Chapter 6

First Order Ordinary Differential Equation

In gentle arcs and flowing streams,
Differential equations weave their dreams.
A first-order DE so pure, so true,
In math's embrace, it guides us through.

6.1 Introduction

Many laws of nature and engineering principles are expressed not in terms of explicit functions, but as relationships between physical quantities and their rates of change. Such relationships arise whenever change, motion, growth, or flow must be understood, and they lead naturally to the mathematical formulation known as a *differential equation*. Differential equations therefore occupy a central position in applied mathematics, physics, and engineering.

A differential equation expresses a relationship between changing quantities through one or more derivatives, which describe how a variable varies with respect to time or space. Solving a differential equation means finding a function that satisfies this relationship. Unlike algebraic equations, whose solutions are numbers, the solutions of differential equations are functions. These solutions typically contain arbitrary constants, reflecting the fact that a system may evolve in many possible ways depending on how it is initialized or constrained.

Ordinary differential equations arise throughout science and engineering. They appear in equations of motion in mechanics, growth and decay processes, electrical circuits, heat transfer, fluid flow, and oscillatory systems. In many cases, fundamental physical laws—such as Newton's laws, conservation principles, or constitutive relations—lead directly to differential equations whose solutions describe how a system evolves over time or space.

This chapter introduces the basic concepts of ordinary differential equations and the methods used to solve them. Emphasis is placed on first-order equations and selected higher-order equations that admit analytical solutions. In addition to solution techniques, attention is given to the role of auxiliary conditions, which are essential for determining physically meaningful and unique solutions.

The aim of this chapter is not only to develop computational skill, but also to cultivate an understanding of how differential equations model real-world phenomena. Mastery of these ideas provides a foundation for advanced topics in physics, engineering analysis, and applied mathematics.

6.2 Differential Equations

A *differential equation* (DE) is an equation involving an **unknown function and its derivatives**. A differential equation is called an **ordinary differential equation (ODE)** if the unknown function depends on a single independent variable. If the unknown function depends on two or more independent variables, the equation is called a **partial differential equation (PDE)**.

A differential equation alone usually admits infinitely many solutions. To select a specific solution that corresponds to a physical or practical situation, additional conditions must be imposed. A differential equation together with conditions specified at the same value of the independent variable forms an *initial-value problem* (IVP). These additional constraints are called the *initial conditions*.

If the conditions are specified at two or more distinct values of the independent variable, the problem is called a *boundary-value problem* (BVP), and the constraints are referred to as the *boundary conditions*. The distinction between initial-value and boundary-value problems is fundamental, as it influences both the methods of solution and the behavior of the resulting solutions.

6.3 Standard & Differential forms of an ODE

The **Standard form** for first order DE is:

$$\boxed{\frac{dy}{dx} = f(x, y)}$$

and the **differential form** is:

$$\boxed{M(x, y)dx + N(x, y)dy = 0}$$

6.4 Order & Degree of a Differential Equation

The order of a differential equation is the **order of the highest derivative** which is also known as the differential coefficient. E.g.,

$$\frac{d^3x}{dx} + 3x\frac{dy}{dx} = e^y$$

the order of the above differential equation is 3. A first order differential equation is of the form:

$$\boxed{\frac{dy}{dx} + Py = Q} \tag{6.1}$$

where P & Q are constants or functions of independent variables. E.g.,

$$\frac{dy}{dx} + (x^2 + 5)y = \frac{x}{5}$$

The **degree** of the differential equation is represented by the **power of the highest order derivative** in the given differential equation.

$$\left[\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right]^4 = k^2\left(\frac{d^3y}{dx^3}\right)^2$$ the degree of the above differential equation is 2.

For the equation:

$$tan\left(\frac{dy}{dx}\right) = x + y$$ the degree is undefined.

6.5 Solving ODE- Method of Separation of Variables

Through algebraic manipulations, some ODEs can be reduced to:

$$\boxed{g(y)\frac{dy}{dx} = f(x)} \tag{6.2}$$

By integrating both sides:

$$\int g(y)dy = \int f(x)dx + c$$

Example:

$$\frac{dy}{dx} = 1 + y^2 \implies \frac{dy}{1 + y^2} = dx$$

Let $y = tan\,\theta \implies \frac{dy}{d\theta} = sec^2\,\theta \implies \frac{sec^2\,\theta}{1 + tan^2\,\theta}d\theta = dx \implies x = \theta + c \implies x = \tan^{-1} y + c$

