MATH4301 Lecture Notes

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MATH4301 Advanced Algebra
The University of Queensland

Version 0.7.0

September 20, 2013

Contents

0.1 Meta

It is the author's intention that these notes, and the LaTeX source code, be publicly released and made available for future reference, so this section records notes about these notes.

At time of writing, this document can be found at `http://jgat.github.io/math4301-notes/lectures.pdf`, and its source code can be found at `https://github.com/jgat/math4301-notes`.

The 2013 offering of MATH4301 Advanced Algebra was divided into two halves, the first half on Galois Theory, taught by Victor Scharaschkin, and the second half on Coxeter Groups, taught by Ole Warnaar. Each half of the course was accompanied by an assignment worth 20% of the grade, and an exam worth 30% of the grade.

Victor has provided typed notes to supplement the first half of the course, not included here. This document contains detailed lecture notes taken during the second half of the course. The notes correspond closely to what was written and said in lectures.

Each section within this document corresponds to notes taken in a single lecture.

Chapter 1

Introduction

1.1 Presentations

Let A be an alphabet, the free group $F(A)$ consists of all words over $A \cup A^{-1}$ in which the pairs aa^{-1} and $a^{-1}a$ are forbidden (i.e. $aa^{-1} = a^{-1}a = 1$). The group multiplication corresponds to concatenation of words and removal of forbidden pairs.

Example: if $A = \{a\}$, $F(A) = \{a^k \mid k \in \mathbb{Z}\} \cong (\mathbb{Z}, +)$. If $w_1 = a^4$, $w_2 = a^{-2}$, then $w_1 w_2 = aaaaa^{-1}a^{-1} = a^2$.

To make life more interesting we need relations. For example, $A = \{a, b\}$ with relation $b = 1$ gives $(\mathbb{Z}, +)$.

A *presentation* (of a group) $\langle A \mid R\rangle$ consists of a set A of *generators* and a set of relations R between the generators (and their inverses). Elements of the group are again words in A, but two words represent the same element in the group if they can be transformed into each other by the use of R. More formally, $G \cong F(A)/N$ where N is the normal subgroup generated by R.

Example: $\langle a \mid a^k = 1\rangle \cong \mathbb{Z}/k\mathbb{Z} = \mathbb{Z}_k$ (for $k = 1, 2, \dots$). Formally, $\langle a \mid a^k = 1\rangle \cong F(a)/\langle a^k\rangle$.

Example: $\langle a, b \mid a^2 = b^2 = (ab)^2 = 1\rangle$ contains elements $1, a, b, ab, ba, \dots$, however note that $ba = (ab)^{-1} = ab$. Simply guessing which words are distinct is not going to work. The multiplication table of the group is (Exercise: Show that this is all of the elements in the group):

G	1	a	b	ab
1	1	a	b	ab
a	a	1	ab	b
b	b	ab	1	a
ab	ab	b	a	1

Note that $bab = a^{-1}abab = a^{-1} = a$. This is the Klein 4-group $\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Geometrically it is the symmetry group of the (non-square) rectangle and a rhombus, where a and b are reflections and ab is rotation by π.

The *word problem* is to decide if two *distinct* words in the generators represent the same/different elements in the group. In 1955, Novikov showed that the word problem is undecidable. This is not the case for Coxeter groups.

1.2 Coxeter Groups

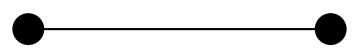
References:

- Bjorner & Brenti: Combinatorics of Coxeter groups (Springer GTM231, '05)
- Bourbaki: Lie groups & Lie algebras (Chap 4-6)
- Cohen: Coxeter groups
- Humphreys: Reflection Groups and Coxeter Groups
- Davis: The Geometry and Topolology of Coxeter Groups

Let M be an $r \times r$ symmetric matrix with entries m_{ij} in $\{1, 2, 3, \dots\} \cup \{\infty\}$ with $m_{ii} = 1$ and $m_{ij} = m_{ji} > 1$ for $i \neq j$. Such a matrix is called a *Coxeter matrix.* For example,

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

Coxeter matrices are often represented as a graph with r labelled vertices $(1, 2, \dots, r)$, and if $m_{ij} \geq 3$, an edge between i and j with a labelling of the edge by m_{ij}. It is standard to drop edge labels which are 3. Hence the above example can be expressed as

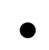


Given a Coxeter matrix M (or graph), a *Coxeter system* (W, S) of type M is a set $S = \{s_1, \dots, s_r\}$ and a group

$$W = \langle S \mid (s_i s_j)^{m_{ij}} = 1, \quad 1 \leq i, j \leq r, m_{ij} \neq \infty \rangle.$$

(That is, whenever $m_{ij} \neq \infty$, impose a relation $(s_i s_j)^{m_{ij}} = 1$). The group W is called a *Coxeter group* (of type M). The number r is known as the *rank* of W. Note that $s_i^2 = 1$ for all $1 \leq i \leq r$.

