Chapter 2
Function

Function, function, everywhere,
Through science and engineering, in the air.
From gravity's pull to currents that flow,
They shape the truths we seek to know.

2.1 Introduction

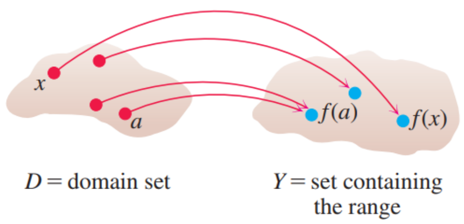
Function, in mathematics, is an expression, rule, or law that defines a relationship between one variable, the independent variable and another variable, which is the dependent variable. Functions are essential for formulating physical relationships in the sciences and are ubiquitous in mathematics.

2.2 Function, Domain & Range

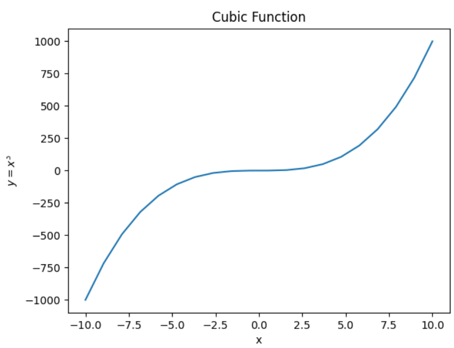
When a value of one variable x depends on another variable y, we say that y *is a function of* x and it is written symbolically as:

$\boxed{y = f(x)}$ and pronounced as **"y equals f of x"**

Formally, a function f, is a rule that assigns an *unique* value $f(x)$ for each x in D where D is known as the ***Domain*** and the set of $y = f(x)$, or Y, is known as the ***Range***.



cb Example:
$y = x^2$
Domain $= [-\infty, +\infty]$
Range $= [0, +\infty]$

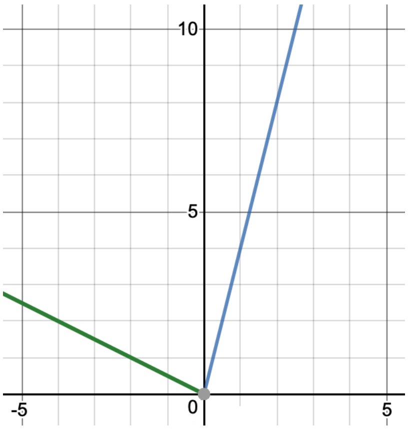


2.3 Piecewise Continuous & Discontinuous Functions

Sometimes a function is described in pieces by using different formulas on different parts of its domain.

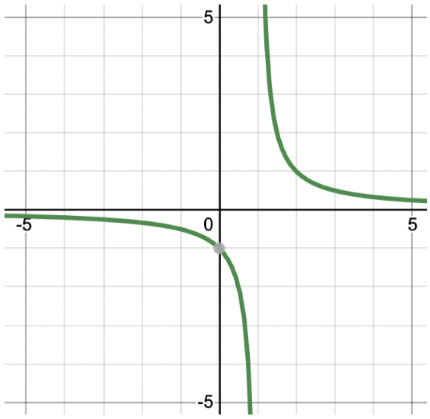
$$|y| = \begin{cases} 4x & \text{if } x \geq 0 \\ -0.5x & \text{if } x < 0 \end{cases}$$

y is ***unique*** for a given x. Such functions are ***piecewise continous*** as there are no ”gaps”.



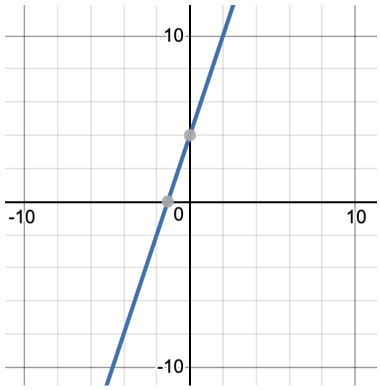
$$y = \frac{1}{x-1}$$

y does not exist for $x = 1$; The curve is not continuous at $x = 1$ and the function is ***discontinous***.