6.6 Solving ODE - Reduction to Separable Form

Consider the ODE of the form:

$$\boxed{\frac{dy}{dx} = f\left(\frac{y}{x}\right)} \tag{6.3}$$

Example: $\dfrac{dy}{dx} = \left(\dfrac{y}{x}\right)^2$

Substitution: Let $y = vx$, so that $v = \dfrac{y}{x}$. Then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting into the original equation gives

$$v + x\frac{dv}{dx} = v^2$$

Hence,

$$x\frac{dv}{dx} = v^2 - v = v(v-1)$$

Separating variables:

$$\frac{dv}{v(v-1)} = \frac{dx}{x}$$

Integrating both sides:

$$\int \frac{dv}{v(v-1)} = \int \frac{dx}{x}$$

Decompose into partial fractions:

$$\frac{1}{v(v-1)} = -\frac{1}{v} + \frac{1}{v-1}$$

Thus,

$$\int \left(-\frac{1}{v} + \frac{1}{v-1}\right) dv = \int \frac{dx}{x}$$

Integration yields:

$$\ln\left|\frac{v-1}{v}\right| = \ln|x| + C$$

Simplifying:

$$\frac{v-1}{v} = Cx \implies \frac{1}{v} = 1 - Cx \implies v = \frac{1}{1-Cx}$$

Back-substitute $v = \frac{y}{x}$:

$$\frac{y}{x} = \frac{1}{1-Cx}$$

Hence, the general solution is

$$\boxed{y = \frac{x}{1-Cx}}$$

Special (constant) solutions: The substitution excluded $v = 0$ and $v = 1$, which correspond to $y = 0 \quad \text{and} \quad y = x$, both of which satisfy the original differential equation.

Therefore, the complete solution set is:

$$\boxed{y = \frac{x}{1-Cx}}, \quad y = 0, \quad y = x.$$

6.7 Solving ODE - Exact ODE & Integrating Factor

An implicit solution of a differential equation is a relationship between the variables, usually x and y, that defines the solution without explicitly solving for one variable in terms of the other.

If an ODE has an **implicit solution** :

$$\boxed{u(x, y) = c = constant}$$

then,

$$\boxed{du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0}$$
$$\boxed{M(x, y)dx + N(x, y)dy = 0}$$
$$M = \frac{\partial u}{\partial x} \quad N = \frac{\partial u}{\partial y}$$

A 1st order ODE is an **exact DE** if:

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}}$$
$$\boxed{u = \int M dx + k(y) = \int N dx + l(x)}$$

Example:

$$cos(x+y)dx + (3y^2 + 2y + cos(x+y))dy = 0$$

$$M = \frac{\partial u}{\partial x} = cos(x+y)$$
$$u(x,y) = sin(x+y) + k(y)$$
$$\frac{\partial u}{\partial y} = cos(x+y) + \frac{dk}{dy}$$

$$N = \frac{\partial u}{\partial y} = 3y^2 + 2y + cos(x+y)$$

$$cos(x+y) + \frac{dk}{dy} = 3y^2 + 2y + cos(x+y)$$
$$\frac{dk}{dy} = 3y^2 + 2y$$
$$k = y^3 + y^2 + c^*$$
$$u(x,y) = sin(x+y) + y^3 + y^2 + c^* = Constant$$
$$sin(x+y) + y^3 + y^2 = C$$

Think of u(x,y) as a "potential surface". A solution curve is a curve along which the potential does not change. So each solution curve lies on a "level curve" of u:

$$u(x,y) = C$$

6.8 Inexact ODE

Consider the ODE:

$$-ydx + xdy = 0$$

Here the above approach will not work, because:

$$M = \frac{\partial u}{\partial x} = -y \;\; N = \frac{\partial u}{\partial y} = x \;\; \frac{\partial M}{\partial y} = \frac{\partial^2 M}{\partial x \partial y} = -1 \;\; \frac{\partial N}{\partial x} = \frac{\partial^2 N}{\partial x \partial y} = 1 \;\; \frac{\partial^2 M}{\partial x \partial y} \neq \frac{\partial^2 N}{\partial x \partial y} \text{ (inexact)}$$
$$u = -y \int dx + k(y) = -xy + k(y)$$
$$\frac{\partial u}{\partial y} = -x + \frac{dk}{dy}$$

But $N = \frac{\partial u}{\partial y} = x$ which contradicts the above equation

6.9 Integrating Factor to transform to an exact ODE

Multiply the equation by a factor $F(x, y)$ to make it exact.

$$FMdx + FNdy = 0$$

and impose the conditions:

$$\frac{\partial}{\partial y}(FM) = \frac{\partial}{\partial x}(FN) \rightarrow F_y M + FM_y = F_x N + FN_x$$

Let F depend only on x,

$$FM_y = F' N + FN_x$$
$$\frac{M_y}{N} = \frac{F'}{F} + \frac{N_x}{N}$$
$$\int \frac{df}{F} dx = \int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dx$$

