One Dimensional (1D) and Two-Dimensional (2D) Spring Mass Chains

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Wave Propagation in Linear and Nonlinear Periodic Media: Analysis and Applications
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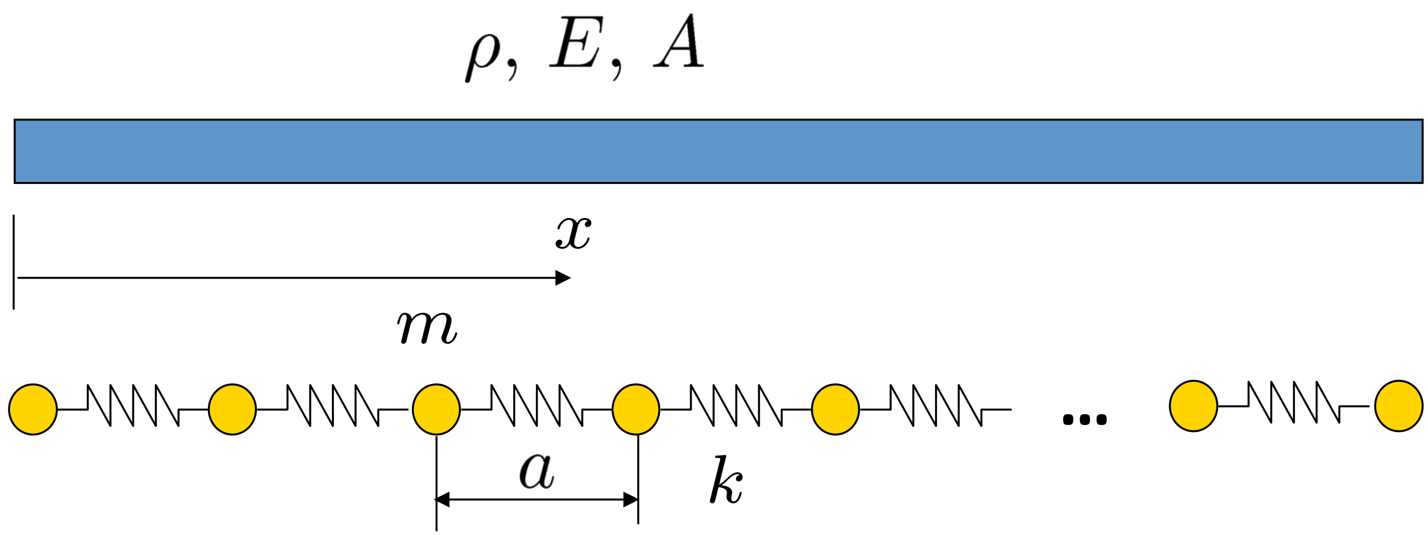
Coarse approximation of a uniform rod:



- Rod is discretized into N elements of length a;
- Mass and stiffness distributions are described as lumped parameters;

$$k = \frac{EA}{a} \qquad m = \rho A a$$

- Location of *n*-th mass:

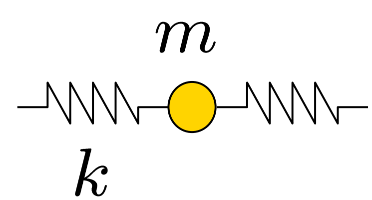
$$x_n = na$$

Notes

- System under consideration is the first, simplest example of a PERIODIC structure:
  - Here obtained considering a "dumb" discretization of a continuous rod;
  - Can be considered as a simple academic exercise
  - System initially studied by Newton (1686) to calculate the speed of sound in air:
    - Newton, *Principia*, Book II, 1686.
  - System is used by John Bernoulli and son Daniel (1727) to demonstrate that a system of N masses is characterized by N modes of vibration and associated frequencies
  - Configuration considered by Baden-Powell (1841) to calculate the velocity of wave propagation along one axis of a cubic lattice structure
  - Results later corrected and expanded by Lord Kelvin (1881)
    - *Popular Lecture,* Vol. I, p. 185.
  - Detailed discussions can be found in:
    - L. Brillouin, *Wave Propagation In Periodic Structures*, Dover 1946.
    - C. Kittel*, Introduction to Solid Sate Physics,* 8th ed. John Wiley & Sons, Inc., 2005.

- System's behavior is governed by *N* equations of the kind:

$$m\ddot{u}_n + 2ku_n - k(u_{n+1} + u_{n-1}) = 0 \qquad n = 1, .., N$$



- Impose a harmonic solution

$$u(x_n, t) = u_n(t) = u_n(\omega)e^{-j\omega t}$$

- Impose a wave solution

$$u_n(\omega) = u_0[\kappa(\omega)]e^{j\kappa x_n}$$

$$u_n(\omega) = u_0[\mu(\omega)]e^{j\mu n}$$

where

$$\mu(\omega) = \kappa(\omega)a$$

- Under the assumption that no external forces are applied:
  - Free wave propagation

- Substitute wave solution at frequency ω in *n*-th equation:

$$[(-\omega^2 m + 2k)e^{i\mu n} - k(e^{i\mu(n-1)} + e^{i\mu(n+1)})]u_0(\mu) = 0$$

$$[(-\omega^2 m + 2k) - k(e^{-i\mu} + e^{i\mu})]u_0(\mu) = 0$$

$$\boxed{[-\omega^2 m + 2k(1-\cos\mu)]\underbrace{u_0(\mu)}_{\neq 0} = 0} \qquad (1)$$

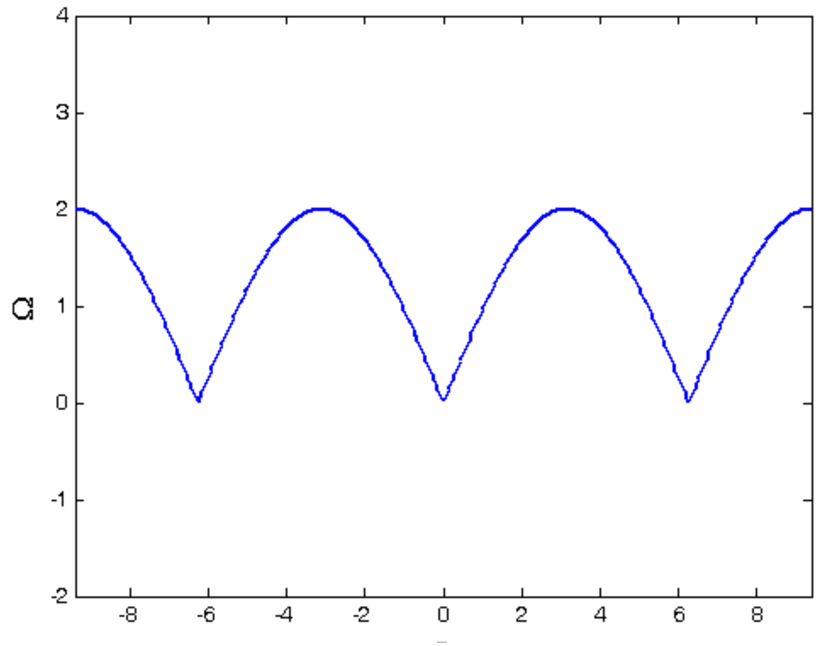
Non-trivial solutions

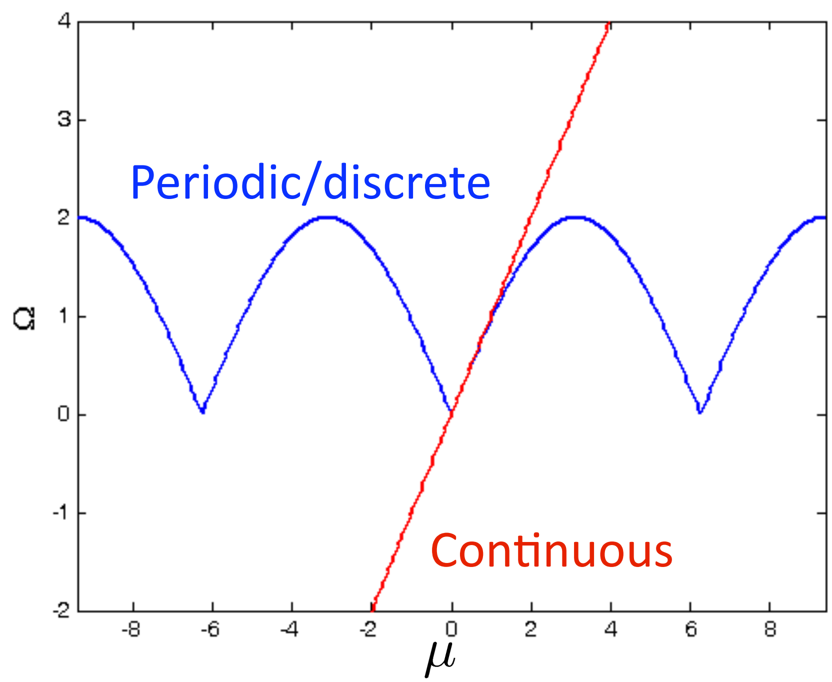
- Dispersion relation (frequency – wavenumber relations):

$$\boxed{-\omega^2 m + 2k(1-\cos\mu) = 0} \qquad \omega_0^2 = \frac{k}{m}$$

$$\boxed{\Omega^2 = 2(1-\cos\mu)} \qquad \Omega = \frac{\omega}{\omega_0}$$

$\omega = \omega(\mu)$:direct solution





- Continuous rod vs. discrete:
  - Discretization process can be described in terms of FINITE DIFFERENCE formalism

$$2u_n - u_{n+1} - u_{n-1} \approx a^2 \frac{\partial^2 u(x)}{\partial x^2}$$

  - This approximation is used in deriving equivalent continuum systems for discrete assemblies
  - Discretization causes the system to be dispersive

$$\frac{\partial \omega}{\partial k} \neq \frac{\omega}{k} \qquad c_g \neq c_p$$

Notes

- Dispersion relation is PERIODIC in the wavenumber space:

$$\Omega(\mu) = \Omega(\mu + 2\pi p) \qquad\qquad p \text{ integer}$$

$$\omega(\kappa) = \omega(\kappa + p\tfrac{2\pi}{a})$$

  – *k*-space is periodic of period 2p/a

- As a result, displacements are also periodic in the wavenumber space

$$\underbrace{[-\omega^2 m + 2k(1 - \cos(\mu + 2\pi p))]}u_0(\mu + 2\pi p) = 0$$

=

$$\overbrace{[-\omega^2 m + 2k(1 - \cos\mu)]}u_0(\mu) = 0$$

$$\Rightarrow \boxed{u_0(\mu + 2\pi p) = u_0(\mu)}$$

- Result is due to the SAMPLING of a continuous system:

$$u(x) \rightarrow u(na)$$

  - Spatial sampling occurs at a frequency

$$\kappa_s = \frac{2\pi}{a}$$

  - The result

$$u_0(\mu + 2\pi p) = u_0(\mu)$$

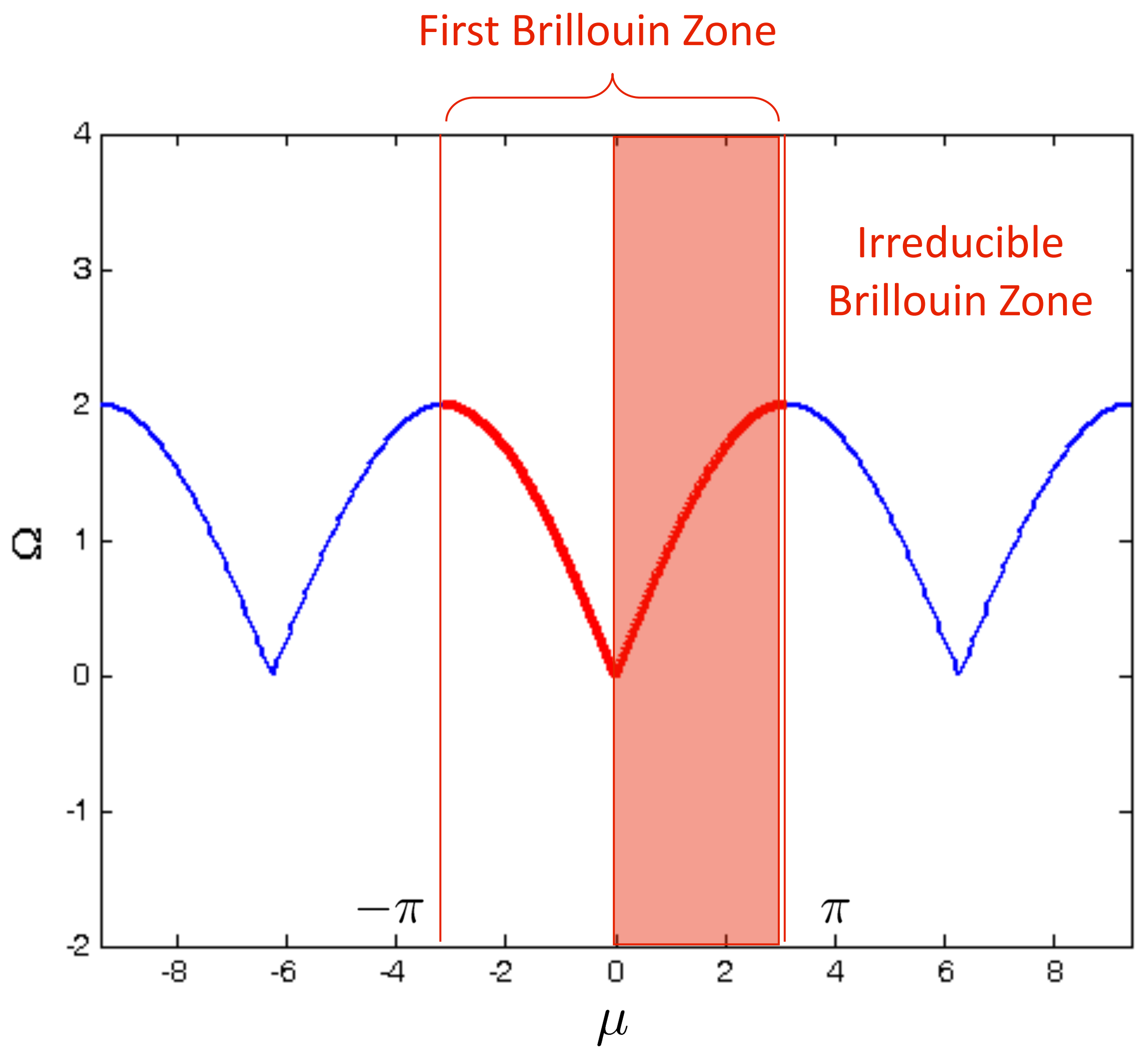
$$u_0(\kappa + p\frac{2\pi}{a}) = u_0(\kappa)$$

    is an expression of the Sampling Theorem (Shannon) theorem, for a system sampled in space

- A single period of the wavenumber/frequency relation for a periodic system is called:

FIRST BRILLOUIN ZONE

$\omega = \omega(\mu)$:direct solution



- Analogy with time-domain signals can be used to obtain a good guess about the NATURAL FREQUENCIES of a FINITE PERIODIC system with N masses (free-free for simplicity):
  - Finite system can be considered as a truncation of an infinite one
  - Truncation causes the system to be DISCRETE instead of continuous

$$u_0(\kappa) \rightarrow u_0(p\Delta\kappa)$$

where wavenumber resolution is:

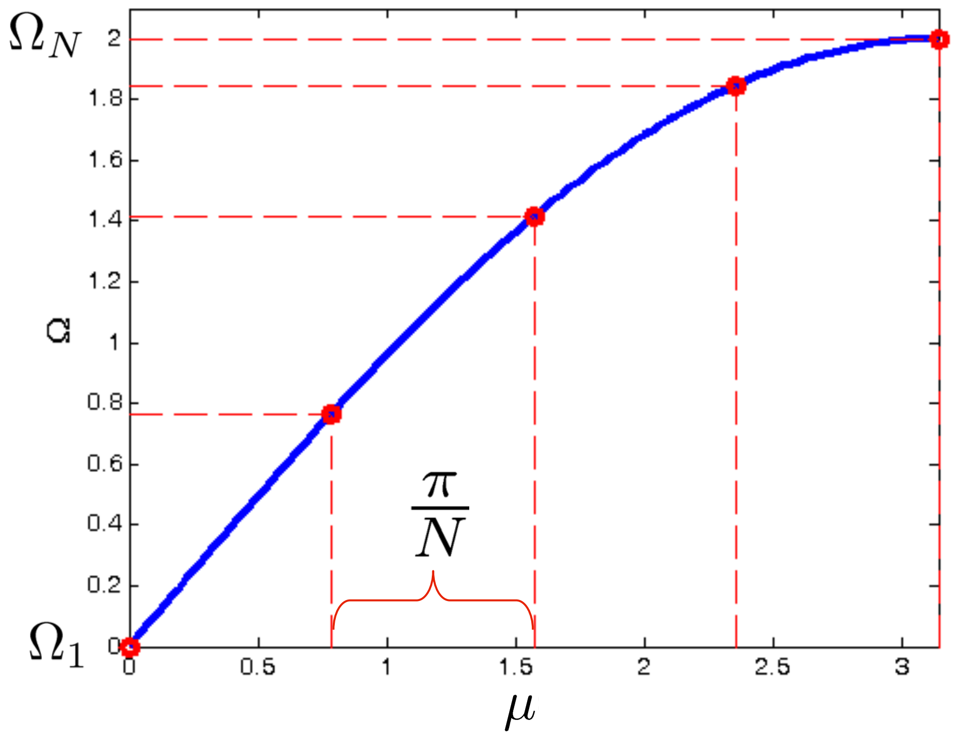
and

$$\Delta\kappa = \frac{\pi}{Na}$$

$$p = -N, -N+1, ...N-1, N$$

  - Discrete wavenumber values correspond to *N* values of frequencies
  - Natural frequencies can be read directly on the dispersion curve, given the number of masses and boundary conditions

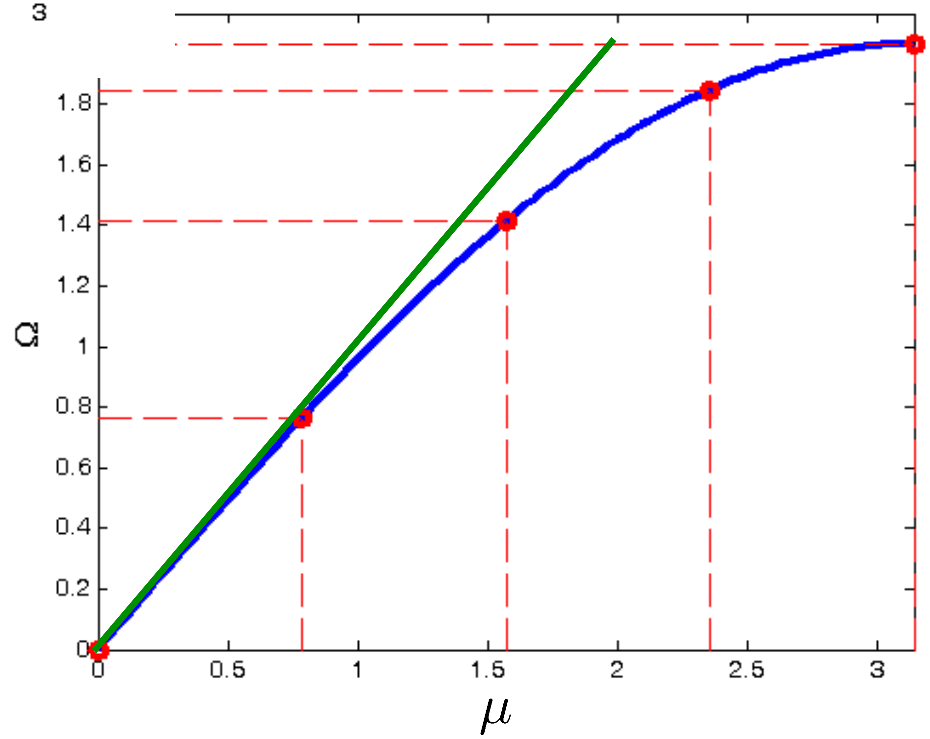
Natural frequencies (*N=5*)



Is parallel with finite time signal completely true?

