Chapter 26

Tensors

Tensors speak where coordinates fall away,
Their truths remain though frames may sway.
From curved spacetime to fields untold,
They bind the laws in forms concise and bold.

Tensors form a rigorous mathematical framework for describing physical quantities in a way that does not depend on the choice of coordinate system. By extending the familiar notions of scalars and vectors, tensors provide a unified language for expressing laws of physics, and are central to continuum mechanics, electromagnetism, relativity, and much of modern physics.

26.1 Scalars and Vectors

26.1.1 Scalars

A scalar is a quantity that is fully described by a single number and remains unchanged under coordinate transformations.

Examples include mass, temperature, and energy.

26.1.2 Vectors

A vector is a physical quantity characterized by both **magnitude** (how large it is) and **direction** (which way it points). Unlike scalars, which have only magnitude (e.g., temperature or mass), vectors describe quantities like displacement, velocity, or force.

In a 3D Cartesian coordinate system, we represent a vector $\mathbf{A}$ using its components along the basis vectors $\mathbf{e}_1 = \hat{i}$, $\mathbf{e}_2 = \hat{j}$, $\mathbf{e}_3 = \hat{k}$:

$$\mathbf{A} = A^i \mathbf{e}_i = A^1 \mathbf{e}_1 + A^2 \mathbf{e}_2 + A^3 \mathbf{e}_3 = (A^1, A^2, A^3)$$

Here, A^i are the *contravariant components* (with the Einstein summation convention implying summation over $i = 1, 2, 3$), and the parentheses denote the standard component tuple.

Example

The vector pointing 3 units east, 4 units north, and 0 units up is $\mathbf{A} = (3, 4, 0)$, with magnitude

$$\|\mathbf{A}\| = \sqrt{3^2 + 4^2 + 0^2} = 5.$$

Under a change of coordinates, the components transform linearly.

26.2 Coordinate Transformations

Consider two coordinate systems related by a linear transformation

$$\boxed{\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j} x^j}$$

The transformation matrix is defined as

$$\tilde{\mathbf{x}} = \Lambda \mathbf{x}$$

where:

- $\mathbf{x} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$ (original coordinates)
- $\tilde{\mathbf{x}} = \begin{pmatrix} \tilde{x}^1 \\ \tilde{x}^2 \\ \tilde{x}^3 \end{pmatrix}$ (new coordinates)
- $\Lambda = \left(\frac{\partial \tilde{x}^i}{\partial x^j} \right)$ is the Jacobian matrix

$$\Lambda^i_j = \frac{\partial \tilde{x}^i}{\partial x^j}$$

Explicit matrix form

$$\begin{pmatrix} \tilde{x}^1 \\ \tilde{x}^2 \\ \tilde{x}^3 \end{pmatrix} = \begin{pmatrix} \frac{\partial \tilde{x}^1}{\partial x^1} & \frac{\partial \tilde{x}^1}{\partial x^2} & \frac{\partial \tilde{x}^1}{\partial x^3} \\ \frac{\partial \tilde{x}^2}{\partial x^1} & \frac{\partial \tilde{x}^2}{\partial x^2} & \frac{\partial \tilde{x}^2}{\partial x^3} \\ \frac{\partial \tilde{x}^3}{\partial x^1} & \frac{\partial \tilde{x}^3}{\partial x^2} & \frac{\partial \tilde{x}^3}{\partial x^3} \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

26.3 Meaning of Superscript and Subscript Indices

In tensor notation, the position of an index—whether it appears as a superscript or a subscript—is not merely a typographical convention. It encodes how the components of an object transform under a change of coordinates and reflects their underlying geometric role. Indices written as superscripts, such as V^i are called contravariant indices. indices written as subscripts, such as V_i are called covariant indices.

26.4 Contravariant Vectors

A contravariant vector V^i transforms according to the chain rule:

$$\boxed{\tilde{V}^i = \frac{\partial \tilde{x}^i}{\partial x^j} V^j}$$

Geometrically, contravariant vectors represent directions or displacements in space. Their components "stretch" or "compress" in the same manner as the coordinate axes themselves. Tangent vectors to curves and velocity vectors are typical examples of contravariant objects.

26.5 Covariant Vectors

A covariant vector (or covector) W_i transforms oppositely to a contravariant vector under a change of coordinates. Its transformation law is given by:

$$\boxed{\tilde{W}_i = \frac{\partial x^j}{\partial \tilde{x}^i} W_j}$$

This transformation ensures that the contraction of a covariant vector with a contravariant vector, $W_i V^i$, remains invariant under coordinate changes.

