Exact Solution of the Younan-Veletsos Problem

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This notebook details and displays the results contained in “Younan-Veletsos retaining wall: The exact solution”. Here one can find some steps of the algebra leading to the exact solution, as well as the data used to generate the plots in the body of the manuscript. References to the actual text are included.

Details on derivations

The transform of the load

It follows, for display purposes, the transform of the sign function

In[1]:=
```
FourierTransform[2 * HeavisideTheta[ξ] - 1, ξ, k]
```

Out[1]=
$$\frac{i\sqrt{\frac{2}{\pi}}}{k}$$

The matrix ***D*** and the corresponding eigensystem

The matrix D appears in eq.(C.20) and its actual expression is given in (18):

In[2]:=
$$Dm := \left\{\left\{0,\ c^2 * k^2 - r^2,\ i * k * \left(1 - c^2\right),\ 0\right\},\ \{1, 0, 0, 0\},\ \left\{i * k * \frac{\left(1 - c^2\right)}{c^2},\ 0,\ 0,\ \frac{\left(k^2 - r^2\right)}{c^2}\right\},\ \{0, 0, 1, 0\}\right\}$$

In[3]:= `Eigensystem[Dm]`

Out[3]=
$$\left\{\left\{-\sqrt{k^2-r^2},\ \sqrt{k^2-r^2},\ -\frac{\sqrt{c^2\left(c^2 k^2-r^2\right)}}{c^2},\ \frac{\sqrt{c^2\left(c^2 k^2-r^2\right)}}{c^2}\right\},\right.$$
$$\left\{\left\{\frac{\mathbb{i}\left(k^2-r^2\right)}{k},\ -\frac{\mathbb{i}\sqrt{k^2-r^2}}{k},\ -\sqrt{k^2-r^2},\ 1\right\},\ \left\{\frac{\mathbb{i}\left(k^2-r^2\right)}{k},\ \frac{\mathbb{i}\sqrt{k^2-r^2}}{k},\ \sqrt{k^2-r^2},\ 1\right\},\right.$$
$$\left\{\mathbb{i}\,k,\ -\frac{\mathbb{i}\,k\sqrt{c^2\left(c^2 k^2-r^2\right)}}{c^2 k^2-r^2},\ -\frac{\sqrt{c^2\left(c^2 k^2-r^2\right)}}{c^2},\ 1\right\},$$
$$\left.\left.\left\{\mathbb{i}\,k,\ \frac{\mathbb{i}\,k\sqrt{c^2\left(c^2 k^2-r^2\right)}}{c^2 k^2-r^2},\ \frac{\sqrt{c^2\left(c^2 k^2-r^2\right)}}{c^2},\ 1\right\}\right\}\right\}$$

■ Solving for the constants

Next, it follows the system of equations displayed in eq.(27), and the explicit solution of it, eqs.(28) and eqs.(29):

In[4]:=
$$\mathbf{A = MatrixForm}\left[\left\{\{1,1,1,1\},\ \left\{-\frac{\mathbb{i}*\alpha}{k},\ \frac{\mathbb{i}*\alpha}{k},\ -\frac{\mathbb{i}*k}{\beta},\ \frac{\mathbb{i}*k}{\beta}\right\},\right.\right.$$
$$\left\{\left(k+\frac{\alpha^2}{k}\right)*\mathrm{Exp}[-\alpha],\ \left(k+\frac{\alpha^2}{k}\right)*\mathrm{Exp}[\alpha],\ 2*k*\mathrm{Exp}[-\beta],\ 2*k*\mathrm{Exp}[\beta]\right\},$$
$$\left.\left.\left\{-2\,\alpha*\mathrm{Exp}[-\alpha],\ 2\,\alpha*\mathrm{Exp}[\alpha],\ -\left(\frac{c^2\left(\beta^2-k^2\right)+2\,k^2}{\beta}\right)\mathrm{Exp}[-\beta],\ \left(\frac{c^2\left(\beta^2-k^2\right)+2\,k^2}{\beta}\right)\mathrm{Exp}[\beta]\right\}\right\}\right]$$