Not quite....

A factor 2 is missing!!!!



$\mu = \mu(\omega)$: inverse solution

Alternatively, the solution of the dispersion relation:

$$\Omega^2 = 2(1 - \cos\mu)$$

can be found by imposing frequency:

$$\mu = \cos^{-1}(1 - \frac{\Omega^2}{2})$$

where

μ is real for $\Omega \leq 2$ $\qquad$ μ is imaginary for $\Omega > 2$

The wave solution to the governing equation:

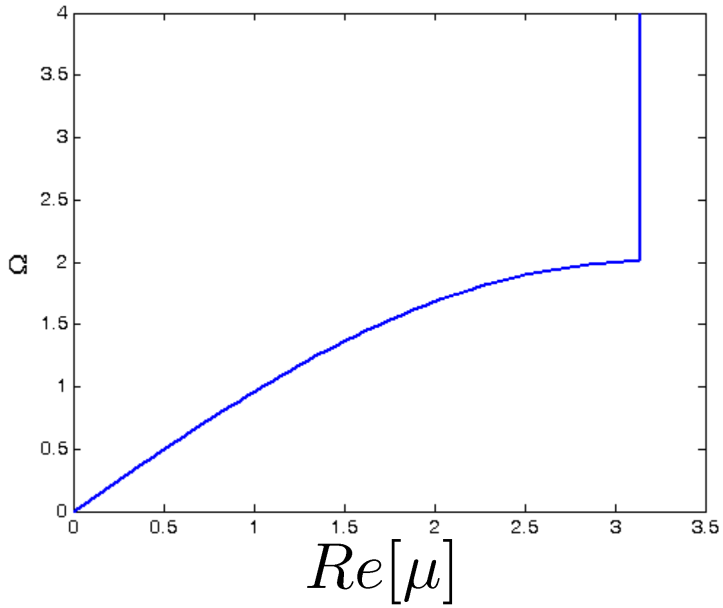
$$m\ddot{u}_n + 2ku_n - k(u_{n+1} + u_{n-1}) = 0$$

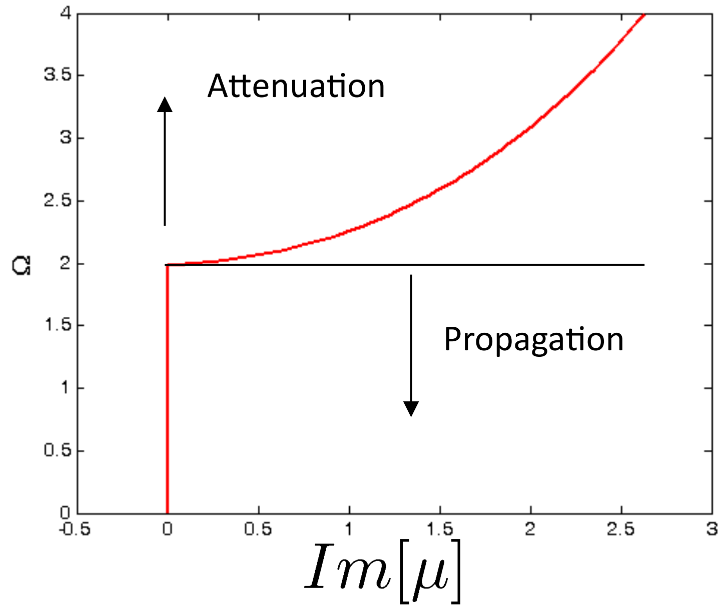
should be expressed as follows

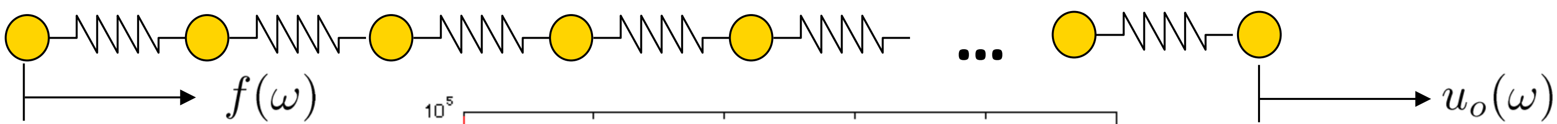
$$u_n(\omega) = u_0(\omega)e^{j\mu n}$$

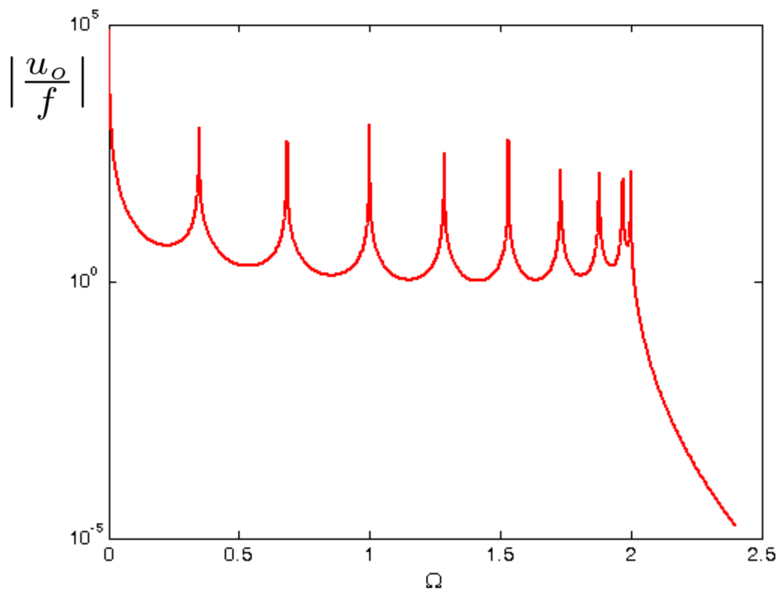
$Im[\mu]$ Attenuation constant

Harmonic response of a finite system (N masses)

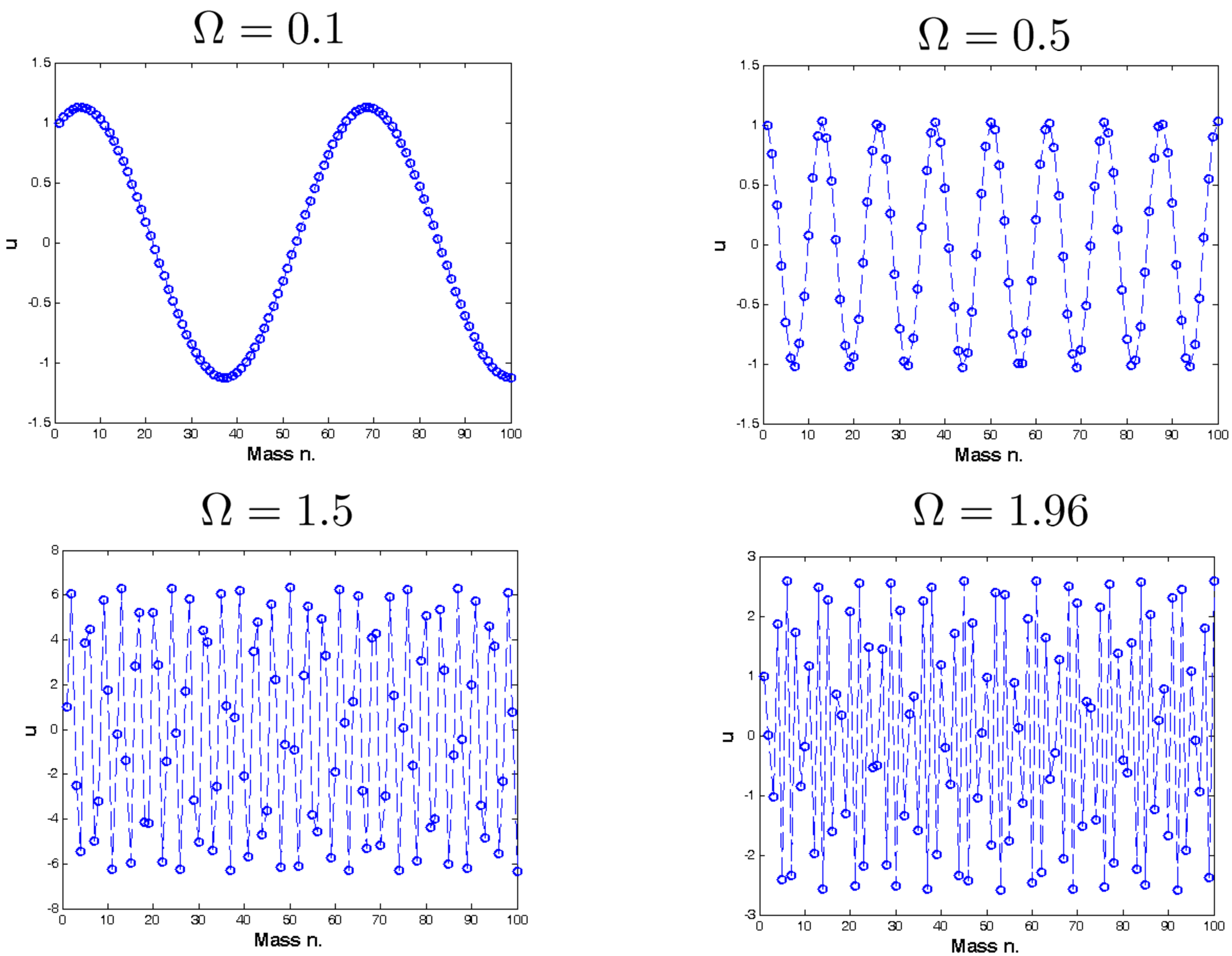






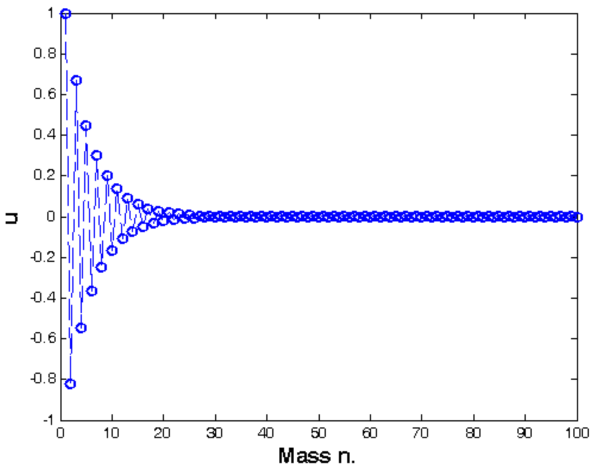


Harmonic response of a finite system (N=100 masses)

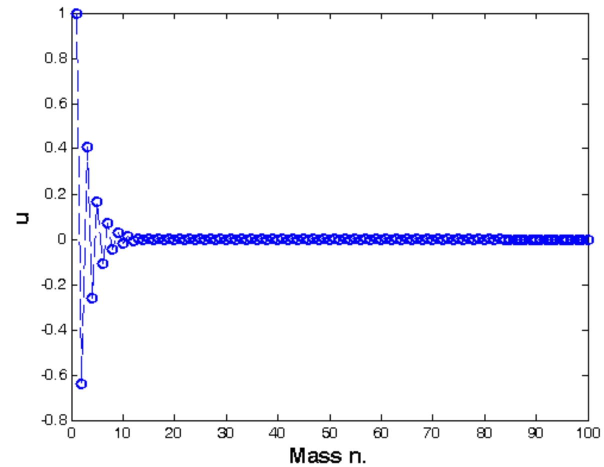


Harmonic response of a finite system (N=100 masses)

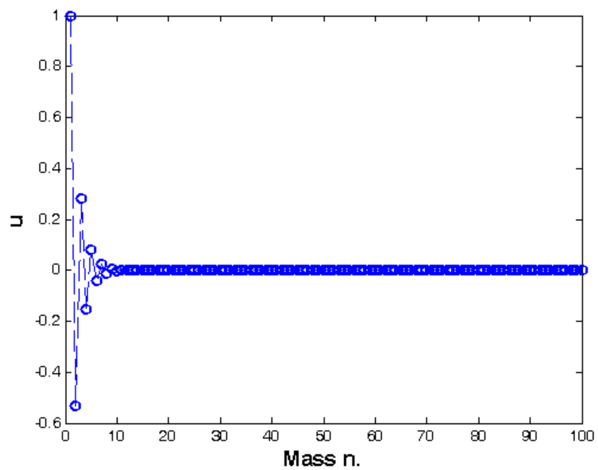
$\Omega = 2.01$



$\Omega = 2.05$



$\Omega = 2.1$



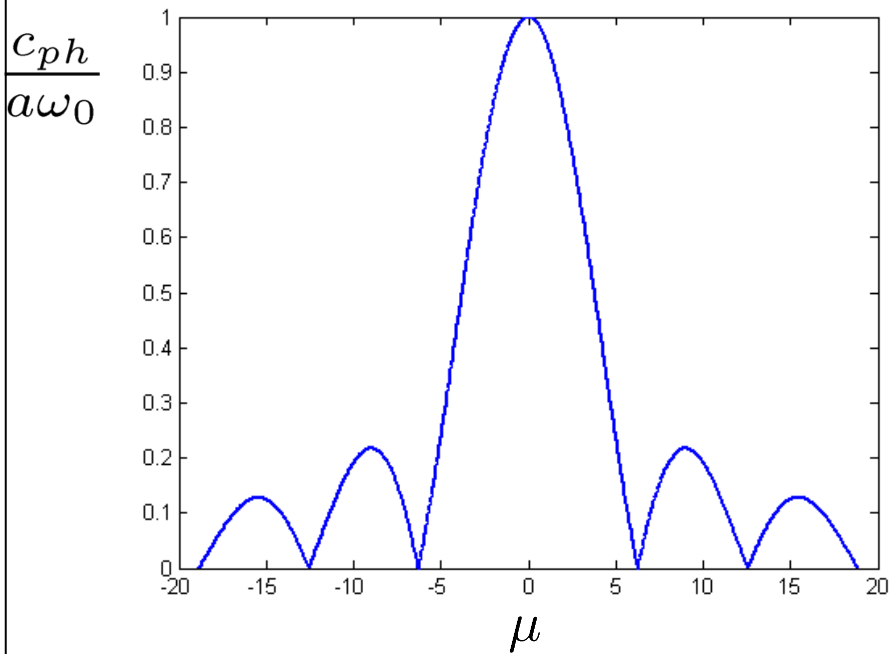
Phase velocity:

