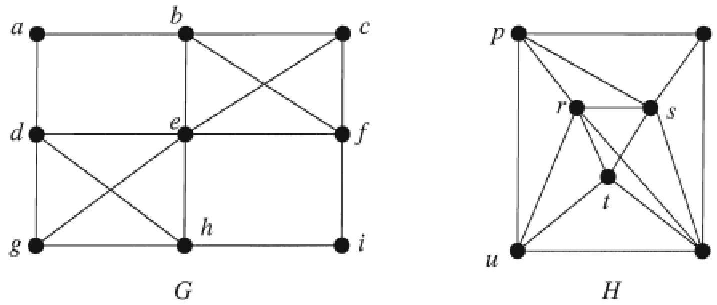
MATH1061/7861 — Discrete Mathematics
Selection from 2011 Semester 1 Exam — Week 11 Tutorial (T09)

These questions comprise about one-quarter of the available marks in the exam; hence you should be able to complete these questions within approximately 30 minutes.

Q4. (a) Let $A = \{a, \emptyset\}$, $B = \{\emptyset\}$, $C = \{A, B, a\}$.
Complete the following (and remember curly braces where appropriate).
(i) $B - A = \ldots$
(ii) $A \cup C = \ldots$
(iii) $\mathcal{P}(A) = \ldots$
(iv) $\mathcal{P}(A) \cap B = \ldots$
(v) Is it true that $B \subseteq C$?

Q11. (a) For each of the following graphs G and H, write down either an Euler circuit or an Euler path, if either of these exist. If neither exists, explain briefly why.



(c) A tree has 11 vertices. Exactly four of the vertices are leaves, and six of the vertices have the same degree. Find the degree of all the vertices. Show your working.[1]

Q12. Let $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ be defined by

$$f(x) = \left\lfloor \frac{x+1}{2} \right\rfloor, \quad g(x) = 2x + 1.$$

Verify your answers below.

(a) What is the range of f?
(b) Is f one-to-one?
(c) Is f onto $\mathbb{Z}$?
(d) What is the range of g?
(e) Calculate $(g \circ f)(-4)$ and $(f \circ g)(-4)$.
(f) Calculate $(g \circ f)(x)$ and $(f \circ g)(x)$, and then show that if x is any *odd* integer then $(f \circ g)(x) = (g \circ f)(x) - 1$.

Q13. (c) A binary relation α is defined on the positive integers by: $x \,\alpha\, y$ if and only if $x + 3y$ is even. Is α reflexive? symmetric? transitive? Justify your answers.[2]
Is α an equivalence relation? If so, give the equivalence classes; if not, explain why not.

[1] The original exam gave a different wording for this question.
[2] The original exam question only asked to simply state whether or not α was reflexive, symmetric, and transitive. However, you should still know how to justify your response.

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Q4. (a) (i) $B - A = \{\} = \emptyset$.

(ii) $A \cup C = \{a, \emptyset, A, B\}$.

(iii) $\mathcal{P}(A) = \{\emptyset, \{a\}, \{\emptyset\}, \{a, \emptyset\}\}$.

(iv) $\mathcal{P}(A) \cap B = \{\emptyset\}$.

(v) Is it true that $B \subseteq C$? *No.*

Q11. (a) An Euler path for G is $c, b, a, d, e, b, f, c, e, f, i, h, d, g, e, h, g$.[3]
H does not have an Euler circuit or an Euler path, since it has more than two vertices of odd degree (namely q, r, s, v).

(c) A tree on 11 leaves has 10 edges, hence total degree 20 (by the Handshake Theorem). Precisely four vertices have degree 1, six share the same degree, call it a, then there is one more vertex, let the degree of that vertex be b. Then, $a, b \geq 2$, and

$$20 = 4 \cdot 1 + 6a + b \implies 16 = 6a + b.$$

In particular, $6a < 16$, so $a < \frac{16}{6} < 3$, and since $a \geq 2$, we have $a = 2$. Hence $b = 16 - 6a = 4$. Thus there are four vertices of degree 1, six of degree 2, and one of degree 4.

Q12. (a) The range of f is $\mathbb{Z}$: for any $y \in \mathbb{Z}$, we have $f(2y) = \left\lfloor y + \frac{1}{2} \right\rfloor = y$, so every $y \in \mathbb{Z}$ is the image of some $x \in \mathbb{Z}$ (where $x = 2y$).

(b) f is not one-to-one: for example, $f(1) = \lfloor 1 \rfloor = 1$ and $f(2) = \left\lfloor \frac{3}{2} \right\rfloor = 1 = f(1)$.

(c) f is onto, since the range is equal to the codomain (by part (a) above).

(d) The range of g is the set of all odd numbers (by definition of an odd number).

(e) $(g \circ f)(-4) = g\left(\left\lfloor \frac{-3}{2} \right\rfloor\right) = g(-2) = -3, \quad (f \circ g)(-4) = f(-7) = \lfloor -3 \rfloor = -3.$

(f) $(g \circ f)(x) = 2\left\lfloor \frac{x+1}{2} \right\rfloor + 1$, and $(f \circ g)(x) = \left\lfloor \frac{(2x+1)+1}{2} \right\rfloor = \lfloor x + 1 \rfloor = x + 1$. Now, suppose x is odd. To show that $(f \circ g)(x) = (g \circ f)(x) - 1$, we need to show that $(g \circ f)(x) = x + 2$.
If x is odd, then $x = 2k + 1$ for some $k \in \mathbb{Z}$, therefore

$$(g \circ f)(x) = 2\left\lfloor \frac{2k+2}{2} \right\rfloor + 1 = 2(k+1) + 1 = 2k + 3 = x + 2.$$

Alternatively: if x is odd, then $x = g(k)$ for some $k \in \mathbb{Z}$, so

$$(g \circ f)(x) = ((g \circ f) \circ g)(k) = (g \circ (f \circ g))(k) = g(k+1) = 2k + 3 = x + 2.$$

Q13. (c)
- α is reflexive: for all $x \in \mathbb{Z}$, $x + 3x = 4x$ is even, so $x \,\alpha\, x$.
- α is symmetric: for all $x, y \in \mathbb{Z}$, if $x \,\alpha\, y$, then $x + 3y$ is even, hence $y + 3x = (x + 3y) + 2(x - y)$ is the sum of two even numbers, so is even.
- α is transitive: for all $x, y, z \in \mathbb{Z}$, if $x \,\alpha\, y$ and $y \,\alpha\, z$, then $x + 3y$, $y + 3z$ are even, and so $x + 3z = (x + 3y) + (y + 3z) - 4y$ is the sum of even numbers, hence is even, so $x \,\alpha\, z$.

Therefore, α is an equivalence relation. Note that $x \,\alpha\, y$ iff x and y have the same parity (i.e. both even or both odd), hence the equivalence classes are $[0] = \{\ldots, -2, 0, 2, 4, \ldots\}$ and $[1] = \{\ldots, -3, -1, 1, 3, \ldots\}$.

[3] Many Euler paths exist, however they must start and finish at c and g (or vice versa).