Decomposing Complete Hypergraphs

Jackson Gatenby

Supervisor: Barbara Maenhaut

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Problem (Steiner, 1853)

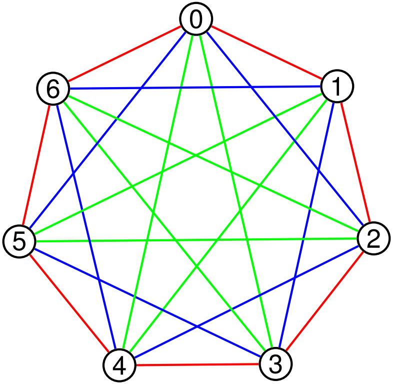
Given a set of v elements, take subsets of size 3 so that every pair of elements occurs in precisely one of the subsets.

Example: every pair of elements from $\{0,1,2,3,4,5,6\}$ occurs in exactly one of the following triples:

$$
\begin{array}{ccc}
\{0,1,3\} & \{1,2,4\} & \{2,3,5\} \\
\{3,4,6\} & \{0,4,5\} & \{1,5,6\} \\
 & \{0,2,6\} &
\end{array}
$$

Definition

A *Steiner Triple System* of order v, denoted $STS(v)$, is a set V of v *points* with a collection $\mathscr{B}$ of 3-element subsets of V, called *blocks*, so that every 2-element subset of V is contained in exactly one block.



$\Rightarrow$ We can draw the complete graph K_7 as a number of copies of K_3.

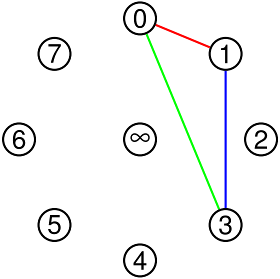
Definition

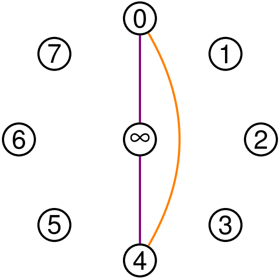
Let G, H be graphs. A G-decomposition of H is a collection of graphs isomorphic to G whose edges partition $E(H)$.

Lemma

An $STS(v)$ is equivalent to a K_3-decomposition of K_v.

An $STS(9)$, expressed as a K_3-decomposition of K_9:





$\{0,1,3\}$, $\{1,2,4\}$, $\{2,3,5\}$, $\ldots$

$\{0,4,\infty\}$, $\{1,5,\infty\}$, $\ldots$

We also have necessary conditions for the existence of a K_3-decomposition of K_v:

1. K_v has $\binom{v}{2}$ edges, which are partitioned into parts of size $|E(K_3)| = 3$, so 3 must divide $\binom{v}{2} = \frac{v(v-1)}{2}$.
2. Consider a vertex $x \in V(K_v)$. Whenever x appears in a copy of K_3, there are 2 edges incident with x. There are a total of $\deg_{K_v}(x) = v - 1$ edges incident with x, so 2 must divide $v - 1$.

Theorem (Kirkman, 1847)

An $STS(v)$ exists iff $v \equiv 1$ or $3 \pmod 6$.

Proof.

($\Rightarrow$) Obvious necessary conditions above.

($\Leftarrow$) Generalise the constructions shown before. $\square$

We can generalise “3-element blocks” to “k-element blocks”, and from “covering every pair of points” to “covering every t-set of points”:

Definition

A *Steiner system*, denoted $S(t, k, v)$, consists of a set of v points, and sets of k points (*blocks*), so that every set of t points occurs in precisely one block.

Problem

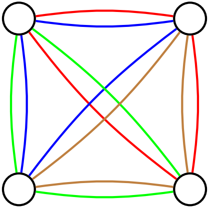
Given k and t, for which v does there exist an $S(t, k, v)$?

We would like to form an equivalent ‘graph theoretic’ question.

Definition

A *hypergraph* is a pair (V, E), where V is a set of *vertices* and E is a set of nonempty subsets of V called *hyperedges* (or *edges*).
A hypergraph is *t-uniform* if every edge has size t.
The *complete t-uniform hypergraph of order v*, denoted $K_v^{(t)}$, consists of v vertices and edges given by every t-set of vertices.

Example: $K_4^{(3)}$:



Lemma

An $S(t,k,v)$ is a $K_k^{(t)}$-decomposition of $K_v^{(t)}$.

