Chapter 14

Complex Analysis

There is a number $\boldsymbol{i}$, quite imaginary,
Yet it gets things done, quite extraordinary.
Who would imagine such a possibility?
But that is indeed the reality!

14.1 Introduction

Complex analysis is the study of complex numbers and complex-valued functions, together with their differentiation, integration, and associated properties. Although complex numbers involve the imaginary unit $\boldsymbol{i} = \sqrt{-1}$, the theory developed around them is remarkably rich and powerful, providing deep insights and elegant solutions to many problems in mathematics, physics, and engineering.

This chapter begins with the algebra and geometry of complex numbers, including their representation in Cartesian and polar forms, complex conjugates, moduli, arguments, and roots. Fundamental properties of complex arithmetic and De Moivre's theorem are established, forming the basis for later analytical developments.

The concept of complex differentiation is then introduced, leading to analytic functions and the Cauchy–Riemann equations. The intimate connection between analytic functions and harmonic functions is explored through Laplace's equation, along with representations of trigonometric and hyperbolic functions in the complex plane.

Subsequent sections develop the theory of complex integration, emphasizing path dependence, the ML–inequality, and Cauchy's integral theorem. These results culminate in Cauchy's integral formula, which provides powerful expressions for analytic functions and their derivatives. The chapter further examines power series representations, including Taylor and Laurent series, and applies them to the study of zeros, singularities, and their classification.

Finally, the residue theorem is introduced as a practical and efficient method for evaluating complex integrals, reducing complicated contour integrals to algebraic computations involving residues. Computational illustrations using symbolic tools are included to demonstrate the applicability of complex analysis techniques in both theoretical and practical contexts.

14.2 Complex Numbers

Complex numbers are the numbers that are expressed in the form of $x + \boldsymbol{i}\,y$ where x, y are real numbers and $\boldsymbol{i}$ is the imaginary unit.

$$\boxed{z = x + \boldsymbol{i}\,y \quad \text{where } \boldsymbol{i} = \sqrt{-1}}$$

Just as with real numbers, we can perform arithmetic operations on complex numbers. To add or subtract complex numbers, we combine the real parts and combine the imaginary parts. Addition, multiplication, and division of complex numbers are given below:

$$z_1 + z_2 = (x_1 + x_2) + \boldsymbol{i}\,(y_1 + y_2)$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + \boldsymbol{i}\,(x_1 y_2 + x_2 y_1)$$

$$\frac{z_1}{z_2} = \frac{x_1 + \boldsymbol{i}\,y_1}{x_2 + \boldsymbol{i}\,y_2}, \quad z_2 \neq 0 = \frac{(x_1 + \boldsymbol{i}\,y_1)(x_2 - \boldsymbol{i}\,y_2)}{(x_2 + \boldsymbol{i}\,y_2)(x_2 - \boldsymbol{i}\,y_2)}$$

$$= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \boldsymbol{i}\,\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

14.2.1 Complex Conjugate

The **complex conjugate** of z is defined as:

$$\boxed{\bar{z} = x - \boldsymbol{i}\,y}$$

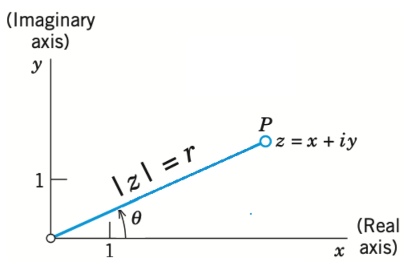
14.2.2 Polar Representation

$$\boxed{z = r\cos\theta + \boldsymbol{i}\,r\sin\theta = re^{\boldsymbol{i}\,\theta}, \quad r \geq 0} \qquad \text{(Polar representation)}$$

$$\boxed{z^n = r^n\big(\cos n\theta + \boldsymbol{i}\,\sin n\theta\big), \quad n \in \mathbb{Z}} \qquad \text{(De Moivre's theorem)}$$

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

$$x = r\cos\theta, \quad y = r\sin\theta \quad \text{(radians, measured counterclockwise).}$$



The complex plane

θ is the **argument** of z, denoted by $\arg z$. Its **principal value** is:

$$\boxed{-\pi < \arg z \leq \pi, \quad z \neq 0}$$

The xy-plane is the complex plane, also known as the **Argand diagram**.

