Faster Intersection Via Vector Trees

Adam Gerber
adam.gerber@jhu.edu
Johns Hopkins University
3400 North Charles Street
Baltimore, MD 21210, USA

Juri Ganitkevitch
j.ganitkevitch@jhu.edu
Johns Hopkins University
3400 North Charles Street
Baltimore, MD 21210, USA

Abstract

This paper implements and extends a theoretical framework for set intersection, developed earlier. This theoretical framework is implemented by means of a tree structure called a VectorTree, which is explained here. This approach is especially effective in dealing with sparsity and locality, while maintaining state of the art performance for dense sets over small ranges. Here, worst-case scenarios are resolved by introducing additional data structures which provide insight into the distributional characteristics of the descendants of each node.

1 Introduction

For two sets A and B, the set intersection problem consists of finding the set that contains all elements of A that also appear in B, but no other elements.

Formally:

$$x \in A \cap B \iff x \in A \wedge x \in B$$

This definition can be generalized to the intersection of an arbitrary, nonempty collection of sets. If M is a nonempty set whose elements are themselves sets, then x is an element of the intersection of M, if and only if, for every element A of M, x is and element of A.

Formally:

$$x \in \cap M \iff \forall A \in M, x \in A$$

Many approaches cast intersection as a search problem, modifying familiar search algorithms to find the common elements in all sets. Common strategies exploit disparities in sizes of sets, asymmetries in the distributional characteristics, and statistics gathered during the lifetime of a set.

Many authors consider the intersection problem for ordered sets isomorphic to the intersection for sets in general. Further, it is almost universally assumed that members are of integer type.

An early approach to this problem, Hwang, et al (0), minimizes searching by linearly searching for each element from a smaller set in a larger set. More recently, Demaine et al. (0) proposed an algorithm which they called Adaptive, which searches for an element of a particular set within all other sets using a combination of "galloping search" and binary search. Barbay, et al (0) introduced a variant of the Adaptive algorithm, called Sequential, which cycles through the sets, "performing one entire gallop search at a time in each (as opposed to a single galloping step in Adaptive)."

One of the most recent algorithms is due to Baeza-Yates (0). This algorithm obtains the median element of the smaller set and attempts to locate that value within the larger set, dividing the problem into two sub-problems for each median element, each of which is solved recursively. Although much of the work in intersection is concerned with search, Udamchaiporn et al. propose a new approach to the intersection of sorted sets using a comparison-and-elimination approach that is "search free"; this method is described in Section 2.

Here, we implement an intersection strategy previously proposed, which represents the ordered sets by multi-bit offsets, organized hierarchically, as in a search tree. This structure gives rise to an efficient intersection procedure which is described below.

The rest of the paper is organized as follows: related works are described in Section 2, the algorithm and associated data structures are described in Section 3, significant cases are discussed in Section 4, future work is discussed in Section 5, and the conclusion in Section 6.

2 Related Work

Below, we discuss several approaches to the intersection problem that were relevant to the theoretical framework, which, here, is implemented and extended.

Demain, et al. proposed the Adaptive algorithm which attempts to match elements using a combination of "galloping search" and binary search. This algorithm performs the so-called "gallop" in parallel through all the sets from both the "low side" and "high side" to find eliminators (elements potentially in the intersection). According to Demain, "Galloping consists of doubling the jump in position each iteration, until it overshoots the current eliminator which will be on the low side. Upon overshooting, the other parallel processes pause while the overshooter does a binary search to find the largest eliminatable element and chooses the next higher element as the new eliminator.“ Adapting to numerous sets of data proved hazardous to its complexity however. Barbay, et al.'s Sequential approach is similar, except for its use of the galloping search as described above, and performs fewer comparisons than the Adaptive algorithm on average.

The Baeza-Yates algorithm finds the median element of the smaller of two ordered sets and uses a binary search to locate the value in the larger set. If the element is found in the larger set, it is added to the intersection set. This algorithm then recursively solves the equivalent sub-problem for each pair of subsets.

The Search-Free algorithm described by Udomchaiporn, et al. takes a different approach to the intersection problem. For each ordered set, they identify the maximal element among the m sets' lowest values and the minimal element among the m sets' highest values, checking for matches while iteratively pruning all other sets of members which fall outside of this range, terminating when a set becomes empty. This process proceeds in linear time with the size of the non-matching elements.