Out[4]//MatrixForm=
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -\frac{\mathbb{i}\,\alpha}{k} & \frac{\mathbb{i}\,\alpha}{k} & -\frac{\mathbb{i}\,k}{\beta} & \frac{\mathbb{i}\,k}{\beta} \\ \mathbb{e}^{-\alpha}\left(k+\frac{\alpha^2}{k}\right) & \mathbb{e}^{\alpha}\left(k+\frac{\alpha^2}{k}\right) & 2\,\mathbb{e}^{-\beta}\,k & 2\,\mathbb{e}^{\beta}\,k \\ -2\,\mathbb{e}^{-\alpha}\,\alpha & 2\,\mathbb{e}^{\alpha}\,\alpha & -\frac{\mathbb{e}^{-\beta}\left(2\,k^2+c^2\left(-k^2+\beta^2\right)\right)}{\beta} & \frac{\mathbb{e}^{\beta}\left(2\,k^2+c^2\left(-k^2+\beta^2\right)\right)}{\beta} \end{pmatrix}$$

In[5]:= `Inverse[A]`

Out[5]=
$$\mathrm{Inverse}\left[\begin{pmatrix} 1 & 1 & 1 & 1 \\ -\frac{\mathbb{i}\,\alpha}{k} & \frac{\mathbb{i}\,\alpha}{k} & -\frac{\mathbb{i}\,k}{\beta} & \frac{\mathbb{i}\,k}{\beta} \\ \mathbb{e}^{-\alpha}\left(k+\frac{\alpha^2}{k}\right) & \mathbb{e}^{\alpha}\left(k+\frac{\alpha^2}{k}\right) & 2\,\mathbb{e}^{-\beta}\,k & 2\,\mathbb{e}^{\beta}\,k \\ -2\,\mathbb{e}^{-\alpha}\,\alpha & 2\,\mathbb{e}^{\alpha}\,\alpha & -\frac{\mathbb{e}^{-\beta}\left(2\,k^2+c^2\left(-k^2+\beta^2\right)\right)}{\beta} & \frac{\mathbb{e}^{\beta}\left(2\,k^2+c^2\left(-k^2+\beta^2\right)\right)}{\beta} \end{pmatrix}\right]$$

In[6]:=

In[7]:=
```
FullSimplify[Inverse[{{1, 1, 1, 1}, {-(i*α)/k, (i*α)/k, -(i*k)/β, (i*k)/β},
    {(k + α^2/k)*Exp[-α], (k + α^2/k)*Exp[α], 2*k*Exp[-β], 2*k*Exp[β]}, {-2α*Exp[-α],
    2α*Exp[α], -((c^2 (β^2 - k^2) + 2 k^2)/β) Exp[-β], ((c^2 (β^2 - k^2) + 2 k^2)/β) Exp[β]}}].
  {0, Sqrt[2/π] i/(k*c^2*β^2), 0, -Sqrt[2/π] (c^2 - 2)/(c^2*β^2)}]
```

Out[7]=

$$\Big\{\Big((-2+c^2)(2+e^{\alpha})(-1+e^{\beta})^2 k^4 - c^2 e^{\alpha}(1+e^{2\beta})\alpha^2\beta^2 + k^2\big(-2(-2+c^2)e^{\alpha+\beta}\alpha^2 - 2(-2+c^2)\alpha\beta + 2(-2+c^2)e^{2\beta}\alpha\beta + 4c^2 e^{\beta}\beta^2 + e^{\alpha}((-2+c^2)\alpha^2 - 4\alpha\beta - c^2\beta^2) + e^{\alpha+2\beta}((-2+c^2)\alpha^2 + 4\alpha\beta - c^2\beta^2)\big)\Big) \Big/ \Big(c^2\sqrt{2\pi}\,\beta\big((1+e^{2\beta})\alpha\beta(-(-6+c^2)k^4 - (-2+c^2)k^2\alpha^2 + c^2(k^2+\alpha^2)\beta^2)\,\mathrm{Cosh}[\alpha] + 2e^{\beta}k^2(-2\alpha\beta(-(-3+c^2)k^2+\alpha^2+c^2\beta^2) + ((-2+c^2)k^2(k^2+\alpha^2) - (c^2k^2+(4+c^2)\alpha^2)\beta^2)\,\mathrm{Sinh}[\alpha]\,\mathrm{Sinh}[\beta])\big)\Big),$$