$$z_1 z_2 = r_1 r_2 e^{\boldsymbol{i}\,(\theta_1 + \theta_2)} = r_1 r_2 \left[\cos(\theta_1 + \theta_2) + \boldsymbol{i}\,\sin(\theta_1 + \theta_2)\right]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{\boldsymbol{i}\,(\theta_1 - \theta_2)}, \quad z_2 \neq 0 = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + \boldsymbol{i}\,\sin(\theta_1 - \theta_2)\right]$$

$$\arg\left(\frac{z_1}{z_2}\right) \equiv \arg z_1 - \arg z_2 \pmod{2\pi}, \quad z_1 \neq 0,\ z_2 \neq 0$$

14.2.3 Properties

$z_1 z_2 = z_2 z_1$ (commutative)

$(z_1 z_2) z_3 = z_1 (z_2 z_3)$ (associative)

$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$(distributive)

$|z_1 z_2| = |z_1||z_2|$ and $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad z_2 \neq 0$

14.2.4 Roots

$z = re^{i\theta}$

$$\boxed{z_k = r^{1/n} e^{i\left(\frac{\theta + 2k\pi}{n}\right)}, \quad k = 0, 1, \ldots, n-1}$$

Example: $4i = 4e^{i\frac{\pi}{2}}$

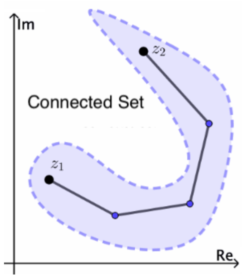
$$\sqrt{4i} = \sqrt{4e^{i\frac{\pi}{2}}} = 2e^{i\left(\frac{\pi}{4} + k\pi\right)}, \quad k = 0, 1$$

$$2e^{i\frac{\pi}{4}} = \sqrt{2}(1 + i), \qquad 2e^{i\frac{5\pi}{4}} = -\sqrt{2}(1 + i)$$

$$\Rightarrow \quad \sqrt{4i} = \sqrt{2}(1 + i), \ -\sqrt{2}(1 + i)$$

14.3 Point Set & Path

A **point set** is simply a collection of points in the complex plane. A set S is called **open** if it contains no boundary points. A set S is called **closed** if it contains all of its boundary points. A set S is called **connected** if it is all in one piece and cannot be broken into two separate open parts.



14.4 Complex Differentiation

Complex analysis is the study of complex-valued functions that are complex differentiable in a domain. The concepts of limits, derivatives, and integrals are similar in spirit to those in real calculus, but they possess much stronger consequences in the complex case.

A function $f(z)$ of a complex variable z is called **analytic** in a domain D if it is **defined** and **complex differentiable** at every point of D.

14.4.1 Cauchy-Riemann Equations

A necessary condition that $f(z) = u(x, y) + i\,v(x, y)$ be analytic in a region R is that u and v satisfy the Cauchy–Riemann equations as stated below

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y+\Delta y) + \boldsymbol{i}\, v(x+\Delta x, y+\Delta y) - [u(x,y) + \boldsymbol{i}\, v(x,y)]}{\Delta x + \boldsymbol{i}\, \Delta y}$$

with $\Delta y = 0$ $\quad f'(z) = \lim_{\Delta x \to 0} \dfrac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + \dfrac{\boldsymbol{i}\, v(x+\Delta x, y) - \boldsymbol{i}\, v(x,y)}{\Delta x} = u_x + \boldsymbol{i}\, v_x$

with $\Delta x = 0$ $\quad f'(z) = \lim_{\Delta y \to 0} \dfrac{u(x, y+\Delta y) - u(x,y)}{\boldsymbol{i}\, \Delta y} + \dfrac{\boldsymbol{i}\, v(x, y+\Delta y) - \boldsymbol{i}\, v(x,y)}{\boldsymbol{i}\, \Delta y} = v_y - \boldsymbol{i}\, u_y$

The Cauchy-Riemann equations are:

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}\ , \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

In polar coordinates:

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}},\quad \boxed{\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}, \quad r \neq 0}$$

Example

$z = x + \boldsymbol{i}\, y \quad \Rightarrow \quad u = x,\ v = y, \quad u_x = 1,\ v_y = 1,\ u_y = 0,\ v_x = 0$