These approaches all have linear average-case time complexity, and also have best-case scenarios that may be constant time for degenerate cases or for very idiosyncratic datasets. Nevertheless, the sets are represented sequentially and so are unable to capture the distributional properties of the data. The theoretical framework built upon here leverages a structured but tractable representation of the state space as well as the ability to process large ranges of data with great efficiency.

3 Method

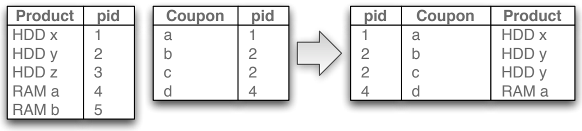
3.1 Context

While many authors consider the intersection problem in the abstract setting, we propose an architecture within a component framework, for utilization by any application, especially solvers such as Prolog or database management systems (DBMS), such as MySQL.

After presenting the original Vector Tree framework in the abstract, we present the algorithm fully integrated as a Java package, accessible to all interested parties.

The package provides a means of representing ordered sets, which is abstracted away from the user. It presents an interface for adding and removing records, and for executing the intersection, which provides access to this potentially very large set by means of an iterator, provides the requester with those values in the intersection (or, if it is preferred, the records to which those keys belong, as may be useful in the context of a DBMS).

A simple example of this would be a query on two tables to find all products that have a coupon, as in Figure 1. In this case, the algorithm would return an iterator either (1) over `pid` values within the intersection or (2) over rows, which correspond



Product	pid
HDD x	1
HDD y	2
HDD z	3
RAM a	4
RAM b	5

Coupon	pid
a	1
b	2
c	2
d	4

pid	Coupon	Product
1	a	HDD x
2	b	HDD y
2	c	HDD y
4	d	RAM a

Figure 1: A simple database join.

to records. This distinction is determined by the requester.

3.2 Definitions

The Vector Tree framework introduce a tree structure for storing m-bit integer keys, where m must be of the form $m = Di$, where $i = 2^n$ for some small integers D and n.

For example, to represent 32-bit integers, one such assignment would be $n = 3$ and $D = 4$, with $i = 2^n = 8$. From the definition of m, it is clear that it is possible to segment every m-bit integer into d contiguous i-bit sequences.

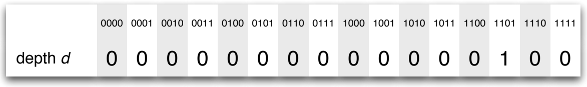
We will use the notation $sig_i(key, j)$ to denote a key's j^{th} most significant i-bit sequence. In other words, for any key k_q in the range 2^{Di}, the following serves as an identity, where $\oplus$ is the append operator:

$$k_q = sig_i(k_q, 0) \oplus sig_i(k_q, 1) \oplus ... \oplus sig_i(k_q, D-1)$$

Formally, a Vector Tree T consists of a hash table called `nodes`, which maps bitvectors to a finite set of tuples, each corresponding to a node in the tree. These tuples are of the form (c, m, p), where $c \in [0, 2^{2^i}]$ indicates the current bit configuration, m the number of leaf nodes (equivalent to the number of unique keys that have been inserted into that node's subtrees), and p is the total number of non-leaf nodes beneath that node. A Vector Tree has an additional hash table called `registrants`, which maps full keys to lists of objects that are associated with them.

The Vector Tree support several operations:

- Insertion
- Deletion
- Intersection
  - BitwiseIntersect
  - SurvivorMap



	0000	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	1110	1111
depth *d*	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0

Figure 2: An example bit-vector for $i = 4$.

3.2.1 Similar Structures

Similarities can be drawn to a van Emde Boas tree (0), although without the root-size reduction in range at each level, or to a suffix tree, in which the bit-vectors are symbols in alphabet of size 2^i.

3.3 Encoding

For a Vector Tree T over the universe 0 to $2^m - 1$, the root node encodes $sig_i(key, 0)$ in a bit-vector of length 2^i, with each bit corresponding to one of the 2^i combinations possible among i bits. For example, where $i = 4$, the 4-bit sequence `0000` would correspond to the first of the vector's 16 elements. This is illustrated in Figure 2.

