

# LIMIT

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*A limit's a dream in sight,  
 We chase it with all our might!  
 But just as we draw near,  
 It grins, and disappears.*

In mathematics, a limit is the value that a function or a sequence approaches as the input approaches some value. Limits are essential to calculus and mathematical analysis, and are used to define continuity, derivatives, and integrals.

## 3.1 DEFINITION OF LIMIT

Consider a function  $f(x)$  that is defined in a domain  $D$  which includes the point  $c$ . The function may or may not be defined at  $c$ . If, for all  $x$  that is *close* to  $c$  except for  $c$ ,  $f(x)$  is arbitrarily close to a number  $L$  (as close to  $L$  as we like), then it is said that  $f$  approaches the limit  $L$  as  $x$  approaches  $c$  and is written as:

$$\lim_{x \rightarrow c} f(x) = L$$

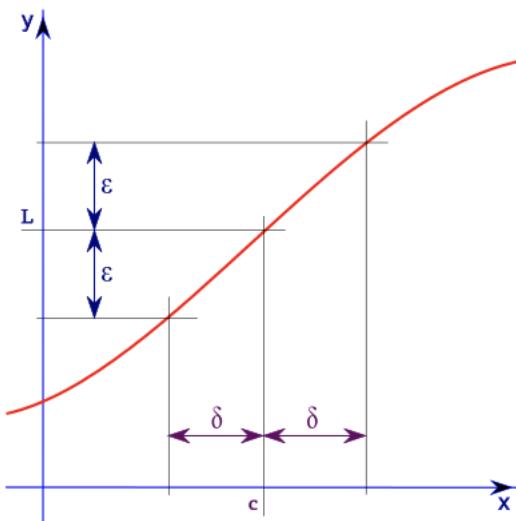
If the function can be evaluated at  $c$ , the limit  $L$  is simply  $f(c)$ . But there can be situations where the function is not evaluable at  $c$ . E.g., the following function cannot be evaluated at  $x = 1$ .

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x - 1}$$

But this function can be easily simplified to:

$$\begin{aligned} f(x) &= \frac{(x-1)(x+2)}{x-1} = x+2 \\ &\rightarrow \lim_{x \rightarrow 1} f(x) = 3. \end{aligned}$$

## 3.2 FORMAL DEFINITION OF LIMIT



Let  $f(x)$  be a function that is defined on an interval that contains  $x = c$ , except possibly at  $c$ . Then,  $\lim_{x \rightarrow c} f(x) = L$  if for every number  $\epsilon > 0$ , there is some number  $\delta > 0$  such that, when  $0 < |x - c| < \delta$ ,  $|f(x) - L| < \epsilon$ .

This means that for any number  $\epsilon > 0$  that we pick, one can go to the graph and sketch two horizontal lines at  $L + \epsilon$  and  $L - \epsilon$ . Then there must be another number  $\delta > 0$  that can be determined to enable us to add in two vertical lines in the graph  $a + \delta$  and  $a - \delta$ .

### 3.3 LAWS OF LIMIT

Given  $L, M, c, k$  are real numbers such that  $\lim_{x \rightarrow c} f(x) = L$  and  $\lim_{x \rightarrow c} g(x) = M$ . Then,

Sum Rule	$\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$
Difference Rule	$\lim_{x \rightarrow c} (f(x) - g(x)) = L - M$
Constant Rule	$\lim_{x \rightarrow c} (kf(x)) = kL$
Product Rule	$\lim_{x \rightarrow c} (f(x)g(x)) = LM$
Quotient Rule	$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M}$
Power Rule	$\lim_{x \rightarrow c} [f(x)]^n = L^n (n > 0)$
Root Rule	$\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{L} (n > 0)$

Examples:

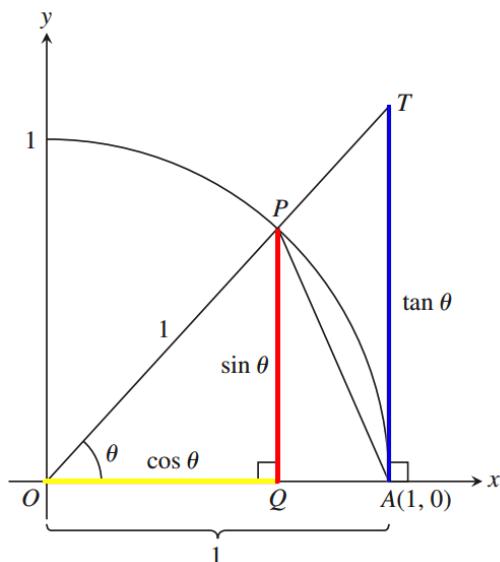
$$\lim_{x \rightarrow 3} \sqrt{(2x^3 + 10)} = 8$$

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 9} - 3}{x^2}$$

The above function is not evaluable at  $x = 0$ . The standard trick is to multiply both numerator and denominator by the conjugate radical expression.

$$\frac{\sqrt{x^2 + 9} - 3}{x^2} = \frac{\sqrt{x^2 + 9} - 3}{x^2} \cdot \frac{\sqrt{x^2 + 9} + 3}{\sqrt{x^2 + 9} + 3} = \frac{1}{\sqrt{x^2 + 9} + 3} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{x^2 + 9} + 3} = \frac{1}{6}$$

