MATH4301 Lecture Notes

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MATH4301 Advanced Algebra
The University of Queensland

Version 0.3.0

September 10, 2013

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0.1 Meta

It is the author's intention that these notes, and the LaTeX source code, be publicly released and made available for future reference, so this section records notes about these notes.

At time of writing, this document can be found at `http://jgat.github.io/math4301-notes/lectures.pdf`, and its source code can be found at `https://github.com/jgat/math4301-notes`.

The 2013 offering of MATH4301 Advanced Algebra was divided into two halves, the first half on Galois Theory, taught by Victor Scharaschkin, and the second half on Coxeter Groups, taught by Ole Warnaar. Each half of the course was accompanied by an assignment worth 20% of the grade, and an exam worth 30% of the grade.

Victor has provided typed notes to supplement the first half of the course, not included here. This document contains detailed lecture notes taken during the second half of the course. The notes correspond closely to what was written and said in lectures.

Each section within this document corresponds to notes taken in a single lecture.

Chapter 1

Introduction

1.1 Presentations

Let A be an alphabet, the free group $F(A)$ consists of all words over $A \cup A^{-1}$ in which the pairs aa^{-1} and $a^{-1}a$ are forbidden (i.e. $aa^{-1} = a^{-1}a = 1$). The group multiplication corresponds to concatenation of words and removal of forbidden pairs.

Example: if $A = \{a\}$, $F(A) = \{a^k \mid k \in \mathbb{Z}\} \cong (\mathbb{Z}, +)$. If $w_1 = a^4$, $w_2 = a^{-2}$, then $w_1 w_2 = aaaaa^{-1}a^{-1} = a^2$.

To make life more interesting we need relations. For example, $A = \{a, b\}$ with relation $b = 1$ gives $(\mathbb{Z}, +)$.

A *presentation* (of a group) $\langle A \mid R\rangle$ consists of a set A of *generators* and a set of relations R between the generators (and their inverses). Elements of the group are again words in A, but two words represent the same element in the group if they can be transformed into each other by the use of R. More formally, $G \cong F(A)/N$ where N is the normal subgroup generated by R.

Example: $\langle a \mid a^k = 1\rangle \cong \mathbb{Z}/k\mathbb{Z} = \mathbb{Z}_k$ (for $k = 1, 2, \dots$). Formally, $\langle a \mid a^k = 1\rangle \cong F(a)/\langle a^k\rangle$.

Example: $\langle a, b \mid a^2 = b^2 = (ab)^2 = 1\rangle$ contains elements $1, a, b, ab, ba, \dots$, however note that $ba = (ab)^{-1} = ab$. Simply guessing which words are distinct is not going to work. The multiplication table of the group is (Exercise: Show that this is all of the elements in the group):

G	1	a	b	ab
1	1	a	b	ab
a	a	1	ab	b
b	b	ab	1	a
ab	ab	b	a	1

Note that $bab = a^{-1}abab = a^{-1} = a$. This is the Klein 4-group $\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Geometrically it is the symmetry group of the (non-square) rectangle and a rhombus, where a and b are reflections and ab is rotation by π.

The *word problem* is to decide if two *distinct* words in the generators represent the same/different elements in the group. In 1955, Novikov showed that the word problem is undecidable. This is not the case for Coxeter groups.

1.2 Coxeter Groups

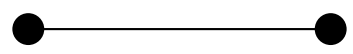
References:

- Bjorner & Brenti: Combinatorics of Coxeter groups (Springer GTM231, '05)
- Bourbaki: Lie groups & Lie algebras (Chap 4-6)
- Cohen: Coxeter groups
- Humphreys: Reflection Groups and Coxeter Groups
- Davis: The Geometry and Topolology of Coxeter Groups

Let M be an $r \times r$ symmetric matrix with entries m_{ij} in $\{1, 2, 3, \dots\} \cup \{\infty\}$ with $m_{ii} = 1$ and $m_{ij} = m_{ji} > 1$ for $i \neq j$. Such a matrix is called a *Coxeter matrix*. For example,

