Chapter 5

Integration

Integration sounds lofty and grand,
Yet it's just adding, hand over hand.
Cloaked in fine symbols, solemn, sublime -
Still it's just summing, one step at a time!

5.1 Introduction

Integration is one of the central operations of calculus and plays a fundamental role in mathematics, physics, and engineering. Where differentiation focuses on local rates of change, integration addresses the inverse problem: the reconstruction of a quantity from its rate of change, as well as the accumulation of continuously varying quantities.

In mathematics, integration may be understood as the technique of finding a function $g(x)$ whose derivative is equal to a given function $f(x)$. That is,

$$\frac{d}{dx}g(x) = f(x).$$

This process is denoted by the integral symbol $\int$, and the expression

$$\int f(x)\,dx$$

is called the *indefinite integral* of the function $f(x)$. The symbol dx represents an infinitesimal displacement along the variable x, and the integral may be interpreted as the continuous summation of the product of $f(x)$ and these infinitesimal increments.

Beyond its interpretation as an antiderivative, integration also admits a geometric and physical meaning. The *definite integral*, written as

$$\int_a^b f(x)\,dx,$$

where a and b are called the limits of integration, represents the accumulated value of $f(x)$ over the interval $[a, b]$. When $f(x)$ is integrable and $g(x)$ is an antiderivative of $f(x)$, the definite integral is given by

$$\int_a^b f(x)\,dx = g(b) - g(a).$$

This fundamental result establishes the deep connection between differentiation and integration and forms the basis for many applications.

Integration arises naturally in problems involving areas under curves, total displacement from velocity, work done by a force, charge distribution, mass, probability, and numerous other quantities encountered in physics and engineering. In this chapter, we develop the concept of integration from its intuitive interpretation as accumulation to its formal mathematical expression, and illustrate its use through representative examples relevant to scientific and engineering applications.

5.2 Integral

Given a function $f(x)$, an ***anti-derivative*** of $f(x)$ is any function $g(x)$ such that $g'(x) = f(x)$. The most general anti-derivative is called the ***indefinite integral***.

$$\boxed{\int f(x)dx = g(x) + c \text{ where c is a constant of integration}}$$

Note the following inequalities.

$$\int f(x)g(x)dx \neq \int f(x)dx \int g(x)dx$$

$$\int \frac{f(x)}{g(x)}dx \neq \frac{\int f(x)dx}{\int g(x)dx}$$

5.3 Common Integrals

$$\int x^n dx = \frac{x^{(n+1)}}{(n+1)} + c \tag{5.1}$$

$$\int e^x dx = e^x + c \tag{5.2}$$

$$\int a^x dx = \frac{a^x}{ln\ a} + c \tag{5.3}$$

$$\int \frac{1}{x} dx = ln|x| + c \tag{5.4}$$

$$\int cos(x)dx = sin(x) + c \tag{5.5}$$

5.4 Substitution Technique

$$\int 18x^2\sqrt[4]{(6x^3+5)}dx$$

Let $u = 6x^3 + 5$

$$\rightarrow du = 18x^2 dx$$

$$\rightarrow \int \sqrt[4]{u}\, du = \frac{u^{\left(\frac{1}{4}+1\right)}}{\frac{1}{4}+1} = \frac{4}{5}u^{\frac{5}{4}} = \frac{4}{5}\left(6x^3+5\right)^{\frac{5}{4}}$$

5.5 Integration by Parts

$$[f(x)g(x)]' = f(x)g'(x) + f'(x)g(x)$$

$$f(x)g'(x) = [f(x)g(x)]' - f'(x)g(x)$$

$$\int f(x)g'(x)\,dx = \int [f(x)g(x)]'\,dx - \int f'(x)g(x)\,dx$$

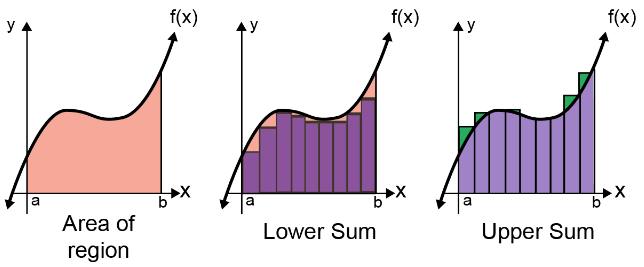
$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$$\boxed{\int u\,dv = uv - \int v\,du} \tag{5.6}$$

Hence, integral of two functions = first function × integral of second function − integral of (differntiation of the first function × integral of the second function).

5.6 Definite Integral

A definite integral is a the area under its curve .



$$\boxed{\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x = g(x)\Big|_a^b = g(b) - g(a)}$$

where $f(x_i^*)$ is the value at the middle of the strip Δx.

$$\boxed{f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx}$$

$$\boxed{\int_a^b f(x)dx = f(c)(b-a) \text{ where c is in [a,b]}}$$

5.7 Some Integration Strategies

1. Simplify the integrand. E.g., $cos^2(\theta) = \frac{1}{2}(1 + cos(2\theta))$
2. Check if simple substitution will work
3. If integrand is a rational expression, partial functions may work
4. If integrand is polynomial x, $trig$, exp, ln function, integration by parts may work
5. If integrand involves $\sqrt{b^2x^2 + a^2}$, trigonometric substitution may work
6. If integrand has a quadratic in it, completing the square may work.

5.8 SymPy

Integrate the following functions:

$$\int (x^2 + x + 1)\,dx$$

$$\int e^{-x^2} \operatorname{erf}(x) dx \quad \text{where} \quad \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

$$\int y = e^{-x}\,dx$$

```python
import sympy as sp
import numpy as np
import matplotlib.pyplot as plt
from IPython.display import display, Math
from sympy import sqrt, diff, integrate, oo
from sympy import sin, cos, tan, ln, exp, erf, trigsimp, expand_trig, simplify
from sympy import sinh, cosh

x = sp.symbols('x')

y = x**2 + x + 1
```

```
int_expr = integrate(y,x) # integrate y wrt x
print(sp.latex(int_expr))
display(int_expr)

y = exp(-x**2)*erf(x)
int_expr = integrate(y,x) # integrate y wrt x
print(sp.latex(int_expr))
display(int_expr)

y = exp(-x)
int_expr = integrate(y, (x, 0, oo)) # definite integral, limits 0 & infinity
print(sp.latex(int_expr))
display(int_expr)
```

$$\frac{x^{3}}{3} + \frac{x^{2}}{2} + x$$

$$\frac{\sqrt{\pi} \operatorname{erf}^{2}{\left(x \right)}}{4}$$

$$1$$