Q_1:

i. $f(x) = \sin(6x-1)$

Chain rule : $\frac{\partial}{\partial x} f(g(x)) = f'(g(x))\, g'(x)$

$= \cos(6x-1)\cdot(6) = 6\cos(6x-1) + c$

ii. $f(x) = x^8 + 30 + \frac{1}{x^4}$; $f(x) = x^8 + 30 + x^{-4}$

$f'(x) = \frac{\partial}{\partial x} x^8 + \frac{\partial}{\partial x} 30 + \frac{\partial}{\partial x} x^{-4}$

$f'(x) = 8x^7 + 0 + (-4x^{-5})$

$f'(x) = 8x^7 - \frac{4}{x^5}$

iii. $f(x) = e^{\frac{1}{x} + \frac{1}{x^2}}$

chain rule , $u = \frac{1}{x} + \frac{1}{x^2}$; $y = e^u$

$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u} e^u = e^u$

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{x} + \frac{1}{x^2}\right) = \frac{\partial}{\partial x}(x^{-1}) + \frac{\partial}{\partial x} x^{-2}$

$\frac{\partial u}{\partial x} = -1x^{-2} + (-2x^{-3}) = -\frac{1}{x^2} - \frac{2}{x^3}$

$\frac{\partial y}{\partial u} \times \frac{\partial u}{\partial x} = \left(e^{\frac{1}{x} + \frac{1}{x^2}}\right) \times \left(-\frac{1}{x^2} - \frac{2}{x^3}\right)$

$\frac{\partial y}{\partial y} = e^{\left(\frac{1}{x} + \frac{1}{x^2}\right)} \times \left(-\frac{1}{x^2} - \frac{2}{x^3}\right)$

$= -\frac{e^{\left(\frac{1}{x} + \frac{1}{x^2}\right)}}{x^2} - \frac{2e^{\left(\frac{1}{x} + \frac{1}{x^2}\right)}}{x^3}$

iv. $f(x) = \sin^2(6x-1)$

$$f(x) = [\sin(6x-1)]^2$$

$$f'(x) = 2[\sin(6x-1)] \cdot [\cos(6x-1) \cdot 6] = 12 \cdot \sin(6x-1)\cos(6x-1)$$

Q2: $f(x) = 2x^3 + 24x^2 - 54x$

f is decreasing: $f'(x) = 6x^2 + 48x - 54$

$f'(x) = 0$

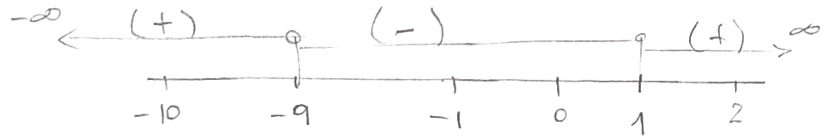
$$0 = 6(x^2 + 8x - 9)$$

$$0 = 6(x+9)(x-1)$$

$x+9=0, \quad x=-9$
$x-1=0 \quad x=1$ } Critical Numbers

Check when increase $f'(x) > 0$, or decrease $f'(x) < 0$

$f'(x) = 6(x+9)(x-1)$



$f'(-10) = 6(-10+9)(-10-1) = 6(-1)(-11) = 66$

$f'(-1) = 6(-1+9)(-1-1) = 6(8)(-2) = -96$

$f'(2) = 6(2+9)(2-1) = 6(11)(1) = 66$

$\therefore$ Answer 1:
increasing $(-\infty, -9) \cup (1, \infty)$
decreasing $(-9, 1)$

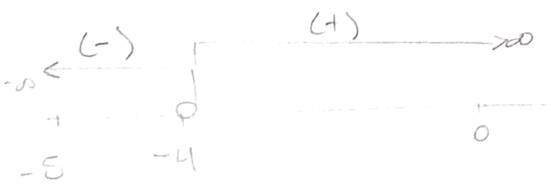
Check when is it concave up $f''(x) > 0$

$f''(x) = \frac{\partial}{\partial x}(6x^2 + 48x - 54) = 12x + 48$, set $f''(x) = 0$; $0 = 12x + 48$;

