Chapter 17
Calculus of Variations

Rates of change in space and time,
Linked by rules both clear and prime,
Heat, waves, flow—one form, one frame,
PDEs describe how nature evolves the same.

17.1 Introduction

The calculus of variations is a branch of mathematical analysis concerned with finding functions that extremize (that is, minimize or maximize) quantities expressed as integrals, known as *functionals*. Unlike ordinary calculus, which deals with functions of numbers, the calculus of variations studies functions whose arguments are themselves functions. This framework arises naturally in problems where one seeks an optimal curve, surface, or trajectory subject to given constraints.

Many classical problems in mathematics and physics can be formulated as variational problems. Typical examples include determining the shortest path between two points, finding curves of minimal surface area, and identifying motion that minimizes action in mechanics. In such problems, the objective is to determine a function that makes a given integral attain an extremum while satisfying prescribed boundary conditions.

This chapter introduces the fundamental concepts of the calculus of variations, beginning with the definition of functionals and extremals. The derivation of the Euler–Lagrange equation is presented as a necessary condition for a function to yield an extremum of a functional. Both the strong form, with fixed boundary conditions, and the weak (natural boundary) form, where endpoint values are not fixed, are developed systematically.

The theory is illustrated through classical examples, including the problem of finding the shortest path between two points in a plane. These examples demonstrate how variational principles lead to differential equations whose solutions recover familiar geometric and physical results. Through these developments, the chapter lays the foundation for variational methods that play a central role in mechanics, physics, engineering, and modern applied mathematics. :contentReference[oaicite:0]index=0

17.2 Functional

A **functional** is a function that accepts one or more functions as inputs and produces a real-valued number as an output. The calculus of variations, or variational calculus, is a field of mathematical analysis that uses variations, which are small changes in functions, to find maxima and minima of functionals.

One of the main problems of the calculus of variations is to determine the curve connecting two given points that either minimizes or maximizes a given integral. Consider a curve connecting two points. Its length, S, is given by

$$S = \int \sqrt{(dx)^2 + (dy)^2}$$

If the curve is written as $y = y(x)$, this becomes

$$S = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

The problem of determining the curve connecting two points (x_1, y_1) and (x_2, y_2) whose length is a minimum is therefore equivalent to finding the function $y = y(x)$, where $y(x_1) = y_1$ and $y(x_2) = y_2$, such that

$$\int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{is a minimum.}$$

In general, we seek the curve $y = y(x)$, with $y(x_1) = y_1$ and $y(x_2) = y_2$, such that for a given function $F(x, y, y')$,

$$\boxed{\int_{x_1}^{x_2} F(x, y, y')\, dx}$$

is either a **minimum** or a **maximum**, also referred to as an **extremum** or a **stationary** value. A function that satisfies this property is called an **extremal**. The above integral, which assigns a numerical value to each admissible function $y(x)$, is called a **functional**.

17.3 Derivation of the Euler-Lagrange Equation

Consider the functional

$$J[y] = \int_{x_1}^{x_2} F\big(x, y(x), y'(x)\big)\, dx,$$

where y is a sufficiently smooth function with fixed end values

$$y(x_1) = y_1, \qquad y(x_2) = y_2$$

We introduce a family of varied functions

$$y_\varepsilon(x) = y(x) + \varepsilon\eta(x)$$

where ε is a real parameter and $\eta(x)$ is an arbitrary smooth function satisfying

$$\eta(x_1) = 0, \qquad \eta(x_2) = 0$$

The boundary conditions are fixed, so the variations vanish at the endpoints.

The functional evaluated on the varied function is

$$J[\varepsilon] \ := J[y_\varepsilon] = \int_{x_1}^{x_2} F\big(x, y_\varepsilon(x), y_\varepsilon'(x)\big)\, dx$$

A necessary condition for y to be an extremum of J is that

$$\left.\frac{dJ[\varepsilon]}{d\varepsilon}\right|_{\varepsilon=0} = 0$$

We compute this derivative:

$$\frac{dJ[\varepsilon]}{d\varepsilon} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} \frac{\partial y_\varepsilon}{\partial \varepsilon} + \frac{\partial F}{\partial y'} \frac{\partial y_\varepsilon'}{\partial \varepsilon} \right) dx$$

Since $y_\varepsilon = y + \varepsilon\eta$, we have

$$\frac{\partial y_\varepsilon}{\partial \varepsilon} = \eta \qquad \frac{\partial y_\varepsilon'}{\partial \varepsilon} = \eta'$$

Evaluating at $\varepsilon = 0$ gives

$$\left.\frac{dJ[\varepsilon]}{d\varepsilon}\right|_{\varepsilon=0} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} \eta + \frac{\partial F}{\partial y'} \eta' \right) dx$$