Let $\boxed{R = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}$

$$ln(F) = \int Rdx$$
$$\boxed{F(x) = e^{\int R(x)dx}}$$

Similarly,

$$R^* = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$
$$\boxed{F(y) = e^{\int R^*(y)dy}}$$

E.g., solve:

$$(e^{x+y} + ye^y)dx + (xe^y - 1)dy = 0$$

$$M = \frac{\partial u}{\partial x} = e^{x+y} + ye^y$$
$$\frac{\partial M}{\partial y} = e^{x+y} + ye^y + e^y$$
$$N = \frac{\partial u}{\partial y} = xe^y - 1$$
$$\frac{\partial N}{\partial x} = e^y$$
$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = e^{x+y} + ye^y$$

$$R = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xe^y - 1}\left(e^{x+y} + ye^y\right)$$

R does not work as it is a function of both x and y. So we try with R^*

$$R^* = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{-1}{e^{x+y} + ye^y}\left(e^{x+y} + ye^y\right) = -1$$

$e^{\int R^* dy} = e^{-y}$ this works as it is a function y only

Multiplying the ODE by $e^{R^*} = e^{-y}$

$$(e^x + y)dx + (x - e^{-y})dy = 0$$
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = 1 \text{ (exact ODE!)}$$
$$M = \frac{\partial u}{\partial x} = e^x + y$$
$$u = e^x + xy + k(y)$$
$$\frac{\partial u}{\partial y} = x + \frac{dk}{dy} = x - e^{-y}$$
$$k = e^y + c^*$$
$$u(x, y) = e^x + xy + e^y = C$$

6.10 1st Order Linear ODE - Homogeneous

A first order ODE is **linear** if it is of the following form:

$$\boxed{\frac{dy}{dx} + p(x)y = r(x)}$$

and is **non-linear** if it cannot be brought to the above form. The above ODE is linear in both y and y' where p and q are any function of x. When $r(x) = 0$, the ODE is called **homogeneous.**

$$\boxed{\frac{dy}{dx} + p(x)y = 0}$$

By the method of separation of variables we have,

$$\int \frac{dy}{y} = -\int p(x)dx$$

$$\ln|y| = -\int p(x)dx + c^*$$

$$y = e^{c^*} e^{\int p(x)dx}$$

$$\boxed{y_h = Ce^{-\int p(x)dx} \text{ (homogeneous solution)}}$$

6.11 1st Order ODE - Non Homogeneous

When $r(x) \neq 0$, the ODE is called **non homogeneous**. We multiply the ODE by a function $F(x)$.

$$Fy' + Fp(x)y = Fr(x)$$

$$\text{Let } Fp(x) = F'$$

$$\frac{F'}{F} = p(x)$$

$$\ln|F| = \int p(x)dx$$

$$\text{Let } h = \int p(x)dx$$

$$F = e^h$$

$$(Fy)' = Fy' + F'y = Fr(x)$$

$$(e^h y)' = r(x)e^h$$

$$e^h y = \int e^h r(x)dx + c$$

$$\boxed{y_p = e^{-h}\int e^h r dx + c}$$

$$\boxed{y = y_h + y_p = ce^{-h} + e^{-h}\int e^h r dx + c}$$

6.12 Reduction to Linear Form - Bernoulli Equation

The **Bernoulli equation, a non-linear ODE** is given by:

$\boxed{y' + p(x)y = r(x)y^n}$ where n is any real number

Let $u = y^{1-n}$

$u' = (1-n)y^{-n}y' = (1-n)y^{-n}(ry^n - py) = (1-n)(r - py^{1-n}) = (1-n)(r - pu)$

$u' = (1-n)r - (1-n)pu$

$u' + (1-n)pu = (1-n)r$ (Linear ODE)

6.13 SymPy

Differential equations and their solutions

```
from sympy import Function, dsolve, Eq, diff, Derivative, sin, cos, symbols,
    ↪ pprint

x = Function('x')
t = symbols('t')
deq = Eq(diff(x(t),t), x(t)) # Eq(LHS, RHS)
print(sp.latex(deq))
display(deq)
xsol = dsolve(deq, x(t)) # dsolve wrt x(t)
print(sp.latex(xsol))
display(xsoln)

y = Function('y')
x = symbols('x')
deq = Eq(diff(y(x), x), 1 + y(x)**2)
print(sp.latex(deq))
display(deq)
y_sol = dsolve(deq)
print(sp.latex(y_sol))
display(y_sol)
```

$$\frac{d}{dt}x(t) = x(t)$$

$$x(t) = C_1e^t$$

$$\frac{d}{dx}y(x) = y^2(x) + 1$$

$$y(x) = -\tan\left(C_1 - x\right)$$