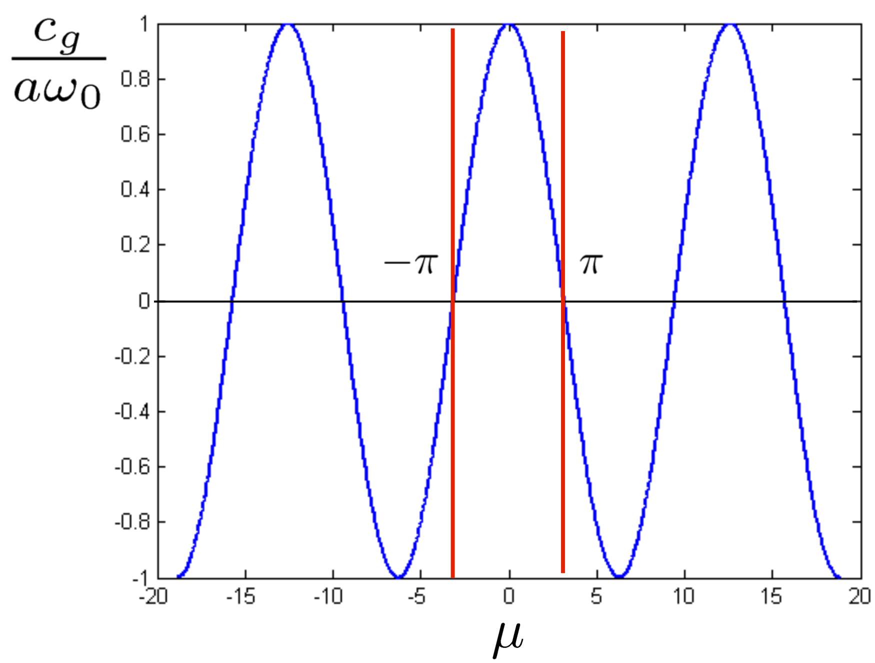
$$c_{ph} = \frac{\omega}{\kappa} = a\omega_0 \frac{|\sin \mu/2|}{|\mu/2|}$$

Group velocity:

$$c_g = \frac{\partial \omega}{\partial \kappa} = a\frac{\partial \omega}{\partial \mu}$$

$$c_g = a\omega_0 \cos \frac{\mu}{2}$$





- Average energy density: sum of average potential and kinetic energy of the unit cell:
  - Average potential energy

$$< U >= \frac{U}{a} = \frac{1}{2a} k \mathcal{R}e[(u_n - u_{n-1})^2]$$

$$u_n - u_{n-1} = A\mathcal{R}e[1 - e^{j\mu}]$$

$$< U >= \frac{1}{2a} A^2 k (1 - e^{j\mu})(1 + e^{-j\mu})$$

$$< U >= A^2 \frac{k}{a} \sin^2 \frac{\mu}{2}$$

  - Average kinetic energy

$$< K >= \frac{K}{a} = \frac{1}{2a} m \mathcal{R}e[\dot{u}_n^2]$$

$$< K >= \frac{mA^2}{4a} \omega^2 \qquad \dot{u}_n = \mathcal{R}e[j\omega u_n]$$

$$< K >= A^2 \frac{k}{a} \sin^2 \frac{\mu}{2} =< U >$$

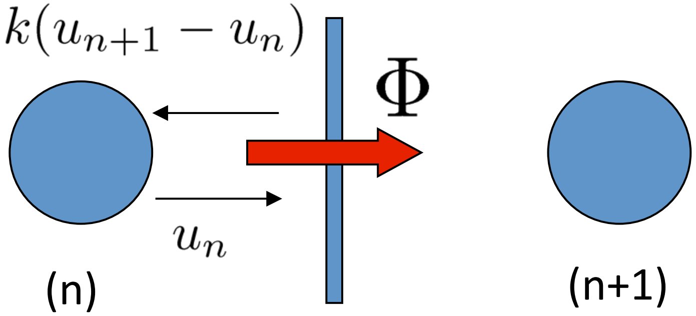
  - Total energy

$$< \mathcal{E} >=< K > + < U >$$

$$\boxed{< \mathcal{E} >= 2A^2 \frac{k}{a} \sin^2 \frac{\mu}{2}}$$

- <u>Energy flow</u> from one cell to the next is the AVERAGE POWER flowing from one cell to the next

$$f_{n,n+1} = k(u_{n+1} - u_n)$$



$$\Phi = - < \mathcal{R}e[f_{n,n+1}] \times \mathcal{R}e[\dot{u}_n] >$$

$$\Phi = -kA^2\omega_0 \sin\tfrac{\mu}{2} \sin\mu$$

- <u>Energy velocity</u>: rate at which energy flows along the lattice

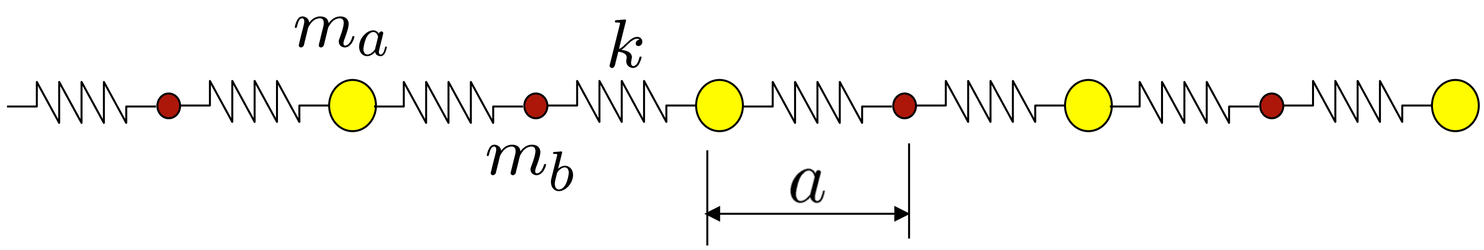
$$c_{\mathcal{E}} = \frac{\Phi}{<\mathcal{E}>} \qquad c_{\mathcal{E}} = \frac{A^2 k\omega_0 \sin\mu \sin\frac{\mu}{2}}{\frac{2kA^2}{a}\sin^2\frac{\mu}{2}}$$

$$c_{\mathcal{E}} = a\omega_0 \cos\frac{\mu}{2}$$

- <u>The energy velocity equals the group velocity</u>

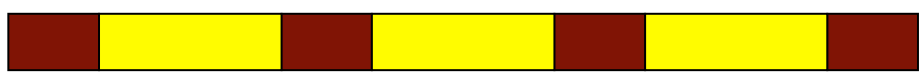
$$\boxed{c_{\mathcal{E}} = c_g}$$

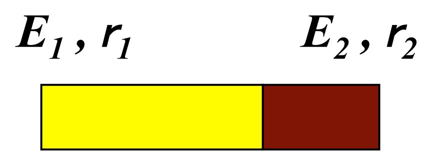
Diatomic lattice



System is representative of:

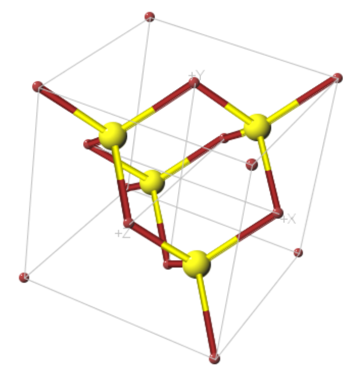
- Bi-material rod:



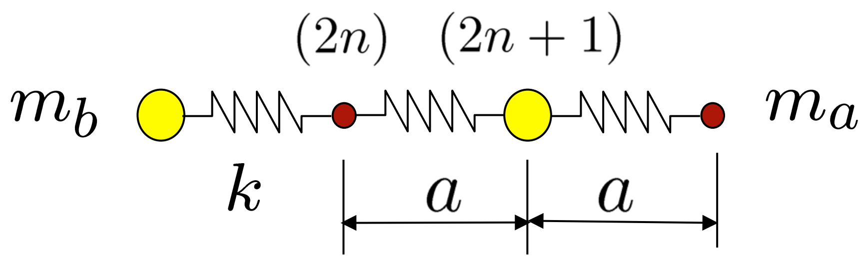


- NaCl crystal along one of lattice directions

- GaAs zincblende crystal: vibration of the (1 0 0) plane



- Governing equations for 2n and (2n+1) masses:



- Impose a solution of the kind:

$$-\omega^2 m_a u_{2n} + 2k u_{2n} - k(u_{2n+1} + u_{2n-1}) = 0$$

$$-\omega^2 m_b u_{2n+1} + 2k u_{2n+1} - k(u_{2n+2} + u_{2n}) = 0$$

$$u_{2n}(\omega) = u_a(\omega) e^{j\kappa x_{2n}} = u_a(\omega) e^{j2a\kappa n}$$

$$u_{2n+1}(\omega) = u_b(\omega) e^{j\kappa x_{2n+1}} = u_b(\omega) e^{j(2n+1)a\kappa}$$

This solution describes waves propagating only through particles (a) and (b). Wavelength and frequencies are the same, but the amplitudes of the two waves are not equal

- Substituting gives:

$$(2k - \omega^2 m_a)u_a - k(e^{j\kappa a} + e^{-j\kappa a})u_b = 0$$
$$(2k - \omega^2 m_b)u_b - k(e^{j\kappa a} + e^{-j\kappa a})u_a = 0$$

- In matrix form:

$$\begin{pmatrix} -\omega^2 m_a + 2k & -2k\cos\kappa a \\ -2k\cos\kappa a & -\omega^2 m_b + 2k \end{pmatrix} \begin{pmatrix} u_a \\ u_b \end{pmatrix} = 0$$

- Characteristic equation

$$\omega^4 - 2k(\frac{1}{m_a} + \frac{1}{m_b})\omega^2 + 4\frac{k^2}{m_a m_b}\sin^2\kappa a = 0$$

- Solution identifies TWO BRANCHES:

$$\omega = \omega_1(\kappa) \text{ and } \omega = \omega_2(\kappa)$$

- where:

$$\omega_1^2(\kappa) = k(\frac{1}{m_a} + \frac{1}{m_b}) - \sqrt{(\frac{1}{m_a} + \frac{1}{m_b})^2 - \frac{4}{m_a m_b}\sin^2 \kappa a}$$

ACOUSTIC BRANCH

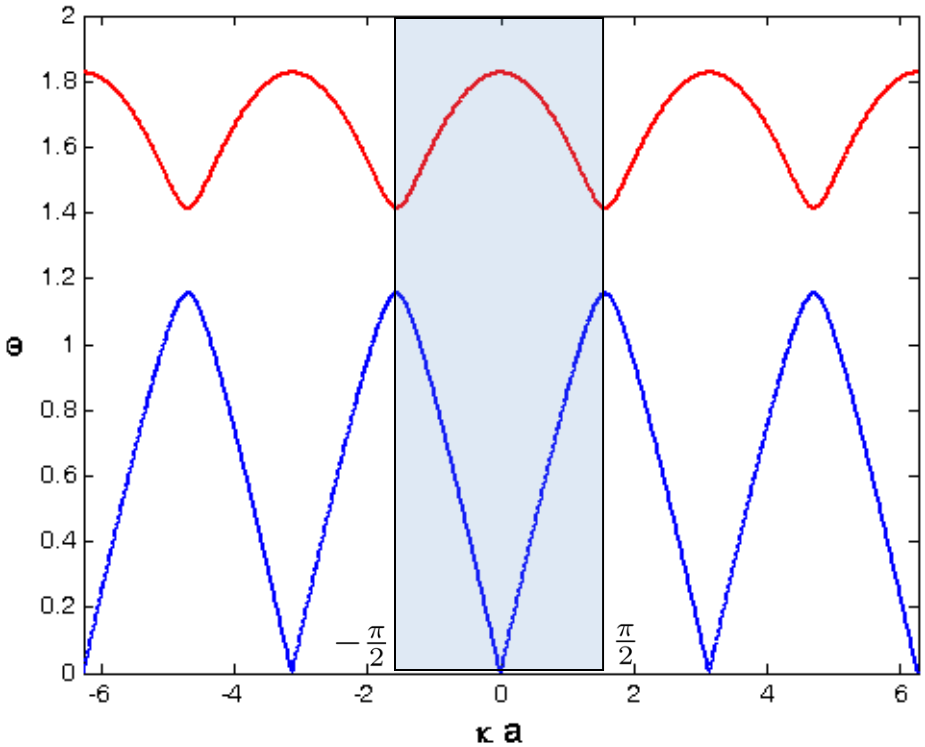
$$\omega_2^2(\kappa) = k(\frac{1}{m_a} + \frac{1}{m_b}) + \sqrt{(\frac{1}{m_a} + \frac{1}{m_b})^2 - \frac{4}{m_a m_b}\sin^2 \kappa a}$$

OPTICAL BRANCH

- Both branches are PERIODIC in the wavenumber domain:

$$\kappa a \in \left[-\frac{\pi}{2}, \quad \frac{\pi}{2}\right]$$

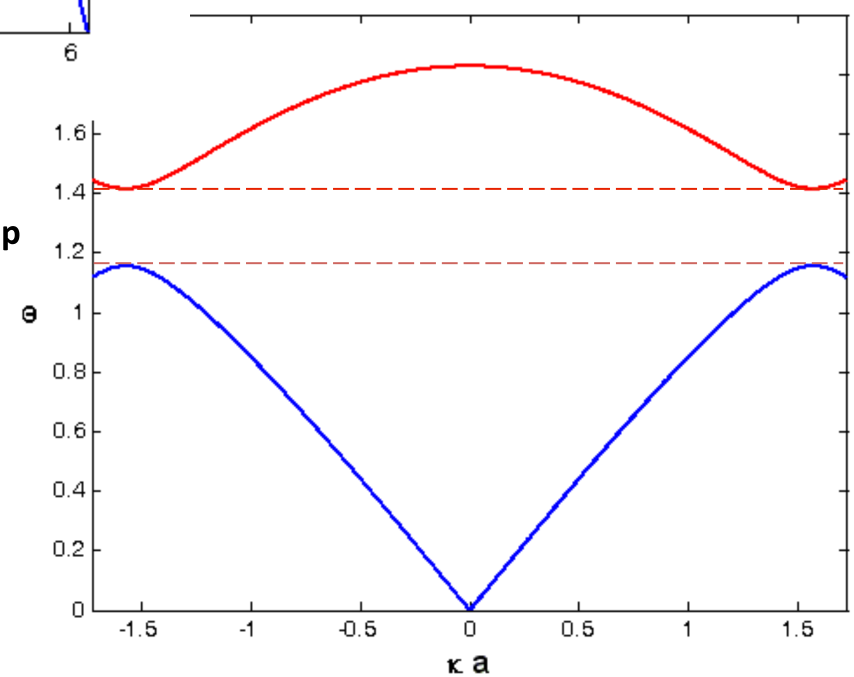
$\omega = \omega(\kappa)$ Direct solution



OPTICAL BRANCH

ACOUSTIC BRANCH

Band gap

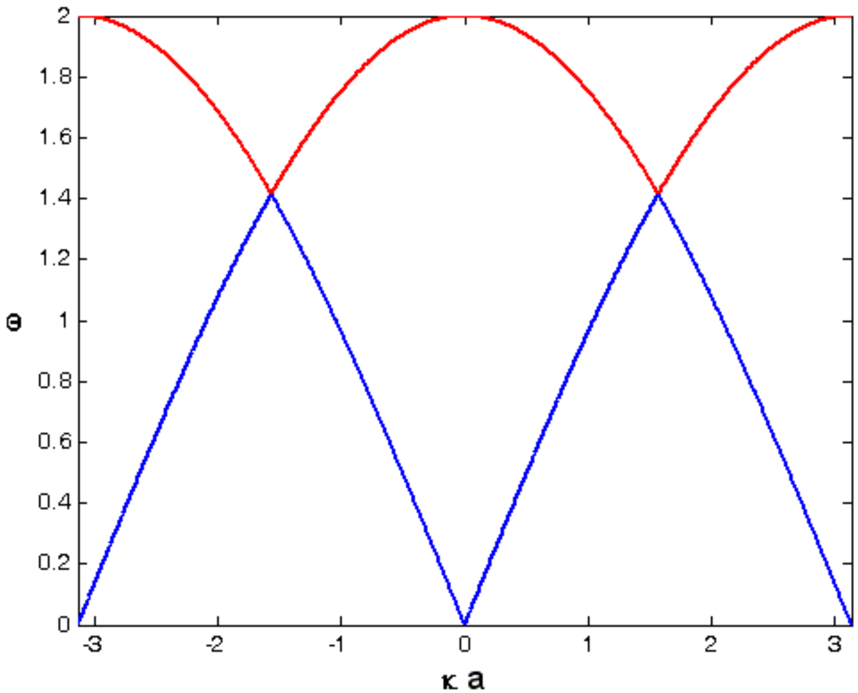


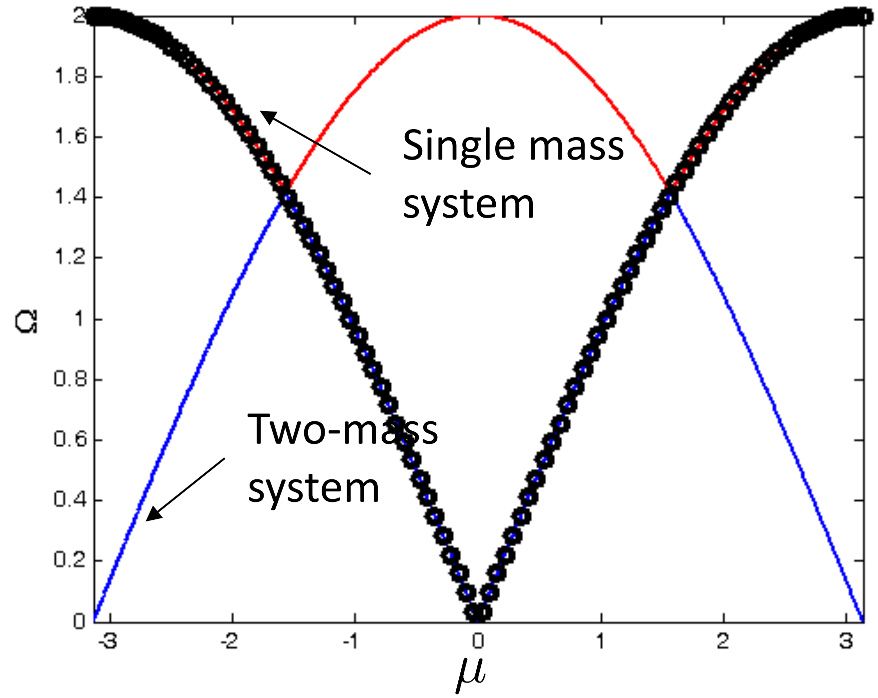
$$\sqrt{\frac{2k}{m_a}}$$

$$\sqrt{\frac{2k}{m_b}}$$

$\omega = \omega(\kappa)$ Direct solution

$$m_a = m_b$$



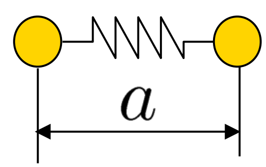


Band-gap disappears

- Period of wavenumber/frequency domain is:
  - Single mass system
$$\kappa a \in [-\pi, \quad +\pi]$$
  - Two-mass system
$$\kappa a \in [-\tfrac{\pi}{2}, \quad +\tfrac{\pi}{2}] \Longrightarrow 2\kappa a \in [-\pi, \quad +\pi]$$
- The period of the dispersion relation is always given by:

