Solution of seismic response of smooth rigid retaining wall

Joaquin Garcia-Suarez, 2020 (All rights reserved)

This notebook details and displays the results contained in “Exact dynamic response of smooth rigid retaining walls resting on bedrock”. Here one can find some steps of the algebra leading to the exact solution, as well as the data used to generate the plots in the body of the manuscript. References to the actual text are included.

Details on derivations

The transform of the load

It follows, for display purposes, the transform of the sign function

```
FourierTransform[2 * HeavisideTheta[ξ] - 1, ξ, k]
```

$$\frac{i\sqrt{\frac{2}{\pi}}}{k}$$

The matrix ***D*** and the corresponding eigensystem

The matrix D appears in eq.(C.20) and its actual expression is given in (18):

```
Dm := {{0, c^2 * k^2 - r^2, i * k * (1 - c^2), 0},
  {1, 0, 0, 0}, {i * k * (1 - c^2)/c^2, 0, 0, (k^2 - r^2)/c^2}, {0, 0, 1, 0}}
```

```
Eigensystem[Dm]
```

$$\left\{\left\{-\sqrt{k^2-r^2},\ \sqrt{k^2-r^2},\ -\frac{\sqrt{c^2\left(c^2k^2-r^2\right)}}{c^2},\ \frac{\sqrt{c^2\left(c^2k^2-r^2\right)}}{c^2}\right\},\right.$$
$$\left\{\left\{\frac{i\left(k^2-r^2\right)}{k},\ -\frac{i\sqrt{k^2-r^2}}{k},\ -\sqrt{k^2-r^2},\ 1\right\},\ \left\{\frac{i\left(k^2-r^2\right)}{k},\ \frac{i\sqrt{k^2-r^2}}{k},\ \sqrt{k^2-r^2},\ 1\right\},\right.$$
$$\left\{i\,k,\ -\frac{i\,k\sqrt{c^2\left(c^2k^2-r^2\right)}}{c^2k^2-r^2},\ -\frac{\sqrt{c^2\left(c^2k^2-r^2\right)}}{c^2},\ 1\right\},$$
$$\left.\left.\left\{i\,k,\ \frac{i\,k\sqrt{c^2\left(c^2k^2-r^2\right)}}{c^2k^2-r^2},\ \frac{\sqrt{c^2\left(c^2k^2-r^2\right)}}{c^2},\ 1\right\}\right\}\right\}$$

Solving for the constants

Next, it follows the system of equations displayed in eq.(27), and the explicit solution of it, eqs.(28) and eqs.(29):

$$A = \text{MatrixForm}\left[\left\{\{1,1,1,1\},\ \left\{-\frac{i*\alpha}{k},\ \frac{i*\alpha}{k},\ -\frac{i*k}{\beta},\ \frac{i*k}{\beta}\right\},\right.\right.$$
$$\left\{\left(k+\frac{\alpha^2}{k}\right)*\text{Exp}[-\alpha],\ \left(k+\frac{\alpha^2}{k}\right)*\text{Exp}[\alpha],\ 2*k*\text{Exp}[-\beta],\ 2*k*\text{Exp}[\beta]\right\},$$
$$\left.\left.\left\{-2\,\alpha*\text{Exp}[-\alpha],\ 2\,\alpha*\text{Exp}[\alpha],\ -\left(\frac{c^2\left(\beta^2-k^2\right)+2\,k^2}{\beta}\right)\text{Exp}[-\beta],\ \left(\frac{c^2\left(\beta^2-k^2\right)+2\,k^2}{\beta}\right)\text{Exp}[\beta]\right\}\right\}\right]$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -\frac{i\,\alpha}{k} & \frac{i\,\alpha}{k} & -\frac{i\,k}{\beta} & \frac{i\,k}{\beta} \\ e^{-\alpha}\left(k+\frac{\alpha^2}{k}\right) & e^{\alpha}\left(k+\frac{\alpha^2}{k}\right) & 2\,e^{-\beta}\,k & 2\,e^{\beta}\,k \\ -2\,e^{-\alpha}\,\alpha & 2\,e^{\alpha}\,\alpha & -\frac{e^{-\beta}\left(2k^2+c^2\left(-k^2+\beta^2\right)\right)}{\beta} & \frac{e^{\beta}\left(2k^2+c^2\left(-k^2+\beta^2\right)\right)}{\beta} \end{pmatrix}$$

```
Inverse[A]
```

$$\text{Inverse}\left[\begin{pmatrix} 1 & 1 & 1 & 1 \\ -\frac{i\,\alpha}{k} & \frac{i\,\alpha}{k} & -\frac{i\,k}{\beta} & \frac{i\,k}{\beta} \\ e^{-\alpha}\left(k+\frac{\alpha^2}{k}\right) & e^{\alpha}\left(k+\frac{\alpha^2}{k}\right) & 2\,e^{-\beta}\,k & 2\,e^{\beta}\,k \\ -2\,e^{-\alpha}\,\alpha & 2\,e^{\alpha}\,\alpha & -\frac{e^{-\beta}\left(2k^2+c^2\left(-k^2+\beta^2\right)\right)}{\beta} & \frac{e^{\beta}\left(2k^2+c^2\left(-k^2+\beta^2\right)\right)}{\beta} \end{pmatrix}\right]$$

```
FullSimplify[Inverse[{{1, 1, 1, 1}, {-ⅈ*α/k, ⅈ*α/k, -ⅈ*k/β, ⅈ*k/β},
    {(k + α^2/k) * Exp[-α], (k + α^2/k) * Exp[α], 2*k*Exp[-β], 2*k*Exp[β]}, {-2α*Exp[-α],
     2α*Exp[α], -((c^2 (β^2 - k^2) + 2 k^2)/β) Exp[-β], ((c^2 (β^2 - k^2) + 2 k^2)/β) Exp[β]}}].
   {0, Sqrt[2/π] ⅈ/(k*c^2*β^2), 0, -Sqrt[2/π] (c^2 - 2)/(c^2*β^2)}]
```