$0 = x + 4$; $x = -4$

$f''(-5) = 12(x+4) = 12(-5+4) = -12$

$f''(0) = 12(4) = 48$



Answer 2:
Concave up $(-4, \infty)$
Concave down $(-\infty, -4)$

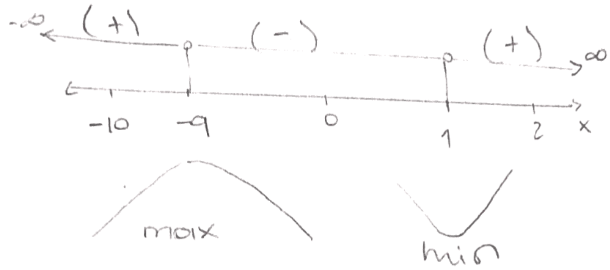
Q3. $f(x) = 2x^3 + 24x^2 - 54x$

$f'(x) = 6x^2 + 48x - 54$

$0 = 6[x^2 + 8x - 9]$

$0 = (x + 9)(x - 1)$

- Critical points $\begin{cases} x_1 = -9 \\ x_2 = 1 \end{cases}$



$f'(x) = 6(x+9)(x-1)$

- y_1 local max: $f(-9) = 2(-9)^3 + 24(-9^2) - 54(-9) = -1458 + 1944 + 486 = 972$

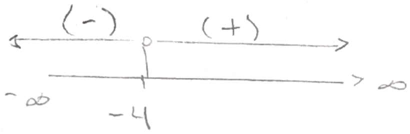
- y_2 local min $f(1) = 2(1)^3 + 24(1)^2 - 54(1) = -28$

Local max $(-9, 972)$; local min $(1, -28)$

- Points of inflection: $f''(x)$ switches signs

$f'(x) = 6x^2 + 48x - 54$

$f''(x) = 12x + 48$



$0 = 12x + 48$

$x = -4$

$f(-4) = 2(-4)^3 + 24(-4)^2 - 54(-4) = -128 + 384 + 216 = 472$

Point of inflection: $(-4, 472)$

- Global max/min $[-3, 3]$ $(-\infty, 0)$

. Before obtained critical points $(-9, 172)$; $(1, -28)$

* Interval $[-3, 3]$

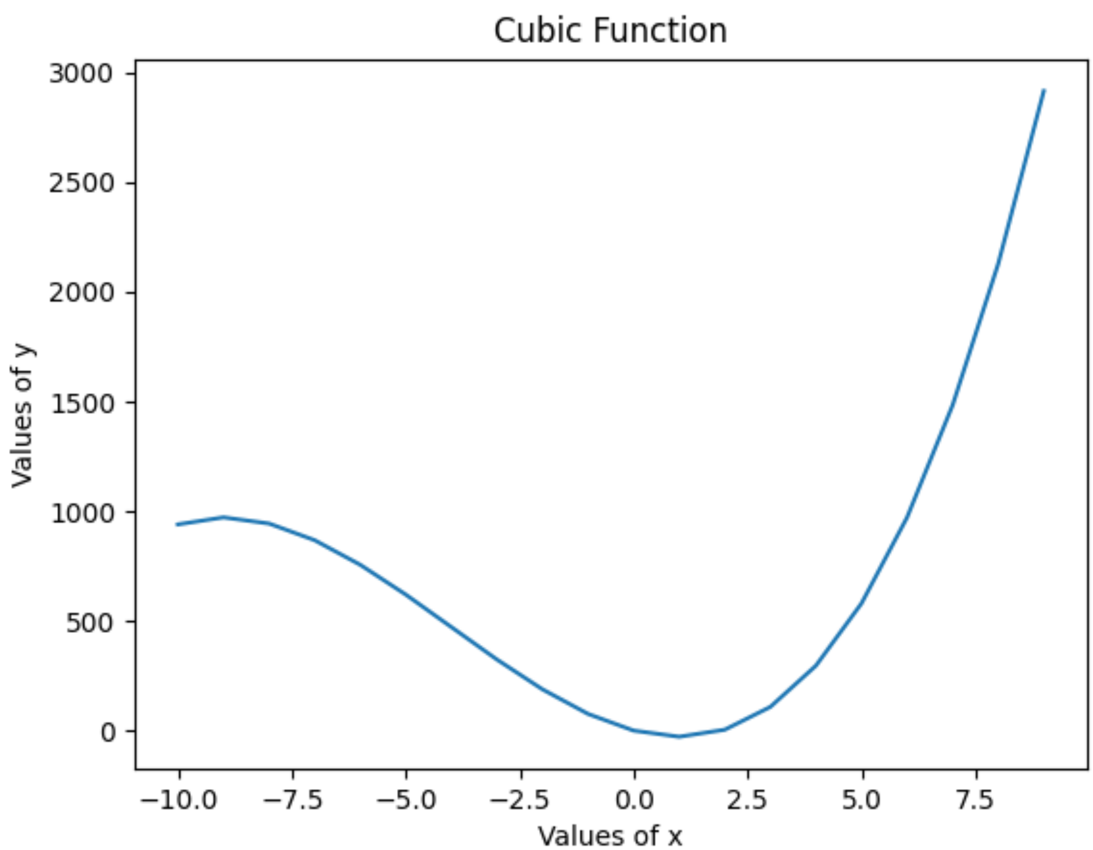
$f(3) = 2(27) + 24(9) - 54(3) = 54 + 216 - 162 = 108$