Example: For rank 1, there is only one Coxeter group, $M = (1)$, with the trivial graph:



and the corresponding Coxeter group $W = \langle s \mid s^2 = 1 \rangle \cong \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.
For rank 2, we have first,

$$M = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

with corresponding graph

and corresponding group

$$W = \langle s, t \mid s^2 = t^2 = (st)^2 = 1 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

(Note that $(s_i s_j)^{m_{ij}} = 1$ implies that $(s_j s_i)^{m_{ij}} = 1$. Why: $(s_j s_i)^{m_{ij}} = (s_j s_i)^{m_{ij}} s_j^2 = s_j (s_i s_j)^{m_{ij}} s_j = s_j^2 = 1$) We will later show that if a Coxeter system has a disconnected graph, then the Coxeter group will be the direct product of the corresponding groups for each component; hence we will focus on connected graphs. We also have

$$M = \begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix}, \quad m \geq 3,$$

and

$$W = \langle s, t \mid s^2 = t^2 = (st)^m = 1 \rangle.$$

This is known as the *dihedral* group of order $2m$ (D_m / D_{2m} / $I_2(m)$). The dihedral group is the symmetry group of the regular m-gon. For example, $I_2(3)$ is the symmetry group of the triangle, where $s, t, sts = tst$ are reflections and st, ts are rotations. $I_2(4)$ has reflections $s, t, sts = s(ts)^2 = (st)^2 s, tst = t(st)^2 = (ts)^2 t$.

Note that a word of odd length corresponds to a reflection, and a word of even length corresponds to a rotation; also note that the relation $(st)^m = 1$ embodies the "rotate m times to get the identity" property of the m-gon.

1.3 Dihedral Groups

Recall that a Coxeter group is of the form

$$W = \langle S \mid (s_i s_j)^{m_{ij}} = 1, 1 \leq i \leq j \leq r, m_{ij} \neq \infty \rangle,$$

with the associated matrix $M = (m_{ij})$ where $m_{ii} = 1$ and $M^T = M$. A special case is, for $m > 2$,

$$I_2(m) = \langle s, t \mid s^2 = t^2 = (st)^m = 1 \rangle,$$

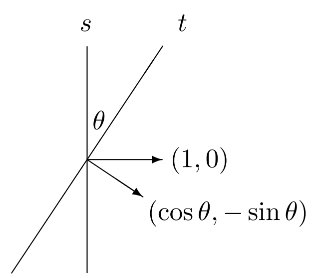
a Coxeter group of rank 2. We saw that $I_2(3)$ is the group of symmetries of the equilateral triangle, and $I_2(4)$ is the group of symmetries of the square. We claim that $I_2(m)$ is a group of order $2m$ consisting of m reflections and m rotations of the regular m-gon.

First, for a vector $\alpha \in \mathbb{R}^n$, let H_α denote the hyperplane with normal α, and denote reflection in H_α by r_α. Now, for any vector λ,

$$r_\alpha(\lambda) = \lambda - \frac{2(\alpha, \lambda)\alpha}{(\alpha, \alpha)}, \tag{1.1}$$

where (a, b) denotes the inner product (vector dot product). Note that $r_\alpha(\lambda) = \lambda$ for every $\lambda \in H_\alpha$, and $r_\alpha(\alpha) = \alpha - \frac{2(\alpha,\alpha)\alpha}{(\alpha,\alpha)} = -\alpha$ as expected. Since we have verified this for a hyperplane of codimension 1 and for a vector normal to the hyperplane, the result is true for all vectors (by Linear Algebra).

Proof. Let s and t be reflections, where the axes of symmetry have an angle of $\theta = \frac{\pi}{m}$, i.e. $s := r_{(1,0)}$ and $t := r_{(\cos\theta, -\sin\theta)}$:



Then, $s(1,0) = (-1,0)$ and $s(0,1) = (0,1)$, so

$$\hat{s} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

is a matrix representation of s, and

$$\hat{t} = \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

is a matrix representation of t (Exercise).