$$\Big(e^{-\alpha-\beta}\big(-(-2+c^2)(1+2e^{\alpha})(-1+e^{\beta})^2k^4 + c^2(1+e^{2\beta})\alpha^2\beta^2 + k^2(-(-2+c^2)(-1+e^{\beta})^2\alpha^2 + 2(2+(-2+c^2)e^{\alpha})(-1+e^{2\beta})\alpha\beta + c^2(1+e^{2\beta}-4e^{\alpha+\beta})\beta^2)\big)\Big) \Big/ \Big(2c^2\sqrt{2\pi}\,\beta\big(\alpha\beta(-(-6+c^2)k^4 - (-2+c^2)k^2\alpha^2 + c^2(k^2+\alpha^2)\beta^2)\,\mathrm{Cosh}[\alpha]\,\mathrm{Cosh}[\beta] + k^2(-2\alpha\beta(-(-3+c^2)k^2+\alpha^2+c^2\beta^2) + ((-2+c^2)k^2(k^2+\alpha^2) - (c^2k^2+(4+c^2)\alpha^2)\beta^2)\,\mathrm{Sinh}[\alpha]\,\mathrm{Sinh}[\beta])\big)\Big),$$

$$\Big(2\alpha((1-(-2+c^2)e^{\beta})k^2+\alpha^2)\beta + \alpha(-4e^{\beta}k^2 + (-2+c^2)(k^2+\alpha^2))\beta\,\mathrm{Cosh}[\alpha] - (k^2+\alpha^2)((-2+c^2)(-1+e^{\beta})k^2 - c^2e^{\beta}\beta^2)\,\mathrm{Sinh}[\alpha]\Big) \Big/ \Big(c^2\sqrt{2\pi}\,\beta\big(\alpha\beta(-(-6+c^2)k^4 - (-2+c^2)k^2\alpha^2 + c^2(k^2+\alpha^2)\beta^2)\,\mathrm{Cosh}[\alpha]\,\mathrm{Cosh}[\beta] + k^2(-2\alpha\beta(-(-3+c^2)k^2+\alpha^2+c^2\beta^2) + ((-2+c^2)k^2(k^2+\alpha^2) - (c^2k^2+(4+c^2)\alpha^2)\beta^2)\,\mathrm{Sinh}[\alpha]\,\mathrm{Sinh}[\beta])\big)\Big),$$

$$\Big(e^{-\alpha-\beta}\big((-2+c^2)(-1+e^{2\alpha})(-1+e^{\beta})k^4 + \alpha^2\beta(c^2(-1+e^{2\alpha})\beta - 2e^{\alpha+\beta}\alpha(2+(-2+c^2)\,\mathrm{Cosh}[\alpha])) + k^2((-2+c^2)(-1+e^{2\alpha})(-1+e^{\beta})\alpha^2 + 4\alpha\beta + c^2(-1+e^{2\alpha})\beta^2 - 2e^{\alpha}\alpha\beta(-2(-2+c^2+e^{\alpha}) + e^{\beta}(2+(-2+c^2)\,\mathrm{Cosh}[\alpha])))\big)\Big) \Big/ \Big(2c^2\sqrt{2\pi}\,\beta\big(\alpha\beta(-(-6+c^2)k^4 - (-2+c^2)k^2\alpha^2 + c^2(k^2+\alpha^2)\beta^2)\,\mathrm{Cosh}[\alpha]\,\mathrm{Cosh}[\beta] + k^2(-2\alpha\beta(-(-3+c^2)k^2+\alpha^2+c^2\beta^2) + ((-2+c^2)k^2(k^2+\alpha^2) - (c^2k^2+(4+c^2)\alpha^2)\beta^2)\,\mathrm{Sinh}[\alpha]\,\mathrm{Sinh}[\beta])\big)\Big)\Big\}$$

Evaluation of the solution and plotting the results

Auxiliary Parameters

Some basic parameters that appear, over and over, in the solution: respectively, the dimensionless frequencies *r* and r_c=r/c (both including the damping), the dimensionless wavenumber of S waves α and of P waves β.

```
In[8]:= c[ν_] := Sqrt[(2 (1 - ν))/(1 - 2 ν)];

In[9]:= rd[r_, δ_] := r/Sqrt[1 + ⅈ * δ];

In[10]:= rc[r_, δ_, ν_] := (r / c[ν])/Sqrt[1 + ⅈ * δ];

In[11]:= α[k_, r_, δ_, ν_] := Sqrt[k^2 - rd[r, δ]^2];

In[12]:= β[k_, r_, δ_, ν_] := Sqrt[k^2 - rc[r, δ, ν]^2];
```