=> Global min: $(1, -28)$

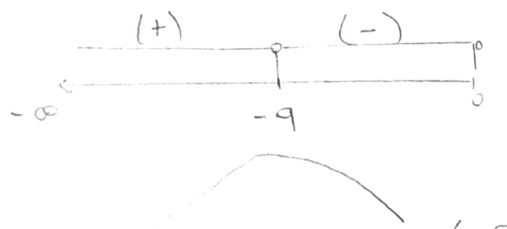
$f(1) = -28$

$f(-3) = 2(-3)^3 + 24(-3)^2 - 54(-3) = -54 + 216 + 162 = 324$ => Global max $(-3, 324)$

$f(-9)$: Not included on interval


Cubic Function
3000
2500
2000
1500
1000
500
0
Values of y
−10.0
−7.5
−5.0
−2.5
0.0
2.5
5.0
7.5
Values of x

* Interval $(-\infty, 0)$



$f(-9) = 972$

Global max = $(-9, 972)$

Q.4: $f(x,y) = x^2 + y^2$

- Gradient vector $\nabla f = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right)$

$\frac{\partial f}{\partial x} = 2x$

$\frac{\partial f}{\partial y} = 2y$

$\nabla f(x,y) = \langle 2x, 2y \rangle$

$\nabla f(1,2) = \langle 2, 4 \rangle$

$\nabla f(2,1) = \langle 4, 2 \rangle$

$\nabla f(0,0) = \langle 0, 0 \rangle$

Q5.