If we can show that st is a rotation over $\frac{2\pi}{m}$, then $(st)^k$ will be a rotation over $\frac{2\pi k}{m}$, which will give m distinct rotations. Now,

$$\hat{s}\hat{t} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix},$$

which is a rotation matrix for rotation over 2θ (Exercise). Then $(\hat{s}\hat{t})^k$ is a rotation by $2k\theta$ (easy to see geometrically, or show inductively that $(\hat{s}\hat{t})^k$ is a rotation matrix).

It remains to show that there are m distinct words of odd length, and they are all reflections. WLOG, we can say that all words of odd length are of the form $t(st)^{k-1}$ for some $k = 1, 2, \ldots, m$, hence there are m distinct words of odd length. Now,

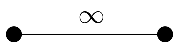
$$\hat{t}(\hat{s}\hat{t})^k = \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} \cos 2\theta(k-1) & -\sin 2\theta(k-1) \\ \sin 2\theta(k-1) & \cos 2\theta(k-1) \end{pmatrix} = \begin{pmatrix} -\cos(2k\theta) & \sin 2k\theta \\ \sin 2k\theta & \cos 2k\theta \end{pmatrix}$$

(Exercise), so $t(st)^{k-1} := r_{(\cos k\theta, -\sin k\theta)}$. Hence the m words of odd length are all reflections. $\square$

Finally, we consider the remaining Coxeter group of rank 2, given by

$$M = \begin{pmatrix} 1 & \infty \\ \infty & 1 \end{pmatrix},$$

with corresponding graph



and corresponding group

$$I_2(\infty) = \langle s, t \mid s^2 = t^2 = 1 \rangle.$$

Elements of this group, known as the ∞-dihedral group are the words of the form: $1, s, t, st, ts, sts, tst, \ldots$. Again this group has a geometric interpretation in terms of reflections (etc.). Before we can describe this, we need some more notation.

Let $V = \mathbb{R}^n$ and for $\alpha \in V$ let H_α denote the hyperplane perpendicular to α. Algebraically, $H_\alpha = \{\lambda \in V : (\lambda, \alpha) = 0\}$. As we have seen, the reflextion r_α in H_α is given by its action on $\lambda \in V$ as

$$r_\alpha(\lambda) = \lambda - \frac{2(\alpha,\lambda)}{(\alpha,\alpha)}\alpha = \lambda - \frac{2(\lambda,\alpha)}{||\alpha||^2}\alpha = \lambda - (\lambda, \alpha^V)\alpha,$$

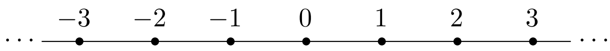
where $\alpha^V = \frac{2\alpha}{||\alpha||^2}$.

1.4 ...?

I was absent from this lecture (Thursday 12 September) (because I slept in) :(. Anyone have notes I can copy?

1.5 Lecture 5

Consider the integer number line,



Let $s = r_0$, $t = r_1$, in the group $W = \langle s, t \mid s^2 = t^2 = 1\rangle$. From last time, we had the following claims:

Claim 1: $r_k = s(st)^k$, $t_{2k} = (ts)^k$, $k \in \mathbb{Z}$.

Claim 2: $\tilde{A}_1 = A_1 \ltimes T$, where $A_1 = \{1, r_0\}$, $T = \{t_{2k} : k \in \mathbb{Z}\}$.

Proof. Recall that
$$r_{a,k}(\lambda) = \lambda - \{(\lambda, \alpha) - k\}\alpha^V,$$
if $\alpha = 1$, then $\alpha^V = 2$ (since the dot product must be 2), so
$$r_k = \lambda - 2\{\lambda - k\} = 2k - \lambda.$$
Then,
$$ts(\lambda) = r_1 r_0(\lambda) = r_1(-\lambda) = 2 + \lambda = t_2(\lambda),$$
thus $ts = t_2$, so $(ts)^k = t_{2k}$. Furthermore,
$$s(st)^k(\lambda) = r_0 t_{-2k}(\lambda) = r_0(\lambda - 2k) = -\lambda + 2k = r_k(\lambda),$$
so $s(st)^k = r_k$ $\square$

