Chapter 7

Second Order Ordinary Differential Equation

A second-order equation, quite a mess,
With derivatives causing plenty of distress.
But Euler, Cauchy, and Lagrange helped evolve,
Elegant methods to solve and resolve!

As noted in the previous chapter, the order of a differential equation is defined as the order of the highest derivative of the unknown function that appears in the equation. Therefore, a second-order differential equation is one in which the second derivative of the unknown function occurs, and no derivative of higher order is present.

Second order differential equations have a variety of applications in science and engineering such as vibrations and electric circuits. There are a host of multi dimensional engineering models that incorporate second order differential equations including wave motion, flow mechanics, Maxwell's electro-magnetic equations and Schroedinger equation in Nuclear Physics.

7.1 2nd Order Linear ODE

The **standard form** of a **2nd Order ODE** is:

$$\boxed{y'' + p(x)y' + q(x)y = r(x)} \quad \text{It is linear in } y,\ y' \text{ and } y'' \tag{7.1}$$

If $r(x) = 0$, the ODE is homogeneous, else it is non-homogeneous. When the coefficients a and b are constant:

$$\boxed{y'' + ay' + by = 0} \tag{7.2}$$

Choose $e^{\lambda x}$ as a solution and substitute in the homogeneous ODE.

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0 \implies \lambda^2 + a\lambda + b = 0 \implies \lambda = \frac{1}{2}\left(-a \pm \sqrt{a^2 - 4b}\right)$$

$$\boxed{y_h = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}} \quad \text{(general solution to the homogeneous ODE)}$$

y_1, corresponding to λ_1, and y_2, corresponding to λ_2, are **linearly independent** and are called **basis of solution**
The **superposition principle** also called the **linearity principle,** i.e., the homogeneous solution is a combination of y_1 and y_2 **is true only for linear homogeneous ODE.** The arbitrary constants c_1 and c_2 are determined from the **initial conditions:**

$$y(x_0) = k_0 \quad y^{'}(x_0) = k_1$$

A **particular solution** is obtained if we assign specific values to c_1 and c_2.

7.2 Lagrange's Method of Reduction of Order

Consider a linear homogeneous 2nd Order ODE in its **standard form**:

$$\boxed{y^{''} + p(x)y^{'} + q(x)y = 0} \tag{7.3}$$

If y_1 is a **basis solution**, we can find y_2 as follows:

$$y_1 = e^{\lambda x}$$

Let $y_2 = u\, y_1$

$$y_2' = u' y_1 + u y_1'$$

$$y_2'' = u'' y_1 + 2u' y_1' + u y_1''$$

Substituting into $y'' + py' + qy = 0$:

$$\left(u'' y_1 + 2u' y_1' + u y_1''\right) + p\left(u' y_1 + u y_1'\right) + q(u y_1) = 0$$

$$u'' y_1 + (2y_1' + p y_1)u' + (y_1'' + p y_1' + q y_1)u = 0$$

$$u'' + u' \frac{2y_1' + p y_1}{y_1} = 0$$

Let $U = u'$

$$U' + U\left(\frac{2y_1'}{y_1} + p\right) = 0$$

$$\frac{U'}{U} = -\left(\frac{2y_1'}{y_1} + p\right)$$

$$\int \frac{U'}{U}\, dx = -\int \left(\frac{2y_1'}{y_1} + p\right) dx$$

$$\ln |U| = -2 \ln |y_1| - \int p\, dx$$

$$\ln |U y_1^2| = -\int p\, dx$$

$$U y_1^2 = e^{-\int p\, dx}$$

$$\boxed{U = \frac{1}{y_1^2} e^{-\int p\, dx}}$$

$$\boxed{u = \int U dx}$$

$$\boxed{y_2 = y_1 \int U dx}$$

$$\boxed{y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int p dx} dx}$$

7.3 Homogeneous Linear ODE with Constant Coefficients

$$\begin{cases} \text{Case 1: 2 Real Roots when} & a^2 - 4b > 0 \\ \text{Case 2: Double Root when} & a^2 - 4b = 0 \\ \text{Case 3: Complex Conjugate Roots when} & a^2 - 4b < 0 \end{cases}$$