Since $u_x = v_y$ and $u_y = -v_x$, $\quad z' = 1$

$\bar{z} = x - \boldsymbol{i}\, y \quad \Rightarrow \quad u = x,\ v = -y, \quad u_x = 1,\ v_y = -1,\ u_y = 0,\ v_x = 0$

Since $u_x \neq v_y$ and $u_y \neq -v_x$, $\quad \bar{z}$ is not analytic

14.4.2 Laplace's Equation

Using the Cauchy–Riemann equations

$$u_x = v_y, \qquad u_y = -v_x$$

and assuming that u and v have continuous second partial derivatives, we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(u_x) = \frac{\partial}{\partial x}(v_y) = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(u_y) = \frac{\partial}{\partial y}(-v_x) = -\frac{\partial^2 v}{\partial y \partial x}$$

Since the mixed partial derivatives of v are equal,

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$$

it follows that

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.}$$

Similarly,

$$\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.}$$

Thus both u and v are harmonic in D. The function v is called the **harmonic conjugate** of u in D (not to be confused with $\bar{z}$) when u and v satisfy the Cauchy–Riemann equations in D.

14.4.3 Trigonometric & Hyperbolic Functions

$$e^{ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos x - i \sin x$$

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$$

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

$$\cosh z = \frac{1}{2}(e^{z} + e^{-z})$$

$$\sinh z = \frac{1}{2}(e^{z} - e^{-z})$$

$$(\cosh z)' = \sinh z$$

$$(\sinh z)' = \cosh z$$

14.5 Complex Integration

Let $f(z) = u(x, y) + i\, v(x, y)$ be continuous on a piecewise smooth curve C. Then

$$\int_C f(z)\, dz = \int_C (u + i\, v)(dx + i\, dy) = \left[\int_C u\, dx - \int_C v\, dy\right] + i \left[\int_C u\, dy + \int_C v\, dx\right]$$

Using a parametric representation,

$$z(t) = x(t) + i\, y(t), \qquad a \le t \le b$$

$$\dot{z}(t) = \frac{dz}{dt}$$

we obtain

$$\boxed{\int_C f(z)\, dz = \int_a^b f[z(t)]\, \dot{z}(t)\, dt}$$

Example 1

$$z = 3t - i\, t^2$$

$$\frac{dz}{dt} = 3 - i\, 2t$$

$$\int f(z)\, dz = \int (3t - i\, t^2)(3 - i\, 2t)\, dt$$

$$= \int (9t - 2t^3 - i\, 9t^2)\, dt$$

$$= \left(-\frac{t^4}{2} + \frac{9t^2}{2}\right) - i\, 3t^3$$

Example 2

$$\oint_C \frac{dz}{z}$$

$$z = re^{i\theta}, \quad dz = ire^{i\theta}d\theta$$

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}}\, d\theta$$

$$= i\int_0^{2\pi} d\theta$$

$$= 2\pi i$$

Example 3

$$\oint_C (z - z_0)^m\, dz$$

$$z(t) = z_0 + re^{it}, \quad dz = ire^{it}dt$$

$$\oint_C (z - z_0)^m\, dz = \int_0^{2\pi} r^m e^{imt}\, ire^{it}\, dt$$

$$= ir^{m+1}\int_0^{2\pi} e^{i(m+1)t}\, dt$$

$$= \begin{cases} 2\pi i, & m = -1 \\ 0, & m \neq -1 \end{cases}$$

14.5.1 Path Dependence

If we integrate a given function $f(z)$ from a point z_1 to a point z_2 along different paths, the integrals will in general have different values. **A complex line integral depends not only on the end points of the path but also, in general, on the path itself.**

14.5.2 ML-Inequality

$$\boxed{\left|\oint_C f(z)\, dz\right| \leq ML, \quad \text{where } |f(z)| \leq M \text{ on } C}$$

Here L is the length of the contour C and $|f(z)| \leq M$, where M is a constant. This follows from the fact that $|f(z)|$ is bounded on the contour C, and its maximum value on C is denoted by M.

