Supplementary Information

Universal path decomposition
of multilayer transfer and scattering matrices

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Supplementary Note 1: Extra details on derivation and fields of application

Shear waves polarized in plane are used henceforth to illustrate. Either by taking the Fourier transform or by assuming a plane-wave *ansatz*, the frequency-domain version of the equilibrium equation and the constitutive law are, in the case of shear waves polarized out of plane

$$\left.\begin{array}{rl}\text{Constitutive relation:} & \frac{d}{dx}(\hat{u}) = \frac{\hat{\tau}}{G} \\ \text{Equilibrium equation:} & \frac{d}{dx}(\hat{\tau}) = -\rho\omega^2\hat{u}\end{array}\right\} \quad \frac{d}{dx}\begin{bmatrix}\hat{u}\\ \hat{\tau}\end{bmatrix} = \begin{bmatrix}0 & 1/G\\ -\rho\omega^2 & 0\end{bmatrix}\begin{bmatrix}\hat{u}\\ \hat{\tau}\end{bmatrix} \Rightarrow \boxed{\frac{d\boldsymbol{f}}{dx} = \boldsymbol{A}\boldsymbol{f}(z)}$$

and recasting in matrix ODE form, where $\hat{u}$ and $\hat{\tau}$ represent the amplitude of the displacement and the stress, respectively, G is the shear modulus and ω is the circular frequency; $\boldsymbol{A}$ is called "layer matrix". For the k-th layer, stretching from x_k to x_{k+1}, its transfer matrix $\boldsymbol{T}(x_{k+1}, x_k)$ is then

(Supplementary Equation 1)

$$\begin{bmatrix}\hat{u}\\ \hat{\tau}\end{bmatrix}_{x=x_{k+1}} = \exp\left(\boldsymbol{A}_k h_k\right)\begin{bmatrix}\hat{\tau}\\ \hat{u}\end{bmatrix}_{x=x_k} = \boldsymbol{T}(x_{k+1}, x_k)\begin{bmatrix}\hat{u}\\ \hat{\tau}\end{bmatrix}_{x=x_k} = \begin{bmatrix}\cos\left(\frac{\omega h_k}{c_k}\right) & \frac{\sin\left(\frac{\omega h_k}{c_k}\right)}{\omega Z_k}\\ -\omega Z_k \sin\left(\frac{\omega h_k}{c_k}\right) & \cos\left(\frac{\omega h_k}{c_k}\right)\end{bmatrix}\begin{bmatrix}\hat{u}\\ \hat{\tau}\end{bmatrix}_{x=x_k},$$

and $c_k = \sqrt{G_k/\rho_k}$ is the shear wave speed in the layer and $Z_k = \sqrt{\rho_k G_k}$ is its (elastic) impedance, while ρ_k is its density, ω is the frequency of the wave, $h = x_{k+1} - x_k$ is the thickness of the layer. This example fits in the theory outlined in the paper simply by considering the stress $\hat{\tau}$ (and not just the gradient of the displacement amplitude $\hat{u}'$) to be the "gradient", what requires replacing $k_i \to \mu_i k_i = Z_i$ in the path coefficients; having done this relabeling, the formulae in the paper are ready to be used. See that the frequency simplifies away from all ratios, thus is not mentioned in the main text. Supplementary Table 1 details the transfer matrices and amplitude gradient variables that should be used in each case to fit with the path-sum decomposition.

The transfer matrix formulation relates amplitudes at different cross-sections; conversely, the scattering matrix looks at amplitudes propagating in and out of a stack. To find the relation between the two decompose the fields as $f = Ae^{-\mathrm{i}kx} + Be^{\mathrm{i}kx}$, where A, B are the amplitudes of the right-propagating wave and the left-propagating one. In the i-th layer:

$$\begin{bmatrix}A\\ B\end{bmatrix}_{x=x_i} = \begin{bmatrix}e^{\mathrm{i}kx_i}/2 & \mathrm{i}e^{\mathrm{i}kx_i}/2k\\ e^{-\mathrm{i}kx_i}/2 & -\mathrm{i}e^{-\mathrm{i}kx_i}/2k\end{bmatrix}\begin{bmatrix}f\\ f'\end{bmatrix}_{x=x_i} = \boldsymbol{E}(x_i)\begin{bmatrix}f\\ f'\end{bmatrix}_{x=x_i},$$

hence, the amplitudes of the outgoing waves (at x_N) in terms of the incoming waves (at x_0):

$$\begin{bmatrix}A_R\\ B_R\end{bmatrix}_{x=x_N} = \boldsymbol{E}(x_N)\boldsymbol{T}(x_N,\, x_0)\boldsymbol{E}(x_0)^{-1}\begin{bmatrix}A_L\\ B_L\end{bmatrix}_{x=x_0} = \boldsymbol{S}(x_N,\, x_0)\begin{bmatrix}A_L\\ B_L\end{bmatrix}_{x=x_0},$$

where $\boldsymbol{S}$ is the scattering matrix and the subcripts R and L denote wave amplitudes at the right end of the stack and at the left one, the inputs to the stack are A_L coming from the left into the first layer at $x_1 = 0$ and B_R coming from the right into the last layer at x_N. Using the prior relations, $\boldsymbol{S}$