We now integrate the second term by parts:

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \eta'\, dx = \left[\frac{\partial F}{\partial y'} \eta \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \eta\, dx$$

Because $\eta(x_1) = \eta(x_2) = 0$, the boundary term vanishes:

$$\left[\frac{\partial F}{\partial y'} \eta \right]_{x_1}^{x_2} = 0.$$

Thus

$$\left.\frac{dJ[\varepsilon]}{d\varepsilon}\right|_{\varepsilon=0} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \right) \eta\, dx$$

For y to be an extremal, we require

$$\left.\frac{dJ[\varepsilon]}{d\varepsilon}\right|_{\varepsilon=0} = 0 \quad \text{for all admissible } \eta(x) \text{ with } \eta(x_1) = \eta(x_2) = 0.$$

The only way this integral can vanish for all such η is if the integrand itself is zero:

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0 \quad \text{for all } x \in [x_1, x_2].$$

This differential equation is called the **Euler-Lagrange equation**:

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0}$$

17.4 Weak (Natural Boundary) Form of the Euler-Lagrange Equation

Consider the functional

$$J[y] = \int_{x_1}^{x_2} F\big(x, y(x), y'(x)\big)\, dx$$

but now assume that the endpoint values of y are *not fixed*. We again introduce a variation

$$y_\varepsilon(x) = y(x) + \varepsilon \eta(x)$$

where $\eta(x)$ is an arbitrary smooth function with

$$\eta(x_1) \neq 0, \qquad \eta(x_2) \neq 0$$

Proceeding as in the strong-form derivation, the first variation is

$$\left.\frac{dJ[\varepsilon]}{d\varepsilon}\right|_{\varepsilon=0} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\right) dx$$

Integrating the second term by parts gives

$$\left.\frac{dJ[\varepsilon]}{d\varepsilon}\right|_{\varepsilon=0} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)\eta\, dx + \left[\frac{\partial F}{\partial y'}\eta\right]_{x_1}^{x_2}$$

Since $\eta(x)$ is now arbitrary *including at the endpoints*, both the integral term and the boundary term must vanish independently. Hence we obtain the interior (weak) Euler-Lagrange equation

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0, \quad \text{for } x \in (x_1, x_2)}$$

together with the corresponding *natural boundary conditions*

$$\boxed{\frac{\partial F}{\partial y'}(x_1) = 0, \qquad \frac{\partial F}{\partial y'}(x_2) = 0}$$

The pair consisting of the interior Euler-Lagrange equation and the natural boundary conditions is collectively referred to as the **weak (natural) form** of the **Euler-Lagrange** equations.

17.5 Example: Shortest Path Between Two Points in the Plane

We now illustrate the Euler-Lagrange equation with the classical problem of finding the shortest curve between two points in the plane. As before, consider two fixed points (x_1, y_1) and (x_2, y_2), and curves of the form $y = y(x)$ joining them. The length of such a curve is

$$S[y] = \int_{x_1}^{x_2} \sqrt{1 + \big(y'(x)\big)^2}\, dx$$

This is a functional of y, and we seek the function $y(x)$ that makes $S[y]$ an extremum (in this case, a minimum).

Here the integrand plays the role of $F(x, y, y')$:

$$F(x, y, y') = \sqrt{1 + (y')^2}.$$

We compute the partial derivatives required by the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0.$$

First, note that F does not depend explicitly on y, only on y':

$$\frac{\partial F}{\partial y} = 0.$$

Next,

$$\frac{\partial F}{\partial y'} = \frac{1}{2}\left(1 + (y')^2\right)^{-1/2} \cdot 2y' = \frac{y'}{\sqrt{1 + (y')^2}}.$$

The Euler-Lagrange equation becomes

$$0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + (y')^2}}\right) = 0,$$

or equivalently,

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1 + (y')^2}}\right) = 0.$$

This implies that

$$\frac{y'}{\sqrt{1 + (y')^2}} = C,$$

where C is a constant. Solving for y',

$$y' = \frac{C}{\sqrt{1 - C^2}},$$

which is itself a constant (assuming $|C| < 1$). Therefore

$$y'(x) = m \quad \text{for some constant } m,$$

and integrating once more gives

$$y(x) = mx + b,$$

where b is another constant of integration. The constants m and b are uniquely determined by the endpoint conditions $y(x_1) = y_1$ and $y(x_2) = y_2$.

Thus, the curve of shortest length between two points in the plane is a straight line. This example shows how the Euler-Lagrange equation recovers the intuitive geometric fact that geodesics in Euclidean space are straight lines.