MATH4301 Lecture Notes

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Contents

1 Introduction

1.1 Presentations

Let A be an alphabet, the free group $F(A)$ consists of all words over $A \cup A^{-1}$ in which the pairs aa^{-1} and $a^{-1}a$ are forbidden (i.e. $aa^{-1} = a^{-1}a = 1$). The group multiplication corresponds to concatenation of words and removal of forbidden pairs.

Example: if $A = \{a\}$, $F(A) = \{a^k \mid k \in \mathbb{Z}\} \cong (\mathbb{Z}, +)$. If $w_1 = a^4$, $w_2 = a^{-2}$, then $w_1 w_2 = aaaaa^{-1}a^{-1} = a^2$.

To make life more interesting we need relations. For example, $A = \{a, b\}$ with relation $b = 1$ gives $(\mathbb{Z}, +)$.

A *presentation* (of a group) $\langle A \mid R \rangle$ consists of a set A of *generators* and a set of relations R between the generators (and their inverses). Elements of the group are again words in A, but two words represent the same element in the group if they can be transformed into each other by the use of R. More formally, $G \cong F(A)/N$ where N is the normal subgroup generated by R.

Example: $\langle a \mid a^k = 1 \rangle \cong \mathbb{Z}/k\mathbb{Z} = \mathbb{Z}_k$ (for $k = 1, 2, \dots$). Formally, $\langle a \mid a^k = 1 \rangle \cong F(a)/\langle a^k \rangle$.

Example: $\langle a, b \mid a^2 = b^2 = (ab)^2 = 1 \rangle$ contains elements $1, a, b, ab, ba, \dots$, however note that $ba = (ab)^{-1} = ab$. Simply guessing which words are distinct is not going to work. The multiplication table of the group is (Exercise: Show that this is all of the elements in the group):

G	1	a	b	ab
1	1	a	b	ab
a	a	1	ab	b
b	b	ab	1	a
ab	ab	b	a	1

Note that $bab = a^{-1}abab = a^{-1} = a$. This is the Klein 4-group $\cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Geometrically it is the symmetry group of the (non-square) rectangle and a rhombus, where a and b are reflections and ab is rotation by π.

The *word problem* is to decide if two *distinct* words in the generators represent the same/different elements in the group. In 1955, Novikov showed that the word problem is undecidable. This is not the case for Coxeter groups.

1.2 Coxeter Groups

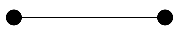
References:

- Bjorner & Brenti: Combinatorics of Coxeter groups (Springer GTM231, '05)
- Bourbaki: Lie groups & Lie algebras (Chap 4-6)
- Cohen: Coxeter groups
- Humphreys: Reflection Groups and Coxeter Groups
- Davis: The Geometry and Topolology of Coxeter Groups

Let M be an $r \times r$ symmetric matrix with entries m_{ij} in $\{1, 2, 3, \dots\} \cup \{\infty\}$ with $m_{ii} = 1$ and $m_{ij} = m_{ji} > 1$ for $i \neq j$. Such a matrix is called a *Coxeter matrix.* For example,

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

Coxeter matrices are often represented as a graph with r labelled vertices $(1, 2, \dots, r)$, and if $m_{ij} \geq 3$, an edge between i and j with a labelling of the edge by m_{ij}. It is standard to drop edge labels which are 3. Hence the above example can be expressed as



Given a Coxeter matrix M (or graph), a *Coxeter system* (W, S) of type M is a set $S = \{s_1, \dots, s_r\}$ and a group

$$W = \langle S \mid (s_i s_j)^{m_{ij}} = 1, \quad 1 \leq i, j \leq r, m_{ij} \neq \infty \rangle.$$

(That is, whenever $m_{ij} \neq \infty$, impose a relation $(s_i s_j)^{m_{ij}} = 1$). The group W is called a *Coxeter group* (of type M). The number r is known as the *rank* of W. Note that $s_i^2 = 1$ for all $1 \leq i \leq r$.

Example: For rank 1, there is only one Coxeter group, $M = (1)$, with the trivial graph:

and the corresponding Coxeter group $W = \langle s \mid s^2 = 1 \rangle \cong \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

For rank 2, we have first,

$$M = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

with corresponding graph

and corresponding group

$$W = \langle s, t \mid s^2 = t^2 = (st)^2 = 1 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

(Note that $(s_i s_j)^{m_{ij}} = 1$ implies that $(s_j s_i)^{m_{ij}} = 1$. Why: $(s_j s_i)^{m_{ij}} = (s_j s_i)^{m_{ij}} s_j^2 = s_j (s_i s_j)^{m_{ij}} s_j = s_j^2 = 1$) We will later show that if a Coxeter system has a disconnected graph, then the Coxeter group will be the direct product of the corresponding groups for each component; hence we will focus on connected graphs. We also have

$$M = \begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix}, \quad m \geq 3,$$

and

$$W = \langle s, t \mid s^2 = t^2 = (st)^m = 1 \rangle.$$

This is known as the *dihedral* group of order $2m$ (D_m / D_{2m} / $I_2(m)$). The dihedral group is the symmetry group of the regular m-gon. For example, $I_2(3)$ is the symmetry group of the triangle, where $s, t, sts = tst$ are reflections and st, ts are rotations. $I_2(4)$ has reflections $s, t, sts = s(ts)^2 = (st)^2 s, tst = t(st)^2 = (ts)^2 t$.

Note that a word of odd length corresponds to a reflection, and a word of even length corresponds to a rotation; also note that the relation $(st)^m = 1$ embodies the "rotate m times to get the identity" property of the m-gon.

1.3 Dihedral Groups

Recall that a Coxeter group is of the form

$$W = \langle S \mid (s_i s_j)^{m_{ij}} = 1, 1 \leq i \leq j \leq r, m_{ij} \neq \infty \rangle,$$

with the associated matrix $M = (m_{ij})$ where $m_{ii} = 1$ and $M^T = M$. A special case is, for $m > 2$,

$$I_2(m) = \langle s, t \mid s^2 = t^2 = (st)^m = 1 \rangle,$$

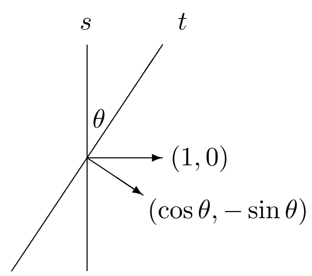
a Coxeter group of rank 2. We saw that $I_2(3)$ is the group of symmetries of the equilateral triangle, and $I_2(4)$ is the group of symmetries of the square. We claim that $I_2(m)$ is a group of order $2m$ consisting of m reflections and m rotations of the regular m-gon.

First, for a vector $\alpha \in \mathbb{R}^n$, let H_α denote the hyperplane with normal α, and denote reflection in H_α by r_α. Now, for any vector λ,

$$r_\alpha(\lambda) = \lambda - \frac{2(\alpha, \lambda)\alpha}{(\alpha, \alpha)}, \tag{1.1}$$

where (a, b) denotes the inner product (vector dot product). Note that $r_\alpha(\lambda) = \lambda$ for every $\lambda \in H_\alpha$, and $r_\alpha(\alpha) = \alpha - \frac{2(\alpha,\alpha)\alpha}{(\alpha,\alpha)} = -\alpha$ as expected. Since we have verified this for a hyperplane of codimension 1 and for a vector normal to the hyperplane, the result is true for all vectors (by Linear Algebra).

Proof. Let s and t be reflections, where the axes of symmetry have an angle of $\theta = \frac{\pi}{m}$, i.e. $s := r_{(1,0)}$ and $t := r_{(\cos\theta, -\sin\theta)}$:



Then, $s(1,0) = (-1,0)$ and $s(0,1) = (0,1)$, so

$$\hat{s} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

is a matrix representation of s, and

$$\hat{t} = \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

is a matrix representation of t (Exercise).

If we can show that st is a rotation over $\frac{2\pi}{m}$, then $(st)^k$ will be a rotation over $\frac{2\pi k}{m}$, which will give m distinct rotations. Now,

$$\hat{s}\hat{t} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix},$$

which is a rotation matrix for rotation over 2θ (Exercise). Then $(\hat{s}\hat{t})^k$ is a rotation by $2k\theta$ (easy to see geometrically, or show inductively that $(\hat{s}\hat{t})^k$ is a rotation matrix).

It remains to show that there are m distinct words of odd length, and they are all reflections. WLOG, we can say that all words of odd length are of the form $t(st)^{k-1}$ for some $k = 1, 2, \ldots, m$, hence there are m distinct words of odd length. Now,

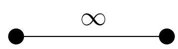
$$\hat{t}(\hat{s}\hat{t})^k = \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} \cos 2\theta(k-1) & -\sin 2\theta(k-1) \\ \sin 2\theta(k-1) & \cos 2\theta(k-1) \end{pmatrix} = \begin{pmatrix} -\cos(2k\theta) & \sin 2k\theta \\ \sin 2k\theta & \cos 2k\theta \end{pmatrix}$$

(Exercise), so $t(st)^{k-1} := r_{(\cos k\theta, -\sin k\theta)}$. Hence the m words of odd length are all reflections. $\square$

Finally, we consider the remaining Coxeter group of rank 2, given by

$$M = \begin{pmatrix} 1 & \infty \\ \infty & 1 \end{pmatrix},$$

with corresponding graph



and corresponding group

$$I_2(\infty) = \langle s, t \mid s^2 = t^2 = 1 \rangle.$$

Elements of this group, known as the ∞-dihedral group are the words of the form: $1, s, t, st, ts, sts, tst, \ldots$. Again this group has a geometric interpretation in terms of reflections (etc.). Before we can describe this, we need some more notation.

Let $V = \mathbb{R}^n$ and for $\alpha \in V$ let H_α denote the hyperplane perpendicular to α. Algebraically, $H_\alpha = \{\lambda \in V : (\lambda, \alpha) = 0\}$. As we have seen, the reflextion r_α in H_α is given by its action on $\lambda \in V$ as

$$r_\alpha(\lambda) = \lambda - \frac{2(\alpha, \lambda)}{(\alpha, \alpha)}\alpha = \lambda - \frac{2(\lambda, \alpha)}{||\alpha||^2}\alpha = \lambda - (\lambda, \alpha^\vee)\alpha,$$

where $\alpha^\vee = \frac{2\alpha}{||\alpha||^2}$.

1.4 Affine Hyperplanes

Apart from reflections, we also need translations. We denote by t_λ a translation by $\lambda \in V(= \mathbb{R})$, such that

$$t_\lambda(\mu) = \lambda + \mu = t_\mu(\lambda).$$

It is not difficult to see that translations are "normalised" by reflections, since

$$r_\alpha t_\lambda r_\alpha^{-1} = r_\alpha t_\lambda r_\alpha = t_{r_\alpha(\lambda)}.$$

Proof.

$$r_\alpha t_\lambda r_\alpha(\mu) = r_\alpha(\lambda + r_\alpha(\mu)) = r_\alpha(\lambda) + r_\alpha^2(\mu) = r_\alpha(\lambda) + \mu.$$

$\square$

A hyperplane that does not necessarily contain the origin is called an *affine* hyperplane. The affine hyperplane $H_{\alpha,\kappa}$ parallel to $H_\alpha = H_{\alpha,0}$ is defined by the equation

$$H_{\alpha,\kappa} = \{\lambda \in V \mid (\lambda, \alpha) = \kappa\}.$$

The distance between H_α and $H_{\alpha,\kappa}$ is given by

$$d(H_\alpha, H_{\alpha,\kappa}) = \frac{|(\alpha,\kappa)|}{||\alpha||^2} = \frac{|\kappa|}{||\alpha||} = \frac{1}{2}|\kappa|\alpha^\vee.$$

In fact, $H_{\alpha,\kappa} = H_{\alpha,0} + \frac{1}{2}\kappa\alpha^\vee$.

Proof. Let $\lambda \in H_{\alpha,\kappa}$. We need to show that $\mu := \lambda - \frac{1}{2}\kappa\alpha^\vee \in H_{\alpha,0}$. But,

$$(\mu,\alpha) = (\lambda,\alpha) - \frac{1}{2}\kappa(\alpha^\vee,\alpha) = \kappa - \frac{1}{2}\kappa \cdot 2 = 0,$$

so $\mu \in H_\alpha$. □

It is also easy to check the following formula for affine reflections:

$$r_{\alpha,\kappa}(\lambda) = \lambda - ((\lambda,\alpha) - \kappa)\alpha^\vee.$$

Proof. Let $\lambda \in H_{\alpha,\kappa}$. Then,

$$r_{\alpha,\kappa}(\lambda) = \lambda - (\kappa - \kappa)\alpha^\vee = \lambda.$$

Also,

$$r_{\alpha,\kappa}(0) = \kappa\alpha^\vee = 2\left(\frac{1}{2}\kappa\alpha^\vee\right).$$

□

More generally, $r_{\alpha,\kappa}(H_{\alpha,\tau}) = H_{\alpha,2\kappa-\tau}$.

