Chapter 10

Vector

Vectors were first used to express the laws of electromagnetism. Since that time, vectors have become essential in physics, mechanics, electrical engineering, and other sciences to describe forces mathematically.

Some quantities in physics are characterized by both magnitude and direction, such as displacement, velocity, force and acceleration. To describe such quantities, we introduce the concept of a **vector** as a directed line segment. There are other quantities in physics that are characterized by magnitude only, such as mass, length and temperature. Such a quantity is called a **scalars.**

For example, speed, say 10 KM/Hr is a scalar whereas velocity, say 10 KM/Hr towards north-east is a vector and is denoted as:

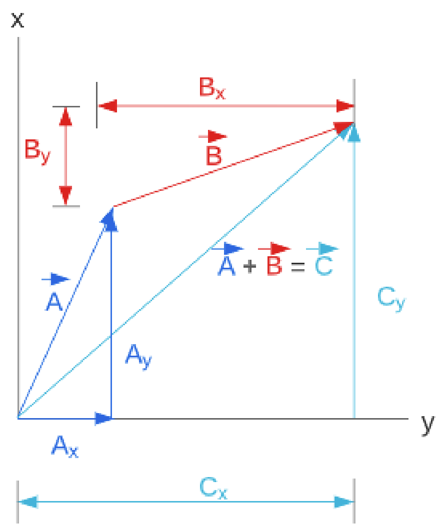
$$\vec{v} = 10 \cdot \frac{1}{\sqrt{2}}\mathbf{i} + 10 \cdot \frac{1}{\sqrt{2}}\mathbf{j}$$

where **i** is an unit vector along the x direction and **j** is an unit vector along the y direction.

10.1 Vector Algebra

10.1.1 Vector Addition & Subtraction

Vector addtion (or subtraction) is performed by adding (or subtracting) their components.



$$\vec{A} + \vec{B} = \vec{C}$$

$$A_x + B_x = C_x$$

$$A_y + B_y = C_y$$

10.1.2 Scalar Multiplication

Multiplication of a vector $\vec{A}$ by a scalar m produces a vector $m\vec{A}$ with magnitude $m \times \|A\|$ where $\|A\|$ is the magnitude of $\vec{A}$.

10.1.3 Unit Vector

Unit vectors are vectors having unit length.

$$A = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$$

$$\|A\| = \sqrt{A_1^2 + A_2^2 + A_3^2} = 1$$

10.1.4 Linear Independence & Dependence

Vectors $\vec{A}_1, \vec{A}_2, \cdots, \vec{A}_n$ are linearly dependent if there exist scalars $\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n$, not all zero, such that:

$$a_1\vec{A}_1 + a_2\vec{A}_2 + \cdots + a_n\vec{A}_n = 0$$

Otherwise, the vectors are linearly independent.

10.1.5 Scalar & Vector Fields

For each point (x, y, z) of a region D in space, if there corresponds a number (scalar) $\phi(x, y, z)$, then ϕ is called a scalar function of position and we say that a scalar field f has been defined on D. A scalar field ϕ, which is independent of time, is called a stationary or steady-state scalar field.

For each point (x, y, z) of a region D in space, if there corresponds a vector $V(x, y, z)$, then $\vec{V}$ is called a vector function of position, and we say that a vector field $\vec{V}$ has been defined on D. A vector field $\vec{V}$ which is independent of time is called a stationary or steady-state vector field.

10.1.6 Vector Space R^n

Let $V = R^n$ where R^n consists of all n-element sequences $u = (a_1, a_2, \cdots, a_n)$ of real numbers called the components of u. The term vector is used for the elements of V and we denote them using the letters u, v, and w, with or without a subscript. The real numbers arel scalars and we denote them using letters other than u, v, or w.

We define two operations on $V = R^n$:

$$\vec{u} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad \vec{v} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad \vec{u} + \vec{v} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

where a_i, b_i are the components of vectors $\vec{u}$ and $\vec{v}$ and,

$$k\vec{u} = \begin{bmatrix} ku_1 \\ ku_2 \\ \vdots \\ ku_n \end{bmatrix}$$

10.2 Vector Spaces

Vectors with n real numbers as components are elements of **real n dimensional vector space R^n.** Each vector in R^n is an ordered **n-tuple** of real numbers. Instead of real numbers, we can have complex numbers to obtain the **complex vector space.**

10.2.1 Dimension

For a non-empty set $\vec{V}$ of vectors where each vector has the same number of components. If, for any two vectors $\vec{a}$ and $\vec{b}$ in $\vec{V}$, all linear combinations $\alpha a + \beta b$ where α, β are real numbers, are also elements of $\vec{V}$.