i) $m = 3$

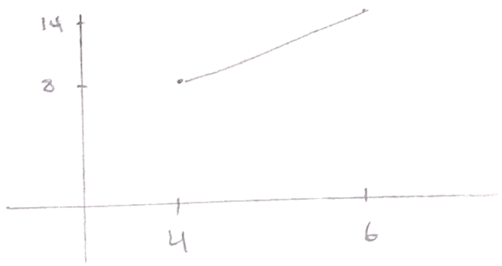
y-intercept = $(0, -0.5)$

$y = mx + b$

$y = 3x - \frac{1}{2}$

ii) $P_1\ (4, 8)$

$P_2\ (6, 14)$



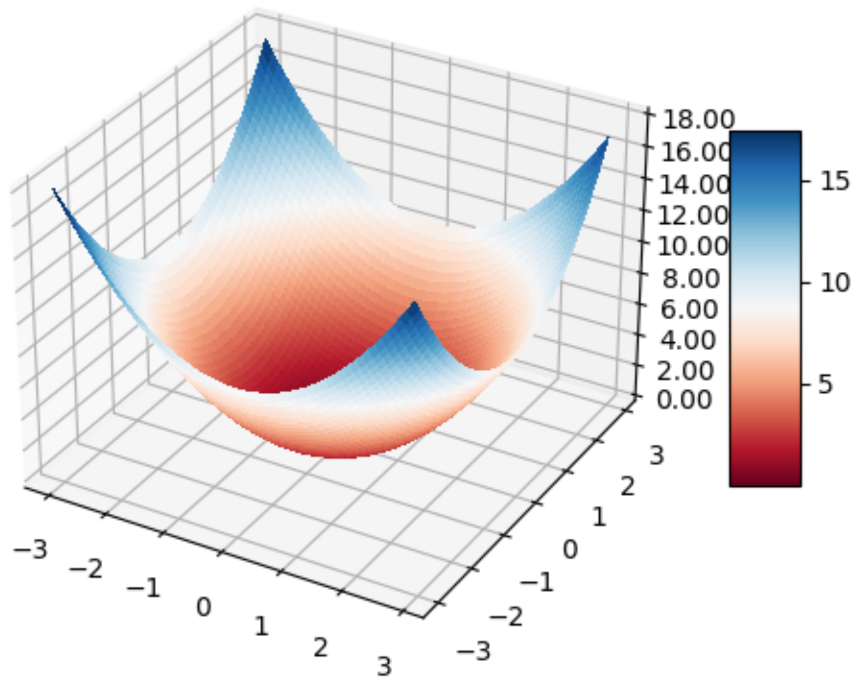
$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{14 - 8}{6 - 4} = \frac{6}{2} = 3$

$y = 3x + b$

$P_1)\ 8 = 3(4) + b \ ,\ b = -4$

$P_2)\ 14 = 3(6) + b \ ,\ b = -4$

$y = 3x - 4$



18.00
16.00
14.00
12.00
10.00
8.00
6.00
4.00
2.00
0.00
15
10
5
−3
−2
−1
0
1
2
3
−3
−2
−1
0
1
2
3

iii) $P_1(3,2) \rightarrow y_1 = m_1 x + b_1$

$y_2 = 5x+3 \rightarrow y_2 = 5x+3$

$m_2 = 5 \rightarrow m_1 = -\frac{1}{5}$

$y_1 = -\frac{1}{5}x + b_1 \,; \quad 2 = -\frac{1}{5}(3) + b_1 \,; \quad b_1 = 2 + \frac{3}{5} = \frac{13}{5}$

$y_1 = -\frac{1}{5}x + \frac{13}{5}$

iv) $b=3$

$P(2,1)$

$y = mx + 3$

$1 = m(2) + 3$

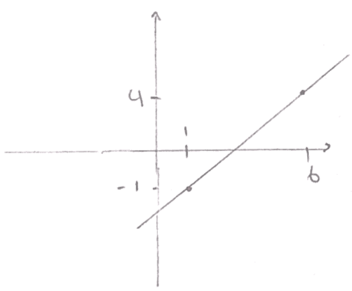
$-2 = 2m$

$m = -1$

$y = -x + 3$

v) $P_1(6,4)$

$P_2(1,-1)$



$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-4}{1-6} = \frac{-5}{-5} = 1$

$P_1(6,4)$.

$4 = 1(6) + b$

$b = -2$

$y = x - 2$

E. 1.

$$y_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad y_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad y_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Basis vectors -> independent
- Set $\{y_1, y_2, y_3\}$ convert $\{v_1, v_2, v_3\}$

$$y_1 = v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_2 = y_2 - a v_1$$

$$a = \frac{(v_1, y_2)}{(v_1, v_1)} ; \quad v_1 \cdot y_2 = v_1^T y_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \qquad \therefore \; a = \frac{1}{1} = 1$$

$$v_1 \; v_1 = v_1^T v_1 = \|v_1\|^2 = 1^2 + 0^2 + 0^2 = 1$$

$$v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$v_3 = y_3 - b v_2$$

$$b = \frac{(v_2, y_3)}{(v_2, v_2)} ; \quad v_2 \; y_3 = v_2^T y_3 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \qquad \therefore \; b = \frac{1}{1} = 1$$

$$v_2 \; v_2 = v_2^T v_2 = \|v_2\|^2 = 0^2 + 1^2 + 0^2 = 1$$

$$v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

New set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

E.2. $x = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}^T$

$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ Basis vectors

1 Check for orthogonality if $(x, y) = 0$

$v_1 \cdot v_2 = v_1^T v_2 = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = (-1) + 1 + 0 = 0$

$v_1 v_3 = v_1^T v_3 = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = -1 + 1 + 0 = 0$