$$
\begin{aligned}
&\Big\{\Big((-2+c^2)(2+e^{\alpha})(-1+e^{\beta})^2 k^4 - c^2 e^{\alpha}(1+e^{2\beta})\alpha^2\beta^2 + \\
&\quad k^2\big(-2(-2+c^2)e^{\alpha+\beta}\alpha^2 - 2(-2+c^2)\alpha\beta + 2(-2+c^2)e^{2\beta}\alpha\beta + 4c^2 e^{\beta}\beta^2 + \\
&\qquad e^{\alpha}\big((-2+c^2)\alpha^2 - 4\alpha\beta - c^2\beta^2\big) + e^{\alpha+2\beta}\big((-2+c^2)\alpha^2 + 4\alpha\beta - c^2\beta^2\big)\big)\Big)\Big/ \\
&\quad \Big(c^2\sqrt{2\pi}\,\beta\Big((1+e^{2\beta})\alpha\beta\big(-(-6+c^2)k^4 - (-2+c^2)k^2\alpha^2 + c^2(k^2+\alpha^2)\beta^2\big)\mathrm{Cosh}[\alpha] + \\
&\qquad 2e^{\beta}k^2\Big(-2\alpha\beta\big(-(-3+c^2)k^2+\alpha^2+c^2\beta^2\big) + \\
&\qquad\quad \big((-2+c^2)k^2(k^2+\alpha^2) - (c^2k^2+(4+c^2)\alpha^2)\beta^2\big)\mathrm{Sinh}[\alpha]\,\mathrm{Sinh}[\beta]\Big)\Big)\Big), \\
&\Big(e^{-\alpha-\beta}\Big(-(-2+c^2)(1+2e^{\alpha})(-1+e^{\beta})^2k^4 + c^2(1+e^{2\beta})\alpha^2\beta^2 + \\
&\quad k^2\big(-(-2+c^2)(-1+e^{\beta})^2\alpha^2 + 2\big(2+(-2+c^2)e^{\alpha}\big)(-1+e^{2\beta})\alpha\beta + c^2(1+e^{2\beta}-4e^{\alpha+\beta})\beta^2\big)\Big)\Big)\Big/ \\
&\quad \Big(2c^2\sqrt{2\pi}\,\beta\Big(\alpha\beta\big(-(-6+c^2)k^4 - (-2+c^2)k^2\alpha^2 + c^2(k^2+\alpha^2)\beta^2\big)\mathrm{Cosh}[\alpha]\,\mathrm{Cosh}[\beta] + \\
&\qquad k^2\Big(-2\alpha\beta\big(-(-3+c^2)k^2+\alpha^2+c^2\beta^2\big) + \\
&\qquad\quad \big((-2+c^2)k^2(k^2+\alpha^2) - (c^2k^2+(4+c^2)\alpha^2)\beta^2\big)\mathrm{Sinh}[\alpha]\,\mathrm{Sinh}[\beta]\Big)\Big)\Big), \\
&\Big(2\alpha\big((1-(-2+c^2)e^{\beta})k^2+\alpha^2\big)\beta + \alpha\big(-4e^{\beta}k^2 + (-2+c^2)(k^2+\alpha^2)\big)\beta\,\mathrm{Cosh}[\alpha] - \\
&\quad (k^2+\alpha^2)\big((-2+c^2)(-1+e^{\beta})k^2 - c^2e^{\beta}\beta^2\big)\mathrm{Sinh}[\alpha]\Big)\Big/ \\
&\quad \Big(c^2\sqrt{2\pi}\,\beta\Big(\alpha\beta\big(-(-6+c^2)k^4 - (-2+c^2)k^2\alpha^2 + c^2(k^2+\alpha^2)\beta^2\big)\mathrm{Cosh}[\alpha]\,\mathrm{Cosh}[\beta] + \\
&\qquad k^2\Big(-2\alpha\beta\big(-(-3+c^2)k^2+\alpha^2+c^2\beta^2\big) + \\
&\qquad\quad \big((-2+c^2)k^2(k^2+\alpha^2) - (c^2k^2+(4+c^2)\alpha^2)\beta^2\big)\mathrm{Sinh}[\alpha]\,\mathrm{Sinh}[\beta]\Big)\Big)\Big), \\
&\Big(e^{-\alpha-\beta}\Big((-2+c^2)(-1+e^{2\alpha})(-1+e^{\beta})k^4 + \alpha^2\beta\big(c^2(-1+e^{2\alpha})\beta - 2e^{\alpha+\beta}\alpha\big(2+(-2+c^2)\mathrm{Cosh}[\alpha]\big)\big) + \\
&\quad k^2\Big((-2+c^2)(-1+e^{2\alpha})(-1+e^{\beta})\alpha^2 + 4\alpha\beta + c^2(-1+e^{2\alpha})\beta^2 - \\
&\qquad 2e^{\alpha}\alpha\beta\big(-2(-2+c^2+e^{\alpha}) + e^{\beta}\big(2+(-2+c^2)\mathrm{Cosh}[\alpha]\big)\big)\Big)\Big)\Big)\Big/ \\
&\quad \Big(2c^2\sqrt{2\pi}\,\beta\Big(\alpha\beta\big(-(-6+c^2)k^4 - (-2+c^2)k^2\alpha^2 + c^2(k^2+\alpha^2)\beta^2\big)\mathrm{Cosh}[\alpha]\,\mathrm{Cosh}[\beta] + \\
&\qquad k^2\Big(-2\alpha\beta\big(-(-3+c^2)k^2+\alpha^2+c^2\beta^2\big) + \\
&\qquad\quad \big((-2+c^2)k^2(k^2+\alpha^2) - (c^2k^2+(4+c^2)\alpha^2)\beta^2\big)\mathrm{Sinh}[\alpha]\,\mathrm{Sinh}[\beta]\Big)\Big)\Big)\Big\}
\end{aligned}
$$

Evaluation of the solution and plotting the results

Auxiliary Parameters

Some basic parameters that appear, over and over, in the solution: respectively, the dimensionless frequencies *r* and r_c=r/c (both including the damping), the dimensionless wavenumber of S waves α and of P waves β.

```
c[ν_] = Sqrt[2 (1 - ν) / (1 - 2 ν)];

rd[r_, δ_] = r / Sqrt[1 + ⅈ * δ];

rc[r_, δ_, ν_] = (r / c[ν]) / Sqrt[1 + ⅈ * δ];

α[k_, r_, δ_, ν_] = Sqrt[k^2 - rd[r, δ]^2];

β[k_, r_, δ_, ν_] = Sqrt[k^2 - rc[r, δ, ν]^2];
```

When it comes to define the branch cut correspond to this square roots, the standard settings implemented in Mathematica are used, these are:

```
ComplexAnalysis`BranchCuts[Sqrt[z], z]
```

```
Re[z] < 0 && Im[z] == 0
```

Hence *Mathematica* assigns, by default, a value to the phase between -π and π.

