Chapter 27

Differential Geometry

Curves whisper truth where flat maps fail,
Angles bend as distances unveil;
Through metrics, forms, and spaces wide,
Geometry learns how worlds abide.

27.1 Manifolds

Differential geometry begins by generalizing the idea of familiar geometric spaces. In elementary geometry, we work with lines, planes, and volumes embedded in ordinary Euclidean space. Differential geometry removes this restriction and studies spaces that may be curved, higher-dimensional, or not globally describable by a single coordinate system. Such spaces are called manifolds.

Informally, a manifold is a space that may be complicated when viewed as a whole, but which looks simple when examined locally. Just as the surface of the Earth appears flat over small regions despite being globally curved, a manifold resembles ordinary Euclidean space in the neighborhood of every point. This local resemblance allows us to use the tools of calculus—derivatives, integrals, and smooth functions—even when the global structure is non-Euclidean.

The essential idea is that while a manifold need not be a subset of $\mathbb{R}^n$, each small patch of it can be described using coordinates drawn from $\mathbb{R}^n$. These local coordinate descriptions are called charts, and collections of compatible charts form an atlas. Smoothness refers to the requirement that transitions between overlapping charts are smooth functions.

27.1.1 Definition of a Manifold

With the intuition in place, we can now state the formal definition. A smooth manifold $\mathscr{M}$ of dimension n is a topological space that locally resembles $\mathbb{R}^n$ and admits smooth coordinate charts.

Each point $p \in \mathscr{M}$ has a neighborhood with coordinates

$$(x^1, x^2, \dots, x^n)$$

27.1.2 Coordinate Transformations

On overlapping coordinate charts, the coordinates are related by smooth transformations

$$\tilde{x}^i = \tilde{x}^i(x^1, \dots, x^n)$$

where the functions $\tilde{x}^i$ are smooth and invertible on their domains.

Differential geometry studies geometric and physical structures whose definitions and properties are invariant under such coordinate transformations.

27.2 Curves and Tangent Vectors

27.2.1 Curves on a Manifold

A curve on a manifold $\mathscr{M}$ represents a smooth path through the space. Formally, a curve is defined as a smooth mapping

$$\gamma : \mathbb{R} \to \mathscr{M}$$

which assigns to each real parameter value t a point $\gamma(t)$ on the manifold.

The parameter t serves only as a label that orders points along the curve. It may represent time, arc length, or any other convenient parameter, but it has no intrinsic geometric significance. As t varies smoothly, the image of $\gamma(t)$ traces out a continuous path on $\mathscr{M}$.

Since a manifold locally resembles $\mathbb{R}^n$, the curve can be described in terms of coordinates within a local chart. In such a coordinate system, the curve is represented by a set of smooth functions

$$x^i = x^i(t), \qquad i = 1, \dots, n$$

Although the coordinate functions $x^i(t)$ depend on the chosen chart, the curve $\gamma(t)$ itself is a geometric object independent of coordinates. Different coordinate systems provide different representations of the same underlying curve.

27.2.2 Tangent Vectors

Let $\gamma : \mathbb{R} \to \mathscr{M}$ be a smooth curve, and let $p = \gamma(t_0)$ be a point on the curve. The tangent vector to the curve at p is defined by the derivatives of the coordinate functions at t_0:

$$v^i = \left.\frac{dx^i}{dt}\right|_{t=t_0}$$

These components define a tangent vector v at the point p. The collection of all tangent vectors at p forms a vector space called the *tangent space*, denoted by $T_p\mathscr{M}$.

27.3 Cotangent Space and Differential Forms

At each point $p \in \mathscr{M}$, the dual space to the tangent space $T_p\mathscr{M}$ is called the *cotangent space* and is denoted by $T_p^*\mathscr{M}$. Elements of $T_p^*\mathscr{M}$ are linear maps that take tangent vectors at p to real numbers.

A basis for the cotangent space is given by the differentials $\{dx^i\}$, which are dual to the coordinate basis vectors $\left\{\frac{\partial}{\partial x^i}\right\}$ and satisfy

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j.$$

A *differential one-form* (or simply a one-form) is a covariant tensor of type $(0,1)$. In local coordinates, a one-form ω can be written as

$$\omega = \omega_i \, dx^i,$$

where the functions ω_i are the components of the one-form in the chosen coordinate system.

Differential forms provide a coordinate-independent framework for integration.