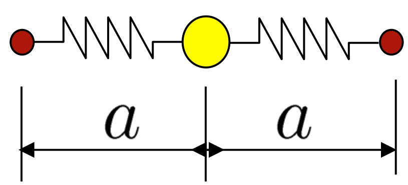
where:
$$\kappa d \in [-\pi, \quad +\pi]$$

  - Single mass system:



$$d = a$$

  - Two mass system:



$$d = 2a$$

- For any 1D periodic system, the frequency/wavenumber spectrum is periodic in the domain:

where $$\kappa d \in [-\pi, \ \ +\pi]$$

d - Spatial period of the structure

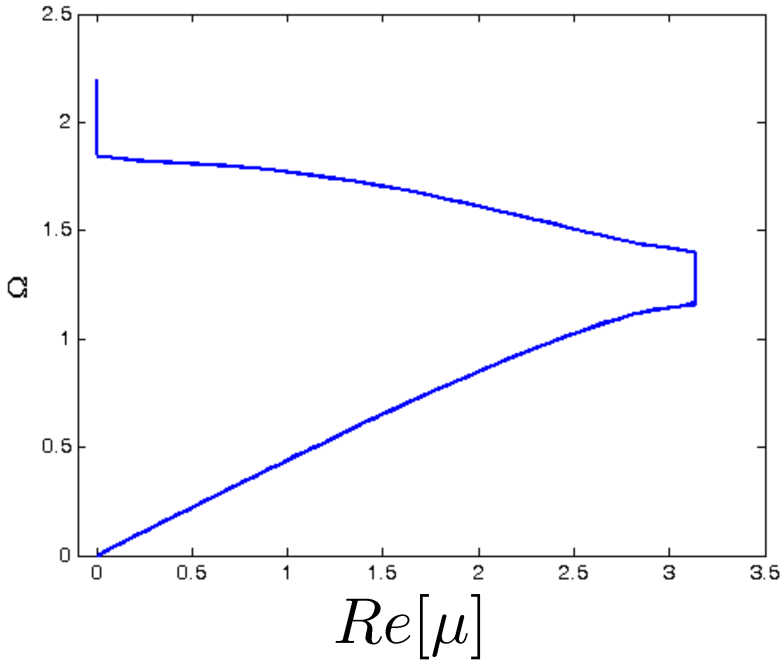
- Propagation constant:

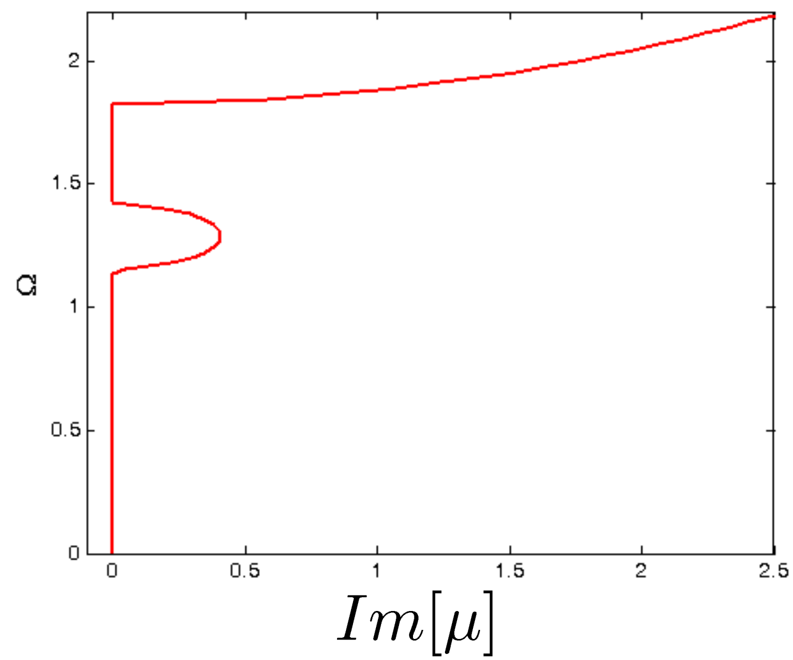
$$\mu = \kappa d$$

- First Brillouin zone: $\mu \in [-\pi, \ \ +\pi]$

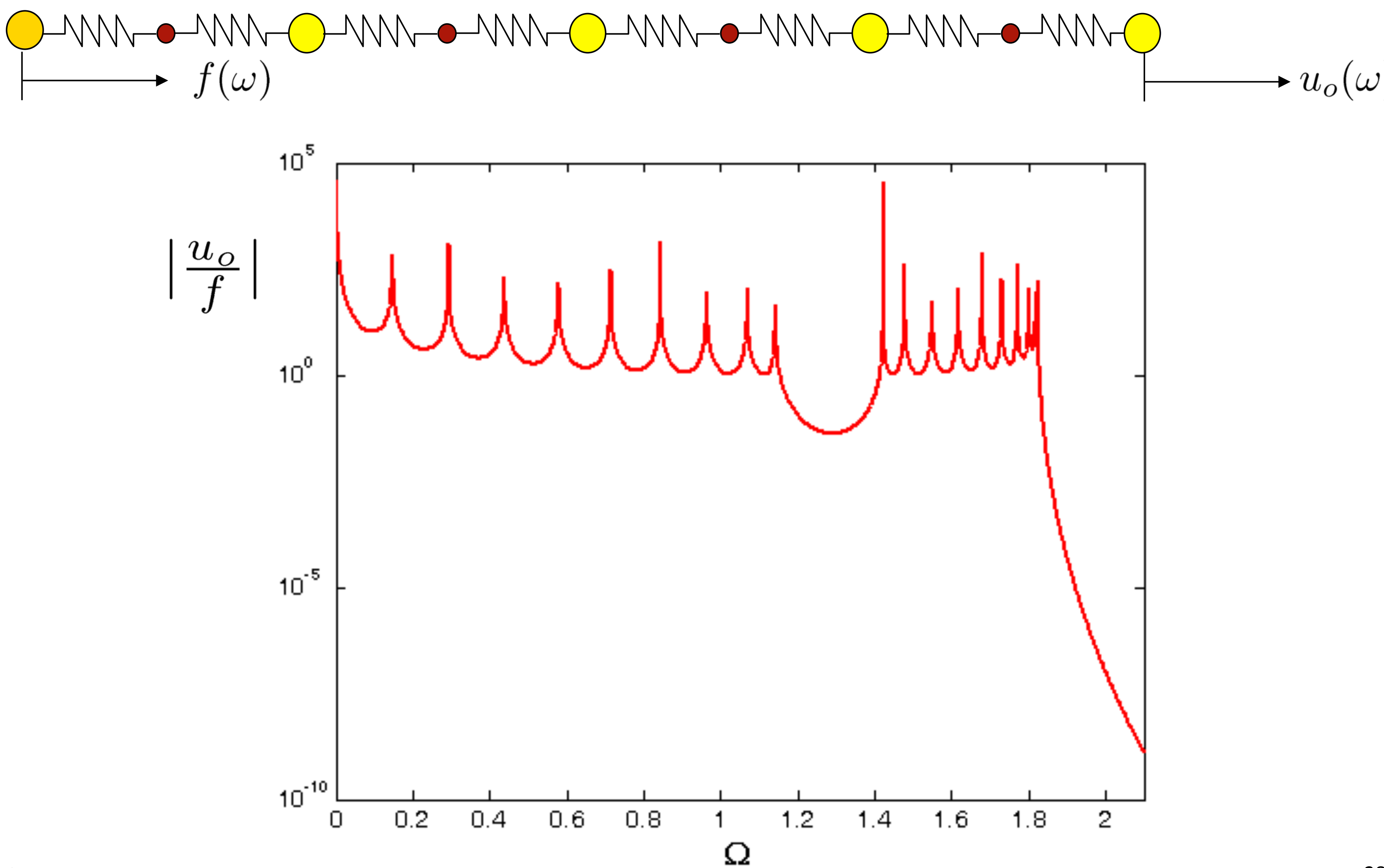
The definition of the Brillouin zone can be used to define unequivocally the SPATIAL PERIOD of the system

$\mu = \mu(\omega)$: inverse solution



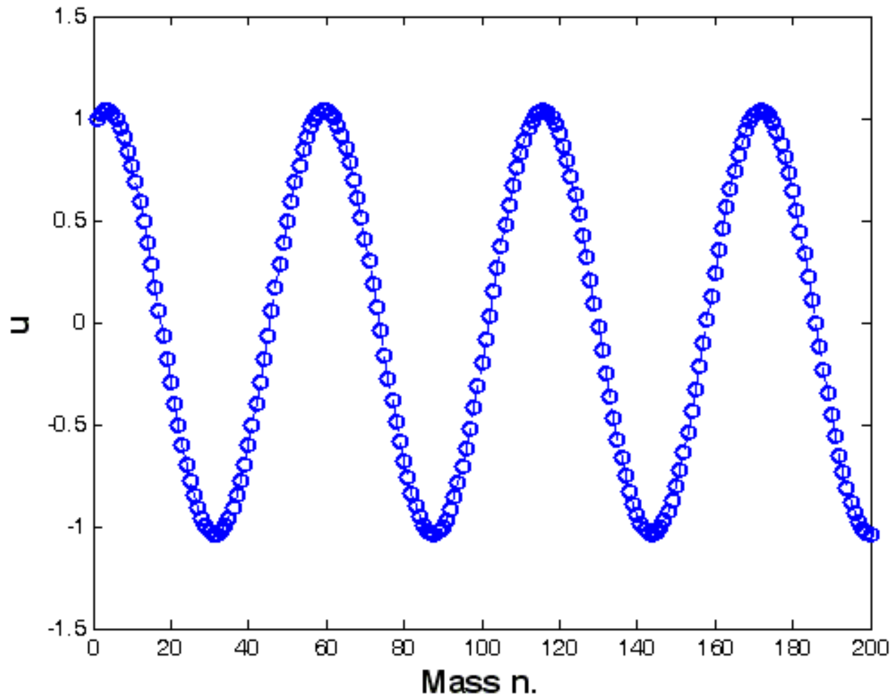


Response

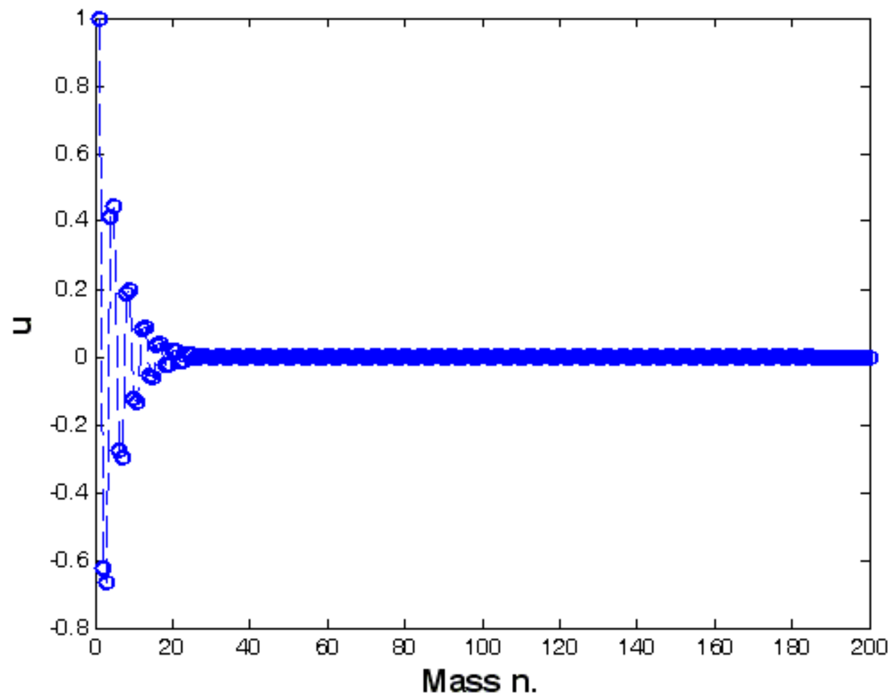

$f(\omega)$
$u_o(\omega)$
$\left|\frac{u_o}{f}\right|$
10^5
10^0
10^{-5}
10^{-10}
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Ω