Coefficients

Express the solution of the system (A, B, C, D) given in, eqs.(28) and eqs.(29), in a way that can be easily evaluated within the notebook.

```
Ac[k_, r_, δ_, ν_] =
  ((-2 + c[ν]^2) (2 + ⅇ^α[k,r,δ,ν]) (-1 + ⅇ^β[k,r,δ,ν])^2 k^4 - c[ν]^2 ⅇ^α[k,r,δ,ν] (1 + ⅇ^(2 β[k,r,δ,ν]))
      α[k, r, δ, ν]^2 β[k, r, δ, ν]^2 + k^2 (-2 (-2 + c[ν]^2) ⅇ^(α[k,r,δ,ν]+β[k,r,δ,ν]) α[k, r, δ, ν]^2 -
        2 (-2 + c[ν]^2) α[k, r, δ, ν] β[k, r, δ, ν] + 2 (-2 + c[ν]^2) ⅇ^(2 β[k,r,δ,ν]) α[k, r, δ, ν] β[k,
          r, δ, ν] + 4 c[ν]^2 ⅇ^β[k,r,δ,ν] β[k, r, δ, ν]^2 + ⅇ^α[k,r,δ,ν] ((-2 + c[ν]^2) α[k, r, δ, ν]^2 -
           4 α[k, r, δ, ν] β[k, r, δ, ν] - c[ν]^2 β[k, r, δ, ν]^2) + ⅇ^(α[k,r,δ,ν]+2 β[k,r,δ,ν])
         ((-2 + c[ν]^2) α[k, r, δ, ν]^2 + 4 α[k, r, δ, ν] β[k, r, δ, ν] - c[ν]^2 β[k, r, δ, ν]^2))) /
   (c[ν]^2 Sqrt[2 * π] β[k, r, δ, ν] ((1 + ⅇ^(2 β[k,r,δ,ν])) α[k, r, δ, ν] β[k, r, δ, ν]
        (-(-6 + c[ν]^2) k^4 - (-2 + c[ν]^2) k^2 α[k, r, δ, ν]^2 + c[ν]^2 (k^2 + α[k, r, δ, ν]^2)
           β[k, r, δ, ν]^2) Cosh[α[k, r, δ, ν]] + 2 ⅇ^β[k,r,δ,ν] k^2 (-2 α[k, r, δ, ν]
           β[k, r, δ, ν] (-(-3 + c[ν]^2) k^2 + α[k, r, δ, ν]^2 + c[ν]^2 β[k, r, δ, ν]^2) +
          ((-2 + c[ν]^2) k^2 (k^2 + α[k, r, δ, ν]^2) - (c[ν]^2 k^2 + (4 + c[ν]^2) α[k, r, δ, ν]^2)
             β[k, r, δ, ν]^2) Sinh[α[k, r, δ, ν]] Sinh[β[k, r, δ, ν]])));
```

```
Bc[k_, r_, δ_, ν_] =
  (ⅇ^(-α[k,r,δ,ν]-β[k,r,δ,ν]) (-(-2 + c[ν]^2) (1 + 2 ⅇ^α[k,r,δ,ν]) (-1 + ⅇ^β[k,r,δ,ν])^2 k^4 + c[ν]^2
        (1 + ⅇ^(2 β[k,r,δ,ν])) α[k, r, δ, ν]^2 β[k, r, δ, ν]^2 + k^2 (-(-2 + c[ν]^2) (-1 + ⅇ^β[k,r,δ,ν])^2
          α[k, r, δ, ν]^2 + 2 (2 + (-2 + c[ν]^2) ⅇ^α[k,r,δ,ν]) (-1 + ⅇ^(2 β[k,r,δ,ν])) α[k, r, δ, ν]
          β[k, r, δ, ν] + c[ν]^2 (1 + ⅇ^(2 β[k,r,δ,ν]) - 4 ⅇ^(α[k,r,δ,ν]+β[k,r,δ,ν])) β[k, r, δ, ν]^2))) /
   (2 c[ν]^2 √(2 π) β[k, r, δ, ν] (α[k, r, δ, ν] β[k, r, δ, ν] (-(-6 + c[ν]^2) k^4 -
          (-2 + c[ν]^2) k^2 α[k, r, δ, ν]^2 + c[ν]^2 (k^2 + α[k, r, δ, ν]^2) β[k, r, δ, ν]^2)
        Cosh[α[k, r, δ, ν]] Cosh[β[k, r, δ, ν]] + k^2 (-2 α[k, r, δ, ν] β[k, r, δ, ν]
           (-(-3 + c[ν]^2) k^2 + α[k, r, δ, ν]^2 + c[ν]^2 β[k, r, δ, ν]^2) +
          ((-2 + c[ν]^2) k^2 (k^2 + α[k, r, δ, ν]^2) - (c[ν]^2 k^2 + (4 + c[ν]^2) α[k, r, δ, ν]^2)
             β[k, r, δ, ν]^2) Sinh[α[k, r, δ, ν]] Sinh[β[k, r, δ, ν]])));

Cc[k_, r_, δ_, ν_] =
  (2 α[k, r, δ, ν] ((1 - (-2 + c[ν]^2) ⅇ^β[k,r,δ,ν]) k^2 + α[k, r, δ, ν]^2) * β[k, r, δ, ν] +
      α[k, r, δ, ν] (-4 ⅇ^β[k,r,δ,ν] k^2 + (-2 + c[ν]^2) (k^2 + α[k, r, δ, ν]^2))
       β[k, r, δ, ν] Cosh[α[k, r, δ, ν]] - (k^2 + α[k, r, δ, ν]^2)
       ((-2 + c[ν]^2) (-1 + ⅇ^β[k,r,δ,ν]) k^2 - c[ν]^2 ⅇ^β[k,r,δ,ν] β[k, r, δ, ν]^2) Sinh[α[k, r, δ, ν]]) /
   (c[ν]^2 Sqrt[2 π] β[k, r, δ, ν] (α[k, r, δ, ν] β[k, r, δ, ν] (-(-6 + c[ν]^2) k^4 -
          (-2 + c[ν]^2) k^2 α[k, r, δ, ν]^2 + c[ν]^2 (k^2 + α[k, r, δ, ν]^2) β[k, r, δ, ν]^2)
        Cosh[α[k, r, δ, ν]] Cosh[β[k, r, δ, ν]] + k^2 (-2 α[k, r, δ, ν] β[k, r, δ, ν]
           (-(-3 + c[ν]^2) k^2 + α[k, r, δ, ν]^2 + c[ν]^2 β[k, r, δ, ν]^2) +
          ((-2 + c[ν]^2) k^2 (k^2 + α[k, r, δ, ν]^2) - (c[ν]^2 k^2 + (4 + c[ν]^2) α[k, r, δ, ν]^2)
             β[k, r, δ, ν]^2) Sinh[α[k, r, δ, ν]] Sinh[β[k, r, δ, ν]])));

Dc[k_, r_, δ_, ν_] =
  (ⅇ^(-α[k,r,δ,ν]-β[k,r,δ,ν]) ((-2 + c[ν]^2) (-1 + ⅇ^(2 α[k,r,δ,ν])) (-1 + ⅇ^β[k,r,δ,ν]) k^4 + α[k, r, δ, ν]^2
        β[k, r, δ, ν] (c[ν]^2 (-1 + ⅇ^(2 α[k,r,δ,ν])) β[k, r, δ, ν] - 2 ⅇ^(α[k,r,δ,ν]+β[k,r,δ,ν])
          α [k, r, δ, ν] (2 + (-2 + c[ν]^2) Cosh[α[k, r, δ, ν]])) +
       k^2 ((-2 + c[ν]^2) (-1 + ⅇ^(2 α[k,r,δ,ν])) (-1 + ⅇ^β[k,r,δ,ν]) α[k, r, δ, ν]^2 +
          4 α[k, r, δ, ν] β[k, r, δ, ν] + c[ν]^2 (-1 + ⅇ^(2 α[k,r,δ,ν])) β[k, r, δ, ν]^2 -
          2 ⅇ^α[k,r,δ,ν] α[k, r, δ, ν] β[k, r, δ, ν]
           (-2 (-2 + c[ν]^2 + ⅇ^α[k,r,δ,ν]) + ⅇ^β[k,r,δ,ν] (2 + (-2 + c[ν]^2) Cosh[α[k, r, δ, ν]]))))) /
   (2 c[ν]^2 * Sqrt[2 π] * β[k, r, δ, ν] (α[k, r, δ, ν] β[k, r, δ, ν] (-(-6 + c[ν]^2) k^4 -
          (-2 + c[ν]^2) k^2 α[k, r, δ, ν]^2 + c[ν]^2 (k^2 + α[k, r, δ, ν]^2) β[k, r, δ, ν]^2)
        Cosh[α[k, r, δ, ν]] Cosh[β[k, r, δ, ν]] + k^2 (-2 α[k, r, δ, ν] β[k, r, δ, ν]
           (-(-3 + c[ν]^2) k^2 + α[k, r, δ, ν]^2 + c[ν]^2 β[k, r, δ, ν]^2) +
          ((-2 + c[ν]^2) k^2 (k^2 + α[k, r, δ, ν]^2) - (c[ν]^2 k^2 + (4 + c[ν]^2) α[k, r, δ, ν]^2)
             β[k, r, δ, ν]^2) Sinh[α[k, r, δ, ν]] Sinh[β[k, r, δ, ν]])));
```

Variables

Vertical displacement at the top of the wall

The vertical displacement at the top of the wall is expressed as an inverse transform as:

```
vtop[r_, δ_, ν_] := 2/Sqrt[2 π]
   NIntegrate[Ac[k, r, δ, ν] * Exp[-α[k, r, δ, ν]] + Bc[k, r, δ, ν] * Exp[α[k, r, δ, ν]] +
     Cc[k, r, δ, ν] * Exp[-β[k, r, δ, ν]] + Dc[k, r, δ, ν] * Exp[β[k, r, δ, ν]], {k, 0, 10}];
```

See that this expression is just eq.(30b) in the body of the text.