Note that
$$r_{\alpha,\kappa}(\lambda + \mu) \neq r_{\alpha,\kappa}(\lambda) + r_{\alpha,\kappa}(\mu),$$
instead,
$$\begin{aligned} r_{\alpha,\kappa}(\lambda + \mu) &= \lambda + \mu - \{(\lambda + \mu, \alpha)\alpha^V - \kappa\} \\ &= \lambda + \mu - \{(\lambda, \alpha)\alpha^V + (\mu, \alpha)\alpha^V - \kappa\} \\ &= r_{\alpha,\kappa}(\lambda) + r_{\alpha,\kappa}(\mu) - \kappa\alpha^V \end{aligned}$$
Now,
$$\begin{aligned} r_{\alpha,\kappa} t_\lambda r_{\alpha,\kappa}(\mu) &= r_{\alpha,\kappa}(\lambda + r_{\alpha,\kappa}(\mu)) \\ &= r_{\alpha,\kappa}(\lambda) + \mu - \kappa\alpha^V \\ &= r_{\alpha,0}(\lambda) + \mu, \end{aligned}$$
so $r_{\alpha,\kappa} t_\lambda r_{\alpha,\kappa} = t_{r_\alpha(\lambda)}$.

1.5.1 Semi-direct products

Recall that, for a group G, if $K \leq G$, $N \triangleleft G$, $K \cap N = \{1\}$, and $G = NK$, then we say $G = K \ltimes N$ or $G = N \rtimes K$. Now, consider claim 2:

$$\tilde{A}_1 = \{t_{2k}, r_k \mid k \in \mathbb{Z}\},$$

and consider the subgroups

$$N = T = \{t_{2k} : k \in \mathbb{Z}\}, K = \{1, r_0\} = A_1.$$

Note that $r_k = r_0 t_{-2k}$, so $\tilde{A}_1 = A_1 T$. Thus, claim 2 holds.

1.5.2 Symmetric Groups

Before moving to the general theory, we will discuss one more important example of a finite reflection group (Coxeter group), the symmetric group S_n, also denoted A_{n-1} (not to be confused with the alternating group!) Let us define S_n to be

$$S_n = \langle s_1, \ldots, s_{n-1} \mid s_i^2 = 1, (s_i s_{i+1})^3 = 1, (s_i s_j)^2 = 1, |i - j| > 1\rangle.$$

Clearly the associated Coxeter matrix is

$$M = \begin{pmatrix} 1 & 3 & 2 & \cdots & 2 \\ 3 & 1 & 3 & \cdots & 2 \\ 2 & 3 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 3 \\ 2 & 2 & \cdots & 3 & 1 \end{pmatrix}$$

with Coxeter graph

The most common description of S_n is as the group of permutations on n letters with s_i acting as "adjacent" transpositions, interchanging the letters in positions i and $i + 1$ (i.e. the cycle $(i \;\; i+1)$). For example, $s_2(2, 3, 6, 4, 5, 1) = (2, 6, 3, 4, 5, 1)$. The relation $(s_i s_j)^2 = 1$ for $|i - j| > 1$ may be recast as the commutation relation $s_i s_j = s_j s_i$. Finally, $(s_i s_{i+1})^3 = 1$ can be restated as $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, known as Artin's braid relation.

Of course, S_n can also be interpreted as a reflection group. Let $V = \mathbb{R}^n$ and s_i the reflection in the hyperplane $H_{\varepsilon_i - \varepsilon_{i+1}} = H_{\alpha_i}$, where ε_i is the i^{th} basis vector in the standard basis. Then, it holds that

$$s_i(\varepsilon_k) = \begin{cases} \varepsilon_k & k \neq i, i+1 \\ \varepsilon_{i+1} & k = i \\ \varepsilon_i & k = i+1 \end{cases}$$

so that indeed $s_i^2 = 1$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, and $s_i s_j = s_j s_i$ for $|i - j| > 1$. (Exercise: show this).

Note that $\varepsilon_1 + \cdots + \varepsilon_n$ is fixed by the action of $A_{n-1}(= S_n)$, and no other vector linearly independent to this is fixed, hence A_{n-1} only acts on an $(n-1)$-dimensional subspace of $\mathbb{R}^n$, namely $\{\lambda \in \mathbb{R}^n \mid (\lambda, \varepsilon_1 + \cdots + \varepsilon_n) = 0\}$. For example in the case of $S_3 = A_2$,

Now, A_{n-1} is the symmetry group of the $(n-1)$-simplex, where for instance the 2-simplex is the triangle, the 3-simplex is the tetrahedron, and so on.

