

# SECOND ORDER ORDINARY DIFFERENTIAL EQUATION

*A second-order equation, quite a mess,  
With derivatives causing plenty of distress.  
But Euler, Cauchy, and Lagrange helped evolve,  
Elegant methods to solve and resolve!*

As noted in the previous chapter, the order of a differential equation is defined as the order of the highest derivative of the unknown function that appears in the equation. Therefore, a second-order differential equation is one in which the second derivative of the unknown function occurs, and no derivative of higher order is present.

Second order differential equations have a variety of applications in science and engineering such as vibrations and electric circuits. There are a host of multi dimensional engineering models that incorporate second order differential equations including wave motion, flow mechanics, Maxwell's electro-magnetic equations and Schroedinger equation in Nuclear Physics.

## 7.1 2ND ORDER LINEAR ODE

The **standard form** of a **2nd Order ODE** is:

$$y'' + p(x)y' + q(x)y = r(x) \quad \text{It is linear in } y, y' \text{ and } y'' \quad (7.1.1)$$

If  $r(x) = 0$ , the ODE is homogeneous, else it is non-homogeneous. When the coefficients  $a$  and  $b$  are constant:

$$y'' + ay' + by = 0 \quad (7.1.2)$$

Choose  $e^{\lambda x}$  as a solution and substitute in the homogeneous ODE.

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0 \implies \lambda^2 + a\lambda + b = 0 \implies \lambda = \frac{1}{2}(-a \pm \sqrt{a^2 - 4b})$$

$$y_h = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \quad (\text{general solution to the homogeneous ODE})$$

$y_1$ , corresponding to  $\lambda_1$ , and  $y_2$ , corresponding to  $\lambda_2$ , are **linearly independent** and are called **basis of solutions**. The **superposition principle** also called the **linearity principle**, i.e., the homogeneous solution is a combination of  $y_1$  and  $y_2$  is true only for linear homogeneous ODE. The arbitrary constants  $c_1$  and  $c_2$  are determined from the **initial conditions**:

$$y(x_0) = k_0 \quad y'(x_0) = k_1$$

A **particular solution** is obtained if we assign specific values to  $c_1$  and  $c_2$ .

## 7.2 LAGRANGE'S METHOD OF REDUCTION OF ORDER

Consider a linear homogeneous 2nd Order ODE in its standard form :

$$y'' + p(x)y' + q(x)y = 0 \quad (7.2.1)$$

If  $y_1$  is a basis solution, we can find  $y_2$  as follows:

$$\text{Let } y = y_2 = uy_1$$

$$y_2' = u' y_1 + u y_1'$$

$$y_2'' = u'' y_1 + 2u' y_1' + u y_1''$$

Substituting,

$$(u'' y_1 + 2u' y_1' + u y_1'') + p(u' y_1 + u y_1') + q(u y_1) = 0$$

$$y_1 u'' + (2y_1' + p y_1) u' + (y_1'' + p y_1' + q y_1) u = 0$$

$$u'' + u' \frac{2y_1' + p y_1}{y_1} = 0$$

$$\text{Let } U = u'$$

$$U' + U \left( \frac{2y_1'}{y_1} + p \right) = 0$$

$$\frac{U'}{U} = - \left( \frac{2y_1'}{y_1} + p \right)$$

$$\int \frac{U'}{U} dx + \int \left( \frac{2y_1'}{y_1} \right) dx = - \int p dx$$

$$\ln |U| + 2 \ln |y_1| = - \int p dx$$

$$\ln |U y_1^2| = - \int p dx$$

$$U y_1^2 = e^{\int -p dx}$$

$$U = \frac{1}{y_1^2} e^{\int -p dx}$$

$$u = \int U dx$$

$$y_2 = y_1 \int U dx$$

$$y_2 = y_1 \int \frac{1}{y_1^2} e^{\int -p dx} dx$$

## 7.3 HOMOGENEOUS LINEAR ODE WITH CONSTANT COEFFICIENTS

$$\begin{cases} \text{Case 1: 2 Real Roots when} & a^2 - 4b > 0 \\ \text{Case 2: Double Root when} & a^2 - 4b = 0 \\ \text{Case 3: Complex Conjugate Roots when} & a^2 - 4b < 0 \end{cases}$$