Plots

For Figure 7, fix ν=0.1 and δ_d=0.16

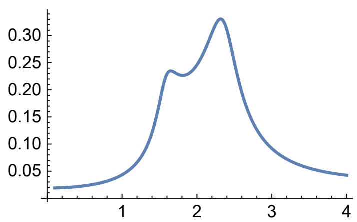
```
NU = 0.1; DE = 0.16;
```

This first plot will allow to compare different ways of generating the plots. The **ParallelTable** option works better in my personal workstation, but a reader can change the method to suit particular necessities.

Let us create an auxiliary vector of values of "r" for the listlineplots:

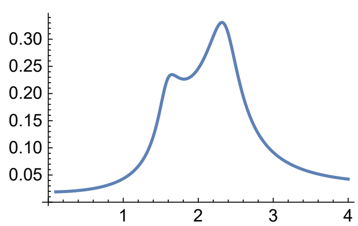
```
rlist = Table[ii/100., {ii, 10, 400}];
```

```
ListLinePlot[Transpose@{rlist,
    Abs[ParallelTable[vtop[rlist[[jj]], DE, NU], {jj, 1, Length[rlist]}]]}] // Timing
```

{0.393847, 

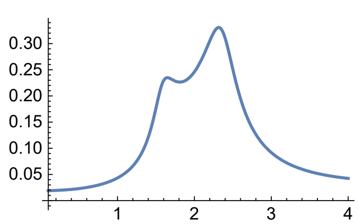
}

```
ListLinePlot[Transpose@
   {rlist, Abs[Table[vtop[rlist[[jj]], DE, NU], {jj, 1, Length[rlist]}]]}] // Timing
```

{19.4955, 

}

```
Plot[{Abs[vtop[r, DE, NU]]}, {r, 0.1, 4}] // Timing
```

{31.7041, 

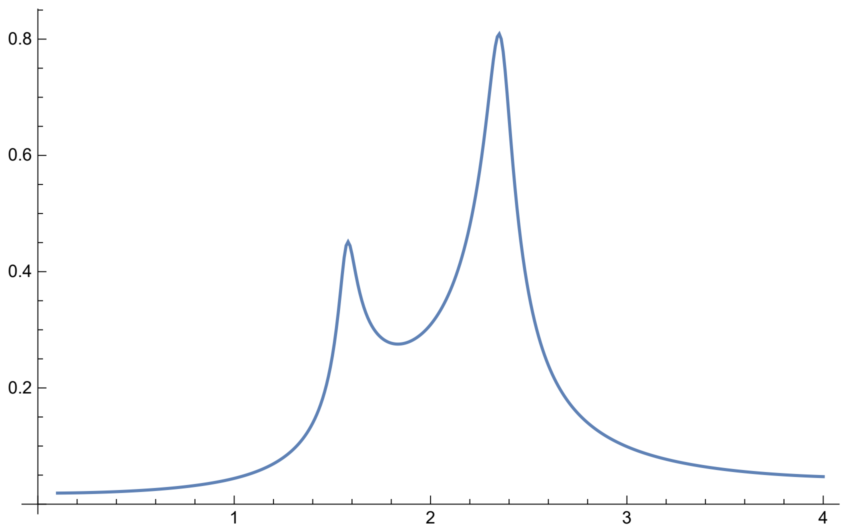
}

See how **ParallelTable** is dramatically faster.

For Figure 8, fix ν=0.1 and δ_d=0.05

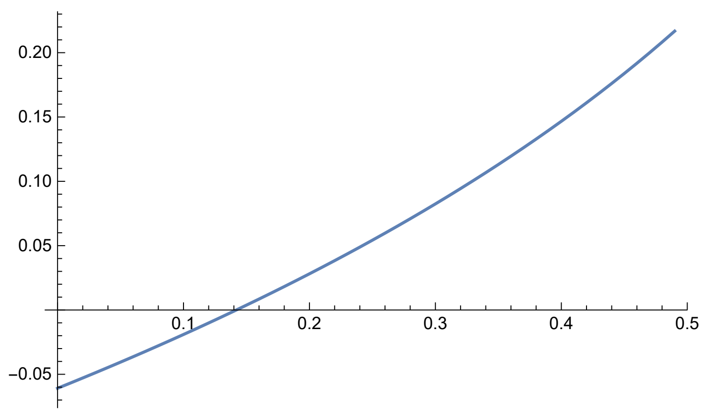
```
NU = 0.1; DE = 0.05;
```

```
ListLinePlot[
 Transpose@{rlist, Abs[ParallelTable[vtop[rlist[[jj]], DE, NU], {jj, 1, Length[rlist]}]]}]
```



For Figure 11, just pick a r<<1 and plot as function of ν (the change of sign is due to using different sign of the load in static simulations):

```
nulist = Table[(ii - 1)/100, {ii, 1, 50}];
ListLinePlot[Transpose@
  {nulist, -Re[ParallelTable[vtop[0.1, DE, nulist[[jj]]], {jj, 1, Length[nulist]}]]}]
```



Thrust

Likewise, next expression corresponds to the dimensionless thrust as in eq.(32):

```
Q[r_, δ_, ν_] :=
  2/Sqrt[2 π] NIntegrate[Sqrt[2/π]/β[k, r, δ, ν]^2 + Ac[k, r, δ, ν] * (c[ν]^2 - 2 * Exp[-α[k, r, δ, ν]]) +
     Bc[k, r, δ, ν] * (c[ν]^2 - 2 * Exp[α[k, r, δ, ν]]) + (c[ν]^2 (1 - (k/β[k, r, δ, ν])^2) - 2) *
      (Cc[k, r, δ, ν] * Exp[-β[k, r, δ, ν]] + Dc[k, r, δ, ν] * Exp[β[k, r, δ, ν]]) +
     ((c[ν] * k)/β[k, r, δ, ν])^2 * (Cc[k, r, δ, ν] + Dc[k, r, δ, ν]), {k, 0, 10}];
```

Plots

To generate Figure 9 y Figure 14 we also need the expression of the thrust provided by Veletsos and Younan and by Kloukinas before we can compare them:

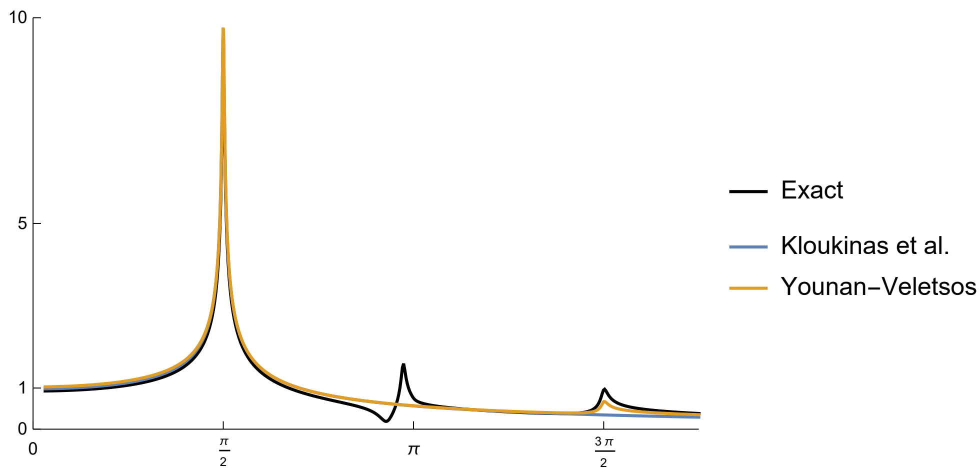
```
QVY[r_, δ_, ν_] := (32/π^3)/Sqrt[(1 - ν) (2 - ν)] Sum[1/(2 n - 1)^3 1/Sqrt[1 - ((r/(π/2 (2 n - 1)))/Sqrt[1 + i * δ])^2], {n, 1, 5}]

QK[r_, δ_, ν_] := 2/Sqrt[(1 - ν) (2 - ν)] (8/π^2)/Sqrt[(π/2)^2 - r^2/(1 + i * δ)]
```