Coefficients

Express the solution of the system (A, B, C, D) given in, eqs.(28) and eqs.(29), in a way that can be easily evaluated within the notebook.

```
In[13]:= Ac[k_, r_, δ_, ν_] :=
  ((-2 + c[ν]^2) (2 + ⅇ^α[k,r,δ,ν]) (-1 + ⅇ^β[k,r,δ,ν])^2 k^4 - c[ν]^2 ⅇ^α[k,r,δ,ν] (1 + ⅇ^(2 β[k,r,δ,ν]))
      α[k, r, δ, ν]^2 β[k, r, δ, ν]^2 + k^2 (-2 (-2 + c[ν]^2) ⅇ^(α[k,r,δ,ν]+β[k,r,δ,ν]) α[k, r, δ, ν]^2 -
        2 (-2 + c[ν]^2) α[k, r, δ, ν] β[k, r, δ, ν] + 2 (-2 + c[ν]^2) ⅇ^(2 β[k,r,δ,ν]) α[k, r, δ, ν] β[k,
          r, δ, ν] + 4 c[ν]^2 ⅇ^β[k,r,δ,ν] β[k, r, δ, ν]^2 + ⅇ^α[k,r,δ,ν] ((-2 + c[ν]^2) α[k, r, δ, ν]^2 -
           4 α[k, r, δ, ν] β[k, r, δ, ν] - c[ν]^2 β[k, r, δ, ν]^2) + ⅇ^(α[k,r,δ,ν]+2 β[k,r,δ,ν])
         ((-2 + c[ν]^2) α[k, r, δ, ν]^2 + 4 α[k, r, δ, ν] β[k, r, δ, ν] - c[ν]^2 β[k, r, δ, ν]^2))) /
   (c[ν]^2 Sqrt[2 * π] β[k, r, δ, ν] ((1 + ⅇ^(2 β[k,r,δ,ν])) α[k, r, δ, ν] β[k, r, δ, ν]
       (-(-6 + c[ν]^2) k^4 - (-2 + c[ν]^2) k^2 α[k, r, δ, ν]^2 + c[ν]^2 (k^2 + α[k, r, δ, ν]^2)
          β[k, r, δ, ν]^2) Cosh[α[k, r, δ, ν]] + 2 ⅇ^β[k,r,δ,ν] k^2 (-2 α[k, r, δ, ν]
          β[k, r, δ, ν] (-(-3 + c[ν]^2) k^2 + α[k, r, δ, ν]^2 + c[ν]^2 β[k, r, δ, ν]^2) +
         ((-2 + c[ν]^2) k^2 (k^2 + α[k, r, δ, ν]^2) - (c[ν]^2 k^2 + (4 + c[ν]^2) α[k, r, δ, ν]^2)
            β[k, r, δ, ν]^2) Sinh[α[k, r, δ, ν]] Sinh[β[k, r, δ, ν]])));

In[14]:= Bc[k_, r_, δ_, ν_] :=
  (ⅇ^(-α[k,r,δ,ν]-β[k,r,δ,ν]) (-(-2 + c[ν]^2) (1 + 2 ⅇ^α[k,r,δ,ν]) (-1 + ⅇ^β[k,r,δ,ν])^2 k^4 + c[ν]^2
        (1 + ⅇ^(2 β[k,r,δ,ν])) α[k, r, δ, ν]^2 β[k, r, δ, ν]^2 + k^2 (-(-2 + c[ν]^2) (-1 + ⅇ^β[k,r,δ,ν])^2
           α[k, r, δ, ν]^2 + 2 (2 + (-2 + c[ν]^2) ⅇ^α[k,r,δ,ν]) (-1 + ⅇ^(2 β[k,r,δ,ν])) α[k, r, δ, ν]
           β[k, r, δ, ν] + c[ν]^2 (1 + ⅇ^(2 β[k,r,δ,ν]) - 4 ⅇ^(α[k,r,δ,ν]+β[k,r,δ,ν])) β[k, r, δ, ν]^2))) /
   (2 c[ν]^2 √(2 π) β[k, r, δ, ν] (α[k, r, δ, ν] β[k, r, δ, ν] (-(-6 + c[ν]^2) k^4 -
          (-2 + c[ν]^2) k^2 α[k, r, δ, ν]^2 + c[ν]^2 (k^2 + α[k, r, δ, ν]^2) β[k, r, δ, ν]^2)
        Cosh[α[k, r, δ, ν]] Cosh[β[k, r, δ, ν]] + k^2 (-2 α[k, r, δ, ν] β[k, r, δ, ν]
          (-(-3 + c[ν]^2) k^2 + α[k, r, δ, ν]^2 + c[ν]^2 β[k, r, δ, ν]^2) +
         ((-2 + c[ν]^2) k^2 (k^2 + α[k, r, δ, ν]^2) - (c[ν]^2 k^2 + (4 + c[ν]^2) α[k, r, δ, ν]^2)
            β[k, r, δ, ν]^2) Sinh[α[k, r, δ, ν]] Sinh[β[k, r, δ, ν]])));
```

