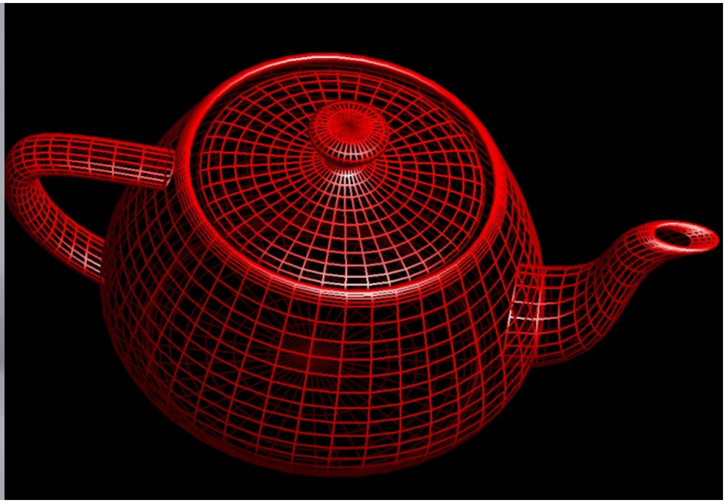


Foundations of Computer Graphics

Saurabh Ray

Reading

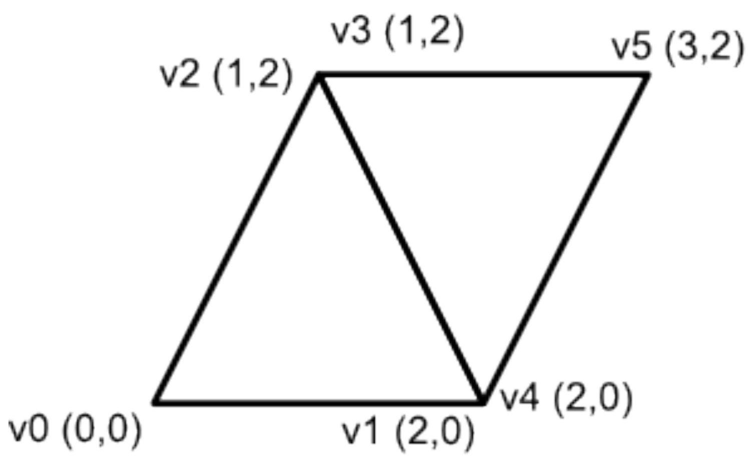
Reading for Lecture 5: Sections 4.1.9, 4.1.10, 4.3.

Reading for Lecture 6: Sections 4.7-4.10.

Please practice WebGL programming with Exercise 1.

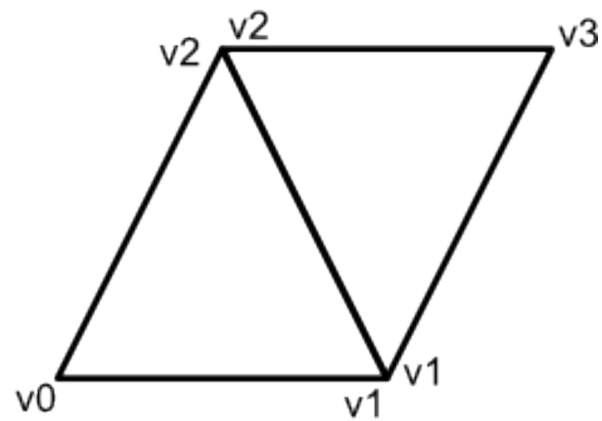
Index Buffers

Without indexing



[0,0, 2,0, 1,2, 1,2, 2,0, 3,2]

With indexing



[0,1,2, 2,1,3]

[0,0, 2,0, 1,2, 3,2]

Vertices reused twice

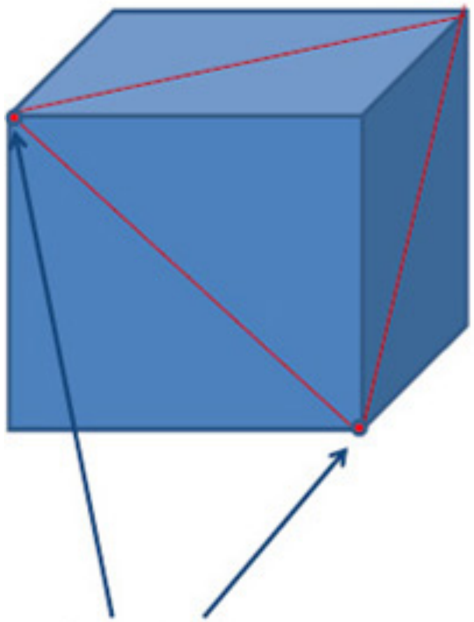
Index Buffers

Drawing a square with index buffers.