### 3.4 AN IMPORTANT LIMIT



Consider the circle with a unit radius.

$$\text{Area } \triangle OAP < \text{area sector OAP} < \text{area } \triangle OAT$$

$$\frac{1}{2} \sin \theta \leq \pi 1^2 \left( \frac{\theta}{2\pi} \right) \leq \frac{1}{2} \tan \theta \quad (\theta \text{ is in radians})$$

$$1 \leq \frac{\theta}{\sin \theta} \leq \frac{1}{\cos \theta}$$

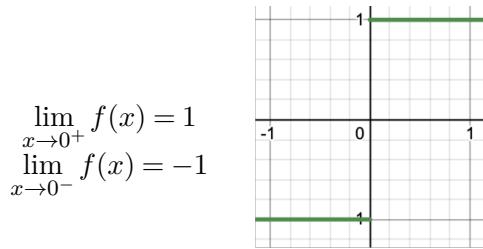
$$\rightarrow 1 \geq \frac{\sin \theta}{\theta} \geq \cos \theta$$

Hence,

$$\boxed{\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1} \quad \text{where } \theta \text{ is in radians} \quad (3.4.1)$$

Now consider the function  $f(\theta) = \frac{1}{\sin \theta}$ . Does it have a limit as  $t \rightarrow \theta$  from either side? As  $\theta$  approaches 0, its reciprocal,  $1/\sin \theta$ , grows without bound and the values of function cycle repeatedly from -1 to 1. There is no single number  $L$  that the function values stay increasingly close to as  $\theta \rightarrow 0$ . The function has neither a right-hand limit nor a lefthand limit at  $\theta = 0$ .

## 3.5 ONE SIDED LIMITS



## 3.6 CONTINUOUS FUNCTION

Function is right-continuous at  $c$  (continuous from right) if  $\lim_{x \rightarrow c^+} f(x) = f(c)$

Function is left-continuous at  $c$  (continuous from left) if  $\lim_{x \rightarrow c^-} f(x) = f(c)$

A function is continuous at  $c$  if  $\lim_{x \rightarrow c} f(x) = f(c)$

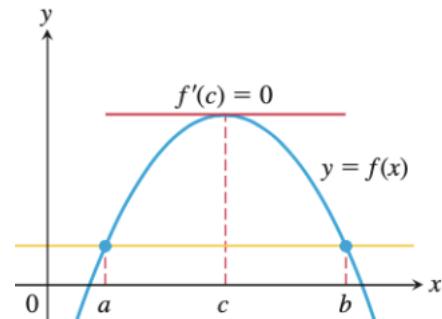
If a function is discontinuous at one or more points of its domain, it is called a discontinuous function.

## 3.7 INFINITE LIMITS

$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty, \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$  Note that this does not mean that the limit exists as there is no real number such as  $\infty$ . It is simply a concise way of saying that the limit does not exist.

## 3.8 ROLLE'S THEOREM

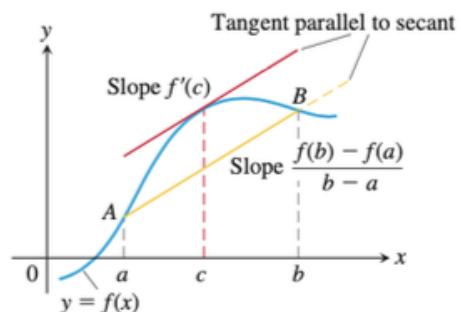
If  $f$  is a continuous function on a closed interval  $[a, b]$  and If  $f(a) = f(b)$ , then there is at least one point  $c$  in  $(a, b)$  where  $f'(c) = 0$ .



## 3.9 MEAN VALUE THEOREM

There is at least one number  $c$  in the interval  $(a, b)$  such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



## 3.10 SymPy Code

Determine the limits of:

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{(x - 1)}$$

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 9} - 3}{x^2}$$

$$\lim_{x \rightarrow 0} \sin(x)$$

```

1 import sympy as sp
2 import numpy as np
3 import matplotlib.pyplot as plt
4 from IPython.display import display, Math
5 from sympy import sin, cos, tan, trigsimp, expand_trig
6 from sympy import oo
7 from sympy import limit
8
9 x = sp.symbols('x')
10 y = (x**2 + x - 2) / (x - 1)
11 lim = limit(y, x, 1)
12 display(lim)
13
14 y = ( (x**2 + 9)**0.5 - 3 ) / x**2
15 lim = limit(y, x, 0)
16 display(lim)
17
18 y = sin(x)/x
19 lim = limit(y, x, 0)
20 display(lim)

```

3

$\frac{1}{6}$

$\cos(x)$

If you are copying code from pdf into your editor or python terminal, please replace the following in your editor.  $^{\wedge}\backslash s^{*}\backslash d+\backslash s+$  with empty string to remove the line numbers and uncode characters such as ` with ' Python source code must use ASCII-compatible syntax characters, even though it fully supports Unicode text for identifiers, strings, and comments.