Proof. Let $\lambda \in H_{\alpha,\tau}$. Then, $r_{\alpha,\kappa}(\lambda) = \lambda + (\kappa - \tau)\alpha^\vee$. Hence,

$$(r_{\alpha,\kappa}(\lambda),\alpha) = (\lambda,\alpha) + (\kappa-\tau)(\alpha^\vee,\alpha) = \tau + 2(\kappa-\tau) = 2\kappa - \tau.$$

□

To describe $\widetilde{A}_1 = \langle s,t \mid s^2 = t^2 = 1\rangle$, let $V = \mathbb{R}$ and $\alpha = (1) = 1$. Write H_k and r_k for the affine hyperplane and affine reflection in the integer point k.

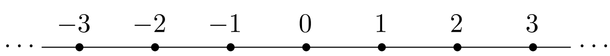
Claim: Take $s = r_0$, $t = r_1$. Then $r_k = s(st)^k$ and $t_{2k} = (ts)^k$ for $k \in \mathbb{Z}$, so that $\widetilde{A}_1 = A_1 \ltimes T$, where $A_1 = \langle s \mid s^2 = 1\rangle \simeq \mathbb{Z}_2$ and T is the group of translations over $2\mathbb{Z}$.

A large class of affine reflection groups (a special class of Coxeter groups) have this structure: $\widetilde{W} = W \ltimes T$ with W a finite/non-affine reflection group and T a lattice.

1.5 $I_2(\infty)$ and Symmetric Groups

1.5.1 Affine Reflections

Consider the integer number line,



Let $s = r_0$, $t = r_1$, in the group $W = \langle s, t \mid s^2 = t^2 = 1\rangle$. From last time, we had the following claims:

Claim 1: $r_k = s(st)^k$, $t_{2k} = (ts)^k$, $k \in \mathbb{Z}$, where r_k is reflection about k, and t_{2k} is a transation by $2k$.

Claim 2: $\tilde{A}_1 = A_1 \ltimes T$, where $A_1 = \{1, r_0\}$, $T = \{t_{2k} : k \in \mathbb{Z}\}$.

Proof. Recall that

$$r_{\alpha,\kappa}(\lambda) = \lambda - \{(\lambda, \alpha) - \kappa\}\alpha^\vee,$$

if $\alpha = 1$, then $\alpha^\vee = 2$ (since the dot product between α and $\alpha^\vee$ must be 2), so

$$r_k = \lambda - 2\{\lambda - k\} = 2k - \lambda.$$

Then,

$$ts(\lambda) = r_1 r_0(\lambda) = r_1(-\lambda) = 2 + \lambda = t_2(\lambda),$$

thus $ts = t_2$, so $(ts)^k = t_{2k}$. Furthermore,

$$s(st)^k(\lambda) = r_0 t_{-2k}(\lambda) = r_0(\lambda - 2k) = -\lambda + 2k = r_k(\lambda),$$

so $s(st)^k = r_k$. □

Note that

$$r_{\alpha,\kappa}(\lambda + \mu) \neq r_{\alpha,\kappa}(\lambda) + r_{\alpha,\kappa}(\mu),$$

instead it holds that,

$$\begin{aligned} r_{\alpha,\kappa}(\lambda + \mu) &= \lambda + \mu - \{(\lambda + \mu, \alpha)\alpha^\vee - \kappa\} \\ &= \lambda + \mu - \{(\lambda, \alpha)\alpha^\vee + (\mu, \alpha)\alpha^\vee - \kappa\} \\ &= r_{\alpha,\kappa}(\lambda) + r_{\alpha,\kappa}(\mu) - \kappa\alpha^\vee \end{aligned}$$

Now,

$$\begin{aligned} r_{\alpha,\kappa} t_\lambda r_{\alpha,\kappa}(\mu) &= r_{\alpha,\kappa}(\lambda + r_{\alpha,\kappa}(\mu)) \\ &= r_{\alpha,\kappa}(\lambda) + \mu - \kappa\alpha^\vee \\ &= r_{\alpha,0}(\lambda) + \mu, \end{aligned}$$

so $r_{\alpha,\kappa} t_\lambda r_{\alpha,\kappa} = t_{r_\alpha(\lambda)}$.

1.5.2 Semi-direct products

Recall that, for a group G, if $K \leq G$, $N \triangleleft G$, $K \cap N = \{1\}$, and $G = NK$, then we say $G = K \ltimes N$ or $G = N \rtimes K$. Now, consider claim 2:

$$\tilde{A}_1 = \{t_{2k}, r_k \mid k \in \mathbb{Z}\},$$

and consider the subgroups

$$N = T = \{t_{2k} : k \in \mathbb{Z}\}, K = \{1, r_0\} = A_1.$$

Note that $r_k = r_0 t_{-2k}$, so $\tilde{A}_1 = A_1 T$. Thus, claim 2 holds.

1.5.3 Symmetric Groups

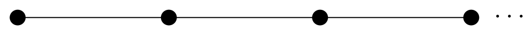
Before moving to the general theory, we will discuss one more important example of a finite reflection group (Coxeter group), the symmetric group S_n, also denoted A_{n-1} (not to be confused with the alternating group!) Let us define S_n to be

$$S_n = \langle s_1, \ldots, s_{n-1} \mid s_i^2 = 1, (s_i s_{i+1})^3 = 1, (s_i s_j)^2 = 1, |i-j| > 1\rangle.$$

Clearly the associated Coxeter matrix is

$$M = \begin{pmatrix} 1 & 3 & 2 & \cdots & 2 \\ 3 & 1 & 3 & \cdots & 2 \\ 2 & 3 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 3 \\ 2 & 2 & \cdots & 3 & 1 \end{pmatrix}$$

with Coxeter graph



The most common description of S_n is as the group of permutations on n letters with s_i acting as "adjacent" transpositions, interchanging the letters in positions i and $i+1$ (i.e. the cycle $(i\ \ i+1)$). For example, $s_2(2,3,6,4,5,1) = (2,6,3,4,5,1)$. The relation $(s_i s_j)^2 = 1$ for $|i-j| > 1$ may be recast as the commutation relation $s_i s_j = s_j s_i$. Finally, $(s_i s_{i+1})^3 = 1$ can be restated as $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, known as Artin's braid relation.

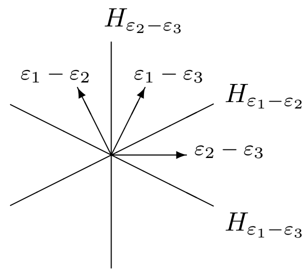
Of course, S_n can also be interpreted as a reflection group. Let $V = \mathbb{R}^n$ and s_i the reflection in the hyperplane $H_{\varepsilon_i - \varepsilon_{i+1}} = H_{\alpha_i}$, where ε_i is the i^{th} basis vector in the standard basis. Then, it holds that

$$s_i(\varepsilon_k) = \begin{cases} \varepsilon_k & k \neq i, i+1 \\ \varepsilon_{i+1} & k = i \\ \varepsilon_i & k = i+1 \end{cases}$$

That is, elements of A_{n-1} permute the basis elements of $\mathbb{R}^n$, so that indeed $s_i^2 = 1$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, and $s_i s_j = s_j s_i$ for $|i-j| > 1$. (Exercise: show this).

Note that $\varepsilon_1 + \cdots + \varepsilon_n$ is fixed by the action of $A_{n-1}(= S_n)$, and no other vector linearly independent to this is fixed, hence A_{n-1} only acts on an $(n-1)$-dimensional subspace of $\mathbb{R}^n$, namely $\{\lambda \in \mathbb{R}^n \mid (\lambda, \varepsilon_1 + \cdots + \varepsilon_n) = 0\}$.

For example, in the case of $S_3 = A_2$, we have reflection in the three hyperplanes (planes) below (note that this is a 2-dimensional projection of $\mathbb{R}^3$, where the lines H_α are planes in $\mathbb{R}^3$):



Now, A_{n-1} is the symmetry group of the $(n-1)$-simplex, the generalisation of the tetrahedron to $n-1$ dimensions, where for instance the 2-simplex is the triangle, the 3-simplex is the tetrahedron, and so on.

2 Finite Reflection Groups

2.1 The General Theory of Finite Reflection Groups

Let V be an Euclidean space, that is, a vector space over $\mathbb{R}$ with a positive definite symmetric bilinear form $(\cdot,\cdot) : V \times V \to \mathbb{R}$, for example the dot product $(a, b) = a \cdot b$ in $\mathbb{R}^n$. (Here it suffices to think of $\mathbb{R}^n$ when using Euclidean spaces).

A *reflection* in V is a linear map s_α ($\alpha \in V$) such that

$$s_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha,$$

where $\alpha^\vee := \frac{2\alpha}{(\alpha,\alpha)}$ (cf. Equation (1.1)). Note that s_α is an *involution* as well as an *orthogonal transformation*:

Proof.

$$\begin{aligned} s_\alpha^2(\lambda) &= s_\alpha(\lambda - (\lambda, \alpha^\vee)\alpha) \\ &= s_\alpha(\lambda) - (\lambda, \alpha^\vee)s_\alpha(\alpha) \\ &= \lambda - (\lambda, \alpha^\vee)\alpha + (\lambda, \alpha^\vee)\alpha \\ &= \lambda, \end{aligned}$$

so s_α is an involution. Also,

$$\begin{aligned} (s_\alpha(\lambda), s_\alpha(\mu)) &= (\lambda - (\lambda, \alpha^\vee)\alpha, \mu - (\mu, \alpha^\vee)\alpha) \\ &= (\lambda, \mu) - (\mu, \alpha^\vee)(\lambda, \alpha) - (\lambda, \alpha^\vee)(\mu, \alpha) + (\lambda, \alpha^\vee)(\mu, \alpha^\vee)(\alpha, \alpha) \\ &= (\lambda, \mu) - 2(\mu, \alpha)(\lambda, \alpha)\frac{\alpha}{(\alpha, \alpha)} + 4(\lambda, \alpha)(\mu, \alpha)\frac{(\alpha, \alpha)}{(\alpha, \alpha)^2} \\ &= (\lambda, \mu), \end{aligned}$$

so s_α is an orthogonal transformation. (Exercise: convince yourself of these results.) □

A *finite reflection group* is a finite subgroup of the group of orthogonal transformations on V generated by reflections. As we shall see (I hope), all finite reflection groups are Coxeter groups. The converse also holds, however we will not show this.

Lemma 1. *Let $O(V)$ be the group of orthogonal transformations on V, and $W < O(V)$ be a finite reflection group. If $s_\alpha \in W$ is a reflection and $g \in O(V)$ then $gs_\alpha g^{-1} = s_{g(\alpha)}$.*

(Note that there may be elements in W which are not reflections. Hence this does not say that W is normal in $O(V)$, since this lemma does not necessarily hold for all elements of W.)

Proof. Let $\beta = g(\alpha) \in V$. Then first,

$$gs_\alpha g^{-1}(\beta) = gs_\alpha g^{-1}g(\alpha) = gs_\alpha(\alpha) = g(-\alpha) = -g(\alpha) = -\beta.$$

If we can show that $gs_\alpha g^{-1}(H_\beta) = H_\beta$ pointwise, then we are done, because $gs_\alpha g^{-1}$ must then be the reflection s_β.