	Layer Matrix $\boldsymbol{A}$	Transfer Matrix $\boldsymbol{T}$	Amplitude-Gradient $\boldsymbol{f}$
Elasticity (1D shear)	$\begin{bmatrix} 0 & 1/G \\ -\rho\omega^2 & 0 \end{bmatrix}$	$\begin{bmatrix} \cos(k_e h) & \frac{\sin(k_e h)}{\omega Z_e} \\ -\omega Z_e \sin(k_e h) & \cos(k_e h) \end{bmatrix}$	$[\hat{u} \quad \hat{\tau}]^\top$
Electromagnetism (normal incidence)	$\begin{bmatrix} 0 & 1 \\ -\mu\varepsilon\omega^2 & 0 \end{bmatrix}$	$\begin{bmatrix} \cos(k_o h) & \frac{\sin(k_o h)}{k_o} \\ -k_o \sin(k_o h) & \cos(k_o h) \end{bmatrix}$	$[\hat{E} \quad \hat{E}']^\top$
Acoustics (longitudinal)	$\begin{bmatrix} 0 & \mathrm{i}\omega/B \\ \mathrm{i}\omega\rho & 0 \end{bmatrix}$	$\begin{bmatrix} \cos(k_a h) & \frac{\mathrm{i}\sin(k_a h)}{Z_a} \\ \mathrm{i}Z_a \sin(k_a h) & \cos(k_a h) \end{bmatrix}$	$[\hat{v} \quad \hat{P}]^\top$
Quantum (Schrödinger eq.)	$\begin{bmatrix} 0 & 1 \\ -\frac{2m(E-V)}{\hbar^2} & 0 \end{bmatrix}$	$\begin{bmatrix} \cos(kh) & \frac{\sin(kh)}{k} \\ -k \sin(kh) & \cos(kh) \end{bmatrix}$	$[\psi \quad \psi']^\top$

Supplementary Table 1. General structure ($\boldsymbol{f}' = \boldsymbol{A}\boldsymbol{f}$, $\boldsymbol{T} = \exp(\boldsymbol{A}h)$) of 1D problems in different layered media: layer matrices, transfer matrices and amplitude-gradient vectors for elastic shear waves (density ρ, shear modulus G, frequency ω, $\hat{\tau}$ amplitude of shear stress, $\hat{u}$ amplitude of displacements, the elastic (shear) impedance and the wavenumber are $Z_e = \sqrt{\rho G}$, $k_e = \omega/\sqrt{G/\rho}$, respectively), electromagnetic waves (medium permeability μ, medium permittivity ε, $\hat{E}$ amplitude of the electric field, $\hat{E}' = \mathrm{d}\hat{E}/\mathrm{d}z$, $k_o = \omega\sqrt{\mu\varepsilon}$), acoustic waves (medium bulk modulus B, particle velocity $\hat{v}$, pressure $\hat{P}$, the acoustic ones $Z_a = \sqrt{\rho B}$, $k_a = \omega/\sqrt{B/\rho}$.) and quantum particles (energy of the quantum wave-particle E, potential well intensity V, mass m, $\hbar$ is the reduced Planck constant, $k = \sqrt{2m(E-V)/\hbar^2}$ is the quantum wavenumber, ψ is the time-independent wavefunction amplitude and $\psi' = \mathrm{d}\psi/\mathrm{d}x$ its gradient). The imaginary unit is $\mathrm{i} = \sqrt{-1}$.

can be written in terms of the N-layer transfer matrix:

$$\begin{aligned}
(S_{11})2k_N e^{-\mathrm{i}k_N x_N} &= k_N T_{11} + k_1 T_{22} + \mathrm{i}\left(T_{21} - k_1 k_N T_{12}\right) \, , \\
(S_{22})2k_N e^{\mathrm{i}k_N x_N} &= k_N T_{11} + k_1 T_{22} - \mathrm{i}\left(T_{21} - k_1 k_N T_{12}\right) \, , \\
(S_{12})2k_N e^{-\mathrm{i}k_N x_N} &= k_N T_{11} - k_1 T_{22} + \mathrm{i}\left(T_{21} + k_1 k_N T_{12}\right) \, , \\
(S_{21})2k_N e^{\mathrm{i}k_N x_N} &= k_N T_{11} - k_1 T_{22} - \mathrm{i}\left(T_{21} + k_1 k_N T_{12}\right) \, .
\end{aligned}$$

Since all its entries can be written in terms of those of $\boldsymbol{T}$, $\boldsymbol{S}$ also admits a path decomposition. Taking $x_N = 0$ as well, substituting the expressions and gathering terms we find:

$$\begin{aligned}
S_{11} &= \sum_{j=1}^{2^{N-1}} \mathcal{S}_j e^{-\mathrm{i}\varphi_j} \, , \quad S_{22} = \sum_{j=1}^{2^{N-1}} \mathcal{S}_j e^{\mathrm{i}\varphi_j} \, , \\
S_{12} &= \sum_{j=1}^{2^{N-1}} \mathcal{R}_j e^{-\mathrm{i}\varphi_j} \, , \quad S_{21} = \sum_{j=1}^{2^{N-1}} \mathcal{R}_j e^{\mathrm{i}\varphi_j} \, ,
\end{aligned}$$

where the amplitudes $\mathcal{S}_j$ and $\mathcal{R}_j$ depend on the impedance of the interfaces and the wavenumber ratio between the first and the last layer:

$$\mathcal{S}_j = \frac{1}{2}\left(\mathcal{A}_j + \mathrm{e}_{jN}\frac{k_1}{k_N}\mathcal{A}'_j\right) \, , \quad \mathcal{R}_j = \frac{1}{2}\left(\mathcal{A}_j - \mathrm{e}_{jN}\frac{k_1}{k_N}\mathcal{A}'_j\right) \, .$$