Case 1: 2 Real Roots when $a^2 - 4b > 0$. The general solution is given by:

$$y_1 = e^{\lambda_1 x} \quad y_2 = e^{\lambda_2 x} \qquad \lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right) \quad \lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right)$$

$$\boxed{y_h = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$$

Case 2: $\boxed{\lambda_1 = -\frac{a}{2},\ y_1 = e^{-\frac{ax}{2}}}$ Determine y_2 using the method of reduction of order.

$$y_1 = e^{\frac{-ax}{2}}$$

$$y_2 = y_1 \int \frac{1}{y_1^2} e^{\int -pdx} dx = e^{\frac{-ax}{2}} \int \frac{1}{\left(e^{\frac{-ax}{2}}\right)^2} e^{\int -adx} dx = e^{-\frac{ax}{2}} \int e^{ax} e^{-ax} dx = xe^{-\frac{ax}{2}}$$

$$y_h = c_1 e^{-ax/2} + c_2 x e^{-\frac{a}{2}x}$$

$$\boxed{y_h = (c_1 + c_2 x) e^{-ax/2}}$$

Case 3: $\boxed{\lambda = -\frac{a}{2} \pm iw,\ w = \sqrt{|a^2 - 4b|}}$

$$y_1 = e^{\lambda_1 x} = e^{(-\frac{a}{2} + iw)x} = e^{-\frac{ax}{2}} e^{iwx}$$

$$y_2 = e^{\lambda_2 x} = e^{(-\frac{a}{2} - iw)x} = e^{-\frac{ax}{2}} e^{-iwx}$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \ldots = (1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots) + i(x - \frac{x^3}{3!} - \frac{x^5}{5|} + \ldots) = \cos x + i \sin x$$

$\boxed{e^{iwx} = \cos wx + i \sin wx}$ (de Moivre's theorem) and $\boxed{e^{i\pi} = -1}$ (Euler's Identity)

Any real solution is a linear combination of the real and imaginary parts:

$\boxed{y = e^{-\frac{ax}{2}}(c_1 \cos wx + c_2 \sin wx)}$ c_1, c_2 are constants

7.4 Euler-Cauchy Equations

The **Euler-Cauchy** equation is of the form:

$$\boxed{x^2y'' + axy' + by = 0} \text{ where } a, b \text{ are constants} \tag{7.4}$$

Let $y = x^m \implies y' = mx^{m-1} \implies y'' = m(m-1)x^{m-2}$ and substituting
$x^2m(m-1)x^{m-2} + axmx^{m-1} + bx^m = 0 \implies m^2 + (a-1)m + b = 0$

$$\boxed{m = \frac{1}{2}(1-a) \pm \sqrt{\frac{1}{4}(a-1)^2 - b}}$$

Case 1: Roots are distinct. The basis solutions are :

$y_1(x) = x^{m_1}$ $y_2(x) = x^{m_2}$, the general solution is given by, $\boxed{y = c_1x^{m_1} + c_2x^{m_2}}$

Case 2: Double roots.

$$\boxed{y_1 = x^{\frac{1}{2}(1-a)}}$$

$$y'' + \frac{a}{x}y' + \frac{(1-a)^2}{4x^2}y = 0$$

Use method of reduction of order, $y_2 = uy_1$ and with $\boxed{p = \frac{a}{x}}$

$$\boxed{U = \frac{1}{y_1^2}e^{\int -pdx}} \quad \boxed{u = \int Udx} \quad \boxed{y_2 = y_1\int Udx} \quad \boxed{y_2 = y_1\int \frac{1}{y_1^2}e^{\int -pdx}dx}$$

$$\int pdx = \int \frac{a}{x}dx = alnx \implies e^{\int -pdx} = e^{-alnx} = e^{lnx^{-a}} = x^{-a} = \frac{1}{x^a}$$

$$U = \frac{1}{y_1^2}\frac{1}{x^a} = \frac{1}{x^{1-a}}\frac{1}{x^a} = \frac{1}{x} \implies u = \int Udx = \int \frac{1}{x}dx = lnx$$

$$y_2 = y_1\int Udx = x^{\frac{1}{2}(1-a)}lnx$$

$$\boxed{y_h = (c_1 + c_2lnx)x^{\frac{1}{2}(1-a)}} \quad c_1, c_2 \text{ are constants}$$

7.5 The Wronskian

Two solutions y_1 and y_2 are **linearly dependent** if their **Wronkskian** W is 0.

