MATH1061 — Week 3 Tutorial (T9)

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(v 1.1, 8 August 2013)

For each of these claims, state the negation, then prove either the original claim or its negation.

Example: "The product of any two odd integers is odd":

Restated: $\forall n, m \in \mathbb{Z}, (n \text{ is odd} \wedge m \text{ is odd}) \rightarrow nm$ is odd.
Negation: $\exists n, m \in \mathbb{Z} :\sim (n \text{ is odd} \wedge m \text{ is odd}) \wedge nm$ is even.

Proof. Let n and m be odd integers. Then, $\exists a, b \in \mathbb{Z}$ such that $n = 2a + 1$ and $m = 2b + 1$. Then,

$$nm = (2a + 1)(2b + 1) = 4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1,$$

which is odd (since $2ab + a + b \in \mathbb{Z}$). $\square$

Exercise:

1. The sum of any two irrational numbers is irrational.

2. There is a prime number n such that $2^n - 1$ is composite.

3. $\forall r, s \in \mathbb{Q}$, if $r < s$, then $\exists t \in \mathbb{Q}$ such that $r < t < s$.

4. $\forall a, b, c \in \mathbb{Z}$, if $a \mid c$ and $b \mid c$, then $(a + b) \mid c$.

5. $\forall x \in \mathbb{R}$, $\lfloor x \rfloor \leq \lceil x \rceil$.

Hints:

1. The claim is true — Give an example.

2. The claim is true — Substitute $d = 2$ into the Quotient-Remainder Theorem (p. 58 workbook); we want to show that every integer satisfies the definition of 'even' or 'odd' (p. 43 workbook).

3. The claim is false — Find a counter-example.

4. The claim is true — The proof will start with something like "Let $r, s \in \mathbb{Q}$ such that $r < s$, then $r = \frac{a}{b}, s = \frac{c}{d}$ where a, b, c, d are ...". Then use r and s to make a new rational number (call it t), and show that $r < t < s$ and t is rational.

5. The claim is false — Find a counter-example.

6. The claim is false — We need to prove that for any integer n, that $n^2 \not\equiv 2 \pmod 3$. We prove this by division into cases, using the fact that for an integer n, either $n \equiv 0$, or $n \equiv 1$, or $n \equiv 2 \pmod 3$. For each of these three cases, work out what n^2 is congruent to (mod 3).

7. The claim is true — Apply the definitions of $\lfloor x \rfloor$ and $\lceil x \rceil$ (p. 64 workbook), i.e. "$\lfloor x \rfloor$ is an integer n such that ..., and $\lceil x \rceil$ is an integer m such that .... Then, ...". Don't use the same letter for $\lfloor x \rfloor$ and $\lceil x \rceil$!

Solutions:

1. *False* — Counter-example: $\sqrt{2}$ and $-\sqrt{2}$ are irrational, but $\sqrt{2} + (-\sqrt{2}) = 0$ is rational.

2. *True* — Example: For $n = 11$, $2^{11} - 1 = 2047 = 23 \cdot 89$.

3. *True* — Let $r, s \in \mathbb{Q}$ such that $r < s$, then $r = \frac{a}{b}, s = \frac{c}{d}$ for some $a, b, c, d \in \mathbb{Z}$ where $b, d \neq 0$. Then, let $t = \frac{r+s}{2} = \frac{ad+bc}{2bd}$. Now, since $r < s$, then $\frac{r+r}{2} < \frac{r+s}{2} < \frac{s+s}{2}$, so $r < t < s$. Also, $ad + bc, 2bd \in \mathbb{Z}$, and $2bd \neq 0$, so t is rational.

4. *False* — Counter-example: $a = 2, b = 3, c = 6$. Observe that $2 \mid 6$ and $3 \mid 6$, but $(2 + 3) = 5 \nmid 6$.

5. *True* — Let $n = \lfloor x \rfloor$ and $m = \lceil x \rceil$, then $n \leq x < n+1$ and $m-1 < x \leq m$. Therefore, $n \leq x \leq m$, so $n \leq m$. Substituting in the definitions of n and m, $\lfloor x \rfloor \leq \lceil x \rceil$.