Supplementary Note 2: Algebraic proof of path decomposition

This appendix focuses on rederiving the main result (path-sum decomposition of transfer and scattering matrices) on purely algebraic grounds, in contrast to the main text that uses a physically-motivated path-enumeration construction. This proof is adapted from the one by M. Lemm [2], with three salient changes that do not modify its general structure: (1) we work with the wavenumber instead of the impedance; (2) we will introduce the path-defining binary vector $\mathbf{e}_j$ at the end to simplify the expressions that would otherwise be expressed using multi-indices, (3) as a last step, we show that the physically derived expression for the path amplitudes in terms of $\mathbf{e}_j$ is mathematically equivalent to the more formal multi-index expression obtained via the algebraic expansion. Multi-indices are kept until the second-to-last step because they greatly simplify the necessary bookkeeping. Below an outline of the proof:

Proof Structure

The path decomposition formula is proven in five main steps:

(1) Expand the total transfer matrix as a product of atomic matrices using trigonometric identities.
(2) Group terms by parity into symmetric (diagonal) and antisymmetric (off-diagonal) forms, representing the overall combinatorial structure with multi-indices.
(3) Using the product-to-sum formula, identify each term with a unique cosine or sine function and encode each term's phase structure (φ_j) with a binary sign vector.
(4) Match algebraic expression of amplitudes ($\mathcal{A}_j$) in terms of multi-indices to path amplitudes as defined in the main text.

A small difference with respect to the paper: we shall use h instead of l to denote the thickness of the layer, as l will be used as a summation index eventually during the derivation.
We begin with the i-th transfer matrix, which can be written as

$$\boldsymbol{T}_i = \cos(k_i h_i)\left(\boldsymbol{I} + \tan(k_i h_i)\begin{bmatrix} 0 & 1/k_i \\ -k_i & 0 \end{bmatrix}\right),$$

hence the cumulative propagator for a N-layer representative cell:

$$\boldsymbol{T} = \prod_{i=1}^{N} \boldsymbol{T}_i = \prod_{i=1}^{N} \cos(k_i h_i)\left(\boldsymbol{I} + \tan(k_i h_i)\begin{bmatrix} 0 & 1/k_i \\ -k_i & 0 \end{bmatrix}\right). \qquad \text{(Supplementary Equation 4)}$$

See how the product of the cosine factors can be effectuated and passed to the left-hand side

$$\left(\prod_{i=1}^{N} \cos(k_i h_i)\right)^{-1} \boldsymbol{T} = \prod_{i=1}^{N}\left(\boldsymbol{I} + \tan(k_i h_i)\begin{bmatrix} 0 & 1/k_i \\ -k_i & 0 \end{bmatrix}\right)$$

thus we would have to multiply over all the terms that appear in the N binomials

$$= \left(\boldsymbol{I} + \tan(r_1)\begin{bmatrix} 0 & 1/k_1 \\ -k_1 & 0 \end{bmatrix}\right) \dots \left(\boldsymbol{I} + \tan(r_N)\begin{bmatrix} 0 & 1/k_N \\ -k_N & 0 \end{bmatrix}\right)$$

this product can be expressed in compact form using multi-index notation, but some of the terms in the expansion are simple enough

$$
\begin{aligned}
&= \sum_{j=0}^{N} \sum_{|\mathsf{b}|=j} \tan(kh)^{\mathsf{b}} \begin{bmatrix} 0 & 1/k \\ -k & 0 \end{bmatrix}^{\mathsf{b}} , \\
&= \boldsymbol{I} + \sum_{i=1}^{N} \tan(k_i h_i) \begin{bmatrix} 0 & 1/k_i \\ -k_i & 0 \end{bmatrix} \\
&\quad + \sum_{j=2}^{N-1} \sum_{|\mathsf{b}|=j} \tan(kh)^{\mathsf{b}} \begin{bmatrix} 0 & 1/k \\ -k & 0 \end{bmatrix}^{\mathsf{b}} \\
&\quad + \prod_{i=1}^{N} \tan(k_i h_i) \begin{bmatrix} 0 & 1/k_i \\ -k_i & 0 \end{bmatrix} ,
\end{aligned}
$$

where the binary multi-index $\mathsf{b} \in (\{0,\ 1\})^N$ is an N-tuple of numbers, each being either 0 or 1. For the tangent terms, a zero entry in the multi-index means getting back 1, a one entry means that we keep that tangent term:

$$
\tan(kl)^{\mathsf{b}} \equiv \prod_{m=1}^{N} \tan(k_m l_m)^{\mathsf{b}_m}
$$

For matrices, zero means that entry would correspond to the identity matrix while one would yield the matrix itself. Let us do $\mathsf{b} = (0, 1)$ as an example:

$$
\begin{bmatrix} 0 & 1/k \\ -k & 0 \end{bmatrix}^{(0,1)} = \begin{bmatrix} 0 & 1/k_1 \\ -k-1 & 0 \end{bmatrix}^{0} \begin{bmatrix} 0 & 1/k_2 \\ -k_2 & 0 \end{bmatrix}^{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1/k_2 \\ -k_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/k_2 \\ -k_2 & 0 \end{bmatrix} .
$$

Therefore, the last term indicates taking the sum over all possible multi-indices of degree ($|\mathsf{b}|$) from 2 to $N-1$ (degree zero, one and N have been shown explicitly and correspond to the first three addends) and for each degree j, taking a sum over all possible binary multi-indices of such degree; there are $N!/(j!(N-j)!)$ binary multi-indices of degree $|\mathsf{b}|$, that is, whose N entries sum up to $|\mathsf{b}|$.