Just as the root node, with depth 0, encodes the sequence $sig_i(key, 0)$, nodes at depth 1 encode the sequence $sig_i(key, 1)$, and, in general, nodes at depth d encode $sig_i(key, d)$. Nodes at the same depth, but which do not share a parent encode the same i-bits of the key, but they necessarily differ in the bit sequence which precedes the sequence they encode. This preceding sequence is called the node's bit prefix.

An "on" (`1`) bit in the root node's bit-vector guarantees the existence of at least one key whose first i bits match the i-bit combination indicated by the bit-vector. Correspondingly, an "off" bit (`0`) indicates the absence of any such element. For example, in the case $i = 4$, if the root node's bit-vector is `1000000000000001`, then the tree is guaranteed to contain at least one key k_1 with $sig_4(k_1, 0) =$ `0000` and at least one key k_2 with $sig_4(k_2, 0) =$ `1111`. Similarly, a node's bitvector will never contain only zeros, because this would indicate there are no values in this range, in which case the node does not exist (it may be created in a subsequent `Insert` operation).

3.3.1 Node Hierarchy

Unlike typical tree structures, in which a parent node contains pointers to its child nodes, nodes in the Vector Tree, parents and children have no direct connection. Instead, the hash table `Nodes` is used for random-access to any node in tree. This is possible because every node in the tree (and, more strongly, every possible node) can be uniquely specified by its bit-vector. For example, assuming $i = 4$, the bit-prefix `00000000` specifies the left-most node at a depth of 2. The empty bit-prefix specifies the root (which is the initial position and has no prefix).

3.3.2 Leaves

Unlike other nodes, a materialized leaf node contains a table, which, for all keys in the node's 2^i value range, maps each key to a list of object references. This allows for the constant-time recovery of all associated objects, for example, the record of which this key is a column-value or the class instance of which this key is a member.

3.4 Operations

3.4.1 Insertion

Figure 3 outlines the insertion of the key `111011010000000` along with the object of which this key is a component. As described in A, the full key is used with `nodes` to access the proper leaf node, and a reference to the object is added to this key's list in `registrants`.

If the corresponding bit is already on, then the insertion operation concludes. If the bit is off, then it is flipped, and the node's member count is incremented. Then, the leaf node's parent is accessed (also via `nodes` by splicing off the least-significant i-bits from the key. It may be the case that this node does not yet exist, in which case, an empty node is initialized. In either case, the parent's progeny count is incremented and the bit corresponding to the leaf node is queried.

If the bit is already on, then the process terminates; otherwise, the bit is flipped and this process advances up the tree, terminating after reaching (and, if necessary, updating) the root node.

3.4.2 Deletion

Deletion requires both a key and an object as parameters, as multiple objects may be registered to the same key. The full bit-sequence of the key is used with `nodes` to obtain its containing leaf node. The list of objects registered to the key is obtained via `registrants` and is scanned for a reference to an object that matches the object parameter. If a match is found, then the reference is removed. If a removal occurred, then the list is queried to determine whether any objects remain registered to this key, if not, the bit corresponding to the key is turned off, and the member count for the node is decremented.

If this update results in a member count of zero, then the node is decommissioned and the tree's node hash for this key is set to null.

If the member count was decremented, then the parent node is loaded and its member count is decremented. If the child node was decommissioned, then the bit corresponding to that child is turned off and the parent node's progeny count is decremented. As a bit change occurred, the vector is passed as a parameter to `SurvivorMap`; if the mapping function returns an empty list, then the node has no children remaining, and so it too is decommissioned. The process repeats for each parent node, up to and including the root node.

3.5 Intersection Algorithm

3.5.1 `Bitwise Intersection`

This method takes, as input, m vectors, efficiently perform a bitwise `AND` operation, and returns the result. This operation is constant in the size of the vector and can also be constant in the size of m, however, in the worst case, is $O(\lg m)$.

