CHAPTER 6

First Order Ordinary Differential Equation

In gentle arcs and flowing streams,
Differential equations weave their dreams.
A first-order DE so pure, so true,
In math's embrace, it guides us through.

A differential equation expresses a relationship between changing quantities — it involves one or more derivatives, which describe how things vary over time or space. Such equations appear everywhere in science and engineering, wherever change and motion must be understood. Solving a differential equation means finding a function that captures how one variable depends on others. The result often includes a few constants, reflecting the many possible ways a system might begin or evolve. In essence, a differential equation gives us a window into how nature's patterns unfold, allowing us to predict behavior within the boundaries of the model we choose.

6.1 Differential Equation

A differential equation (DE) is an equation involving an unknown function and its derivatives. A DE is an ordinary differential equation (ODE) if the unknown function depends on only one variable. If the unknown function depends on 2 or more independent variables, the DE is a partial differential equation.

A differential equation (DE) together with additional conditions on the unknown function and its derivatives — all specified at the same value of the independent variable — forms an initial-value problem (IVP). These additional conditions are called the initial conditions.

If, instead, the conditions are specified at two or more distinct values of the independent variable, the problem is called a boundary-value problem (BVP), and the conditions are referred to as the boundary conditions.

6.2 Standard & Differential forms of an ODE

The Standard form for first order DE is:

$$\boxed{\frac{dy}{dx} = f(x,y)}$$

and the differential form is:

$$\boxed{M(x,y)dx + N(x,y)dy = 0}$$

6.3 Order & Degree of a Differential Equation

The order of a differential equation is the order of the highest derivative which is also known as the differential coefficient. E.g.,

$$\frac{d^3x}{dx} + 3x\frac{dy}{dx} = e^y$$

the order of the above differential equation is 3. A first order differential equation is of the form:

$$\boxed{\frac{dy}{dx} + Py = Q} \tag{6.3.1}$$

where P & Q are constants or functions of independent variables. E.g.,

$$\frac{dy}{dx} + (x^2+5)y = \frac{x}{5}$$

The degree of the differential equation is represented by the power of the highest order derivative in the given differential equation.

$$\left[\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right]^4 = k^2\left(\frac{d^3y}{dx^3}\right)^2$$ the degree of the above differential equation is 2.

For the equation:

$$tan\left(\frac{dy}{dx}\right) = x + y$$ the degree is undefined.

6.4 Solving ODE- Method of Separation of Variables

Through algebraic manipulations, some ODEs can be reduced to:

$$\boxed{g(y)\frac{dy}{dx} = f(x)} \tag{6.4.1}$$

By integrating both sides:

$$\int g(y)dy = \int f(x)dx + c$$

Example:

$$\frac{dy}{dx} = 1 + y^2 \implies \frac{dy}{1+y^2} = dx$$

$$\text{Let } y = tan\,\theta \implies \frac{dy}{d\theta} = sec^2\theta \implies \frac{sec^2\theta}{1+tan^2\theta}d\theta = dx \implies x = \theta + c \implies x = \tan^{-1}y + c$$

6.5 Solving ODE - Reduction to Separable Form

Consider the ODE of the form:

$$\boxed{\frac{dy}{dx} = f\left(\frac{y}{x}\right)} \tag{6.5.1}$$

Example: $$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2$$

Substitution: Let $y = vx$, so that $v = \frac{y}{x}$. Then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting into the original equation gives

$$v + x\frac{dv}{dx} = v^2$$

Hence,

$$x\frac{dv}{dx} = v^2 - v = v(v-1)$$

Separating variables:

$$\frac{dv}{v(v-1)} = \frac{dx}{x}$$

Integrating both sides:

$$\int \frac{dv}{v(v-1)} = \int \frac{dx}{x}$$

Decompose into partial fractions:

$$\frac{1}{v(v-1)} = -\frac{1}{v} + \frac{1}{v-1}$$

Thus,

$$\int \left(-\frac{1}{v} + \frac{1}{v-1}\right) dv = \int \frac{dx}{x}$$

Integration yields:

$$\ln\left|\frac{v-1}{v}\right| = \ln|x| + C$$

Simplifying:

$$\frac{v-1}{v} = Cx \quad \Longrightarrow \quad \frac{1}{v} = 1 - Cx \quad \Longrightarrow \quad v = \frac{1}{1-Cx}$$

Back-substitute $v = \frac{y}{x}$:

$$\frac{y}{x} = \frac{1}{1-Cx}$$

Hence, the general solution is

$$\boxed{y = \frac{x}{1-Cx}}$$

Special (constant) solutions: The substitution excluded $v = 0$ and $v = 1$, which correspond to $y = 0 \quad$ and $\quad y = x$, both of which satisfy the original differential equation.

