Chapter 4
Differentiation

All things change — a timeless creed,
Yet how they change is thought's true seed.
From motion's pulse to nature's chart,
Derivatives trace the living heart.

4.1 Introduction

Change lies at the heart of both mathematics and the natural sciences. Many phenomena of interest—motion, growth, decay, optimization, and equilibrium—are not described merely by static quantities, but by how those quantities vary with respect to one another. The mathematical framework that formalizes and analyzes such variation is *differentiation*.

In mathematics, the derivative of a function of a real variable measures the sensitivity to change of the function value, that is, the change in the output value with respect to a change in its argument, the input value. Derivatives are fundamental to calculus, providing a precise language for describing rates of change, local linear behavior, and the notion of instantaneous variation.

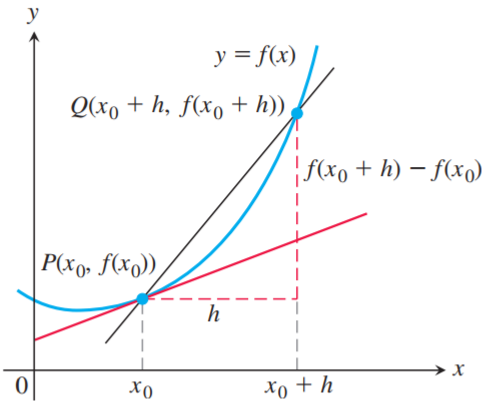
This chapter develops the theory of differentiation in a systematic manner. We begin with the intuitive idea of average rate of change and its limiting process, leading to the formal definition of the derivative. Geometric interpretations are emphasized, particularly the relationship between derivatives and tangents to curves, as well as physical interpretations such as velocity and acceleration.

Building on the definition, we study the algebra of derivatives, including linearity, product and quotient rules, and the chain rule, which together form the computational backbone of differential calculus. Special attention is given to derivatives of elementary functions, whose structural simplicity makes them foundational building blocks for more complex models.

The chapter further explores higher–order derivatives and their significance in understanding curvature, concavity, and local approximation. These concepts naturally lead to applications of differentiation, including curve sketching, optimization problems, and the analysis of extrema and inflection points.

By the end of this chapter, the reader will have acquired both conceptual clarity and practical proficiency in differentiation, preparing the ground for subsequent topics in integral calculus, differential equations, and mathematical modeling across the sciences and engineering.

4.2 Definition of a Derivative



Consider the limit: $\lim\limits_{h\to 0} \dfrac{f(x_0+h)-f(x_0)}{h}$

This limit is called the *derivative* and is written as:

$$\boxed{\frac{df}{dx} = \frac{dy}{dx} = f^{'}(x)}$$

Its value at a is represented as: $\boxed{f^{'}(a) = \frac{dy}{dx}\Big|_{x=a}}$

A derivative is rate of change, it is the *tangent* at the point .

A function $f(x)$ is differentiable at $x = a$ if $f^{'}(a)$ exists and $f(x)$ is called differentiable on an interval if the derivative exists for each point in that interval. If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$. $\boxed{\frac{d}{dx}}$ is known as the **Differential Operator** .

A piecewise continuous function can be **differentiated** at the intersection of its pieces, but the differentiability at the point of intersection depends on whether the two pieces meet smoothly (i.e., whether the function is continuous and has the same derivative from both sides at the intersection point).

For a piecewise continuous function $f(x)$ to be differentiable at the intersection point x_0 of its pieces:

1. **Continuity at x_0**: The function must be continuous at x_0. This means the left-hand limit and right-hand limit of $f(x)$ at x_0 must be equal. This ensures there is no jump or hole in the graph of $f(x)$ at x_0.

   .

   $$\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = f(x_0)$$

2. **Equal Derivatives at x_0**: The derivatives of the two pieces of the function must be the same at x_0 for the function to be differentiable at that point. This is a stronger condition - it ensures the slope (rate of change) from both sides matches.That's what makes the graph not only joined but also smooth (no sharp corner or cusp). Specifically:

   $$\lim_{x\to x_0^-} f'(x) = \lim_{x\to x_0^+} f'(x)$$

If either of these conditions is violated, the function will **not be differentiable** at x_0.

