Chapter 10

Vector

Vectors were first used to express the laws of electromagnetism. Since then, they have become essential in physics, mechanics, electrical engineering, and many other sciences for describing forces and motion mathematically.

Some physical quantities are characterized by both **magnitude** and **direction**, such as displacement, velocity, force, and acceleration. To describe such quantities, we introduce the concept of a **vector**, represented as a directed line segment. Other quantities in physics are characterized by **magnitude only**, such as mass, length, and temperature. Such a quantity is called a **scalar**. For example, speed—say, 10 km/h—is a scalar, whereas velocity—say, 10 km/h toward the north-east—is a vector and may be written as:

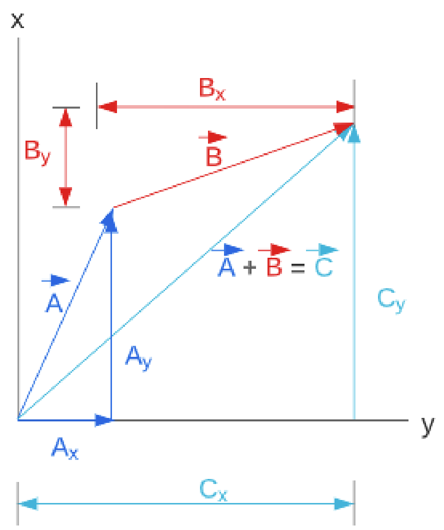
$$\vec{v} = 10 \cdot \frac{1}{\sqrt{2}}\hat{i} + 10 \cdot \frac{1}{\sqrt{2}}\hat{j},$$

where $\hat{i}$ is the unit vector along the x -direction and $\hat{j}$ is the unit vector along the y-direction.

10.1 Vector Algebra

10.1.1 Vector Addition & Subtraction

Vector addtion (or subtraction) is performed by adding (or subtracting) their components.



$$\vec{A} + \vec{B} = \vec{C}$$

$$A_x + B_x = C_x$$

$$A_y + B_y = C_y$$

10.1.2 Scalar Multiplication

Multiplication of a vector $\vec{A}$ by a scalar m produces a vector $m\vec{A}$ with magnitude $m \times \|A\|$ where $\|A\|$ is the magnitude of $\vec{A}$.

10.1.3 Unit Vector

Unit vectors are vectors having unit length.

$$A = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$$

$$\|A\| = \sqrt{A_1^2 + A_2^2 + A_3^2} = 1$$

10.1.4 Linear Independence & Dependence

Vectors $\vec{A}_1, \vec{A}_2, \cdots, \vec{A}_n$ are linearly dependent if there exist scalars $\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n$, not all zero, such that:

$$a_1\vec{A}_1 + a_2\vec{A}_2 + \cdots + a_n\vec{A}_n = 0$$

Otherwise, the vectors are linearly independent.

10.1.5 Scalar & Vector Fields

Let $D \subset \mathbb{R}^3$ be a region in space. If to each point $(x, y, z) \in D$ there corresponds a real number

$$\phi = \phi(x, y, z),$$

then ϕ is called a *scalar function of position*, and the assignment defines a *scalar field* over D. Typical examples of scalar fields include temperature, pressure, and electric potential. If the scalar field does not depend explicitly on time, it is called a *stationary* or *steady-state* scalar field.

Similarly, if to each point $(x, y, z) \in D$ there corresponds a vector

$$\mathbf{V} = \mathbf{V}(x, y, z),$$

then $\mathbf{V}$ is called a *vector function of position*, and the assignment defines a *vector field* over D. Common examples include velocity fields in fluid flow and electromagnetic force fields. If the vector field is independent of time, it is called a *stationary* or *steady-state* vector field.