14.6 Cauchy's Integral Theorem

A **simple closed path** is a closed path that does not intersect or touch itself

An **open and connected** set is called a **domain**. In a **simply connected domain** D, any simple closed curve C is the boundary of some region R which is contained in D. In simple words, a region is simply connected if every closed curve within it can be shrunk continuously to a point that is within the region. That means, a simply connected region is one that has no holes

If f(z) is analytic in a simply connected domain D, then for every simple closed path C in D,

$$\boxed{\oint_C f(z)\,dz = 0}$$

Since $f(z)$ is analytic in D, $f'(z)$ exists in D. Assume $f'(z)$ to be continuous, i.e., u and v have continuous partial derivatives in D[1]

$$\int_C f(z)\,dz = \int_C (u+\boldsymbol{i}v)(dx+\boldsymbol{i}dy) = \left[\int_C u\,dx - \int_C v\,dy\right] + \boldsymbol{i}\left[\int_C u\,dy + \int_C v\,dx\right]$$

Applying Green's theorem to the vector field $(P,Q) = (u,-v)$,

$$\boxed{\oint_C u(x,y)\,dx - \oint_C v(x,y)\,dy = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\,dx\,dy}$$

and using the Cauchy-Riemann equations

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \qquad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

$$\left[\int_C u\,dx - \int_C v\,dy\right] = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\,dx\,dy = 0$$

$$\left[\int_C u\,dy + \int_C v\,dx\right] = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\,dx\,dy = 0$$

$$\boxed{\oint_C f(z)\,dz = 0}$$

[1]Goursat proved this without the condition that $f'(z)$ is continuous, but the proof is more involved.

14.6.1 Path Independence

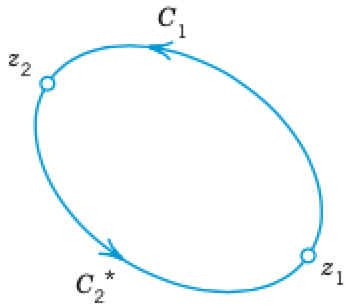
If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of the path in D. This follows from Cauchy's Integral Theorem.

$$\oint_C f(z)\,dz = 0$$

$$\int_{C_1} f(z)\,dz + \int_{C_2^*} f(z)\,dz = 0$$

$$\int_{C_1} f(z)\,dz = -\int_{C_2^*} f(z)\,dz$$

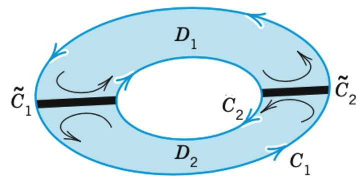
$$\boxed{\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz}$$



14.6.2 Cauchy's Integral Theorem for Multiply Connected Domains

Suppose $f(z)$ is analytic in the region between the curves (and on the curves themselves). Then:

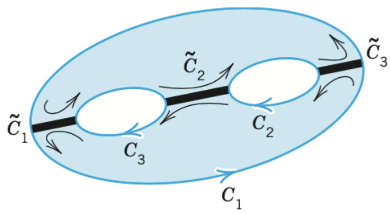
$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$



Doubly connected domain

and, in the triply connected case,

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz$$



Triply connected domain

Here C_1 is the outer boundary oriented counterclockwise, and C_2 (and C_3) are the inner boundaries oriented clockwise, so that $C_1 + C_2$ $(+C_3)$ is the positively oriented boundary of the region.

14.6.3 Existence of Indefinite Integral

If $f(z)$ is analytic in a simply connected domain D, then its complex integral is **path independent**. This means that the value of the integral from a fixed point z_0 to any point z is the same for all paths lying in D. Hence the function

$$F(z) = \int_{z_0}^{z} f(\zeta)\,d\zeta$$

is a **single-valued** function in D. Moreover, this function $F(z)$ is itself analytic and satisfies

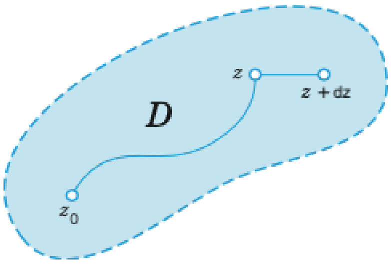
$$F'(z) = f(z)$$

Thus, every analytic function possesses an **antiderivative** in a simply connected domain. Consequently, complex integration is the inverse process of complex differentiation. If $f(z)$ is analytic in a simply connected domain D, then the integral

$$F(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta$$

is independent of the path in D and hence defines a single-valued function $F(z)$. It is analytic in D and hence $F'(z) = f(z)$. The definite integral can be evaluated as

