of notation, but a profound reorganization of the geometric and physical foundations of classical mechanics.

29.1 The Geometric Unity of Space and Time

In classical physics, space and time are treated as fundamentally distinct. Space is the domain of geometry, endowed with distances and angles, while time acts as an external parameter ordering physical processes. This separation, though intuitive, is incompatible with the symmetries revealed by modern physics.

The invariance of physical laws under Lorentz transformations shows that spatial lengths and temporal durations are not absolute. Different observers assign different spatial and temporal coordinates to the same physical process, yet agree on certain invariant quantities. What is preserved is not space alone, nor time alone, but a specific combination of the two.

The fundamental objects of description therefore become *events*, each specified by both spatial and temporal coordinates. Physics is reformulated as the study of relations among events, independent of the coordinate system used to label them.

The invariant interval between events replaces Euclidean distance as the primary geometric quantity. This structure classifies separations as timelike, spacelike, or null, embedding causality directly into geometry.

29.2 Spacetime as a Manifold

Spacetime is modeled as a four-dimensional smooth manifold $\mathscr{M}$ whose points represent events. Around every event, spacetime appears locally flat, allowing the use of differential geometry.

Local coordinate charts label events but have no intrinsic physical meaning. Physical laws must therefore be expressed in a coordinate-independent form.

Each event may be labeled by coordinates

$$x^\mu = (ct, x, y, z)$$

where $\mu = 0, 1, 2, 3$.

29.3 Metric Structure of Spacetime

A metric tensor defined on spacetime encodes invariant intervals, causal structure, and geometric relations among events.

29.3.1 Minkowski Metric

In special relativity, spacetime is flat and equipped with the Minkowski metric

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The spacetime interval between nearby events is

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

This interval is invariant under Lorentz transformations.

29.3.2 Timelike, Spacelike, and Null Intervals

Intervals are classified by the sign of ds^2.

For timelike intervals

$$ds^2 > 0$$

For spacelike intervals

$$ds^2 < 0$$

For null intervals

$$ds^2 = 0$$

This classification determines causal relationships between events.

29.4 Four-Vectors and Worldlines

Physical quantities are expressed as four-vectors to ensure Lorentz invariance.

29.4.1 Position and Velocity Four-Vectors

The position four-vector is

$$x^\mu = (ct, x, y, z)$$

A particle's motion through spacetime is represented by a **worldline**

$$x^\mu = x^\mu(\tau)$$

parameterized by the proper time τ.

The four-velocity is defined as

$$u^\mu = \frac{dx^\mu}{d\tau}$$

and satisfies the normalization condition

$$\eta_{\mu\nu} u^\mu u^\nu = c^2$$

29.4.2 Energy–Momentum Four-Vector

The energy–momentum four-vector is

$$p^\mu = (E/c, \mathbf{p})$$

It satisfies the invariant relation

$$\eta_{\mu\nu} p^\mu p^\nu = m^2 c^2$$

29.5 Proper Time and Free Motion

The proper time interval along a worldline is defined by

$$d\tau = \frac{1}{c}\sqrt{ds^2}$$

In flat spacetime, free particles move along straight worldlines, corresponding to inertial motion.

29.6 Lorentz Transformations as Isometries

Lorentz transformations preserve the Minkowski metric

$$\eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu = \eta_{\mu\nu}$$

They are therefore isometries of spacetime geometry.

29.7 Curved Spacetime and Gravitation

In the presence of gravitation, spacetime becomes curved. The metric is generalized to

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$$

where the metric components vary with position.

Gravitation is the interaction by which mass–energy influences the geometry of spacetime. In general relativity, gravitation is not described as a force acting at a distance, but as the curvature of spacetime produced by energy and momentum. Free particles move along geodesics of the curved spacetime, and their apparent acceleration arises from the geometric structure rather than from a gravitational force. The source of spacetime curvature is the energy–momentum tensor, which encodes mass, energy, momentum, pressure, and stress. This relationship is expressed by Einstein's field equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

which state that spacetime curvature is determined by the distribution of energy and momentum.

29.8 Geodesics in Spacetime

The motion of free particles in curved spacetime follows geodesics. The geodesic equation is

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

where $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols associated with the metric.

29.9 Curvature and Gravitation

Spacetime curvature is described by the Riemann curvature tensor

$$R^\mu{}_{\nu\alpha\beta}$$

Curvature determines the relative acceleration of nearby worldlines and encodes the gravitational interaction.

29.10 Spacetime Geometry and Physics

Spacetime geometry unifies kinematics and gravitation:

- Lorentz invariance arises from flat spacetime geometry
- Gravitation is interpreted as spacetime curvature
- Physical laws are expressed in covariant tensor form

29.11 Closing Remarks

Spacetime geometry replaces the notion of absolute space and time with an intrinsic geometric structure. By describing physics in terms of invariant intervals, worldlines, and curvature, geometry becomes an essential component of physical law rather than a passive background.