10.1.6 Vector Space $\mathbb{R}^n$

Let $V = \mathbb{R}^n$, where $\mathbb{R}^n$ denotes the set of all ordered n-tuples of real numbers

$$\mathbf{u} = (a_1, a_2, \dots, a_n),$$

called the *components* of the vector $\mathbf{u}$. The elements of V are called *vectors*, and we typically denote them by $\mathbf{u}, \mathbf{v}, \mathbf{w}$, with or without subscripts. Real numbers are called *scalars* and are denoted by letters such as k, λ, μ.

Two fundamental operations are defined on $V = \mathbb{R}^n$:

Vector Addition.
If

$$\mathbf{u} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

then their sum is defined componentwise by

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}.$$

Scalar Multiplication.
For any scalar $k \in \mathbb{R}$ and vector $\mathbf{u} \in \mathbb{R}^n$,

$$k\mathbf{u} = \begin{bmatrix} ka_1 \\ ka_2 \\ \vdots \\ ka_n \end{bmatrix}.$$

These two operations satisfy all the axioms of a vector space, making $\mathbb{R}^n$ the standard n-dimensional real vector space.

10.2 Vector Spaces

A **real n-dimensional vector space**, denoted by R^n, is the set of all vectors whose components are real numbers. Each vector in R^n is written as an ordered **n-tuple** of real numbers:

$$(x_1, x_2, \ldots, x_n).$$

The word *ordered* means that the position of each component matters. If, instead of real numbers, the components are complex numbers, we obtain a **complex vector space**, denoted by C^n.

10.2.1 Dimension

A non-empty set V of vectors is called a **vector space** if all vectors in V have the same number of components and if V is closed under linear combinations, i.e., for any two vectors $\vec{a}$ and $\vec{b}$ in V, and for any real numbers α and β, the vector

$$\alpha\vec{a} + \beta\vec{b} \quad \text{also belongs to } V.$$

The **dimension** of a vector space V is the number of vectors in any basis of V. A basis is a minimal set of linearly independent vectors that can generate every vector in V using linear combinations. Thus, the space R^n has **dimension n**.

The maximum number of linearly independent vectors in $\vec{V}$ is called the dimension of $\vec{V}$ and is denoted as $\mathbf{dim}\vec{V}$. Hence, a vector space having vectors with **n** components has the dimension **n**.

10.2.2 Span, Subspaces, and Bases

Span

The **span** of a set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ in a vector space V is the set of all linear combinations of these vectors:

$$\operatorname{span}\{\vec{v}_1, \dots, \vec{v}_k\} = \left\{\alpha_1 \vec{v}_1 + \cdots + \alpha_k \vec{v}_k \;\middle|\; \alpha_i \in \mathbb{R}\right\}.$$

Thus, the span is the set of all vectors that can be generated from the given vectors.

Subspace

A **subspace** W of a vector space V is a non-empty subset of V that is closed under vector addition and scalar multiplication. That is, if $\vec{u}, \vec{v} \in W$ and $\alpha \in \mathbb{R}$, then

$$\vec{u} + \vec{v} \in W, \qquad \alpha \vec{u} \in W.$$

A set is said to be *closed* under an operation if applying that operation to elements of the set always produces an element of the same set. Every subspace must contain the zero vector and is itself a vector space.

Basis

A **basis** for a vector space V is a set of vectors in V that is both *linearly independent* and *spans* V. Equivalently, every vector in V can be written uniquely as a linear combination of the basis vectors. Note that basis vectors need not be unit vectors unless the basis is required to be orthonormal.