Case 1: 2 Real Roots when  $a^2 - 4b > 0$ . The general solution is given by:

$$y_1 = e^{\lambda_1 x} \quad y_2 = e^{\lambda_2 x}$$

$$\lambda_1 = \frac{1}{2} \left( -a + \sqrt{a^2 - 4b} \right) \quad \lambda_2 = \frac{1}{2} \left( -a - \sqrt{a^2 - 4b} \right)$$

$$y_h = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \quad (7.3.1)$$

Case 2:  $\lambda_1 = -\frac{a}{2}, y_1 = e^{-\frac{ax}{2}}$  Determine  $y_2$  using the method of reduction of order.

$$y_1 = e^{-\frac{ax}{2}}$$

$$y_2 = y_1 \int \frac{1}{y_1^2} e^{\int -p dx} dx = e^{-\frac{ax}{2}} \int \frac{1}{\left(e^{-\frac{ax}{2}}\right)^2} e^{\int -a dx} dx = e^{-\frac{ax}{2}} \int e^{ax} e^{-ax} dx = x e^{-\frac{ax}{2}}$$

$$y_h = c_1 e^{-ax/2} + c_2 x e^{-\frac{a}{2}x} \implies y_h = (c_1 + c_2 x) e^{-ax/2}$$

Case 3:  $\lambda = -\frac{a}{2} \pm iw, w = \sqrt{|a^2 - 4b|}$

$$y_1 = e^{\lambda_1 x} = e^{(-\frac{a}{2} + iw)x} = e^{-\frac{ax}{2}} e^{iwx} \quad y_2 = e^{\lambda_2 x} = e^{(-\frac{a}{2} - iw)x} = e^{-\frac{ax}{2}} e^{-iwx}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots = \cos x$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \dots = \left(1 - \frac{x^2}{2!} + \dots\right) + i\left(x - \frac{x^3}{3!} + \dots\right) = \sin x$$

$$\implies e^{iwx} = \cos wx + i \sin wx \quad (\text{de Moivre's theorem}) \quad \text{and} \quad e^{i\pi} = -1 \quad (\text{Euler's Identity})$$

The general solution is given by,  $y = e^{-\frac{ax}{2}} (c_1 \cos wx + c_2 \sin wx)$   $c_1, c_2$  are constants

## 7.4 EULER-CAUCHY EQUATIONS

The Euler-Cauchy equation is of the form:

$$x^2 y'' + ax y' + by = 0 \quad \text{where } a, b \text{ are constants} \quad (7.4.1)$$

$$\text{Let } y = x^m \implies y' = mx^{m-1} \implies y'' = m(m-1)x^{m-2}$$

$$\text{Substituting, } x^2 m(m-1)x^{m-2} + ax m x^{m-1} + b x^m = 0 \implies m^2 + (a-1)m + b = 0$$

$$m = \frac{1}{2}(1-a) \pm \sqrt{\frac{1}{4}(a-1)^2 - b}$$

Case 1: Roots are distinct. The basis solutions are :

$$y_1(x) = x^{m_1} \quad y_2(x) = x^{m_2}, \text{ the general solution is given by, } y = c_1 x^{m_1} + c_2 x^{m_2}$$

Case 2: Double roots.

$$b = \frac{1}{4}(1-a)^2 \quad m = \frac{1}{2}(1-a) \quad \boxed{y_1 = x^{\frac{1}{2}(1-a)}}$$

$$y'' + \frac{a}{x}y' + \frac{(1-a)^2}{4x^2}y = 0$$

Use method of reduction of order,  $y_2 = uy_1$  and with  $\boxed{p = \frac{a}{x}}$

$$\boxed{U = \frac{1}{y_1^2} e^{\int -p dx}}$$

$$\boxed{u = \int U dx}$$

$$\boxed{y_2 = y_1 \int U dx}$$

$$\boxed{y_2 = y_1 \int \frac{1}{y_1^2} e^{\int -p dx} dx}$$

$$\int p dx = \int \frac{a}{x} dx = a \ln x \implies e^{\int -p dx} = e^{-a \ln x} = e^{\ln x^{-a}} = x^{-a} = \frac{1}{x^a}$$

$$U = \frac{1}{y_1^2} \frac{1}{x^a} = \frac{1}{x^{1-a}} \frac{1}{x^a} = \frac{1}{x} \implies u = \int U dx = \int \frac{1}{x} dx = \ln x$$

$$y_2 = y_1 \int U dx = x^{\frac{1}{2}(1-a)} \ln x$$

$$\boxed{y_h = (c_1 + c_2 \ln x) x^{\frac{1}{2}(1-a)}} \quad c_1, c_2 \text{ are constants}$$

## 7.5 THE WRONSKIAN

Two solutions  $y_1$  and  $y_2$  are linearly dependent if their Wronskian  $W$  is 0.