For Figure 9, fix ν=1/3 and δ_d=0.01

```
NU = 1/3; DE = 0.01;
```

Note: in some of the following plots the function **Quiet** is in use to remove messages concerning sporadic low convergence rates in the numerical integration (which do not have an impact on the results).

```
rlist = Table[ii/100., {ii, 10, Ceiling[350. * π/2]}];
fig1 = ListLinePlot[Transpose@{rlist, ParallelTable[Abs[Quiet[Q[rlist[[jj]], DE, NU]]],
     {jj, 1, Length[rlist]}]}, PlotRange → {{0, 3.5 * π / 2}, {0, 10}},
  Ticks → {{0, Pi/2, Pi, 3 Pi/2}, {0, 1, 5, 10}}, PlotStyle → Black, PlotLegends → {"Exact"}];
fig2 = Plot[{Abs[QK[r, DE, NU]], Abs[QVY[r, DE, NU]]}, {r, 0.1, 3.5 * π / 2},
  PlotRange → {{0, 3.5 * π / 2}, {0, 10}}, Ticks → {{0, Pi/2, Pi, 3 Pi/2}, {0, 1, 5, 10}},
  PlotLegends → {"Kloukinas et al.", "Younan-Veletsos"}]; Show[fig1, fig2]
```

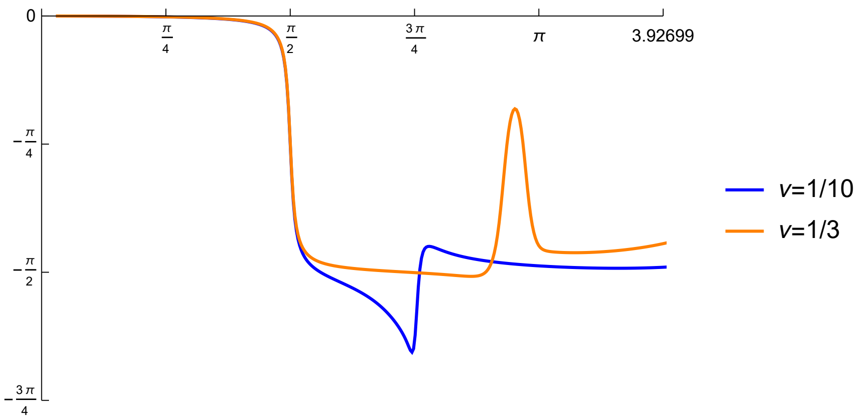


In order to generate the results in Figure 10, Figure 14 and Figure 15, just tweak the values of NU and DE.

For Figure 13, get the phase of this variable, fixing v=0.1 and δ_d=0.03

```
NU1 = 0.1; NU2 = 1 / 3; DE = 0.03;
```

```
rlist = Table[ii/100., {ii, 10, Ceiling[350. * π/2]}];
ListLinePlot[{Transpose@{rlist, ParallelTable[
     Arg[Quiet[Q[rlist[[jj]], DE, NU1]]], {jj, 1, Length[rlist]}]}, Transpose@
   {rlist, ParallelTable[Arg[Quiet[Q[rlist[[jj]], DE, NU2]]], {jj, 1, Length[rlist]}]}},
 PlotRange → {{0, 2.5 * π/2}, {-3 π / 4, 0}}, Ticks → {{0, Pi/4, Pi/2, 3 Pi/4, Pi, 2.5 Pi/2}, {-3 π /
     4, -π/2, -π / 4, 0}}, PlotStyle → {Blue, Orange}, PlotLegends → {"ν=1/10", "ν=1/3"}]
```



Analysis of intermediate resonance

To generate the plots in this section, we split the thrust in two contributions: the integral of horizontal gradient of horizontal displacement at the vertical ("*intdudxw*") and the vertical displacement at the top (already implemented).

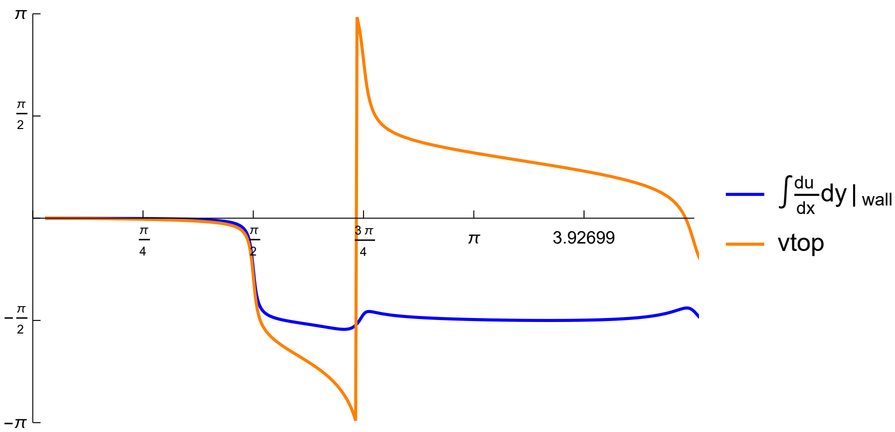
```
intdudxw[r_, δ_, ν_] := 2/Sqrt[2 π]
  NIntegrate[Sqrt[2/π] * 1/(c[ν]^2 β[k, r, δ, ν]^2) - Ac[k, r, δ, ν] * (Exp[-α[k, r, δ, ν]] - 1) -
    Bc[k, r, δ, ν] * (Exp[α[k, r, δ, ν]] - 1) - (k/β[k, r, δ, ν])^2 * (Cc[k, r, δ, ν] *
        (Exp[-β[k, r, δ, ν]] - 1) + Dc[k, r, δ, ν] * (Exp[β[k, r, δ, ν]] - 1)), {k, 0, 10}];
```

Plots

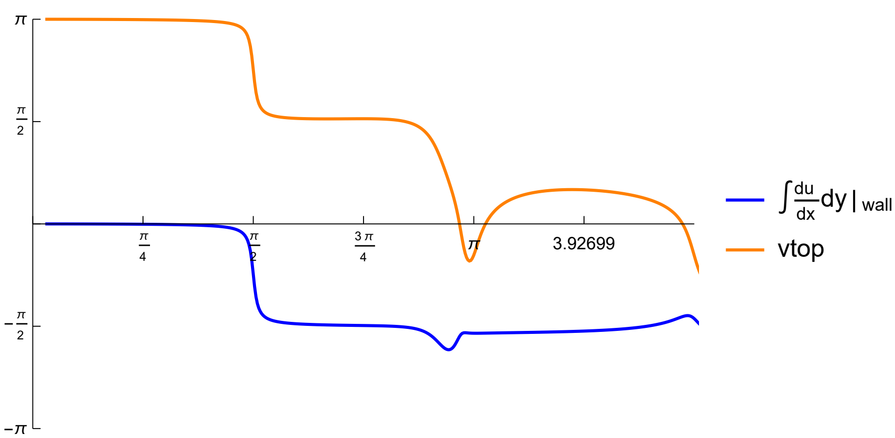
Compare the phases of these two components: Figure 12 (a) for v=0.1, and Figure 12 (b) for v=1/3.

```
NU = 0.1; DE = 0.03;
```

```
rlist = Table[ii/100., {ii, 10, Ceiling[400. * π/2]}];
ListLinePlot[{Transpose@{rlist, ParallelTable[
    Arg[Quiet[intdudxw[rlist[[jj]], DE, NU]]], {jj, 1, Length[rlist]}]}, Transpose@
  {rlist, ParallelTable[Arg[Quiet[vtop[rlist[[jj]], DE, NU]]], {jj, 1, Length[rlist]}]}},
 PlotRange → {{0, 3 * π/2}, {-π, π}}, Ticks → {{0, Pi/4, Pi/2, 3 Pi/4, Pi, 2.5 Pi/2}, {-π,
   -π/2, 0, π/2, π}}, PlotStyle → {Blue, Orange}, PlotLegends → {"∫du/dx dy|wall", "vtop"}]
```




```
NU = 1/3;
ListLinePlot[{Transpose@{rlist, ParallelTable[
    Arg[Quiet[intdudxw[rlist[[jj]], DE, NU]]], {jj, 1, Length[rlist]}]}, Transpose@
  {rlist, ParallelTable[Arg[Quiet[vtop[rlist[[jj]], DE, NU]]], {jj, 1, Length[rlist]}]}},
 PlotRange → {{0, 3 * π/2}, {-π, π}},
 Ticks → {{0, Pi/4, Pi/2, 3 Pi/4, Pi, 2.5 Pi/2}, {-π, -π/2, 0, π/2, π}},
 PlotStyle → {Blue, Orange}, PlotLegends → {"∫du/dx dy|wall", "vtop"}]
```