The maximum number of linearly independent vectors in $\vec{V}$ is called the dimension of $\vec{V}$ and is denoted as **dim**$\vec{V}$. Hence, a vector space having vectors with **n** components has the dimension **n**.

10.2.2 Basis

A linearly independent set in $\vec{V}$ consisting of a maximum possible number of vectors in $\vec{V}$ is called the **basis** for $\vec{V}$.

10.2.3 Span

Span (a vector space) is the set of all linear combinations of the vectors.

10.2.4 Subspace

Subspace of A is a non-empty subset of **V** including **V** itself.

10.3 Vector Products

10.3.1 Dot Product

The dot or scalar product of two vectors A and B, denoted by $A \cdot B$, is defined as the product of the magnitudes of A and B and the cosine of the angle θ between them.

$$\boxed{A\cdot = |A||B|cos\theta, \quad 0 \leq \theta \leq \pi} \tag{10.1}$$

10.3.2 Inner Product

An inner product is a generalization of the dot product. In a vector space, it is a way to multiply vectors together, with the result of this multiplication being a scalar. an inner product $< \cdot, \cdot >$ satisfies the following four properties. Let u, v, and w be vectors and α be a scalar, then:

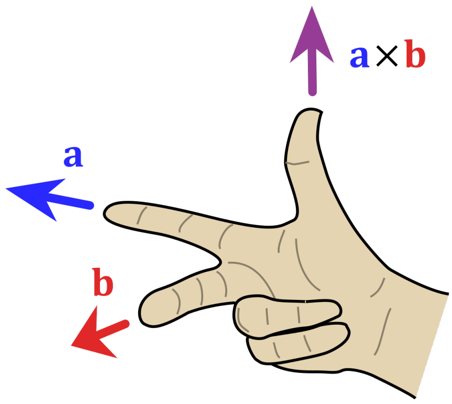
1. $< u + v, w >=< u, w > + < v, w >$
2. $< \alpha v, w >= \alpha < v, w >$
3. $< v, w >=< w, v >$
4. $< v, v >\geq 0$ and equal if and only if $v = 0$.

10.3.3 Cross Product

The cross product of vectors $\vec{A}$ and $\vec{B}$ is a vector $\vec{C} = \vec{A} \times \vec{B}$ (read as $\vec{A}$ *cross* $\vec{B}$) defined as follows.

$$\boxed{\vec{C} = \vec{A} \times \vec{B} = |A||B| sin\,\theta\, \hat{u}, \quad 0 \leq \theta \leq \pi}$$

The magnitude of $\vec{C} = \vec{A} \times \vec{B}$ is equal to the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the *sine* of the angle θ between them. The direction of $\vec{C}$ is perpendicular to the plane of $\vec{A}$ and $\vec{B}$ so that $\vec{A}$, $\vec{B}$, and $\vec{C}$ form a right-handed system. where $\hat{u}$ is a unit vector indicating the direction of $\vec{A} \times \vec{B}$.



The cross product of two vectors can be expressed in terms of *determinant* as follows:

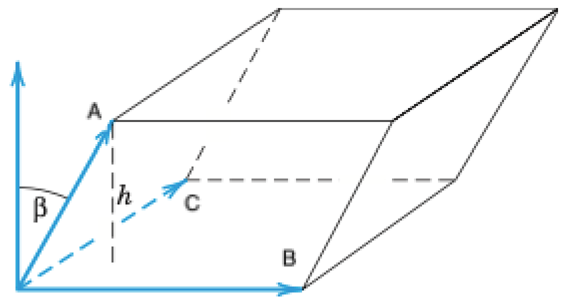
$$\vec{A} \times \vec{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \begin{vmatrix} A_2 & A_3 \\ B_2 & B_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} A_1 & A_3 \\ B_1 & B_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix} \mathbf{k}$$