$$\boxed{F(z_2) - F(z_1) = \int_{z_1}^{z_2} f(\zeta)\, d\zeta}$$



$$\frac{F(z+\Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(\zeta)\, d\zeta$$

$$= \frac{1}{\Delta z} \int_{z}^{z+\Delta z} \left[f(\zeta) - f(z) + f(z) \right] d\zeta$$

$$= \frac{f(z)}{\Delta z} \int_{z}^{z+\Delta z} d\zeta + \frac{1}{\Delta z} \int_{z}^{z+\Delta z} \left[f(\zeta) - f(z) \right] d\zeta = f(z) + \frac{1}{\Delta z} \int_{z}^{z+\Delta z} \left[f(\zeta) - f(z) \right] d\zeta$$

Hence

$$\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| = \frac{1}{|\Delta z|} \left| \int_{z}^{z+\Delta z} \left[f(\zeta) - f(z) \right] d\zeta \right|$$

Since f is continuous at z, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(\zeta) - f(z)| < \epsilon \quad \text{whenever } |\zeta - z| < \delta$$

For $|\Delta z| < \delta$, and taking the straight-line path from z to $z + \Delta z$, the path length is $|\Delta z|$, so

$$\left| \int_{z}^{z+\Delta z} \left[f(\zeta) - f(z) \right] d\zeta \right| \le \epsilon |\Delta z|.$$

Therefore

$$\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| \le \frac{1}{|\Delta z|} \epsilon |\Delta z| = \epsilon.$$

Since ϵ is arbitrary,

$$\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = F'(z) = f(z).$$

14.7 Cauchy's Integral Formula

Let f be analytic in a simply connected domain D, and let C be a positively oriented simple closed curve in D such that z_0 lies in the interior of C and the interior of C is contained in D. Then

$$\boxed{f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz} \quad \text{Equivalently,} \quad \boxed{\oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi i\, f(z_0)}$$

Let C_r be the circle $|z - z_0| = r$ contained in D. For z on C_r,

$$f(z) = f(z_0) + [f(z) - f(z_0)],$$

hence

$$\oint_{C_r} \frac{f(z)}{z - z_0}\, dz = f(z_0) \oint_{C_r} \frac{1}{z - z_0}\, dz + \oint_{C_r} \frac{f(z) - f(z_0)}{z - z_0}\, dz.$$

We compute the first integral by the parametrization $z = z_0 + re^{it}$, $0 \le t \le 2\pi$:

$$\oint_{C_r} \frac{1}{z - z_0}\, dz = \int_0^{2\pi} \frac{1}{re^{it}} \cdot \boldsymbol{i} re^{it}\, dt = \boldsymbol{i} \int_0^{2\pi} dt = 2\pi \boldsymbol{i}.$$

Therefore,

$$f(z_0) \oint_{C_r} \frac{1}{z - z_0}\, dz = 2\pi \boldsymbol{i}\; f(z_0).$$

We now show that the second integral is zero. Since f is continuous at z_0, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon \quad \text{whenever } |z - z_0| < \delta.$$

Choose r with $0 < r < \delta$, so that on C_r we have $|z - z_0| = r$ and hence $|f(z) - f(z_0)| < \epsilon$. Then

$$\left| \oint_{C_r} \frac{f(z) - f(z_0)}{z - z_0}\, dz \right| \le \max_{z \in C_r} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| \cdot \text{length}(C_r) \le \frac{\epsilon}{r} \cdot 2\pi r = 2\pi\epsilon.$$

Since $\epsilon > 0$ is arbitrary, this implies

$$\oint_{C_r} \frac{f(z) - f(z_0)}{z - z_0}\, dz = 0.$$

Combining the two parts, we obtain

$$\oint_{C_r} \frac{f(z)}{z - z_0}\, dz = 2\pi \boldsymbol{i}\; f(z_0).$$

Now let C be any positively oriented simple closed curve in D with z_0 in its interior, and let C_r be a small circle around z_0 contained entirely in the interior of C. The function

$$g(z) = \frac{f(z)}{z - z_0}$$

is analytic in the region between C and C_r, since z_0 is outside that annular region. By Cauchy's theorem, the integrals of g over C and C_r are equal:

$$\oint_C \frac{f(z)}{z - z_0}\, dz = \oint_{C_r} \frac{f(z)}{z - z_0}\, dz = 2\pi \boldsymbol{i}\; f(z_0).$$