Eccentricity

To obtain the eccentricity one needs to evaluate the integral factor in eq.(37) evaluated at η=1:

```
intQ[r_, δ_, ν_] := 2/Sqrt[2 π] NIntegrate[
   Sqrt[2/π]/(2 β[k, r, δ, ν]^2) + Ac[k, r, δ, ν] * (c[ν]^2 + 2/α[k, r, δ, ν] * (Exp[-α[k, r, δ, ν]] - 1)) +
    Bc[k, r, δ, ν] * (c[ν]^2 - 2/α[k, r, δ, ν] * (Exp[α[k, r, δ, ν]] - 1)) +
    (c[ν]^2 (1 - (k/β[k, r, δ, ν])^2) - 2) * 1/β[k, r, δ, ν] (-Cc[k, r, δ, ν] *
        (Exp[-β[k, r, δ, ν]] - 1) + Dc[k, r, δ, ν] * (Exp[β[k, r, δ, ν]] - 1)) +
    ((c[ν] * k)/β[k, r, δ, ν])^2 * (Cc[k, r, δ, ν] + Dc[k, r, δ, ν]), {k, 0, 10}];
```

Once the integral and the thrust itself are ready, the eccentricity can be computed as in eq.(38):

```
ecc[r_, δ_, ν_] := Abs[(Q[r, δ, ν] - intQ[r, δ, ν])/Q[r, δ, ν]]
```

The corresponding expression for the eccentricity derived by Younan and Veletsos is given for comparison purposes (the one derived by Kloukinas et al is simply $2/\pi$):

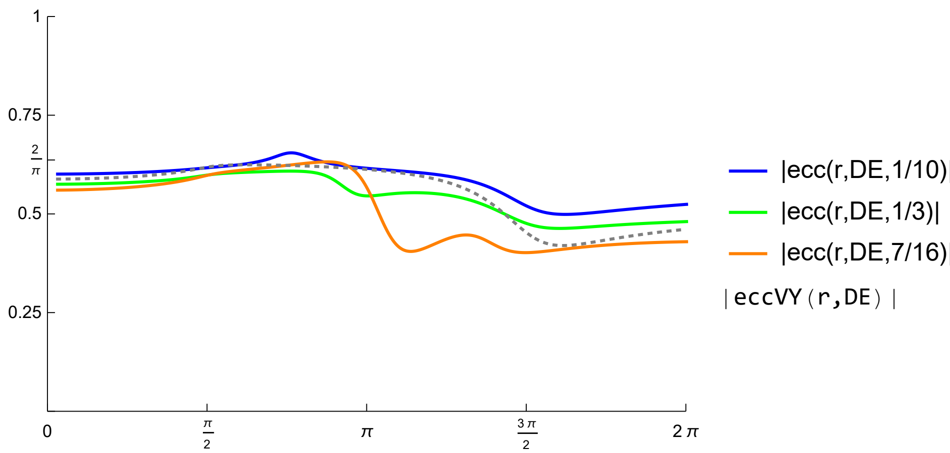
```
eccVY[r_, δ_] := 2/π (Sum[(-1)^(n+1)/(2 n-1)^3 1/Sqrt[1-(2 rd[r,δ]/π/(2 n-1))^2], {n, 1, 5}] /
    Sum[1/(2 n-1)^3 1/Sqrt[1-(2 rd[r,δ]/π/(2 n-1))^2], {n, 1, 5}])
```

Plots

For Figure 16(a), vary ν=0.1,1/3,7/16 while fixing δ_d=0.20 so as to plot the corresponding eccentricities.

```
DE = 0.20;
```

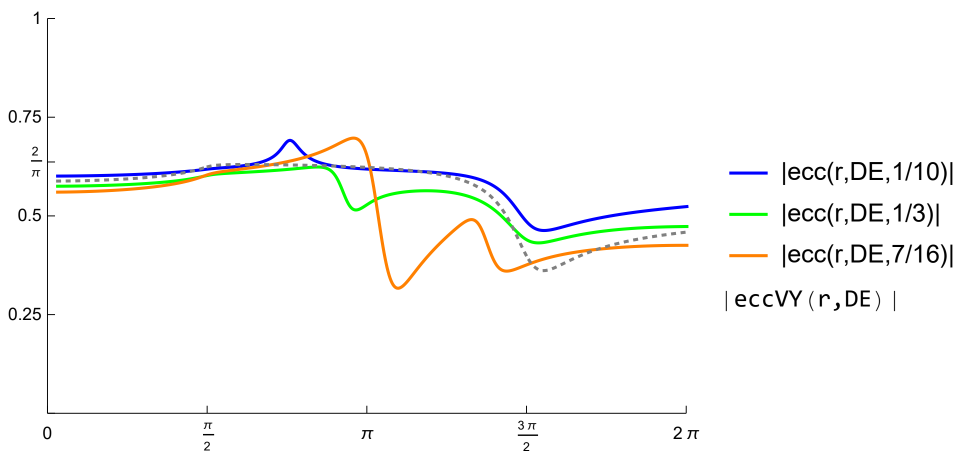
```
fig1 = ListLinePlot[{Transpose@{rlist,
      ParallelTable[Quiet[ecc[rlist[[jj]], DE, 0.1]], {jj, 1, Length[rlist]}]}, Transpose@
     {rlist, ParallelTable[Quiet[ecc[rlist[[jj]], DE, 1/3]], {jj, 1, Length[rlist]}]},
    Transpose@{rlist, ParallelTable[Quiet[ecc[rlist[[jj]], DE, 7/16]],
       {jj, 1, Length[rlist]}]}}, PlotRange → {{0, 4 * π/2}, {0, 1}},
  Ticks → {{0, Pi/2, Pi, 3 Pi/2, 2 π}, {0, 0.25, 0.5, 2/π, 0.75, 1}},
  PlotStyle → {Blue, Green, Orange},
  PlotLegends → {"|ecc(r,DE,1/10)|", "|ecc(r,DE,1/3)|", "|ecc(r,DE,7/16)|"}];
fig2 = Plot[{Abs[eccVY[r, DE]]}, {r, 0.1, 4 * π/2}, PlotRange → {{0, 2 * π}, {0, 1}},
  Ticks → {{0, Pi/2, Pi, 3 Pi/2, 2 Pi}, {0, 0.25, 0.5, 2/π, 0.75, 1}},
  PlotLegends → "|eccVY(r,DE)|", PlotStyle → {{Gray, Dotted}}]; Show[fig1, fig2]
```