4.3 Derivative of a Polynomial Term

$$
\begin{aligned}
f(x) &= x^n \\
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^n-a^n}{x-a} \\
x^n - a^n &= (x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}) \\
f'(a) &= \lim_{x\to a}(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots a^{n-2}x + a^{n-1}) = na^{n-1}
\end{aligned}
$$

$\boxed{\dfrac{dx^n}{dx} = nx^{n-1}}$ and, obviously, $\dfrac{d}{dx}(constant) = 0$

4.4 Derivatives of a Trigonometric Function

$$
\begin{aligned}
\frac{d}{dx}(sin(x)) &= \lim_{h\to 0} \frac{sin(x+h) - sin(x)}{h} \\
&= \lim_{h\to 0} \frac{sin(x)\,cos(h) + cos(x)\,sin(h) - sin(x)}{h} \\
&= sin(x) \lim_{h\to 0} \frac{cos(h)-1}{h} + cos(x) \lim_{h\to 0} \frac{sin(h)}{h} \\
&= sin(x) \lim_{h\to 0} \frac{cos(h)-1}{h} + cos(x) \cancelto{1}{\lim_{h\to 0} \frac{sin(h)}{h}}
\end{aligned}
$$

From basic trigonometric identities we have,

$$1 - \cos h = 2\sin^2\left(\frac{h}{2}\right)$$

$$
\begin{aligned}
\frac{d}{dx}(\sin x) &= \sin x \lim_{h\to 0} \frac{1-\cos h}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} \\
&= \sin x \lim_{h\to 0} \frac{2\sin^2(h/2)}{h} + \cos x \cancelto{1}{\lim_{h\to 0} \frac{\sin h}{h}} \\
&= \sin x \lim_{h\to 0} \left[2\sin\left(\frac{h}{2}\right) \cancelto{1}{\frac{\sin(h/2)}{h/2}} \cdot \frac{1}{2} \right] + \cos x \\
&= \sin x \cdot 1 + \cos x = \cos x
\end{aligned}
$$

$\boxed{\dfrac{d}{dx}(\sin x) = \cos x}$

4.5 Derivative of a log function

Compute $\frac{d}{dx}(ln\ x)$

$$\frac{d}{dx}ln\ x = \lim_{h\to 0}\frac{ln(x+h)-ln(x)}{h} = \lim_{h\to 0}\frac{ln\frac{(x+h)}{x}}{h} = \lim_{h\to 0}\frac{1}{h}ln\left(1+\frac{h}{x}\right) = \lim_{h\to 0}ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}}$$

Let $h = nx$

$$\lim_{h\to 0}ln(1+n)^{\frac{1}{n}\frac{1}{x}} = \lim_{h\to 0}ln\left((1+n)^{\frac{1}{n}}\right)^{\frac{1}{x}} = \frac{1}{x}ln\left(\lim_{n\to 0}(1+n)^{\frac{1}{n}}\right) = \frac{1}{x}ln\left[\lim_{n\to\infty}(1+\left(\frac{1}{n}\right)^{n}\right] = ln(e)$$

Euler's number, e, is defined as:

$$e \quad = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = \sum_{n=0}^{\infty}\frac{1}{n!} = 2.718271\ldots$$

The natural logarithm function $ln(x)$ is the inverse of the exponential function e^x.
That means, $ln(e^x) = x$ and $e^{ln(x)} = x$. Setting $x = 1$, $ln(e) = 1$. Hence,

$$\boxed{\frac{d}{dx}ln\ x = \frac{1}{x}}$$

4.6 Chain Rule

Compute $\frac{d}{dx}(v(u(x)))$

$$\frac{dv}{dx} = \lim_{x\to 0}\frac{\Delta v}{\Delta x} = \lim_{x\to 0}\left(\frac{\Delta v}{\Delta u}\times\frac{\Delta u}{\Delta x}\right) = \lim_{x\to 0}\left(\frac{\Delta v}{\Delta u}\right)\times\lim_{x\to 0}\left(\frac{\Delta u}{\Delta x}\right)$$

$$\boxed{\frac{dv}{dx} = \frac{dv}{du}\times\frac{du}{dx}}$$

4.7 Derivative of an exponential function

Compute $\frac{d}{dx}(a^x)$

$$\begin{aligned}
&\text{Let} && y = a^x \\
&\ln y && = x\ln a \\
&\frac{1}{y}\frac{dy}{dx} && = \ln a \\
&\frac{dy}{dx} && = y\ln a
\end{aligned}$$

$$\boxed{\frac{d}{dx}a^x = a^x\ln a}$$

4.8 Implicit Differentiation

In implicit differentiation, we differentiate each side of an equation with two variables (usually x and y) by treating one of the variables as a function of the other. This calls for using the chain rule. Example:

$$
\begin{aligned}
x^2 + y^2 &= 1 \\
\frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) \\
&= 2x + 2y\frac{dy}{dx} = 0 \\
\implies \frac{dy}{dx} &= -\frac{x}{y}
\end{aligned}
$$

