Chapter 30

Einstein's Theory of Relativity

Einstein's theory of relativity provides a unified description of space, time, motion, and gravitation. It consists of special relativity, which governs physics in inertial frames, and general relativity, which incorporates gravitation through the geometry of spacetime.

30.1 Special Relativity

30.1.1 Postulates

Special relativity is founded on two postulates:

- The laws of physics are identical in all inertial frames
- The speed of light in vacuum is the same in all inertial frames

These postulates require a revision of classical notions of space and time.

30.1.2 Spacetime Interval

Events are described by spacetime coordinates

$$x^\mu = (ct, x, y, z)$$

The invariant spacetime interval is

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

or in tensor form

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

30.1.3 Lorentz Transformations

Invariance of the spacetime interval leads to the Lorentz transformation

$$x' = \gamma(x - vt)$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

with

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

30.1.4 Relativistic Energy and Momentum

The energy–momentum four-vector is

$$p^\mu = (E/c, \mathbf{p})$$

Its invariant magnitude yields

$$E^2 = p^2c^2 + m^2c^4$$

For a particle at rest

$$E = mc^2$$

30.2 Motivation for General Relativity

Special relativity applies only to inertial frames and does not describe gravitation. The equivalence of inertial and gravitational mass suggests that gravitation is not a force in the Newtonian sense.

30.3 Equivalence Principle

30.3.1 Statement

The equivalence principle states that locally, the effects of gravitation are indistinguishable from those of acceleration.

This implies that free-fall motion corresponds to inertial motion in curved spacetime.

30.4 Spacetime Geometry

30.4.1 Metric Tensor

Spacetime is modeled as a four-dimensional manifold equipped with a metric

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$$

The metric determines distances, time intervals, and causal structure.

30.4.2 Geodesic Motion

Free particles follow geodesics that extremize the spacetime interval

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

30.5 Spacetime Curvature

30.5.1 Curvature Tensors

The Riemann curvature tensor measures spacetime curvature

$$R^\rho{}_{\sigma\mu\nu}$$

From it one constructs the Ricci tensor

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$$

and the scalar curvature

$$R = g^{\mu\nu}R_{\mu\nu}$$

30.6 Matter and Energy

Matter and non-gravitational fields are described by the stress–energy tensor $T_{\mu\nu}$.

Local conservation of energy and momentum requires

$$\nabla^\mu T_{\mu\nu} = 0$$

30.7 Derivation of Einstein's Field Equations

30.7.1 Guiding Requirements

The gravitational field equations must:

- Be generally covariant
- Reduce to Newtonian gravity in the weak-field limit
- Respect local energy–momentum conservation
- Depend on the metric and its derivatives

30.7.2 Einstein Tensor

The only symmetric rank-2 tensor constructed from the metric and curvature with vanishing covariant divergence is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

It satisfies the identity

$$\nabla^\mu G_{\mu\nu} = 0$$

30.7.3 Field Equation Ansatz

To relate geometry to matter, the field equations must take the form

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

where κ is a constant.

30.7.4 Newtonian Limit

In the weak-field limit

$$g_{00} \approx 1 + \frac{2\Phi}{c^2}$$

The 00 component of the field equations reduces to

$$\nabla^2\Phi = \frac{\kappa c^4}{2}\rho$$

Comparison with Poisson's equation

$$\nabla^2\Phi = 4\pi G\rho$$

yields

$$\kappa = \frac{8\pi G}{c^4}$$

30.7.5 Einstein Field Equations

The final form of Einstein's field equations is

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

These equations determine how matter and energy curve spacetime.

30.8 Physical Consequences

General relativity predicts:

- Gravitational time dilation
- Bending of light
- Precession of planetary orbits
- Gravitational waves

All have been confirmed experimentally.

30.9 Closing Remarks

Special relativity unifies space and time through invariant geometry. General relativity extends this framework by identifying gravitation with spacetime curvature. Einstein's equations complete this synthesis, expressing gravity as geometry governed by energy and momentum.