Chapter 18

Weighted Residual Method

Errors whisper where the trial solution stands,
Weighted echoes answer, tuned by chosen hands.
Balance sought where residuals quietly fade,
Truth emerges where approximations are weighed.

18.1 Introduction

The *Weighted Residual Method* (WRM) is a general and powerful framework for obtaining approximate solutions to differential equations, particularly boundary value problems arising in physics and engineering. Instead of seeking an exact solution—which is often impossible for complex geometries or nonlinear equations—the method constructs an approximate solution that satisfies the governing equation in an *average* or *integral* sense.

Many well-known numerical methods, including the Galerkin method, least-squares method, collocation method, and finite element method, can be viewed as special cases of the weighted residual approach.

18.2 Basic Idea

Consider a differential equation defined over a domain Ω:

$$\mathscr{L}(u) = f \quad \text{in } \Omega \tag{18.1}$$

subject to appropriate boundary conditions on $\partial\Omega$, where $\mathscr{L}$ is a differential operator.

We approximate the unknown solution $u(x)$ by a finite expansion:

$$u_N(x) = \sum_{i=1}^{N} a_i \phi_i(x) \tag{18.2}$$

where:

- $\phi_i(x)$ are known trial (or basis) functions,
- a_i are unknown coefficients to be determined.

In general, u_N does not satisfy the differential equation exactly. Substituting u_N into the governing equation produces a *residual*:

$$R(x) = \mathscr{L}(u_N) - f \tag{18.3}$$

18.3 Weighted Residual Statement

The central idea of WRM is to require that the residual be orthogonal to a set of weighting functions $w_j(x)$:

$$\int_\Omega w_j(x)\, R(x)\, d\Omega = 0, \quad j = 1, 2, \ldots, N \tag{18.4}$$

This yields N algebraic equations for the N unknown coefficients a_i.

The choice of weighting functions w_j defines the particular weighted residual method.

18.4 Common Variants of the Weighted Residual Method

18.4.1 Galerkin Method

In the Galerkin method, the weighting functions are chosen to be the same as the trial functions:

$$w_j(x) = \phi_j(x) \tag{18.5}$$

The Galerkin method is widely used because it often preserves symmetry and conservation properties of the original differential equation. It forms the theoretical foundation of the finite element method.

18.4.2 Least-Squares Method

In the least-squares method, the weighting functions are chosen as:

$$w_j(x) = \frac{\partial R}{\partial a_j} \tag{18.6}$$

This choice minimizes the integrated square of the residual:

$$\int_\Omega R^2\, d\Omega \tag{18.7}$$

leading to stable formulations even for problems where Galerkin methods may fail.

18.4.3 Collocation Method

In the collocation method, the weighting functions are Dirac delta functions centered at selected points x_j:

$$w_j(x) = \delta(x - x_j) \tag{18.8}$$

This enforces the residual to vanish exactly at the collocation points:

$$R(x_j) = 0. \tag{18.9}$$

18.4.4 Subdomain Method

In the subdomain method, the domain Ω is partitioned into subdomains Ω_j, and the weighting functions are chosen as:

$$w_j(x) = \begin{cases} 1, & x \in \Omega_j, \\ 0, & \text{otherwise.} \end{cases} \tag{18.10}$$

This enforces the average residual over each subdomain to be zero.

18.5 Treatment of Boundary Conditions

Boundary conditions play a crucial role in the weighted residual method.

18.5.1 Essential (Dirichlet) Boundary Conditions

Essential boundary conditions are typically enforced directly by choosing trial functions that satisfy them identically.

18.5.2 Natural (Neumann) Boundary Conditions

Natural boundary conditions arise naturally when integrating by parts in Galerkin-type formulations and are incorporated into the weak form of the problem.

18.6 Weak Formulation

The weak form structure arises for a large class of second-order partial differential equations, with appropriate modifications. In particular, consider the general second-order elliptic problem written in divergence form:

$$-\nabla \cdot \big(\mathbf{A}(x)\,\nabla u\big) = f \quad \text{in } \Omega \tag{18.11}$$

subject to the Neumann boundary condition

$$\mathbf{A}(x)\,\nabla u \cdot \mathbf{n} = g \quad \text{on } \partial\Omega \tag{18.12}$$

where $\mathbf{A}(x)$ is a symmetric, positive-definite tensor, f is a given source term, g is the prescribed boundary flux, and $\mathbf{n}$ denotes the outward unit normal to $\partial\Omega$.

Multiplying the governing equation by a weighting (test) function w and integrating over the domain Ω yields the weighted residual statement:

$$-\int_\Omega w\nabla \cdot \big(\mathbf{A}\nabla u\big)\, d\Omega = \int_\Omega w f\, d\Omega \tag{18.13}$$

Applying the divergence theorem and assuming sufficient regularity of u and w, we obtain

$$\int_\Omega \nabla w \cdot \mathbf{A}\nabla u\, d\Omega = \int_\Omega w f\, d\Omega + \int_{\partial\Omega} w\mathbf{A}\nabla u \cdot \mathbf{n}\, d\Gamma \tag{18.14}$$

Invoking the Neumann boundary condition, the weak (variational) form of the problem can be written as:

$$\int_{\Omega} \nabla w \cdot \mathbf{A} \nabla u \, d\Omega = \int_{\Omega} w f \, d\Omega + \int_{\partial\Omega} w g \, d\Gamma \tag{18.15}$$

This formulation reduces the order of spatial derivatives on the trial solution and naturally incorporates Neumann boundary conditions through boundary integrals. It serves as the foundational weak form for a wide range of elliptic and parabolic problems and underpins Galerkin and finite element discretizations.

18.7 Advantages and Limitations

18.7.1 Advantages

- Provides a unified framework for many numerical methods.
- Allows systematic approximation of complex problems.
- Naturally accommodates variational and conservation principles.

18.7.2 Limitations

- Accuracy depends strongly on the choice of trial functions.
- Nonlinear problems require iterative solution strategies.
- Implementation complexity increases for higher dimensions.

18.8 Summary

The weighted residual method transforms differential equations into algebraic systems by enforcing the residual to vanish in an averaged sense. By appropriate choice of weighting and trial functions, it forms the foundation of many modern numerical techniques for solving partial differential equations, most notably the finite element method.