Chapter 2

Finite Reflection Groups

2.1 The General Theory of Finite Reflection Groups

Let V be an Euclidean space, that is, a vector space over $\mathbb{R}$ with a positive definite symmetric bilinear form $(\cdot,\cdot) : V \times V \to \mathbb{R}$, for example the dot product $(a,b) = a \cdot b$ in $\mathbb{R}^n$. (Here it suffices to think of $\mathbb{R}^n$ when using Euclidean spaces).

A *reflection* in V is a linear map s_α $(\alpha \in V)$ such that

$$s_\alpha(\lambda) = \lambda - (\lambda, \alpha^V)\alpha,$$

where $\alpha^V := \frac{2\alpha}{(\alpha,\alpha)}$ (cf. Equation (1.1)). Note that s_α is an *involution* as well as an *orthogonal transformation*:

Proof.

$$\begin{aligned} s_\alpha^2(\lambda) &= s_\alpha(\lambda - (\lambda, \alpha^V)\alpha) \\ &= s_\alpha(\lambda) - (\lambda, \alpha^V)s_\alpha(\alpha) \\ &= \lambda - (\lambda, \alpha^V)\alpha + (\lambda, \alpha^V)\alpha \\ &= \lambda, \end{aligned}$$

so s_α is an involution. Also,

$$\begin{aligned} (s_\alpha(\lambda), s_\alpha(\mu)) &= (\lambda - (\lambda, \alpha^V)\alpha, \mu - (\mu, \alpha^V)\alpha) \\ &= (\lambda,\mu) - (\mu,\alpha^V)(\lambda,\alpha) - (\lambda,\alpha^V)(\mu,\alpha) + (\lambda,\alpha^V)(\mu,\alpha^V)(\alpha,\alpha) \\ &= (\lambda,\mu) - 2(\mu,\alpha)(\lambda,\alpha)\frac{\alpha}{(\alpha,\alpha)} + 4(\lambda,\alpha)(\mu,\alpha)\frac{(\alpha,\alpha)}{(\alpha,\alpha)^2} \\ &= (\lambda,\mu), \end{aligned}$$

so s_α is an orthogonal transformation. (Exercise: convince yourself of these results.) $\square$

A *finite reflection group* is a finite subgroup of the group of orthogonal transformations on V generated by reflections. As we shall see (I hope), all finite reflection groups are Coxeter groups. The converse also holds, however we will not show this.

Lemma 1. *Let $O(V)$ be the group of orthogonal transformations on V, and $W < O(V)$ be a finite reflection group. If $s_\alpha \in W$ is a reflection and $g \in O(V)$ then $g s_\alpha g^{-1} = s_{g(\alpha)}$.*

(Note that there may be elements in W which are not reflections. Hence this does not say that W is normal in $O(V)$, since this lemma does not necessarily hold for all elements of W.)

Proof. Let $\beta = g(\alpha) \in V$. Then first,

$$gs_\alpha g^{-1}(\beta) = gs_\alpha g^{-1}g(\alpha) = gs_\alpha(\alpha) = g(-\alpha) = -g(\alpha) = -\beta.$$

If we can show that $gs_\alpha g^{-1}(H_\beta) = H_\beta$ pointwise, then we are done, because $gs_\alpha g^{-1}$ must then be the reflection s_β.

Let $\lambda \in H_\alpha$. Note that $\lambda \in H_\alpha \iff g(\lambda) \in H_\beta$, since $0 = (\lambda, \alpha) = (g(\lambda), \beta)$. Now,

$$gs_\alpha g^{-1}(g(\lambda)) = gs_\alpha(\lambda) = g(\lambda),$$

so indeed $g(\lambda)$ is fixed by $gs_\alpha g^{-1}$. □

Corollary 2. *If $s_\alpha, w \in W$ then $s_{w(\alpha)} \in W$. i.e. if H_α is a reflection hyperplane, so is $H_{w(\alpha)}$.*

Proof. Set $g = w$ in Lemma 1. □

We conclude that reflecting hyperplanes are permuted by the action of w. For example, take $W = A_2 (= S_3)$