3.5.2 `SurvivorMap`

This method takes, as input a vector of length 2^i and, in constant-time, returns a list of the "on" bits, in their i-bit encoding. For example, for $i = 4$, the call `SurvivorMap(1100000010000001)` would return the following:

$$[\mathtt{0000}, \mathtt{0001}, \mathtt{1000}, \mathtt{1111}]$$

Constant-time (as opposed to time proportional to the size of the input) is achieved by precomputing the entire mapping of 2^{2^i} configurations of 2^i. For example, for $i = 4$, node vectors must be of length $2^4 = 16$, which means this method must store a range of $2^1 6 = 65536$ values.

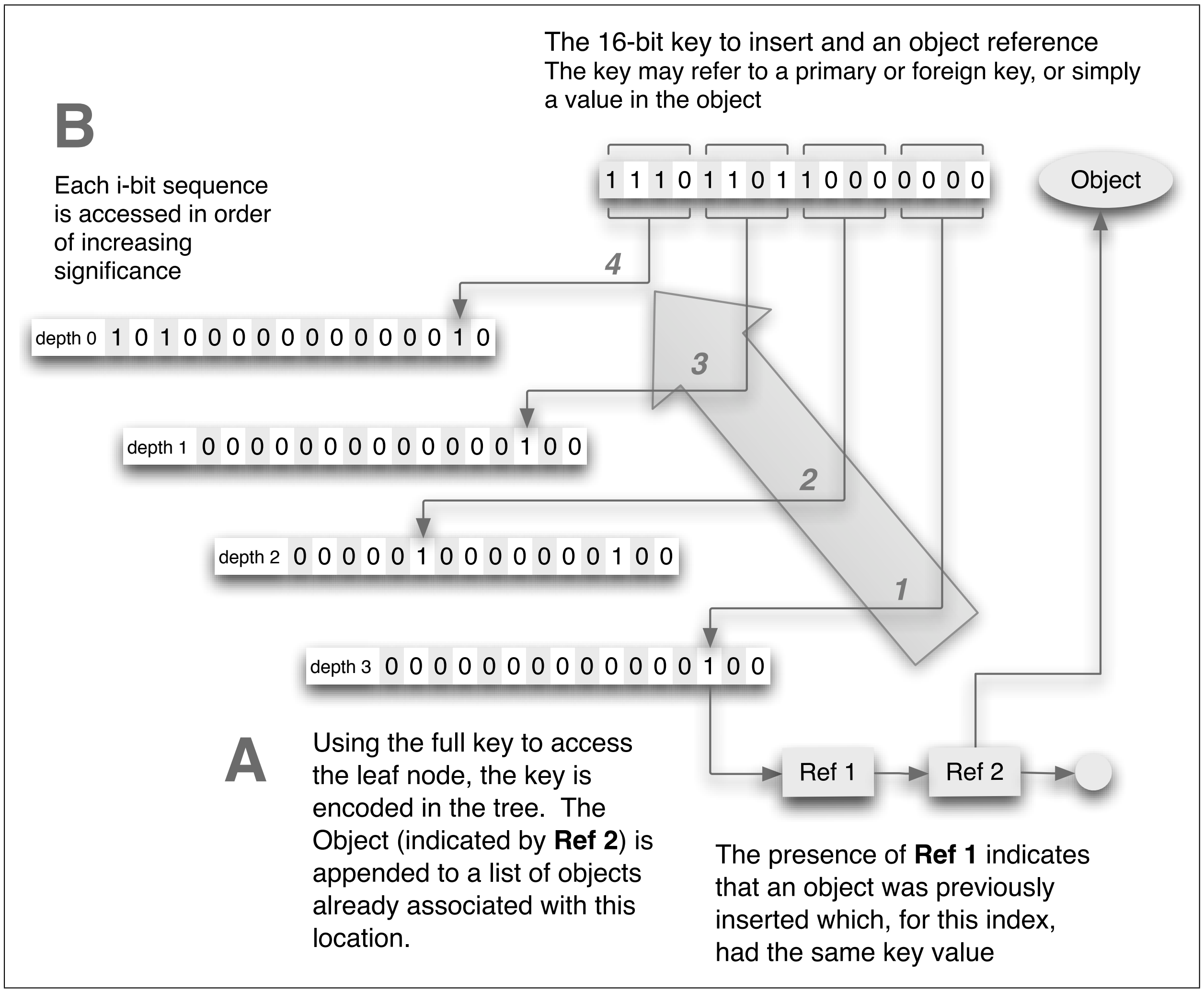

The 16-bit key to insert and an object reference
The key may refer to a primary or foreign key, or simply a value in the object
B
Each i-bit sequence is accessed in order of increasing significance
1 1 1 0 1 1 0 1 1 0 0 0 0 0 0 0
Object
4
3
2
1
depth 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0
depth 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0
depth 2 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0
depth 3 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0
A
Using the full key to access the leaf node, the key is encoded in the tree. The Object (indicated by **Ref 2**) is appended to a list of objects already associated with this location.
Ref 1
Ref 2
The presence of **Ref 1** indicates that an object was previously inserted which, for this index, had the same key value

Figure 3: Insertion of a key.

An example of this can be seen in B and C in Figure 4.

3.5.3 Intersection

The intersection procedure begins by initializing an iterator and a queue. The iterator will read from the queue whenever a caller requests an element of the intersection. The queue is the structure onto which elements in the intersection will be pushed.

The process continues by obtaining the bit-vector at the root node of each of the m sets involved in the intersection and passing them to the method `BitwiseIntersection`, which produces the intersection of the bit-vectors. Rather than scan the bit-vector for "on" bits, we pass the result as a parameter to the method `SurvivorMap`, which returns a list of indices that correspond to the "on" bits in the bitwise-intersection parameter and, more importantly, to the surviving subtrees. This process occurs in constant time for each node. It should be noted that intersection is not guaranteed at this stage; the only assurance is that all sets contain members in the indicated range.

If this list is empty, then we move to the parent of the current node; however, if the current node is the root node, then the operation terminates. If the list is not empty, then each index is used to create a key to access the bit-vectors at next search location. To create the key, the index is appended to the the bit-prefix of the current location. For each of the m sets, the key is then used to access the surviving