10.3.4 Scalar Triple Product

The scalar triple product of three vectors $\vec{A}, \vec{B}, \vec{C}$ is defined as:

$$\boxed{(\vec{A}\,\vec{B}\,\vec{C}) = \vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = det[\vec{A}\,\vec{B}\,\vec{C}]}$$

Geometrically, the absolute value of $\|(\vec{A}\ \vec{B}\ \vec{C})\|$ is the volume of the parallelepiped with $\vec{A}, \vec{B}, \vec{C}$ as edge vectors. The three vectors in R^3 are linearly independent if and only if their scalar triple product is not zero.



Properties of scalar triple product are as follows:

$$(\vec{A}\ \vec{B}\ \vec{C}) = \vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$$
$$\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$$

10.3.5 Reciprocal Set

A reciprocal set $a^{'}$ satisfies the following:

$$a \cdot a^{'} = 1$$

10.3.6 Vector Properties

Given three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$; they satisfy the following properties:

1. Commutative: $\vec{A} + \vec{B} = \vec{B} + \vec{A}$
2. Associative: $(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$
3. Distributative: $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$
4. Distributative: $\vec{A} \times (\vec{B} + \vec{C}) = (\vec{A} \times \vec{B}) + (\vec{A} \times \vec{C})$
5. Distributive: $(\vec{A} + \vec{B}) \times \vec{C} = (\vec{A} \times \vec{C}) + (\vec{B} \times \vec{C})$
6. Zero Vector: $\vec{A} + 0 = \vec{A}$
7. Scalar Multiplication by m: $m(\vec{A} + \vec{B}) = m\vec{A} + m\vec{B}$
8. Inner Product: $(\vec{A}, \vec{B}) = \vec{A} \cdot \vec{B} = \vec{A}^{\,T}\vec{B}$ (n-Dimensional Euclidean Space)
9. Linear Transformation of Space R^n into Space R^m:
   $X = R^n, Y = R^m, Y = AX$ where A is an $m \times n$ matrix.

10.3.7 Gram–Schmidt orthonormalization

The Gram–Schmidt orthonormalization process is a procedure for orthonormalizing a set of vectors in an inner product space. Let $\{v_1, v_2, \cdots, v_k\}$ to be a non-orthonormal basis for V. Then, we need to determine $\{u_1, u_2, \cdots, u_k\}$ an orthonormal basis for the span of $\{v_1, v_2, \cdots v_p\}$. We define the projection operator by:

$$proj_u(v) = \frac{< u, v >}{< u, u >} u$$

where $< u, v >$ denotes the inner product of the vectors u and v. This operator projects the vector v orthogonally onto the line spanned by vector u. The Gram–Schmidt process is given by:

$$\begin{aligned}
u_1 &= v_1 & e_1 &= \frac{u_1}{||u_1||} \\
u_2 &= v_2 - proj_{u_1}(v_2) & e_2 &= \frac{u_2}{||u_2||} \\
u_3 &= v_3 - proj_{u_1}(v_3) - proj_{u_2}(v_3) & e_3 &= \frac{u_3}{||u_3||} \\
&\vdots \\
u_k &= v_k - \sum_{j=1}^{k-1} proj_{u_j}(v_k) & e_k &= \frac{u_k}{||u_k||}
\end{aligned}$$

The sequence $u_1, u_2, \cdots, u_k$ is the required system of orthogonal vectors, and the normalized vectors $e_1, e_2, \cdots, e_k$ form an orthonormal set. The calculation of the sequence $u_1, u_2, \cdots, u_k$ is known as Gram–Schmidt orthogonalization, while the calculation of the sequence $e_1, e_2, \cdots, e_k$ is known as Gram–Schmidt orthonormalization as the vectors are normalized.