10.3 Vector Products

10.3.1 Dot Product

The dot (or scalar) product of two vectors $\vec{A}$ and $\vec{B}$, denoted by $\vec{A} \cdot \vec{B}$, is defined as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the cosine of the angle θ between them:

$$\boxed{\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos\theta, \qquad 0 \le \theta \le \pi} \tag{10.1}$$

10.3.2 Inner Product

An **inner product** is a generalization of the dot product. It defines a way to multiply two vectors so that the result is a scalar. An inner product on a vector space V is a function

$$\langle \cdot, \cdot \rangle \,:\, V \times V \to \mathbb{R} \text{ (or } \mathbb{C})$$

that satisfies the following four properties. For all vectors $u, v, w \in V$ and any scalar α, we have:

Property	**Mathematical Statement**
Linearity	$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
Homogeneity	$\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$
Symmetry (real case)	$\langle v, w \rangle = \langle w, v \rangle$ (For complex vector spaces: $\langle v, w \rangle = \overline{\langle w, v \rangle}$)
Positive definiteness	$\langle v, v \rangle \ge 0$, with equality if and only if $v = 0$

Table 10.1 – Axioms of an Inner Product

10.3.3 Cross Product

The cross product of vectors $\vec{A}$ and $\vec{B}$ is a vector $\vec{C} = \vec{A} \times \vec{B}$ (read as $\vec{A}$ *cross* $\vec{B}$) defined as follows.

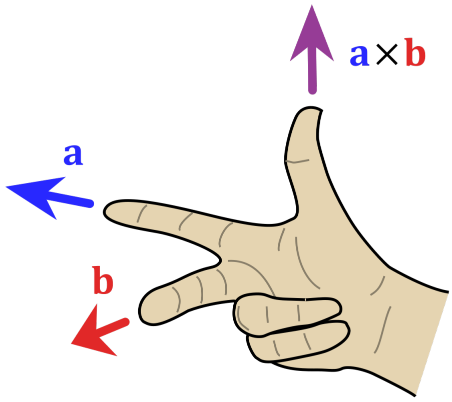
$$\boxed{\vec{C} = \vec{A} \times \vec{B} = |A||B| sin\,\theta\,\hat{u}, \quad 0 \le \theta \le \pi}$$

The magnitude of the cross product

$$\vec{C} = \vec{A} \times \vec{B}$$

is equal to the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the sine of the angle θ between them:

$$|\vec{C}| = |\vec{A}|\,|\vec{B}|\,\sin\theta.$$



The direction of $\vec{C}$ is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$, such that the vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ form a right-handed system. The unit vector $\hat{u}$ denotes the direction of $\vec{A} \times \vec{B}$.

The cross product of two vectors can be expressed in terms of *determinant* as follows:

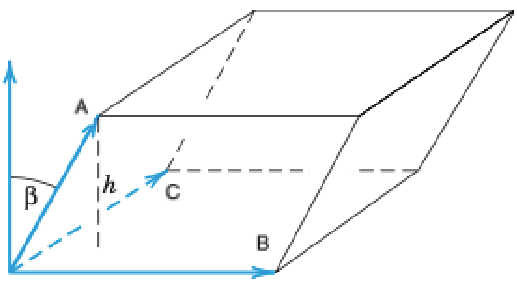
$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \begin{vmatrix} A_2 & A_3 \\ B_2 & B_3 \end{vmatrix} \hat{i} - \begin{vmatrix} A_1 & A_3 \\ B_1 & B_3 \end{vmatrix} \hat{j} + \begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix} \hat{k}$$

10.3.4 Scalar Triple Product

The **scalar triple product** of three vectors $\vec{A}, \vec{B}, \vec{C}$ is defined by

$$(\vec{A}, \vec{B}, \vec{C}) = \vec{A} \cdot (\vec{B} \times \vec{C}).$$

In component form, it is equal to the determinant



$$\boxed{(\vec{A}, \vec{B}, \vec{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \det[\vec{A}\ \vec{B}\ \vec{C}].}$$

The scalar triple product satisfies the following fundamental identity:

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}.$$

Geometrically, the absolute value $\left|(\vec{A}, \vec{B}, \vec{C})\right|$ represents the volume of the parallelepiped formed by the vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$. Moreover, the three vectors in $\mathbb{R}^3$ are linearly independent if and only if $(\vec{A}, \vec{B}, \vec{C}) \neq 0$.