4.9 Product Rule

$$
\begin{aligned}
(fg)' &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} f(x+h)\frac{(g(x+h) - g(x))}{h} + \lim_{h\to 0} g(x)\frac{f(x+h) - f(x)}{h}
\end{aligned}
$$

$$
\boxed{(fg)' = f(x)g'(x) + g(x)f'(x)} \tag{4.1}
$$

4.10 Quotient Rule

$$
\begin{aligned}
\left(\frac{f}{g}\right)' &= \lim_{h\to 0} \frac{f'g - fg'}{g^2} = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h\to 0} \frac{1}{h} \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)} \\
&= \lim_{h\to 0} \frac{1}{g(x+h)g(x)} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h} \\
&= \lim_{h\to 0} \left(g(x)\frac{f(x+h) - f(x)}{h} - f(x)\frac{g(x+h) - g(x)}{h} \right)
\end{aligned}
$$

$$
\boxed{\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}} \tag{4.2}
$$

4.11 L'Hôpital's rule

First, need to do mathematical manipulations to get the limit into a l'Hôpital form, i.e., $0/0$ or ∞/∞ form. Let $f(x)$ and $g(x)$ be continuous functions on an interval containing $x = a$, with $f(a) = g(a) = 0$. Suppose that f and g are differentiable, and that f' and g' are continuous. and, suppose that $g'(a) \neq 0$.

Then,

$$\begin{aligned}\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}\\ &= \lim_{x\to a}\frac{(f(x)-f(a))/(x-a)}{(g(x)-f(a))/(x-a)}\\ &= \frac{\lim\limits_{x\to a}(f(x)-f(a))/(x-a)}{\lim\limits_{x\to a}(g(x)-f(a))/(x-a)}\end{aligned}$$

$$\boxed{\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}} \tag{4.3}$$

4.12 Concave Up (Convex) & Concave Down

Let $y = f(x)$ be twice-differentiable on an interval I. If f" > 0 on I, the graph of f over I is concave up (also called convex). If f" < 0 on I, the graph of f over I is concave down.

4.13 Power Series, Taylor Series & Maclaurin Series

Consider the following function that is represented as a **power series.**

$$\begin{aligned}&f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n\\ &f(a) = c_0\\ &f'(a) = c_1\\ &f''(a) = 2c_2 \to c_2 = \frac{1}{2}f''(a)\\ &f'''(a) = 3\times 2c_3 \to c_3 = \frac{1}{3!}f'''(a)\\ &\vdots\\ &f^n(a) = n(n-1)(n-2)\ldots 1 \to c_n = \frac{1}{n!}f^n(a)\end{aligned}$$

If $f^n(x)$ exists at $x = a$, the **Taylor series** for $f(x)$ at a is given by:

$$\boxed{\sum_{n=0}^{\infty}\frac{f^n(a)}{n!}(x-a) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^n(a)}{n!}(x-a)^n + \ldots} \tag{4.4}$$

A **Maclaurin series** is a Taylor series expansion about 0.

$$\boxed{f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots\frac{f^n(0)}{n!}x^n + \ldots} \tag{4.5}$$

The series may or may not converge at $x = x_p$. To converge, for any ϵ, there is exists an N that satisfies:

$$|R_n(x_p)| = |s(x_p) - s_n(x_p)| < \epsilon \; \forall n > N \text{ (for all } n > N)$$

where $s_n(x)$ is the nth partial sum:

$$s_n(x) = a_0 + a_a(x - x_0) + \cdot + a_n(x - a_0)^n$$

and $R_n(x_p)$ is the remainder.

$$R_n(x) = a_{n+1}(x - x_0)^{n+1} + a_{n+2}(x - x_0)^{n+2} + \dots$$

The **convergence interval** is $|x - x_0| < R$ (radius of convergence). This means that in the case of convergence, we can approximate the sum $s(x_1)$ by $s_n(x_1)$ as accurately as we want by taking a large enough n.

$f(x)$ is called analytic at a point $x = x_0$ if it can be represented by a power series in powers of $x - x_0$ with a positive radius of convergence R. This means that a real analytic function has to be an infinitely differentiable function.