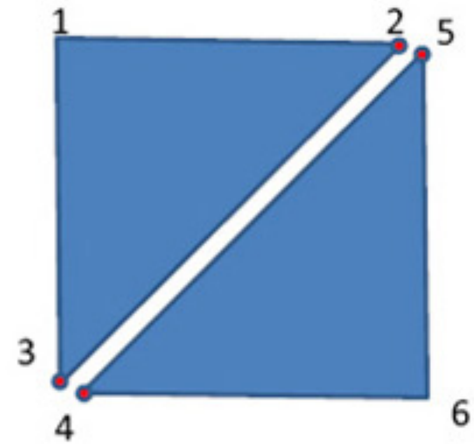
```
var s = 0.2;
var a = vec2(-s,-s);
var b = vec2(s, -s);
var c = vec2(s,s);
var d = vec2(-s,s);

var vertices = [a,b,c,d];
var indices = [0,1,2,0,2,3];

var buffer = gl.createBuffer();
gl.bindBuffer(gl.ARRAY_BUFFER, buffer);
gl.bufferData(gl.ARRAY_BUFFER, flatten(vertices), gl.STATIC_DRAW);

var vPosition = gl.getAttribLocation(program, "vPosition");
gl.vertexAttribPointer(vPosition, 2, gl.FLOAT, false, 0, 0);
gl.enableVertexAttribArray(vPosition);

// set up index buffer
var ibuffer = gl.createBuffer();
gl.bindBuffer(gl.ELEMENT_ARRAY_BUFFER, ibuffer);
gl.bufferData(gl.ELEMENT_ARRAY_BUFFER, new Uint8Array(indices), gl.STATIC_DRAW);

//Draw
gl.drawElements(gl.TRIANGLES, 6, gl.UNSIGNED_BYTE, 0);
```

Index Buffers

3D Cube



Each Cube Face



For the side above...
Vertex 3 and 4 are the same
Vertex 2 and 5 are the same

Operations on Points and Vectors

$$Point + Vector = Point$$

$$Vector + Point = Point$$

$$Vector + Vector = Vector$$

$$Point + Point : Undefined$$

$$Point - Point = Vector$$

$$Scalar \times Vector = Vector$$

$$Scalar \times Point : Undefined$$

Linear Combinations of Vectors

$\vec{v}_1, \vec{v}_2$: two vectors $\qquad$ α_1, α_2 : two scalars

Then $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2$ is a vector.

linear combination of $\vec{v}_1$ and $\vec{v}_2$

More generally,

Vectors: $\vec{v}_1, \cdots, \vec{v}_n$ $\qquad$ Scalars: $\alpha_1, \cdots, \alpha_n$

Then, $\alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n$ is a linear combination of $\vec{v}_1, \cdots, \vec{v}_n$.

Linear Dependence

Vectors: $\vec{v}_1, \cdots, \vec{v}_n$

We say that the vectors are **linearly dependent** if one of them can be written as a linear combination of the others.

Otherwise, they are **linearly independent**.

Example: Are the vectors $(2, 0, -1)$, $(1, -2, -3)$ and $(1.5, -5, -7)$ in 3D linearly dependent?

Yes! $(2, 0, -1) = 5 \times (1, -2, -3) - 2 \times (1.5, -5, -7)$

Some Linear Algebra

Linear Algebra Fact 1: $\leq d$ *linearly independent vectors in d-dimensional Euclidean space.*

$\Longrightarrow$ If we have d linearly independent vectors in d dimensions then any can other vector can be written as a linear combination of those.

Such a collection of vectors is called a *basis*.

Linear Algebra Fact 2: *Any basis of the d-dimensional Euclidean space has exactly d vectors.*

Coordinate Systems

Basis in two dimensional space:

any two linearly independent vectors

Any other vector in the plane can be written **uniquely** as: $\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2$.

WHY?

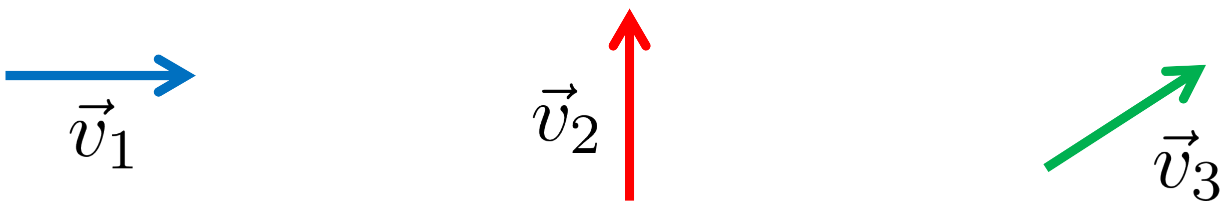
Once the basis is fixed, we can write this vector as: (α_1, α_2).

Typically basis vectors are chosen to be mutually perpendicular and unit length.

Such a basis is called an **Orthonormal** basis.

Coordinate Systems

Similarly, in three dimensions, any three vectors $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ which don't lie on the same plane can be used as a basis.



