On the fundamental resonant mode of inhomogeneous soil deposits
Numerical computation of base-to-top transfer function for layered site

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The authors would like to detail the implementation of a simple version of Haskell-Thompson method, as presented in Kramer (1996), utilized in the *Jupyter* notebook used for calculations (see Supplementary Material section in the paper for more information).

Wave propagation fundamentals

Beginning from the wave equation (for each i-th layer)

$$\frac{\partial^2 u_i}{\partial z^2} = \frac{1}{V_i^2}\frac{\partial^2 u_i}{\partial t^2}\,. \tag{1}$$

A potential solution (for harmonic loading) for two consecutive layers

$$u_i = A_i e^{i(\varpi t + k_i z_i)} + B_i e^{-i(\varpi t + k_i z_i)}\,, \tag{2a}$$

$$u_{i+1} = A_{i+1} e^{i(\varpi t + k_{i+1} z_{i+1})} + B_{i+1} e^{-i(\varpi t + k_{i+1} z_{i+1})} \tag{2b}$$

where $k_i = \varpi / V_i$ represents the wavelengths in the i-th layer. Solving the problem reduces to finding the coefficients A (amplitude of the upwards-propagating wave in each layer), B (amplitude of the downwards-propagating wave in each layer) that satisfy the boundary conditions of the problem (note: there are no considerations in terms of initial conditions, since the system is assumed to have reached steady-state conditions). The boundary conditions to consider are the following.
Free surface:

$$\tau_{xy}(z_1 = 0) = 0 \rightarrow A_1 = B_1\,, \tag{3a}$$

Displacement continuity and stresses equilibrium at the lower interface between i-th and $i+1$-th:

$$\tau_{xz}(z_i = h_i) = \tau_{xz}(z_{i+1} = 0\,, \tag{3b}$$

$$u(z_i = h_i) = u(z_{i+1} = 0)\,. \tag{3c}$$

Eq.(3b) and Eq.(3c) effectively translate into

$$A_{i+1} = \frac{A_i}{2}(1+\alpha_i)e^{ik_i h_i} + \frac{B_i}{2}(1-\alpha)e^{-ik_i h_i}\,, \tag{4a}$$

$$B_{i+1} = \frac{A_i}{2}(1-\alpha_i)e^{ik_i h_i} + \frac{B_i}{2}(1+\alpha)e^{-ik_i h_i}\,, \tag{4b}$$

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where α_i represents the impedance ratio:

$$\alpha_i = \frac{\sqrt{\mu_i \rho_i}}{\sqrt{\mu_{i+1}\rho_{i+1}}} = \frac{\rho_i V_i}{\rho_{i+1} V_{i+1}} \,. \tag{5}$$

Implementation (considering rigid bedrock)

Starting from

$$A_{m+1} = \frac{A_m}{2}(1+\alpha)e^{ik_m h_m} + \frac{B_m}{2}(1-\alpha)e^{-ik_m h_m} \tag{6a}$$

$$B_{m+1} = \frac{A_m}{2}(1-\alpha)e^{ik_m h_m} + \frac{B_m}{2}(1+\alpha)e^{-ik_m h_m} \tag{6b}$$

$k_m = \varpi / V_m$ for $m = 1 \ldots N-1$. See this can be written in matrix form

$$\boldsymbol{A}_{m+1} = \begin{bmatrix} A_{m+1} \\ B_{m+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1+\alpha_m)e^{ik_m h_m} & \frac{1}{2}(1-\alpha_m)e^{-ik_m h_m} \\ \frac{1}{2}(1-\alpha_m)e^{ik_m h_m} & \frac{1}{2}(1+\alpha_m)e^{-ik_m h_m} \end{bmatrix} \begin{bmatrix} A_m \\ B_m \end{bmatrix} = \boldsymbol{L}_m \boldsymbol{A}_m \,, \tag{7}$$

thus there is a clear recurrence:

$$\boldsymbol{A}_{m+1} = \boldsymbol{L}_m \boldsymbol{A}_m = \boldsymbol{L}_m(\boldsymbol{L}_{m-1}\boldsymbol{A}_{m-1}) = \ldots = \prod_{j=1}^{m} \boldsymbol{L}_j \boldsymbol{A}_1 = \prod_{j=1}^{m} \boldsymbol{L}_j \begin{bmatrix} 1 \\ 1 \end{bmatrix} A_1 \,. \tag{8}$$

note that the matrices of deeper layer multiply from the left.