$W(y_1, y_2) = y_1y_2' - y_2y_1' = 0$
Because if the solutions are dependent, $y_1 = ky_2$, where k is a constant
$\implies W(y_1, y_2) = y_1y_2' - y_2y_1' = ky_2y_2' - y_2ky_2' = 0$

The Wronksian is expressed as a **Wronkski Determinant:**

$$\boxed{W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \tag{7.5}$$

7.6 Non-homogeneous ODE

Consider the following non-homogeneous ODE:

$$y'' + p(x)y' + q(x)y = r(x)$$

The complete solution is the sum of homogeneous (y_h) and particular (y_p)solutions.

$$y(x) = y_h(x) + y_p(x) \text{ where } y_h = c_1 y_1 + c_2 y_2 \quad \text{(general solution)}$$

y_p is a solution of the non-homogeneous equation without any constants. A particular solution is obtained by assigning specific values to the constants. The **Method of Undetermined Coeffcients** is an approach to finding a particular solution to nonhomogeneous ODEs. If the term in $r(x)$ contains the following term, the choice for $y_p(x)$ is given by:

Term in r(x)	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$Kx^n (n = 0, 1, \ldots)$	$K_n x^n + K_{n-1}x^{n-1} + \ldots + K_1 x + K_0$
$k\cos wx$ or $k\sin wx$	$K\cos wx + M\sin wx$
$ke^{\alpha x}\cos wx$ or $ke^{\alpha x}\sin wx$	$e^{\alpha x}(K\cos wx + M\sin wx)$

7.7 Particular Solution by Variation of Parameters (Lagrange)

The particular solution for the standard form ODE is derived as follows:

$$y'' + p(x)y' + q(x)y = r(x)$$

Find a pair of functions $u_1(x)$ and $u_2(x)$ such that:

$y_p(x) = u_1(x)y_1(x)+u_2(x)y_2(x) \implies y_p'(x) = u_1'y_1+u_1y_1'+u_2'y_2+u_2y_2'$Set constraint, $\boxed{u_1'y_1 + u_2'y_2 = 0}$ $y_p'(x) = u_1 y$

7.8 SymPy

Solve the following non-homogeneous ordinary differential equation:

$$2y(x) - 3\frac{d}{dx}y(x) + \frac{d^2}{dx^2}y(x) = x^2$$

```
import sympy as sp
from IPython.display import display, Math

# Variables and function
x = sp.symbols('x')
```

```python
y = sp.Function('y')

# Nonhomogeneous ODE: y'' - 3 y' + 2 y = x^2
ode = sp.Eq(sp.diff(y(x), x, 2) - 3*sp.diff(y(x), x) + 2*y(x), x**2)

# Solve full ODE
soln_full = sp.dsolve(ode)

# Extract complementary (homogeneous) and particular parts
# dsolve returns: Eq(y(x), C1*exp…()+C2*exp…() + particular)
rhs = soln_full.rhs
C1, C2 = sp.symbols('C1 C2')

# The homogeneous part is the expression containing C1, C2
soln_homo = rhs.expand().coeff(C1)*C1 + rhs.expand().coeff(C2)*C2

# The particular solution is the rest (remove terms with C1, C2)
soln_part = sp.simplify(rhs - soln_homo)

# Display results
print(sp.latex(ode))
print(sp.latex(soln_homo))
print(sp.latex(soln_part))
print(sp.latex(soln_full))

display(Math(r"\textbf{Homogeneous solution: } y_h = " + sp.latex(soln_homo)))
display(Math(r"\textbf{Particular solution: } y_p = " + sp.latex(soln_part)))
display(Math(r"\textbf{General solution: } y = " + sp.latex(rhs)))

print(sp.latex(soln_full))
soln_full
```

General solution:

$$y = C_1 e^{x} + C_2 e^{2x} + \frac{x^2}{2} + \frac{3x}{2} + \frac{7}{4}$$

Homogeneous solution:

$$y_h = C_1 e^{x} + C_2 e^{2x}$$

Particular solution:

$$y_p = \frac{x^2}{2} + \frac{3x}{2} + \frac{7}{4}$$

General (or Full) solution:

$$y(x) = C_1 e^{x} + C_2 e^{2x} + \frac{x^2}{2} + \frac{3x}{2} + \frac{7}{4}$$