For Figure 16(b), likewise, vary ν=0.1,1/3,7/16 while fixing δ_d=0.10.

```
DE = 0.10;
```

```
fig1 = ListLinePlot[{Transpose@{rlist,
      ParallelTable[Quiet[ecc[rlist[[jj]], DE, 0.1]], {jj, 1, Length[rlist]}]}, Transpose@
     {rlist, ParallelTable[Quiet[ecc[rlist[[jj]], DE, 1/3]], {jj, 1, Length[rlist]}]},
    Transpose@{rlist, ParallelTable[Quiet[ecc[rlist[[jj]], DE, 7/16]],
       {jj, 1, Length[rlist]}]}}, PlotRange → {{0, 4 * π/2}, {0, 1}},
   Ticks → {{0, Pi/2, Pi, 3 Pi/2, 2 π}, {0, 0.25, 0.5, 2/π, 0.75, 1}},
   PlotStyle → {Blue, Green, Orange},
   PlotLegends → {"|ecc(r,DE,1/10)|", "|ecc(r,DE,1/3)|", "|ecc(r,DE,7/16)|"}];
fig2 = Plot[{Abs[eccVY[r, DE]]}, {r, 0.1, 4 * π/2}, PlotRange → {{0, 2 * π}, {0, 1}},
   Ticks → {{0, Pi/2, Pi, 3 Pi/2, 2 Pi}, {0, 0.25, 0.5, 2/π, 0.75, 1}},
   PlotLegends → "|eccVY(r,DE)|", PlotStyle → {{Gray, Dotted}}]; Show[fig1, fig2]
```



Stress distribution on the wall

The following implements eq.(33):

```
sigmaxx[η_, r_, δ_, ν_] := 1/Sqrt[2 π] NIntegrate[Sqrt[2/π]/β[k, r, δ, ν]^2 + 2 α[k, r, δ, ν]
     (Ac[k, r, δ, ν] * Exp[-α[k, r, δ, ν] * η] - Bc[k, r, δ, ν] * Exp[α[k, r, δ, ν] * η]) -
    β[k, r, δ, ν] (c[ν]^2 (1 - (k/β[k, r, δ, ν])^2) - 2) * (Cc[k, r, δ, ν] *
       Exp[-β[k, r, δ, ν] * η] - Dc[k, r, δ, ν] * Exp[β[k, r, δ, ν] * η]), {k, -10, 10}];
```

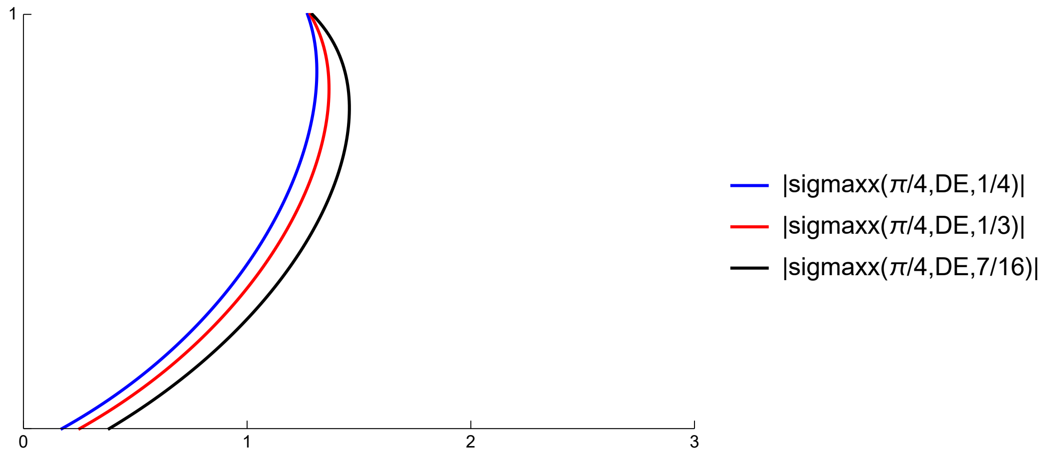
Once the stress distribution on the wall is available, it can be evaluated to generate the results in Figure 14.

The following line fixes the spatial discretization used to evaluate stresses along the height, and the value of damping is assigned ($\delta_d = 0.10$)

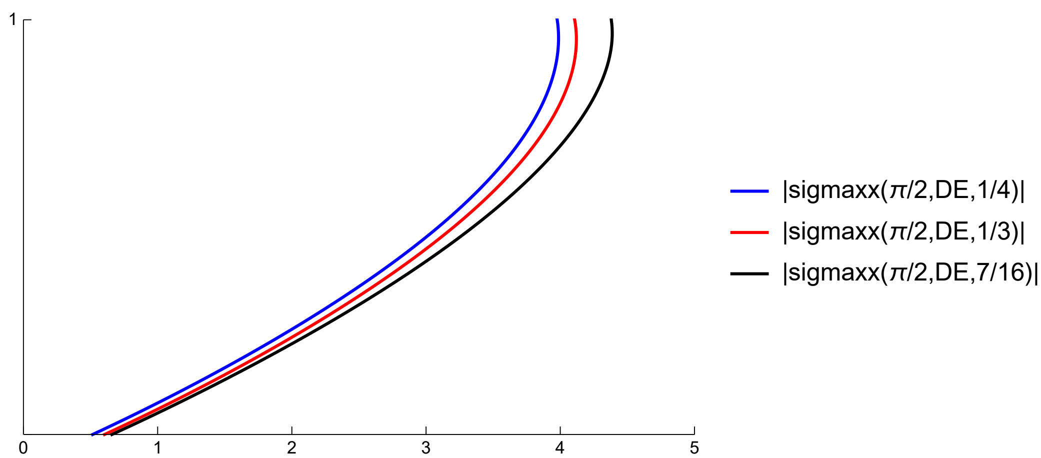
```
ylist = Table[ii/100., {ii, 0, 100}]; DE = 0.1;
```

For Figure 17, we used 6 different values of *r* to graph stresses: from π/4 to 3π/2 in increments of π/4.

```
RR = π/4;
FIG1 = ListLinePlot[{Transpose@{ParallelTable[Quiet[Abs[sigmaxx[ylist[[jj]], RR, DE,
        0.25]]], {jj, 1., Length[ylist]}], ylist}, Transpose@{ParallelTable[
      Quiet[Abs[sigmaxx[ylist[[jj]], RR, DE, 0.33333]]], {jj, 1., Length[ylist]}], ylist},
    Transpose@{ParallelTable[Quiet[Abs[sigmaxx[ylist[[jj]], RR, DE, 0.4375]]],
      {jj, 1., Length[ylist]}], ylist}}, PlotRange → {{0, 3}, {0, 1}},
   Ticks → {{0, 1, 2, 3, 4, 5}, {0, 1}}, PlotStyle → {Blue, Red, Black}, PlotLegends →
    {"|sigmaxx(π/4,DE,1/4)|", "|sigmaxx(π/4,DE,1/3)|", "|sigmaxx(π/4,DE,7/16)|"}]
```