14.7.1 Multiply Connected Domain

Let D be a multiply connected domain whose boundary consists of the outer positively oriented simple closed curve C_0 and the inner negatively oriented simple closed curves $C_1, C_2, \ldots, C_n$. If $f(z)$ is analytic in D and $z_0 \in D$, then

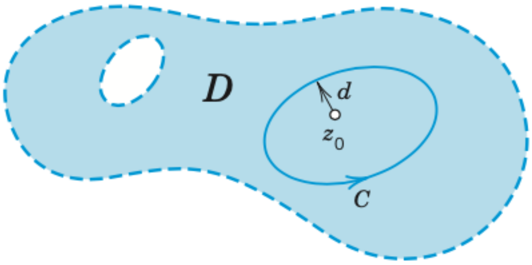
$$\oint_{C_0} \frac{f(z)}{z-z_0}\,dz - \sum_{k=1}^{n} \oint_{C_k} \frac{f(z)}{z-z_0}\,dz = 2\pi i\; f(z_0).$$

14.8 Derivatives of Analytic Functions

$$\boxed{f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2}\,dz}$$

$$\boxed{f''(z_0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^3}\,dz}$$

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz}$$



We derive the formula for $f'(z_0)$; the higher derivatives follow similarly.

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}.$$

Let C be a simple closed curve contained in the domain of analyticity of f and enclosing both z_0 and $z_0 + \Delta z$ for all sufficiently small Δz. By Cauchy's integral formula,

$$f(z_0+\Delta z) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-(z_0+\Delta z)}\,dz \qquad f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0}\,dz$$

Hence

$$\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{\Delta z}\left[\frac{1}{z-z_0-\Delta z} - \frac{1}{z-z_0}\right] dz = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}\,dz$$

We compare this with

$$\frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2}\,dz$$

Their difference is

$$\frac{1}{2\pi i} \oint_C \left[\frac{f(z)}{(z-z_0-\Delta z)(z-z_0)} - \frac{f(z)}{(z-z_0)^2}\right] dz = \frac{1}{2\pi i} \oint_C \frac{f(z)\,\Delta z}{(z-z_0-\Delta z)(z-z_0)^2}\,dz$$

Let

$$M = \max_{z\in C} |f(z)|, \quad L = \text{length}(C), \quad d = \min_{z\in C} |z-z_0| > 0$$

Then on C we have $|z-z_0| \geq d$, so

$$\frac{1}{|z-z_0|^2} \leq \frac{1}{d^2}$$

Moreover,

$$|z - z_0| \le |z - z_0 - \Delta z| + |\Delta z| \implies |z - z_0 - \Delta z| \ge d - |\Delta z|$$

If $|\Delta z| \le d/2$, then $|z - z_0 - \Delta z| \ge d/2$ and hence

$$\frac{1}{|z - z_0 - \Delta z|} \le \frac{2}{d}$$

Therefore,

$$\left| \oint_C \frac{f(z)\,\Delta z}{(z - z_0 - \Delta z)(z - z_0)^2}\,dz \right| \le M|\Delta z| \cdot \frac{2}{d} \cdot \frac{1}{d^2} \cdot L = \frac{2ML}{d^3}|\Delta z| \xrightarrow[\Delta z \to 0]{} 0$$

Thus

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi \boldsymbol{i}} \oint_C \frac{f(z)}{(z - z_0)^2}\,dz$$

that is,

$$f'(z_0) = \frac{1}{2\pi \boldsymbol{i}} \oint_C \frac{f(z)}{(z - z_0)^2}\,dz$$

Repeated differentiation of Cauchy's integral formula yields the general result

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi \boldsymbol{i}} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz}$$

14.8.1 Cauchy's Inequality

Let f be analytic inside and on the circle $|z - z_0| = r$, and let

$$M = \max_{|z - z_0| = r} |f(z)|.$$

Then for all $n \ge 0$,

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi} \left| \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz \right| \le \frac{n!}{2\pi} M \frac{1}{r^{n+1}} 2\pi r = \frac{n!M}{r^n}.$$

$$\boxed{|f^{(n)}(z_0)| = \frac{n!}{2\pi} \left| \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz \right| \le \frac{n!}{2\pi} M \frac{1}{r^{n+1}} 2\pi r = \frac{n!M}{r^n}}$$

Entire Functions

A function $f(z)$ is called **entire** if it is analytic for all $z \in \mathbb{C}$, that is, if it is complex differentiable at every point of the complex plane.