$s_\alpha, s_\beta, s_\alpha s_\beta, s_\alpha s_\beta s_\alpha$. Now,

$$\begin{aligned}
H_\alpha &= s_\alpha(H_\alpha) = s_\beta(H_{\alpha+\beta}) = s_\alpha s_\beta(H_{\alpha+\beta}) = s_\beta s_\alpha(H_\beta) = s_\alpha s_\beta s_\alpha(H_\beta) \\
H_\beta &= s_\alpha(H_{\alpha+\beta}) = s_\beta(H_\beta) = s_\alpha s_\beta(H_\alpha) = s_\beta s_\alpha(H_{\alpha+\beta}) = s_\alpha s_\beta s_\alpha(H_\alpha) \\
H_{\alpha+\beta} &= s_\alpha(H_\beta) = s_\beta(H_\alpha) = s_\alpha s_\beta(H_\beta) = s_\beta s_\alpha(H_\alpha) = s_\alpha s_\beta s_\alpha(H_{\alpha+\beta})
\end{aligned}$$

We see that each hyperplane occurs twice as a permutation of each other hyperplane. What we don't see here is that the group elements will also permute the normals of the hyperplane.

To better understand the structure of W, we introduce the notion of a *root system*.

Definition: Let Φ be a finite subste of V. Φ is called a root system if, for all $\alpha \in \Phi$,

1. The only multiples of α in Φ are $\alpha, -\alpha$ (so that for each normal vector α, we only have it and its negative).
2. $s_\alpha(\Phi) = \Phi$.

Several remarks are in order:

- Sometimes, condition 1 is dropped from the definition, allowing for *non-reduced* root systems.
- Sometimes a third condition is assumed, that $(\alpha, \beta^V) \in \mathbb{Z}$ for $\alpha, \beta \in \Phi$. (We will not assume this.) This leads to *crystallographic* root systems, important in Lie theory.

Lemma 3. *The classification of finite reflection groups boils down to the classification of root systems.*

Proof. Homework. □

2.2 Root Systems

Claim: The classification of finite reflection groups is the same as the classification of root systems.

Recall that a root system $\Phi \subset V$ is a finite set such that for all $\alpha \in \Phi$, the only multiples of α are $\alpha, -\alpha$, and $s_\alpha(\Phi) = \Phi$.

For example, we have seen that A_2 is generated by reflection about three hyperplanes in $\mathbb{R}^3$; the six unit normals of these hyperplanes form a root system.

The direction $W \to \Phi$ follows from everything that has been said so far. (Since we observed that the elements of the reflection groups permute the hyperplanes and the normals, and this lead to the definition of a root system.)

For the direction $\Phi \to W$, we need to consider, if we start with a finite root system, could we get an infinite group? We need to show that the reflections induced by the hyperplanes perpendicular to the roots α form a finite group. (Recall that the group is generated by finitely many reflections, including things which are not reflections. Are there infinitely many of these?)

Let $\phi : W \to S_k$ be the natural homomorphism from W into the symmetric group on Φ. Since only $w = 1$ fixes all of Φ (in particular, $s_\alpha \in W$ sends α to $-\alpha$), $\ker \Phi = 1$, so W is finite.

We have the remarkable fact each root system admits a decomposition into a positive and negative part,

$$\Phi = \Phi^+ \cup \Phi^-,$$

such that $\Phi^- = -\Phi^+$ and $\Phi^+ \cap \Phi^- = \emptyset$, where Φ^+ admits a unique basis (known as a *simple system* or *base*) $\Delta \subseteq \Phi^+$ such that for all $\beta \in \Phi^+$,

$$\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha, \quad c_\alpha \geq 0.$$

(The key here is that the coefficients are all non-negative.) For example, in the root system we have $\Delta = \{\alpha_1, \alpha_2\}$, $\Phi^- = -\Phi^+$.

Put formally,

Definition: Let Φ be a root system in V. $\Delta \subset \Phi$ is called a *simple system* (or base) if its elements are linearly independent (over $\mathbb{R}$), spans Φ, and each root $\beta \in \Phi$ can be written as

$$\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$$

with all $c_\alpha \geq 0$ or all $c_\alpha \leq 0$. Roots in Δ are called *simple roots*, and roots with positive/negative height are called positive/negative rppts. Here, height$(\beta) = \mathrm{ht}(\beta) = \sum_{\alpha \in \Delta} c_\alpha$. Roots with positive height form a set Φ^+, and roots with negative height form a set Φ^-.

