

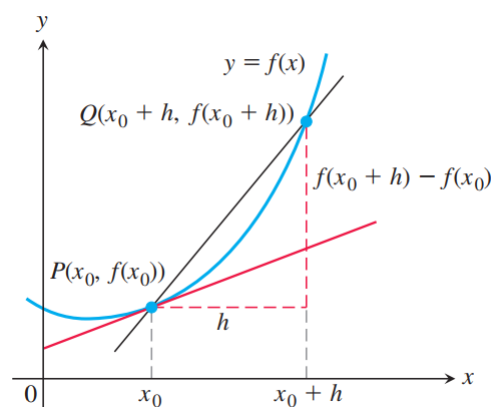
# CHAPTER 4

## DIFFERENTIATION

*All things change — a timeless creed,  
Yet how they change is thought's true seed.  
From motion's pulse to nature's chart,  
Derivatives trace the living heart.*

In mathematics, the derivative of a function of a real variable measures the sensitivity to change of the function value, i.e., the output value with respect to a change in its argument, i.e., the input value. Derivatives are a fundamental to calculus.

### 4.1 DEFINITION OF A DERIVATIVE



Consider the limit:  $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$

This limit is called the *derivative* and is written as:

$$\frac{df}{dx} = \frac{dy}{dx} = f'(x)$$

Its value at  $a$  is represented as:  $f'(a) = \left. \frac{dy}{dx} \right|_{x=a}$

**A derivative is rate of change, it is the *tangent* at the point.**

A function  $f(x)$  is differentiable at  $x = a$  if  $f'(a)$  exists and  $f(x)$  is called differentiable on an interval if the derivative exists for each point in that interval. If  $f(x)$  is differentiable at  $x = a$ , then  $f(x)$  is continuous at  $x = a$ .  $\frac{d}{dx}$  is known as the **Differential Operator**.

A piecewise continuous function can be **differentiated** at the intersection of its pieces, but the differentiability at the point of intersection depends on whether the two pieces meet smoothly (i.e., whether the function is continuous and has the same derivative from both sides at the intersection point).

For a piecewise continuous function  $f(x)$  to be differentiable at the intersection point  $x_0$  of its pieces:

1. **Continuity at  $x_0$ :** The function must be continuous at  $x_0$ . This means the left-hand limit and right-hand limit of  $f(x)$  at  $x_0$  must be equal. This ensures there is no jump or hole in the graph of  $f(x)$  at  $x_0$ .

$$\lim_{x \rightarrow x_0^-} f(x) = \lim_{x \rightarrow x_0^+} f(x) = f(x_0)$$

2. **Equal Derivatives at  $x_0$ :** The derivatives of the two pieces of the function must be the same at  $x_0$  for the function to be differentiable at that point. This is a stronger condition - it ensures the slope (rate of change) from both sides matches. That's what makes the graph not only joined but also smooth (no sharp corner or cusp). Specifically:

$$\lim_{x \rightarrow x_0^-} f'(x) = \lim_{x \rightarrow x_0^+} f'(x)$$

If either of these conditions is violated, the function will **not be differentiable** at  $x_0$ .

## 4.2 DERIVATIVE OF A POLYNOMIAL TERM

$$\begin{aligned} f(x) &= x^n \\ f'(a) &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} \\ x^n - a^n &= (x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-2}x + a^{n-1}) \\ f'(a) &= \lim_{x \rightarrow a} (x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-2}x + a^{n-1}) = na^{n-1} \end{aligned}$$

$$\boxed{\frac{dx^n}{dx} = nx^{n-1}} \text{ and, obviously, } \frac{d}{dx}(\text{constant}) = 0$$

## 4.3 DERIVATIVES OF A TRIGONOMETRIC FUNCTION

$$\begin{aligned} \frac{d}{dx}(\sin(x)) &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(x) \cos(h) + \cos(x) \sin(h) - \sin(x)}{h} \\ &= \sin(x) \lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\ &= \sin(x) \lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \end{aligned}$$