$v_2 v_3 = v_2^T v_3 = \begin{bmatrix} 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 + 1 + 0 = 2$

2 Obtain reciprocal basis vectors

$$R^T = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & -2 & 0 \end{bmatrix}^{-1} = \left[\begin{array}{ccc|ccc} -1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_1 + r_2} \left[\begin{array}{ccc|ccc} -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_2 + r_3}$$

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 \end{array}\right] \xrightarrow{r_2 - r_3} \left[\begin{array}{ccc|ccc} -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 1 & 1 & 1 \end{array}\right] \xrightarrow{2r_1 - r_3} \left[\begin{array}{ccc|ccc} -2 & 2 & 0 & 1 & -1 & -1 \\ 0 & 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{r_1 - r_2} \left[\begin{array}{ccc|ccc} -2 & 0 & 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 1 & 1 & 1 \end{array}\right] \xrightarrow{(-1)r_1} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 1 & 1 & 1 \end{array}\right] \times \frac{1}{2} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$R^T = \begin{bmatrix} -1/2 & 1/2 & 0 \\ 0 & 0 & -1/2 \\ 1/2 & 1/2 & 1/2 \end{bmatrix}$ $r_1 = \begin{bmatrix} -0{,}5 \\ 0 \\ 0{,}5 \end{bmatrix}$, $r_2 = \begin{bmatrix} 0{,}5 \\ 0 \\ 0{,}5 \end{bmatrix}$, $r_3 = \begin{bmatrix} 0 \\ -0{,}5 \\ 0{,}5 \end{bmatrix}$

$x_1 = r_1^T x = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = -\frac{1}{2} + 0 + 1 = \frac{1}{2}$

$x_2 = r_2^T x = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{2} + 0 + 1 = \frac{3}{2}$

$x_3 = r_3^T x = \begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 0 - 1 + 1 = 0$

$$x = \frac{1}{2} v_1 + \frac{3}{2} v_2 + 0 v_3$$

E.B: $f_1(t) = 1 \quad 0 \leq t \leq 1$

$f_2(t) = \begin{cases} 1 & 0 \leq t \leq 0{,}25 \\ -1 & 0{,}25 \leq t \leq 1 \end{cases}$

$f_3(t) = \begin{cases} 1 & 0 \leq t \leq 0{,}75 \\ -1 & 0{,}75 \leq t \leq 1 \end{cases}$

i) $a_1 f_1 + a_2 f_2 + a_3 f_3 = 0$

$a_1 = a_2 = a_3 = 0$

ii) $v_1 = f_1$

$v_2 = f_2 + a y_1$; $\quad a = \frac{(v_1, f_2)}{v_1, v_1} = (v_1, f_2) = \int_0^1 f_1(t)\, f_2(t)\, dt = \int_0^1 f_2(t) dt = \int_0^{0{,}25} dt - \int_{0{,}25}^1 dt$

$a = \Big|_0^{0{,}25} t - \Big|_{0{,}25}^1 t = 0{,}25 - 0 - (1 - 0{,}25) = 0{,}25 - 1 + 0{,}25 = -0{,}5$

$v_2 = f_2 - 0{,}5 f_1 = \begin{cases} 0{,}5 & 0 \leq t \leq 0{,}25 \\ -1{,}5 & 0{,}25 \leq t \leq 1 \end{cases}$

$v_3 = v_2 - b\cancel{y_2} = v_2 - b f_2$, $\quad b = \frac{(v_2, f_3)}{(v_2, v_2)} = (v_2, f_3) = \int_0^1 v_2\, f_3\, dt = \int_0^{0{,}25} 0{,}5 dt + \int_{0{,}25}^1 1{,}5 dt$

$b = \Big|_0^{0{,}25} \frac{1}{2} t + \Big|_{0{,}25}^1 \frac{3}{2} t = \left(\frac{1}{2} \cdot \frac{1}{4}\right) + \frac{3}{2} - \left(\frac{3}{2} \cdot \frac{1}{4}\right) = \frac{1}{8} + \frac{3}{2} - \frac{3}{8} = \frac{1 + 12 - 3}{8}$