In[15]:=
```
Cc[k_, r_, δ_, ν_] :=
  (2 α[k, r, δ, ν] ((1 - (-2 + c[ν]^2) e^β[k,r,δ,ν]) k^2 + α[k, r, δ, ν]^2) * β[k, r, δ, ν] +
     α[k, r, δ, ν] (-4 e^β[k,r,δ,ν] k^2 + (-2 + c[ν]^2) (k^2 + α[k, r, δ, ν]^2))
      β[k, r, δ, ν] Cosh[α[k, r, δ, ν]] - (k^2 + α[k, r, δ, ν]^2)
      ((-2 + c[ν]^2) (-1 + e^β[k,r,δ,ν]) k^2 - c[ν]^2 e^β[k,r,δ,ν] β[k, r, δ, ν]^2) Sinh[α[k, r, δ, ν]]) /
   (c[ν]^2 Sqrt[2 π] β[k, r, δ, ν] (α[k, r, δ, ν] β[k, r, δ, ν] (-(-6 + c[ν]^2) k^4 -
          (-2 + c[ν]^2) k^2 α[k, r, δ, ν]^2 + c[ν]^2 (k^2 + α[k, r, δ, ν]^2) β[k, r, δ, ν]^2)
        Cosh[α[k, r, δ, ν]] Cosh[β[k, r, δ, ν]] + k^2 (-2 α[k, r, δ, ν] β[k, r, δ, ν]
          (-(-3 + c[ν]^2) k^2 + α[k, r, δ, ν]^2 + c[ν]^2 β[k, r, δ, ν]^2) +
         ((-2 + c[ν]^2) k^2 (k^2 + α[k, r, δ, ν]^2) - (c[ν]^2 k^2 + (4 + c[ν]^2) α[k, r, δ, ν]^2)
            β[k, r, δ, ν]^2) Sinh[α[k, r, δ, ν]] Sinh[β[k, r, δ, ν]])));
```

In[16]:=
```
Dc[k_, r_, δ_, ν_] :=
  (e^(-α[k,r,δ,ν]-β[k,r,δ,ν]) ((-2 + c[ν]^2) (-1 + e^(2 α[k,r,δ,ν])) (-1 + e^β[k,r,δ,ν]) k^4 + α[k, r, δ, ν]^2
        β[k, r, δ, ν] (c[ν]^2 (-1 + e^(2 α[k,r,δ,ν])) β[k, r, δ, ν] -
          2 e^(α[k,r,δ,ν]+β[k,r,δ,ν]) α [k, r, δ, ν] (2 + (-2 + c[ν]^2) Cosh[α[k, r, δ, ν]])) +
       k^2 ((-2 + c[ν]^2) (-1 + e^(2 α[k,r,δ,ν])) (-1 + e^β[k,r,δ,ν]) α[k, r, δ, ν]^2 +
          4 α[k, r, δ, ν] β[k, r, δ, ν] + c[ν]^2 (-1 + e^(2 α[k,r,δ,ν])) β[k, r, δ, ν]^2 -
          2 e^α[k,r,δ,ν] α[k, r, δ, ν] β[k, r, δ, ν]
           (-2 (-2 + c[ν]^2 + e^α[k,r,δ,ν]) + e^β[k,r,δ,ν] (2 + (-2 + c[ν]^2) Cosh[α[k, r, δ, ν]]))))) /
   (2 c[ν]^2 * Sqrt[2 π] * β[k, r, δ, ν] (α[k, r, δ, ν] β[k, r, δ, ν] (-(-6 + c[ν]^2) k^4 -
          (-2 + c[ν]^2) k^2 α[k, r, δ, ν]^2 + c[ν]^2 (k^2 + α[k, r, δ, ν]^2) β[k, r, δ, ν]^2)
        Cosh[α[k, r, δ, ν]] Cosh[β[k, r, δ, ν]] + k^2 (-2 α[k, r, δ, ν] β[k, r, δ, ν]
          (-(-3 + c[ν]^2) k^2 + α[k, r, δ, ν]^2 + c[ν]^2 β[k, r, δ, ν]^2) +
         ((-2 + c[ν]^2) k^2 (k^2 + α[k, r, δ, ν]^2) - (c[ν]^2 k^2 + (4 + c[ν]^2) α[k, r, δ, ν]^2)
            β[k, r, δ, ν]^2) Sinh[α[k, r, δ, ν]] Sinh[β[k, r, δ, ν]])));
```