It remains to be seen that every root system admits a simple system. (Assignment task.) We will look instead at simple consequences of the existence of a base.

Lemma 4. *Let $\Delta \subset \Phi$ be a base, and $\alpha, \beta \in \Delta$, $\alpha \neq \beta$. Then, $(\alpha, \beta) \leq 0$. (Then, angles between simple roots are $\frac{\pi}{2}$ or obtuse.)*

When we return to Coxeter groups, we will see that the distinction between $(\alpha, \beta) = 0$ and $(\alpha, \beta) < 0$ relates to whether or not α and β are connected in the Coxeter graph. (In particular, they are connected iff $(\alpha, \beta) < 0$.)

Proof. Assume towards contradiction that $(\alpha, \beta) > 0$, i.e. $(\beta, \alpha^V) > 0$. Then,

$$s_\alpha(\beta) = \beta - (\beta, \alpha^V)\alpha = \beta - k\alpha,$$

for some $k > 0$. However, $s_\alpha(\beta) \in \Phi$, but its expression in terms of base elements contains positive and negative coefficients. Thus, $s_\alpha(\beta)$ is neither in Φ^+ nor in Φ^-. Contradiction. $\square$

Lemma 5. *Let $\Delta \subset \Phi$ be a base. Then, for $\alpha \in \Delta$, $s_\alpha(\Phi^+ \setminus \{\alpha\}) = \Phi^+ \setminus \{\alpha\}$.*

Note that clearly $s_\alpha(\alpha) - \alpha$, so the exclusion of α is necessary. In other words, Φ^+ and $s_\alpha(\Phi^+)$ differ in a simple root. This is also not true for reflections not corresponding to simple roots. For example, in A_2 above,

$$s_{\alpha_1}(\Phi^+) = \{-\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}, \quad s_{\alpha_1+\alpha_2}(\Phi^+) = \Phi^-.$$

Proof. Let $\beta \in \Phi^+ \setminus \{\alpha\}$. Then,

$$\begin{aligned} s_\alpha(\beta) &= \beta - (\beta, \alpha^V)\alpha \\ &= \sum_{\gamma \in \Delta, \gamma \neq \alpha} c_\gamma \gamma + k\alpha \quad (k \in \mathbb{R}, \text{ at least one } c_\gamma \neq 0), \end{aligned}$$

hence $s_\alpha(\beta) \notin \Phi^-$, so $s_\alpha(\beta) \in \Phi^+$. Since at least one $c_\gamma \neq 0$, $s_\beta(\alpha) \neq \alpha$. $\square$

2.3 Lecture 8

Recall from last time the notion of a simple system $\Delta \subset \Phi$, Φ^+ the set of positive roots, and height$(\gamma) = \sum_{\alpha \in \Delta} c_\alpha$ where $\gamma = \sum_{\alpha \in \Delta} c_\alpha \alpha$. Recall also Lemmas 4, 5.

Lemma 6. *Let Δ and $\hat{\Delta}$ be simple systems with corresponding positive root sets Φ^+ and $\hat{\Phi}^+$, then $\hat{\Phi}^+ = w(\Phi^+)$ and $\hat{\Delta} = w(\Delta)$ for some $w \in \Phi$.*

Recall that when we fix Φ^+, then we have a unique simple system, so the statements $\hat{\Delta} = w(\Delta)$ and $\hat{\Phi}^+ = w(\Phi^+)$ are equivalent.

Note: clearly if Δ is a simple system then $s_\alpha(\Delta)$ is also simple: if $\beta = \sum_{\gamma \in \Delta} c_\gamma \gamma$ then

$$s_\alpha(\beta) = \sum_{\gamma \in \Delta} c_\gamma s_\alpha(\gamma) = \sum_{s_\alpha(\tau) \in \Delta} c_{s_\alpha(\tau)} \tau = \sum_{\tau \in s_\alpha(\Delta)} c_{s_\alpha(\tau)} \tau,$$

and since the coefficients $c_{s_\alpha(\tau)}$ have the same sign as c_γ, $s_\alpha(\beta)$ has the same sign in this new root system.

Hence, if Δ is simple with set of positive roots Φ^+, then $s_\alpha(\Delta)$ is simple with set of positive roots $s_\alpha(\Phi^+)$.