10.3.5 Reciprocal Vector

Given a nonzero vector $\vec{a}$, its **reciprocal vector** $\vec{a}\,'$ is defined by

$$\vec{a} \cdot \vec{a}\,' = 1.$$

In Euclidean space, this implies

$$\vec{a}' = \frac{\vec{a}}{|\vec{a}|^2}.$$

10.3.6 Basic Vector Properties

Given three vectors $\vec{A}, \vec{B}, \vec{C} \in \mathbb{R}^n$ and a scalar m, the following properties hold:

1. **Commutative (Addition):** $\vec{A} + \vec{B} = \vec{B} + \vec{A}$
2. **Associative (Addition):** $(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$
3. **Distributive (Dot Product):** $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$
4. **Distributive (Cross Product, Left):** $\vec{A} \times (\vec{B} + \vec{C}) = (\vec{A} \times \vec{B}) + (\vec{A} \times \vec{C})$
5. **Distributive (Cross Product, Right):** $(\vec{A} + \vec{B}) \times \vec{C} = (\vec{A} \times \vec{C}) + (\vec{B} \times \vec{C})$
6. **Zero Vector:** $\vec{A} + \vec{0} = \vec{A}$
7. **Distributive (Scalar Multiplication):** $m(\vec{A} + \vec{B}) = m\vec{A} + m\vec{B}$
8. **Inner Product (Euclidean Space):** $(\vec{A}, \vec{B}) = \vec{A} \cdot \vec{B} = \vec{A}^T\vec{B}$
9. **Linear Transformation:** Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$. A linear transformation $T : X \to Y$ is given by $\vec{Y} = A\vec{X}$ where A is an $m \times n$ matrix.

10.3.7 Gram–Schmidt Orthonormalization

The **Gram–Schmidt orthonormalization process** is a systematic procedure for converting a linearly independent set of vectors in an inner product space into an orthonormal set that spans the same subspace.

Let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ be a linearly independent (but not necessarily orthonormal) basis for a subspace V. The goal is to construct an orthonormal basis $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_k\}$ for

$$\operatorname{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}.$$

The **projection** of a vector $\vec{v}$ onto a nonzero vector $\vec{u}$ is defined by:

$$\operatorname{proj}_{\vec{u}}(\vec{v}) = \frac{\langle \vec{u}, \vec{v} \rangle}{\langle \vec{u}, \vec{u} \rangle}\, \vec{u},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product. This gives the orthogonal projection of $\vec{v}$ onto the line spanned by $\vec{u}$.

The Gram–Schmidt process is given by:

$$
\begin{aligned}
u_1 &= v_1 & e_1 &= \frac{u_1}{||u_1||} \\
u_2 &= v_2 - proj_{u_1}(v_2) & e_2 &= \frac{u_2}{||u_2||} \\
u_3 &= v_3 - proj_{u_1}(v_3) - proj_{u_2}(v_3) & e_3 &= \frac{u_3}{||u_3||} \\
&\vdots \\
u_k &= v_k - \sum_{j=1}^{k-1} proj_{u_j}(v_k) & e_k &= \frac{u_k}{||u_k||}
\end{aligned}
$$

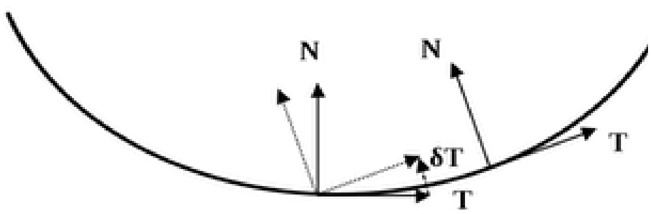
The sequence $u_1, u_2, \cdots, u_k$ is the required system of orthogonal vectors, and the normalized vectors $e_1, e_2, \cdots, e_k$ form an orthonormal set. The calculation of the sequence $u_1, u_2, \cdots, u_k$ is known as Gram–Schmidt orthogonalization, while the calculation of the sequence $e_1, e_2, \cdots, e_k$ is known as Gram–Schmidt orthonormalization as the vectors are normalized.