Any other vector $\vec{v}$ in three dimensions can be written **uniquely** as:

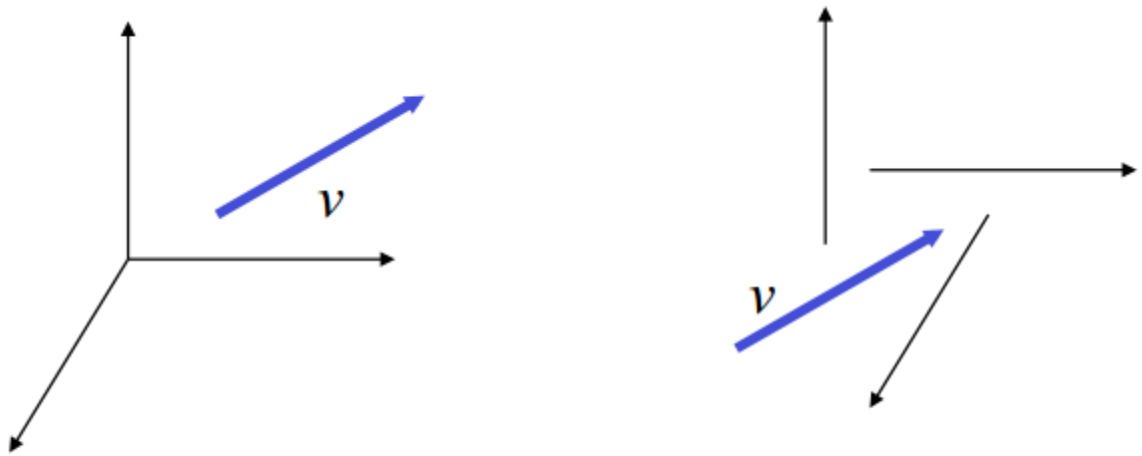
$$\vec{v} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3,$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.

Once the basis is fixed, we can represent $\vec{v}$ as $(\alpha_1, \alpha_2, \alpha_3)$.

But don't think of vectors as tuples of numbers!

Coordinate Systems and Frames



Typically we draw the basis vectors as emanating from a common point.

However vectors really don't have a location.

So, the picture on the right is equivalent to the one on the left.

Note: A coordinate system is insufficient for representing points.

To represent points, we need to add a reference point: the origin.

Coordinate Frame = Coordinate System + Origin

Coordinate Systems and Frames

Given a Coordinate Frame $(\vec{v}_1, \vec{v}_2, \vec{v}_3, P_0)$:

- Vectors are written as $\vec{v} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3$

- Points are written as $\vec{v} = \beta_1 \vec{v}_1 + \beta_2 \vec{v}_2 + \beta_3 \vec{v}_3 + P_0$

Operations on Points and Vectors

Let p and q be two points.

Then, $q - p$ is a vector and so is $\alpha(q - p)$.

Lets add this vector to p. So, $p + \alpha(q - p)$ is a point.

i.e., $(1 - \alpha)p + \alpha q$ is a point.

Here we seem to be violating the rules. We are multiplying points by scalars and adding them!

So, we need to extend our definition.

Affine Combinations of Points

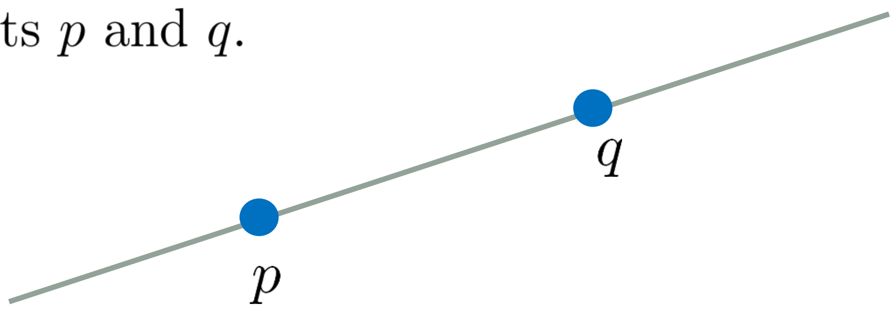
For any two points p and q and two scalars α_1 and α_2 s.t. $\alpha_1 + \alpha_2 = 1$, we call $\alpha_1 p + \alpha_2 q$, an **affine combination** of p and q.

The meaning of this affine combination is: $p + \alpha_2(q - p)$.

The affine combination of two points gives another point.

How can you describe the set of all affine combinations of p and q?

Ans: Its the line through the points p and q.



Affine Combinations of Points

Let p and q be two points and let $x = (1-\alpha)p + \alpha q$

Now lets take an affine combination of x and another point r.

Let $y = \beta x + (1-\beta)r \ = \beta(1-\alpha)p + \beta\alpha q + (1-\beta)r$.

What do the coefficients of p, q and r add up to? $\beta(1-\alpha) + \beta\alpha + \beta = 1$.

So, we can extend our definition to more than two points.

Given points $p_1, p_2, \cdots, p_n$, and scalars $\alpha_1, \cdots, \alpha_n$ s.t. $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, $\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n$ is an affine combination of the points $p_1, \cdots, p_n$.

Affine Combinations of Points

Another way to think about it:

Fix an arbitrary reference point o.

For any point p call the vector $\tilde{p} = p - o$ the **position vector** of p.