Let $\lambda \in H_\alpha$. Note that $\lambda \in H_\alpha \iff g(\lambda) \in H_\beta$, since $0 = (\lambda, \alpha) = (g(\lambda), \beta)$. Now,

$$gs_\alpha g^{-1}(g(\lambda)) = gs_\alpha(\lambda) = g(\lambda),$$

so indeed $g(\lambda)$ is fixed by $gs_\alpha g^{-1}$. □

Corollary 2. *If $s_\alpha, w \in W$ then $s_{w(\alpha)} \in W$. i.e. if H_α is a reflection hyperplane, so is $H_{w(\alpha)}$.*

Proof. Set $g = w$ in Lemma 1. □

We conclude that reflecting hyperplanes are permuted by the action of w. For example, take $W = A_2(= S_3)$, which contains the non-identity elements $s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, s_\alpha s_\beta s_\alpha$. Now,

$$\begin{aligned}
H_\alpha &= s_\alpha(H_\alpha) = s_\beta(H_{\alpha+\beta}) = s_\alpha s_\beta(H_{\alpha+\beta}) = s_\beta s_\alpha(H_\beta) = s_\alpha s_\beta s_\alpha(H_\beta) \\
H_\beta &= s_\alpha(H_{\alpha+\beta}) = s_\beta(H_\beta) = s_\alpha s_\beta(H_\alpha) = s_\beta s_\alpha(H_{\alpha+\beta}) = s_\alpha s_\beta s_\alpha(H_\alpha) \\
H_{\alpha+\beta} &= s_\alpha(H_\beta) = s_\beta(H_\alpha) = s_\alpha s_\beta(H_\beta) = s_\beta s_\alpha(H_\alpha) = s_\alpha s_\beta s_\alpha(H_{\alpha+\beta})
\end{aligned}$$

We see that each hyperplane occurs twice as a permutation of each other hyperplane. What we don't see here is that the group elements will also permute the normals of the hyperplane.

To better understand the structure of W, we introduce the notion of a *root system.*

Definition: Let Φ be a finite subste of V. Φ is called a root system if, for all $\alpha \in \Phi$,

1. The only multiples of α in Φ are $\alpha, -\alpha$ (so that for each normal vector α, we only have it and its negative).
2. $s_\alpha(\Phi) = \Phi$.

Several remarks are in order:

- Sometimes, condition 1 is dropped from the definition, allowing for *non-reduced* root systems.
- Sometimes a third condition is assumed, that $(\alpha, \beta^\vee) \in \mathbb{Z}$ for $\alpha, \beta \in \Phi$. (We will not assume this.) This leads to *crystallographic* root systems, important in Lie theory.

Lemma 3. *The classification of finite reflection groups boils down to the classification of root systems.*

Proof. Homework. □

2.2 Root Systems

Claim: The classification of finite reflection groups is the same as the classification of root systems.

Recall that a root system $\Phi \subset V$ is a finite set such that for all $\alpha \in \Phi$, the only multiples of α are $\alpha, -\alpha$, and $s_\alpha(\Phi) = \Phi$.

For example, we have seen that A_2 is generated by reflection about three hyperplanes in $\mathbb{R}^3$; the six unit normals of these hyperplanes form a root system.

The direction $W \to \Phi$ follows from everything that has been said so far. (Since we observed that the elements of the reflection groups permute the hyperplanes and the normals, and this lead to the definition of a root system.)

For the direction $\Phi \to W$, we need to consider, if we start with a finite root system, could we get an infinite group? We need to show that the reflections induced by the hyperplanes perpendicular to the roots α form a finite group. (Recall that the group is generated by finitely many reflections, including things which are not reflections. Are there infinitely many of these?)

Let $\phi : W \to S_k$ be the natural homomorphism from W into the symmetric group on Φ. Since only $w = 1$ fixes all of Φ (in particular, $s_\alpha \in W$ sends α to $-\alpha$), $\ker \Phi = 1$, so W is finite.

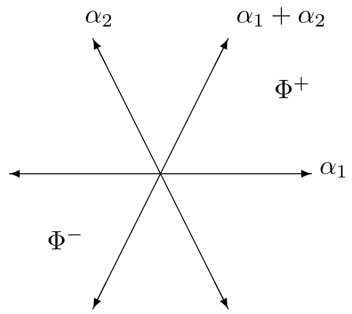
We have the remarkable fact each root system admits a decomposition into a positive and negative part,

$$\Phi = \Phi^+ \cup \Phi^-,$$

such that $\Phi^- = -\Phi^+$ and $\Phi^+ \cap \Phi^- = \emptyset$, where Φ^+ admits a unique basis (known as a *simple system* or *base*) $\Delta \subseteq \Phi^+$ such that for all $\beta \in \Phi^+$,

$$\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha, \quad c_\alpha \geq 0.$$

(The key here is that the coefficients are all non-negative.) For example, in the root system



we have $\Delta = \{\alpha_1, \alpha_2\}$, $\Phi^- = -\Phi^+$.

Put formally,

Definition: Let Φ be a root system in V. $\Delta \subset \Phi$ is called a *simple system* (or base) if its elements are linearly independent (over $\mathbb{R}$), spans Φ, and each root $\beta \in \Phi$ can be written as

$$\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$$

with all $c_\alpha \geq 0$ or all $c_\alpha \leq 0$. Roots in Δ are called *simple roots*, and roots with positive/negative height are called positive/negative roots. Here, $\text{height}(\beta) = \text{ht}(\beta) = \sum_{\alpha \in \Delta} c_\alpha$. Roots with positive height form a set Φ^+, and roots with negative height form a set Φ^-.

It remains to be seen that every root system admits a simple system. (Assignment task.) We will look instead at simple consequences of the existence of a base.

Lemma 4. *Let $\Delta \subset \Phi$ be a base, and $\alpha, \beta \in \Delta$, $\alpha \neq \beta$. Then, $(\alpha, \beta) \leq 0$. (Then, angles between simple roots are $\frac{\pi}{2}$ or obtuse.)*

When we return to Coxeter groups, we will see that the distinction between $(\alpha, \beta) = 0$ and $(\alpha, \beta) < 0$ relates to whether or not α and β are connected in the Coxeter graph. (In particular, they are connected iff $(\alpha, \beta) < 0$.)

Proof of Lemma 4. Assume towards contradiction that $(\alpha, \beta) > 0$, i.e. $(\beta, \alpha^\vee) > 0$. Then,

$$s_\alpha(\beta) = \beta - (\beta, \alpha^\vee)\alpha = \beta - k\alpha,$$

for some $k > 0$. However, $s_\alpha(\beta) \in \Phi$, but its expression in terms of base elements contains positive and negative coefficients. Thus, $s_\alpha(\beta)$ is neither in Φ^+ nor in Φ^-. Contradiction. $\square$

Lemma 5. *Let $\Delta \subset \Phi$ be a base. Then, for $\alpha \in \Delta$, $s_\alpha(\Phi^+ \setminus \{\alpha\}) = \Phi^+ \setminus \{\alpha\}$.*

Note that clearly $s_\alpha(\alpha) - \alpha$, so the exclusion of α is necessary. In other words, Φ^+ and $s_\alpha(\Phi^+)$ differ in a simple root. This is also not true for reflections not corresponding to simple roots. For example, in A_2 above,

$$s_{\alpha_1}(\Phi^+) = \{-\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}, \quad s_{\alpha_1+\alpha_2}(\Phi^+) = \Phi^-.$$

Proof of Lemma 5. Let $\beta \in \Phi^+ \setminus \{\alpha\}$. Then,

$$\begin{aligned} s_\alpha(\beta) &= \beta - (\beta, \alpha^\vee)\alpha \\ &= \sum_{\gamma \in \Delta, \gamma \neq \alpha} c_\gamma \gamma + k\alpha \quad (k \in \mathbb{R}, \text{ at least one } c_\gamma \neq 0), \end{aligned}$$

hence $s_\alpha(\beta) \notin \Phi^-$, so $s_\alpha(\beta) \in \Phi^+$. Since at least one $c_\gamma \neq 0$, $s_\beta(\alpha) \neq \alpha$. $\square$

2.3 Groups Generated by Simple Systems

Recall from last time the notion of a simple system $\Delta \subset \Phi$, Φ^+ the set of positive roots, and $\text{height}(\gamma) = \sum_{\alpha \in \Delta} c_\alpha$ where $\gamma = \sum_{\alpha \in \Delta} c_\alpha \alpha$. Recall also Lemmas 4, 5.

Lemma 6. *Let Δ and $\hat{\Delta}$ be simple systems with corresponding positive root sets Φ^+ and $\hat{\Phi}^+$, then $\hat{\Phi}^+ = w(\Phi^+)$ and $\hat{\Delta} = w(\Delta)$ for some $w \in \Phi$.*

Recall that when we fix Φ^+, then we have a unique simple system, so the statements $\hat{\Delta} = w(\Delta)$ and $\hat{\Phi}^+ = w(\Phi^+)$ are equivalent.

Note: clearly if Δ is a simple system then $s_\alpha(\Delta)$ is also simple: if $\beta = \sum_{\gamma \in \Delta} c_\gamma \gamma$ then

$$s_\alpha(\beta) = \sum_{\gamma \in \Delta} c_\gamma s_\alpha(\gamma) = \sum_{s_\alpha(\tau) \in \Delta} c_{s_\alpha(\tau)} \tau = \sum_{\tau \in s_\alpha(\Delta)} c_{s_\alpha(\tau)} \tau,$$

and since the coefficients $c_{s_\alpha(\tau)}$ have the same sign as c_γ, $s_\alpha(\beta)$ has the same sign in this new root system.

Hence, if Δ is simple with set of positive roots Φ^+, then $s_\alpha(\Delta)$ is simple with set of positive roots $s_\alpha(\Phi^+)$.

Proof of Lemma 6. We proceed by induction on $n = |\Phi^+ \cap \hat{\Phi}^-|$. If $n = 0$, then $\Phi^+ = \hat{\Phi}^+$ and $\Delta = \hat{\Delta}$, so we can take $w = 1$.

If statement is true for $0 \leq n \leq N - 1$, and suppose $|\Phi^+ \cap \hat{\Phi}^-| = N$. Then, $\Delta \not\subset \hat{\Phi}^+$, so there exists an $\alpha \in \Delta$ such that $\alpha \in \hat{\Phi}^-$. But then we can apply induction on $s_\alpha(\Phi^+)$ and $\hat{\Phi}^+$ since by Lemma 5, $s_\alpha(\Phi^+) = (\Phi^+ \setminus \{\alpha\}) \cup \{-\alpha\}$, so

$$\left| s_\alpha(\Phi^+) \cap \hat{\Phi}^- \right| = N - 1,$$

hence there exists a $v \in W$ such that $v(s_\alpha(\Phi^+)) = \hat{\Phi}^+$. Now take $w = vs_\alpha$. $\square$

Theorem 7. *Let W be a finite reflection group with simple system Δ. Then, W is generated by the simple reflections s_α, for $\alpha \in \Delta$.*

Proof. Let $V < W$ be the subgroup generated by the s_α, $\alpha \in \Delta$. The aim is to show that $V = W$ (by showing $W \subset V$). For $\beta \in \Phi^+$, let $\gamma \in V(\beta) \cap \Phi^+$ be of minimal height, where $V(\beta)$ is the V-orbit of β. Note that $\beta \in V(\beta) \cap \Phi^+$, so this set is indeed nonempty.

Then, we claim that $ht(\gamma) = 1$. To see this, write $\gamma = \sum_{\alpha\in\Delta} c_\alpha \alpha$. Then,

$$0 < (\gamma, \gamma) = ||\gamma||^2 = \sum_{\alpha\in\Delta} c_\alpha(\alpha, \gamma),$$

so that $(\gamma, \alpha^\vee) > 0$ for some $\alpha \in \Delta$. Now, assume towards contradiction that $ht(\gamma) > 1$. Then, $s_\alpha(\gamma) \in \Phi^+$ (by Lemma 5 and the fact that $\gamma \neq \alpha$). Also note that $s_\alpha(\gamma) = V(\beta)$ as $s_\alpha \in V$ and $\gamma \in V(\beta)$. But, $s_\alpha(\gamma) = \gamma - (\gamma, \alpha^\vee)\alpha$ so that $ht(s_\alpha(\gamma)) < ht(\gamma)$, contradicting minimality.

In other words, the V-orbit of $\beta \in \Phi^+$ contains a simple root $\alpha \in \Delta$. Hence, if we consider the above for β simple, we see $\Phi^+ \subset V(\Delta)$.

Similarly for $\beta \in \Phi^-$ there exists a $v \in V$ such that $-\beta = v(\alpha)$ for some $\alpha \in \Delta$, then $\beta = (vs_\alpha)(\alpha)$ so that $\Phi^- \subset V(\Delta)$. Therefore, $\Phi \subset V(\Delta)$.

