# Decomposing Complete Hypergraphs

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$$\begin{array}{ccc} \{0,1,3\} & \{1,2,4\} & \{2,3,5\} \\ \{3,4,6\} & \{0,4,5\} & \{1,5,6\} \\ & \{0,2,6\} & \end{array}$$

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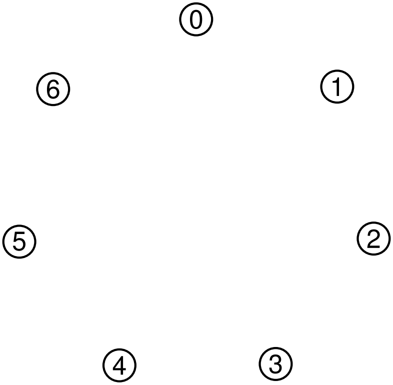
**Definition**

A *Steiner Triple System* of order $v$, denoted $STS(v)$, is a set $V$ of $v$ *points* with a collection $\mathscr{B}$ of 3-element subsets of $V$, called *blocks*, so that every 2-element subset of $V$ is contained in exactly one block.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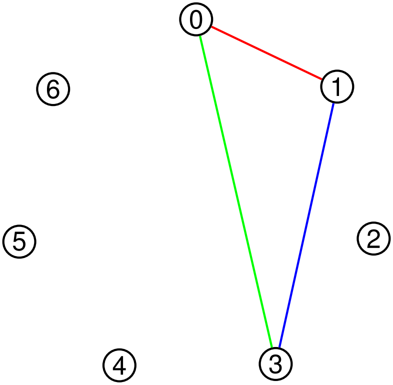
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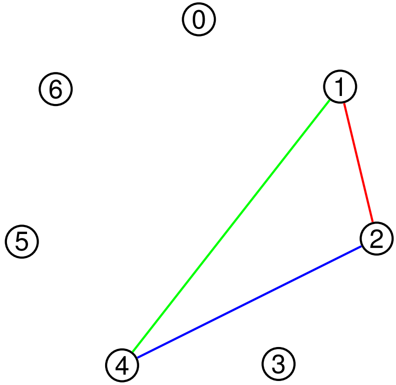
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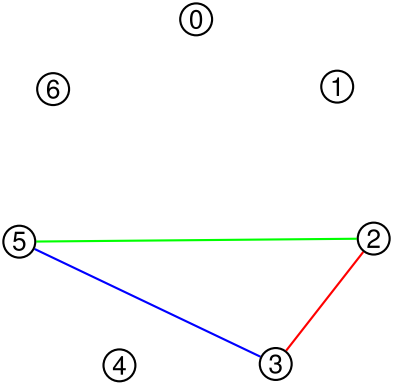
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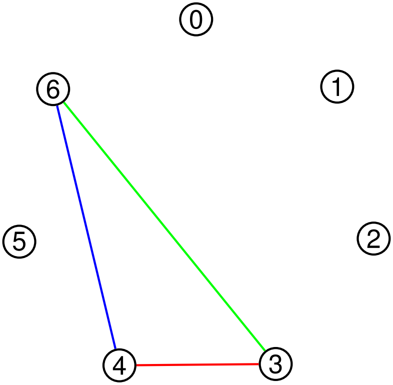
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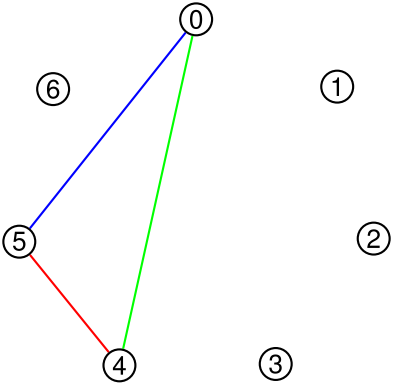
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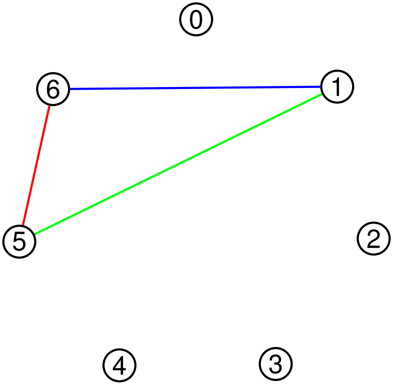
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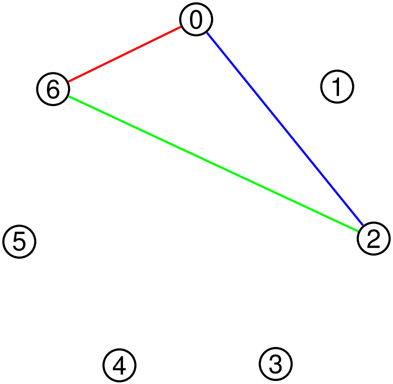
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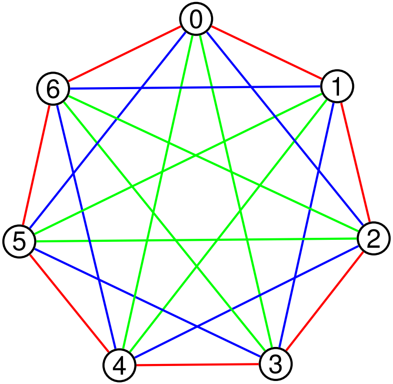