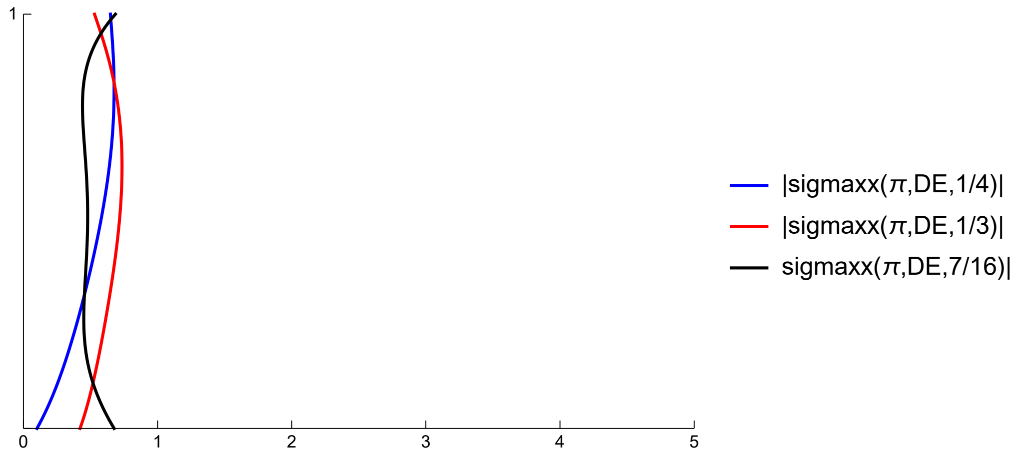



```
RR = π/2;
FIG2 = ListLinePlot[{Transpose@{ParallelTable[Quiet[Abs[sigmaxx[ylist[[jj]], RR, DE,
        0.25]]], {jj, 1., Length[ylist]}], ylist}, Transpose@{ParallelTable[
      Quiet[Abs[sigmaxx[ylist[[jj]], RR, DE, 0.33333]]], {jj, 1., Length[ylist]}], ylist},
    Transpose@{ParallelTable[Quiet[Abs[sigmaxx[ylist[[jj]], RR, DE, 0.4375]]],
      {jj, 1., Length[ylist]}], ylist}}, PlotRange → {{0, 5}, {0, 1}},
   Ticks → {{0, 1, 2, 3, 4, 5}, {0, 1}}, PlotStyle → {Blue, Red, Black}, PlotLegends →
    {"|sigmaxx(π/2,DE,1/4)|", "|sigmaxx(π/2,DE,1/3)|", "|sigmaxx(π/2,DE,7/16)|"}]
```

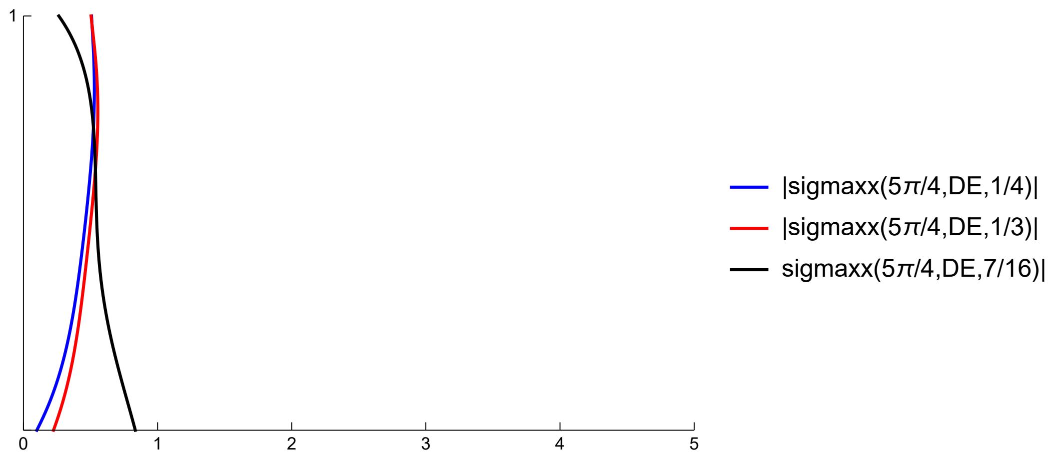


```
RR = 3 π / 4;
FIG3 = ListLinePlot[{Transpose@{ParallelTable[Quiet[Abs[sigmaxx[ylist[[jj]], RR, DE,
          0.25]]], {jj, 1., Length[ylist]}], ylist}, Transpose@{ParallelTable[
       Quiet[Abs[sigmaxx[ylist[[jj]], RR, DE, 0.33333]]], {jj, 1., Length[ylist]}], ylist},
    Transpose@{ParallelTable[Quiet[Abs[sigmaxx[ylist[[jj]], RR, DE, 0.4375]]],
       {jj, 1., Length[ylist]}], ylist}}, PlotRange → {{0, 5}, {0, 1}},
   Ticks → {{0, 1, 2, 3, 4, 5}, {0, 1}}, PlotStyle → {Blue, Red, Black}, PlotLegends →
    {"|sigmaxx(3π/4,DE,1/4)|", "|sigmaxx(3π/4,DE,1/3)|", "sigmaxx(3π/4,DE,7/16)|"}];
```

```
RR = π;
FIG4 = ListLinePlot[{Transpose@{ParallelTable[Quiet[Abs[sigmaxx[ylist[[jj]], RR, DE,
          0.25]]], {jj, 1., Length[ylist]}], ylist}, Transpose@{ParallelTable[
       Quiet[Abs[sigmaxx[ylist[[jj]], RR, DE, 0.33333]]], {jj, 1., Length[ylist]}], ylist},
    Transpose@{ParallelTable[Quiet[Abs[sigmaxx[ylist[[jj]], RR, DE, 0.4375]]],
       {jj, 1., Length[ylist]}], ylist}}, PlotRange → {{0, 5}, {0, 1}},
   Ticks → {{0, 1, 2, 3, 4, 5}, {0, 1}}, PlotStyle → {Blue, Red, Black},
   PlotLegends → {"|sigmaxx(π,DE,1/4)|", "|sigmaxx(π,DE,1/3)|", "sigmaxx(π,DE,7/16)|"}]
```



```
RR = 5 π / 4;
FIG5 = ListLinePlot[{Transpose@{ParallelTable[Quiet[Abs[sigmaxx[ylist[[jj]], RR, DE,
          0.25]]], {jj, 1., Length[ylist]}], ylist}, Transpose@{ParallelTable[
       Quiet[Abs[sigmaxx[ylist[[jj]], RR, DE, 0.33333]]], {jj, 1., Length[ylist]}], ylist},
    Transpose@{ParallelTable[Quiet[Abs[sigmaxx[ylist[[jj]], RR, DE, 0.4375]]],
       {jj, 1., Length[ylist]}], ylist}}, PlotRange → {{0, 5}, {0, 1}},
   Ticks → {{0, 1, 2, 3, 4, 5}, {0, 1}}, PlotStyle → {Blue, Red, Black}, PlotLegends →
    {"|sigmaxx(5π/4,DE,1/4)|", "|sigmaxx(5π/4,DE,1/3)|", "sigmaxx(5π/4,DE,7/16)|"}]
```




```
RR = 3 π / 2;
FIG6 = ListLinePlot[{Transpose@{ParallelTable[Quiet[Abs[sigmaxx[ylist[[jj]], RR, DE,
          0.25]]], {jj, 1., Length[ylist]}], ylist}, Transpose@{ParallelTable[
       Quiet[Abs[sigmaxx[ylist[[jj]], RR, DE, 0.33333]]], {jj, 1., Length[ylist]}], ylist},
    Transpose@{ParallelTable[Quiet[Abs[sigmaxx[ylist[[jj]], RR, DE, 0.4375]]],
       {jj, 1., Length[ylist]}], ylist}}, PlotRange → {{0, 5}, {0, 1}},
   Ticks → {{0, 1, 2, 3, 4, 5}, {0, 1}}, PlotStyle → {Blue, Red, Black}, PlotLegends →
    {"|sigmaxx(3π/2,DE,1/4)|", "|sigmaxx(3π/2,DE,1/3)|", "sigmaxx(3π/2,DE,7/16)|"}]
```