Response of a system of N=200 masses

$$\Omega = 0.1$$

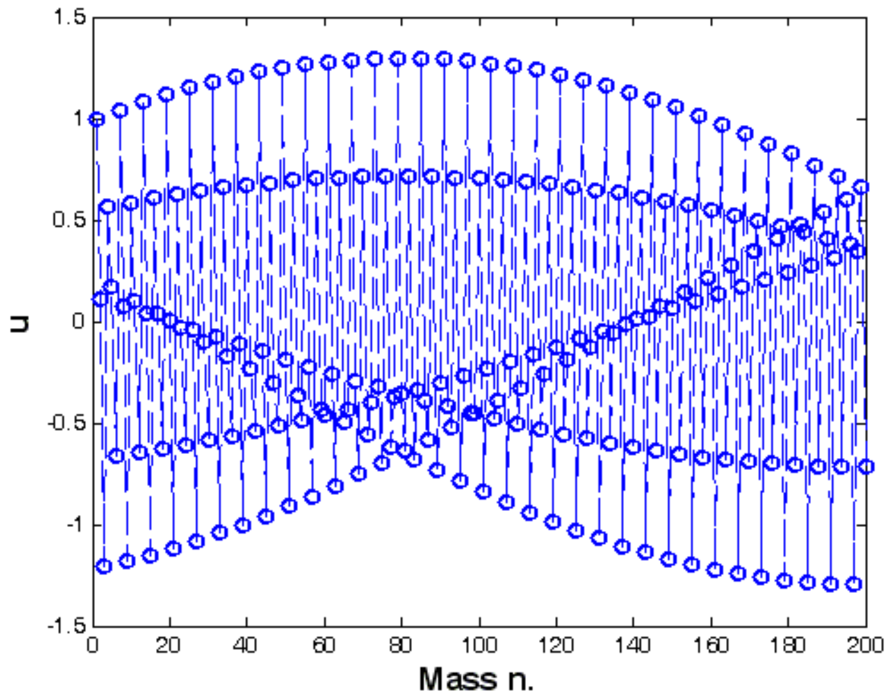


$$\Omega = 1.3$$

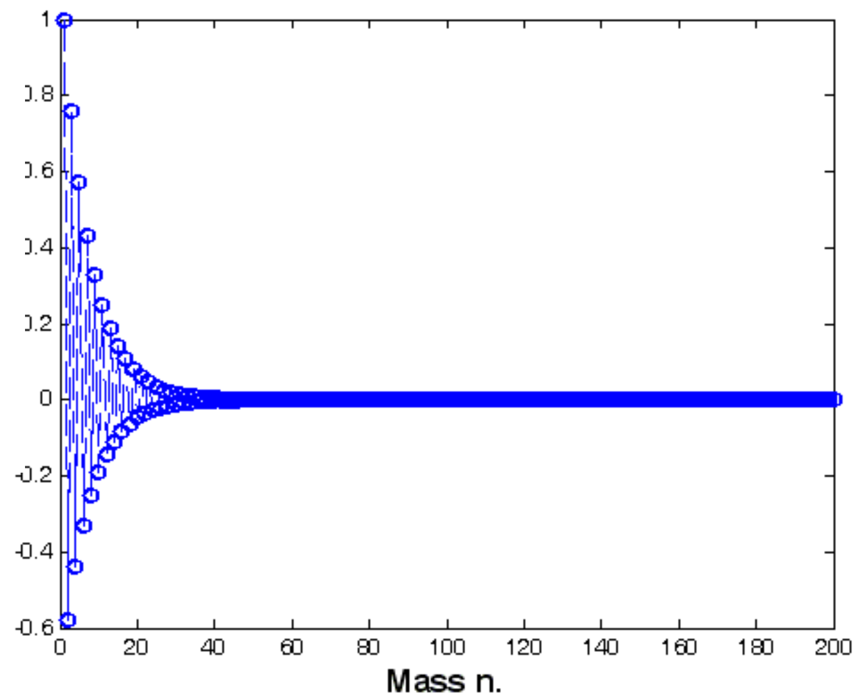


Response of a system of N=200 masses

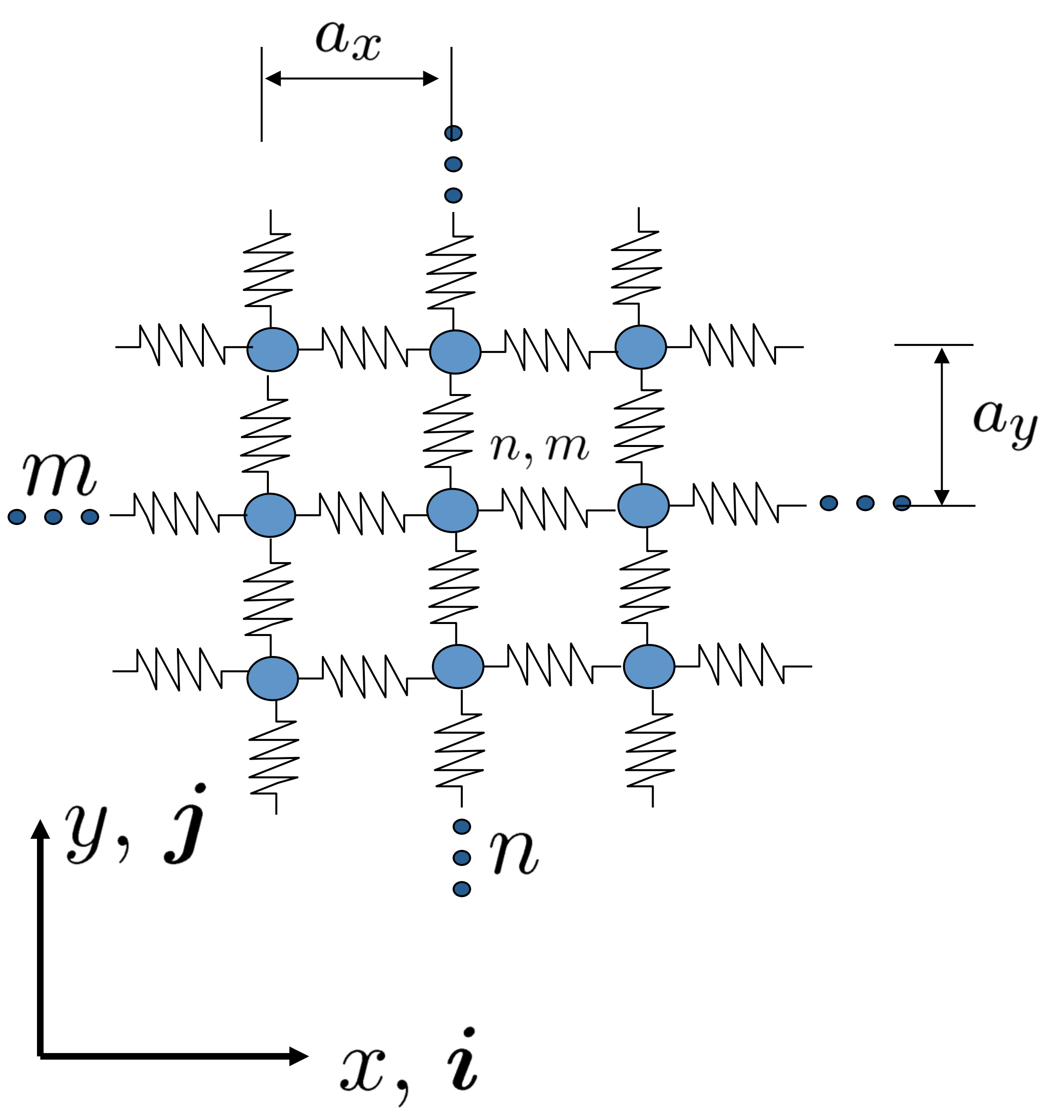
$$\Omega = 1.6$$



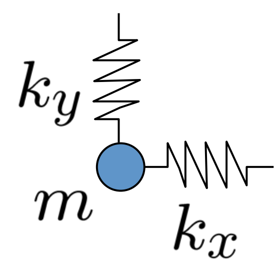
$$\Omega = 1.83$$



Spring Mass System



Unit Cell:



Governing equations & wave solution

- Equation of harmonic motion for mass *n,m*:

$$-\omega^2 m u_{n,m} + (2k_x + 2k_y)u_{n,m} ...$$
$$... - k_x(u_{n+1,m} + u_{n-1,m}) - k_y(u_{n,m+1} + u_{n,m-1}) = 0$$

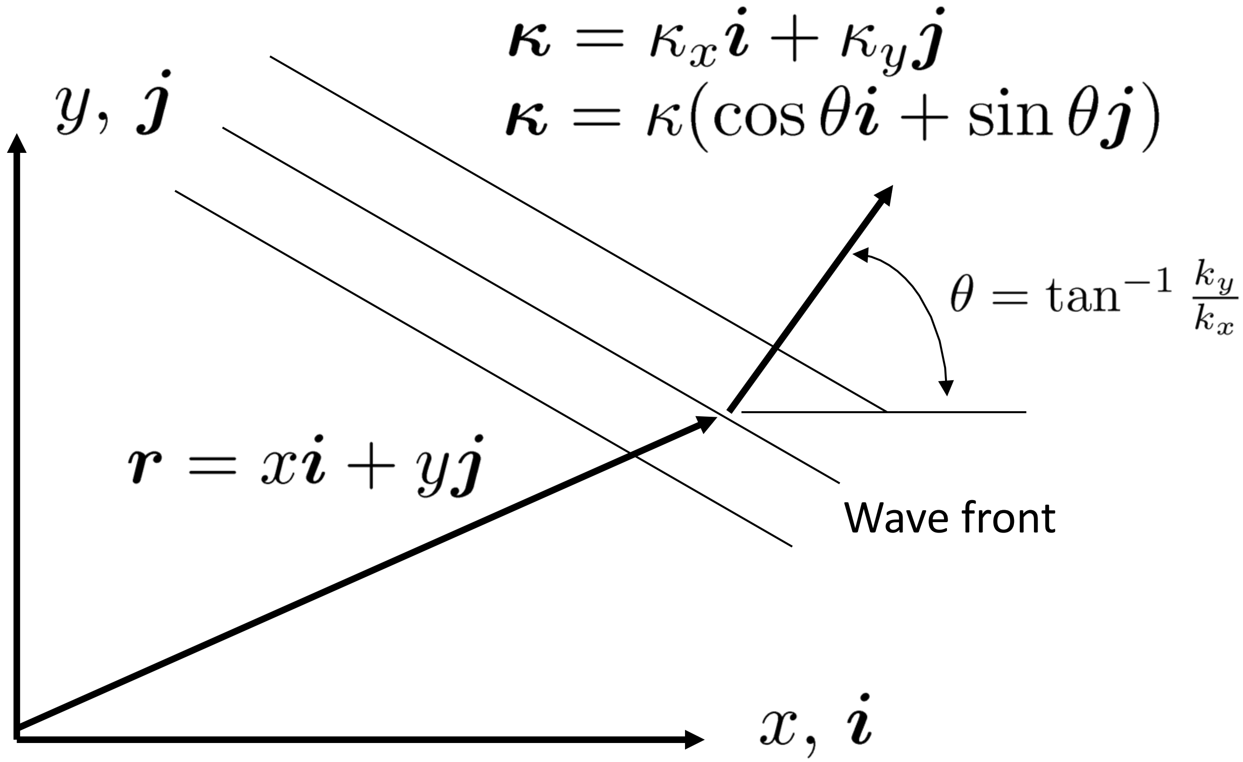
- Wave propagation solution:

$$u_{n,m}(\omega) = u_0[\kappa(\omega)]e^{j\boldsymbol{\kappa}\cdot\boldsymbol{r}}$$

  – where

$$\boldsymbol{\kappa} = \kappa_x \boldsymbol{i} + \kappa_y \boldsymbol{j}$$
$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$$
$$\boldsymbol{r} = n a_x \boldsymbol{i} + m a_y \boldsymbol{j}$$

- Direction of wave propagation



- Rewrite solution as:

$$\begin{aligned} u_{n,m}(\omega) &= u_0[\kappa(\omega)]e^{j\boldsymbol{\kappa}\cdot\boldsymbol{r}} \\ &= u_0[\kappa(\omega)]e^{j(\kappa_x a_x n+\kappa_y a_y m)} \\ &= u_0[\kappa(\omega)]e^{j(\mu_x n+\mu_y m)} \end{aligned}$$

- and

$$u_{n\pm1,m\pm1} = u_0[\kappa(\omega)]e^{j(\mu_x(n\pm1)+\mu_y(m\pm1))}$$

$$u_{n\pm1,m\pm1} = u_{n,m}(\omega)e^{j(\pm\mu_x\pm\mu_y)}$$

- Substituting in governing equation leads to:

$$[(-\omega^2 m + 2k_x + 2k_y) - k_x(e^{-j\mu_x} + e^{j\mu_x}) - k_y(e^{-j\mu_y} + e^{j\mu_y})]u_0(\mu_x, \mu_y) = 0$$

$$[-\omega^2 m + 2k_x(1 - \cos\mu_x) + 2k_y(1 - \cos\mu_y)]u_0(\mu_x, \mu_y) = 0$$

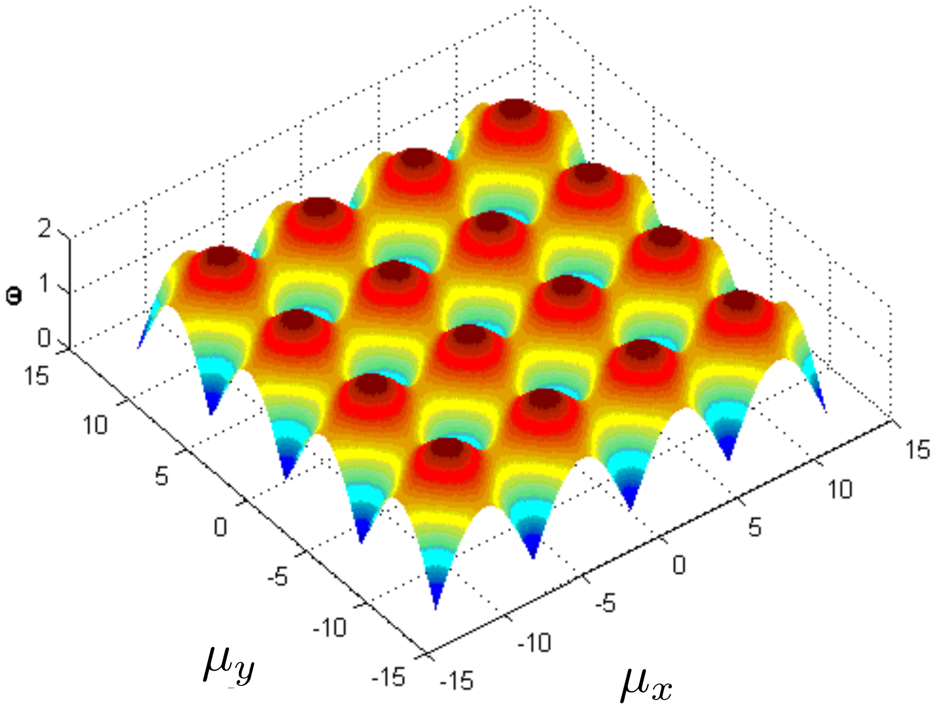
- 2D dispersion relation

$$\boxed{-\omega^2 m + 2k_x(1 - \cos\mu_x) + 2k_y(1 - \cos\mu_y) = 0}$$

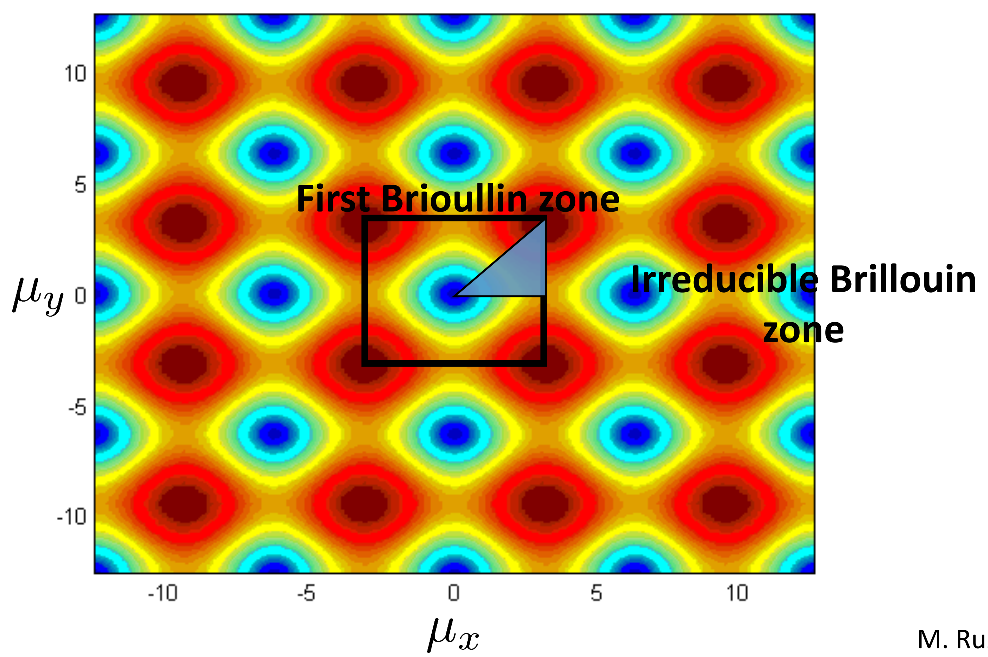
$$\boxed{\omega = \omega(\mu_x, \mu_y)}$$

Surface in the wavenumber domain

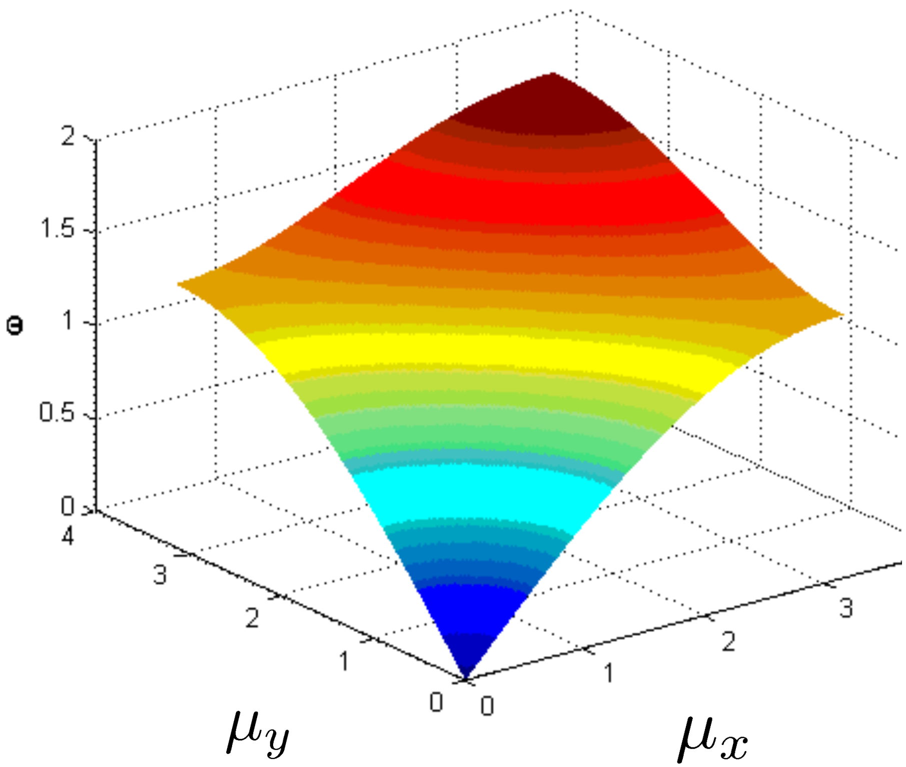
2D Dispersion relation



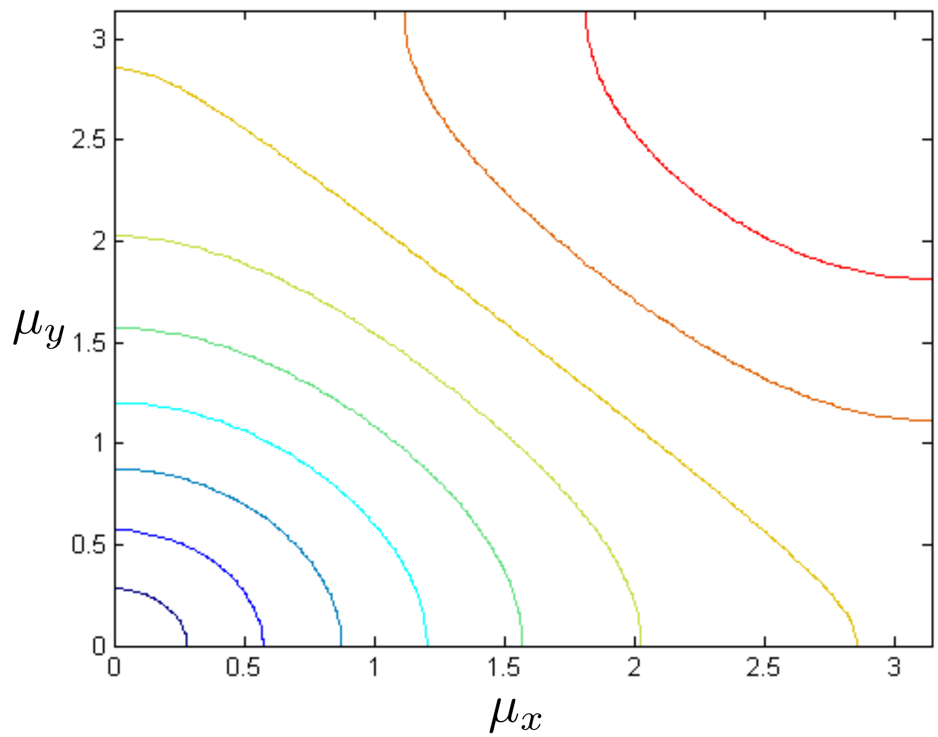
$$k_x = k_y = 1$$
$$m = 1$$



$$k_x = k_y = 1$$
$$m = 1$$



2
1.5
1
0.5
0
4
3
2
1
0
0
1
2
3
ω
μ_y
μ_x



3
2.5
2
1.5
1
0.5
0
0
0.5
1
1.5
2
2.5
3
μ_y
μ_x

- According to definition:

$$\boldsymbol{c}_g = \boldsymbol{\nabla}\omega(\mu_x, \mu_y)$$

  - Where

$$\boldsymbol{c}_g = c_{g_x}\boldsymbol{i} + c_{g_y}\boldsymbol{j}$$

$$c_{g_x} = \frac{\partial \omega}{\partial \kappa_x} = a_x \frac{\partial \omega}{\partial \mu_x}$$

$$c_{g_y} = \frac{\partial \omega}{\partial \kappa_y} = a_y \frac{\partial \omega}{\partial \mu_y}$$