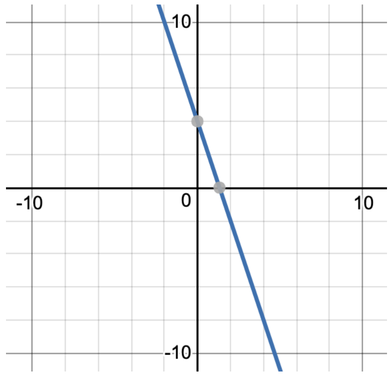


2.4 Increasing & Decreasing Functions

Increasing function:
$f(x_2) > f(x_1)$ when $x_2 > x_1$
Example: $y = 3x + 4$

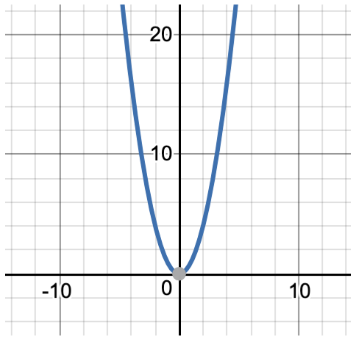


Decreasing function
$f(x_2) < f(x_1)$ when $x_2 > x_1$
Example: $y = -3x + 4$

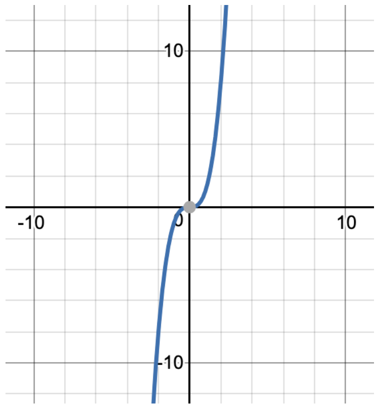


2.5 Even & Odd Functions

Even function $f(-x) = f(x)$
Example: $y = x^2$



Odd function $f(-x) = -f(x)$
Example: $y = x^3$



2.6 Types of Functons

Following are some types of functions.

1. **Linear** Functions $f(x) = mx + b$
2. **Polynomial** Functions $f(x) = a_0 + a_1x + a_2x^2 + \cdots a_{n-1}x^{n-1} + a_nx^n$
   $n = 2 \rightarrow$ Quadratic, $n = 3 \rightarrow$ Cubic.
3. **Rational** Functions $f(x) = p(x)/q(x)$
4. **Algebraic** Functions - constructed from polynomials using algebraic operations ($+, -, \times, \div$, and roots)
5. **Trigonometric** functions, e.g., $f(x) = sin(x)$
6. **Exponential** Functions, e.g., $y = 2^x$, Logarithmic Functions $y = log_5^x$

7. **Transcendental** Functions - functions that are **not expressible as a finite combination of algebraic operations** of addition, subtraction, multiplication, division, raising to a power, and extracting a root. E.g., *log x*, *sin x*, e^x and any functions containing them. Such functions are expressible in algebraic terms only as infinite series. In general, the term **transcendental means nonalgebraic**.

2.7 Sums, Differences, Products & Quotients of Functions

Much like numbers, functions can be added, subtracted, multiplied, and divided. By defnition:

$$
\begin{aligned}
(f+g)(x) &= f(x)+g(x) \\
(f-g)(x) &= f(x)-g(x) \\
(fg)(x) &= f(x)g(x) \\
\frac{f}{g}(x) &= \frac{f(x)}{g(x)} \text{ where } g(x) \neq 0
\end{aligned}
$$

2.8 Function Composition

The output from is one function is the input to the second function.