Now, if we have points $p_1, \cdots, p_n$, we can take linear combinations of their position vectors.

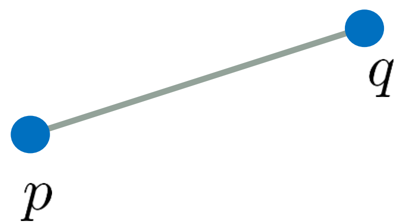
Let $q = \alpha_1(p_1 - o) + \cdots + \alpha_n(p_n - o)$.

$$= \alpha_1 p_1 + \cdots + \alpha_n p_n - (\alpha_1 + \cdots + \alpha_n)o.$$

q is the position vector of another point **iff** $\alpha_1 + \cdots + \alpha_n = 1$.

Convex Combinations of Points

Let p and q be two points.



Consider affine combinations $\alpha p + \beta q$ where $\alpha + \beta = 1$.

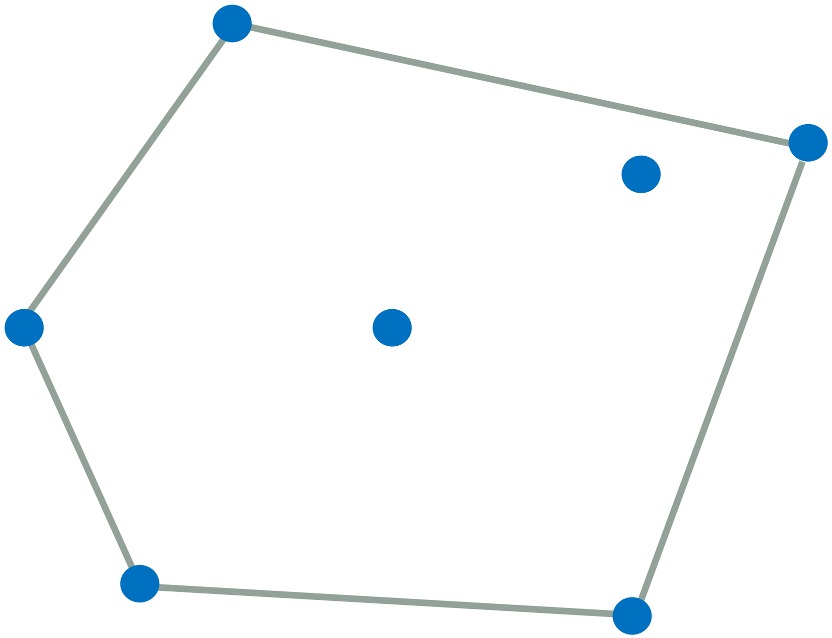
What is the set of affine combinations if we restict both α and β to be non-negative?

Such affine combinations are called **convex combinations**.

Convex Combinations of Points

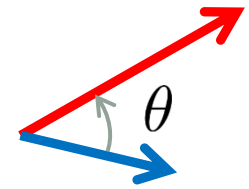
Points: $p_1, \cdots, p_n$ Scalars: $\alpha_1, \cdots, \alpha_n \geq 0$ s.t. $\sum_{i=1}^{n} \alpha_i = 1$

Then, $\alpha_1 p_1 + \cdots + \alpha_n p_n$ is a convex combination of the points.



The set of convex combinations of a set of points is its convex hull.

Dot Product of Vectors



$$\vec{v}_1 \cdot \vec{v}_2 := \ \|\vec{v}_1\| \ \|\vec{v}_2\| \ cos(\theta)$$

What is $\vec{v}_1 \cdot \vec{v}_1$? $\|\vec{v}_1\|^2$

If $\vec{v}_1$ and $\vec{v}_2$ are perpendicular to each other,

what is $\vec{v}_1 \cdot \vec{v}_2$? 0

Let $\vec{v}_1 = (3, 4)$ and $\vec{v}_2 = (2, -1)$ w.r.t. an orthonormal basis.
Calculate $\vec{v}_1 \cdot \vec{v}_2$.

Dot Product of Vectors

An easier way to calculate dot products:

Let $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ w.r.t. an orthonormal basis.

Then $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$.

$(3, 4) \cdot (2, -1) = 3 \times 2 + 4 \times (-1) = 2$

Why does it work?

Dot Product of Vectors

Let $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$

Then $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$

$$\vec{w} = \vec{v} - \vec{u} \quad = (v_1 - u_1, v_2 - u_2)$$

Let $a = \|\vec{u}\| = \sqrt{u_1^2 + u_2^2}$

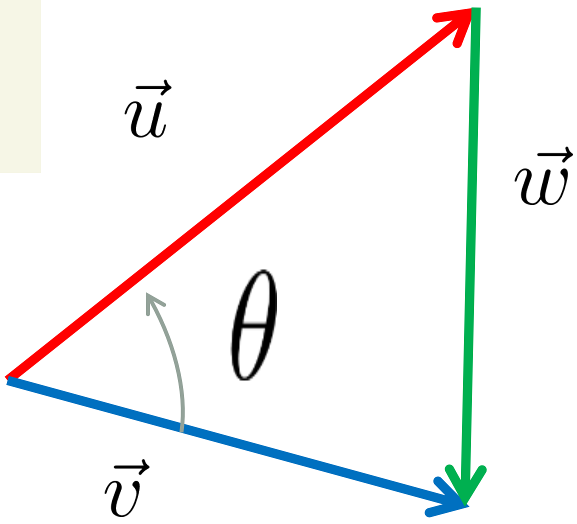
$$b = \|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$$

