CHAPTER 3

Limit

A limit's a dream in sight,
We chase it with all our might!
But just as we draw near,
It grins, and disappears.

In mathematics, a limit is the value that a function or a sequence approaches as the input approaches some value. Limits are essential to calculus and mathematical analysis, and are used to define continuity, derivatives, and integrals.

1 Definition of Limit

Consider a function $f(x)$ that is defined in a domain D which includes the point c. The function may or may not be defined at c. If, for all x that is *close* to c except for c, $f(x)$ is arbitrarily close to a number L (as close to L as we like), then it is said that f approaches the limit L as x approaches c and is written as:

$$\boxed{\lim_{x \to c} f(x) = L}$$

If the function can be evaluated at c, the limit L is simply $f(c)$. But there can be situations where the function is not evaluable at c. E.g., the following function cannot be evaluated at $x = 1$.

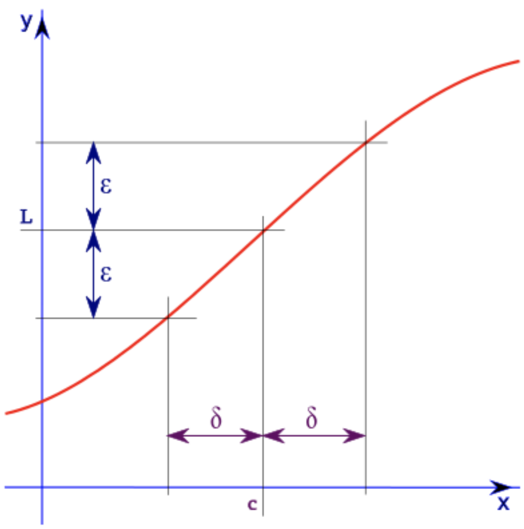
$$\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1}$$

But this function can be easily simplified to:

$$f(x) = \frac{(x-1)(x+2)}{x-1} = x + 2$$

$$\to \lim_{x \to 1} f(x) = 3.$$

2 Formal Definition of Limit



Let $f(x)$ be a function that is defined on an interval that contains $x = c$, except possibly at c. Then, $\lim_{x \to c} f(x) = L$ if for every number $\epsilon > 0$, there is some number $\delta > 0$ such that, when $0 < |x - a| < \delta$, $|f(x) - L| < \epsilon$.

This means that for any number $\epsilon > 0$ that we pick, one can go to the graph and sketch two horizontal lines at $L + \epsilon$ and $L - \epsilon$. Then there must be another number $\delta > 0$ that can be determined to enable us to add in two vertical lines in the graph $a + \delta$ and $a - \delta$.

3 Laws of Limit

Given L, M, c, k are real numbers such that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. Then,

Sum Rule	$\lim_{x\to c}(f(x)+g(x)) = L+M$
Difference Rule	$\lim_{x\to c}(f(x)-g(x)) = L-M$
Constant Rule	$\lim_{x\to c}(kf(x)) = kL$
Product Rule	$\lim_{x\to c}(f(x)g(x)) = LM$
Quotient Rule	$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M}$
Power Rule	$\lim_{x\to c}[f(x)]^n = L^n \; (n>0)$
Root Rule	$\lim_{x\to c}\sqrt[n]{(f(x)} = \sqrt[n]{(L)} \; (n>0)$

Examples:

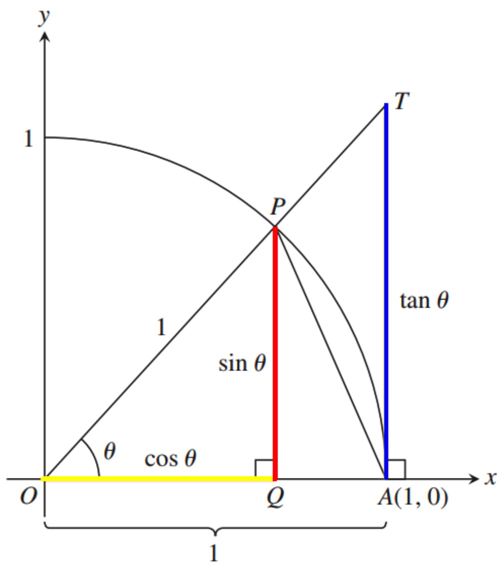
$$\lim_{x\to 3}\sqrt{(2x^3+10} = 8$$

$$\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}$$

The above function is not evaluable at $x=0$. The standard trick is to multiply both numerator and denominator by the conjugate radical expression.

$$\frac{\sqrt{x^2+9}-3}{x^2} = \frac{\sqrt{x^2+9}-3}{x^2}\cdot\frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3} = \frac{1}{\sqrt{x^2+9}+3} = \lim_{x\to 0}\frac{1}{\sqrt{x^2+9}+3} = \frac{1}{6}$$

4 An Important Limit



Consider the circle with a unit radius.

