Chapter 27

Differential Geometry

Curves whisper truth where flat maps fail,
Angles bend as distances unveil;
Through metrics, forms, and spaces wide,
Geometry learns how worlds abide.

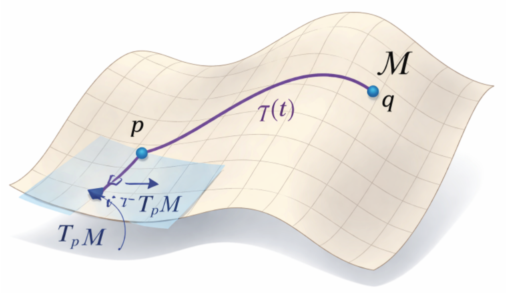
27.1 Manifolds

Differential geometry generalizes familiar geometric spaces. In elementary geometry, we work with lines, planes, and volumes embedded in Euclidean space. Differential geometry removes this restriction and studies spaces that may be curved, higher-dimensional, or not globally describable by a single coordinate system. Such spaces are called *manifolds*.

Informally, a manifold is a space that may be complicated when viewed as a whole, but which appears simple when examined locally. Just as the surface of the Earth looks flat over sufficiently small regions despite being globally curved, a manifold resembles ordinary Euclidean space in the neighborhood of every point. This local resemblance allows the tools of calculus—derivatives, integrals, and smooth functions—to be applied even when the global structure is non-Euclidean.

More precisely, while a manifold need not be a subset of $\mathbb{R}^n$, each small neighborhood of it can be described using coordinates drawn from $\mathbb{R}^n$. These local descriptions are called *charts*, and collections of compatible charts form an *atlas*. Smoothness refers to the requirement that transitions between overlapping charts are smooth functions.

In standard treatments, manifolds are assumed to satisfy mild topological conditions such as being Hausdorff and second countable. These assumptions ensure well-behaved global properties and will be taken for granted throughout.



27.1.1 Definition of a Manifold

With the intuition in place, we can state the formal definition.

A *smooth manifold* $\mathscr{M}$ of dimension n is a topological space such that each point $p \in \mathscr{M}$ has a neighborhood that is smoothly parameterized by coordinates

$$(x^1, x^2, \dots, x^n)$$

drawn from $\mathbb{R}^n$, with smooth and invertible transition maps between overlapping coordinate charts.

27.1.2 Coordinate Transformations

On overlapping charts, the coordinates are related by smooth transformations

$$\tilde{x}^i = \tilde{x}^i(x^1, \dots, x^n)$$

where the functions $\tilde{x}^i$ are smooth and invertible on their domains.

Differential geometry studies geometric and physical structures whose definitions and properties remain invariant under such coordinate transformations.

27.2 Curves and Tangent Vectors

27.2.1 Curves on a Manifold

A curve on a manifold $\mathscr{M}$ represents a smooth path through the space. Formally, a curve is a smooth mapping

$$\gamma : \mathbb{R} \to \mathscr{M}$$

that assigns to each real parameter value t a point $\gamma(t)$ on the manifold.

The parameter t merely orders points along the curve. It may represent time, arc length, or any other convenient parameter, but it has no intrinsic geometric meaning. As t varies smoothly, the image of $\gamma(t)$ traces out a path on $\mathscr{M}$.

Within a local chart, the curve is described by coordinate functions

$$x^i = x^i(t), \qquad i = 1, \dots, n$$

Although these coordinate functions depend on the chosen chart, the curve itself is a geometric object independent of coordinates.

27.2.2 Tangent Vectors

Let $\gamma : \mathbb{R} \to \mathscr{M}$ be a smooth curve and let $p = \gamma(t_0)$ be a point on the curve. The tangent vector to the curve at p is defined by the derivatives of the coordinate functions at t_0:

$$v^i = \left.\frac{dx^i}{dt}\right|_{t=t_0}$$

These components define a tangent vector v at p. The collection of all tangent vectors at p forms a vector space called the *tangent space*, denoted $T_p\mathscr{M}$.

Equivalent, coordinate-independent definitions of tangent vectors exist (for example, as derivations acting on smooth functions), but the curve-based definition suffices for our purposes.

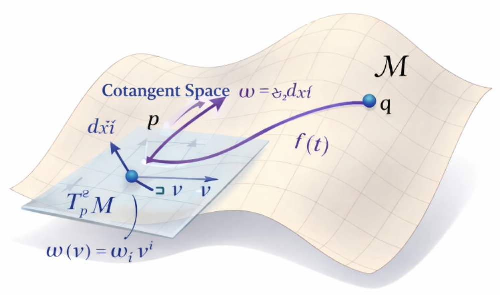
27.3 Cotangent Space

At each point $p \in \mathscr{M}$, the tangent space $T_p\mathscr{M}$ describes possible directions of motion through the point.

The *dual space* of the tangent space, denoted $T_p^*\mathscr{M}$, consists of all linear maps from the tangent space to the real numbers:

$$T_p^*\mathscr{M} = \left\{ \omega : T_p\mathscr{M} \to \mathbb{R} \;\middle|\; \omega \text{ is linear} \right\}$$

This space is called the *cotangent space*. Its elements, known as *covectors* or *one-forms*, act on tangent vectors to produce real numbers. While tangent vectors encode directions of motion, cotangent vectors measure rates of change of scalar quantities along those directions.