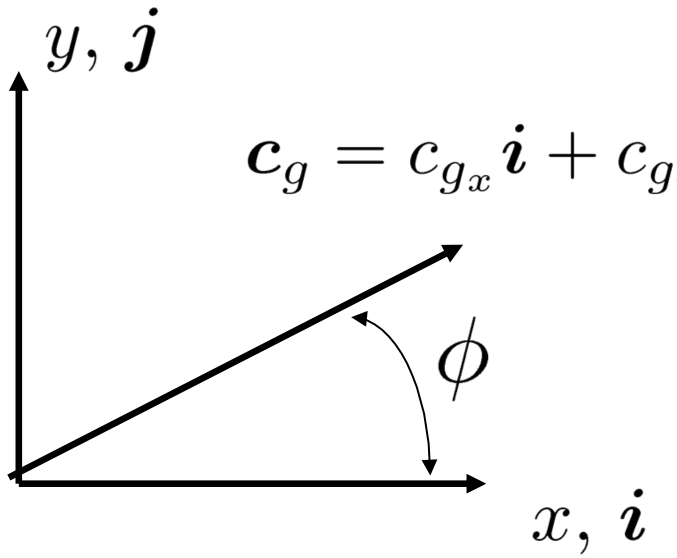
- Recall that:
  - Velocity of energy flow equals the group velocity
  - Energy flows in the direction corresponding to the group velocity

- In this case:

$$c_{g_x} = a_x \frac{\partial \omega}{\partial \mu_x} = \frac{a_x}{2} \frac{k_x/m \sin \mu_x}{[k_x/m(1-\cos \mu_x)+k_y/m(1-\cos \mu_y)]^{1/2}}$$

$$c_{g_x} = a_y \frac{\partial \omega}{\partial \mu_y} = \frac{a_y}{2} \frac{k_y/m \sin \mu_y}{[k_x/m(1-\cos \mu_x)+k_y/m(1-\cos \mu_y)]^{1/2}}$$

$$\boldsymbol{c}_g = c_{g_x}\boldsymbol{i} + c_{g_y}\boldsymbol{j}$$



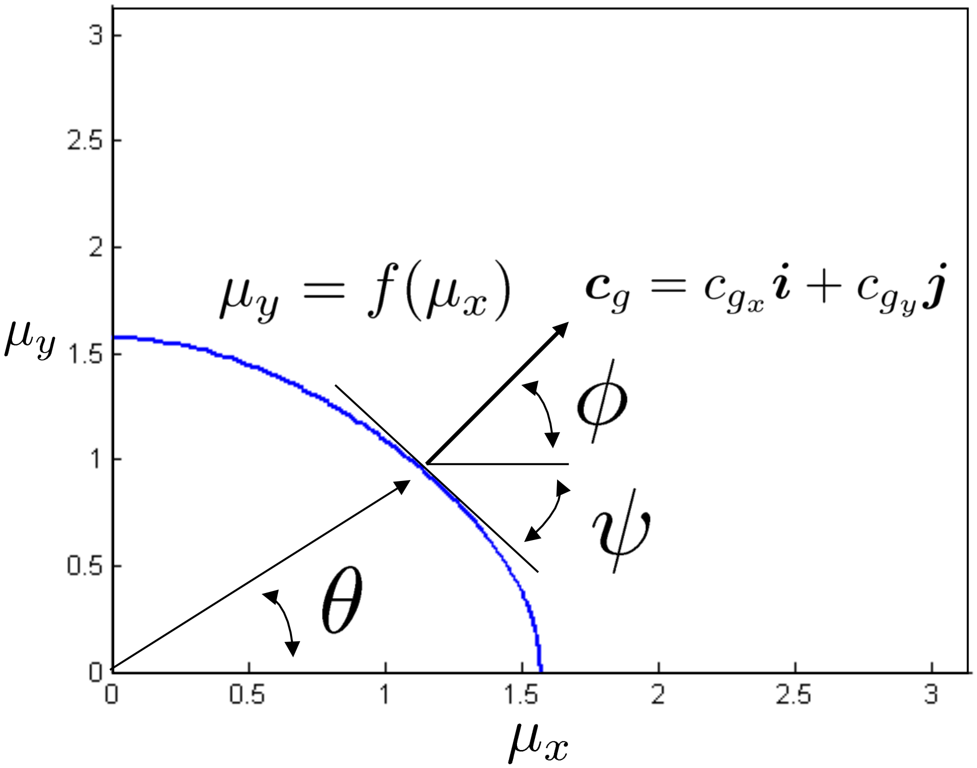
$$\frac{k_y \sin \mu_y}{k_x \sin \mu_x} = \tan \phi$$

Assume

$$a_y = a_x$$

- Contour at a single frequency

$$\omega_0 = 1$$



From dispersion relations:

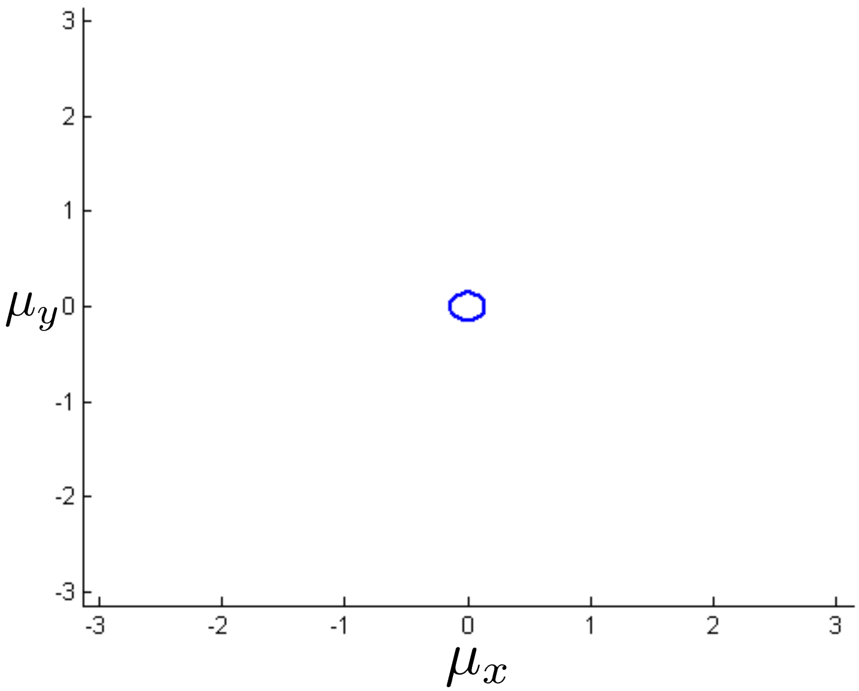
$$\frac{\partial \mu_y}{\partial \mu_x} = -\frac{a_x k_x \sin \mu_x}{a_y k_y \sin \mu_y} = \tan \psi$$

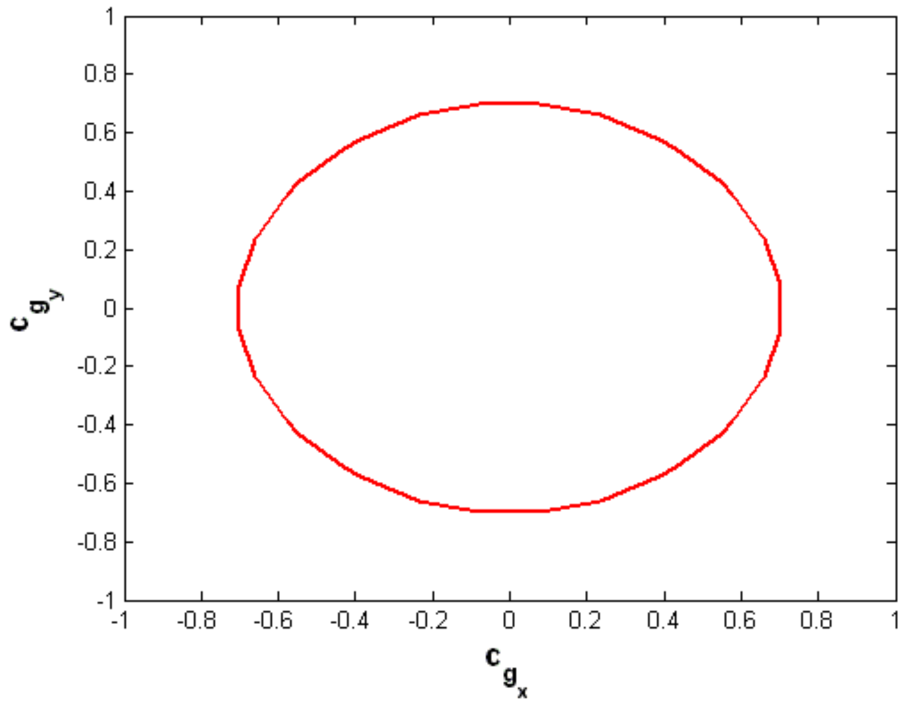
Direction of energy flow at a given frequency and direction is perpendicular to isofrequency contour