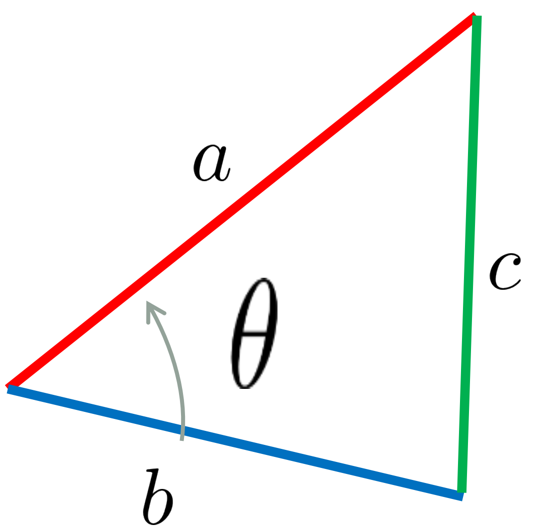
$$c = \|\vec{w}\| = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2}$$

Note: $\vec{u} \cdot \vec{v} = ab\ cos\theta.$

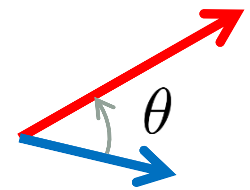
Law of Cosines: $c^2 = a^2 + b^2 - 2ab\ cos\theta.$

$$\Longrightarrow \quad \vec{u} \cdot \vec{v} = (a^2 + b^2 - c^2)/2.$$
$$= u_1 v_1 + u_2 v_2$$





Dot Product of Vectors



$$\vec{u} \cdot \vec{v} := \quad \|\vec{u}\| \; \|\vec{v}\| \; cos(\theta)$$

If $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$ in a coordinate system with an orthonormal basis, then $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$.

Note that the dot product is not dependent on the coordinate system.

Dot Product of Vectors

Easy exercise 1: $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

dot product distributes over addition

Easy exercise 2: $\vec{u} \cdot (\alpha \vec{v}) = \alpha \; \vec{u} \cdot \vec{v}, \quad$ where $\alpha \in \mathbb{R}$

We will mostly use the easier rule to calculate dot products.

But we should keep in mind the meaning.

The dot product is very useful in many calculations in graphics.

E.g. In computing the intensity of light falling on a surface.

Dot Product of Vectors in 3D

We described dot products in 2D but it works the same way in 3D.
(in fact, any dimension)

$$\vec{u} \cdot \vec{v} = \|u\|\|v\| cos\theta$$

where θ is the angle between the vectors

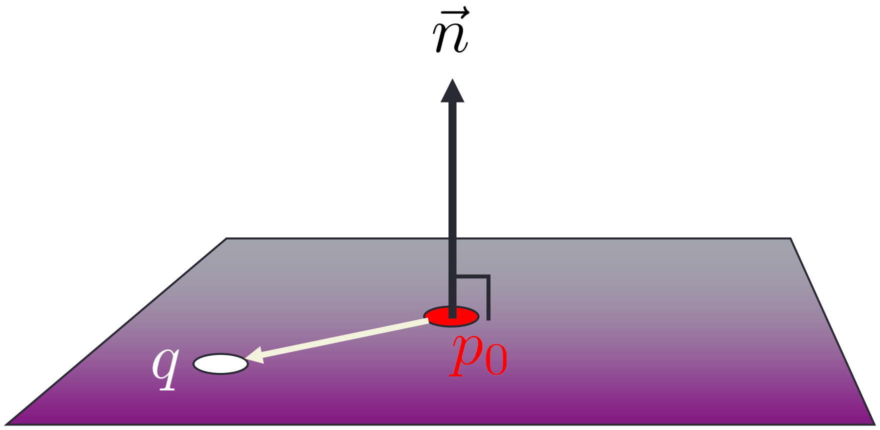
$$(u_1, u_2, u_3) \cdot (v_1, v_2, v_3) = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Example: Compute $(2, 7, -8) \cdot (-6, 4, 2)$

Planes

To define a plane in $3D$ we need a point p_0 on the plane and a vector $\vec{n}$ perpendicular to the plane.

$\vec{n}$ is called a normal vector for the plane.