To complete the proof, let s_γ be a generator of W. By above, $\gamma = v(\alpha)$ by some $v \in V$ and $\alpha \in \Delta$. Recall from Corollary 2 that $s_{v(\alpha)} \in V$ if $s_\alpha \in V$. Hence, $s_\gamma \in V$, so $V = W$. $\square$

Now, for a simple system Δ, the group generated by $s_{\alpha_1}, \dots, s_{\alpha_r}$ has the relations $s_\alpha^2 = 1$, $(s_\alpha, s_\beta)^{m_{\alpha,\beta}} = 1$, where $m_{\alpha,\beta} = m_{\beta,\alpha}$, so these relations look like those for a Coxeter group. It remains to show that there are no other relations within this group.

2.4 Reduced Words and Word Length

2.4.1 Linear Algebra Intermezzo

Let $g \in O(V)$. Then, g viewed as a linear transformation on V has determinant ± 1; transformations with determinant 1 are orientation preserving (e.g. rotations), and reflections correspond to determinant -1.

Recall that a matrix is orthogonal iff its columns form an orthonormal basis, or equivalently, $A^{-1} = A^t$. If this is the case, then $\det(AA^t) = \det(A)^2 = 1$, so $\det(A) = \pm 1$. Similarly, we could also define $A^* = A^{-1}$, where A^* is the adjoint matrix.

2.4.2 Reduced Words and Word Length

Definition: Let W be a finite reflection group generated by a simple system Δ. We say that a word $w = s_1 s_2 \dots s_r \in W$ (where s_i are simple reflections) is *reduced* if there does not exist a shorter word in the generators representing w.[1] If $w = s_1 \dots s_r$ is reduced, then the *length* of w, denoted $l(w)$, is r. By definition, $l(1) = 0$.

For example, in $A_2 = \langle s, t \mid s^2 = t^2 = 1, sts = tst \rangle$, we know that there are six words,

[1]There may be multiple words of shortest length.

$$\begin{tikzcd}[row sep=small]
\text{length:} & 0 & & 1 \arrow[dl, dash] \arrow[dr, dash] & \\
& 1 & s \arrow[d, dash] & & t \arrow[d, dash] \\
& 2 & ts \arrow[dr, dash] & & st \arrow[dl, dash] \\
& 3 & & sts(=tst) & \\
& \vdots & & &
\end{tikzcd}$$

This is an example of a 'Strong Bruhat graph' (we may not have time to cover these in the course).

Some simple facts about the length function:

- $l(w) = 1$ iff w is a simple reflection.
- $l(w^{-1}) = l(w)$, since if $w = s_1 \dots s_r$, $w^{-1} = s_r \dots s_1$.
- $\det(w) = (-1)^{l(w)}$, since if $w = s_1 \dots s_r$ then $\det(w) = \det(s_1) \dots \det(s_r) = (-1)^r$.
- $|l(s_\alpha w) - l(w)| = 1$, we can see this in the example above.

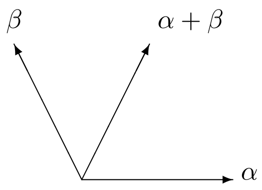
If $w = s_1 \dots s_r$ is reduced, can we have that $s_\alpha s_1 \dots s_l = 1$ for some l with $1 < l < r$, and so $s_\alpha s_1 \dots s_r$ is a lot shorter? If this were the case, then

$$w = s_\alpha^2 \dots s_r = s_\alpha(s_\alpha s_1 \dots s_l)s_{l+1} \dots s_r = s_\alpha s_{l+1} \dots s_r,$$

so $l(w) \leq l(w) - l + 1$, so $l = 1$ and $s_1 = s_\alpha$. So no, prepending a reduced word with a simple reflection will cancel out at most the first simple reflection of the word.

Let $\Phi^+(w) = \Phi^+ \cap w^{-1}(\Phi^-)$, that is, the set of positive roots which are sent to negative roots by the action w; and let $n(w) = |\Phi^+(w)|$. We aim to show that $n(w) = l(w)$.

In the familiar example of A_2, we have



$$\begin{aligned}
\Phi^+(1) &= \Phi^+, \\
\Phi^+(s_\alpha) &= \{\alpha\}, \\
\Phi^+(s_\beta) &= \{\beta\}, \\
\Phi^+(s_\alpha s_\beta) &= \{\beta, \alpha + \beta\}, \\
\Phi^+(s_\beta s_\alpha) &= \{\alpha, \alpha + \beta\}, \\
\Phi^+(s_\alpha s_\beta s_\alpha) &= \Phi^+.
\end{aligned}$$

Claim: $n(w) = n(w^{-1})$:

$$\Phi^+(w) = w^{-1}w\left(\Phi^+ \cap w^{-1}(\Phi^-)\right) = w^{-1}\left(w(\Phi^+) \cap \Phi^-\right) = -w^{-1}\left(w(\Phi^-) \cap \Phi^+\right) = -w^{-1}\left(\Phi^+(w^{-1})\right),$$

and since $-w^{-1}$ is a bijection, the cardinalities of $\Phi^+(w)$ and $\Phi^-(w^{-1})$ are the same. $\square$

Proposition 8. *Given $\alpha \in \Delta$ and $w \in W$, if $w(\alpha) \in \Phi^{\pm}$ then $n(ws_\alpha) = n(w) \pm 1$ – i.e. $n(w)+1$ if $w(\alpha) \in \Phi^+$ and $n(w)-1$ if $w(\alpha) \in \Phi^-$.*

Before proving this we note that equivalently, if $w^{-1}(\alpha) \in \Phi^{\pm}$ then $n(s_\alpha w) = n(w) \pm 1$. Indeed, $n((s_\alpha w)^{-1}) = n(w^{-1}s_\alpha) = n(w^{-1}) \pm 1$, so $n(s_\alpha w) = n(w) \pm 1$ by above.

Proof. If $w(\alpha) \in \Phi^-$, then $\alpha \in \Phi^+(w)$ in which case $\Phi^+(ws_\alpha) = \Phi^+(w) \setminus \{\alpha\}$ so that $n(ws_\alpha) = n(w) - 1$.

Similarly, if $w(\alpha) \in \Phi^+$, then $\alpha \notin \Phi^+(w)$, in which case $\Phi^+(ws_\alpha) = \Phi^+(w) \cup \{\alpha\}$ so that $n(ws_\alpha) = n(w) + 1$. □

2.5 Proof that $l(w) = n(w)$

Lemma 9. *Let $w \in W$. We have $n(w) \leq l(w)$.*

Proof. Let w admit the reduced expression $w = s_1 \dots s_r$ so that $l(w) = r$. But $n(s_1 \dots s_{r-1}s_r) = n(s_1 \dots s_{r-1}) \pm 1$ so that $n(w)$ is at most r. (Remember that $n(1) = 0$ and $n(s_i) = 1$.) □

We of course want to show that $n(w) = l(w)$. This will lead to the *deletion* and *exchange conditions.*

Given a word $w = s_1 s_2 \dots s_i \dots s_k$, write $s_1 \dots \hat{s_i} \dots s_k$ for the word $s_1 \dots s_{i-1}s_{i+1} \dots s_k$. For example, $s_1 s_2 \hat{s_1} s_2 s_3 = s_1 s_2^2 s_3 = s_1 s_3$.

Theorem 10 (Deletion condition)**.** *Let $w = s_1 \dots s_k$, for s_i simple reflections (w.r.t. some simple system), such that $n(w) < k$. Then there exists $1 \leq i < j \leq k$ such that*

(1) $s_i \dots s_{j-1} = s_{i+1} \dots s_j$

(2) $w = s_1 \dots \hat{s_i} \dots \hat{s_j} \dots s_k$

For example, in A_2 with $\Delta = \{\alpha, \beta\}$, write $s_\alpha = s$, $s_\beta = t$, and let $w = stst (= ts)$. (Note that $n(w) = 2 < 4$). Now, $w = (sts)t = s(tst)$, where $(sts) = (tst)$, and $w = (s)ts(t) = ts$.

Proof of Theroem 10. Warning: We will "identify" s_i with s_{α_i} for $\alpha_i \in \Delta$, meaning nothing more than that s_i is the simple reflection wrt some root we will denote by α_i. The subscripts of α should not be interpreted as a labelling of the simple roots.

According to Proposition 8, if $w(\alpha) \in \Phi^+$ then $n(ws_\alpha) = n(w) + 1$. Hence if $s_1(\alpha_2) \in \Phi^+$ then $n(s_1 s_2) = n(s_1) + 1 = 2$. Then, if it also holds that $s_1 s_2(\alpha_3) \in \Phi^+$, then $n(s_1 s_2 s_3) = n(s_1 s_2) + 1 = 3$, and so on. If this continues to s_k, it would hold that $n(s_1 \dots s_k) = k$.

Since $n(w) < k$, this must break at some point, so that there exists a $2 \leq j \leq k$ so that

$$s_1 \dots s_{j-1}(\alpha_j) \in \Phi^-.$$

But $1(\alpha_j) \in \Phi^+$, and if $s_{j-1} \neq s_j$ then $s_{j-1}(\alpha_j) \in \Phi^+$, so $n(s_{j-1}s_j) = 2$, etc.

So, there must be an $i < j$ so that

$$s_{i+1} \dots s_{j-1}(\alpha_j) \in \Phi^+ \quad \text{and} \quad s_i s_{i+1} \dots s_{j-1}(\alpha_j) \in \Phi^-.$$

This means that s_i maps $\lambda = s_{i+1} \dots s_{j-1}(\alpha_j)$ from Φ^+ to Φ^-, hence $\lambda = \alpha_i$ (by Lemma 5). We can summarise this as $\alpha_i = w(\alpha_j)$, $w = s_{i+1} \dots s_{j-1}$ for some $1 \leq i < j \leq k$).

By Lemma 1, $ws_{\alpha_j}w^{-1} = s_{w(\alpha_j)} = s_{\alpha_i} = s_i$, hence $ws_j = s_i w$. This is result (1).

(2) is essentially equivalent to (1):

$$(s_i \dots s_{j-1})s_j = (s_{i+1} \dots s_j)s_j = s_{i+1} \dots s_{j-1},$$
$$\implies w = s_1 \dots s_k = s_1 \dots s_{i-1}(s_{i+1} \dots s_{j-1})s_{j+1} \dots s_k.$$

□

Theorem 11. *We have $l(w) = n(w)$.*

i.e. the length of w is the number of positive roots which are mapped to negative roots by w.

Proof. We already know that $n(w) \leq l(w)$. Let $w = s_1 \dots s_r$ be reduced. Assume that $n(w) < r$, then by Theorem 10, we can delete two letters from w, a contradiction. □

Note that as a consequence of Theorem 10, words with different parity cannot be equal.

2.6 Construction of $\Phi^+(w)$, Exchange Condition

Note that Theorem 10 can be restated as "Let $w = s_1 \dots s_k$ (s_i simple) such that w is not reduced. Then there exists ...".

Proposition 12. *Let $w = s_1 \dots s_r$ be reduced. Then, $\Phi^+(w) = \{s_r \dots s_{i+1}(\alpha_i) \mid 1 \leq i \leq r\}$.*

For example, in A_2 with simple roots $s = s_\alpha, t = s_\beta$,

- For $w = 1$, $\Phi^+(w) = \emptyset$,
- For $w = s$, $\Phi^+(w) = \{\alpha\}$,
- For $w = t$, $\Phi^+(w) = \{\beta\}$,
- For $w = ts = ts \cdot 1$, we have $1(\alpha) = \alpha$ and $s(\beta) = \alpha + \beta$, so $\Phi^+(w) = \{\alpha, \alpha + \beta\}$,
- For $w = st$, we have $1(\beta) = \beta$ and $t(\alpha) = \alpha + \beta$, so $\Phi^+(w) = \{\beta, \alpha + \beta\}$,
- For $w = sts$, we have $1(\alpha) = \alpha$, $s(\beta) = \alpha + \beta$, and $st(\alpha) = \beta$, so $\Phi^+(w) = \{\alpha, \alpha + \beta, \beta\}$.

Proof of Proposition 12. Since $n(w) = l(w) = r$, if we can show that

$$\Phi^+(w) \subseteq \{s_r \dots s_{i+1}(\alpha_i) \mid 1 \leq i \leq r\},$$

then we are done.