Diagonal elements. Obviously, the first term (identity matrix) is a diagonal matrix, the second one cannot be diagonal, and the last one will be diagonal when N is an even number. Concerning the two last terms, let us put them together in multi-index notation and then we can split into two pieces, one that will yield a diagonal matrix as result and another one that does not:

$$
\begin{aligned}
\sum_{j=2}^{N} \sum_{|\mathsf{b}|=j} \tan(kh)^{\mathsf{b}} \begin{bmatrix} 0 & 1/k \\ -k & 0 \end{bmatrix}^{\mathsf{b}} &= \sum_{j\in\mathcal{I}_e} \sum_{|\mathsf{b}|=j} \tan(kh)^{\mathsf{b}} \begin{bmatrix} 0 & 1/k \\ -k & 0 \end{bmatrix}^{\mathsf{b}} \\
&\quad + \sum_{j\in\mathcal{I}_o} \sum_{|\mathsf{b}|=j} \tan(kh)^{\mathsf{b}} \begin{bmatrix} 0 & 1/k \\ -k & 0 \end{bmatrix}^{\mathsf{b}} ,
\end{aligned}
$$

where $\mathcal{I}_e = \{j \in [2, 3, \ldots, N] : j \text{ is even}\}$ $\mathcal{I}_o = \{j \in [2, 3, \ldots, N] : j \text{ is odd}\}$. See that the set of values of j to consider in this case can be also expressed as

$$
\mathcal{I}_e = \{2, 4, \ldots, 2\lfloor N/2 \rfloor\} = 2\,\{1, 2, \ldots, \lfloor N/2 \rfloor\} \ ,,
$$

the use of the floor function $\lfloor \cdot \rfloor$ accounts for the possibility of N being itself even or odd. Likewise, the odd indices can be expressed as

$$
\mathcal{I}_e = \{1, 3, \ldots, 1 + 2\lfloor (N-1)/2 \rfloor\} \ .
$$

Hence, the terms that yield a diagonal result can be expressed as

$$\sum_{j\in\mathcal{I}_e}\sum_{|\mathsf{b}|=j}\tan(kh)^{\mathsf{b}}\begin{bmatrix}0 & 1/k_i\\ -k_i & 0\end{bmatrix}^{\mathsf{b}} = \sum_{l=1}^{\lfloor N/2\rfloor}\sum_{|\mathsf{b}|=2l}\tan(kh)^{\mathsf{b}}\begin{bmatrix}0 & 1/k\\ -k & 0\end{bmatrix}^{\mathsf{b}}.$$

For a given b, $|\mathsf{b}|=2l$, take the a-th position to be the first non-zero entry, the a-th to be the second one, the a'-th one the prior to last and b'-th the last one. Hence, let us expand the matrix multiplication defined via this multi-index. Assume that the first 1-entry in b is in the a-th position, while the next 1 is in the b-th one, the second to last was in the a'-th and the last 1-entry corresponded to the b'-th entry:

$$\begin{bmatrix}0 & 1/k\\ -k & 0\end{bmatrix}^{\mathsf{b}} = \underbrace{\begin{bmatrix}0 & 1/k_a\\ -k_a & 0\end{bmatrix}\begin{bmatrix}0 & 1/k_b\\ -k_b & 0\end{bmatrix}}_{\text{first pair}}\cdots\underbrace{\begin{bmatrix}0 & 1/k_{a'}\\ -k_{a'} & 0\end{bmatrix}\begin{bmatrix}0 & 1/k_{b'}\\ -k_{b'} & 0\end{bmatrix}}_{\text{last pair}},$$

grouping pairs,

$$= \underbrace{\begin{bmatrix}-k_b/k_a & 0\\ 0 & -k_a/k_b\end{bmatrix}\cdots\begin{bmatrix}-k_{b'}/k_{a'} & 0\\ 0 & -k_{a'}/k_{b'}\end{bmatrix}}_{l\text{ factors}},$$

using the notation presented in [1]

$$= (-1)^{|\mathsf{b}|/2}\begin{bmatrix}k^{\mathrm{f}(\mathsf{b})} & 0\\ 0 & k^{-\mathrm{f}(\mathsf{b})}\end{bmatrix}, \qquad \text{(Supplementary Equation 6)}$$

where the auxiliary function $\mathrm{f}:\{0,1\}^N\to\{-1,0,1\}^N$ takes the multi-index b, entry-wise, to the multi-index $\mathrm{f}(\mathsf{b})$ defined as:

- if $\mathsf{b}_i=0$, then $\mathrm{f}(\mathsf{b}_i)=0$,
- if $\mathsf{b}_i=1$ and it is the first instance of an entry being equal to 1, then $\mathrm{f}(\mathsf{b}_i)=1$.
- if $\mathsf{b}_i=1$ and the previous value assigned by f to the prior 1-entry in b was -1, then $\mathrm{f}(\mathsf{b}_i)=1$, else $\mathrm{f}(\mathsf{b}_i)=-1$.