$= \frac{10}{8} = \frac{5}{4}$

$v_3 = v_2 - \frac{5}{4} f_2 = \begin{cases} 0{,}5 - \frac{5}{4} & 0 \leq t \leq 0{,}25 \\ -1{,}5 + \frac{5}{4} & 0{,}25 \leq t \leq 1 \end{cases}$

orthogonal $= \left\{ 1 \quad 0 \leq t \leq 1 \; ; \; \begin{cases} 0{,}5 & 0 \leq t \leq 0{,}25 \\ -1{,}5 & 0{,}25 \leq t \leq 1 \end{cases} ; \; \begin{cases} 0{,}5 - \frac{5}{4} & 0 \leq t \leq 0{,}25 \\ -1{,}5 + \frac{5}{4} & 0{,}25 \leq t \leq 1 \end{cases} \right\}$

E.4 $\|x - ay\| = 0$; $x - ay = 0$

$x = ay$, $a = \frac{x}{y}$

Orthogonality $(z, y) = 0$

$$z = x - ay = x - \frac{x}{\cancel{y}}\cancel{y} = 0$$

$$(0, y) = 0$$

$$\|x - ay\|^2 + \|ay\|^2 = \|x\|^2$$

$$\|x - x\|^2 + \|x\|^2 = \|x\|^2$$

$$0 + \|x\|^2 = \|x\|^2$$

E.5 i) $f(0)=0$

set $X=\{x, 2x, 3x, \frac{x}{2}\}$; X vector space if satisfy conditions

1) $x \in X$, $y \in X$, $x+y \in X$

$x = 2x$ $\quad x+y = 5x \in X$

$y = 3x$

2) $x+y = y+x$

$2x+3x = 3x+2x$

$5x = 5x$

3) $(x+y)+z = x+(y+z)$; $z=\frac{x}{2}$

$(2x+3x)+\frac{x}{2} = \frac{x}{2}+(2x+3x)$

$\frac{11}{2}x = \frac{11}{2}x$

4) $0 \in X$ $\quad X=\{x+0, 2x+0, 3x+0, \frac{x}{2}+0\} = \{x, 2x, 3x, \frac{x}{2}\}$

5) $-x \in X$ $\quad X=\{x-x, 2x-2x, 3x-3x, \frac{x}{2}-\frac{x}{2}\} = \{0, 0, 0, 0\}$

6) $x = 3x \in X$, $a=\frac{1}{3}$; $ax = \frac{1}{3} \times 3x = x \in X$

7) $X=\{x \cdot 1, 2x \cdot 1, 3x \cdot 1, \frac{1}{2}x \cdot 1\} = \{x, 2x, 3x, \frac{1}{2}x\}$

8) $a=2$

$b=3$

$a(bx) = (ab)x$

$2(3(3x)) = (2 \cdot 3)(3x)$

$18x = 18x$

9) $(a+b)x = ax+bx$

$(2+3)(3x) = 2(3x)+3(3x)$

$15x = 15x$

10) $a(x+y) = ax+ay$, $2(x+2x) = 2x+4x = 6x$

ii) Set 2x2 matrices

Set $\left\{ \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right\}$

1) Sum of two 2x2 matrces is still a 2x2 matrix

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

2) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

3) $\left(\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) + \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} + \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right)$

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

4) By choosing $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $\left\{ \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} + 0, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + 0, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + 0 \right\}$

$$= \left\{ \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \right\}$$

5) $x = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $-x = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$, $x + (-x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

6) $a = 2$ $\quad ax = 2 \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \in X$

7) $a = 1$ $\quad ax = 1 \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in X$

8) $a = 2$
$b = 1$

$a(bx) = 2 \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$(ab)x = 2 \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

9) $(a+b)x = ax + bx = 3 \times \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

10) $a(x+y) = ax + ay$ $\quad 2\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right) = 2\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$$\begin{bmatrix} 4 & 2 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 6 & 4 \end{bmatrix}$$

E.6: Check for linear independence

i. $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}$ $x = \{x_1, x_2, x_3\}$

$$a_1x_1 + a_2x_2 + a_3x_3 = 0$$

$$a_1\begin{bmatrix}1\\2\\3\end{bmatrix} + a_2\begin{bmatrix}1\\0\\1\end{bmatrix} + a_3\begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}; \begin{bmatrix}a_1\\2a_1\\3a_1\end{bmatrix} + \begin{bmatrix}a_2\\0\\a_2\end{bmatrix} + \begin{bmatrix}a_3\\2a_3\\a_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$a_1 + a_2 + a_3 = 0$$
$$2a_1 + 2a_3 = 0$$
$$3a_1 + a_2 + a_3 = 0$$