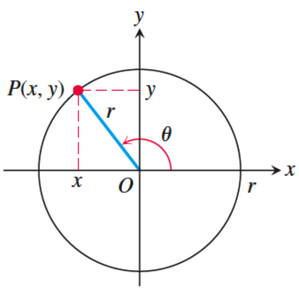
$$\boxed{(f \circ g)(x) = f(g(x))}$$

2.9 Vertical & Horizontal Scaling, Reflecting a Function

Following are the transformation equations:

$$
\begin{aligned}
y = &\quad cf(x) && \text{for c > 1, stretch vertically} \\
y = &\quad \tfrac{1}{c}f(x) && \text{for c > 1, compress vertically} \\
y = &\quad f(cx) && \text{for c > 1, stretch horizontally} \\
y = &\quad f\left(\tfrac{x}{c}\right) && \text{for c > 1, compress horizontally} \\
y = &\quad -f(x) && \text{for c = -1, reflect across x axis} \\
y = &\quad f(-x) && \text{for c = -1, reflect across y axis}
\end{aligned}
$$

2.10 Basic Trigonometric Function Definitions



b = base
p = perpendicular
r = h (hypoetenuse)
$b^2 + p^2 = r^2$
(Pythagoras)

$$\sin\theta = \frac{p}{h} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{b}{h} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{p}{b} = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

2.11 Basic Trigonometric Identities

Considering a triangle with side lengths a, b, and c, and angles A, B, and C, opposite to these sides respectively, the following identities can be readily derived using the fundamental definitions of trigonometric relationships. These identities are not only foundational in geometry but also serve as powerful tools in solving a wide range of problems in mathematics, physics, and engineering.

$$\boxed{sin^2\ \theta + \cos^2\ \theta = 1}$$

$$\boxed{sec^2\ \theta = 1 + tan^2\ \theta}$$

$$\boxed{csc^2\ \theta = 1 + cot^2\ \theta}$$

$$\boxed{sin\ (\theta_1 + \theta_2) = sin\ \theta_1\ cos\ \theta_2 + cos\ \theta_1\ sin\ \theta_2}$$

$$\boxed{cos\ (\theta_1 + \theta_2) = cos\ \theta_1\ cos\ \theta_2 - sin\ \theta_1\ sin\ \theta_2}$$

$$\boxed{c^2 = a^2 + b^2 - 2ab\ cos\ \theta \qquad \text{(Law of Cosine)}}$$

$$\boxed{\left(\frac{sin\ A}{a}\right) = \left(\frac{sin\ B}{b}\right) = \left(\frac{sin\ C}{c}\right) \qquad \text{(Law of Sine)}}$$

Additional trigonometric identities can be derived from the above fundamental identities.

2.12 SymPy

Plot the function $y = x^3$.

```python
import sympy as sp
import numpy as np
import matplotlib.pyplot as plt
from IPython.display import display, Math
from sympy import sqrt, diff, integrate, oo
from sympy import sin, cos, tan, ln, exp, erf, trigsimp, expand_trig, simplify
from sympy import sinh, cosh

x = sp.symbols('x')
y = x**3

lamb_y = lamb_y = sp.lambdify(x,y)
x_num = np.linspace(start = -10, stop = 10, num = 20)
y_num = lamb_y(x_num)
display(x_num)
display(y_num)

plt.plot(x_num, y_num)
plt.xlabel('x')
plt.ylabel('$y = x^3$')
plt.title('Cubic Function')
```

```python
display(y)
plt.savefig("plot.png")
plt.show()
```

```
array([-10.       , -8.94736842, -7.89473684, -6.84210526,
-5.78947368, -4.73684211, -3.68421053, -2.63157895,
-1.57894737, -0.52631579, 0.52631579, 1.57894737,
2.63157895, 3.68421053, 4.73684211, 5.78947368,
6.84210526, 7.89473684, 8.94736842, 10.        ])
array([-1.00000000e+03, -7.16285173e+02, -4.92054235e+02, -3.20309083e+02,
-1.94051611e+02, -1.06283715e+02, -5.00072897e+01, -1.82242309e+01,
-3.93643388e+00, -1.45793847e-01, 1.45793847e-01, 3.93643388e+00,
1.82242309e+01, 5.00072897e+01, 1.06283715e+02, 1.94051611e+02,
3.20309083e+02, 4.92054235e+02, 7.16285173e+02, 1.00000000e+03])
```