10.4 Vector Differentiation

$$\frac{\Delta\vec{R}}{\Delta\vec{u}} = \frac{\vec{R}(\vec{u} + \Delta\vec{u}) - \vec{R}(\vec{u})}{\Delta\vec{u}}$$

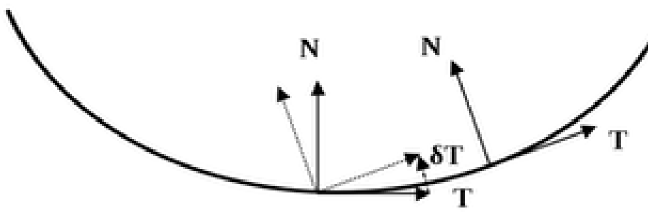
$$\frac{d\vec{R}}{d\vec{U}} = \lim_{\Delta\vec{u}\to 0} \frac{\Delta\vec{R}}{\Delta\vec{u}} = \lim_{\Delta\vec{u}\to 0} \frac{\vec{R}(\vec{u} + \Delta\vec{u}) - \vec{R}(\vec{u})}{\Delta\vec{u}}$$

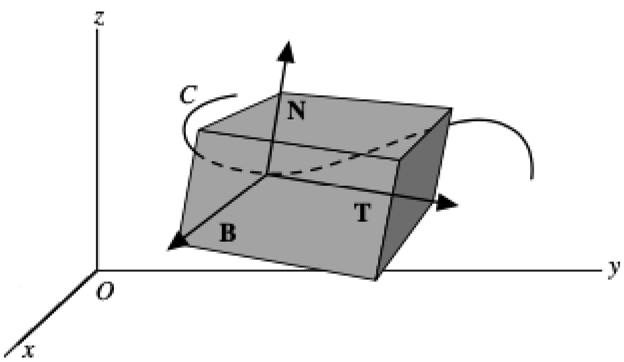
If, $r(u) = x(u)\mathbf{i} + y(u)\mathbf{j} + z(u)\mathbf{k}$

$$\frac{dr}{du} = \frac{dx}{du}\mathbf{i} + \frac{dy}{du}\mathbf{j} + \frac{dz}{du}\mathbf{k}$$

10.5 Differential Geometry

10.5.1 Frenet–Serret Formulae





Consider C to be a space curve defined by the function $r(u)$. Then, dr/du is a vector in the direction of the tangent to C. If the scalar u is taken as the arc length s measured from some fixed point on C, then dr/ds is a unit tangent vector to C and is denoted by T. The rate at which T changes with respect to s is a measure of the curvature of C and is given by dT/ds. The direction of dT/ds at any given point on C is normal to the curve at that point. If N is a unit vector in this normal direction, it is called the ***principal normal*** to the curve. Then $dT/ds = kN$, where k is called the curvature of C at the specified point. The quantity $\rho = 1/k$ is called the ***radius of curvature***.

A unit vector B perpendicular to the plane of T and N and such that $B = T \times N$, is called the ***binormal*** to the curve. It follows that directions T, N, B form a localized right-handed rectangular coordinate system at any specified point of C. This coordinate system is called the trihedral or triad at the point. As s changes, the coordinate system moves and is known as the moving trihedral. The ***Frenet–Serret*** formulae are given by:

$$\boxed{\frac{dT}{ds} = \kappa N} \quad \boxed{\frac{dN}{ds} = \tau B - \kappa T} \quad \boxed{\frac{dB}{ds} = \tau N} \tag{10.2}$$

where τ is a scalar called the **torsion.** The quantity $s = 1/\tau$ is called the **radius of torsion**. The osculating plane to a curve at a point P is the plane containing the tangent and principal normal at P. The normal plane is the plane through P perpendicular to the tangent. The rectifying plane is the plane through P, which is perpendicular to the principal normal.

10.5.2 Gradient

The differential operator **del**, written as ∇ is defined as:

$$\boxed{\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial y}\mathbf{k}}$$

if $\phi(x, y, z)$ be a scalar function defined and differentiable at each point (x, y, z) in a certain region of space, then the gradient of ϕ, written $\nabla\phi$ or $grad\ \phi$ is defined as follows:

$$\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial y}\mathbf{k}\right)\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

10.5.3 Divergence

If $V(x, y, z) = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}$ is defined and differentiable at each point (x, y, z) in a region of space, then the divergence of V, a scalar, is defined as follows:

$$\boxed{\nabla \cdot V = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial y}\mathbf{k}\right) \cdot (V_i\mathbf{i} + V_j\mathbf{j} + V_k\mathbf{k}) = \frac{\partial V_i}{\partial x}\mathbf{i} + \frac{\partial V_j}{\partial y}\mathbf{j} + \frac{\partial V_k}{\partial z}\mathbf{k}\ \text{(scalar)}}$$