1) $2a_1 + 2a_3 = 0, \quad a_1 = -a_3$

2) $a_1 + a_2 + a_3 = 0; \quad -a_3 + a_2 + \cancel{a_3} = 0; \quad a_2 = 0$

3) $3a_1 + a_2 + a_3 = 0, \quad 3(-a_3) + 0 + a_3 = 0, \quad -2a_3 = 0; \quad a_3 = 0 \quad \therefore a_1, a_2, a_3 = 0$

$$\text{Span}(x_1, x_2, x_3) = a_1x + a_2x + a_3x = \begin{bmatrix}m\\n\\o\end{bmatrix}$$

$$a_1\begin{bmatrix}1\\2\\3\end{bmatrix} + a_2\begin{bmatrix}1\\0\\1\end{bmatrix} + a_3\begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}a_1\\2a_1\\3a_1\end{bmatrix} + \begin{bmatrix}a_2\\0\\a_2\end{bmatrix} + \begin{bmatrix}a_3\\2a_3\\a_3\end{bmatrix} = \begin{bmatrix}m\\n\\0\end{bmatrix}$$

$$\begin{bmatrix}m\\n\\0\end{bmatrix} = \begin{bmatrix}a_1 + a_2 + a_3\\2a_1 + 2a_3\\3a_1 + a_2 + a_3\end{bmatrix}$$

Dimension 3: $x = \{a_1x_1 + a_2x_2 + a_3x_3\}$

ii) $\sin(t), \cos(t), \cos(2t)$

Linear independant ?

$$\cos(2t) = \cos^2 t - \sin^2 t$$
$$= 1 - 2\sin^2 t$$
$$= 2\cos^2 t - 1$$

No linear combination

$\sin(t) \neq a\cos(t)$ No linear combination

$\therefore$ Linear Independent

$x = a_1 \sin(t) + a_2 \cos(t) + a_3(\cos(2t)) \rightarrow$ Span

Dimension = 3 $\quad x = \{a_1 \sin(t) + a_2 \cos(t) + a_3 \cos(2t)$

iii) $\{1+t, 1-t\}$

Linear independent? Yes..

Span $\Rightarrow$ $x = \{a_1(1+t) + a_2(1-t)\}$ Dimension = 2

For example, $x = 5-t = 3(1+t) + 2(1-t) = 3+3t+2-2t = 5-t$

iii) $\begin{bmatrix}1\\2\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}3\\4\\4\\3\end{bmatrix}$ $\quad C_1X_1 + C_2X_2 + C_3X_3 = 0$

$$C_1\begin{bmatrix}1\\2\\2\\1\end{bmatrix} + C_2\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + C_3\begin{bmatrix}3\\4\\4\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$C_1 + C_2 - 3C_3 = 0$

$2C_1 + \quad +4C_2 = 0$

$2C_1 + \quad +4C_3 = 0$

$C_1 + C_2 + 3C_3 = 0$

1) $2C_1 = -4C_3$

$C_1 = -2C_3$

2) $C_1 + C_2 + 3C_3 = 0$

$-2C_3 + C_2 + 3C_3 = 0$

$C_3 + C_2 = 0$

$C_3 = -C_2$

3) $C_1 = -2C_3$

$C_1 = -2C_2$

4) $C_1 + C_2 + 3C_3 = 0$

$-2C_2 + C_2 - \cancel{C_2} = 0$

$-2C_2 = 0$

$C_2 = 0, C_3 = 0, C_1 = 0$ $\therefore$ Linear independant

$$\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = C_1\begin{bmatrix}1\\2\\2\\1\end{bmatrix} + C_2\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + C_3\begin{bmatrix}3\\4\\4\\3\end{bmatrix} = \begin{bmatrix}C_1 + C_2 + 3C_3\\2C_1 + 4C_3\\2C_1 + 4C_3\\C_1 + C_2 + 3C_3\end{bmatrix}$$

Dimension = 3