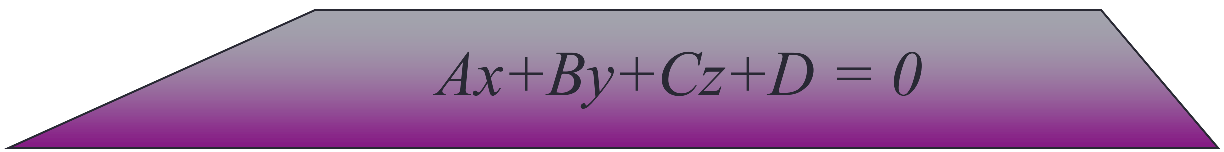


point q lies on the plane $\Leftrightarrow$ $(q - p_0) \cdot \vec{n} = 0$

Implicit equation of a plane in 3D

$$Ax + By + Cz + D = 0,\ A, B, C, D \in \mathbb{R},\ (A, B, C) \neq (0, 0, 0)$$

$Ax+By+Cz+D > 0$

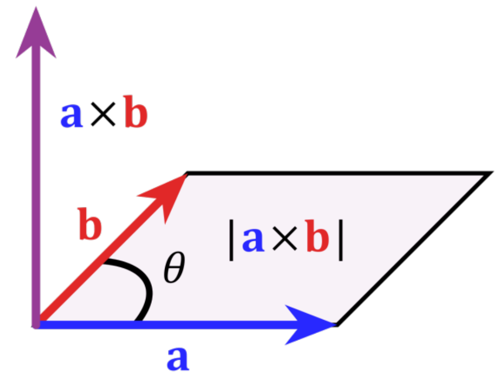


$Ax+By+Cz+D < 0$

(A, B, C) is a vector normal to the plane

Cross Product

$\vec{a}, \vec{b}$: two vectors in <u>*three*</u> dimensions



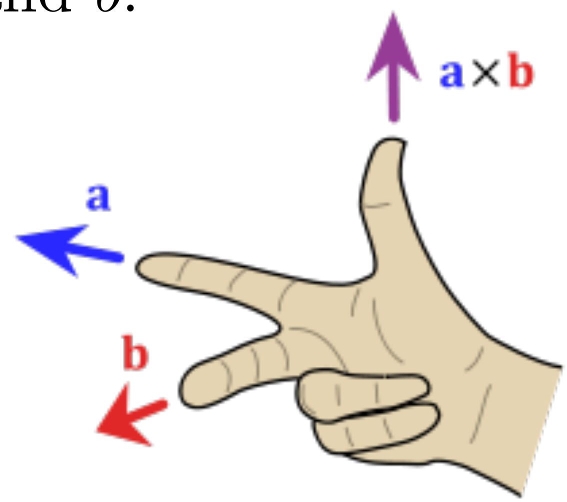
$\vec{a} \times \vec{b}$ is a **vector**.

- *magnitude:* $||\vec{a}||\ ||\vec{b}||\ sin\theta$

  (θ: angle between $\vec{a}$ and $\vec{b}$)

- *direction*: $\perp$ to the plane containing $\vec{a}$ and $\vec{b}$.

  given by the right hand rule



Note: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

cross product is anti-commutative

Properties:

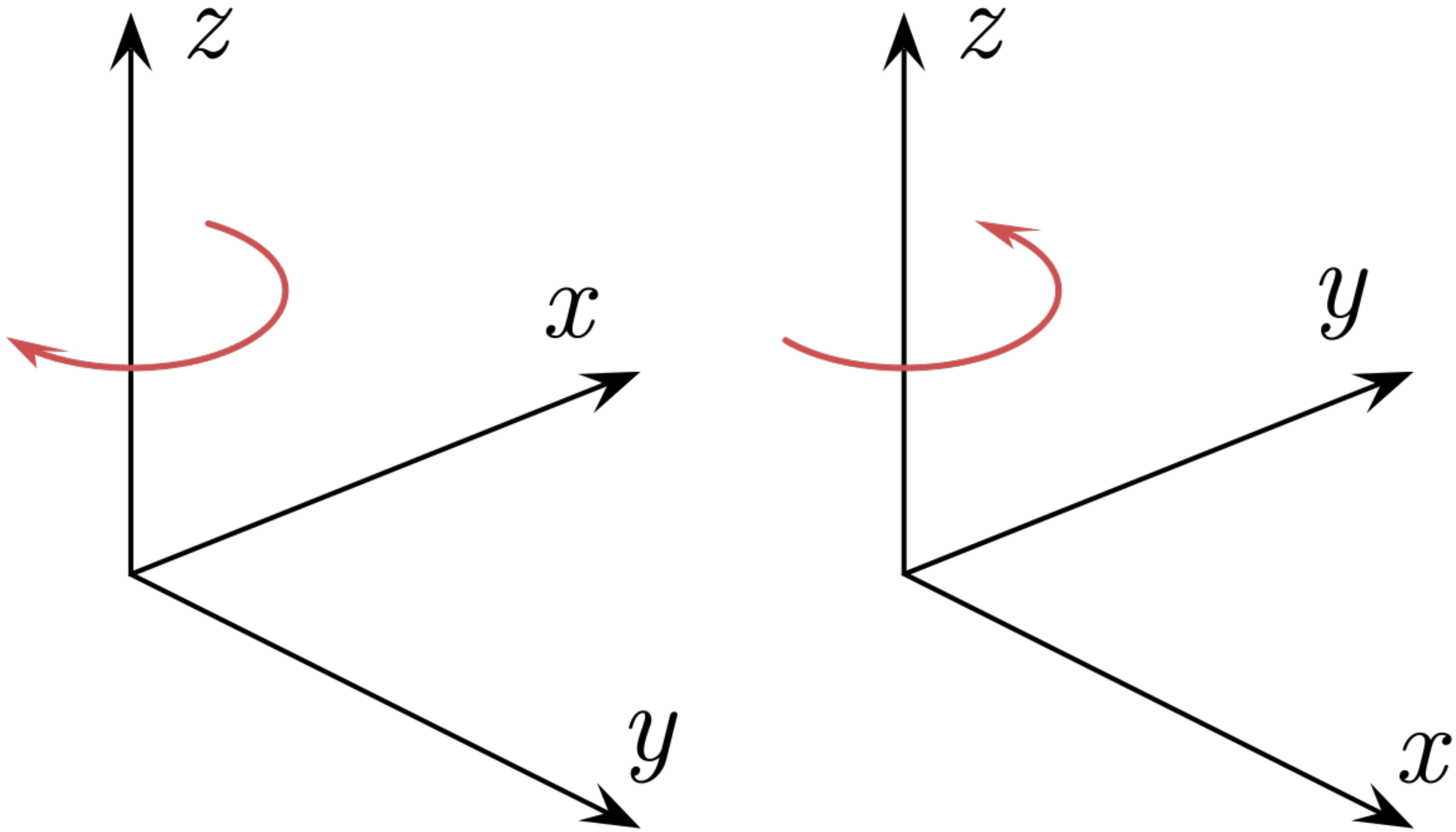
- $\vec{a} \times (\alpha\vec{b}) = \alpha(\vec{a} \times \vec{b})$ for any $\alpha \in \mathbb{R}$.
- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ *cross product distributes over addition*