Variables

Vertical displacement at the top of the wall

The vertical displacement at the top of the wall is expressed as an inverse transform as:

In[17]:=
```
vtop[r_, δ_, ν_] := 2/Sqrt[2 π]
   NIntegrate[Ac[k, r, δ, ν] * Exp[-α[k, r, δ, ν]] + Bc[k, r, δ, ν] * Exp[α[k, r, δ, ν]] +
     Cc[k, r, δ, ν] * Exp[-β[k, r, δ, ν]] + Dc[k, r, δ, ν] * Exp[β[k, r, δ, ν]], {k, 0, 10}];
```

See that this expression is just eq.(30b) in the body of the text.

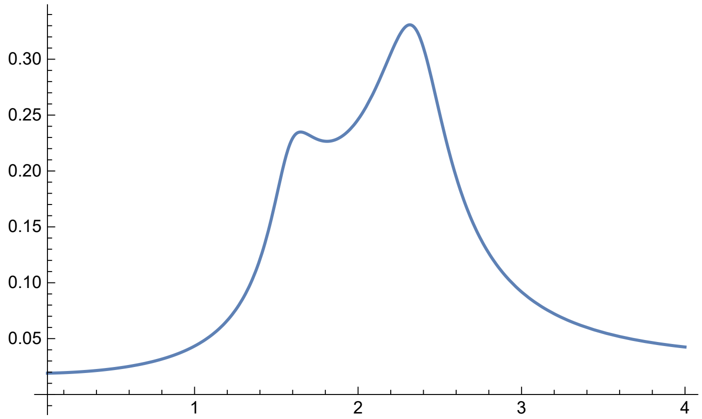
Plots

For Figure 8, fix ν=0.1 and δ_d=0.16

In[18]:=
```
NU = 0.1; DE = 0.16;
```

In[19]:= **Plot[{Abs[vtop[r, DE, NU]]}, {r, 0.1, 4}]**

Out[19]=

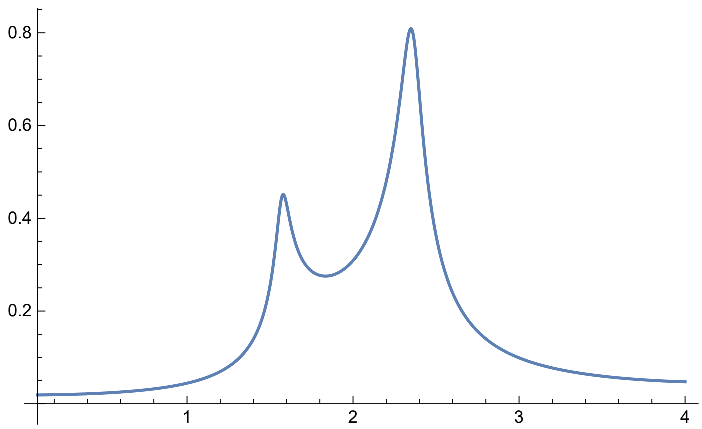


For Figure 9, fix v=0.1 and δ_d=0.05

In[20]:= **NU = 0.1; DE = 0.05;**

In[21]:= **Plot[{Abs[vtop[r, DE, NU]]}, {r, 0.1, 4}]**

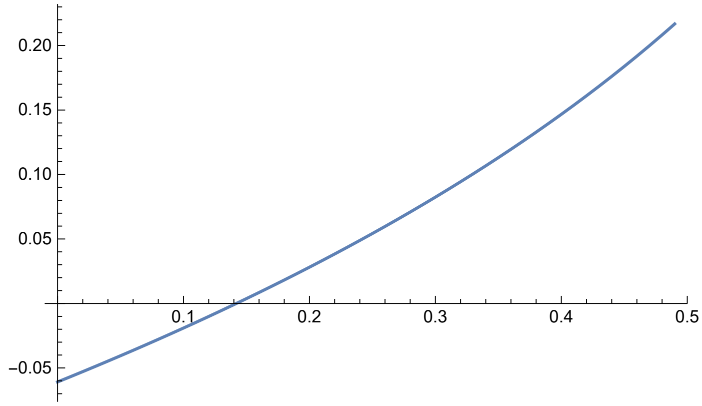
Out[21]=



For Figure 12, just pick a r<<1 and plot as function of v:

In[22]:= **Plot[-Re[vtop[0.1, DE, ν]], {ν, 0, 0.49}]**

Out[22]=



Thrust

Likewise, next expression corresponds to the dimensionless thrust as in eq.(32):

In[23]:=
```
Q[r_, δ_, ν_] :=