As an illustrating example, consider $\mathsf{b}=(0,1,0,1,1,0,0,1)\in\{0,\,1\}^8$ (one of the combinations that would appear in a 8-layer laminate), thus $\mathrm{f}(\mathsf{b})=(0,1,0,-1,1,0,0,-1)$, hence

$$\begin{bmatrix}0 & 1/k\\ -k & 0\end{bmatrix}^{\mathsf{b}} = +\begin{bmatrix}\frac{Z_4\,Z_8}{Z_2\,Z_5} & 0\\ 0 & \frac{Z_2\,Z_5}{Z_4\,Z_8}\end{bmatrix}.$$

Anti-diagonal elements. We can also look into the anti-diagonal terms in similar fashion,

$$\sum_{j\in\mathcal{I}_o}\sum_{|\mathsf{b}|=j}\tan(kh)^{\mathsf{b}}\begin{bmatrix}0 & 1/k\\ -k & 0\end{bmatrix}^{\mathsf{b}} = \sum_{l=1}^{\lfloor (N-1)/2\rfloor}\sum_{|\mathsf{b}|=2l+1}\tan(kh)^{\mathsf{b}}\begin{bmatrix}0 & 1/k\\ -k & 0\end{bmatrix}^{\mathsf{b}},$$

so for a given b such that $|\mathsf{b}|=2l+1$ (taking advantage of the notation introduced in the prior case),

$$\begin{bmatrix}0 & 1/k\\ -k & 0\end{bmatrix}^{\mathsf{b}} = \underbrace{\begin{bmatrix}-k_b/k_b & 0\\ 0 & -k_a/k_b\end{bmatrix}\cdots\begin{bmatrix}-k_{b'}/k_{a'} & 0\\ 0 & -k_{a'}/k_{b'}\end{bmatrix}}_{l\text{ factors}}\begin{bmatrix}0 & 1/k_c\\ -k_c & 0\end{bmatrix},$$

which is the result we got before bar one last factor

$$= (-1)^l \begin{bmatrix} k^{\mathsf{f}(\mathsf{b})} & 0 \\ 0 & k^{-\mathsf{f}(\mathsf{b})} \end{bmatrix} \begin{bmatrix} 0 & 1/k_c \\ -k_c & 0 \end{bmatrix} ,$$

$$= \begin{bmatrix} 0 & \frac{(-1)^l}{k_c} k^{\mathsf{f}(\mathsf{b})} \\ (-1)^{l+1} k_c k^{-\mathsf{f}(\mathsf{b})} & 0 \end{bmatrix} .$$

Thus the addends corresponding to odd-degree multi-indices can simply be calculated as the immediately prior even-degree multi-index multiplied by the extra entry.

Closed-form expression of the entries of the transfer matrix. Using the previous results, Supplementary Equation 6, T_{11} can be written as

$$\text{(Supplementary Equation 8)} \qquad T_{11} = \left(\prod_{i=1}^{N} \cos(k_i h_i) \right) \left(\sum_{l=0}^{\lfloor N/2 \rfloor} \sum_{|\mathsf{b}|=2l} (-1)^{|\mathsf{b}|/2} k^{\mathsf{f}(\mathsf{b})} \tan(kh)^{\mathsf{b}} \right) ,$$

and it is computed easily from the rules outlined above. Performing some algebraic manipulations we find

$$T_{11} = \left(\prod_{i=1}^{N} \cos(k_i h_i) \right) \left(\sum_{l=0}^{\lfloor N/2 \rfloor} \sum_{|\mathsf{b}|=2l} (-1)^{|\mathsf{b}|/2} k^{\mathsf{f}(\mathsf{b})} \tan(kh)^{\mathsf{b}} \right) ,$$

$$= \sum_{l=0}^{\lfloor N/2 \rfloor} \sum_{|\mathsf{b}|=2l} (-1)^{|\mathsf{b}|/2} k^{\mathsf{f}(\mathsf{b})} \cos(\phi^{\mathsf{b}}) ,$$

where the new notation $\cos(\phi^{\mathsf{b}})$ must be explained before continuing the derivation: as we multiply the cosine factors by each element of the sum that makes up the second factor, some of those cosines will become sines when combined with the tangent factors; however, we can think of the sines as cosines with extra phase $-\pi/2$, thus

$$\cos(\phi^{\mathsf{b}}) = \prod_{i=1}^{N} \cos\left(\phi_i - \mathsf{b}_i \frac{\pi}{2} \right) ,$$

and can also apply a well-known trigonometric identity termed "product to sum" formula so as to reach

$$= \frac{1}{2^N} \sum_{\mathbf{e}' \in \{-1,1\}^N} (-1)^{\mathsf{b}^\top \mathbf{e}'} \cos(\mathbf{\Psi}^\top \mathbf{e}') ,$$

where $\mathbf{\Psi}$ is just the vector encompassing all phase gains ϕ_i, from $i = 1$ to $i = N$, while $\mathbf{e}'$ is a new binary multi-index, N entries, this time taking ± 1 values; since $\mathbf{e}'$ appears in the argument of a cosine and $|\mathsf{b}|$ is even, each term in the sum appears exactly twice and thus we can restrict the sum to the elements $\mathbf{e} \in \{-1, 1\}^N$ s.t. $\mathsf{e}_1 = 1$ (thus we encounter the same path-defining vector introduced in the text), so