Therefore, the complete solution set is:

$$\boxed{y = \frac{x}{1-Cx}}, \quad y = 0, \quad y = x.$$

6.6 Solving ODE - Exact ODE & Integrating Factor

If an ODE has an implicit solution :

$$\boxed{u(x,y) = c = constant}$$

then,

$$\boxed{du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0}$$

$$\boxed{M(x,y)dx + N(x,y)dy = 0}$$

$$M = \frac{\partial u}{\partial x}$$
$$N = \frac{\partial u}{\partial y}$$

A 1st order ODE is an exact DE if:

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}}$$

$$\boxed{u = \int M dx + k(y) = \int N dx + l(x)}$$

Example:

$$cos(x+y)dx + (3y^2 + 2y + cos(x+y))dy = 0$$
$$M = \frac{\partial u}{\partial x} = cos(x+y)$$
$$u = sin(x+y) + k(y) \implies \frac{\partial u}{\partial y} = cos(x+y) + \frac{dk}{dy}$$
$$N = \frac{\partial u}{\partial y} = 3y^2 + 2y + cos(x+y) = cos(x+y) + \frac{dk}{dy}$$
$$k = y^3 + y^2 + c^*$$
$$u = sin(x+y) + y^3 + y^2 + c$$

6.7 Inexact ODE

Consider the ODE:

$$-ydx + xdy = 0$$

Here the above approach will not work, because:

$$M = \frac{\partial u}{\partial x} = -y \quad N = \frac{\partial u}{\partial y} = x \quad \frac{\partial M}{\partial y} = \frac{\partial^2 M}{\partial x \partial y} = -1 \quad \frac{\partial N}{\partial x} = \frac{\partial^2 N}{\partial x \partial y} = 1 \quad \frac{\partial^2 M}{\partial x \partial y} \neq \frac{\partial^2 N}{\partial x \partial y} \text{ (inexact)}$$
$$u = -y \int dx + k(y) = -xy + k(y)$$
$$\frac{\partial u}{\partial y} = -x + \frac{dk}{dy}$$

But $N = \frac{\partial u}{\partial y} = x$ which contradicts the above equation

6.8 Integrating Factor to transform to an exact ODE

Multiply the equation by a factor $F(x,y)$ to make it exact.

$$FMdx + FNdy = 0$$

and impose the conditions:

$$\frac{\partial}{\partial y}(FM) = \frac{\partial}{\partial x}(FN) \rightarrow F_y M + FM_y = F_x N + FN_x$$

Let F depend only on x,

$$FM_y = F^{'}N + FN_x$$

$$\frac{M_y}{N} = \frac{F^{'}}{F} + \frac{N_x}{N}$$

$$\int \frac{df}{F}dx = \int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx$$

Let $\boxed{R = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}$

$$ln(F) = \int Rdx \implies \boxed{F(x) = e^{\int R(x)dx}}$$

Similarly,

$$R^* = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \implies \boxed{F(y) = e^{\int R^*(y)dy}}$$

E.g., solve:

$$(e^{x+y} + ye^y)dx + (xe^y - 1)dy = 0$$

$$M = \frac{\partial u}{\partial x} = e^{x+y} + ye^y \quad N = \frac{\partial u}{\partial y} = xe^y - 1$$

$$\frac{\partial M}{\partial y} = e^{x+y} + ye^y + e^y \quad \frac{\partial N}{\partial x} = e^y \quad \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = e^{x+y} + ye^y$$

$$R = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xe^y - 1}\left(e^{x+y} + ye^y\right)$$

