Chapter 28
Lorentz Transformation

The Lorentz transformation relates space and time coordinates measured in different inertial reference frames moving at constant relative velocity. It forms the mathematical foundation of special relativity and replaces the Galilean transformation of classical mechanics.

28.1 Inertial Frames and the Principle of Relativity

An inertial frame is a reference frame in which Newton's first law holds.

The principle of relativity states that the laws of physics are identical in all inertial frames.

28.2 Postulates of Special Relativity

Special relativity is based on two fundamental postulates:

- The laws of physics are the same in all inertial frames
- The speed of light in vacuum has the same value in all inertial frames

These postulates require a new relationship between space and time coordinates.

28.3 Failure of Galilean Transformation

In classical mechanics, coordinates transform as

$$x' = x - vt$$
$$t' = t$$

These transformations do not preserve the constancy of the speed of light and are therefore incompatible with electromagnetic theory.

28.4 Derivation of the Lorentz Transformation

Consider two inertial frames S and $\tilde{S}$ where $\tilde{S}$ moves with constant velocity v along the x-axis relative to S. By homogeneity of space and time, the transformation between coordinates must be linear:

$$\tilde{x} = Ax + Bt, \qquad \tilde{t} = Cx + Dt$$

where A, B, C, D depend only on v.

28.4.1 Relative Velocity Condition

The origin of $\tilde{S}$ moves along the worldline $x = vt$. A worldline is simply the path of an object through spacetime. Since $\tilde{x} = 0$ on this worldline,

$$0 = A(vt) + Bt$$

which implies

$$B = -Av$$

Thus,

$$\tilde{x} = A(x - vt)$$

Defining $A = \gamma$, we obtain

$$\tilde{x} = \gamma(x - vt)$$

28.4.2 Invariance of the Speed of Light

A light signal moving in the $+x$ direction satisfies $x = ct$ in frame S, and must also satisfy $\tilde{x} = c\tilde{t}$ in frame $\tilde{S}$. Substituting $x = ct$,

$$\tilde{x} = \gamma(ct - vt) = \gamma t(c - v)$$

$$\tilde{t} = C(ct) + Dt = t(Cc + D)$$

Requiring $\tilde{x} = c\tilde{t}$ gives

$$\gamma(c - v) = c(Cc + D)$$

For light moving in the $-x$ direction, $x = -ct$. Then

$$\tilde{x} = -\gamma t(c + v) \qquad \tilde{t} = t(-Cc + D)$$

Imposing $\tilde{x} = -c\tilde{t}$ yields

$$\gamma(c + v) = c(-Cc + D)$$

28.4.3 Determination of the Coefficients

Adding the two equations gives

$$2\gamma c = 2cD \quad \Rightarrow \quad D = \gamma$$

Subtracting them gives

$$-2\gamma v = 2c^2C \quad \Rightarrow \quad C = -\gamma\frac{v}{c^2}$$

28.4.3.1 Final Transformation

Substituting C and D into $\tilde{t} = Cx + Dt$, we obtain

$$\boxed{\tilde{t} = \gamma\left(t - \frac{vx}{c^2}\right)}$$

This term encodes the **relativity of simultaneity** and is required by the invariance of the speed of light. Time is not absolute: both the rate at which time passes and the notion of simultaneity depend on the observer's state of motion. A moving observer measures time intervals to be longer than those measured by an observer at rest, a phenomenon known as **time dilation**, often summarized by the statement that moving clocks run more slowly. Moreover, two events that occur simultaneously at different locations in one inertial frame will, in general, not be simultaneous in another frame moving relative to it. In this sense, time in a moving frame is a mixture of time and space as measured in the stationary frame. A useful illustration is the synchronization of clocks along a long train using light signals: an observer on the platform and an observer on the train will disagree on which clocks are synchronized, even though both apply the same physical laws.

28.4.4 Determination of the Lorentz Factor

The spacetime interval between two events is defined as

$$s^2 = c^2t^2 - x^2$$

Requiring invariance of the spacetime interval

$$\tilde{s}^2 = s^2$$

yields

$$\boxed{\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}}$$

28.5 Complete Lorentz Transformation

Including transverse coordinates, the full Lorentz transformation is

$$\tilde{x} = \gamma(x - vt)$$

$$\tilde{y} = y$$

$$\tilde{z} = z$$

$$\tilde{t} = \gamma\left(t - \frac{vx}{c^2}\right)$$

The inverse transformation is

$$x = \gamma(\tilde{x} + v\tilde{t})$$

$$t = \gamma\left(\tilde{t} + \frac{v\tilde{x}}{c^2}\right)$$

28.6 Consequences of the Lorentz Transformation

28.6.1 Time Dilation

A time interval measured in a moving frame is longer than that measured in the rest frame

$$\Delta t = \gamma \Delta t_0$$

28.6.2 Length Contraction

A length measured parallel to the direction of motion contracts according to

$$L = \frac{L_0}{\gamma}$$

28.6.3 Relativity of Simultaneity

Events simultaneous in one inertial frame are not necessarily simultaneous in another

$$\Delta t' = \gamma \left(\Delta t - \frac{v \Delta x}{c^2} \right)$$

28.7 Velocity Transformation

Velocities transform according to

$$u_x' = \frac{u_x - v}{1 - \frac{v u_x}{c^2}}$$

This ensures that no velocity exceeds the speed of light.

28.8 Lorentz Transformation in Tensor Form

Spacetime coordinates are combined into a four-vector

$$x^\mu = (ct, x, y, z)$$

The Lorentz transformation may be written as

$$x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^\nu$$

where $\Lambda^{\mu'}{}_{\nu}$ is the Lorentz transformation matrix satisfying

$$\eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu = \eta_{\mu\nu}$$

with $\eta_{\mu\nu}$ the Minkowski metric.

28.9 Connection to Electromagnetism

Maxwell's equations are invariant under Lorentz transformations, not Galilean transformations. This invariance explains the relativistic unification of electric and magnetic fields.

28.10 Closing Remarks

The Lorentz transformation reshapes classical notions of space and time into a unified spacetime structure. It provides the kinematic framework for special relativity and serves as a bridge between classical field theory and modern relativistic physics.