

# CHAPTER 5

## INTEGRATION

*Integration sounds lofty and grand,  
Yet it's just adding, hand over hand.  
Cloaked in fine symbols, solemn, sublime -  
Still it's just summing, one step at a time!*

Integration, in mathematics, is the technique of finding a function  $g(x)$  the derivative of which is equal to a given function  $f(x)$ . This is indicated by the integral sign  $\int$  as in  $\int f(x)dx$  and is called the indefinite integral of the function. The symbol  $dx$  represents an infinitesimal displacement along  $x$ . Hence,  $\int f(x)dx$  is the summation of the product of  $f(x)$  and  $dx$ . The definite integral, written as  $\int_a^b f(x)dx$  where  $a$  and  $b$  are called the limits of integration, is equal to  $g(b) - g(a)$ , where  $\frac{d}{dx}g(x) = f(x)$ .

### 5.1 INTEGRAL

Given a function  $f(x)$ , an **anti-derivative** of  $f(x)$  is any function  $g(x)$  such that  $g'(x) = f(x)$ . The most general anti-derivative is called the **indefinite integral**.

$$\int f(x)dx = g(x) + c \text{ where } c \text{ is a constant of integration}$$

Note the following inequalities.

$$\int f(x)g(x)dx \neq \int f(x)dx \int g(x)dx$$

$$\int \frac{f(x)}{g(x)}dx \neq \frac{\int f(x)dx}{\int g(x)dx}$$

## 5.2 COMMON INTEGRALS

$$\int x^n dx = \frac{x^{(n+1)}}{(n+1)} + c \quad (5.1)$$

$$\int e^x dx = e^x + c \quad (5.2)$$

$$\int a^x dx = \frac{a^x}{\ln a} + c \quad (5.3)$$

$$\int \frac{1}{x} dx = \ln|x| + c \quad (5.4)$$

$$\int \cos(x) dx = \sin(x) + c \quad (5.5)$$

## 5.3 SUBSTITUTION TECHNIQUE

$$\int 18x^2 \sqrt[4]{(6x^3 + 5)} dx$$

$$\text{Let } u = 6x^3 + 5$$

$$\rightarrow du = 18x^2 dx$$

$$\rightarrow \int \sqrt[4]{u} du = \frac{u^{\left(\frac{1}{4}+1\right)}}{\frac{1}{4}+1} = \frac{4}{5} u^{\frac{5}{4}} = \frac{4}{5} (6x^3 + 5)^{\frac{5}{4}}$$

## 5.4 INTEGRATION BY PARTS

$$[f(x)g(x)]' = f(x)g'(x) + f'(x)g(x)$$

$$f(x)g'(x) = [f(x)g(x)]' - f'(x)g(x)$$

$$\int f(x)g'(x) dx = \int [f(x)g(x)]' dx - \int f'(x)g(x) dx$$

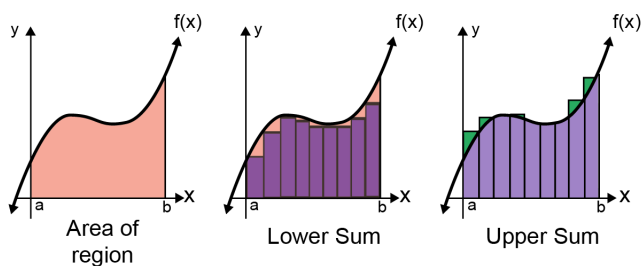
$$\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$$

$$\boxed{\int u dv = uv - \int v du} \quad (5.6)$$

Hence, integral of two functions = first function  $\times$  integral of second function – integral of ( differentiation of the first function  $\times$  integral of the second function ).

## 5.5 DEFINITE INTEGRAL

**A definite integral is a the area under its curve .**



$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*)\Delta x = g(x) \Big|_a^b = g(b) - g(a)$$

where  $f(x_i^*)$  is the value at the middle of the strip  $\Delta x$ .

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x)dx$$

$$\int_a^b f(x)dx = f(c)(b-a) \text{ where } c \text{ is in } [a,b]$$

## 5.6 SOME INTEGRATION STRATEGIES

- ① Simplify the integrand. E.g.,  $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$
- ② Check if simple substitution will work
- ③ If integrand is a rational expression, partial functions may work
- ④ If integrand is polynomial  $x$ , *trig*, *exp*,  $\ln$  function, integration by parts may work
- ⑤ If integrand involves  $\sqrt{b^2x^2 + a^2}$ , trigonometric substitution may work
- ⑥ If integrand has a quadratic in it, completing the square may work.

## 5.7 SYMPY

Integrate the following functions:

$$\int (x^2 + x + 1) dx$$

$$\int e^{-x^2} \operatorname{erf}(x) dx \text{ where } \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$\int y = e^{-x} dx$$

```

1 import sympy as sp
2 import numpy as np
3 import matplotlib.pyplot as plt
4 from IPython.display import display, Math
5 from sympy import sqrt, diff, integrate, oo
6 from sympy import sin, cos, tan, ln, exp, erf, trigsimp, expand_trig, simplify
7 from sympy import sinh, cosh
8
9 x = sp.symbols('x')
10
11 y = x**2 + x + 1
12 int_expr = integrate(y,x) # integrate y wrt x
13 print(sp.latex(int_expr))
14 display(int_expr)
15
16 y = exp(-x**2)*erf(x)
17 int_expr = integrate(y,x) # integrate y wrt x
18 print(sp.latex(int_expr))
19 display(int_expr)
20
21 y = exp(-x)
22 int_expr = integrate(y, (x, 0, oo)) # definite integral, limits 0 & infinity
23 print(sp.latex(int_expr))
24 display(int_expr)

```

$$\frac{x^3}{3} + \frac{x^2}{2} + x$$

$$\frac{\sqrt{\pi} \operatorname{erf}^2(x)}{4}$$

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