$$= \frac{1}{2^{N-1}} \sum_{\mathbf{e} \in \{-1,1\}^N} (-1)^{\mathbf{e}^\top \mathsf{b}} \cos(\mathbf{e}^\top \mathbf{\Psi}) ,$$

so, for each multi-index b, the trigonometric factors can be turned into a sum of 2^{N-1} cosines terms, whose arguments are combinatorially defined in terms of the ϕ_i, always the same ones, the only thing changing from one multi-index to another being the sign with which the cosines appear in the sum. Plugging the later result in the expression of T_{11},

$$T_{11} = \sum_{l=0}^{\lfloor N/2 \rfloor} \sum_{|\mathsf{b}|=2l} (-1)^{|\mathsf{b}|/2} k^{-\mathsf{f}(\mathsf{b})} \cos(\phi^{\mathsf{b}}) \,,$$

$$= \sum_{l=0}^{\lfloor N/2 \rfloor} \sum_{|\mathsf{b}|=2l} \frac{(-1)^{|\mathsf{b}|/2} k^{-\mathsf{f}(\mathsf{b})}}{2^{N-1}} \sum_{\mathbf{e}\in\{-1,1\}^N} (-1)^{\mathbf{e}^\top \mathsf{b}} \cos(\mathbf{e}^\top \mathbf{\Psi}) \,,$$

re-arranging the sums,

$$= \sum_{\mathbf{e}\in\{-1,1\}^N} \left(\frac{1}{2^{N-1}} \sum_{l=0}^{\lfloor N/2 \rfloor} \sum_{|\mathsf{b}|=2l} (-1)^{|\mathsf{b}|/2 + \mathbf{e}^\top \mathsf{b}} k^{-\mathsf{f}(\mathsf{b})} \right) \cos(\mathbf{e}^\top \mathbf{\Psi}) \,,$$

so instead of summing over the values of the multi-index b last, we do it first, and lastly over $\mathbf{e}$ (that contains 2^{N-1} elements),

$$= \sum_{j=1}^{2^{N-1}} \mathcal{A}_j \cos(\tau_j \omega) \,,$$

thus we reach a path decomposition of T_{11}. The phase change along the j-th path is therefore

$$\varphi_j = \mathbf{e}_j^\top \mathbf{\Psi} = \sum_{i=1}^{N} \mathrm{e}_{ji} (k_i l_i) \,,$$

Likewise, if we introduce the angular frequency ω, the "period" of the j-th path, τ_j such that $\varphi_j = \omega \tau_j$, understood as a function of ω, can be computed as

$$\tau_j = \mathbf{e}_j^\top \mathbf{t} \,,$$

with $\mathbf{t} = [l_1/c_1, \ldots, l_N/c_N]^\top$, while the corresponding amplitude change along the said path is

$$\mathcal{A}_j = \frac{1}{2^{N-1}} \sum_{l=0}^{\lfloor N/2 \rfloor} \sum_{|\mathsf{b}|=2l} (-1)^{|\mathsf{b}|/2 + \mathbf{e}_j^\top \mathsf{b}} k^{-\mathsf{f}(\mathsf{b})} \,.$$

The last step is to show that the expression in the text yields the same result as just shown, thus proving that they are identical. The form in the paper is

$$\mathcal{A}_j = \prod_{i=2}^{N} \frac{1}{2} \left(1 + \mathrm{e}_{ji-1} \mathrm{e}_{ji} \frac{k_{i-1}}{k_i} \right) \,.$$

Factor out the global prefactor. Each of the $(N-1)$ brackets contributes a factor $1/2$:

$$\mathcal{A}_j = \frac{1}{2^{N-1}} \prod_{i=2}^{N} \Big(1 + \mathsf{e}_{ji-1}\, \mathsf{e}_{ji}\, \frac{k_{i-1}}{k_i} \Big).$$

The product above must be shown to be equivalent to the sum over even multi-indices. This follows from a simple binomial expansion:

$$\prod_{\alpha=1}^{N-1} \left(1 + \mathsf{e}_{ji+1}\, \mathsf{e}_{ji}\, \frac{k_i}{k_{i+1}} \right) = \sum_{m=0}^{N-1} \sum_{\substack{\mathsf{b}\in\{0,1\}^{N-1} \\ |\mathsf{b}=m|}} \left(\mathsf{e}_{ji+1}\, \mathsf{e}_{ji}\, \frac{k_i}{k_{i+1}} \right)^{\mathsf{b}} \,,$$

which, simplifying the ± 1 in the $\mathbf{e}_j$ and the k factors that appear in the denominator and numerator, lead to the definition of $\mathsf{f}(\mathsf{b})$.

Supplementary Note 3: Path enumeration for three-layer transfer matrix

Recall that $\mathbf{e}_j$ fully defines the path: the sign of each entry spells out if the wave picks negative of positive phase, and the relative signs of two consecutive entries tell if the amplitude increases or decreases (if the signs are equal, increase; if different, decrease).

