Chapter 26

Tensors

Tensors form a rigorous mathematical framework for describing physical quantities in a way that does not depend on the choice of coordinate system. By extending the familiar notions of scalars and vectors, tensors provide a unified language for expressing laws of physics, and are central to continuum mechanics, electromagnetism, relativity, and much of modern physics.

26.1 Scalars and Vectors

26.1.1 Scalars

A scalar is a quantity that is fully described by a single number and remains unchanged under coordinate transformations.

Examples include mass, temperature, and energy.

26.1.2 Vectors

A vector is a physical quantity characterized by both **magnitude** (how large it is) and **direction** (which way it points). Unlike scalars, which have only magnitude (e.g., temperature or mass), vectors describe quantities like displacement, velocity, or force.

In a 3D Cartesian coordinate system, we represent a vector **A** using its components along the basis vectors $\mathbf{e}_1 = \hat{i}$, $\mathbf{e}_2 = \hat{j}$, $\mathbf{e}_3 = \hat{k}$:

$$\mathbf{A} = A^i \mathbf{e}_i = A^1 \mathbf{e}_1 + A^2 \mathbf{e}_2 + A^3 \mathbf{e}_3 = (A^1, A^2, A^3)$$

Here, A^i are the *contravariant components* (with the Einstein summation convention implying summation over $i = 1, 2, 3$), and the parentheses denote the standard component tuple.

Example

The vector pointing 3 units east, 4 units north, and 0 units up is $\mathbf{A} = (3, 4, 0)$, with magnitude

$$\|\mathbf{A}\| = \sqrt{3^2 + 4^2 + 0^2} = 5.$$

Under a change of coordinates, the components transform linearly.

26.2 Coordinate Transformations

Consider two coordinate systems related by a linear transformation

$$\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j} x^j$$

The transformation matrix is defined as

$$\tilde{\mathbf{x}} = \Lambda \mathbf{x} \tag{26.1}$$

where:

- $\mathbf{x} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$ (original coordinates)
- $\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x}^1 \\ \tilde{x}^2 \\ \tilde{x}^3 \end{pmatrix}$ (new coordinates)
- $\Lambda = \left(\frac{\partial \tilde{x}^i}{\partial x^j} \right)$ is the Jacobian matrix

$$\Lambda^i_j = \frac{\partial \tilde{x}^i}{\partial x^j}$$

Explicit matrix form

$$\begin{pmatrix} \tilde{x}^1 \\ \tilde{x}^2 \\ \tilde{x}^3 \end{pmatrix} = \begin{pmatrix} \frac{\partial \tilde{x}^1}{\partial x^1} & \frac{\partial \tilde{x}^1}{\partial x^2} & \frac{\partial \tilde{x}^1}{\partial x^3} \\ \frac{\partial \tilde{x}^2}{\partial x^1} & \frac{\partial \tilde{x}^2}{\partial x^2} & \frac{\partial \tilde{x}^2}{\partial x^3} \\ \frac{\partial \tilde{x}^3}{\partial x^1} & \frac{\partial \tilde{x}^3}{\partial x^2} & \frac{\partial \tilde{x}^3}{\partial x^3} \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{26.2}$$

26.3 Definition of a Tensor

A tensor is a mathematical object defined by how its components transform under coordinate changes. Specifically, a tensor of type (k, l) transforms as:

$$T'^{i_1 \ldots i_k}_{j_1 \ldots j_l} = \frac{\partial x'^{i_1}}{\partial x^{m_1}} \cdots \frac{\partial x'^{i_k}}{\partial x^{m_k}} \frac{\partial x^{n_1}}{\partial x'^{j_1}} \cdots \frac{\partial x^{n_l}}{\partial x'^{j_l}} T^{m_1 \ldots m_k}_{n_1 \ldots n_l}$$

Vectors as special cases

- Contravariant vector $(1, 0)$:

$$V'^i = \frac{\partial x'^i}{\partial x^j} V^j \tag{26.3}$$

- Covariant vector $(0, 1)$:

$$W'_i = \frac{\partial x^j}{\partial x'^i} W_j \tag{26.4}$$

Key insight: Tensors are the objects whose transformation rules are consistent across coordinate systems, distinguishing them from general functions.

26.3.1 Contravariant Vectors

A contravariant vector V^i transforms according to the chain rule:

$$\tilde{V}^i = \frac{\partial \tilde{x}^i}{\partial x^j} V^j \tag{26.5}$$

The components "stretch" with the new coordinate basis.