Area △ OAP < area sector OAP < area △ OAT

$\frac{1}{2}sin\,\theta \le \pi 1^2\left(\frac{\theta}{2\pi}\right) \le \frac{1}{2}\tan\,\theta$ (θ is in radians)

$1 \le \frac{\theta}{sin\,\theta} \le \frac{1}{cos\,\theta}$

$\rightarrow 1 \ge \frac{sin\,\theta}{\theta} \ge cos\,\theta$

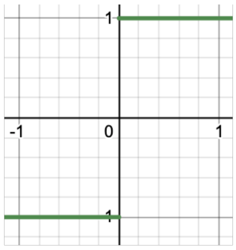
Hence, $\boxed{\lim_{\theta\to 0}\frac{Sin\,\theta}{\theta} = 1}$ where θ is in radians

Now consider the function $f(\theta) = \frac{1}{sin\,\theta}$. Does it have a limit as $t \to \theta$ from either side? As θ approaches 0, its reciprocal, $1/x$, grows without bound and the values of function cycle repeatedly from -1 to 1. There is no single number L that the function values stay increasingly close to as $\theta \to 0$. The function has neither a right-hand limit nor a lefthand limit at $\theta = 0$.

5 One Sided Limits

$$\lim_{x \to 0^+} f(x) = 1$$

$$\lim_{x \to 0^-} f(x) = -1$$



6 Continuous Function

Function is right-continous at c (continuous from right) if $\lim_{x \to c^+} f(x) = f(c)$
Function is left-continous at c (continuous from left) if $\lim_{x \to c^-} f(x) = f(c)$
A function is continous at c if $\lim_{x \to c} f(x) = f(c)$

If a function is discontinuous at one or more points of its domain, it is called a discontinuous function.

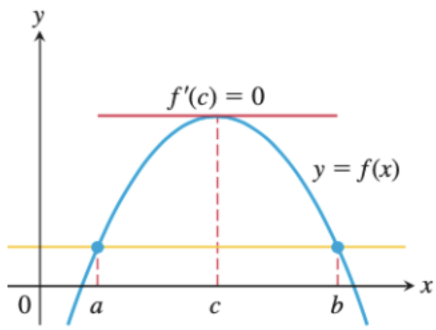
7 Infinite Limits

$$\lim_{x \to 0^+} \frac{1}{x} = \infty, \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

Note that this does not mean that the limit exists as there is no real number such as ∞. It is simply a concise way of saying that the limit does not exist.

8 Rolle's Theorem

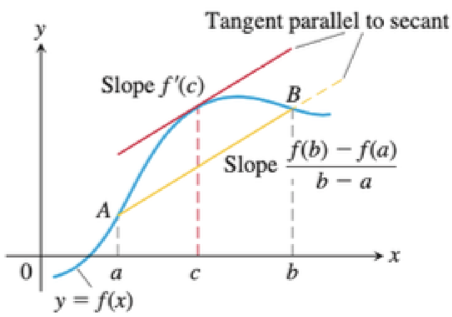
If f is a continuous function on a closed interval $[a, b]$ and If $f(a) = f(b)$, then there is at least one point c in (a, b) where $f'(c) = 0$.



9 Mean Value Theorem

There is at least one number c in the interval (a, b) such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



10 SymPy Code

Determine the limits of:

$$\lim_{x\to 1} \frac{x^2+x-2}{(x-1)}$$

$$\lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2}$$

$$\lim_{x\to 0} sin(x)$$

```python
import sympy as sp
import numpy as np
import matplotlib.pyplot as plt
from IPython.display import display, Math
from sympy import sin, cos, tan, trigsimp, expand_trig
from sympy import oo
from sympy import limit

x = sp.symbols('x')
y = (x**2 + x - 2) / (x - 1)
lim = limit(y, x, 1)
display(lim)

y = ( (x**2 + 9)**0.5 - 3 ) / x**2
lim = limit(y, x, 0)
display(lim)

y = sin(x)/x
lim = limit(y, x, 0)
display(lim)
```

3

$\frac{1}{6}$

$cos(x)$