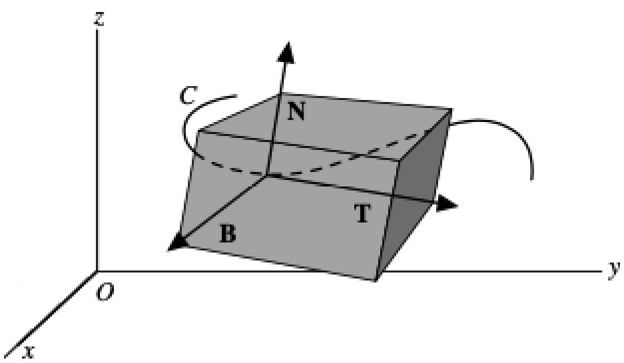
10.4 Vector Differentiation

$$
\begin{aligned}
&\frac{\Delta\vec{R}}{\Delta\vec{u}} = \frac{\vec{R}(\vec{u}+\Delta\vec{u}) - \vec{R}(\vec{u})}{\Delta\vec{u}} \\
&\frac{d\vec{R}}{d\vec{U}} = \lim_{\Delta\vec{u}\to 0}\frac{\Delta\vec{R}}{\Delta\vec{u}} = \lim_{\Delta\vec{u}\to 0}\frac{\vec{R}(\vec{u}+\Delta\vec{u}) - \vec{R}(\vec{u})}{\Delta\vec{u}} \\
&\text{If, } r(u) = x(u)\hat{i} + y(u)\hat{j} + z(u)\hat{k} \\
&\frac{dr}{du} = \frac{dx}{du}\hat{i} + \frac{dy}{du}\hat{j} + \frac{dz}{du}\hat{k}
\end{aligned}
$$

10.5 Differential Geometry

10.5.1 Frenet–Serret Formulae





10.5.2 Tangent, Normal, and the Frenet–Serret Frame

Consider a space curve C defined by the vector-valued function

$$\vec{r}(u).$$

The derivative

$$\frac{d\vec{r}}{du}$$

gives a vector tangent to the curve at each point.

If the parameter u is chosen to be the *arc length* s, measured from a fixed point on C, then

$$\vec{T} = \frac{d\vec{r}}{ds}$$

is a *unit tangent vector* to the curve.

The rate at which the tangent vector changes with respect to arc length,

$$\frac{d\vec{T}}{ds},$$

measures how sharply the curve bends and thus defines the **curvature**. The direction of $\frac{d\vec{T}}{ds}$ is perpendicular to the tangent and lies along the normal direction.

If $\vec{N}$ is the unit vector in this direction, it is called the ***principal normal*** to the curve. The curvature κ is defined by

$$\frac{d\vec{T}}{ds} = \kappa\vec{N}.$$

The reciprocal of curvature,

$$\rho = \frac{1}{\kappa},$$

is called the ***radius of curvature***.

A third unit vector $\vec{B}$, perpendicular to both $\vec{T}$ and $\vec{N}$, is defined by

$$\vec{B} = \vec{T} \times \vec{N}.$$

This vector is called the ***binormal*** to the curve.

At any point on the curve, the three mutually perpendicular unit vectors

$$\vec{T}, \quad \vec{N}, \quad \vec{B}$$

form a right-handed orthonormal coordinate system called the ***Frenet triad*** (or moving trihedral). As the point moves along the curve, this coordinate system moves with it and is therefore called the *moving trihedral.*

10.5.3 Frenet–Serret Formulae

The evolution of the Frenet triad along the curve is governed by the ***Frenet–Serret equations***:

$$\boxed{\frac{d\vec{T}}{ds} = \kappa\vec{N}}$$

$$\boxed{\frac{d\vec{N}}{ds} = -\kappa\vec{T} + \tau\vec{B}}$$

$$\boxed{\frac{d\vec{B}}{ds} = -\tau\vec{N}}$$

Here τ is a scalar called the **torsion**. It measures how rapidly the curve twists out of the plane of curvature. The reciprocal

$$\sigma = \frac{1}{\tau}$$

is called the **radius of torsion**.