14.8.2 Liouville's Theorem

If an entire function is bounded in absolute value, then it must be a constant. Indeed, suppose that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Fix any point $z_0 \in \mathbb{C}$. By Cauchy's inequality applied to the circle $|z - z_0| = r$, we have

$$|f'(z_0)| \leq \frac{M}{r}.$$

Since r can be chosen arbitrarily large, it follows that $f'(z_0) = 0$. Because z_0 is arbitrary, we conclude that $f'(z) = 0$ for all z, and hence $f(z)$ is constant.

14.8.3 Morera's Theorem (Converse of Cauchy's Integral Theorem)

If $f(z)$ is continuous in a simply connected domain D and

$$\oint_C f(z)\,dz = 0$$

for every closed path C in D, then $f(z)$ is analytic in D.

14.9 Power Series

Complex power series are the natural analogs of real power series in calculus. Every analytic function can be represented locally by a power series.

14.9.1 Taylor Series

Let f be analytic inside and on a simple closed curve C enclosing the point z_0. Then f admits a Taylor expansion about z_0 of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz.$$

The remainder after n terms is given by

$$R_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}(\zeta - z)}\,d\zeta.$$

A Maclaurin series is a Taylor series with center $z_0 = 0$.

Assume that $|z - z_0| < r$, where $r = |z^* - z_0|$ for all $z^* \in C$. Then

$$\left| \frac{z - z_0}{z^* - z_0} \right| < 1.$$

We write

$$\frac{1}{z^* - z} = \frac{1}{z^* - z_0 - (z - z_0)} = \frac{1}{z^* - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{z^* - z_0}},$$

and define

$$q = \frac{z - z_0}{z^* - z_0}.$$

Since $|q| < 1$, we use the geometric series expansion

$$\frac{1}{1-q} = 1 + q + q^2 + \cdots + q^n + \frac{q^{n+1}}{1-q}.$$

Therefore,

$$\frac{1}{z^* - z} = \frac{1}{z^* - z_0}\left[1 + \frac{z - z_0}{z^* - z_0} + \left(\frac{z - z_0}{z^* - z_0}\right)^2 + \cdots + \left(\frac{z - z_0}{z^* - z_0}\right)^n\right] + \frac{1}{z^* - z}\left(\frac{z - z_0}{z^* - z_0}\right)^{n+1}.$$

$$\frac{f(z^*)}{z^* - z} = \frac{f(z^*)}{z^* - z_0} + \frac{f(z^*)(z - z_0)}{(z^* - z_0)^2} + \frac{f(z^*)(z - z_0)^2}{(z^* - z_0)^3} + \cdots + \frac{f(z^*)(z - z_0)^n}{(z^* - z_0)^{n+1}} + \frac{f(z^*)}{z^* - z}\left(\frac{z - z_0}{z^* - z_0}\right)^{n+1}$$

Insert the above into

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^* - z}\, dz^*$$

substitution gives

$$\boxed{f(z) = \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^* - z_0}\, dz^* + \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^* - z_0)^2}\, dz^* + \cdots + \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}}\, dz^* + R_n(z)}$$

14.9.2 Laurent's Series

A Laurent series generalizes a Taylor series by allowing both positive and negative integer powers of $(z - z_0)$. It converges in an annulus

$$r < |z - z_0| < R$$

where $0 \le r < R \le \infty$.

If f is analytic in the annulus $r < |z - z_0| < R$, then f can be represented as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

for all z in the annulus.

The first sum,

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is called the **regular part**, while the second sum,

$$\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

is called the **principal part** of the Laurent series.

The coefficients are given by

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz, \qquad n \geq 0,$$

$$b_n = \frac{1}{2\pi i} \oint_C (z-z_0)^{n-1} f(z)\, dz, \qquad n \geq 1,$$

where C is any positively oriented simple closed curve in the annulus $r < |z - z_0| < R$.

14.10 Zeros and Singularities

14.10.1 Zeros

A **zero** of an analytic function f is a point z_0 such that $f(z_0) = 0$. If $f(z) = (z - z_0)^m g(z)$ where $g(z)$ is analytic and $g(z_0) \neq 0$, then z_0 is called a **zero of order m**. In particular, a zero of order 1 is called a **simple zero**.

