Chapter 16

Partial Differential Equation (PDE)

Rates of change in space and time,
Linked by rules both clear and prime,
Heat, waves, flow—one form, one frame,
PDEs describe how nature evolves the same.

Partial differential equations (PDEs) arise naturally in the mathematical modeling of physical, biological, and engineering systems involving functions of several independent variables. They govern fundamental phenomena such as heat conduction, wave propagation, fluid flow, elasticity, electrostatics, electrodynamics, diffusion, and quantum mechanics. A partial differential equation relates an unknown multivariable function to its partial derivatives.

16.1 Concept of Partial Differential Equations

A **partial differential equation** is an equation involving partial derivatives of an unknown function of two or more independent variables. If $u = u(x_1, x_2, \dots, x_n)$ is an unknown function, then a PDE has the general form

$$F\left(x_1, x_2, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial^k u}{\partial x_1^k}, \dots\right) = 0.$$

The **order** of a PDE is the highest order partial derivative appearing in the equation. The **degree** is the power of the highest order derivative when the equation is polynomial in derivatives.

Important model PDEs include:

- **Wave equation:** $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$
- **Heat equation:** $\dfrac{\partial u}{\partial t} = \alpha \dfrac{\partial^2 u}{\partial x^2}$
- **Laplace equation:** $\nabla^2 u = 0$
- **Poisson equation:** $\nabla^2 u = f$

16.2 Methods of Solution

Exact solution of PDEs is generally possible only for special classes of equations. The most important analytical methods include:

- Method of Separation of Variables
- Fourier Series and Eigenfunction Expansions
- Laplace Transform Method
- Green's Function Method
- Transform and Integral Methods

16.3 Solution by Separation of Variables

The method of separation of variables assumes that the solution can be written as a product of single-variable functions:

$$u(x,t) = X(x)T(t).$$

Substitution into the governing PDE leads to ordinary differential equations for $X(x)$ and $T(t)$. These equations are solved subject to boundary and initial conditions, leading to eigenvalue problems.

This method is widely applied to:

- Heat equation
- Wave equation
- Laplace equation

16.4 Fourier Series

Any periodic function $f(x)$ defined on $[-L, L]$ may be expanded as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx, \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx.$$

Fourier series form the basis of eigenfunction expansions for solving PDEs by separation of variables.

16.4.1 D'Alembert's Solution of the Wave Equation

For the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

the general solution is given by D'Alembert's formula:

$$u(x,t) = \frac{1}{2}\big[f(x-ct) + f(x+ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds,$$

where $f(x) = u(x,0)$ and $g(x) = u_t(x,0)$.

This solution describes the propagation of waves traveling in opposite directions.

16.5 Types and Normal Forms of PDEs

Second-order linear PDEs in two variables have the general form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G.$$

The discriminant $\Delta = B^2 - AC$ classifies the equation as:

- **Elliptic** if $\Delta < 0$ (Laplace, Poisson)
- **Parabolic** if $\Delta = 0$ (Heat equation)
- **Hyperbolic** if $\Delta > 0$ (Wave equation)

Each type exhibits fundamentally different physical and mathematical behavior.

16.6 Solution Using Laplace Transforms

The Laplace transform of a function $f(t)$ is defined as

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt.$$

Applying the transform to time-dependent PDEs converts partial derivatives in time into algebraic expressions in s, thereby reducing the PDE to an ordinary differential equation in space. After solving the transformed equation, the inverse Laplace transform yields the solution.

This method is particularly effective for:

- Initial-value heat conduction problems
- Forced vibration problems

16.7 Green's Function

Consider the linear boundary-value problem

$$Lu(x) = f(x), \qquad x \in [0,l]$$

where L is a linear differential operator subject to prescribed boundary conditions. Formally,

$$u(x) = L^{-1} f$$

Let the forcing function be concentrated in a small neighborhood of x_0:

$$f(x) = \begin{cases} f_0(x), & x_0 - \xi < x < x_0 + \xi \\ 0, & \text{elsewhere.} \end{cases}$$

The **impulse** is defined as

$$I(\xi) = \int_0^l f(x)\,dx = \int_{x_0-\xi}^{x_0+\xi} f_0(x)\,dx$$

A specific impulse distribution is given by

$$f(x) = \begin{cases} \dfrac{1}{2\xi}, & x_0 - \xi < x < x_0 + \xi \\ 0, & \text{elsewhere.} \end{cases}$$

Then,

$$I(\xi) = \int_{x_0-\xi}^{x_0+\xi} \frac{1}{2\xi}\,dx = 1$$

The **Dirac delta function** is defined as

$$\delta(x - x_0) = \lim_{\xi \to 0} \begin{cases} \dfrac{1}{2\xi}, & x_0 - \xi < x < x_0 + \xi \\ 0, & \text{elsewhere.} \end{cases}$$

It satisfies the **sifting property**:

$$\int_0^l f(x)\delta(x - x_0)\,dx = f(x_0)$$

The **Green's function** $G(x, \xi)$ for the operator L is defined as the solution of

$$LG(x, \xi) = \delta(x - \xi)$$

subject to the same boundary conditions as u. Once G is known, the solution of $Lu = f$ is given by

$$u(x) = \int_0^l G(x, \xi) f(\xi)\,d\xi$$

Thus Green's functions represent the response of the system to a point impulse and provide a powerful method for solving linear PDEs with arbitrary forcing.