R does not work as it is a function of both x and y. So we try with R^*

$$R^* = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{-1}{e^{x+y} + ye^y}\left(e^{x+y} + ye^y\right) = -1$$

$e^{\int R^* dy} = e^{-y}$ this works as it is a function y only

Multiplying the ODE by $e^{R^*} = e^{-y}$

$$(e^x + y)dx + (x - e^{-y})dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = 1 \text{ (exact ODE!)}$$

$$M = \frac{\partial u}{\partial x} = e^x + y \implies u = e^x + xy + k(y) \implies \frac{\partial u}{\partial y} = x + \frac{dk}{dy} = x - e^{-y} \implies k = e^y + c^*$$

$$u = e^x + xy + e^y + c$$

6.9 1st Order Linear ODE - Homogeneous

A first order ODE is linear if it is of the following form:

$$\boxed{\frac{dy}{dx} + p(x)y = r(x)}$$

and is non-linear if it cannot be brought to the above form. The above ODE is linear in both y and y' where p and q are any function of x. When $r(x) = 0$, the ODE is called homogeneous.

$$\boxed{\frac{dy}{dx} + p(x)y = 0}$$

By the method of separation of variables we have,

$$\int \frac{dy}{y} = -\int p(x)dx \implies \ln|y| = -\int p(x)dx + c^*$$

$$\boxed{y = ce^{-\int p(x)dx} \text{ (homogeneous solution } y_h)}$$

6.10 1st Order ODE - Non Homogeneous

When $r(x) \neq 0$, the ODE is called non homogeneous. We multiply the ODE by a function $F(x)$.

$$Fy' + Fp(x)y = Fr(x)$$

$$\text{Let } Fp(x) = F' \implies \frac{F'}{F} = p(x) \implies \ln|F| = \int p(x)dx \quad \text{Let } h = \int p(x)dx \implies F = e^h$$

$$\text{Now } (Fy)' = Fr(x) \implies (e^h y)' = r(x)e^h \implies e^h y = \int e^h r(x)dx + c$$

$$\boxed{y_p = e^{-h}\int e^h r dx + c} \quad \boxed{y = y_h + y_p = ce^{-h} + e^{-h}\int e^h r dx + c}$$

6.11 Reduction to Linear Form - Bernoulli Equation

The Bernoulli equation, a non-linear ODE is given by:

$$\boxed{y' + p(x)y = r(x)y^n} \tag{6.11.1}$$

where n is any real number.

$$\text{Let } u = y^{1-n}$$
$$u' = (1-n)y^{-n}y'$$
$$u' = (1-n)y^{-n}(ry^n - py)$$
$$u' = (1-n)(r - py^{1-n})$$
$$u' = (1-n)(r - pu)$$
$$u' + (1-n)pu = (1-n)r \text{ (Linear ODE)}$$

6.12 SymPy

Solve the following differential equations:

$$\frac{d}{dt}x(t) = x(t)$$

```python
from sympy import Function, dsolve, Eq, diff, Derivative, sin, cos, symbols, pprint
x = Function('x')
t = symbols('t')
deq = Eq(diff(x(t),t), x(t)) # Eq(LHS, RHS)
display(deq)
xsoln = dsolve(deq, x(t))    # dsolve wrt x(t)
display(xsoln)
```

$$x(t) = C_1 e^t$$

$$\frac{d}{dt}x(t) = \frac{x(t)}{2} - 450$$

```python
from sympy import Function, dsolve, Eq, diff, Derivative, sin, cos, symbols, pprint
x = Function('x')
t = symbols('t')
deq = Eq(diff(x(t),t), x(t)) # Eq(LHS, RHS)
display(deq)
xsoln = dsolve(deq, x(t))    # dsolve wrt x(t)
display(xsoln)
```

$$x(t) = C_1 e^{\frac{t}{2}} + 900$$

$$\frac{d}{dx}y(x) = y^2(x) + 1$$

```python
y = Function('y')
x = symbols('x')
deq = Eq(diff(y(x),x), (1 + y(x)**2))
display(deq)
xsoln = dsolve(deq, y(x))
display(xsoln)
```

$$y(x) = -tan(C_1 - x)$$

b