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

Coxeter matrices are often represented as a graph with r labelled vertices $(1, 2, \dots, r)$, and if $m_{ij} \geq 3$, an edge between i and j with a labelling of the edge by m_{ij}. It is standard to drop edge labels which are 3. Hence the above example can be expressed as



Given a Coxeter matrix M (or graph), a *Coxeter system* (W, S) of type M is a set $S = \{s_1, \dots, s_r\}$ and a group

$$W = \langle S \mid (s_i s_j)^{m_{ij}} = 1, \quad 1 \leq i, j \leq r, m_{ij} \neq \infty \rangle.$$

(That is, whenever $m_{ij} \neq \infty$, impose a relation $(s_i s_j)^{m_{ij}} = 1$). The group W is called a *Coxeter group* (of type M). The number r is known as the *rank* of W. Note that $s_i^2 = 1$ for all $1 \leq i \leq r$.

Example: For rank 1, there is only one Coxeter group, $M = (1)$, with the trivial graph:

and the corresponding Coxeter group $W = \langle s \mid s^2 = 1 \rangle \cong \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

For rank 2, we have first,

$$M = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

with corresponding graph

and corresponding group

$$W = \langle s, t \mid s^2 = t^2 = (st)^2 = 1 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

(Note that $(s_i s_j)^{m_{ij}} = 1$ implies that $(s_j s_i)^{m_{ij}} = 1$. Why: $(s_j s_i)^{m_{ij}} = (s_j s_i)^{m_{ij}} s_j^2 = s_j (s_i s_j)^{m_{ij}} s_j = s_j^2 = 1$) We will later show that if a Coxeter system has a disconnected graph, then the Coxeter group will be the direct product of the corresponding groups for each component; hence we will focus on connected graphs. We also have

$$M = \begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix}, \quad m \geq 3,$$

and

$$W = \langle s, t \mid s^2 = t^2 = (st)^m = 1 \rangle.$$

This is known as the *dihedral* group of order $2m$ (D_m / D_{2m} / $I_2(m)$). The dihedral group is the symmetry group of the regular m-gon. For example, $I_2(3)$ is the symmetry group of the triangle, where $s, t, sts = tst$ are reflections and st, ts are rotations. $I_2(4)$ has reflections $s, t, sts = s(ts)^2 = (st)^2 s, tst = t(st)^2 = (ts)^2 t$.

Note that a word of odd length corresponds to a reflection, and a word of even length corresponds to a rotation; also note that the relation $(st)^m = 1$ embodies the "rotate m times to get the identity" property of the m-gon.

1.3 Dihedral Groups

Recall that a Coxeter group is of the form

$$W = \langle S \mid (s_i s_j)^{m_{ij}} = 1, 1 \leq i \leq j \leq r, m_{ij} \neq \infty \rangle,$$

with the associated matrix $M = (m_{ij})$ where $m_{ii} = 1$ and $M^T = M$. A special case is, for $m > 2$,

$$I_2(m) = \langle s, t \mid s^2 = t^2 = (st)^m = 1 \rangle,$$

a Coxeter group of rank 2. We saw that $I_2(3)$ is the group of symmetries of the equilateral triangle, and $I_2(4)$ is the group of symmetries of the square. We claim that $I_2(m)$ is a group of order $2m$ consisting of m reflections and m rotations of the regular m-gon.

First, for a vector $\alpha \in \mathbb{R}^n$, let H_α denote the hyperplane with normal α, and denote reflection in H_α by r_α. Now, for any vector λ,

$$r_\alpha(\lambda) = \lambda - \frac{2(\alpha, \lambda)\alpha}{(\alpha, \alpha)}, \tag{1.1}$$

where (a, b) denotes the inner product (vector dot product). Note that $r_\alpha(\lambda) = \lambda$ for every $\lambda \in H_\alpha$, and $r_\alpha(\alpha) = \alpha - \frac{2(\alpha,\alpha)\alpha}{(\alpha,\alpha)} = -\alpha$ as expected. Since we have verified this for a hyperplane of codimension 1 and for a vector normal to the hyperplane, the result is true for all vectors (by Linear Algebra).