  1/Sqrt[2 π] NIntegrate[Sqrt[2/π]/β[k, r, δ, ν]^2 + Ac[k, r, δ, ν] * (c[ν]^2 - 2 * Exp[-α[k, r, δ, ν]]) +
     Bc[k, r, δ, ν] * (c[ν]^2 - 2 * Exp[α[k, r, δ, ν]]) + (c[ν]^2 (1 - (k/β[k, r, δ, ν])^2) - 2) *
      (Cc[k, r, δ, ν] * Exp[-β[k, r, δ, ν]] + Dc[k, r, δ, ν] * Exp[β[k, r, δ, ν]]) +
     ((c[ν] * k)/β[k, r, δ, ν])^2 * (Cc[k, r, δ, ν] + Dc[k, r, δ, ν]), {k, -10, 10}];
```

Plots

To generate Figure 10 y Figure 15 we also need the expression of the thrust provided by Veletsos and Younan and Kloukinas before we can compare them:

In[24]:=
```
QVY[r_, δ_, ν_] := (32/π^3)/Sqrt[(1 - ν) (2 - ν)] Sum[1/(2 n - 1)^3 1/Sqrt[1 - ((r/(π/2 (2 n - 1)))/Sqrt[1 + i * δ])^2], {n, 1, 5}]
```

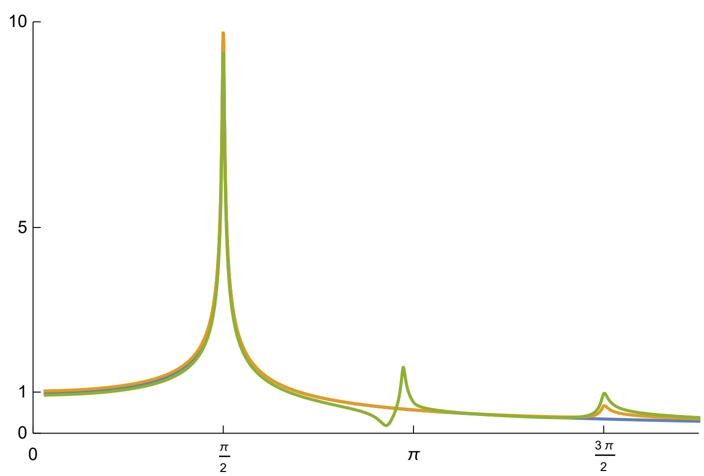
In[25]:=
```
QK[r_, δ_, ν_] := 2/Sqrt[(1 - ν) (2 - ν)] (8/π^2)/Sqrt[(π/2)^2 - r^2/(1 + i * δ)]
```

For Figure 10, fix ν=1/3 and δ_d=0.01

In[26]:=
```
NU = 1/3; DE = 0.01;
```

In[27]:=
```
Plot[{Abs[QK[r, DE, NU]], Abs[QVY[r, DE, NU]], Abs[Q[r, DE, NU]]}, {r, 0.1, 3.5 * π/2},
 PlotRange → {{0, 3.5 * π/2}, {0, 10}}, Ticks → {{0, Pi/2, Pi, 3 Pi/2}, {0, 1, 5, 10}}]
```