10.5.4 Associated Planes

At a given point P on the curve:

- The **osculating plane** is the plane spanned by $\vec{T}$ and $\vec{N}$.
- The **normal plane** is the plane through P perpendicular to $\vec{T}$.
- The **rectifying plane** is the plane through P perpendicular to $\vec{N}$ (spanned by $\vec{T}$ and $\vec{B}$).

10.6 Gradient

The differential operator **del**, written as ∇ is defined as:

$$\boxed{\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial y}\hat{k}}$$

if $\phi(x, y, z)$ be a scalar function defined and differentiable at each point (x, y, z) in a certain region of space, then the gradient of ϕ, written $\nabla\phi$ or *grad* ϕ is defined as follows:

$$\nabla\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial y}\hat{k}\right)\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

10.7 Divergence

If $V(x, y, z) = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ is defined and differentiable at each point (x, y, z) in a region of space, then the divergence of V, a scalar, is defined as follows:

$$\boxed{\nabla \cdot V = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial y}\hat{k}\right) \cdot (V_i\hat{i} + V_j\hat{j} + V_k\hat{k}) = \frac{\partial V_i}{\partial x}\hat{i} + \frac{\partial V_j}{\partial y}\hat{j} + \frac{\partial V_k}{\partial z}\hat{k} \text{ (scalar)}}$$

10.8 Curl

The **curl** or rotation of V, a vector, is defined as:

$$\boxed{curl\, V = \nabla \times V = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial y}\hat{k}\right) \times (V_1\hat{i} + V_2\hat{j} + V_3\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}}$$

10.9 Vector Integration

10.9.1 Line Integration

Let $A(x, y, z) = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ be a vector function of position defined and continuous along C. Then the integral of the tangential component of $\vec{A}$ along C from P_1 to P_2, written as:

$$\int_{P_1}^{P_2} \vec{A} \cdot \vec{dr} = \int_C \vec{A} \cdot \vec{dr} = \int_C (A_1 dx + A_2 dy + A_3 dz)$$

If C is a closed curve (which we shall suppose is a simple closed curve, that is, a curve that does not intersect itself anywhere), the integral around C is often denoted by:

$$\boxed{\oint_C \vec{A} \cdot \vec{dr} = \oint_C (A_1 dx + A_2 dy + A_3 dz)}$$

10.9.2 Surface Integration

Consider a differential of surface area dS a vector **dS** whose magnitude is dS and whose direction is that of $\hat{n}$. Then $\vec{dS} = \hat{n} dS$. The integral is given by:

$$\boxed{\iint_S \vec{A} \cdot \vec{dS} = \iint_S \vec{A} \cdot \hat{n}\, dS}$$

10.9.3 Volume Integration

Consider a closed surface in space enclosing a volume V. The volume integral is given by:

$$\boxed{\iiint_V A\, dV}$$

10.10 Gauss' Divergence Theorem

Suppose V is the volume bounded by a closed surface S and $\vec{F}$ is a vector function of position with continuous derivatives. Then:

$$\iiint_V \nabla\cdot\vec{F}\,dV = \iiint_V \left(\frac{\partial}{\partial x}\hat{\imath}+\frac{\partial}{\partial y}\hat{\jmath}+\frac{\partial}{\partial z}\hat{k}\right)\cdot(F_x\hat{\imath}+F_y\hat{\jmath}+F_z\hat{k})\,dV$$

$$= \iiint_V \frac{\partial F_x}{\partial x}\,dx\,dy\,dz + \iiint_V \frac{\partial F_y}{\partial y}\,dx\,dy\,dz + \iiint_V \frac{\partial F_z}{\partial z}\,dx\,dy\,dz$$