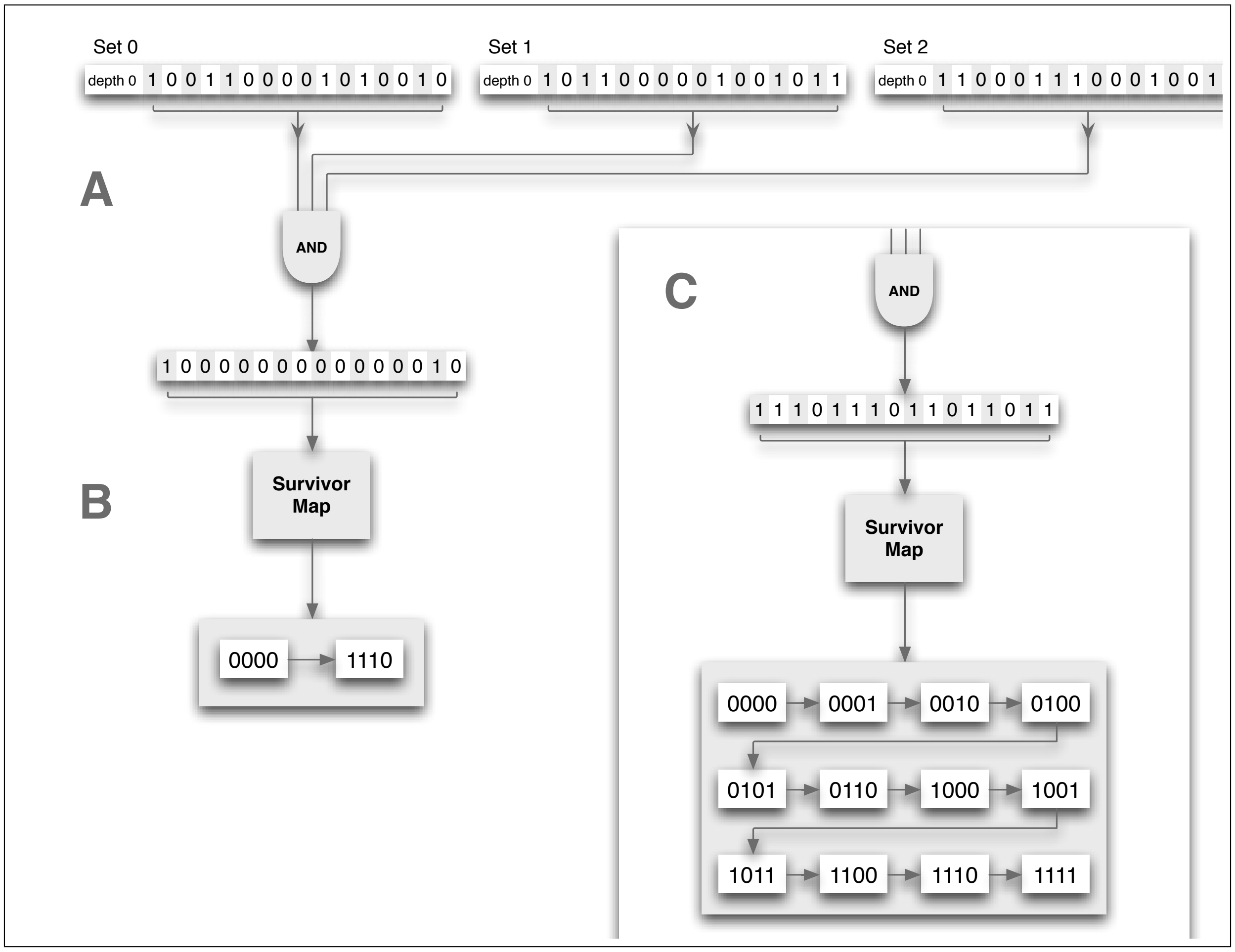


Figure 4: Initial step in the intersection of 3 sets

node and obtain its bit-vector. The process repeats as before, until a leaf node is reached. A depiction of this initial iteration can be seen in Figure 4.

In the leaf node, the procedure is only slightly different. Just as before the bit-vectors are obtained from each of the m sets. Here, too, they are passed to `BitwiseIntersection`, which returns a bit-vector; however this bit-vector does not indicate common ranges as before, but, instead, actual matching keys. We nevertheless want to leverage `SurvivorMap`'s constant-time conversion of vector to list, but we will handle the resulting list differently.

First, we will push onto the queue a metadata object, then we will push the entire list of indices returned by `SurvivorMap`. The metadata object, interpretable by the iterator, will (1) provide access to `Registrants` so that the references or records associated with a key can be accessed over all sets (for example, allowing them to be combined, as in a join operation, and for external access to the potentially numerous objects sharing a key), and (2) provides the iterator with the bit-prefix of the current location. The bit-prefix provides the iterator with sufficient information to recover the matching key at time of user request, allowing us to remain in constant-time for each node.

4 Bloom Filter Guarding

While our method presents an efficient approach to multi-set intersection, there are cases in which an unfortunate distribution of the data causes the algorithm to explore the entire tree only to find that there are no, or only very few, matches. Consider for instance sparse, uniformly distributed sets with little overlap. While the uniformity will for many sets result in a full Vector Tree, the sparsity entails that only very few of the leaf nodes actually contain matching keys.