  See `http://www.math.oregonstate.edu/bridge/papers/dot+cross.pdf`

Combined: $\vec{a} \times (\alpha\vec{b} + \vec{c}) = \alpha(\vec{a} \times \vec{b}) + \vec{a} \times \vec{c}$
for any $\alpha \in \mathbb{R}$.

Easy Exercise: Prove that the combined equation is equivalent to the two equations above.

Handedness of a Coordinate System

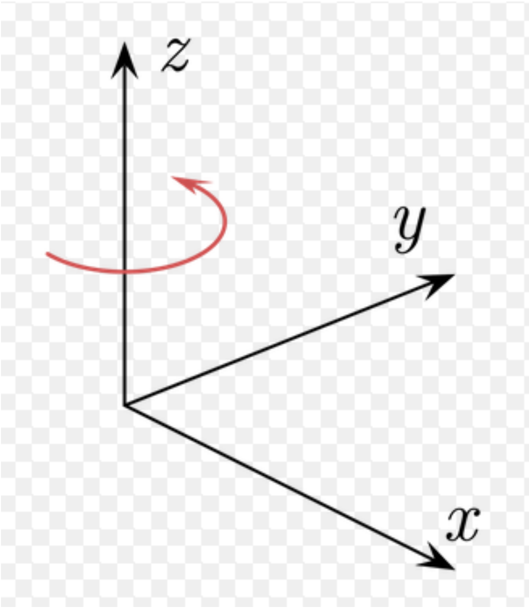


Left-Handed *Right-Handed*

We will use a **right-handed** coordinate system.

Cross Product

Suppose that we have a right-handed coordinate system.



$\hat{i}$: *unit vector in the positive x direction*

$\hat{j}$: *unit vector in the positive y direction*

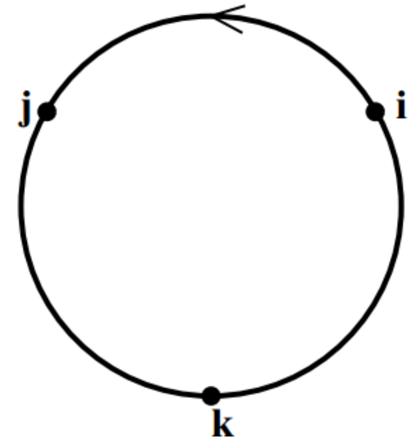
$\hat{k}$: *unit vector in the positive z direction*

Let $\vec{u} = (u_1, u_2, u_3)$. Then, $\vec{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$

$$\hat{i} \times \hat{j} = \hat{k} \qquad \hat{j} \times \hat{k} = \hat{i} \qquad \hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k} \qquad \hat{k} \times \hat{j} = -\hat{i} \qquad \hat{i} \times \hat{k} = -\hat{j}$$

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$



Cross Product

Let $\vec{u} = (u_1, u_2, u_3), \quad \vec{v} = (v_1, v_2, v_3)$.