Now, $\displaystyle\iiint_V \frac{\partial F_z}{\partial z}\,dx\,dy\,dz$

$$= \int_{x_1}^{x_2}\int_{y_1}^{y_2}\left[F_z(x,y,z_2)-F_z(x,y,z_1)\right]\,dx\,dy = \iint_{S_{\text{top}}} \vec{F}\cdot\hat{k}\,dS - \iint_{S_{\text{bottom}}} \vec{F}\cdot\hat{k}\,dS.$$

$$\boxed{\iiint_V \nabla\cdot\vec{F}\,dV = \oiint_S \vec{F}\cdot d\vec{S} = \oiint_S (F_x n_x + F_y n_y + F_z n_z)\,dS.}$$

10.11 Stoke's Theorem

The line integral of a vector field around a closed curve is equal to the flux of its curl through the surface bounded by the curve. Suppose S is an open, two-sided surface bounded by a simple closed curve C, and let $\vec{F}$ be a vector field with continuous partial derivatives. Then,

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot d\vec{S}.}$$

$$\nabla\times\vec{F} = \left(\frac{\partial}{\partial x}\hat{\imath}+\frac{\partial}{\partial y}\hat{\jmath}+\frac{\partial}{\partial z}\hat{k}\right)\times\left(F_x\hat{\imath}+F_y\hat{\jmath}+F_z\hat{k}\right) = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_x & F_y & F_z\end{vmatrix}$$

$$= \begin{vmatrix}\frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_y & F_z\end{vmatrix}\hat{\imath} - \begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial z}\\ F_x & F_z\end{vmatrix}\hat{\jmath} + \begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ F_x & F_y\end{vmatrix}\hat{k}$$

$$= \left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\hat{\imath} - \left(\frac{\partial F_z}{\partial x}-\frac{\partial F_x}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\hat{k}$$

$$= \left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\hat{k}.$$

$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\,dS = \iint_S \left[\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\hat{k}\right]\cdot\mathbf{\hat{n}}\,dS.$$

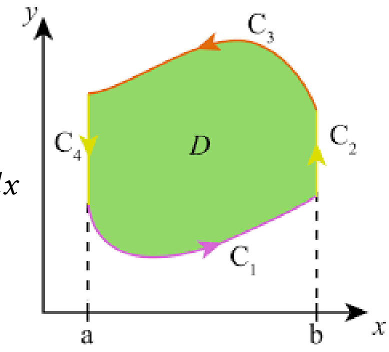
10.12 Green's Theorem

Suppose R is a closed region in the xy plane bounded by a simple closed curve C, and suppose M and N are continuous functions of x and y having continuous derivatives in R. Then,

$$\boxed{\oiint_R \left(\frac{\partial M}{\partial x}-\frac{\partial N}{\partial y}\right)dx\,dy = \oint_C (M\,dx+N\,dy)}$$

This can be proven from the following:

$$
\begin{aligned}
\oiint_R \frac{\partial N}{\partial y} dxdy &= \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial N}{\partial y} dydx \\
&= \int_a^b N(x, g_2(x))dx - \int_b^a N(x, g_1(x))dx \\
&= -\int_{C_3} N(x, g_2(x))dx - \int_{C_1} N(x, g_1(x))dx \\
\int_{C_2} N(x, y)dx &= \int_{C_4} N(x, y)dx = 0 \\
\text{Similarly, } \int_C M(x, y)dy &= \int_{C_2} M(y, h_1(y))dy + \int_{C_4} M(x, h_2(y))dy
\end{aligned}
$$



where C is traversed in the positive (counter clockwise) direction. Green's theorem is a planar case of Stoke's theorem.