We address this issue by introducing Bloom filters, a probabilistic data-structure that provides a compact bit-vector representation of the keys in the sub-tree rooted in each node.

4.1 Bloom Filters

A Bloom filter consists of a bit vector of size m and a set of k hash functions $h_1, \ldots, h_k$ that map any object to an integer $1 \ldots m$. To insert an object o into the filter, we compute $h_i(o)$ for each hash function and set the corresponding bit in the bit vector to 1. To check whether an object is in the Bloom filter, we similarly compute the hash indices for each of the k functions and test whether the corresponding bits are 1. Since the sets of bits set for different objects can overlap (i.e. $h_i(o) = h_j(o')$), querys to the Bloom filter can result in false positives, however (since a bit set to 1 it is never flipped back) never in false negatives.

Bloom filters have been thoroughly studied and bounds on the false positive rate for a given m, expected number n of objects to be inserted into the filter as well as the optimal k for given m and n are readily available in the literature.

4.2 Use of Bloom Filters in the Vector Tree

We augment the Vector Tree structure by adding a Bloom filter to each node in the tree. Each Bloom filter contains the keys in the subtree rooted at its respective node. Furthermore, for a given depth d, all Bloom filters are of the same size and employ the same set of hash functions. This setup makes the resulting bit vectors comparable across two or more Vector Trees[1], enabling us to use the Bloom filters to obtain information useful in the tree intersection process.