Let $\gamma \in \Phi^+(w)$, i.e. $\gamma \in \Phi^+$ and $w(\gamma) \in \Phi^-$. Hence there exists an $i \leq r$ such that

$$\lambda = s_{i+1} \dots s_r(\gamma) \in \Phi^+ \quad \text{but} \quad s_i(\lambda) = s_i \dots s_r(\gamma) \in \Phi^-.$$

Hence, $\lambda = \alpha_i = s_{i+1} \dots s_r(\gamma)$, hence $\gamma = s_r \dots s_{i+1}(\alpha_i)$. □

Theorem 13 (Exchange condition)**.** *Let $w = s_1 \dots s_k$. If $l(ws) < l(w)$ then there exists an i such that $w = s_1 \dots \hat{s}_i \dots s_k s$.*

For example, in A_2, consider the word $w = stst$. We know that $w = ts$, $l(w) = 2$. Now, $ws = ststs (= sttst = t)$, so $l(ws) = 1$. Then, $w = s\hat{t}sts = s^2ts = ts$.

Proof of Theorem 13. Let $w = s_1 \dots s_k$ where $l(ws) < l(w) = n(w)$. By Proposition 8, $w(\alpha) \in \Phi^-$ (where $s = s_\alpha$), so that there must exist an i such that $\lambda := s_{i+1} \dots s_k(\alpha) \in \Phi^+$ but $s_i(\lambda) \in \Phi^-$, so $\lambda = \alpha_i = s_{i+1} \dots s_k(\alpha)$.

Recall $w s_\alpha w^{-1} = s_{w(\alpha)}$, so

$$s_{i+1} \dots s_k \cdot s \cdot s_k \dots s_{i+1} = s_{\alpha_i} = s_i.$$

Then,

$$ws \cdot s_k \dots s_{i+1} = s_1 \dots s_i s_{i+1} \dots s_k \cdot s \cdot s_k \dots s_{i+1} = s_1 \dots s_{i-1},$$

so

$$w = s_1 \dots s_{i-1} s_{i+1} \dots s_k s.$$

□

We now state the main theorem:

Theorem 14. *Every finite reflection group W is a Coxeter group.*

The issue to consider is the following question: Is it possible to have a relation $s_1 s_2 \dots s_k = 1$ that cannot be derived from the relations $(s_\alpha s_\beta)^{m_{\alpha\beta}} = 1$?

2.7 Finite Reflection Groups are Coxeter Groups

Proof of Theorem 14. We can fix a simple system Δ such that $W = \langle s_\alpha, \alpha \in \Delta \mid \text{"relations"} \rangle$. We know that $s_\alpha s_\beta$ must have finite order, say $m_{\alpha,\beta}$, so

$$(s_\alpha s_\beta)^{m_{\alpha,\beta}} = 1. \qquad (*)$$

Since we know s_α is an involution (i.e. $s_\alpha^2 = 1$), we have $m_{\alpha,\beta} = m_{\beta,\alpha}$. We need to prove that any relation in W follows from $(*)$.

We begin by noting that any relation may be written as $s_1 \dots s_k = 1$ (indeed, if $s_1 \dots s_i = s_k \dots s_{i+1}$, then this is equivalent to $s_1 \dots s_k = 1$).

Now let (R) stand for the relation $s_1 \dots s_k = 1$ (where s_i are simple reflections). Taking the determinant on either side gives $(-1)^k = 1$, so k is even.

We will proceed by induction on k. For $k = 0$ we get a tautology. For $k = 2$ we get $s_1 s_2 = 1$, so $s_2 = s_1^{-1} = s_1$, so the only relations on two letters are simply that $s_\alpha^2 = 1$.

Now assume $k \geq 4$ and define $\kappa = \frac{k}{2}$. Then (R) can be written as

$$\underbrace{(s_1 \dots s_{\kappa+1})}_{\kappa+1 \text{ letters}} \underbrace{(s_{\kappa+2} \dots s_{2\kappa})}_{\kappa-1 \text{ letters}} = 1,$$

which is equivalent to

$$s_1 \dots s_{\kappa+1} = s_{2\kappa} \dots s_{\kappa+2}.$$

The length of the word on the right is at most $\kappa-1$ so that the word on the left is *not* reduced. Hence we can apply the deletion condition (Theorem 10), so there exists a pair of indices $1 \leq i < j \leq \kappa+1$ such that

$$s_i \dots s_{j-1} = s_{i+1} \dots s_j, \quad \text{i.e.} \quad s_i \dots s_{j-1} s_j \dots s_{i+1} = 1.$$

The word on the left has $2j - 2i$ letters and $2 \leq 2(j-i) \leq 2\kappa = k$.

Case 1: If $2j - 2i < k$, then $s_i \dots s_{j-1} s_j \dots s_{i+1} = 1$ follows from $(*)$ (by the induction hypothesis), hence (R) can be written as

$$1 = s_1 \dots s_k = s_1 \dots s_i s_{i+1} \dots s_j s_{j+1} \dots s_k = s_1 \dots s_i s_i \dots s_{j-1} s_{j+1} \dots s_k = s_1 \dots \hat{s_i} \dots \hat{s_j} \dots s_k.$$

By induction this follows from $(*)$.

Case 2: $2j - 2i = k$, i.e. $i = 1$, $j = \kappa + 1$. In this case we must have $s_1 \dots s_\kappa = s_2 \dots s_{\kappa+1}$. Write (R) as $s_2 \dots s_k s_1 = 1$ (left-multiply and right-multiply both sides by s_1) and repeat the same steps as before. Again, there are two cases to consider, Case 2^1 and 2^2, and again we are done in case 2^1 and stuck in case 2^2 for which $s_2 \dots s_{\kappa+1} = s_3 \dots s_{\kappa+2}$.

Again, repeat the procedure, now on (R) written as $s_3 \dots s_k s_1 s_2 = 1$; we are stuck in the case of 2^{2^2} for which we get $s_3 \dots s_{\kappa+2} = s_4 \dots s_{\kappa+3}$.

Continuing, we end up with the system of relations:

$$s_1 \dots s_{2\kappa} = 1 \tag{R}$$
$$s_1 \dots s_\kappa = s_2 \dots s_{\kappa+1} \tag{$1\ **$}$$
$$s_2 \dots s_{\kappa+1} = s_3 \dots s_{\kappa+2} \tag{$2\ **$}$$
$$\dots$$
$$s_i \dots s_{\kappa+i-1} = s_{i+1} \dots s_{\kappa+i} \tag{$i\ **$}$$
$$\dots$$
$$s_\kappa \dots s_{2\kappa-1} = s_{\kappa+1} \dots s_{2\kappa} \tag{$\kappa\ **$}$$

Note the relation $s_i \dots s_{\kappa+i-1} = s_{i+1} \dots s_{\kappa+i}$ arises from $s_i \dots s_{2\kappa} s_1 \dots s_{i+1} = 1$, where $s_i \dots s_{2\kappa}$ has $2\kappa - i + 1$ letters, so we must have $\kappa + 1 \leq 2\kappa - i + 1$, so $i \leq \kappa$.

The case (i) can be written as

$$s_{i+1} = s_i \dots s_{\kappa+i-1} s_{\kappa+i} \dots s_{i+2},$$

or as

$$s_{i+1} s_i \dots s_{\kappa+i-1} s_{\kappa+i} \dots s_{i+2} = 1 \tag{$\hat{\imath}$}$$

(We continue the proof in the next lecture.)

2.7.1 cont'd

For the relation $(\hat{\imath})$, we apply the same "trick" to this word, we are once again stuck if

$$s_{i+1} s_i \dots s_{i+\kappa-2} = s_i \dots s_{\kappa+i-1}$$

for all $1 \leq i \leq k$, i.e.

$$s_2 s_1 \dots s_{\kappa-1} = s_1 \dots s_\kappa$$
$$s_3 s_2 \dots s_\kappa = s_2 \dots s_{\kappa+1}$$
$$\dots$$
$$s_{\kappa+1} s_\kappa \dots s_{2\kappa-2} = s_\kappa \dots s_{2\kappa-1}$$

Then, together with $(**)$, we have

$$
\begin{aligned}
s_2 s_1 \dots s_{\kappa-1} &= s_1 \dots s_\kappa \\
s_3 s_2 \dots s_\kappa &= s_2 \dots s_{\kappa+1} \overset{(1\ **)}{=} s_1 s_2 \dots s_\kappa \quad \Longrightarrow\ s_3 = s_1 \\
s_4 s_3 \dots s_{\kappa+1} &= s_3 \dots s_{\kappa+2} \overset{(2\ **)}{=} s_2 s_3 \dots s_{\kappa+1} \quad \Longrightarrow\ s_4 = s_2 \\
&\dots \\
s_{\kappa+1} s_\kappa \dots s_{2\kappa-2} &= s_\kappa \dots s_{2\kappa-1} \overset{(\kappa-1\ **)}{=} s_{\kappa-1} s_\kappa \dots s_{2\kappa-2} \quad \Longrightarrow\ s_{\kappa+1} = s_{\kappa-1}
\end{aligned}
$$

Hence, $s := s_1 = s_3 = s_5 = \cdots = s_{2\lfloor \frac{\kappa}{2} \rfloor + 1}$, and $t := s_2 = s_4 = s_6 = \cdots = s_{2\lfloor \frac{\kappa}{2} \rfloor}$. Then, $(**)$ becomes

$$
\begin{aligned}
stst\dots &= tsts\dots \\
tsts\dots &= (stst\dots)s_{\kappa+2} \\
(stst\dots)s_{\kappa+2} &= (tsts\dots)s_{\kappa+2}s_{\kappa+3} \\
&\dots
\end{aligned}
$$

Hence $s_\kappa = s_{\kappa+2} = s_{\kappa+4} = \dots$ and $s_{\kappa+1} = s_{\kappa+3} = s_{\kappa+5} = \dots$, so

$$
\begin{aligned}
s_1 = s_3 = s_5 = \cdots &= s_{k-1} \\
s_2 = s_4 = s_6 = \cdots &= s_k
\end{aligned}
$$

So, $(**)$ and (R) reduce to

$$
st\dots st = (st)^\kappa = 1.
$$

□

3 The Classification of Irreducible Finite Reflection Groups

3.1 Irreducible Coxeter Groups

Definition: A Coxeter group is irreducible if its Coxeter graph is connected.

It is not hard to show that if the Coxeter graph G of W consists of k connected components $G_1, G_2, \ldots G_k$, and the sets $S_1, \ldots S_k$ as the corresponding sets of generators, then

$$W = W_{S_1} \times W_{S_2} \times \cdots \times W_{S_k}$$

where W_{S_i} is the irreducible Coxeter group $W_{S_i} = \langle S_i \mid (s_\alpha s_\beta)^{m_{\alpha\beta}} = 1, \quad \alpha, \beta \in S_i \rangle$ (excuse the abuse of notation).

(Note that if G_1 and G_2 are separate components, then all the generators in S_1 and S_2 commute, so the group generated by G_1 and G_2 is a direct product of two smaller groups.)

The W_{S_i} are often referred to as *parabolic subgroups.* More generally, a parabolic (subgroup) of W is a subgroup generated by a subgraph of G, or subgroups obtained by conjugating such a subgroup, wW_Sw^{-1} for $w \in W$.

Given a Coxeter system (W, S) corresponding to an irreducible finite reflection group (or irreducible Coxeter matrix M), define the symmetric matrix $C = (c_{ij})_{1 \le i,j \le r}$ where

$$c_{ij} = -2\cos\left(\frac{\pi}{m_{ij}}\right).$$

For example, with A_{n-1}, where the Coxeter graph is a path of length $n-1$ (see §1.5.3), then

$$M = \begin{pmatrix} 1 & 3 & 2 & \cdots & 2 \\ 3 & 1 & 3 & \cdots & 2 \\ 2 & 3 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 3 \\ 2 & 2 & \cdots & 3 & 1 \end{pmatrix} \implies C = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

3.2 Positive Definite Graphs

As another example of the symmetric matrix C from above, consider $I_2(m)$:

$$C = \begin{pmatrix} 2 & -2\cos\frac{\pi}{m} \\ -2\cos\frac{\pi}{m} & 2 \end{pmatrix}$$

Note that if $s_\alpha, s_\beta \in S$ are the reflections in the hyperplanes H_α, H_β in an Euclidean space V, and $||\alpha|| = ||\beta|| = \sqrt{2}$ then

$$\begin{aligned}(\alpha, \beta) &= ||\alpha|| \cdot ||\beta|| \cdot \cos\phi \\ &= 2\cos\left(\frac{m_{\alpha\beta} - 1}{m_{\alpha\beta}}\pi\right) \\ &= -2\cos\left(\frac{\pi}{m_{\alpha\beta}}\right)\end{aligned}$$

(Recall §1.3, where we showed the simple reflections in $I_2(m)$ meet at an angle of $\frac{\pi}{m}$.)