10.5.4 Curl

The **curl** or rotation of V, a vector, is defined as:

$$\boxed{curl\ V = \nabla \times V = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial y}\mathbf{k}\right) \times (V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}}$$

10.6 Vector Integration

10.6.1 Line Integration

Let $A(x, y, z) = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ be a vector function of position defined and continuous along C. Then the integral of the tangential component of $\vec{A}$ along C from P_1 to P_2, written as:

$$\int_{P_1}^{P_2} \vec{A} \cdot d\vec{r} = \int_C \vec{A} \cdot d\vec{r} = \int_C (A_1dx + A_2dy + A_3dz)$$

If C is a closed curve (which we shall suppose is a simple closed curve, that is, a curve that does not intersect itself anywhere), the integral around C is often denoted by:

$$\boxed{\oint_C \vec{A} \cdot d\vec{r} = \oint_C (A_1dx + A_2dy + A_3dz)}$$

10.6.2 Surface Integration

Consider a differential of surface area dS a vector **dS** whose magnitude is dS and whose direction is that of $\hat{n}$. Then $d\vec{S} = \hat{n}dS$. The integral is given by:

$$\boxed{\iint_S \vec{A} \cdot d\vec{S} = \iint_S \vec{A} \cdot \hat{n}\ dS}$$

10.6.3 Volume Integration

Consider a closed surface in space enclosing a volume V. The volume integral is given by:

$$\boxed{\iiint_V A\ dV}$$

10.6.4 Gauss' Divergence Theorem

Suppose V is the volume bounded by a closed surface S and $\vec{F}$ is a vector function of position with continuous derivatives. Then:

$$\iiint_V \nabla\cdot\vec{F}\,dV = \iiint_V \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial y}\mathbf{k}\right)\cdot(F_x\mathbf{i}+F_y\mathbf{j}+F_z\mathbf{k})\,dV$$

$$= \iiint_V \frac{\partial F_x}{\partial x}\,dx\,dy\,dz + \iiint_V \frac{\partial F_y}{\partial y}\,dx\,dy\,dz + \iiint_V \frac{\partial F_z}{\partial z}\,dx\,dy\,dz$$

$$\text{Now, } \iiint_V \frac{\partial F_z}{\partial z}\,dx\,dy\,dz = \int_{x_1}^{x_2}\int_{y_1}^{y_2}[F_z(x,y,z_2)-F_z(x,y,z_1)]\,dx\,dy = \int_{S_1}\int_{S_2} F_z\cdot ds$$

$$\implies \boxed{\iiint_V \nabla\cdot\vec{F}\,dV = \iint_S F_x\cdot ds + \iint_S F_y\cdot ds + \iint_S F_z\cdot ds = \oiint \vec{F}\cdot dS}$$

10.6.5 Stoke's Theorem

The line integral of a vector field over a loop is equal to the flux of its curl through the enclosed surface. Suppose S is an open, two sided surface bounded by a closed, non intersecting curve C (simple closed curve), and suppose $\vec{F}$ is a vector function of position with continuous derivatives. Then,

$$\oint_C \vec{F}\cdot dl = \oiint_S (\nabla\times\vec{F})\cdot\vec{n}\,dS$$

$$curl\,\vec{F} = \nabla\times\vec{F} = \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial y}\mathbf{k}\right)\times(F_x\mathbf{i}+F_y\mathbf{j}+F_z\mathbf{k}) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_x & F_y & F_z\end{vmatrix}$$

$$= \begin{vmatrix}\frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_y & F_z\end{vmatrix}\mathbf{i} - \begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial z}\\ F_x & F_z\end{vmatrix}\mathbf{j} + \begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ F_x & F_y\end{vmatrix}\mathbf{k}$$

$$= \left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\mathbf{i} - \left(\frac{\partial F_z}{\partial x}-\frac{\partial F_x}{\partial z}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\mathbf{k}$$

$$= \left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\mathbf{k}$$

$$\oiint_S (\nabla\times\vec{F})\cdot\vec{n}\,dS = \oiint_S \left[\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\mathbf{k}\right]\cdot\hat{\mathbf{n}}\,dS$$