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = 0$$

Because if the solutions are dependent,  $y_1 = k y_2$ , where  $k$  is a constant

$$\implies W(y_1, y_2) = y_1 y_2' - y_2 y_1' = k y_2 y_2' - y_2 k y_2' = 0$$

The Wronskian is expressed as a Wronski Determinant:

$$\boxed{W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \quad (7.5.1)$$

## 7.6 NON-HOMOGENEOUS ODE

Consider the following non-homogeneous ODE:

$$y'' + p(x)y' + q(x)y = r(x)$$

The complete solution is the sum of homogeneous ( $y_h$ ) and particular ( $y_p$ ) solutions.

$$y(x) = y_h(x) + y_p(x) \text{ where } y_h = c_1 y_1 + c_2 y_2 \text{ (general solution)}$$

$y_p$  is a solution of the non-homogeneous equation without any constants. A particular solution is obtained by assigning specific values to the constants. The Method of Undetermined Coefficients is an approach to finding a particular solution to nonhomogeneous ODEs. If the term in  $r(x)$  contains the following term, the choice for  $y_p(x)$  is given by:

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$Kx^n (n = 0, 1, \dots)$	$K_n x^n + K_{n-1} x^{n-1} + \dots + K_1 x + K_0$
$k \cos wx$ or $k \sin wx$	$K \cos wx + M \sin wx$
$ke^{\alpha x} \cos wx$ or $ke^{\alpha x} \sin wx$	$e^{\alpha x} (K \cos wx + M \sin wx)$

## 7.7 PARTICULAR SOLUTION BY VARIATION OF PARAMETERS (LAGRANGE)

The particular solution for the standard form ODE is derived as follows:

$$y'' + p(x)y' + q(x)y = r(x)$$

Find a pair of functions  $u_1(x)$  and  $u_2(x)$  such that:

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \implies y'_p(x) = u'_1y_1 + u_1y'_1 + u'_2y_2 + u_2y'_2$$

$$\text{Set constraint, } u'_1y_1 + u'_2y_2 = 0$$

$$y'_p(x) = u_1y'_1 + u_2y'_2$$

$$y''_p(x) = u'_1y'_1 + u_1y''_1 + u'_2y'_2 + u_2y''_2$$

Substituting,

$$(u'_1y'_1 + u_1y''_1 + u'_2y'_2 + u_2y''_2) + p(u_1y'_1 + u_2y'_2) + q(u_1y_1 + u_2y_2) = r$$

$$(y''_1 + py'_1 + qy_1)u_1 + (y''_2 + py'_2 + qy_2)u_2 + (u'_1y'_1 + u'_2y'_2) = r$$

Since  $y_1$  and  $y_2$  are solutions to the homogeneous ODE,

$$u'_1y'_1 + u'_2y'_2 = r$$

We now have the following simultaneous equations:

$$u'_1y_1 + u'_2y_2 = 0$$

$$u'_1y'_1 + u'_2y'_2 = r$$

Solving,

$$u'_1 = -\frac{y_2r}{y_1y'_2 - y'_1y_2} = -\frac{y_2r}{W}$$

$$u'_2 = -\frac{y_1r}{y_1y'_2 - y'_1y_2} = -\frac{y_1r}{W}$$

$$y_p(x) = -y_1 \int \frac{y_2r}{W} dx + y_2 \int \frac{y_1r}{W} dx$$

## 7.8 SyMPy

```

1 import sympy as sp
2
3 # Define the variable and the unknown function
4 x = sp.symbols('x')
5 y = sp.Function('y')
6
7 # Define the differential equation
8 ode = sp.Eq(sp.diff(y(x), x, 2) - 3*sp.diff(y(x), x) + 2*y(x), 0)
9
10 # Solve the ODE
11 solution = sp.dsolve(ode)
12 print(sp.latex(ode))
13 display(ode)
14
15 print(sp.latex(solution))
16 display(solution)

```

$$2y(x) - 3\frac{d}{dx}y(x) + \frac{d^2}{dx^2}y(x) = 0$$

$$y(x) = (C_1 + C_2e^x)e^x$$