From basic trigonometric identities we have,

$$\begin{aligned}
 1 - \cos h &= 2 \sin^2\left(\frac{h}{2}\right) \\
 \frac{d}{dx}(\sin x) &= \sin x \lim_{h \rightarrow 0} \frac{1 - \cos h}{h} + \cos x \lim_{h \rightarrow 0} \frac{\sin h}{h} \\
 &= \sin x \lim_{h \rightarrow 0} \frac{2 \sin^2(h/2)}{h} + \cos x \lim_{h \rightarrow 0} \frac{\sin h}{h} \xrightarrow{1} \\
 &= \sin x \lim_{h \rightarrow 0} \left[ 2 \sin\left(\frac{h}{2}\right) \frac{\sin(h/2)}{h/2} \cdot \frac{1}{2} \right] + \cos x \\
 &= \sin x \cdot 1 + \cos x = \cos x
 \end{aligned}$$

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

## 4.4 DERIVATIVE OF A LOG FUNCTION

Compute  $\frac{d}{dx}(\ln x)$

$$\frac{d}{dx} \ln x = \lim_{h \rightarrow 0} \frac{\ln(x+h) - \ln(x)}{h} = \lim_{h \rightarrow 0} \frac{\ln \frac{(x+h)}{x}}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \ln \left(1 + \frac{h}{x}\right) = \lim_{h \rightarrow 0} \ln \left(1 + \frac{h}{x}\right)^{\frac{1}{h}}$$

Let  $h = nx$

$$\lim_{h \rightarrow 0} \ln \left(1 + \frac{h}{x}\right)^{\frac{1}{h}} = \lim_{h \rightarrow 0} \ln \left(1 + \frac{1}{n}\right)^{\frac{1}{nx}} = \frac{1}{x} \ln \left( \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \right) = \frac{1}{x} \ln \left[ \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \right] = \ln(e)$$

Euler's number,  $e$ , is defined as:

$$e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.718271 \dots$$

The natural logarithm function  $\ln(x)$  is the inverse of the exponential function  $e^x$ .

That means,  $\ln(e^x) = x$  and  $e^{\ln(x)} = x$ . Setting  $x = 1$ ,  $\ln(e) = 1$ . Hence,

$$\boxed{\frac{d}{dx} \ln x = \frac{1}{x}}$$

## 4.5 CHAIN RULE

Compute  $\frac{d}{dx}(v(u(x)))$

$$\frac{dv}{dx} = \lim_{x \rightarrow 0} \frac{\Delta v}{\Delta x} = \lim_{x \rightarrow 0} \left( \frac{\Delta v}{\Delta u} \times \frac{\Delta u}{\Delta x} \right) = \lim_{x \rightarrow 0} \left( \frac{\Delta v}{\Delta u} \right) \times \lim_{x \rightarrow 0} \left( \frac{\Delta u}{\Delta x} \right)$$

$$\boxed{\frac{dv}{dx} = \frac{dv}{du} \times \frac{du}{dx}}$$

## 4.6 DERIVATIVE OF AN EXPONENTIAL FUNCTION

Compute  $\frac{d}{dx}(a^x)$

$$\begin{aligned}\text{Let} \quad y &= a^x \\ \ln y &= x \ln a \\ \frac{1}{y} \frac{dy}{dx} &= \ln a \\ \frac{dy}{dx} &= y \ln a\end{aligned}$$

$$\boxed{\frac{d}{dx}a^x = a^x \ln a}$$

## 4.7 IMPLICIT DIFFERENTIATION

In implicit differentiation, we differentiate each side of an equation with two variables (usually  $x$  and  $y$ ) by treating one of the variables as a function of the other. This calls for using the chain rule. Example:

$$\begin{aligned}x^2 + y^2 &= 1 \\ \frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) \\ &= 2x + 2y \frac{dy}{dx} = 0 \\ \implies \frac{dy}{dx} &= -\frac{x}{y}\end{aligned}$$

## 4.8 PRODUCT RULE

$$\begin{aligned}(fg)' &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} f(x+h) \frac{(g(x+h) - g(x))}{h} + \lim_{h \rightarrow 0} g(x) \frac{f(x+h) - f(x)}{h}\end{aligned}$$

$$\boxed{(fg)' = f(x)g'(x) + g(x)f'(x)}$$

(4.1)