⇒ We can draw the complete graph $K_7$ as a number of copies of $K_3$.

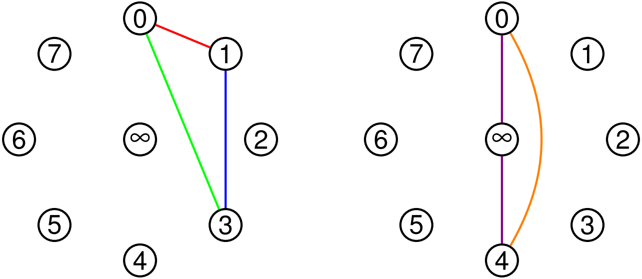
**Definition**

Let $G, H$ be graphs. A $G$-decomposition of $H$ is a collection of graphs isomorphic to $G$ whose edges partition $E(H)$.

**Lemma**

*An $STS(v)$ is equivalent to a $K_3$-decomposition of $K_v$.*

An $STS(9)$, expressed as a $K_3$-decomposition of $K_9$:




$\{0,1,3\}, \{1,2,4\}, \{2,3,5\}, \ldots$

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**Theorem (Kirkman, 1847)**

*An $STS(v)$ exists iff $v \equiv 1$ or $3 \pmod 6$.*

**Proof.**

($\Rightarrow$) Obvious necessary conditions above.
($\Leftarrow$) Generalise the constructions shown before. $\square$

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Given $k$ and $t$, for which $v$ does there exist an $S(t, k, v)$?

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We would like to form an equivalent ‘graph theoretic’ question.

## Definition

A *hypergraph* is a pair $(V, E)$, where $V$ is a set of *vertices* and $E$ is a set of nonempty subsets of $V$ called *hyperedges* (or *edges*).

A hypergraph is $t$*-uniform* if every edge has size $t$.

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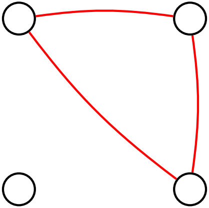

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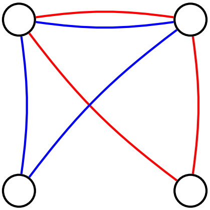

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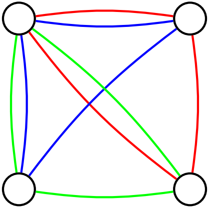

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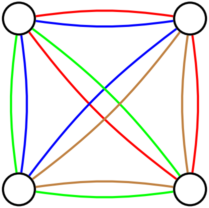

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In summary:
For $i \geq 0$, $\binom{k-i}{t-i}$ must divide $\binom{v-i}{t-i}$.

## Problem

Given a $t$-uniform hypergraph $H$, for which $v$ does there exist an $H$-decomposition of $K_v^{(t)}$?