Out[27]=



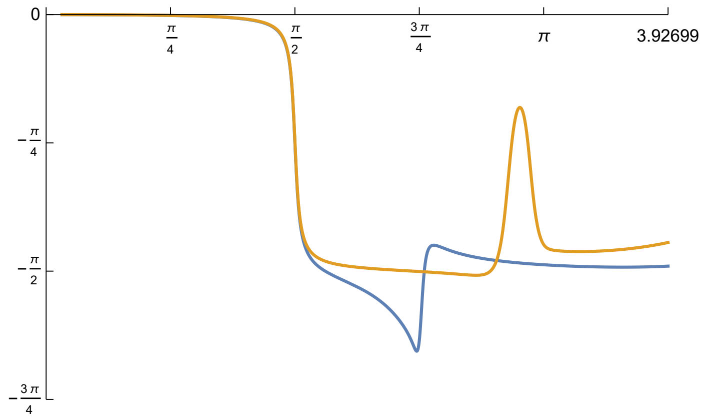
In order to generate the results in Figure 11, Figure 15 and Figure 16, just tweak the values of NU and DE.

For Figure 14, get the phase of this variable, fixing ν=0.1 and δ_d=0.03

In[28]:=
```
NU1 = 0.1; NU2 = 1/3; DE = 0.03;
```

In[29]:=
```
Plot[{Arg[Q[r, DE, NU1]], Arg[Q[r, DE, NU2]]},
 {r, 0.1, 2.5 * π/2}, PlotRange → {{0, 2.5 * π/2}, {-3 π / 4, 0}},
 Ticks → {{0, Pi/4, Pi/2, 3 Pi/4, Pi, 2.5 Pi/2}, {-3 π / 4, -π/2, -π / 4, 0}}]
```

Out[29]=



Analysis of intermediate resonance

To generate the plots in this section, we split the thrust in two contributions: the integral of horizontal gradient of horizontal displacement at the vertical (“*intdudxw*”) and the vertical displacement at the top (already implemented).

In[30]:=
```
intdudxw[r_, δ_, ν_] := 1/Sqrt[2 π]
  NIntegrate[Sqrt[2/π] * 1/(c[ν]^2 β[k, r, δ, ν]^2) - Ac[k, r, δ, ν] * (Exp[-α[k, r, δ, ν]] - 1) -
    Bc[k, r, δ, ν] * (Exp[α[k, r, δ, ν]] - 1) - (k/β[k, r, δ, ν])^2 * (Cc[k, r, δ, ν] *
       (Exp[-β[k, r, δ, ν]] - 1) + Dc[k, r, δ, ν] * (Exp[β[k, r, δ, ν]] - 1)), {k, -10, 10}];
```

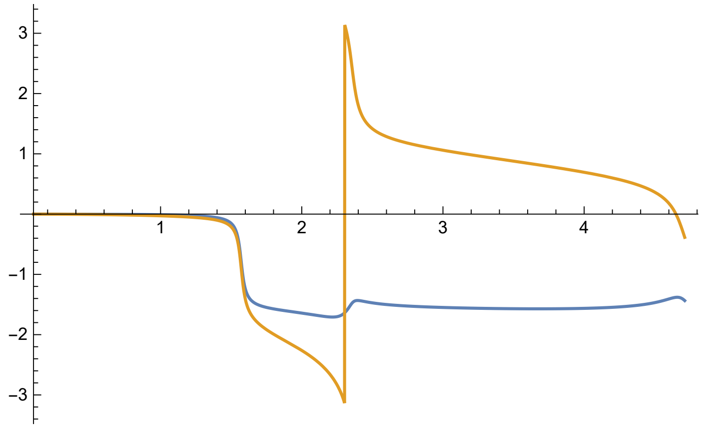
Plots

Compare the phases of these two components: Figure 13 (a) for ν=0.1, and Figure 13 (b) for ν=1/3.

In[31]:=
```
NU = 0.1; DE = 0.03;
```

In[32]:=
```
Plot[{Arg[intdudxw[r, DE, NU]], Arg[vtop[r, DE, NU]]}, {r, 0.1, 3 * π/2}]
```

Out[32]=



In[33]:=
```
NU = 1/3;
```

In[34]:=
```
Plot[{Arg[intdudxw[r, DE, NU]], Arg[vtop[r, DE, NU]]}, {r, 0.1, 3 * π/2}]
```

Out[34]=