27.3.1 Differential Forms

Differential forms generalize covectors to higher dimensions and provide a coordinate-independent framework for integration and differentiation on manifolds.

At each point $p \in \mathscr{M}$, a k-*form* is a totally antisymmetric, multilinear map

$$\omega_p : \underbrace{T_p\mathscr{M} \times \cdots \times T_p\mathscr{M}}_{k \text{ times}} \to \mathbb{R}$$

A *one-form* acts on a single tangent vector. In local coordinates, it can be written as

$$\omega = \omega_i \, dx^i$$

where $\{dx^i\}$ form a basis of the cotangent space. Gradients of scalar functions provide natural examples.

Higher-order forms are constructed using the *wedge product*, which enforces antisymmetry:

$$dx^i \wedge dx^j = -\, dx^j \wedge dx^i, \qquad dx^i \wedge dx^i = 0$$

A two-form has the local expression

$$\omega = \tfrac{1}{2} \, \omega_{ij} \, dx^i \wedge dx^j$$

with $\omega_{ij} = -\omega_{ji}$.

Differential forms are naturally suited for integration. A k-form can be integrated over a k-dimensional oriented manifold. Differentiation of forms is performed using the *exterior derivative*, and the *generalized Stokes' theorem* unifies the fundamental theorems of calculus into a single geometric statement.

27.4 Metric Tensor

A *metric tensor* assigns an inner product to tangent vectors at each point of a manifold. Given two tangent vectors V^i and W^j, their inner product is defined as $g_{ij}V^iW^j$.

The components of the metric are

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

and the squared line element is

$$ds^2 = g_{ij}\, dx^i dx^j$$

The metric determines distances, angles, and volumes on the manifold.

27.5 Connection and Covariant Derivative

27.5.1 Parallel Transport

Tangent vectors at different points belong to different vector spaces and cannot be compared directly. A *connection* provides a rule for relating tangent spaces at nearby points. In Euclidean space, you can slide a vector along a curve without changing it. On a curved manifold, "keeping the vector the same" is ambiguous. Parallel transport defines what "the same" means geometrically.

Parallel transport describes how a vector is moved along a curve while changing as little as possible according to the geometry of the manifold.

Definition 27.1. (Parallel Transport) *Let $(\mathcal{M}, \nabla)$ be a smooth manifold equipped with a connection ∇, and let*

$$\gamma : [a, b] \to \mathcal{M}$$

be a smooth curve. A vector field $V(t)$ along γ is said to be parallel transported *along γ if it satisfies*

$$\nabla_{\dot{\gamma}(t)} V(t) = 0 \quad \textit{for all } t \in [a, b]$$

27.5.2 Covariant Derivative

The covariant derivative measures how tensor fields vary while accounting for changes in the coordinate basis. For a contravariant vector field V^i,

$$\boxed{\nabla_j V^i = \partial_j V^i + \Gamma^i_{jk} V^k}$$

and for a covariant vector field V_i,

$$\boxed{\nabla_j V_i = \partial_j V_i - \Gamma^k_{ji} V_k}$$

27.5.3 Christoffel Symbols

The Christoffel symbols Γ^i_{jk} describe how coordinate basis vectors vary across the manifold:

$$\nabla_j \left(\frac{\partial}{\partial x^k} \right) = \Gamma^i_{jk} \frac{\partial}{\partial x^i}$$

For a metric-compatible, torsion-free connection—the *Levi–Civita connection*—they are uniquely determined by the metric:

$$\boxed{\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk} \right)}$$

27.6 Geodesics

Geodesics generalize straight lines to curved spaces. They extremize the length functional

$$s = \int \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}}\, dt$$

Equivalently, they are curves whose tangent vectors are parallel transported along themselves:

$$\frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0$$

27.7 Curvature

27.7.1 Riemann Curvature Tensor

Curvature measures the failure of covariant derivatives to commute. The Riemann curvature tensor is

$$\boxed{R^i{}_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{km} \Gamma^m_{jl} - \Gamma^i_{lm} \Gamma^m_{jk}}$$

27.7.2 Ricci Tensor and Scalar Curvature

Contracting indices yields the Ricci tensor

$$R_{ij} = R^k{}_{ikj}$$

and the scalar curvature

$$\boxed{R = g^{ij} R_{ij}}$$

27.8 Integration on Manifolds

On an oriented manifold with a metric, the invariant volume element is

$$\boxed{dV = \sqrt{|g|}\, d^n x}$$

27.9 Differential Geometry in Physics

Differential geometry underlies modern physical theories:

- Geodesics describe free particle motion
- Curvature encodes gravitation
- Differential forms unify electromagnetic laws

27.10 Closing Remarks

Differential geometry replaces flat intuition with intrinsic structure. By expressing physical laws in coordinate-independent language, it provides the mathematical foundation for modern theoretical physics.