- Path I $\rightarrow \mathbf{e}_1 = [+1,+1,+1]$: "transmission, transmission", phase change $\omega(t_1+t_2+t_2)$, proportion of total amplitude $1/\mathsf{T}_{1\to 2} \times 1/\mathsf{T}_{2\to 3}$. It would correspond to
$$\frac{1}{\mathsf{T}_{1\to 2}}\frac{1}{\mathsf{T}_{2\to 3}}\cos\left(\omega(t_1+t_2+t_3)\right)\,.$$
- Path II $\rightarrow \mathbf{e}_2 = [+1,+1,-1]$: "transmission, reflection", phase change $\omega(t_1+t_2-t_3)$, proportion of total amplitude $1/\mathsf{T}_{1\to 2} \times \mathsf{R}_{2\to 3}/\mathsf{T}_{2\to 3}$. It would correspond to
$$\frac{1}{\mathsf{T}_{1\to 2}}\frac{\mathsf{R}_{2\to 3}}{\mathsf{T}_{2\to 3}}\cos\left(\omega(t_1+t_2-t_3)\right)\,.$$
- Path III $\rightarrow \mathbf{e}_3 = [+1,-1,+1]$: "reflection, reflection", phase change $\omega(t_1-t_2+t_3)$, proportion of total amplitude $\mathsf{R}_{1\to 2}/\mathsf{T}_{1\to 2} \times \mathsf{R}_{2\to 3}/\mathsf{T}_{2\to 3}$. It would correspond to
$$\frac{\mathsf{R}_{1\to 2}}{\mathsf{T}_{1\to 2}}\frac{\mathsf{R}_{2\to 3}}{\mathsf{T}_{2\to 3}}\cos\left(\omega(t_1-t_2+t_3)\right)\,.$$
- Path IV $\rightarrow \mathbf{e}_4 = [+1,-1,-1]$: "reflection, transmission", phase change $\omega(t_1-t_2-t_3)$, proportion of total amplitude $\mathsf{R}_{1\to 2}/\mathsf{T}_{1\to 2} \times 1/\mathsf{T}_{2\to 3}$. It would correspond to
$$\frac{\mathsf{R}_{1\to 2}}{\mathsf{T}_{1\to 2}}\frac{1}{\mathsf{T}_{2\to 3}}\cos\left(\omega(t_1-t_2-t_3)\right)\,.$$

These are all the four possible paths in a three-piece system. Adding all up:
$$\begin{aligned}T_{11} = &\frac{1}{\mathsf{T}_{1\to 2}}\frac{1}{\mathsf{T}_{2\to 3}}\cos\left(\omega(t_1+t_2+t_3)\right) + \frac{1}{\mathsf{T}_{1\to 2}}\frac{\mathsf{R}_{2\to 3}}{\mathsf{T}_{2\to 3}}\cos\left(\omega(t_1+t_2-t_3)\right)\\ &+\frac{\mathsf{R}_{1\to 2}}{\mathsf{T}_{1\to 2}}\frac{\mathsf{R}_{2\to 3}}{\mathsf{T}_{2\to 3}}\cos\left(\omega(t_1-t_2+t_3)\right) + \frac{\mathsf{R}_{1\to 2}}{\mathsf{T}_{1\to 2}}\frac{1}{\mathsf{T}_{2\to 3}}\cos\left(\omega(t_1-t_2-t_3)\right)\,.\end{aligned}$$
(Supplementary Equation 12)

Let us remark that all the coefficients of the cosines add up to 1:
$$\begin{aligned}&\frac{1}{\mathsf{T}_{1\to 2}}\frac{1}{\mathsf{T}_{2\to 3}} + \frac{1}{\mathsf{T}_{1\to 2}}\frac{\mathsf{R}_{2\to 3}}{\mathsf{T}_{2\to 3}} + \frac{\mathsf{R}_{1\to 2}}{\mathsf{T}_{1\to 2}}\frac{\mathsf{R}_{2\to 3}}{\mathsf{T}_{2\to 3}} + \frac{\mathsf{R}_{1\to 2}}{\mathsf{T}_{1\to 2}}\frac{1}{\mathsf{T}_{2\to 3}}\\ &= \frac{1}{\mathsf{T}_{1\to 2}}\left(\left[\frac{1}{\mathsf{T}_{2\to 3}} + \frac{\mathsf{R}_{2\to 3}}{\mathsf{T}_{2\to 3}}\right] + \mathsf{R}_{1\to 2}\left[\frac{\mathsf{R}_{2\to 3}}{\mathsf{T}_{2\to 3}} + \frac{1}{\mathsf{T}_{2\to 3}}\right]\right)\\ &= \frac{1}{\mathsf{T}_{1\to 2}}\left(1+\mathsf{R}_{1\to 2}\right) = 1\,.\end{aligned}$$

Compare the result to the one reported in the literature [1, 3] (obtained from direct elementary transfer matrix multiplication):
$$\begin{aligned}T_{11} = \cos(\omega t_1)\cos(\omega t_2)\cos(\omega t_3)\Bigg(&1 - \frac{k_2}{k_1}\tan(\omega t_1)\tan(\omega t_2)\\ &+\frac{k_2}{k_3}\tan(\omega t_1)\tan(\omega t_3) - \frac{k_1}{k_3}\tan(\omega t_2)\tan(\omega t_3)\Bigg)\,.\end{aligned}$$

Expand the product to eliminate the tangent terms, use the product-to-sum trigonometric identities to turn the products of sines and cosines into cosines, plus recognize the coefficients in terms of