Proof. Let s and t be reflections, where the axes of symmetry have an angle of $\theta = \frac{\pi}{m}$, i.e. $s := r_{(1,0)}$ and $t := r_{(\cos\theta, -\sin\theta)}$. Then, $s(1, 0) = (-1, 0)$ and $s(0, 1) = (0, 1)$, so

$$\hat{s} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

is a matrix representation of s, and

$$\hat{t} = \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

is a matrix representation of t (Exercise).

If we can show that st is a rotation over $\frac{2\pi}{m}$, then $(st)^k$ will be a rotation over $\frac{2\pi k}{m}$, which will give m distinct rotations. Now,

$$\hat{s}\hat{t} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix},$$

which is a rotation matrix for rotation over 2θ (Exercise). Then $(\hat{s}\hat{t})^k$ is a rotation by $2k\theta$ (easy to see geometrically, or show inductively that $(\hat{s}\hat{t})^k$ is a rotation matrix).

It remains to show that there are m distinct words of odd length, and they are all reflections. WLOG, we can say that all words of odd length are of the form $t(st)^{k-1}$ for some $k = 1, 2, \ldots, m$, hence there are m distinct words of odd length. Now,

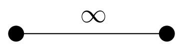
$$\hat{t}(\hat{s}\hat{t})^k = \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} \cos 2\theta(k-1) & -\sin 2\theta(k-1) \\ \sin 2\theta(k-1) & \cos 2\theta(k-1) \end{pmatrix} = \begin{pmatrix} -\cos(2k\theta) & \sin 2k\theta \\ \sin 2k\theta & \cos 2k\theta \end{pmatrix}$$

(Exercise), so $t(st)^{k-1} := r_{(\cos k\theta, -\sin k\theta)}$. Hence the m words of odd length are all reflections. $\square$

Finally, we consider the remaining Coxeter group of rank 2, given by

$$M = \begin{pmatrix} 1 & \infty \\ \infty & 1 \end{pmatrix},$$

with corresponding graph



and corresponding group

$$I_2(\infty) = \langle s, t \mid s^2 = t^2 = 1 \rangle.$$

Elements of this group, known as the ∞-dihedral group are the words of the form: $1, s, t, st, ts, sts, tst, \ldots$. Again this group has a geometric interpretation in terms of reflections (etc.). Before we can describe this, we need some more notation.

Let $V = \mathbb{R}^n$ and for $\alpha \in V$ let H_α denote the hyperplane perpendicular to α. Algebraically, $H_\alpha = \{\lambda \in V : (\lambda, \alpha) = 0\}$. As we have seen, the reflextion r_α in H_α is given by its action on $\lambda \in V$ as

$$r_\alpha(\lambda) = \lambda - \frac{2(\alpha, \lambda)}{(\alpha, \alpha)}\alpha = \lambda - \frac{2(\lambda, \alpha)}{||\alpha||^2}\alpha = \lambda - (\lambda, \alpha^V)\alpha,$$

where $\alpha^V = \frac{2\alpha}{||\alpha||^2}$ (a covector).

Appendix A

Version History

Below we describe briefly the version history of this document, based on versions which have been published at `http://jgat.github.io/math4301-notes/lectures.pdf`. Version numbers will roughly follow the format v0.n.m, where n indicates the number of lectures which notes have been written for, and m indicates the number of minor revisions. Once notes have been taken for all lectures, the version number will increment to v1.0.

v0.3.0: Initial publication with lectures 1 to 3, §1.1-1.3.