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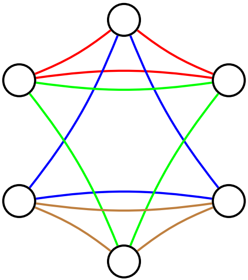
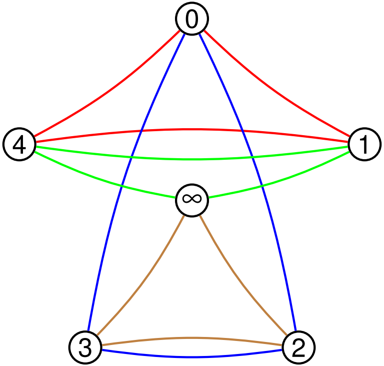
Example: for the hypergraph $H$ below, does there exist an $H$-decomposition of $K_6^{(3)}$?

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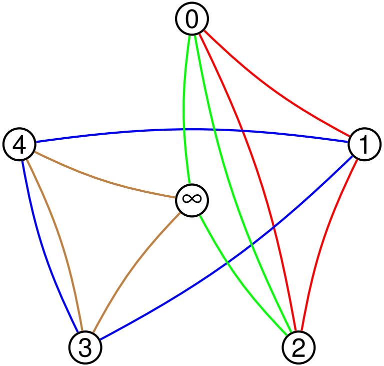
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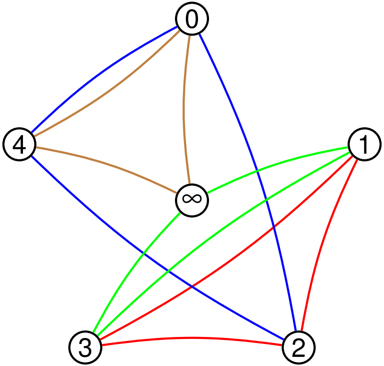

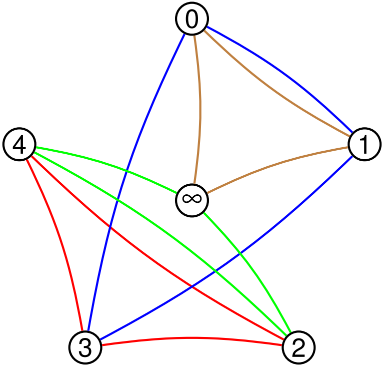
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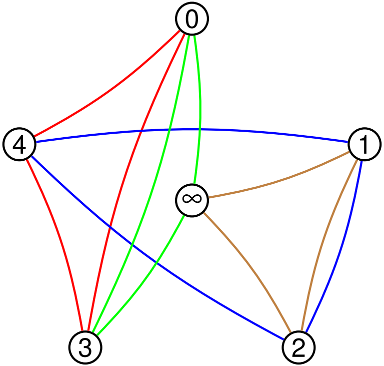
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The necessary divisibility conditions for $H$-decompositions of $K_v^{(t)}$ are:
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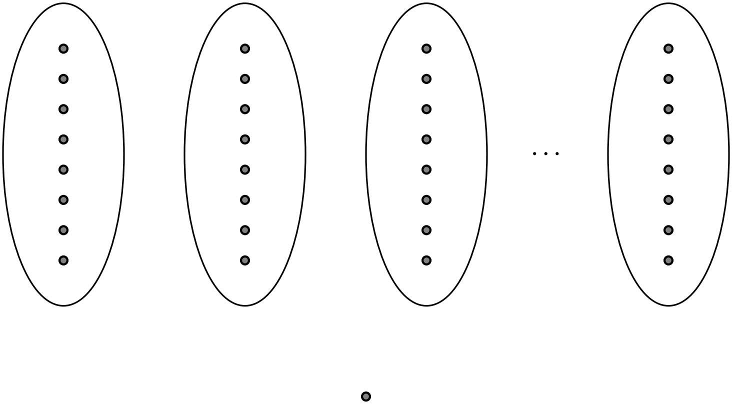
These conditions are equivalent to $v \equiv 1$, 2 or 6 (mod 8).

Does there always exist an $H$-design for these values of $v$?