14.10.2 Singularities

Let f be analytic in a punctured neighborhood of z_0. Then z_0 is called an **isolated singularity** of f if f is not analytic at z_0. Let the Laurent expansion of f about z_0 be

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}.$$

14.10.3 Classification of Isolated Singularities

- If $b_n = 0$ for all $n \geq 1$, then z_0 is a **removable singularity**.
- If the principal part has finitely many terms,

  $$\frac{b_1}{z-z_0} + \cdots + \frac{b_m}{(z-z_0)^m}, \qquad b_m \neq 0,$$

  then z_0 is a **pole of order m**. A pole of order 1 is called a **simple pole**.
- If the principal part has infinitely many terms, then z_0 is called an **essential singularity**.

If $f(z)$ has a pole at z_0, then $|f(z)| \to \infty \quad$ as $z \to z_0$.

14.11 Residue Integration Method

The residue method is a powerful technique for evaluating complex contour integrals by reducing them to the study of the singularities of the function inside the contour.

Definition of a Residue

Let f be analytic in a punctured neighborhood of z_0, and suppose its Laurent expansion about z_0 is

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

The coefficient b_1 of $(z-z_0)^{-1}$ is called the **residue** of f at z_0 and is denoted by

$$\mathrm{Res}_{z=z_0} f(z) = b_1$$

Only this coefficient contributes to the value of a contour integral surrounding z_0.

By Cauchy's coefficient formula,

$$b_1 = \frac{1}{2\pi i} \oint_C f(z)\,dz$$

where C is any positively oriented simple closed curve enclosing z_0.

Residue at a Simple Pole

If z_0 is a simple pole, then the Laurent expansion has the form

$$f(z) = \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

Multiplying by $(z-z_0)$ removes the singularity, hence

$$\boxed{\mathrm{Res}_{z=z_0} f(z) = \lim_{z\to z_0} (z-z_0)f(z)}$$

In particular, if

$$f(z) = \frac{p(z)}{q(z)}$$

where p and q are analytic and z_0 is a simple zero of q, that is, $q(z_0) = 0$ and $q'(z_0) \neq 0$. Since z_0 is a simple pole of f, we have

$$\mathrm{Res}\; z = z_0 f(z) = \lim_{z\to z_0} (z-z_0)f(z)$$

Thus

$$\mathrm{Res}\; z = z_0 \frac{p(z)}{q(z)} = \lim_{z\to z_0} (z-z_0)\frac{p(z)}{q(z)} = \lim_{z\to z_0} \frac{p(z)(z-z_0)}{q(z)}$$

Since $q(z_0) = 0$ and $q'(z_0) \neq 0$, we have

$$\boxed{\mathrm{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}}$$

Residue at a Pole of Order m

If z_0 is a pole of order m, then near z_0 the function can be written as

$$f(z) = \frac{b_m}{(z-z_0)^m} + \cdots + \frac{b_1}{z-z_0} + \text{regular terms}$$

The residue is the coefficient b_1.

Multiplying by $(z-z_0)^m$ gives

$$(z-z_0)^m f(z) = b_m + b_{m-1}(z-z_0) + \cdots + b_1(z-z_0)^{m-1} + \cdots$$

which is analytic at z_0. Differentiating $(m-1)$ times removes all terms except the one containing b_1. Hence

$$\boxed{\operatorname{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]\Big|_{z=z_0}}.$$

Let f be analytic inside and on a simple closed positively oriented contour C, except for finitely many isolated singularities $z_1, z_2, \dots, z_k$ inside C. Then

$$\boxed{\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname{Res}_{z=z_j} f(z)}$$

Thus, a complicated contour integral is reduced to a simple algebraic sum of residues at the singularities inside the contour.

14.12 SymPy

```python
import sympy as sp

# Define symbol
z = sp.symbols('z')

# Define the function
f = (z**3 + 1)/(z*(z**3 + z**2 + 2))

# Find poles
poles = sp.solve(z*(z**3 + z**2 + 2), z)

# Compute residues
residues = [sp.residue(f, z, p) for p in poles]

# Sum of residues
residue_sum = sum(residues)

```

```
# Integral using Residue Theorem
integral = 2 * sp.pi * sp.I * residue_sum

display(poles)
display(residue_sum)
```