The $r \times r$ matrix C, as well as the corresponding Coxeter graph, are called *positive definite* if

$$\sum_{i,j} x_i C_{ij} x_j > 0$$

for all nonzero $x \in \mathbb{R}^r$. There are various characterisations of positive definite symmetric real matrices A:

(1) All its eigenvalues are positive (and real).

(2) All its principal minors are positive (Sylvester's criterion).

(3) $(x, y) := \sum_{i,j} x_i A_{ij} y_j$ $(x, y \in \mathbb{R}^r)$ is an inner product.

Note: if $A = (a_{ij})$ is an $r \times r$ matrix, then the k^{th} (or possibly $(r-k)^{\text{th}}$) principal minor, for $1 \leq k \leq r$, is

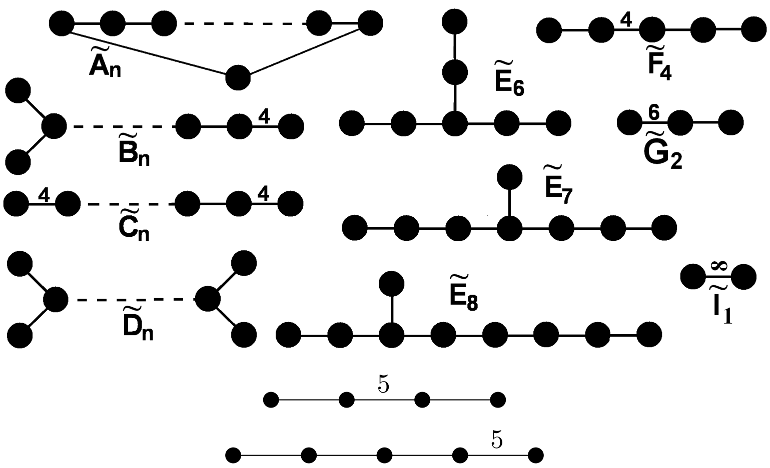
$$\det \begin{pmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kk} \end{pmatrix}.$$

This last characterisation is particularly relevant for us since we are dealing with finite reflection groups for which (3) holds by construction. It is our usual positive definite bilinear symmetric form on V. Moreover $C_{\alpha\beta} = (\alpha, \beta)$ provided we normalise all simple roots as $||\alpha|| = \sqrt{2}$.

Hence, we need to classify all positive definite real symmetric matrices. Our strategy will be to identify graphs to act as a 'boundary' on the space of positive definite graphs, intuitively replacing the condition > 0 with ≥ 0.

Before we do this, we list a collection of graphs that are not positive definite.

Lemma 15. *The following graphs are* not *positive definite.*



(Image courtesy of Wikipedia. The final two are of 'hyperbolic type'.) We will denote $\widetilde{A}_1 = \widetilde{I}_1$, $\widetilde{B}_2 = \widetilde{C}_2$. Note that $\widetilde{A}_n$, $\widetilde{B}_n$, $\widetilde{C}_n$, $\widetilde{D}_n$ contain $n+1$ vertices. These four cases must be checked for all n, the rest can be directly checked.

Proof. For all of the graphs of type $\widetilde{G}$ (affine type) we claim that $\det(C_{\widetilde{G}}) = 0$.

In each case this is an elementary exercise, and we only consider a few examples:

For $\widetilde{A}_{n-1}$, we have

$$C = \begin{pmatrix} 2 & -1 & 0 & \cdots & -1 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & & \vdots \\ \vdots & \vdots & & \ddots & -1 \\ -1 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

and since the sum of the rows is zero, so the determinant is zero. Note that for an arbitrary $n \times n$ circulant matrix, i.e. one which can be determined by shifts of its first column, with first column given by $(c_0, c_1, \ldots, c_{n-1})^t$, we have

$$\det C = \prod_{k=0}^{n-1} \sum_{j=0}^{n-1} c_j e^{\frac{2\pi i \cdot jk}{n}}.$$

For us, $c_0 = 2$, $c_1 = c_{n-1} = -1$, and other $c_i = 0$, so

$$\det C = \prod_{k=0}^{n-1} \left(2 - 2\cos\frac{2\pi k}{n}\right) = 0.$$

□

3.3 Positive Definiteness of Subgraphs

As another example of Lemma 15, consider $\widetilde{B}_n$:

$$C = \begin{pmatrix} 2 & -\sqrt{2} & & & & & \\ -\sqrt{2} & 2 & -1 & & & & \\ & -1 & 2 & \ddots & & & \\ & & \ddots & \ddots & -1 & -1 \\ & & & -1 & 2 & 0 \\ & & & -1 & 0 & 2 \end{pmatrix}$$

The following linear combination of rows is zero:

$$\sqrt{2}R_1 + 2R_2 + 2R_3 + \cdots + 2R_{n-1} + R_n + R_{n+1} = (0, 0, \ldots, 0),$$

so $\det C = 0$.

Definition: By a *subgraph* of a Coxeter graph G, we mean a graph obtained from G by successively removing vertices (and the corresponding incident edges), and/or lowering labels, (including changing $m > 2$ to $m = 2$, hence removing an edge). Note that this may end up disconnecting the graph.

Proposition 16. *Let G be a positive definite graph. Then all of its subgraphs are positive definite.*

(We don't consider the empty subgraph)

Proof. Every subgraph G' is obtained by successively removing vertices and/or lowering labels of G. Hence it suffices to consider the following two cases:

(1) G' obtained from G by removing a single vertex.

(2) G' obtained from G by lowering a single label.

In case (2), assume we lower label m_{kl} to m'_{kl}, where $2 \le m'_{kl} < m_{kl}$. Then $C_{ij} = C'_{ij}$ unless $(i,j) = (k,l)$ or $(i,j) = (l,k)$, and $C_{kl} = C_{lk} < C'_{kl} = C'_{lk}$. By contradiction, assume there exists a nonzero $x \in \mathbb{R}^r$ such that

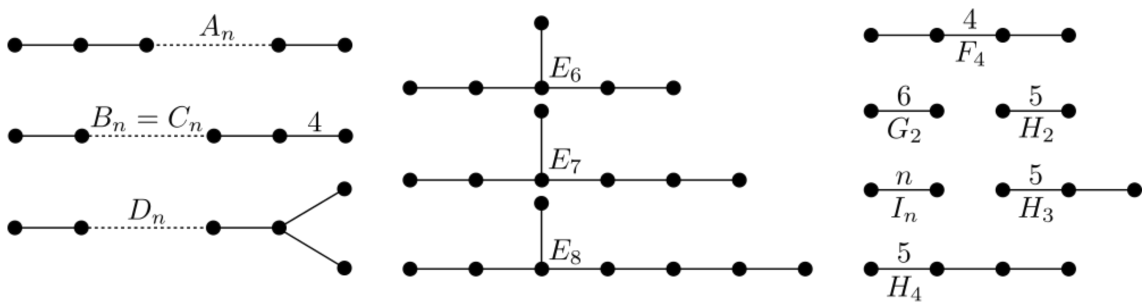
$$\sum_{i,j} x_i C'_{ij} x_j \le 0.$$

Let $y := (|x_1|, \dots, |x_r|)$. Then, by the positive definiteness of C,

$$\begin{aligned}
0 &< \sum_{i,j} y_i C_{ij} y_j \\
&= \sum_{i,j} y_i C'_{ij} y_j + \underbrace{2(\underbrace{C'_{kl} - C'_{lk}}_{<0}) y_k y_l}_{\le 0} \\
&\le \sum_{i,j} y_i C'_{ij} y_j \\
&= 2\sum_i y_i^2 + \underbrace{\sum_{i \ne j} y_i \underbrace{C'_{i,j}}_{\le 0} y_j}_{\le 0} \\
&\le 2\sum_i x_i^2 + \sum_{i \ne j} x_i C'_{i,j} x_j \\
&= \sum_{i,j} x_i C'_{i,j} x_j
\end{aligned}$$

In case (1), assume WLOG we have deleted the vertex labelled r. Then, C' follows from C by removing the r^{th} (last) row and column. Hence C' is positive definite (it is one of the principal minors of C). $\square$

Proposition 17. *The only potential Coxeter graphs corresponding to finite reflection groups are the following.*



(Image via Wikipedia. We denote $I_2(n)$ instead of I_n. Note that $I_2(3) = A_2$, $I_2(4) = B_2$, $I_2(5) = H_2$, $I_2(6) = G_2$; in particular, we need not consider H_2 or G_2. Note that A_n, B_n, D_n each contain n vertices.)

That is, we will show that any graph not in this list must have one of the graphs from Lemma 15 as a subgraph. Moreover, we will see that none of the graphs from Lemma 15 are subgraphs of these.

It will remain to show that these graphs are indeed positive definite.

Proof. Make the following observations:

- Because $\widetilde{D}_n$ is not an admissible subgraph, we can have at most one branch point (vertex of degree > 2).
- Because $\widetilde{A}_{n-1}$ is not admissible, we cannot have cycles or ∞ labels.

This leaves us with two types of graphs: linear chains, and graphs with a single branch point.

We continue the proof in the next lecture.

3.3.1 Proof cont'd

Consider the first case, where the graphs are linear chains:

- Because $\widetilde{G}_2$ is not an admissible subgraph, the only graph with a label 6 (or higher) is $I_2(6)$ (and $I_2(m)$ for $m > 6$).
- Because the two hyperbolic graphs are not admissible subgraphs, the only three graphs containing a label 5 are $I_2(5)$, H_3 or H_4.
- Because $\widetilde{C}_n$ are not admissible subgraphs, we can have at most one label 4. Because $\widetilde{F}_4$, the only graph that does not have a 4 at the end of the chain is F_4, and the graphs with label 4 at the end are B_n.
- There is only one type of linear chain without labels: A_{n-1}.

Now, consider the second case, where the graph has a single branch point.

- Because $\widetilde{B}_n$ is not an admissible subgraph, all labels must be 3 (i.e. unlabelled). Moreover, it is easy to see that we cannot rule out D_n.
- We are left with graphs that have at least two vertices on two of the branches, and at least one vertex in the remaining branch.

  Because of $\widetilde{E}_6$, one of the branches must have size 1.

  Because of $\widetilde{E}_7$, one of the branches must have size 2.

  Because of $\widetilde{E}_8$, the remaining branch has size at most 4. This gives us E_6, E_7 and E_8.

$\square$

3.4 Classification of Finite Reflection Groups

Theorem 18. *The finite reflection groups are classified by the graphs of Proposition 17.*

One can compute (inductively) the determinants of C_G corresponding to the graph G. Since the principal minors can be recognised (by a clever labelling) as corresponding to $C_{G'}$ where G' is a smaller graph in our list. (e.g. by dropping vertex n from D_n, we are left with A_{n-1}, etc.)

One finds that all determinants are positive. The actual values are:

- $\det A_{n-1} = n$
- $\det B_n = 2$
- $\det D_n = 4$
- $\det I_2(m) = 4\sin^2 \frac{\pi}{m}$
- $\det H_3 = -2 + 8\sin^2 \frac{\pi}{5}$
- $\det H_4 = -4 + 12\sin^2 \frac{\pi}{5}$
- $\det F_4 = 1$
- $\det E_6 = 3$
- $\det E_7 = 2$
- $\det E_8 = 1$

However such a proof hides much of the beauty of our collection of graphs. In particular, each graph allows its spectrum to be fully determined in closed form. We will demonstrate this for type A_{n-1}.

Recall the characterisation of positive definite matrices as those whose eigenvalues are all positive.