If $v \equiv 1 \pmod 8$, then we need $n$ sets of 8 points, plus one extra point:



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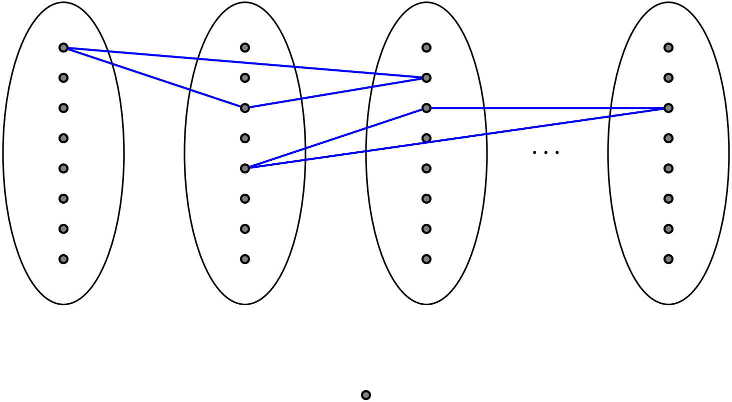

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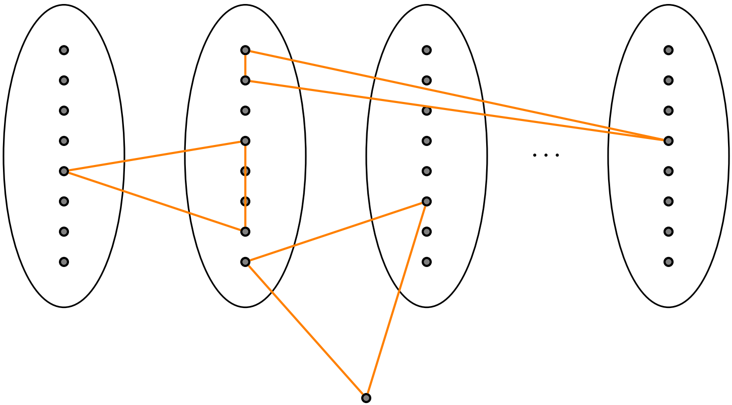

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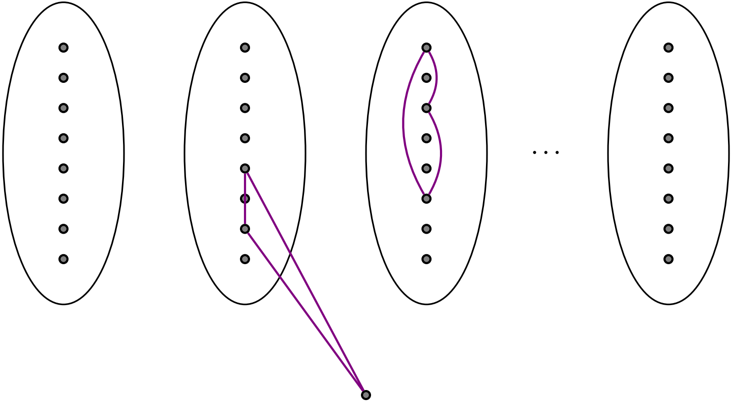

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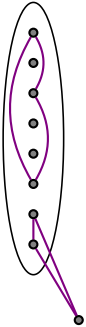

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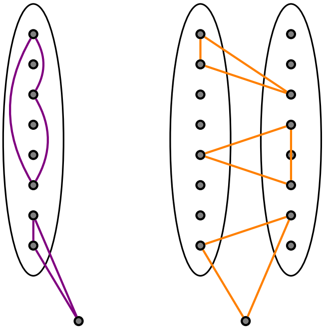

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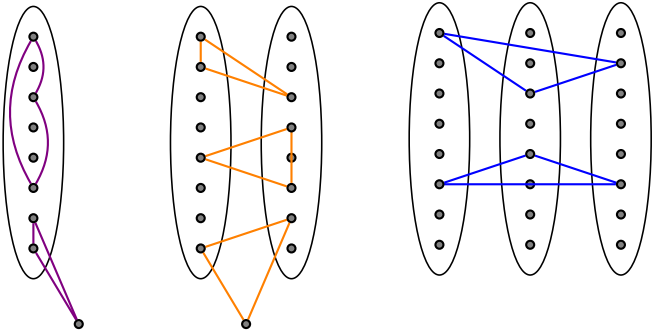

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*For the hypergraph H given before, there is an H-design of order v (H-decomposition of $K_v^{(3)}$) iff $v \equiv 1, 2,$ or $6 \pmod 8$.*

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**Proof.**

($\Rightarrow$) Necessary divisibility conditions.
($\Leftarrow$) Divide into three cases: 1, 2, 6 (mod 8).
Find $H$-decompositions of certain small hypergraphs.
Combine copies of these decompositions as necessary. $\square$