Dispersion surface vs. group velocity

$$k_x = k_y = 1 \qquad m = 1$$

$$\omega_0 = 0.1$$

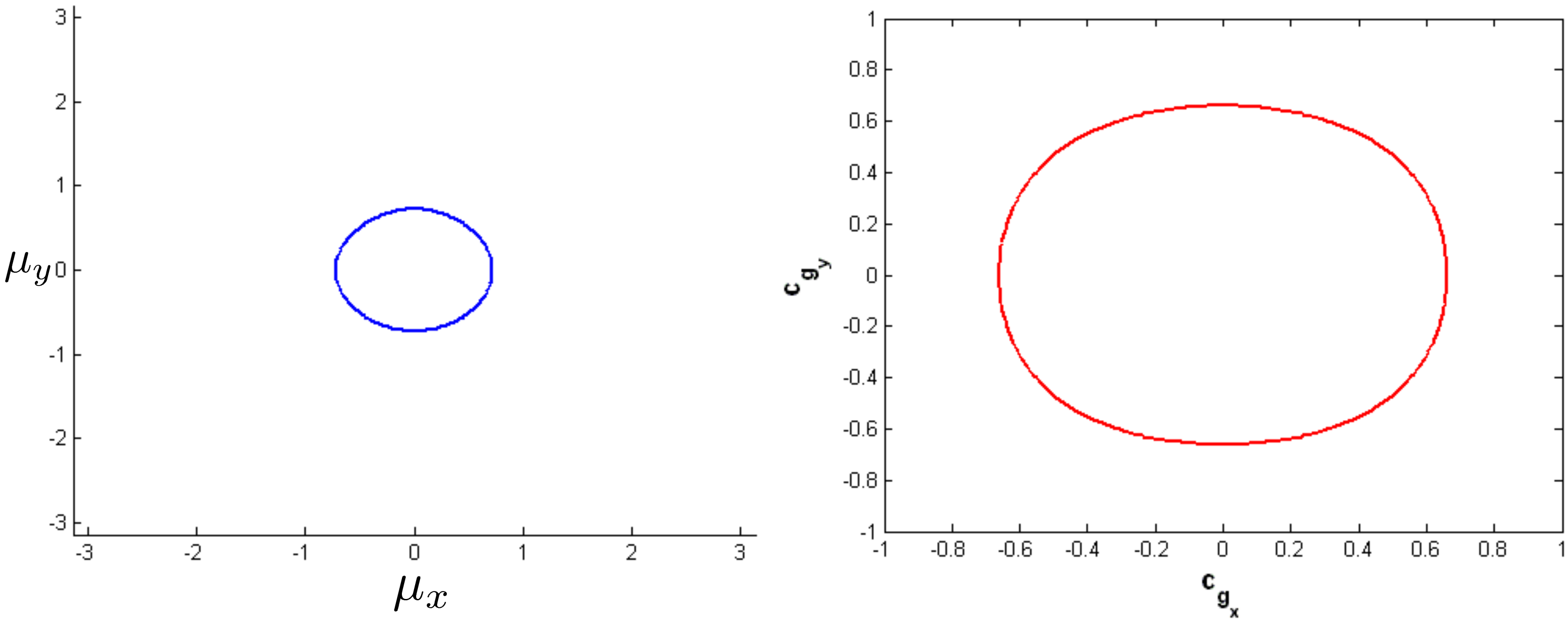




Dispersion surface vs. group velocity

$$k_x = k_y = 1 \qquad m = 1$$

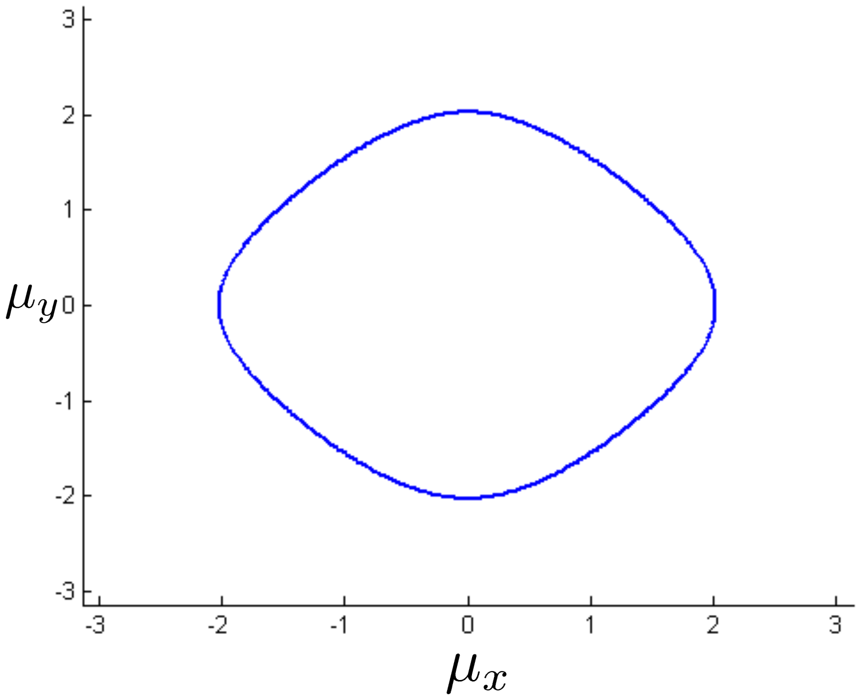
$$\omega_0 = 0.5$$

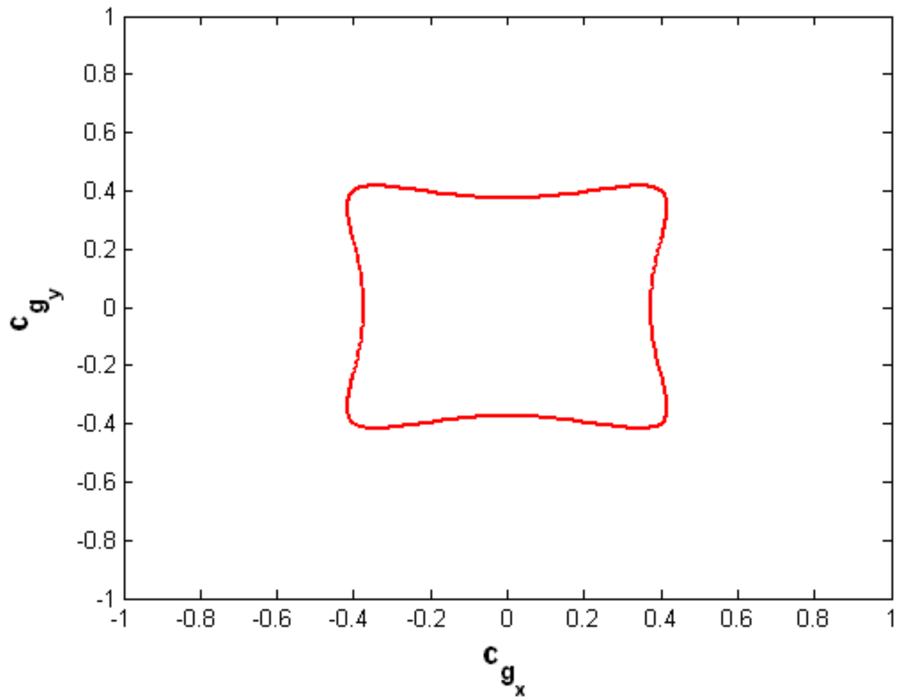


Dispersion surface vs. group velocity

$$k_x = k_y = 1 \qquad m = 1$$

$$\omega_0 = 1.2$$

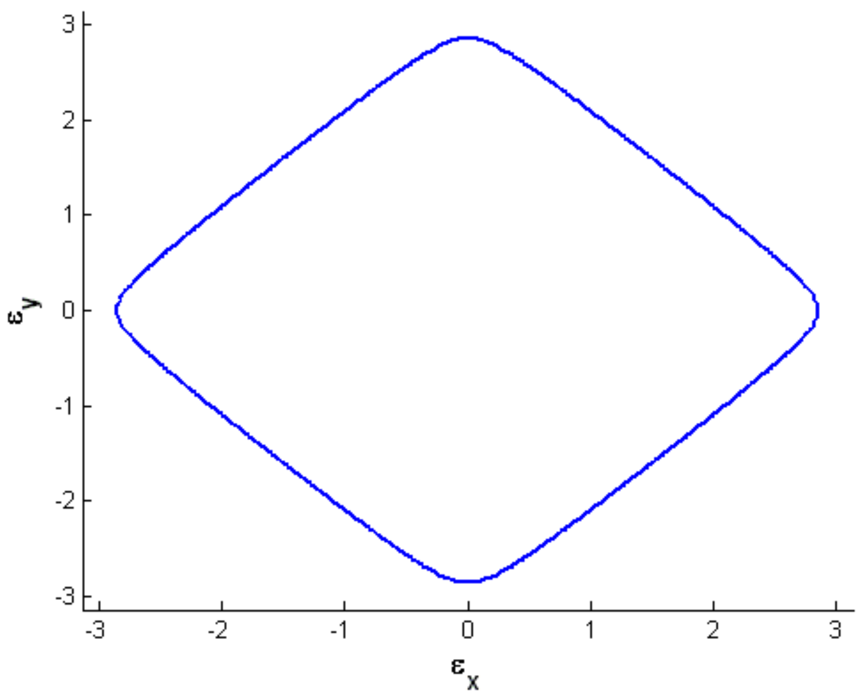


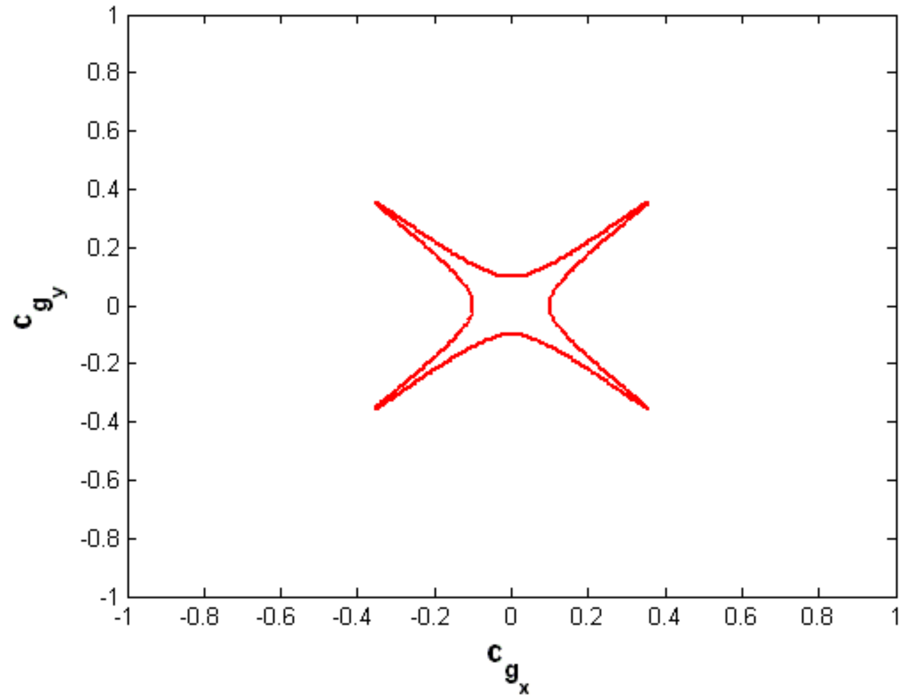


Dispersion surface vs. group velocity

$$k_x = k_y = 1 \qquad m = 1$$

$$\omega_0 = 1.4$$

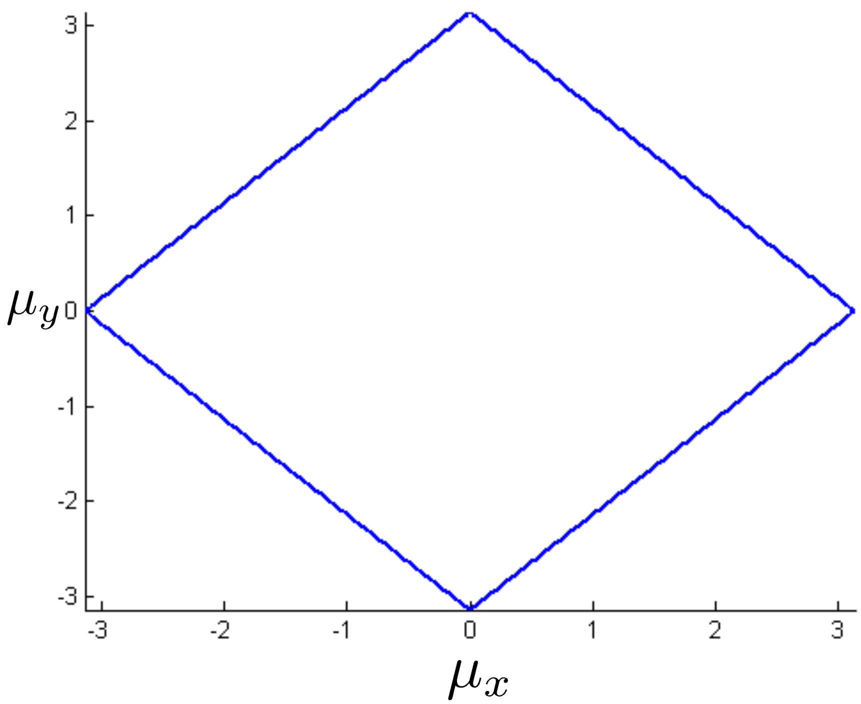


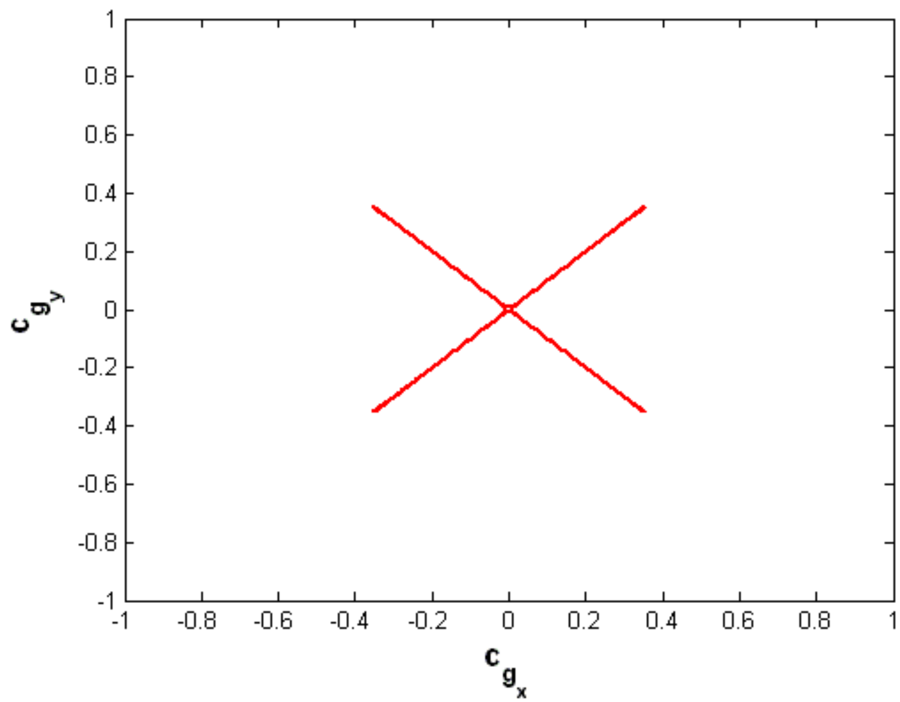


Dispersion surface vs. group velocity

$$k_x = k_y = 1 \qquad m = 1$$

$$\omega_0 = \sqrt{2}$$





Notes

- Propagation of waves is strongly directional at specified frequency
- At those frequencies, waves propagate only in certain directions

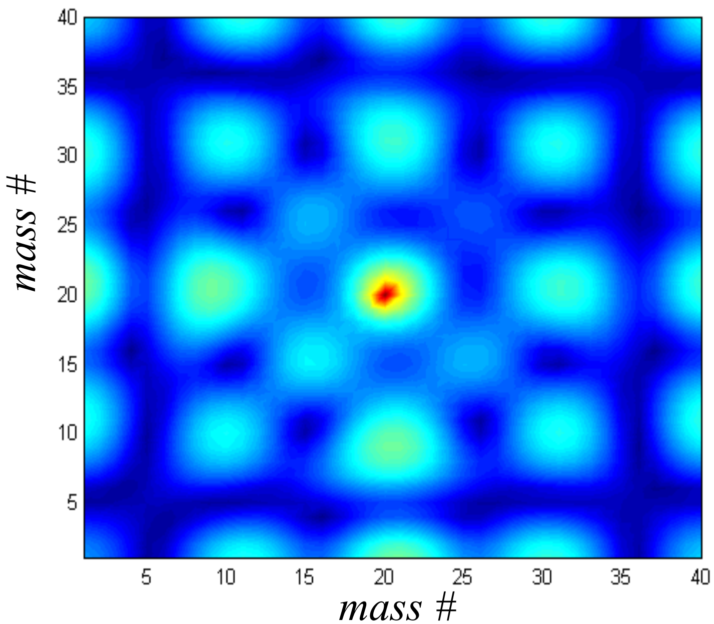
BEAMING PHENOMENA

- For considered configuration beaming is a very focused, but very narrow-band phenomenon

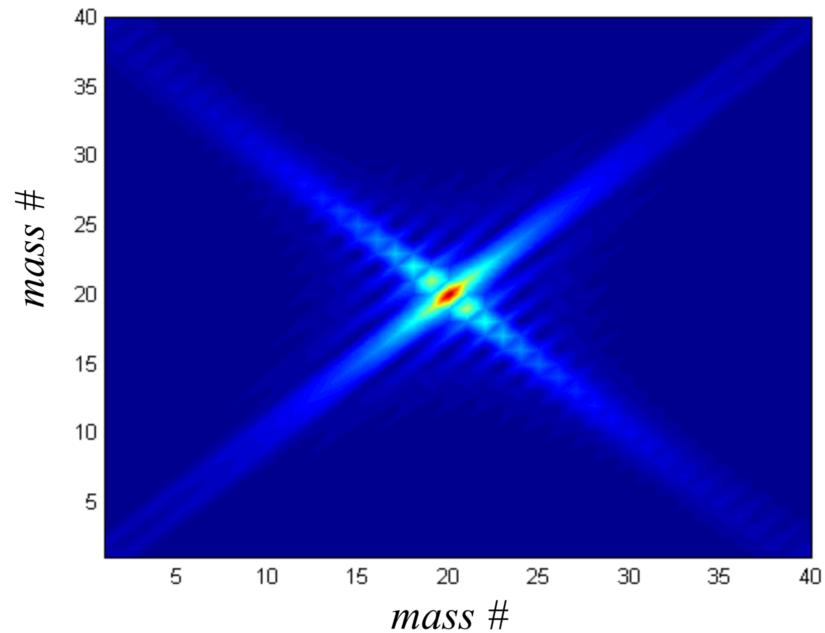
Example: 2D spring-mass lattice

Harmonic response of 40*40 lattice:

$$\omega_0 = 0.5$$



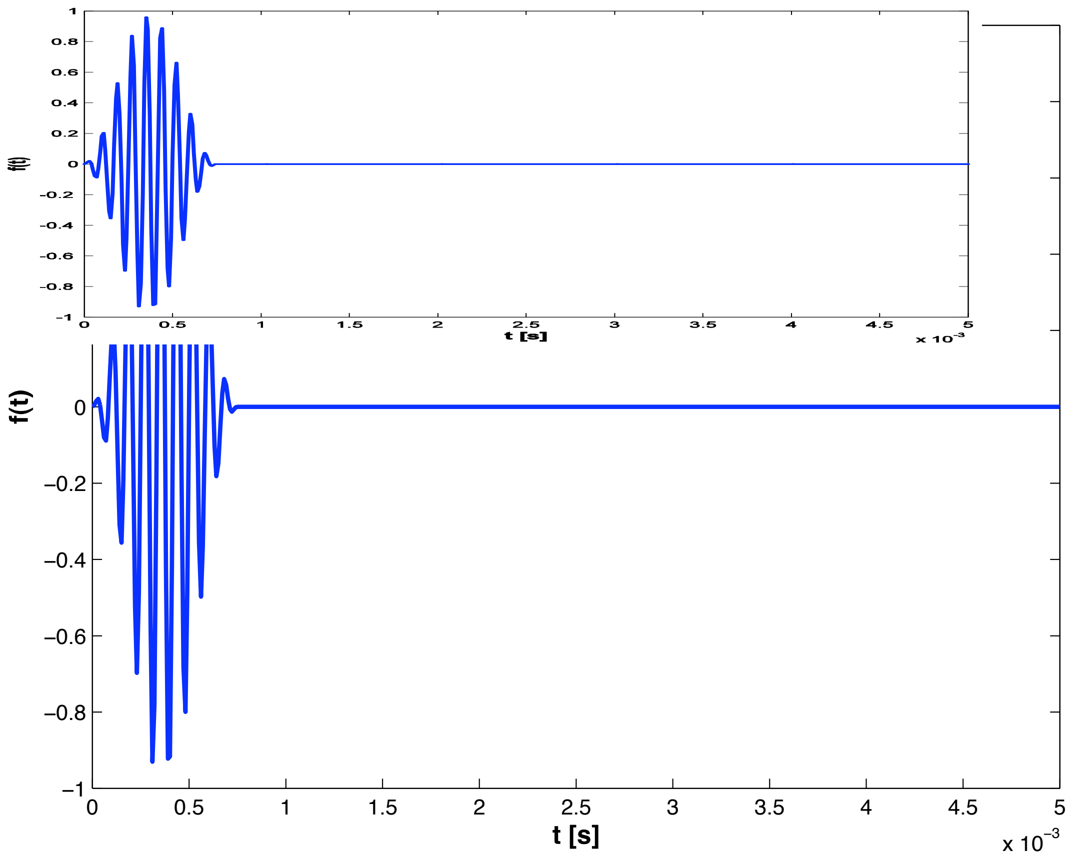
$$\omega_0 = \sqrt{2}$$



Time domain response

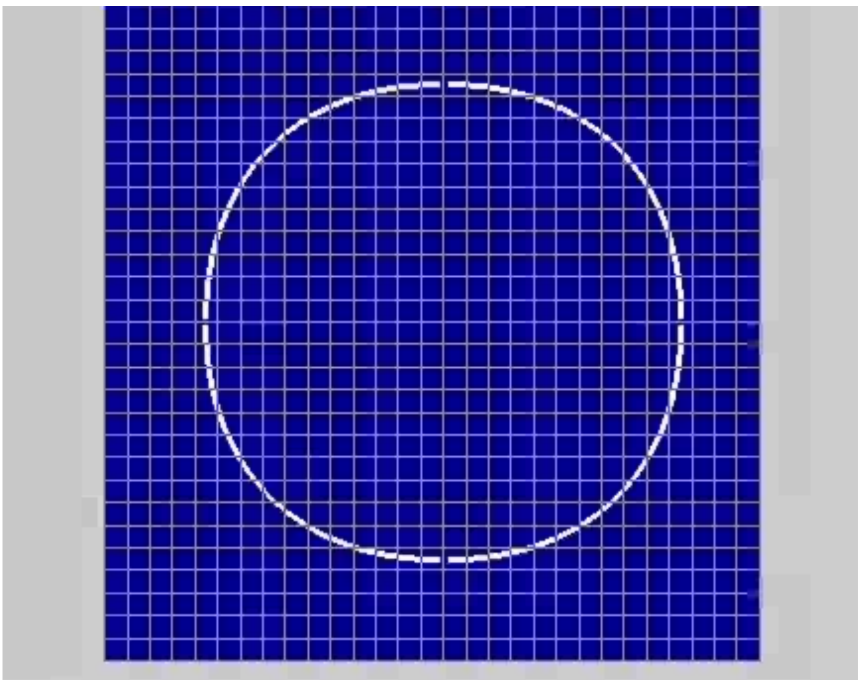
Time domain simulations

Input modulated sine burst at various frequencies w_e

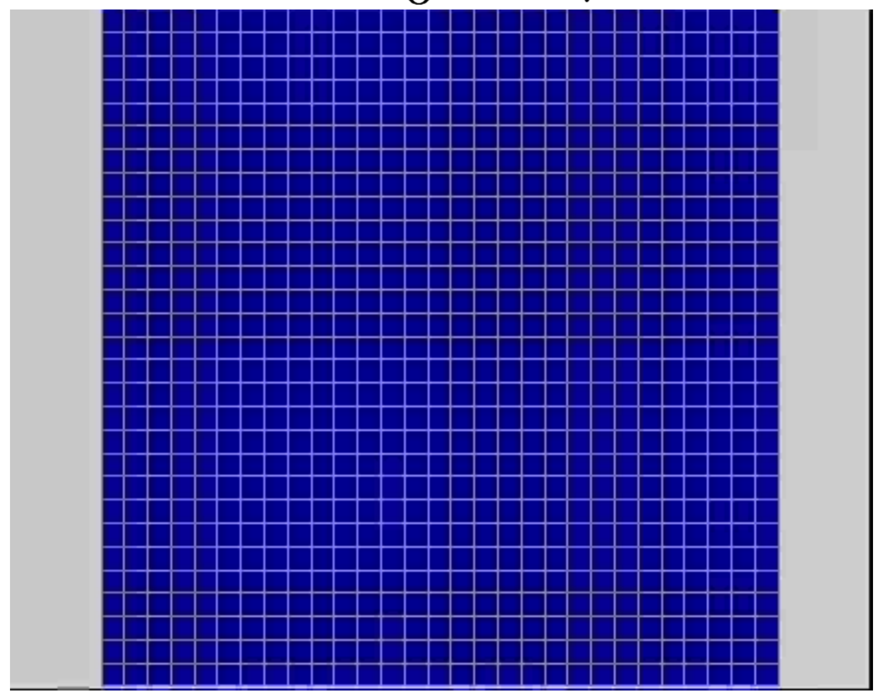


Time domain response

$\omega_0 = 0.5$



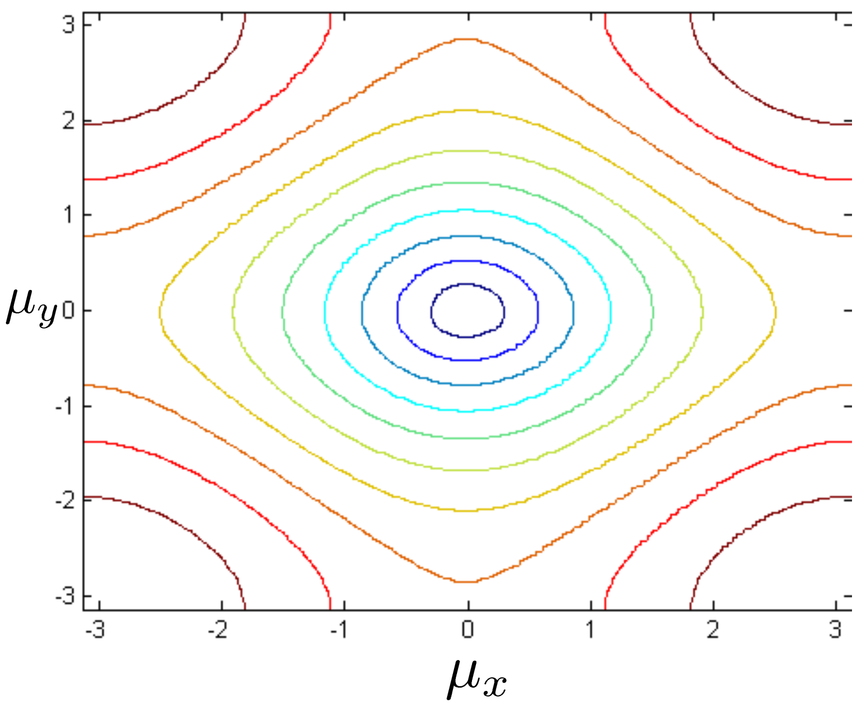
$\omega_0 = \sqrt{2}$

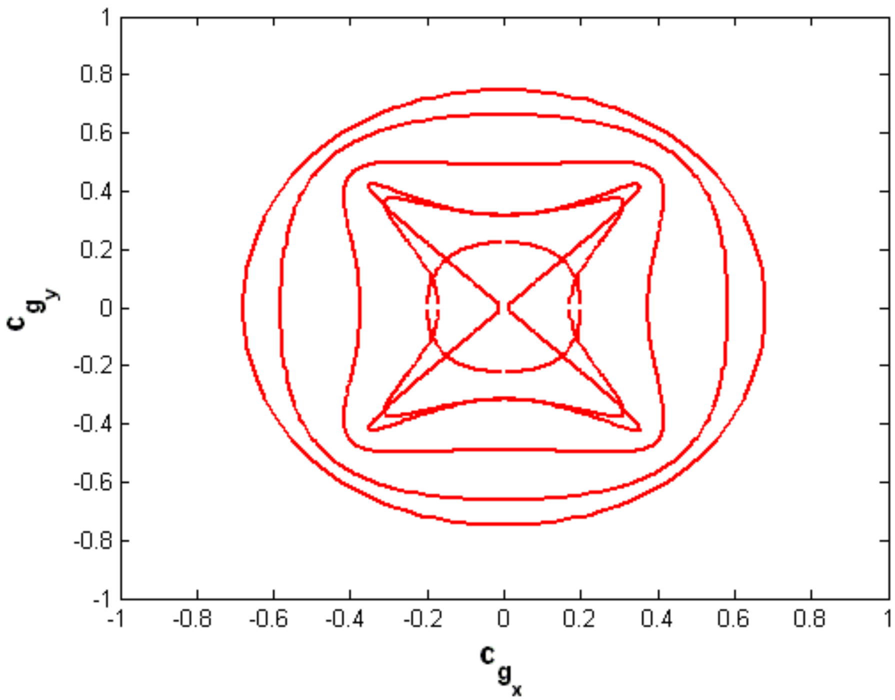


Anisotropic lattice

$$k_x \neq k_y$$

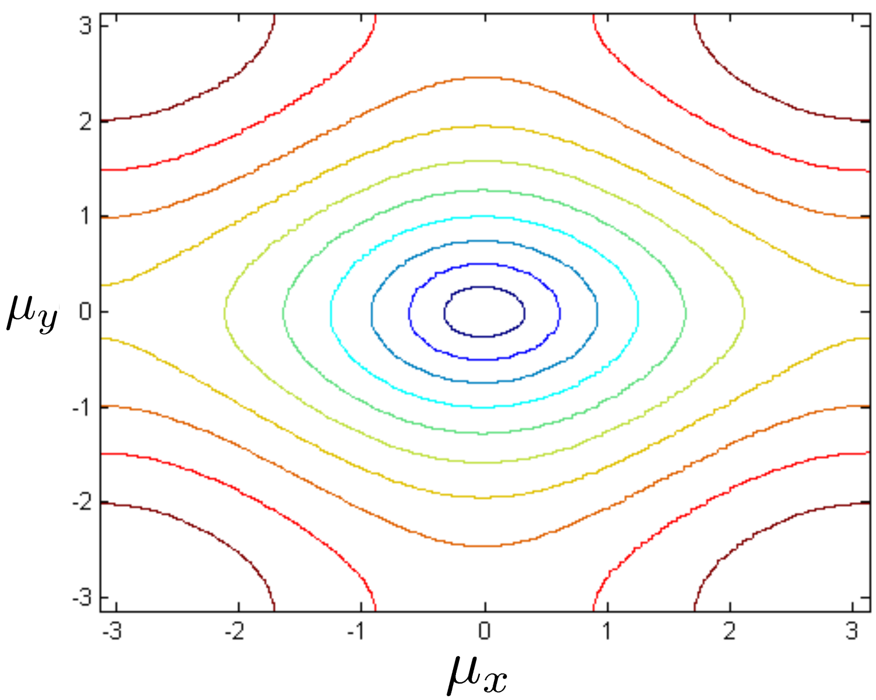
$$k_x = 1, k_y = 1.2$$

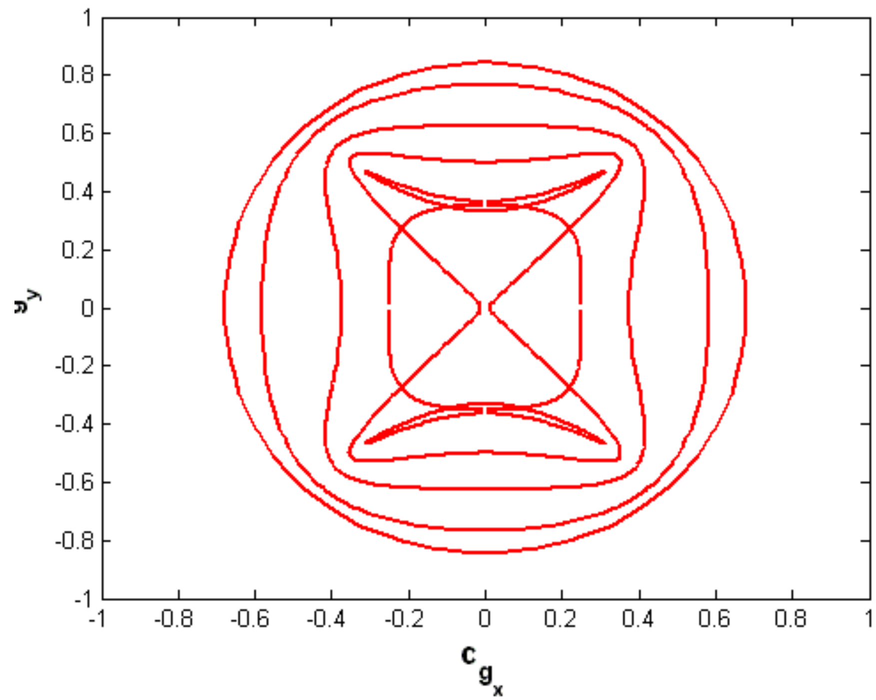




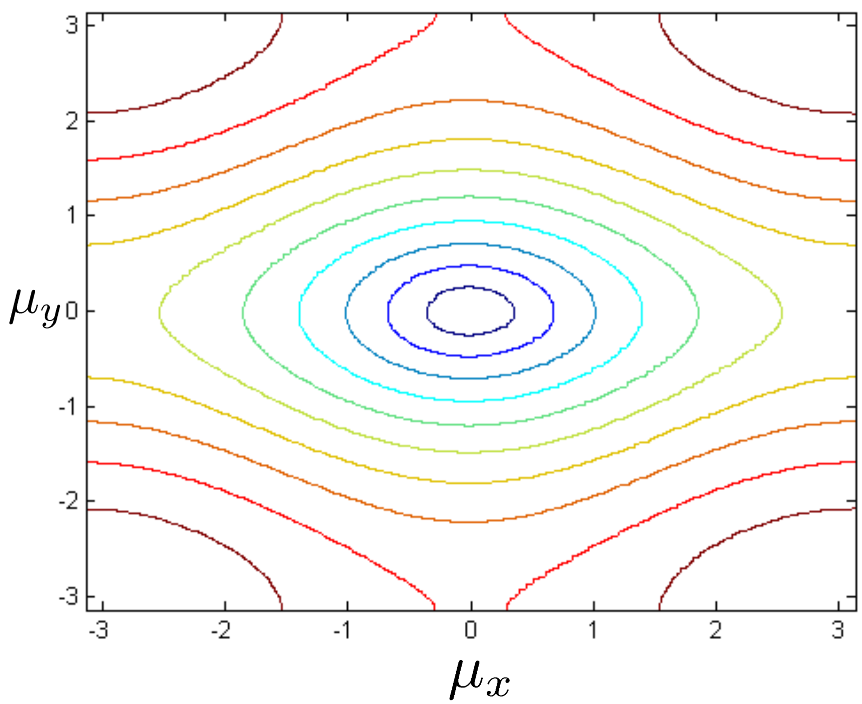
Anisotropic lattice
$k_x \neq k_y$

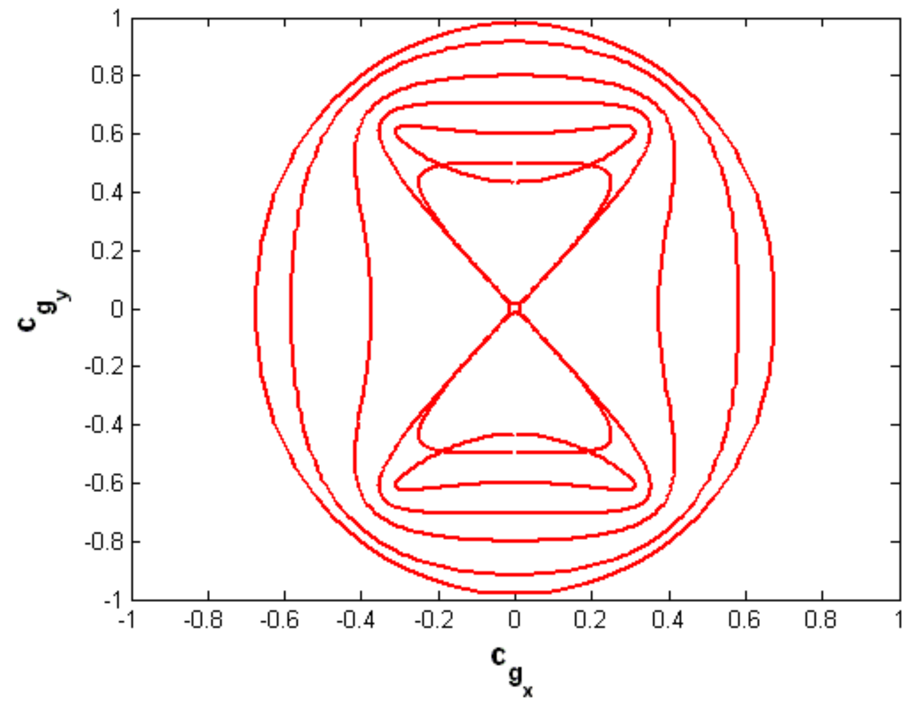
$$k_x = 1, k_y = 1.5$$





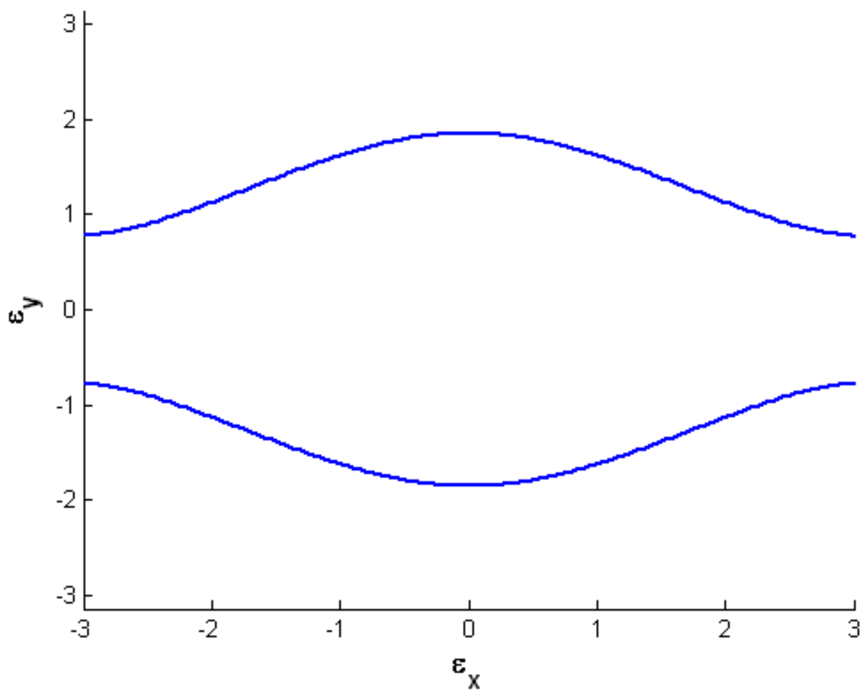
$$k_x = 1, k_y = 2$$

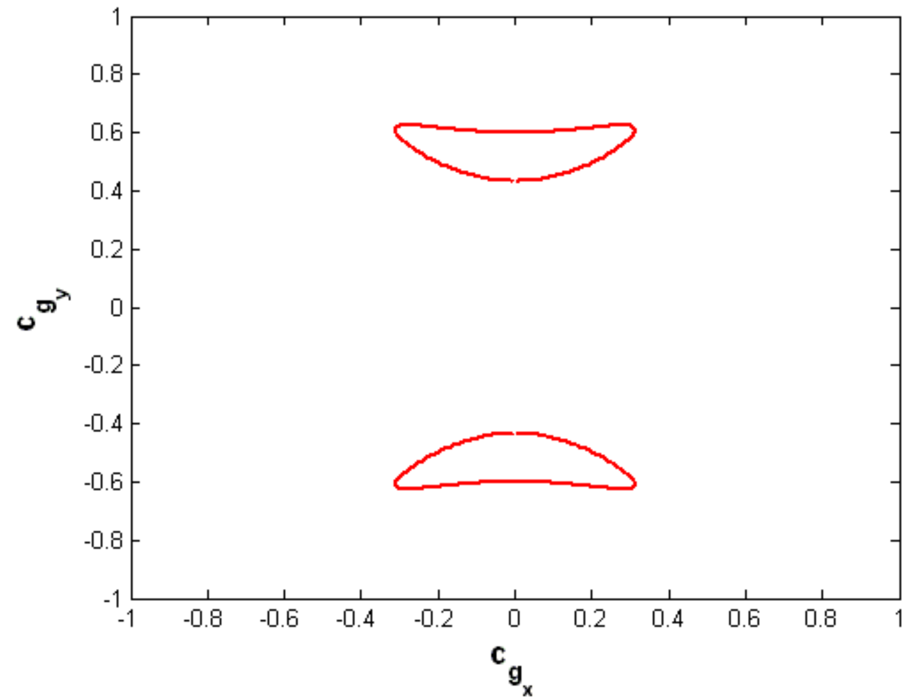

3
2
1
μ_y 0
-1
-2
-3
-3
-2
-1
0
1
2
3
μ_x


1
0.8
0.6
0.4
0.2
c_{g_y} 0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
c_{g_x}

- "Forbidden propagation" zone

$$k_x = 1, k_y = 2$$
$$\omega_0 = 1.6$$





Waves do not propagate along the x direction

Example: 2D spring-mass lattice

Harmonic response of 40*40 lattice:

$$k_x = 2, k_y = 1$$