The necessary divisibility conditions for the existence of an $S(t,k,v)$ are:

1. $|E(K_k^{(t)})| = \binom{k}{t}$ divides $|E(K_v^{(t)})| = \binom{v}{t}$
2. For $x \in V(K_v^{(t)})$: $\deg_{K_k^{(t)}}(x) = \binom{k-1}{t-1}$ divides $\deg_{K_v^{(t)}}(x) = \binom{v-1}{t-1}$
3. For x, y: $\deg_{K_k^{(t)}}(\{x,y\}) = \binom{k-2}{t-2}$ divides $\deg_{K_v^{(t)}}(\{x,y\}) = \binom{v-2}{t-2}$
4. $\ldots$

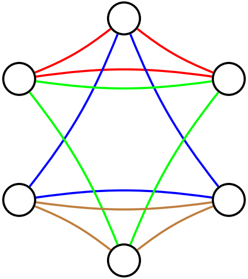
In summary:
For $i \geq 0$, $\binom{k-i}{t-i}$ must divide $\binom{v-i}{t-i}$.

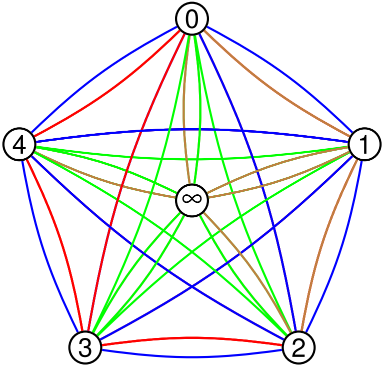
Problem

Given a t-uniform hypergraph H, for which v does there exist an H-decomposition of $K_v^{(t)}$?

These can also be referred to as *H-designs of order v*.

Example: for the hypergraph H below, does there exist an H-decomposition of $K_6^{(3)}$?




0
1
2
3
4
∞

The necessary divisibility conditions for H-decompositions of $K_v^{(t)}$ are:
For all $i \geq 0$, $\gcd\left(\{\deg_H(S) : S \subseteq V(H), |S| = i\}\right)$ must divide $\binom{v-i}{t-i}$.

Example

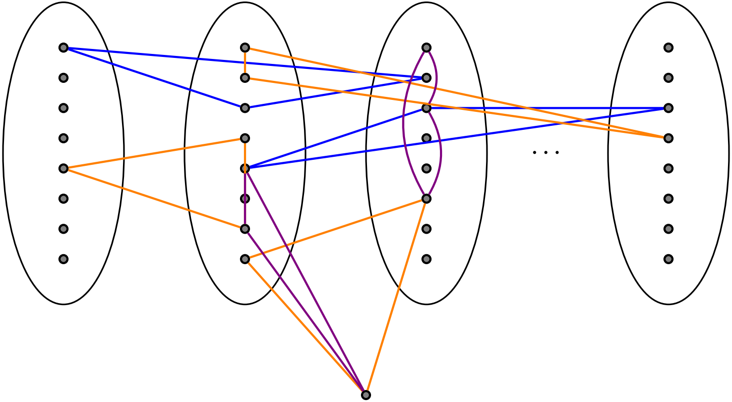
If H is as before, and there is a H-design of order v, then

1. $\deg_H(\emptyset) = |E(H)| = 4$ divides $\binom{v}{3}$
2. For all $x \in V(H)$, $\deg_H(x) = 2$. So 2 divides $\binom{v-1}{2}$
3. $\{\deg_H(\{x,y\}) : x, y \in V(H), x \neq y\} = \{1,2\}$. So $\gcd(\{1,2\}) = 1$ divides $\binom{v-2}{1} = v - 2$.

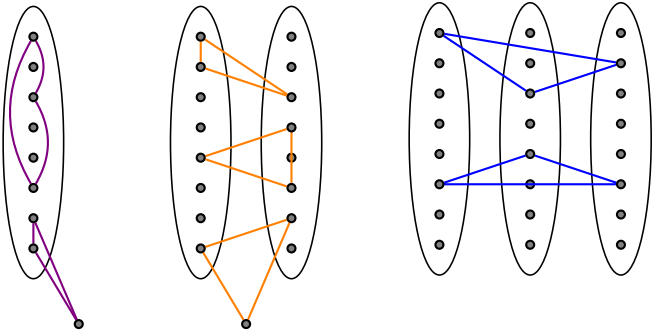
These conditions are equivalent to $v \equiv 1$, 2 or 6 (mod 8).

Does there always exist an H-design for these values of v?

If $v \equiv 1 \pmod 8$, then we need n sets of 8 points, plus one extra point:



If we can find H-decompositions of the following hypergraphs, then multiple copies of them will cover all three cases:



It can be shown that all of these exist, so H-designs of order v exist for all $v \equiv 1 \pmod 8$. Similarly for $v \equiv 2$ and 6, so

Theorem

For the hypergraph H given before, there is an H-design of order v (H-decomposition of $K_v^{(3)}$) iff $v \equiv 1, 2,$ or $6 \pmod 8$.

Proof.

($\Rightarrow$) Necessary divisibility conditions.
($\Leftarrow$) Divide into three cases: 1, 2, 6 (mod 8).
Find H-decompositions of certain small hypergraphs.
Combine copies of these decompositions as necessary. $\square$