When intersecting a set of nodes n_i, we first intersect their Bloom filters b_i by computing a bit vector $b = \bigwedge_i b_i$ through bit-wise intersection of the b_i. Since the Bloom filters are of equal size and use the same set of hash functions, a key is present in all the b_i will be present in b. Conversely, if b is all zeros we are guaranteed that the b_i do not contain any common keys. Furthermore the number of ones in b, divided by the number of hash functions, $\frac{|b|_1}{k}$, gives us an estimate (and upper bound) on the number of common elements in the subtrees. The precision of this estimate depends on the false positive rate of the Bloom filters

This information can be used to avoid descend into subtrees that exhibit identical structure, yet do not contain matching keys. The deeper the node is in the Vector Tree, the smaller the range of potential keys it covers and the smaller the Bloom filter required to maintain an acceptable false positive rate. Additionally, we vary the hash function sets used for each depth level, creating a more diverse set of Bloom filters that are overall less likely to produce false positives over the same data.

[1]Obviously, we only compare the Bloom filters of nodes that govern the same key prefix.

5 Analysis

While describing the asymptotic time complexity of many algorithms is useful, in the case of the intersection problem even the most naive methods will approach linear time. For this reason, we present a number of cases and compare how the Vector Tree algorithm performs in comparison to the Search-Free algorithm and the Baeza-Yates algorithm.

5.1 Case 1

Consider the case of m sets over the range of 0 to $2^{Di} - 1$, with r regions of dense keys (with an average of k keys per region), sparsely distributed over the key range. Let P represent the set of regions that will be in the intersection set and Q be the set of all those regions that will be pruned (i.e. not in the intersection).

For all regions in P, both the Search-Free algorithm and the Baeza-Yates algorithm would require an average of km comparisons per region. However, the Vector Tree approach would require, in the worst case, $\frac{(2^i)^{X+1}}{2^i-1}$ comparisons where $X = \frac{k}{2^i}$. This is because at every level d of the tree, a maximum of 2^{id} comparisons are performed. As the algorithm recurses down the tree, intersections are performed successively, eliminating unnecessary branches until it reaches a leaf node. For each leaf, there are only 2^i candidates for the intersection set; determining the remaining candidates is a constant-time operation at the leaf level, regardless of the number of elements added to the intersection set

For all regions in Q, the Search-Free algorithm would still perform km operations per region r (each key in each set must be set to null), while the Baeza-Yates algorithm would do close to km comparisons over all regions. In this case, the Vector Tree algorithm would perform a maximum of $\frac{(2^i)^{X+1}}{2^i-1}$ comparisons where $X = \frac{k}{2^i}$ to prune that part of the tree. In practice, it is likely to perform significantly fewer comparisons for regions in Q because large blocks of elements would be eliminated closer to the root node. Because of this, on the order of 2^i comparisons are avoided, speeding up the intersection accordingly.

Wasteful comparisons are nevertheless made. The Search-Free algorithm performs approximately km comparisons, as does the Baeza-Yates algorithm, while the Vector Tree performs $\frac{(2^i)^{X+1}}{2^i-1}$ comparisons where $X = \frac{k}{2^i}$. In cases of large r in both P and Q, the Vector Tree prunes the regions of Q in the same time, while the other algorithms (with the exception of those implementing gallop search) are limited to examining individual keys. In P, the Vector Tree performs comparisons of 2^i elements for every single element compared in the other approaches.

6 Experimental Results

The space over which we aim to obtain performance results is a highly parametrized one. We conduct our experiments on synthetic data, i.e. randomly generated sets of keys. This approach provides us with good control over the nature of the task we evaluate on. We describe the data generation procedure in the following sub-section and then proceed to present a selection of performance statistics we obtained in our experiments.