Geometrically, covariant vectors do not represent directions or displacements. Instead, they act as *linear functionals* on contravariant vectors, mapping vectors to scalars. One may think of a covariant vector as defining a family of parallel hypersurfaces; when a vector is applied to it, the result measures how the vector pierces these surfaces.

Gradients of scalar fields provide the most common example of covariant vectors. Given a scalar function $\phi(x)$, its gradient

$$W_i = \frac{\partial \phi}{\partial x^i}$$

is a covariant object. The gradient naturally transforms with the inverse Jacobian because it measures rates of change with respect to coordinates rather than displacements along them.

In curved spaces and general relativity, covariant vectors play a fundamental role in defining differential forms, forces, and momenta, complementing contravariant vectors and enabling the coordinate-independent formulation of physical laws.

Example: Coordinate-Invariant Scalar

Consider a covector W_i and a vector V^i. Their contraction

$$W_i V^i$$

is a scalar.

Under a coordinate transformation, W_i and V^i transform oppositely, and the Jacobian factors cancel exactly. This invariance illustrates why tensors provide a coordinate-independent language for physics.

26.6 Definition of a Tensor

A tensor is a mathematical object defined by how its components transform under coordinate changes. Specifically, a tensor of type (k, l) transforms as:

$$\boxed{T^{i_1 \dots i_k}_{j_1 \dots j_l} \sim \frac{\partial \tilde{x}^{i_1}}{\partial x^{m_1}} \cdots \frac{\partial \tilde{x}^{i_k}}{\partial x^{m_k}} \frac{\partial x^{n_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{n_l}}{\partial \tilde{x}^{j_l}} T^{m_1 \dots m_k}_{n_1 \dots n_l}}$$

Example,

$$\tilde{T}^{abc}{}_{def} = \frac{\partial \tilde{x}^a}{\partial x^m} \frac{\partial \tilde{x}^b}{\partial x^n} \frac{\partial \tilde{x}^c}{\partial x^p} \frac{\partial x^q}{\partial \tilde{x}^d} \frac{\partial x^r}{\partial \tilde{x}^e} \frac{\partial x^s}{\partial \tilde{x}^f} T^{mnp}{}_{qrs}$$

Tensors are the objects whose transformation rules are consistent across coordinate systems, distinguishing them from general functions.

26.7 Tensor Contraction

A fundamental operation on tensors is *contraction*, which consists of summing over one contravariant and one covariant index of the same tensor. Contraction reduces the rank of a tensor by two.

For a tensor $T^i{}_j$, contraction gives a scalar:

$$T^i{}_i$$

More generally, contracting a (k, l) tensor produces a $(k-1, l-1)$ tensor. The key property is that contraction preserves tensorial character: the result transforms as a tensor under coordinate changes.

Example

Consider a mixed tensor $T^i{}_j$ of type $(1, 1)$. Contracting the contravariant index with the covariant index produces a scalar:

$$T^i{}_i$$

Under a change of coordinates, the tensor transforms as

$$\tilde{T}^i{}_j = \frac{\partial \tilde{x}^i}{\partial x^m} \frac{\partial x^n}{\partial \tilde{x}^j} T^m{}_n.$$

Contracting the indices yields

$$\tilde{T}^i{}_i = \frac{\partial \tilde{x}^i}{\partial x^m} \frac{\partial x^n}{\partial \tilde{x}^i} T^m{}_n = \delta^n_m T^m{}_n = T^i{}_i$$

Thus, the contracted quantity is invariant under coordinate transformations and is therefore a scalar.

This invariance follows directly from the cancellation of the Jacobian and inverse Jacobian factors.

26.8 The Metric Tensor

The metric tensor g_{ij} is a symmetric covariant tensor of type $(0, 2)$ that defines lengths and angles in a space. Given two vectors V^i and W^j, their inner product is

$$\langle V, W \rangle = g_{ij} V^i W^j$$

The metric allows one to convert between contravariant and covariant components.

26.9 Raising and Lowering Indices

Using the metric tensor, one can convert vectors between covariant and contravariant forms.

Lowering an index:

$$V_i = g_{ij} V^j$$

Raising an index:

$$V^i = g^{ij} V_j$$

Here, g^{ij} is the inverse metric satisfying

$$g^{ik} g_{kj} = \delta^i_j$$

The position of indices is therefore not arbitrary; it encodes how quantities transform and interact geometrically.