$\mathsf{T}_{1\to2}, \mathsf{R}_{1\to2}$ and $\mathsf{T}_{2\to3}, \mathsf{R}_{2\to3}$ to recover the other expression. Let us proceed: eliminate the tangents by substituting $\tan x = \sin x/\cos x$, noting that each of the $\cos(\omega t_i)$ factors in the denominator is canceled by the prefactor:

$$\begin{aligned} T_{11} &= \cos(\omega t_1)\cos(\omega t_2)\cos(\omega t_3) - \frac{k_2}{k_1}\sin(\omega t_1)\sin(\omega t_2)\cos(\omega t_3) \\ &\quad + \frac{k_2}{k_3}\sin(\omega t_1)\sin(\omega t_3)\cos(\omega t_2) - \frac{k_1}{k_3}\sin(\omega t_2)\sin(\omega t_3)\cos(\omega t_1). \end{aligned}$$

Now apply product-to-sum identities

$$\cos A\cos B = \tfrac{1}{2}[\cos(A{+}B) + \cos(A{-}B)], \qquad \sin A\sin B = \tfrac{1}{2}[\cos(A{-}B) - \cos(A{+}B)],$$

twice on each addend; denoting $\alpha = \omega t_1$, $\beta = \omega t_2$, and $\gamma = \omega t_3$, we obtain:

$$\begin{aligned} T_{11} &= \tfrac{1}{4}\Big[\cos(\alpha{+}\beta{+}\gamma) + \cos(\alpha{+}\beta{-}\gamma) + \cos(\alpha{-}\beta{+}\gamma) + \cos(\alpha{-}\beta{-}\gamma)\Big] \\ &\quad - \frac{k_2}{k_1}\cdot\tfrac{1}{4}\Big[\cos(\alpha{-}\beta{+}\gamma) + \cos(\alpha{-}\beta{-}\gamma) - \cos(\alpha{+}\beta{+}\gamma) - \cos(\alpha{+}\beta{-}\gamma)\Big] \\ &\quad + \frac{k_2}{k_3}\cdot\tfrac{1}{4}\Big[\cos(\alpha{-}\gamma{+}\beta) + \cos(\alpha{-}\gamma{-}\beta) - \cos(\alpha{+}\gamma{+}\beta) - \cos(\alpha{+}\gamma{-}\beta)\Big] \\ &\quad - \frac{k_1}{k_3}\cdot\tfrac{1}{4}\Big[\cos(\beta{-}\gamma{+}\alpha) + \cos(\beta{-}\gamma{-}\alpha) - \cos(\beta{+}\gamma{+}\alpha) - \cos(\beta{+}\gamma{-}\alpha)\Big], \end{aligned}$$

thus all cosine arguments above reduce to one of the four phase combinations:

$$C_{++} = \cos(\alpha+\beta+\gamma), \quad C_{+-} = \cos(\alpha+\beta-\gamma), \quad C_{-+} = \cos(\alpha-\beta+\gamma), \quad C_{--} = \cos(\alpha-\beta-\gamma),$$

and after collecting coefficients, we find:

$$\begin{aligned} T_{11} &= \left(\tfrac{1}{4} + \tfrac{1}{4}\tfrac{k_2}{k_1} + \tfrac{1}{4}\tfrac{k_2}{k_3} + \tfrac{1}{4}\tfrac{k_1}{k_3}\right)C_{++} + \left(\tfrac{1}{4} - \tfrac{1}{4}\tfrac{k_2}{k_1} + \tfrac{1}{4}\tfrac{k_2}{k_3} - \tfrac{1}{4}\tfrac{k_1}{k_3}\right)C_{+-} \\ &\quad + \left(\tfrac{1}{4} + \tfrac{1}{4}\tfrac{k_2}{k_1} - \tfrac{1}{4}\tfrac{k_2}{k_3} - \tfrac{1}{4}\tfrac{k_1}{k_3}\right)C_{-+} + \left(\tfrac{1}{4} - \tfrac{1}{4}\tfrac{k_2}{k_1} - \tfrac{1}{4}\tfrac{k_2}{k_3} + \tfrac{1}{4}\tfrac{k_1}{k_3}\right)C_{--}. \end{aligned}$$

Finally, we express these coefficients in terms of the interface factors:

$$\mathsf{T}_{i\to j} = \frac{2k_j}{k_i + k_j}, \quad \mathsf{R}_{i\to j} = \frac{k_j - k_i}{k_i + k_j},$$

to confirm that:

$$\begin{aligned} T_{11} &= \frac{1}{\mathsf{T}_{1\to2}\,\mathsf{T}_{2\to3}}\cos\big[\omega(t_1+t_2+t_3)\big] + \frac{1}{\mathsf{T}_{1\to2}}\frac{\mathsf{R}_{2\to3}}{\mathsf{T}_{2\to3}}\cos\big[\omega(t_1+t_2-t_3)\big] \\ &\quad + \frac{\mathsf{R}_{1\to2}}{\mathsf{T}_{1\to2}}\frac{\mathsf{R}_{2\to3}}{\mathsf{T}_{2\to3}}\cos\big[\omega(t_1-t_2+t_3)\big] + \frac{\mathsf{R}_{1\to2}}{\mathsf{T}_{1\to2}}\frac{1}{\mathsf{T}_{2\to3}}\cos\big[\omega(t_1-t_2-t_3)\big]. \end{aligned}$$

References

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