26.3.2 Covariant Vectors (Dual Vectors)

A covariant vector W_i (one-form) transforms inversely:

$$\tilde{W}_i = \frac{\partial x^j}{\partial \tilde{x}^i} W_j \tag{26.6}$$

This ensures $W_i V^i$ remains invariant under coordinate changes.

Note the pattern reciprocity: Contravariant uses $\frac{\partial \text{new}}{\partial \text{old}}$, covariant uses $\frac{\partial \text{old}}{\partial \text{new}}$.

Why Covariant Vectors Are Called Dual Vectors

A covariant vector is called a **dual vector** because it belongs to the **dual space** V^* of the original vector space V.

Dual Space Concept

For any vector space V with basis $\{\mathbf{e}_i\}$:

- **Vectors** in V: $V = v^i \mathbf{e}_i$ (contravariant components)
- **Dual vectors** in V^*: $\omega = \omega_i \mathbf{e}^i$ where $\{\mathbf{e}^i\}$ is the **dual basis**

Dual basis definition: $\mathbf{e}^i(\mathbf{e}_j) = \delta^i_j$ (Kronecker delta)

Perfect Pairing

The scalar product is **invariant**:

$$\omega(\mathbf{v}) = \omega_i v^i$$

When coordinates change, contravariant components v^i "stretch" one way, covariant ω_i stretch oppositely—**duality ensures the product stays constant**.

Transformation Duality

$$\tilde{v}^i = \frac{\partial \tilde{x}^i}{\partial x^j} v^j \qquad \text{(stretches with new basis)}$$

$$\tilde{\omega}_i = \frac{\partial x^j}{\partial \tilde{x}^i} \omega_j \qquad \text{(stretches inversely)}$$

Analogy: Vectors point **with** the basis arrows $\rightarrow$, dual vectors measure **across** them $\leftrightarrow$.

26.4 Metric Tensor

The metric tensor g_{ij} defines geometry—distances, angles, and volumes in curved spaces.

The squared infinitesimal distance (line element) is:

$$ds^2 = g_{ij}\,dx^i dx^j$$

g_{ij} is symmetric ($g_{ij} = g_{ji}$) and defines the *inner product*.

Example: In flat 3D Cartesian: $g_{ij} = \delta_{ij}$. In polar: $g_{rr} = 1$, $g_{\theta\theta} = r^2$.

26.5 Raising and Lowering Indices

The metric tensor allows conversion between covariant and contravariant components.

Lowering an index is performed by

$$V_i = g_{ij}V^j$$

Raising an index is performed using the inverse metric g^{ij}

$$V^i = g^{ij}V_j$$

26.6 General Tensors

A tensor of type (p, q) has p contravariant indices and q covariant indices.

Its components transform as

$$\tilde{T}^{i_1\ldots i_p}_{\;j_1\ldots j_q} = \frac{\partial \tilde{x}^{i_1}}{\partial x^{k_1}} \cdots \frac{\partial \tilde{x}^{i_p}}{\partial x^{k_p}} \frac{\partial x^{l_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{l_q}}{\partial \tilde{x}^{j_q}} T^{k_1\ldots k_p}_{\;l_1\ldots l_q}$$

26.7 Tensor Operations

26.7.1 Addition and Scalar Multiplication

Tensors of the same type may be added componentwise.

26.7.2 Tensor Product

The tensor product of two tensors A and B produces a tensor of higher rank

$$(A \otimes B)^{ij} = A^i B^j$$

26.7.3 Contraction

Contraction reduces the rank of a tensor by summing over one upper and one lower index

$$T^i{}_i$$

26.8 Important Physical Tensors

26.8.1 Stress Tensor

In continuum mechanics, the stress tensor σ_{ij} relates force to area

$$F_i = \sigma_{ij} n^j$$

where n^j is the normal vector.

26.8.2 Moment of Inertia Tensor

The moment of inertia tensor is defined as

$$I_{ij} = \sum_k m_k (\delta_{ij} r_k^2 - x_{k,i} x_{k,j})$$

26.8.3 Electromagnetic Field Tensor

In relativistic electrodynamics, the electromagnetic field is represented by a rank-2 tensor.

26.9 Tensor Calculus

26.9.1 Partial Derivatives

The partial derivative of a tensor is generally not a tensor.

26.9.2 Covariant Derivative

To preserve tensorial character, the covariant derivative is introduced

$$\nabla_k V^i = \partial_k V^i + \Gamma^i_{kj} V^j$$

where Γ^i_{kj} are the Christoffel symbols.

26.10 Closing Remarks

Tensors provide a coordinate-independent language for physical laws. Their systematic use unifies diverse areas of physics and enables the formulation of laws valid in arbitrary coordinate systems.