Implementing rigid bedrock condition

Consider a stack of N layers, the one labeled 1 being the top one while N is the lower one in contact to the bedrock. This last layer is assumed to be *infinitely* softer that the rock below, thus at the contact interface between this layer and the rock:

$$\alpha_N = \frac{\rho_N V_N}{\rho_{rock} V_{rock}} = 0 \,, \tag{9}$$

hence, for the last layer considered in the calculations, the corresponding impedance is taken to be zero.

For the rock layer, set $m+1 = N+1$:

$$\boldsymbol{A}_{N+1} = \boldsymbol{L}_N \boldsymbol{A}_N = \underbrace{\prod_{j=1}^{N} \boldsymbol{L}_j}_{\boldsymbol{L}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} A_1 = \boldsymbol{L} \begin{bmatrix} 1 \\ 1 \end{bmatrix} A_1 = \begin{bmatrix} L_{11} + L_{12} \\ L_{21} + L_{11} \end{bmatrix} A_1 = \begin{bmatrix} A_N \\ B_N \end{bmatrix} \,, \tag{10}$$

thus the matrix $\boldsymbol{L}$ relates wave amplitudes in the rigid rock to the displacements at the ground surface (characterized by A_1).

For each fixed ω_k to be considered, the matrix

$$\boldsymbol{L}(\omega_k) = \prod_{j=1}^{N} \boldsymbol{L}_j(\omega_k) \,, \tag{11}$$

is computed by the function `getAmp` in the following manner: the interfaces are looped, but first an auxiliary matrix

$$\boldsymbol{L}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \boldsymbol{L}(\omega_k) \,, \tag{12a}$$

is set up, thus, after computing $\boldsymbol{L}_1$ according to eq.(7),the update

$$\boldsymbol{L}(\omega_k) \to \boldsymbol{L}_1(\omega_k)\boldsymbol{L}(\omega_k)\,, \tag{12b}$$

thence the product can be computed in accumulative manner, for instance, the n-th layer adds

$$\boldsymbol{L}(\omega_k) \to \boldsymbol{L}_n(\omega_k)\boldsymbol{L}(\omega_k) = \boldsymbol{L}_n(\omega_k)\ldots\boldsymbol{L}_1(\omega_k)\boldsymbol{L}_0\,, \tag{12c}$$

until reaching the last impedance of the list (which, recall, is zero)

$$\boldsymbol{L}(\omega_k) \to \boldsymbol{L}_N(\omega_k)\boldsymbol{L}(\omega_k)\,. \tag{12d}$$

The information of wave propagation in the system is consigned, for each frequency (wavelength), within $\boldsymbol{L}_k$. It only remains to utilize it to establish the relation between displacement at the base and at the top.

See that the displacement at the top of the rock, what we call X_g, can be expressed as

$$X_g = u_{N+1} = A_{N+1} + B_{N+1}\,, \tag{13}$$

by following the eqs.(4) and fixing $z_{N+1} = 0$ (what's equivalent to $z_N = h_N$). For each frequency ω_k, eq.(7) in conjunction with eq.(8) implies

$$\begin{aligned}
\boldsymbol{A}_{N+1}(\omega_k) = \begin{bmatrix} A_{N+1}(\omega_k) \\ B_{N+1}(\omega_k) \end{bmatrix} &= \prod_{j=1}^{N} \boldsymbol{L}_j(\omega_k) \begin{bmatrix} 1 \\ 1 \end{bmatrix} A_1 && (14a)\\
&= \boldsymbol{L}(\omega_k) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{u_{top}}{2} && (14b)\\
&= \begin{bmatrix} L_{11}(\omega_k) + L_{12}(\omega_k) \\ L_{21}(\omega_k) + L_{22}(\omega_k) \end{bmatrix} \frac{u_{top}}{2}\,, && (14c)
\end{aligned}$$