Proof. We proceed by induction on $n = |\Phi^+ \cap \hat{\Phi}^-|$. If $n = 0$, then $\Phi^+ = \hat{\Phi}^+$ and $\Delta = \hat{\Delta}$, so we can take $w = 1$.

If statement is true for $0 \leq n \leq N - 1$, and suppose $|\Phi^+ \cap \hat{\Phi}^-| = N$. Then, $\Delta \not\subset \hat{\Phi}^+$, so there exists an $\alpha \in \Delta$ such that $\alpha \in \hat{\Phi}^-$. But then we can apply induction on $s_\alpha(\Phi^+)$ and $\hat{\Phi}^+$ since by Lemma 5, $s_\alpha(\Phi^+) = (\Phi^+ \setminus \{\alpha\}) \cup \{-\alpha\}$, so

$$\left| s_\alpha(\Phi^+) \cap \hat{\Phi}^- \right| = N - 1,$$

hence there exists a $v \in W$ such that

$$v(s_\alpha(\Phi^+)) = \hat{\Phi}^+.$$

Now take $w = v s_\alpha$. $\square$

Theorem 7. *Let W be a finite reflection group with simple system Δ. Then, W is generated by the simple reflections s_α, for $\alpha \in \Delta$.*

Proof. Let $V < W$ be the subgroup generated by the s_α, $\alpha \in \Delta$. The aim is to show that $V = W$ (by showing $W \subset V$). For $\beta \in \Phi^+$, let $\gamma \in V(\beta) \cap \Phi^+$ be of minimal height, where $V(\beta)$ is the V-orbit of β. Note that $\beta \in V(\beta) \cap \Phi^+$, so this set is indeed nonempty.

Then, we claim that $ht(\gamma) = 1$. To see this, write $\gamma = \sum_{\alpha \in \Delta} c_\alpha \alpha$. Then,

$$0 < (\gamma, \gamma) = ||\gamma||^2 = \sum_{\alpha \in \Delta} c_\alpha (\alpha, \gamma),$$

so that $(\gamma, \alpha^V) > 0$ for some $\alpha \in \Delta$. Now, assume towards contradiction that $ht(\gamma) > 1$. Then, $s_\alpha(\gamma) \in \Phi^+$ (by Lemma 5 and the fact that $\gamma \neq \alpha$). Also note that $s_\alpha(\gamma) = V(\beta)$ as $s_\alpha \in V$ and $\gamma \in V(\beta)$. But, $s_\alpha(\gamma) = \gamma - (\gamma, \alpha^V)\alpha$ so that $ht(s_\alpha(\gamma)) < ht(\gamma)$, contradicting minimality.

In other words, the V-orbit of $\beta \in \Phi^+$ contains a simple root $\alpha \in \Delta$. Hence, if we consider the above for β simple, we see $\Phi^+ \subset V(\Delta)$.

Similarly for $\beta \in \Phi^-$ there exists a $v \in V$ such that $-\beta = v(\alpha)$ for some $\alpha \in \Delta$, then $\beta = (vs_\alpha)(\alpha)$ so that $\Phi^- \subset V(\Delta)$. Therefore, $\Phi \subset V(\Delta)$.

To complete the proof, let s_γ be a generator of W. By above, $\gamma = v(\alpha)$ by some $v \in V$ and $\alpha \in \Delta$. Recall from Corollary 2 that $s_{v(\alpha)} \in V$ if $s_\alpha \in V$. Hence, $s_\gamma \in V$, so $V = W$. $\square$

Now, for a simple system Δ, the group generated by $s_{\alpha_1}, \ldots, s_{\alpha_r}$ has the relations $s_\alpha^2 = 1$, $(s_\alpha, s_\beta)^{m_{\alpha,\beta}} = 1$, where $m_{\alpha,\beta} = m_{\beta,\alpha}$, so these relations look like those for a Coxeter group. It remains to show that there are no other relations within this group.

Appendix A

Version History

Below we describe briefly the version history of this document, based on versions which have been published at `http://jgat.github.io/math4301-notes/lectures.pdf`. Version numbers will roughly follow the format v0.n.m, where n indicates the number of lectures which notes have been written for, and m indicates the number of minor revisions. Once notes have been taken for all lectures, the version number will increment to v1.0.

v0.3.0: Initial publication with lectures 1 to 3, §1.1-1.3.

v0.4.0: Lecture 4 placeholder.

v0.7.0: Up to lecture 7. Not cleaned up.