## 4.9 QUOTIENT RULE

$$\begin{aligned}
 \left(\frac{f}{g}\right)' &= \lim_{h \rightarrow 0} \frac{f'g - fg'}{g^2} = \lim_{h \rightarrow 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)} \\
 &= \lim_{h \rightarrow 0} \frac{1}{g(x+h)g(x)} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h} \\
 &= \lim_{h \rightarrow 0} \left( g(x) \frac{f(x+h) - f(x)}{h} - f(x) \frac{g(x+h) - g(x)}{h} \right)
 \end{aligned}$$

$$\boxed{\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}} \quad (4.2)$$

## 4.10 L'HÔPITAL'S RULE

First, need to do mathematical manipulations to get the limit into a l'Hôpital form, i.e.,  $0/0$  or  $\infty/\infty$  form. Let  $f(x)$  and  $g(x)$  be continuous functions on an interval containing  $x = a$ , with  $f(a) = g(a) = 0$ . Suppose that  $f$  and  $g$  are differentiable, and that  $f'$  and  $g'$  are continuous. and, suppose that  $g'(a) \neq 0$ . Then,

$$\begin{aligned}
 \lim_{x \rightarrow a} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{g(x) - g(a)} \\
 &= \lim_{x \rightarrow a} \frac{(f(x) - f(a))/(x - a)}{(g(x) - g(a))/(x - a)} \\
 &= \frac{\lim_{x \rightarrow a} (f(x) - f(a))/(x - a)}{\lim_{x \rightarrow a} (g(x) - g(a))/(x - a)}
 \end{aligned}$$

$$\boxed{\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)}} \quad (4.3)$$

## 4.11 CONCAVE UP (CONVEX) & CONCAVE DOWN

Let  $y = f(x)$  be twice-differentiable on an interval  $I$ . If  $f'' > 0$  on  $I$ , the graph of  $f$  over  $I$  is concave up (also called convex). If  $f'' < 0$  on  $I$ , the graph of  $f$  over  $I$  is concave down.

## 4.12 POWER SERIES, TAYLOR SERIES & MACLAURIN SERIES

Consider the following function that is represented as a **power series**.

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_n(x-a)^n \\
f(a) &= c_0 \\
f'(a) &= c_1 \\
f''(a) &= 2c_2 \rightarrow c_2 = \frac{1}{2}f''(a) \\
f'''(a) &= 3 \times 2c_3 \rightarrow c_3 = \frac{1}{3!}f'''(a) \\
&\vdots \\
f^n(a) &= n(n-1)(n-2)\dots 1 \rightarrow c_n = \frac{1}{n!}f^n(a)
\end{aligned}$$

If  $f^n(x)$  exists at  $x = a$ , the **Taylor series** for  $f(x)$  at  $a$  is given by:

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^n(a)}{n!}(x-a)^n + \dots \quad (4.4)$$

A **Maclaurin series** is a Taylor series expansion about 0.

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^n(0)}{n!}x^n + \dots \quad (4.5)$$

The series may or may not converge at  $x = x_p$ . To converge, for any  $\epsilon$ , there is exists an  $N$  that satisfies:

$$|R_n(x_p)| = |s(x_p) - s_n(x_p)| < \epsilon \quad \forall n > N \quad (\text{for all } n > N)$$

where  $s_n(x)$  is the  $n$ th partial sum:

$$s_n(x) = a_0 + a_1(x - x_0) + \dots + a_n(x - x_0)^n$$

and  $R_n(x_p)$  is the remainder.

$$R_n(x) = a_{n+1}(x - x_0)^{n+1} + a_{n+2}(x - x_0)^{n+2} + \dots$$

The **convergence interval** is  $|x - x_0| < R$  (radius of convergence). This means that in the case of convergence, we can approximate the sum  $s(x_1)$  by  $s_n(x_1)$  as accurately as we want by taking a large enough  $n$ .

$f(x)$  is called analytic at a point  $x = x_0$  if it can be represented by a power series in powers of  $x - x_0$  with a positive radius of convergence  $R$ . This means that a real analytic function has to be an infinitely differentiable function.