$$
\begin{aligned}
\text{Then, } \vec{u} \times \vec{v} = \;& (u_1\hat{i} + u_2\hat{j} + u_3\hat{k}) \times (v_1\hat{i} + v_2\hat{j} + v_3\hat{k}) \\
=\;& (u_2v_3 - u_3v_2)\hat{i} + (u_3v_1 - u_1v_3)\hat{j} + (u_1v_2 - u_2v_1)\hat{k} \\
=\;& (u_2v_3 - u_3v_2,\; u_3v_1 - u_1v_3,\; u_1v_2 - u_2v_1)
\end{aligned}
$$

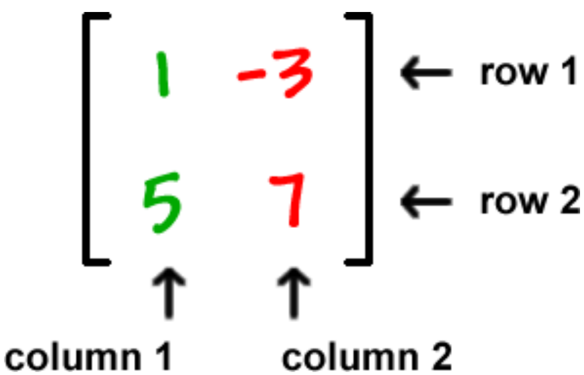
Easy exercise: Find a vector of length 1 that is perpendicular to both the vectors $(1, 1, 1)$ and $(-1, 0, 1)$.

Easy exercise: For vectors $\vec{u} = (2, -1, 7)$ and $v = (-2, 9, 3)$, compute $(\vec{u} + \vec{v}) \cdot (\vec{u} \times \vec{v})$.

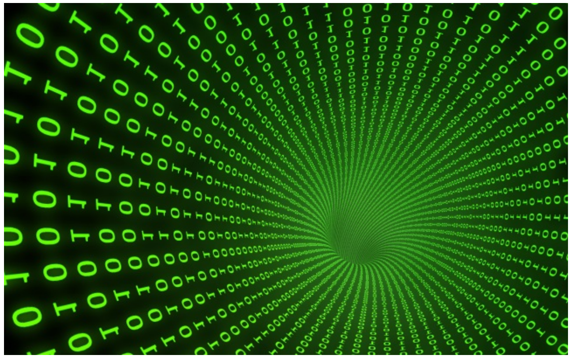
Matrices

A matrix is a two dimensional array of numbers.

Looks like this:



not like this:



We say that a matrix has dimensions $m \times n$ if it has m rows and n columns.

Given a matrix A, we refer to the entry in row i and column j as A_{ij}.

Matrix Multiplication

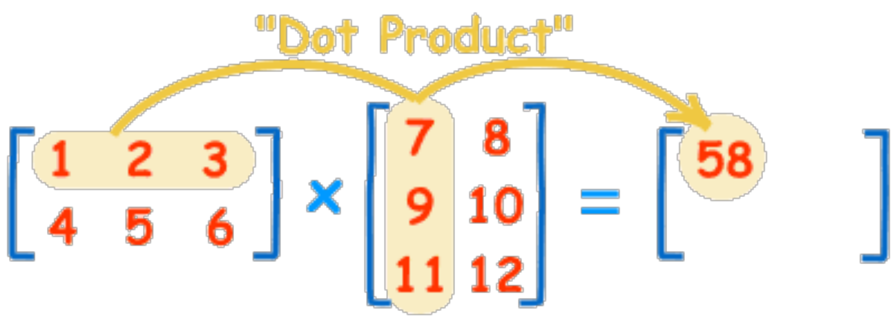
A : matrix of dimension $m \times n$. B : matrix of dimension $n \times p$.

Then we can define $C = A \times B$ as follows:

C has dimension $m \times p$.

$C_{ij} =$ the dot product of the i^{th} row of A and j^{th} column of B.

Example:



A	B	C
2×3	3×2	2×2

We generally skip the symbol '$\times$' and write AB for $A \times B$.

Matrix Multiplication

Matrix multiplication is associative: $A \times (B \times C) = (A \times B) \times C$

It is **not commutative**: $A \times B$ need not equal $B \times A$

Example: $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$A \times B = \begin{bmatrix} 1 & 3 \\ 1 & 0 \end{bmatrix} \qquad B \times A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

Identity Matrix

A square matrix A such that: $A_{ii} = 1$ for all i

all other entries are 0

1×1 identity matrix: $\begin{bmatrix} 1 \end{bmatrix}$

2×2 identity matrix: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

3×3 identity matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We denote an $n \times n$ identity matrix as I_n.

If M is any $n \times n$ matrix, then $I_n M = M I_n = M$

Matrix Inverse

Let A be an $n \times n$ matrix.

A is called **singular** if the columns of A are *linearly dependent* vectors.

Otherwise, we say that A is **non-singular**.

Linear Algebra fact: For any $n \times n$ non-singular matrix M, there exists a unique matrix N such that: $MN = NM = I_n$.

This matrix is denoted by M^{-1}.

Useful fact: $(AB)^{-1} = B^{-1}A^{-1}$ (assuming A and B are invertible)

Transpose of a matrix

Switches rows and columns.

$$\begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

Easy to prove but important fact: $(AB)^T = B^T A^T$