Example: Green's Function for a One-Dimensional Poisson Problem

Consider the boundary-value problem

$$-\frac{d^2u}{dx^2} = f(x), \qquad 0 < x < l,$$

with homogeneous Dirichlet boundary conditions

$$u(0) = 0, \qquad u(l) = 0.$$

Here the linear operator is

$$Lu(x) = -u''(x).$$

Construction of the Green's Function

The Green's function $G(x, \xi)$ is defined as the solution of

$$-\frac{d^2}{dx^2}G(x,\xi) = \delta(x-\xi), \qquad 0 < x < l,$$

subject to

$$G(0,\xi) = 0, \qquad G(l,\xi) = 0.$$

For fixed $\xi \in (0, l)$, the function $G(x, \xi)$ as a function of x satisfies

$$-G_{xx}(x,\xi) = 0 \quad \text{for } x \neq \xi,$$

so G is linear in x on each side of $x = \xi$. We therefore write

$$G(x,\xi) = \begin{cases} A_1 x + B_1, & 0 \le x < \xi, \\ A_2 x + B_2, & \xi < x \le l. \end{cases}$$

The boundary conditions give

$$G(0,\xi) = 0 \;\Rightarrow\; B_1 = 0, \qquad G(l,\xi) = 0 \;\Rightarrow\; A_2 l + B_2 = 0 \;\Rightarrow\; B_2 = -A_2 l.$$

Thus

$$G(x,\xi) = \begin{cases} A_1 x, & 0 \le x < \xi, \\ A_2(x-l), & \xi < x \le l. \end{cases}$$

Next, $G(x, \xi)$ must be continuous at $x = \xi$:

$$\lim_{x\to\xi^-} G(x,\xi) = \lim_{x\to\xi^+} G(x,\xi),$$

which yields

$$A_1\xi = A_2(\xi - l).$$

The presence of the delta function imposes a jump condition on the first derivative. Integrating

$$-G_{xx}(x,\xi) = \delta(x-\xi)$$

from $\xi - \varepsilon$ to $\xi + \varepsilon$ and letting $\varepsilon \to 0$ gives

$$-\left[G_x(\xi^+,\xi) - G_x(\xi^-,\xi)\right] = 1,$$

or equivalently

$$G_x(\xi^+,\xi) - G_x(\xi^-,\xi) = -1.$$

Since

$$G_x(x,\xi) = \begin{cases} A_1, & x < \xi, \\ A_2, & x > \xi, \end{cases}$$

the jump condition becomes

$$A_2 - A_1 = -1.$$

We now solve the system

$$A_1\xi = A_2(\xi - l), \qquad A_2 - A_1 = -1.$$

From the second equation,

$$A_2 = A_1 - 1.$$

Substituting into the first equation,

$$A_1\xi = (A_1 - 1)(\xi - l) \;\Rightarrow\; A_1\xi = A_1(\xi - l) - (\xi - l) \;\Rightarrow\; A_1 l = \xi - l \;\Rightarrow\; A_1 = \frac{\xi - l}{l} = -\frac{l - \xi}{l}.$$

Then

$$A_2 = A_1 - 1 = -\frac{l - \xi}{l} - 1 = -\frac{l - \xi + l}{l} = -\frac{\xi}{l}.$$

Thus the Green's function can be written as

$$G(x,\xi) = \begin{cases} \dfrac{l-\xi}{l}\,x, & 0 \le x < \xi, \\ \dfrac{\xi}{l}\,(l - x), & \xi < x \le l. \end{cases}$$

Since L is self-adjoint with these boundary conditions, $G(x,\xi)$ is symmetric:

$$G(x,\xi) = G(\xi, x),$$

and one often writes the equivalent piecewise form in terms of min and max:

$$G(x,\xi) = \frac{1}{l}\begin{cases} (l-\xi)x, & x \le \xi, \\ (l-x)\xi, & x \ge \xi. \end{cases}$$

Solution Formula for $u(x)$

Once the Green's function is known, the solution of

$$-u''(x) = f(x), \qquad 0 < x < l, \qquad u(0) = u(l) = 0,$$

is given by

$$u(x) = \int_0^l G(x,\xi)\, f(\xi)\, d\xi.$$

For example, if $f(x) = 1$ (a constant source term), then

$$u(x) = \int_0^l G(x,\xi)\, d\xi,$$

which can be evaluated explicitly using the piecewise expression for $G(x,\xi)$ to obtain a quadratic function of x satisfying $u(0) = u(l) = 0$ and $-u''(x) = 1$.