We claim the matrix

$$C_{A_{n-1}} = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}$$

has eigenvalues $\lambda(k) = 2 - 2\cos\frac{\pi k}{n}$ for $k = 1, 2, \ldots, n-1$ with corresponding eigenvectors

$$v(k) = \begin{pmatrix} \sin\frac{\pi k}{n} \\ \sin\frac{2\pi k}{n} \\ \vdots \\ \sin\frac{(n-1)\pi k}{n} \end{pmatrix}.$$

3.5 Structure of Finite Reflection Groups

3.5.1 Eigenvalues and Eigenvectors of $C_{A_{n-1}}$

Consider the above claim. In particular,

$$0 < \lambda(1) < \cdots < \lambda(n-1).$$

Proof. It suffices to check that $Cv(k) = \lambda(k)v(k)$, i.e.

$$\sum_{j=1}^{n-1} C_{ij} v_j(k) = \lambda(k) v_i(k).$$

If A is the adjacency matrix of A_{n-1}, namely

$$\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}$$

and $\mu(k) = 2\cos\frac{\pi k}{n} = 2 - \lambda(k)$, then the eigenvalue equation becomes

$$\sum_{j=1}^{n-1} A_{ij} v_j(k) = \mu(k) v_i(k).$$

Hence we must show that

$$\sum_{j=1}^{n-1} (\delta_{i+1,j} + \delta_{i-1,j}) \sin(j\theta) = 2\cos\theta \sin(i\theta),$$

where $\theta = \frac{\pi k}{m}$ and $\delta_{x,y}$ is the Kronecker delta function. This simplifies to

$$\sin((i+1)\theta) + \sin((i-1)\theta) = 2\cos\theta\sin(i\theta).$$

This follows from the identity $\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\cos\beta\sin\alpha$.

(Note that if $i = 1$, then $\sin((i-1)\theta) = 0$, so this is ok. Similarly, if $i = n-1$, then $\sin((i+1)\theta) = 0$, so this is also ok.) $\square$

Remark: If we ask ourselves "which graphs have largest eigenvalue less than 2?", we will find that A_{n-1}, D_n, E_6, E_7, E_8 are the only ones. If we also allow graphs with largest eigenvalue equal to two, we also get the affine versions of these graphs.

3.5.2 Construction of A_{n-1}, B_n, D_n

To complete our proof, we will explicitly realise each of the Coxeter groups in question as a finite reflection group.

- For A_{n-1}, we have already seen that we can take $V = \{v \in \mathbb{R}^n \mid v \cdot (\varepsilon_1 + \cdots + \varepsilon_n) = 0\}$, $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\}$, and $\Delta = \{\varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq n-1\} = \{\alpha_1, \dots, \alpha_n\}$. Then,

$$s_{\alpha_i}(\varepsilon_k) = \begin{cases} \varepsilon_k & \text{if } k \neq i, i+1 \\ \varepsilon_{i+1} & \text{if } k = i \\ \varepsilon_i & \text{if } k = i+1 \end{cases}$$

  This is enough to conclude that $W \cong S_n$.

  (Checking the Coxeter relations, e.g. that $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, is left as an exercise.)

- For B_n, take $V = \mathbb{R}^n$, $\Phi^+ = \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\} \cup \{\varepsilon_i \mid 1 \leq i \leq n\}$, and

$$\Delta = \{\varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq n-1\} \cup \{\varepsilon_n\} = \{\alpha_1, \dots, \alpha_{n-1}, \alpha_n\}.$$

  Then, if $1 \leq i \leq n-1$, we again have

$$s_{\alpha_i}(\varepsilon_k) = \begin{cases} \varepsilon_k & \text{if } k \neq i, i+1 \\ \varepsilon_{i+1} & \text{if } k = i \\ \varepsilon_i & \text{if } k = i+1 \end{cases}$$

  and finally,

$$s_{\alpha_n}(\varepsilon_k) = \begin{cases} \varepsilon_k & \text{if } k \neq n \\ -\varepsilon_n & \text{if } k = n \end{cases}$$

  Then, $W \cong S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$, where each copy of $\mathbb{Z}/2\mathbb{Z}$ corresponds to a change of sign. Note that conjugating a sign change with a permutation will result in a sign change, so we indeed get a semidirect product.

- For D_n, take $V = \mathbb{R}^n$, $\Phi = \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}$, and

$$\Delta = \{\varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq n-1\} \cup \{\varepsilon_{n-1}, \varepsilon_n\} = \{\alpha_1, \dots, \alpha_{n-1}, \alpha_n\}.$$

  Then, for $1 \leq i \leq n-1$, we again have

$$s_{\alpha_i}(\varepsilon_k) = \begin{cases} \varepsilon_k & \text{if } k \neq i, i+1 \\ \varepsilon_{i+1} & \text{if } k = i \\ \varepsilon_i & \text{if } k = i+1 \end{cases}$$

  Then,

$$s_{\alpha_n}(\varepsilon_k) = \begin{cases} \varepsilon_k & \text{if } k \neq n-1, n \\ -\varepsilon_n & \text{if } k = n-1 \\ -\varepsilon_{n-1} & \text{if } k = n \end{cases}$$

  So, $W \cong S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-1}$ (effectively, we have lost one sign change).

3.6 Structure of Finite Reflection Groups (II)

3.6.1 Construction of $I_2(m)$, F_4, H_4

- For $I_2(m)$, with $\theta = \frac{\pi}{m}$, we take $V = \mathbb{R}^2$, the inner product $(\cdot,\cdot)$ as before,

$$\Phi^+ = \{(\cos k\theta, \sin k\theta) \ : \ k = 0, \dots, m-1\},$$

$$\Delta = \{(1,0), (-\cos\theta, \sin\theta)\} = \{\alpha_1, \alpha_2\}.$$

Note that $(\cos k\theta, \sin k\theta) = \frac{\sin((k+1)\theta)}{\sin\theta}\alpha_1 + \frac{\sin k\theta}{\sin\theta}\alpha_2$.

Note also that $s_{\alpha_1}(\varepsilon_1) = -\varepsilon_1$, $s_{\alpha_1}(\varepsilon_2) = \varepsilon_2$. By standard trig identities, we also have $s_{\alpha_2}(\varepsilon_1) = (-\cos 2\theta, \sin 2\theta)$, $s_{\alpha_2}(\varepsilon_2) = (\sin 2\theta, \cos 2\theta)$, so we can represent

$$s_{\alpha_1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad s_{\alpha_2} = \begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

Giving the construction for the exceptional reflection groups is a tedious linear algebra exercise. We will only consider F_4 and H_4.

- For F_4, we have

$$C = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -\sqrt{2} & 0 \\ 0 & -\sqrt{2} & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Take $V = \mathbb{R}^+$,

$$\Phi^+ = \{\varepsilon_i \mid 1 \le i \le 4\} \cup \{\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le 4\} \cup \left\{\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\right\},$$

$$\Delta = \left\{\varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \varepsilon_4, \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\right\} (=: \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})$$

- For H_4, we have the symmetry group of a regular solid in 4 dimensions which has 120 'faces' given by regular dodecahedra. A regular dodecahedron (one of the 5 platonic solids) has 12 pentagonal faces. The (parabolic) subgroup H_3 (of H_4) is the symmetry group of the regular icosahedron (another platonic solid consisting of 20 triangular faces).

The root system of H_4 can again be realised in $\mathbb{R}^4$, $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, with

$$\alpha_1 = \left(\cos\theta, -\frac{1}{2}, \cos 2\theta, 0\right)$$
$$\alpha_2 = \left(-\cos\theta, -\frac{1}{2}, \cos 2\theta, 0\right)$$
$$\alpha_3 = \left(\frac{1}{2}, \cos 2\theta, -\cos\theta, 0\right)$$
$$\alpha_4 = \left(-\frac{1}{2}, -\cos\theta, 0, \cos\theta\right)$$

where $\theta = \frac{\pi}{5}$. Φ^+ has 30 roots, it would be a boring exercise to list them all.

3.6.2 Quaternions

Much more interesting is that the root system of H_4 arises as a finite reflection subgroup of the quaternions.

Hamilton wondered about higher dimensional analogues of the complex numbers. We now know there are only 4 normed division algebras over the reals, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (quaternions) and $\mathbb{O}$ (octonians).

The quaternions are associative but not commutative; the octonians are neither associative nor commutative.

The quaternions are a 4-dimensional $\mathbb{R}$-algebra (a 4-dimensional vector space over $\mathbb{R}$ equipped with a multiplication) with basis $\{1, i, j, k\}$, the usual addition and scalar multiplication, and with product determined by

$\times$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Note that this table can be reconstructed from the identity

$$i^2 = j^2 = k^2 = ijk = -1.$$

(The story goes that Hamilton's Eureka moment happened while taking a walk with his wife on the Brougham Bridge in 1843.)

For example,

$$\begin{aligned}
&(a_1 + a_2 i + a_3 j + a_4 k)(b_1 + b_2 i + b_3 j + b_4 k)\\
&= (a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4)\\
&\quad + (a_1 b_2 + a_2 b_1 + a_3 b_4 - a_4 b_3) i\\
&\quad + (a_1 b_3 - a_2 b_4 + a_3 b_1 + a_4 b_2) j\\
&\quad + (a_1 b_4 + a_2 b_3 - a_3 b_2 + a_4 b_1) k.
\end{aligned}$$

Conjugation in $\mathbb{H}$ is what you would expect: $\bar{z} = \overline{a + bi + cj + dk} = a - bi - cj - dk$. The *norm* $\in \mathbb{R}$ of $z \in \mathbb{H}$ is defined by

$$||z|| = (z\bar{z})^{\frac{1}{2}} = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

Note the norm is multiplicative, $||zw|| = ||z|| \cdot ||w||$. Division in $\mathbb{H}$ is defined as $z^{-1} = \frac{\bar{z}}{||z||^2}$. The standard inner product in $\mathbb{R}^4$ can easily be translated to $\mathbb{H}$: for $v \in \mathbb{R}^4$, where $v = (a, b, c, d) \mapsto \tilde{v} = a + bi + cj + dk$,

$$(v, w) \mapsto \frac{1}{2}\left(\tilde{v}\overline{\tilde{w}} + \tilde{w}\overline{\tilde{v}}\right)$$

(check for yourself). Clearly, $||v||_{\mathbb{R}^4} = ||\tilde{v}||_{\mathbb{H}}$.

Claim: Fix $\alpha \in \mathbb{R}^4$ such that $||\alpha|| = 1$ and let s_α be the reflection in H_α. Then, $s_\alpha(v) \mapsto -\tilde{\alpha}\tilde{v}\tilde{\alpha}$.

Proof.

$$\begin{aligned}
s_\alpha(v) = v - 2(v, \alpha)\alpha \mapsto \tilde{v} &- (\tilde{v}\overline{\tilde{w}} + \tilde{\alpha}\overline{\tilde{v}})\tilde{\alpha}\\
&= \tilde{v} - \tilde{v}\overline{\tilde{\alpha}}\tilde{\alpha} - \tilde{\alpha}\overline{\tilde{v}}\tilde{\alpha}\\
&= \tilde{v} - \tilde{v} - \tilde{\alpha}\overline{\tilde{v}}\tilde{\alpha}\\
&= -\tilde{\alpha}\overline{\tilde{v}}\tilde{\alpha}.
\end{aligned}$$

□

3.7 Quaternion Groups & Lattices

3.7.1 Every Even Quaternion Group is a Root System

Claim: Let G be a group of even order. Then G contains an involution.

Proof. Elements of order greater than 2 come in pairs, since $|g| = |g^{-1}|$ and $g \neq g^{-1}$. Since the identity is the unique element of order 1, there must be an odd number of involutions. □

The surprising fact is that any subgroup G of $\mathbb{H}$ of even order yields a root system.

To see this, let G be an even order (hence finite) subgroup of $\mathbb{H}$. For $z \in G$, $z^r = 1$ for some finite r. Hence $||z||^r = 1$. But since $||z|| \in \mathbb{R}^+$, we must have $||z|| = 1$. In other words, all elements of G have norm 1. Since $z^{-1} = \frac{\bar{z}}{||z||^2} = \bar{z}$, G is closed under conjugation.

Since G has even order, it must contain an element of order 2. But $z^2 = (a^2 - b^2 - c^2 - d^2) + 2a(bi + cj + dk) = 1$ implies

$$\begin{cases} a^2 - b^2 - c^2 - d^2 = 1 \\ 2ab = 0 \\ 2ac = 0 \\ 2ad = 0, \end{cases}$$

hence $a^2 = 1$ and $z = \pm 1$. Hence there is a unique element of order 2, namely $z = -1$. But, $(-1)z = z(-1) = -z$, so that if G contains z it must also contain $-z$.

If $\alpha \in G$ and $s_\alpha(z) = -\alpha\bar{z}\alpha$ for $z \in G$ then $s_\alpha(z) \in G$, i.e. $s_\alpha(G) = G$. Hence, G satisfies the definition of a root system:

- If $z \in G$ then $-z \in G$, but no other multiples of z are in G (since $||z|| = 1$ for all $z \in G$)
- $s_\alpha(G) = G$ for all $\alpha \in G$.