wherein the fact that $u_{top} = 2A_1$, eq.3a, has been brought to bear. Adding the two rows in the previous matrix equation:

$$X_g = A_{N+1}(\omega_k) + B_{N+1}(\omega_k) = \left(L_{11}(\omega_k) + L_{12}(\omega_k) + L_{21}(\omega_k) + L_{22}(\omega_k)\right)\frac{u_{top}}{2}\,, \tag{15}$$

whence we finally obtain that, for each frequency ω_k,

$$\frac{u_{top}}{X_g} = \frac{2}{L_{11}(\omega_k) + L_{12}(\omega_k) + L_{21}(\omega_k) + L_{22}(\omega_k)} = A(\omega_k)\,, \tag{16}$$

which is the variable to be displayed in the plots:

Quick verification: divide homogeneous layer in K slices

Imagine we wanted to calculate the transfer function of a homogeneous site dividing it in K sub-layers (not necessarily of the same height). The vector of impedances will be $\alpha_i = 1\ \forall i \neq K$ and $\alpha_N = 0$ (last layer - bedrock). Hence (k is being so as to point out that the wavenumber is the same in all layers),

$$\begin{aligned}
\boldsymbol{L}_j =& \frac{1}{2}\begin{bmatrix} 2e^{\mathrm{i}\mathrm{k}h_j} & 0 \\ 0 & 2e^{-\mathrm{i}\mathrm{k}h_j} \end{bmatrix} \qquad \forall\, j \in [1, K-1]\,, && (17a)\\
\boldsymbol{L}_K =& \frac{1}{2}\begin{bmatrix} e^{\mathrm{i}\mathrm{k}h_N} & e^{-\mathrm{i}\mathrm{k}h_N} \\ e^{\mathrm{i}\mathrm{k}h_N} & e^{-\mathrm{i}\mathrm{k}h_N} \end{bmatrix}\,. && (17b)
\end{aligned}$$

Then, see how

$$\prod_{j=1}^{N-1} \boldsymbol{L}_j = \boldsymbol{L}_{N-1}\ldots\boldsymbol{L}_1 = \begin{bmatrix} \exp\left(i\mathrm{k}\sum_{j=1}^{N-1} h_j\right) & 0 \\ 0 & \exp\left(-\mathrm{i}\mathrm{k}\sum_{j=1}^{N-1} h_j\right) \end{bmatrix} \tag{18}$$

thus

$$\prod_{j=1}^{N} \boldsymbol{L}_j = \frac{1}{2}\begin{bmatrix} e^{i\mathrm{k}h_N} & e^{-i\mathrm{k}h_N} \\ e^{i\mathrm{k}h_N} & e^{-i\mathrm{k}h_N} \end{bmatrix} \begin{bmatrix} \exp\left(i\mathrm{k}\sum_{j=1}^{N-1} h_j\right) & 0 \\ 0 & \exp\left(-i\mathrm{k}\sum_{j=1}^{N-1} h_j\right) \end{bmatrix} \tag{19a}$$

$$= \frac{1}{2}\begin{bmatrix} \exp\left(i\mathrm{k}\sum_{j=1}^{N} h_j\right) & \exp\left(-i\mathrm{k}\sum_{j=1}^{N} h_j\right) \\ \exp\left(i\mathrm{k}\sum_{j=1}^{N} h_j\right) & \exp\left(-i\mathrm{k}\sum_{j=1}^{N} h_j\right) \end{bmatrix} \tag{19b}$$

$$= \frac{1}{2}\begin{bmatrix} \exp\left(i\mathrm{k}H\right) & \exp\left(-i\mathrm{k}H\right) \\ \exp\left(i\mathrm{k}H\right) & \exp\left(-i\mathrm{k}H\right) \end{bmatrix} . \tag{19c}$$

Finally, from eq.(15):

$$X_g = \frac{1}{2}\left(2e^{i\mathrm{k}H} + 2e^{-i\mathrm{k}H}\right)\frac{u_{top}}{2} = \cos(\mathrm{k}H)u_{top} , \tag{20a}$$

$$\frac{u_{top}}{X_g} = \frac{1}{\cos(\mathrm{k}H)} . \tag{20b}$$