Using a parametric representation of the surface we have,

$$r(u,v) = [x(u,v), y(u,v), z(u,v)] = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

$$\text{Normal Vector, } \vec{N} = \vec{r}_u\times\vec{r}_v,\quad \text{Unit Normal Vecrtor } \hat{\mathbf{n}} = \frac{1}{\|N\|}\vec{N}$$

$$\|\vec{r}_u\times\vec{r}_v\| = \|N\| \quad \text{(area of the parallelogram with sides } \vec{r}_u \text{ and } \vec{r}_v\text{)}$$

$$\implies \boxed{\vec{n}\,dS = \hat{\mathbf{n}}\,\|N\|\vec{du}\cdot\vec{dv} = \vec{N}\,\vec{du}\cdot\vec{dv}}$$

$$\implies \oiint_S (\nabla\times\vec{F})\cdot\vec{n}\,dS = \oiint_S \left[\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\mathbf{k}\right]\cdot\vec{N}\,du\,dv$$

Setting, $u = x,\ v = y,\ r(u,v) = r(x,y) = x\mathbf{i} + y\mathbf{j} + f\mathbf{k}$

$$N = |r_u \times r_v| = |r_x \times r_y| = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k} \text{ where } f_x = \frac{\partial r(x,y))}{\partial x} \text{ and } f_y = \frac{\partial r(x,y))}{\partial y}$$

$$\oiint_S \left[\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}\right] \cdot [-f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}]\,dx\,dy$$

$$= \oiint_S \left[\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)(-f_x) + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)(-f_y) + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\right] dx\,dy$$

From Chain Rule, $-\frac{\partial}{\partial y}F_x(x,y,f(x,y)) = -\frac{\partial}{\partial y}F_x(x,y,z) - \frac{\partial}{\partial z}F_x(x,y,z)\frac{\partial}{\partial y}f(x,y)$

$$\implies \oiint_S \left[\left(-\frac{\partial F_x}{\partial y} - \frac{\partial F_x}{\partial z}f_y\right)\right] dx\,dy = \oiint_S -\frac{\partial}{\partial y}F_x(x,y,f(x,y))dx\,dy = \oint_C F_x dx$$

Similarly use, $y = g(x,z),\quad z = h(x,y)$ to arrive at $\oint_C F_y dy,\ \oint_C F_z dz$

$$\implies \boxed{\oiint_S (\nabla \times \vec{F}) \cdot \vec{n}\,dS = \oint_C \vec{F} \cdot dl = \oint_C F_x\,dx + \oint_C F_y\,dy + \oint_C F_z\,dz}$$

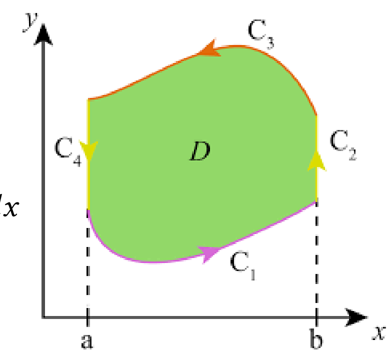
10.6.6 Green's Theorem

Suppose R is a closed region in the xy plane bounded by a simple closed curve C, and suppose M and N are continuous functions of x and y having continuous derivatives in R. Then,

$$\boxed{\oiint_R \left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right) dx\,dy = \oint_C (Mdx + Ndy)}$$

This can be proven from the following:

$$\oiint_R \frac{\partial N}{\partial y} dx dy = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial N}{\partial y} dy dx$$
$$= \int_a^b N(x, g_2(x))dx - \int_b^a N(x, g_1(x))dx$$
$$= -\int_{C_3} N(x, g_2(x))dx - \int_{C_1} N(x, g_1(x))dx$$
$$\int_{C_2} N(x,y)dx = \int_{C_4} N(x,y)dx = 0$$



Similarly, $\int_C M(x,y)dy = \int_{C_2} M(y, h_1(y))dy + \int_{C_4} M(x, h_2(y))dy$

where C is traversed in the positive (counter clockwise) direction. Green's theorem is a planar case of Stoke's theorem.