4.14 Derivative of e^x

Taylor series expansion of e^h near $h = 0$:

$$e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \dots \implies \lim_{h \to 0} \frac{e^h - 1}{h} = 1 \implies \boxed{\frac{d}{dx}e^x = e^x}$$

4.15 Hyperbolic Functions - Definitions

$$\boxed{sinh\ x = \frac{e^x - e^{-x}}{2}}$$

$$\boxed{cosh\ x = \frac{e^x + e^{-x}}{2}}$$

$$\boxed{tanh\ x = \frac{e^x - e^{-x}}{e^x + e^{-x}}}$$

$$\boxed{cosh^2 x - sinh^2 x = 1}$$

4.16 Partial Derivatives

$$\begin{aligned}
f_x(x,y) &= \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}\\
f_y(x,y) &= \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}\\
f_x = f_x(x,y) &= \frac{\partial}{\partial x}f(x,y)\\
f_y = f_x(x,y) &= \frac{\partial}{\partial y}f(x,y)
\end{aligned}$$

Example,

$$f(x,y) = x^2y - 10y^2z^3 + 43x - 7tan(4y)$$

$$\begin{aligned}
\frac{\partial}{\partial x}f(x,y,z) &= 2xy + 43\\
\frac{\partial}{\partial y}f(x,y,z) &= x^2 - 20yz^3 - 28sec^2(4y)\\
\frac{\partial}{\partial z}f(x,y,z) &= -30y^2z^2
\end{aligned}$$

4.17 SymPy Code

Compute the derivative of y with respect to x for some functions in the code below: $y = x^2$

```
import sympy as sp
import numpy as np
import matplotlib.pyplot as plt
from IPython.display import display, Math
from sympy import sqrt, diff
from sympy import sin, cos, tan, ln, trigsimp, expand_trig, simplify
from sympy import sinh, cosh

x = sp.symbols('x')
y = x**2
der = diff(y, (x, 1))  # first derivative of y wrt x
display(der)

y = sin(x)
der = diff(y, (x,2))  # second derivative of y wrt x
display(der)

y = ln(x)
```

```
der = diff(y, (x,1))  # first derivative of y wrt x
display(der)
value_at_4 = der.subs(x, 4) # evaluate value of the derivative at x = 4
display(value_at_4)

a = sp.symbols('a')
y = a**x
der = diff(y, x)   # derivative of y wrt x
display(der)
```

$2x$

$-sin(x)$

$\frac{1}{x}$

$\frac{1}{4}$

$a^x log(a)$

```
import sympy as sp
from IPython.display import display, Math
from sympy import sin, cos, tan, exp, E, I, simplify, integrate, latex
from sympy.abc import x, y, z, t, w

fun_expr = E**(x)
display(sp.latex(fun_expr))
display(sp.latex(fun_expr.series(x,n=10)))
display(fun_expr.series(x,n=10))


fun_expr = E**(I*w*x)
display(sp.latex(fun_expr))
display(sp.latex(fun_expr.series(x,n=10)))
display(fun_expr.series(x,n=10))

fun_expr = cos(w*x)
display(sp.latex(fun_expr))
display(sp.latex(s1))
s1 = fun_expr.series(x,n=10)
display(s1)

fun_expr = I*sin(w*x)
display(sp.latex(fun_expr))
s2 = fun_expr.series(x,n=10)
display(s2)

display(sp.latex(s1+s2))
display(s1+s2)
```

$$1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}+\frac{x^8}{40320}+\frac{x^9}{362880}+O\left(x^{10}\right)=e^x$$

$$1+iwx-\frac{w^2x^2}{2}-\frac{\boldsymbol{i}w^3x^3}{6}+\frac{w^4x^4}{24}+\frac{\boldsymbol{i}w^5x^5}{120}-\frac{w^6x^6}{720}-\frac{\boldsymbol{i}w^7x^7}{5040}+\frac{w^8x^8}{40320}+\frac{\boldsymbol{i}w^9x^9}{362880}+O\left(x^{10}\right)=e^{\boldsymbol{i}wx}$$

$$1-\frac{w^2x^2}{2}+\frac{w^4x^4}{24}-\frac{w^6x^6}{720}+\frac{w^8x^8}{40320}+O\left(x^{10}\right)=cos(wx)$$

$$\boldsymbol{i}wx-\frac{\boldsymbol{i}w^3x^3}{6}+\frac{\boldsymbol{i}w^5x^5}{120}-\frac{\boldsymbol{i}w^7x^7}{5040}+\frac{\boldsymbol{i}w^9x^9}{362880}+O\left(x^{10}\right)=\boldsymbol{i}\,sin(wx)$$

$$1+iwx-\frac{w^2x^2}{2}-\frac{\boldsymbol{i}w^3x^3}{6}+\frac{w^4x^4}{24}+\frac{\boldsymbol{i}w^5x^5}{120}-\frac{w^6x^6}{720}-\frac{\boldsymbol{i}w^7x^7}{5040}+\frac{w^8x^8}{40320}+\frac{\boldsymbol{i}w^9x^9}{362880}+O\left(x^{10}\right)=cos(wx)+\boldsymbol{i}\,sin(wx)$$