27.4 Metric Tensor

A metric tensor assigns an inner product to tangent vectors at each point of a manifold. Given two tangent vectors V^i and W^j, the metric defines their inner product as $g_{ij}V^iW^j$.

The squared line element is

$$ds^2 = g_{ij} \, dx^i dx^j$$

The metric tensor is symmetric,

$$g_{ij} = g_{ji}$$

Distances and angles on the manifold are defined through the metric.

27.5 Connection and Covariant Derivative

27.5.1 Parallel Transport

Vectors at different points on a manifold live in different tangent spaces and cannot be compared directly. To compare them, one must specify a rule that relates tangent spaces at nearby points.

Parallel transport provides such a rule. Given a curve on the manifold, parallel transport describes how a vector attached to one point is moved along the curve so that it changes as little as possible, according to the geometry of the manifold.

The mathematical structure that defines this rule of transport is called a *connection*. A connection specifies how vectors vary as one moves from point to point, allowing derivatives and comparisons of vectors defined at different locations.

27.5.2 Covariant Derivative

Ordinary partial derivatives compare vector components at different points as if they lived in the same vector space. On a curved manifold, this is not valid, because tangent spaces at different points are distinct. A derivative that respects the geometric structure of the manifold is therefore required.

The covariant derivative provides such a notion. It measures how a vector field changes while accounting for the variation of the coordinate basis itself.

For a contravariant vector field V^i, the covariant derivative is defined as

$$\nabla_j V^i = \partial_j V^i + \Gamma^i_{jk} V^k$$

where the Christoffel symbols Γ^i_{jk} encode how the basis vectors change from point to point.

For a covariant vector field V_i, the covariant derivative is

$$\nabla_j V_i = \partial_j V_i - \Gamma^k_{ji} V_k$$

The difference in sign reflects the covariant transformation behavior of lower indices. In each case, the covariant derivative produces a tensor, ensuring that the result is independent of the chosen coordinate system.

27.5.3 Christoffel Symbols

On a manifold, the coordinate basis vectors

$$\left\{ \frac{\partial}{\partial x^i} \right\}$$

vary from point to point. As a result, even a vector with constant components can change direction purely due to the geometry of the coordinate system.

The Christoffel symbols Γ^i_{jk} quantify this variation of the basis vectors. They are defined through the covariant derivative of the basis:

$$\nabla_j \left(\frac{\partial}{\partial x^k} \right) = \Gamma^i_{jk} \frac{\partial}{\partial x^i}$$

Thus, Γ^i_{jk} describe how the coordinate basis changes as one moves along the manifold. They are not components of a tensor, but rather connection coefficients that depend on the chosen coordinate system.

For a metric-compatible, torsion-free connection (the Levi–Civita connection), the Christoffel symbols are uniquely determined by the metric:

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk} \right)$$

27.6 Geodesics

Geodesics are curves that generalize straight lines to curved spaces. They represent paths of extremal length between points.

The length of a curve is given by

$$s = \int \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}}\, dt$$

Equivalently, geodesics are curves whose tangent vectors are parallel transported along themselves.

The geodesic equation is

$$\frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0$$

27.7 Curvature

27.7.1 Riemann Curvature Tensor

Curvature measures the failure of covariant derivatives to commute and reflects the intrinsic geometry of the manifold. Geometrically, it quantifies how parallel transport around an infinitesimal loop fails to preserve a vector.

The Riemann curvature tensor is defined by

$$R^i{}_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{km} \Gamma^m_{jl} - \Gamma^i_{lm} \Gamma^m_{jk}$$

27.7.2 Ricci Tensor and Scalar Curvature

The Ricci tensor is obtained by contracting the Riemann tensor:

$$R_{ij} = R^k{}_{ikj}$$

The scalar curvature is defined as

$$R = g^{ij} R_{ij}$$

27.8 Integration on Manifolds

For a manifold equipped with a metric, the invariant volume element is

$$dV = \sqrt{|g|}\, d^n x$$

where g is the determinant of the metric tensor.

27.9 Differential Geometry in Physics

Differential geometry provides the mathematical structure underlying many physical theories:

- Geodesics describe free particle motion in the absence of non-gravitational forces
- Curvature encodes gravitational effects
- Differential forms provide a unified formulation of electromagnetic laws

27.10 Closing Remarks

Differential geometry replaces flat-space intuition with intrinsic geometric structure. By expressing physical laws in a coordinate-independent form, it provides a universal language for modern theoretical physics.