□

Identifying $a + bi + cj + dk \in \mathbb{H}$ with $(a, b, c, d) \in \mathbb{R}^4$, then the previous result will give us a root system in $\mathbb{R}^4$. In the case of H_4,

$$\begin{aligned} \Delta &= \{\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4\} \\ &= \left\{\cos\theta - \frac{1}{2}i + \cos 2\theta j, -\cos\theta + \frac{1}{2}i + \cos 2\theta j, \frac{1}{2} + \cos\theta i - \cos\theta j, -\frac{1}{2} - \cos\theta i + \cos\theta k\right\} \end{aligned}$$

3.7.2 Crystallographic Root Systems & Lattices

We finally turn to the classification of *crystallographic* root systems, for which $(\alpha, \beta^\vee) \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$. In this case, the corresponding Coxeter groups are also known as Weyl groups.

The definition of crystallographic root systems is perhaps not very insightful and certainly doesn't explain its name.

Definition: Let V be an $\mathbb{F}$-vector space, $B = \{v_1, \ldots, v_r\}$ be a basis of V, and $R \subseteq F$ be a ring. Then,

$$\left\{\sum_{i=1}^{r} \lambda_i v_i \ : \ \lambda_i \in R, v_i \in B\right\}$$

is called an R-lattice generated by B.

For example, take $V=\mathbb{R}^2$, $\mathbb{F}=\mathbb{R}$, $R=\mathbb{Z}\subseteq\mathbb{F}$. Then,

- $B=\{\varepsilon_1,\varepsilon_2\}$ gives the square lattice;
- $B=\{\varepsilon_1,a\varepsilon_2\}$ (for fixed $a\neq 0$) gives a rectangular lattice;
- $B=\left\{\varepsilon_1,\frac{1}{2}(\varepsilon_1+\sqrt{3}\varepsilon_2)\right\}$ gives the hexagonal lattice;
- $B=\left\{\varepsilon_1,\frac{1}{2}(\varepsilon_1+a\varepsilon_2)\right\}$ gives a rhombic lattice; and
- $B=\{\varepsilon_1,a\varepsilon_1+b\varepsilon_2\}$ gives a parallelogramatic lattice.

Taking $V=\mathbb{R}^3$, $\mathbb{F}=\mathbb{R}$, $R=\mathbb{Z}$, we get 14 Bravais lattices.

The *covolume* $d(\Lambda)$ of a lattice Λ is the volume of its fundamental region, given by the volume of an r-dimensional parallelepiped with vertices $0,v_1,\dots,v_r$, where

$$d(\Lambda)=|\det(v_1,\dots v_r)|.$$

A lattice is said to be *unimodular* if it has covolume 1. The most famous unimodular lattice is the 24-dimensional Leech lattice related to the Golay code, sphere packings (if you place 24-dimensional unit balls on each of the points of the Leech lattice, then no balls overlap, and each ball kisses 196560 neighbours), related to Ramanujan's τ function, the Monster, and so on.

If V is a Euclidean space, we can define the dual Λ^* of the lattice Λ as $\Lambda^*=\{\beta\in V:(\alpha,\beta)\in\mathbb{Z}),\alpha\in\Lambda\}$. For example, $\Lambda=\mathbb{Z}\epsilon_1+\mathbb{Z}\epsilon_2=\Lambda^*$, and $\Lambda=\sqrt{2}\left(\mathbb{Z}\varepsilon_1+\mathbb{Z}\left(\frac{1}{2}\varepsilon_1+\sqrt{3}\varepsilon_2\right)\right)=\Lambda^*$.

3.8 Classification of Crystallographic Root Systems

If $G<GL(V)$ such that $g(\Lambda)=\Lambda$ for all $g\in G$ for Λ some lattice $\Lambda\subset V$, then G is said to be crystallographic.

Lemma 19. *Let W be a crystallographic finite reflection group. Then its labels must be in $\{2,3,4,6\}$.*

Proof. Let $\Lambda=\sum_{i=1}^r\mathbb{Z}b_i$. Then the matrix $A=(a_{ij})$ of g wrt $\{b_1,\dots,b_r\}$ is an integer matrix, since $g(b_i)=\sum_{j=1}^r a_{ij}b_j\in\Lambda$. Hence $\mathrm{tr}(A)\in\mathbb{Z}$. Since the trace of a linear operator is an invariant[1], then $\mathrm{tr}(g)\in\mathbb{Z}$.

Now choose a different basis of V. Let $P_{\alpha,\beta}=\mathbb{Z}\alpha+\mathbb{Z}\beta$ for $\alpha,\beta\in\Delta,(\alpha\neq\beta)$, and $B^\perp_{\alpha,\beta}$ a basis of $P^\perp_{\alpha,\beta}$ (the orthogonal space to $P_{\alpha,\beta}$), and $B=\{\alpha,\beta\}\cup B^\perp_{\alpha,\beta}$ (i.e. if $b\in B^\perp_{\alpha,\beta}$, $(b,\alpha)=(b,\beta)=0$).

$s_\alpha s_\beta$ acts as a rotation through $2\theta=\frac{2\pi}{m_{\alpha\beta}}$ and fixes, pointwise $P^\perp_{\alpha,\beta}$. With respect to our new basis B, g has matrix representation

$$\begin{pmatrix}\cos(2\theta) & -\sin(2\theta) & & & & \\ \sin(2\theta) & \cos(2\theta) & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1\end{pmatrix}$$

Hence $\mathrm{tr}(g)=2\cos 2\theta+(r-2)\in\mathbb{Z}$, so $2\cos 2\theta\in\mathbb{Z}$, so $m_{\alpha\beta}\in\{2,3,4,6\}$. $\square$

[1] does not depend on the choice of basis; a change of basis corresponds to a similarity transformation leaving the trace invariant – $\mathrm{tr}(SAS^{-1})=\mathrm{tr}(S)\mathrm{tr}(A)\mathrm{tr}(S^{-1})=\mathrm{tr}(ASS^{-1})=\mathrm{tr}(A)$

Corollary 20. *The finite reflection groups H_3, H_4 and $I_2(m)$, unless $m \in \{3,4,6\}$, are not crystallographic.*

(Note that $I_2(3) = A_2$, $I_2(4) = B_2$. $I_2(6)$ is also called G_2.) The crystallographic condition $(\alpha, \beta^\vee) \in \mathbb{Z}$ now makes perfect sense.

From $(\alpha, \beta^\vee) \in \mathbb{Z}$, it follows that $(\alpha, \beta^\vee)(\alpha^\vee, \beta) \in \mathbb{Z}$, so $4\cos^2\theta \in \mathbb{Z}$, therefore

$$\theta \in \left\{ \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6} \right\}$$

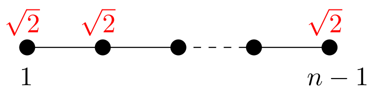
(unless $\alpha = \pm\beta$, in which case $\theta \in \{0, \pi\}$).

Since $\theta = \pi - \frac{\pi}{m_{\alpha\beta}}$, we have $m_{\alpha\beta} \in \{2,3,4,6\}$. Moreover, since $s_\alpha(\beta) = \beta - (\alpha^\vee, \beta)\alpha \in \beta + \mathbb{Z}\alpha$. Therefore, $\beta = \sum_{\alpha\in\Delta} n_\alpha \alpha$, for $n_\alpha \in \mathbb{Z}$. Therefore, $\Lambda = \sum_{i=1}^r \mathbb{Z}\alpha_i$ is fixed by W.

However, not just the angles are fixed, but also the relative length of the roots: $(\alpha, \beta^\vee) = 2\frac{||\alpha||}{||\beta||}\cos\theta \in \mathbb{Z}$, so adjacent roots have relative lengths $1, \sqrt{2}, \sqrt{3}$ (corresponding to $m = 3,4,6$).

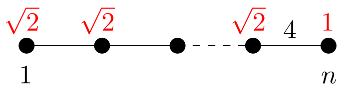
Revisiting our previous description of root systems, we get the following: (note that we also annotate the vertices of the graph with the length of the root, in red)

- A_{n-1},



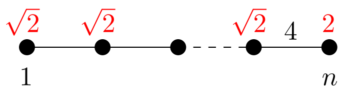
$$\Lambda = \sum_{i=1}^{n-1} \mathbb{Z}(\varepsilon_i - \varepsilon_{i+1}).$$

- B_n,



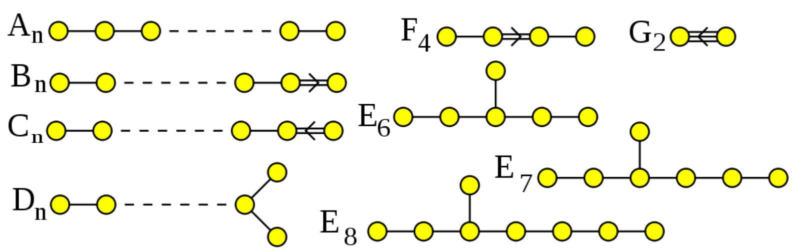
$$\Lambda = \sum_{i=1}^{n-1} \mathbb{Z}(\varepsilon_i - \varepsilon_{i+1}) + \mathbb{Z}\varepsilon_n.$$

- C_n,



$$\Lambda = \sum_{i=1}^{n-1} \mathbb{Z}(\varepsilon_i - \varepsilon_{i+1}) + 2\mathbb{Z}\varepsilon_n.$$

Usually, we do not write the length of the roots in the diagram, instead B_n and C_n are distinguished using a Dynkin diagram (image via Wikipedia):



A double-edge pointing from α to β indicates that $\frac{||\alpha||}{||\beta||} = \sqrt{2}$. A triple-edge pointing from α to β indicates that $\frac{||\alpha||}{||\beta||} = \sqrt{3}$ (i.e. the arrow points from the larger root to the smaller root). A single edge indicates that $||\alpha|| = ||\beta||$.

We also have,

- D_n,
$$\Lambda = \sum_{i=1}^{n-1} \mathbb{Z}(\varepsilon_i - \varepsilon_{i-1}) + \mathbb{Z}(\varepsilon_{n-1} + \varepsilon_n).$$

- F_4,
$$\Lambda = \mathbb{Z}(\varepsilon_2 - \varepsilon_3) + \mathbb{Z}(\varepsilon_3 - \varepsilon_4) + \mathbb{Z}\varepsilon_4 + \frac{1}{2}\mathbb{Z}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4).$$

- $G_2 = I_2(6)$,
$$\Lambda = \mathbb{Z}(\varepsilon_1 - \varepsilon_2) + \mathbb{Z}(-2\varepsilon_1 + \varepsilon_2 + \varepsilon_3).$$

A Appendix

A.1 About

It is the author's intention that these notes, and the LaTeX source code, be publicly released and made available for future reference, so this section records notes about these notes.

At time of writing, this document can be found at `http://jgat.github.io/math4301-notes/lectures.pdf`, and its source code can be found at `https://github.com/jgat/math4301-notes`.

The 2013 offering of MATH4301 Advanced Algebra was divided into two halves, the first half on Galois Theory, taught by Victor Scharaschkin, and the second half on Coxeter Groups, taught by Ole Warnaar. Each half of the course was accompanied by an assignment worth 20% of the grade, and an exam worth 30% of the grade.

This document contains detailed lecture notes taken during the second half of the course. The notes correspond closely to what was written and said in lectures. Each section within this document corresponds to notes taken in a single lecture.

A.2 Version History

Below we describe briefly the version history of this document, based on versions which have been published at `http://jgat.github.io/math4301-notes/lectures.pdf`. Version numbers will roughly follow the format v0.n.m, where n indicates the number of lectures which notes have been written for, and m indicates the number of minor revisions. Once notes have been taken for all lectures, the version number will increment to v1.0.

v0.3.0: Initial publication with lectures 1 to 3, §1.1-1.3.

v0.4.0: Lecture 4 placeholder.

v0.7.0: Lectures 5-7. (Still haven't drawn some pictures).

v0.8.0: Lecture 8.

v0.8.1: Touching up some of lecture 5.

v0.11.0: Through to Lecture 11.

v0.11.1: Formatting, and a basic diagram of A_2 from lecture 5.

v0.12.0: Lecture 12 (First part of proof of Theorem 14).

v0.12.1: Links to Lecture 4 pictures.

v1.0: Completed version of all lectures.