### 4.13 DERIVATIVE OF $e^x$

Taylor series expansion of  $e^h$  near  $h = 0$ :

$$e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \dots \implies \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1 \implies \frac{d}{dx} e^x = e^x$$

### 4.14 HYPERBOLIC FUNCTIONS - DEFINITIONS

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\cosh^2 x - \sinh^2 x = 1$$

### 4.15 PARTIAL DERIVATIVES

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y + h) - f(x, y)}{h}$$

$$f_x = f_x(x, y) = \frac{\partial}{\partial x} f(x, y)$$

$$f_y = f_y(x, y) = \frac{\partial}{\partial y} f(x, y)$$

Example,

$$f(x, y) = x^2y - 10y^2z^3 + 43x - 7\tan(4y)$$

$$\frac{\partial}{\partial x} f(x, y, z) = 2xy + 43$$

$$\frac{\partial}{\partial y} f(x, y, z) = x^2 - 20yz^3 - 28\sec^2(4y)$$

$$\frac{\partial}{\partial z} f(x, y, z) = -30y^2z^2$$

## 4.16 SYMPY CODE

Compute the derivative of  $y$  with respect to  $x$  for some functions in the code below:  $y = x^2$

```
1 import sympy as sp
2 import numpy as np
3 import matplotlib.pyplot as plt
4 from IPython.display import display, Math
5 from sympy import sqrt, diff
6 from sympy import sin, cos, tan, ln, trigsimp, expand_trig, simplify
7 from sympy import sinh, cosh
8
9 x = sp.symbols('x')
10 y = x**2
11 der = diff(y, (x, 1)) # first derivative of y wrt x
12 display(der)
13
14 y = sin(x)
15 der = diff(y, (x,2)) # second derivative of y wrt x
16 display(der)
17
18 y = ln(x)
19 der = diff(y, (x,1)) # first derivative of y wrt x
20 display(der)
21 value_at_4 = der.subs(x, 4) # evaluate value of the derivative at x = 4
22 display(value_at_4)
23
24 a = sp.symbols('a')
25 y = a**x
26 der = diff(y, x) # derivative of y wrt x
27 display(der)
```

$2x$

$-\sin(x)$

$\frac{1}{x}$

$\frac{1}{4}$

$a^x \log(a)$



```

1 import sympy as sp
2 from IPython.display import display, Math
3 from sympy import sin, cos, tan, exp, E, I, simplify, integrate, latex
4 from sympy.abc import x, y, z, t, w
5
6 fun_expr = E**(x)
7 display(sp.latex(fun_expr))
8 display(sp.latex(fun_expr.series(x,n=10)))
9 display(fun_expr.series(x,n=10))
10
11
12 fun_expr = E**(I*w*x)
13 display(sp.latex(fun_expr))
14 display(sp.latex(fun_expr.series(x,n=10)))
15 display(fun_expr.series(x,n=10))
16
17 fun_expr = cos(w*x)
18 display(sp.latex(fun_expr))
19 display(sp.latex(s1))
20 s1 = fun_expr.series(x,n=10)
21 display(s1)
22
23 fun_expr = I*sin(w*x)
24 display(sp.latex(fun_expr))
25 s2 = fun_expr.series(x,n=10)
26 display(s2)
27
28 display(sp.latex(s1+s2))
29 display(s1+s2)

```

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40320} + \frac{x^9}{362880} + O(x^{10}) = e^x$$

$$1 + iwx - \frac{w^2x^2}{2} - \frac{iw^3x^3}{6} + \frac{w^4x^4}{24} + \frac{iw^5x^5}{120} - \frac{w^6x^6}{720} - \frac{iw^7x^7}{5040} + \frac{w^8x^8}{40320} + \frac{iw^9x^9}{362880} + O(x^{10}) = e^{iwx}$$

$$1 - \frac{w^2x^2}{2} + \frac{w^4x^4}{24} - \frac{w^6x^6}{720} + \frac{w^8x^8}{40320} + O(x^{10}) = \cos(wx)$$

$$iwx - \frac{iw^3x^3}{6} + \frac{iw^5x^5}{120} - \frac{iw^7x^7}{5040} + \frac{iw^9x^9}{362880} + O(x^{10}) = i\sin(wx)$$

$$1 + iwx - \frac{w^2x^2}{2} - \frac{iw^3x^3}{6} + \frac{w^4x^4}{24} + \frac{iw^5x^5}{120} - \frac{w^6x^6}{720} - \frac{iw^7x^7}{5040} + \frac{w^8x^8}{40320} + \frac{iw^9x^9}{362880} + O(x^{10}) = \cos(wx) + i\sin(wx)$$