6.1 Data Generation

We focus on two factors that we deem important when conducting multi-set intersections: data sparsity and data set overlap. While in real-world applications keys can't generally be assumed to be uniformly distributed, we generate our synthetic key sets from uniform distributions[2]. The data generation process works as follows: we first determine the number n and size m of key sets to intersect, as well as the sparsity ratio – defined as $\sigma = \frac{keys\ in\ set}{keys\ in\ scope}$, where the scope is the overall set of keys we draw from uniformly. The size of the scope is determined by reversing the rationale and given by $\frac{m}{\sigma}$. We then generate a root key set by uniformly drawing m keys from a set of $\frac{m}{\sigma}$. Next, we set an overlap factor $\omega = \frac{keys\ shared\ with\ root\ set}{keys\ in\ set}$. The remainder of the sets are then constructed by uniformly drawing from the root set until ω is fulfilled and padded with keys drawn uniformly from the set of all possible integers.

6.2 Experiments

Here, we present performance results for multiset intersections using the Vector Tree structure, with and without the Bloom filter augmentation. Aside from the number, size, sparisity and overlap of the data sets, we also investigate two parameters that affect the Bloom filters themselves.

As mentioned earlier, Bloom filters rely on an estimate of the number of keys they will hold to parametrize optimally. In our setup, this estimate is very crude: we simply use the maximum number of keys that a particular node *could* hold. I.e. the set of all integers (2^{32} in Java) at the root node etc. Clearly, in a more realistic setting this estimate could be bounded by the total number of objects in the database and computed from more sophisticated statistics. The result of using an estimate this crude is, that a Bloom filter that were able to hold this many object would itself be unreasonably large. While attemping to partially amend this problem by making the Bloom filter as large as memory permits may be an option it turns out to be a bad choice for two reasons: firstly, the resulting bit vectors are very large and therefore relatively costly to intersect.

[2]In fact, we have implemented a method to generate more complex key distributions by combining multiple local uniform or normal distribution, leading to locally denser key sets. However, we did not use this data in our experiments.

7 Future Work

7.1 Greater Bit Length

In practice, $i = 4$ is the highest feasible bit-length that allows the constant-time strategies we employed here. This is because `SurvivorMap` must encode all 2^{2^i} combinations of vector combinations. With $i = 4$, this means 65536 keys, but for $i = 8$, this would require $2^{2^8} = 2^{256} = 1.1579 \times 10^{77}$ keys. However, there may be intermediate values that are viable.

7.2 Alternate Encodings

In this investigation, we used significant-bits as the partition criteria, however, it may be the case that other encodings have distributional properties that allow for greater sparsity or dissimilarity between regions.

7.3 Alternate Encodings

In this investigation, we used significant-bits as the partition criteria, however, it may be the case that other encodings have distributional properties that allow for greater sparsity or dissimilarity between regions.

8 Conclusion

We have proposed a set-intersection algorithm that performs well on average and is frequently able to eliminate very large regions of the keyspace, as well as exploit both sparsity and data locality. This algorithm has an average case time complexity of $O\left(b^{\frac{k}{b}}\right)$ where b is the size of the bit-vector, and k is a measure of data sparsity. This algorithm is efficient in almost all cases and does not suffer any severe performance losses on any one specific case. There are several improvements were would like to consider and those are further discussed in Future Work.

Acknowledgments

The authors would like to thank Wes Filardo for his time and his insight.

References

Ricardo Baeza-yates, Ro Salinger, and Santiago Chile. Experimental analysis of a fast intersection algorithm for sorted sequences. In *In Proceedings of 12th International Conference on String Processing and Information Retrieval (SPIRE*, pages 13–24. Springer, 2005.

Jérémy Barbay, Alejandro López-Ortiz, Tyler Lu, and Alejandro Salinger. An experimental investigation of set intersection algorithms for text searching. *J. Exp. Algorithmics*, 14:3.7–3.24, 2009.

Erik D. Demaine, Alejandro López-Ortiz, and J. Ian Munro. Adaptive set intersections, unions, and differences. In *SODA '00: Proceedings of the eleventh annual ACM-SIAM symposium on Discrete algorithms*, pages 743–752, Philadelphia, PA, USA, 2000. Society for Industrial and Applied Mathematics.

F. K. HWANG and S. LIN. Optimal merging of 2 elements with n elements. *ACTA Informatica*, page 145158, 1971.

Wikipedia. Van emde boas tree — Wikipedia, the free encyclopedia, 2010. [Online; accessed 10-May-2010].