MATH1061 — Week 9 Tutorial (T5)

30 April 2013

1. Do graphs with the following descriptions exist? If so, draw one, if not, explain why.
   (a) A graph on 4 vertices with vertices of degree 1, 2, 3, 3.
   (b) A graph on 5 vertices with vertices of degree 1, 2, 3, 3, 5.
   (c) A simple graph on 6 vertices, all with degree 3.
   (d) A tree on 5 vertices with no leaves.
   (e) A connected graph with 8 vertices and 6 edges.
   (f) A connected graph with 10 vertices, 9 edges, and at least one cycle.

2. A group of 4 men and 3 women play a tennis tournament in the following way: two people (call them A and B) play a match, then B plays with someone else who is not A (call them C), then C plays with someone who is not B, then that person plays with someone who is not C, and so on.
   (a) The group would like to play in a way that everyone plays against everyone else exactly once. Is it possible to do this?
   (b) The group would like to play in a way that all the men play against each woman exactly once, all the women play against each man exactly once, and no two people of the same gender play against each other. Is it possible to do this?
   (c) The group would like to play in a way that everyone plays exactly twice, and they only play with someone of the opposite gender. Is it possible to do this?
   (d) Suppose now there are n men and m women (where n and m are positive integers). For each of the three situations above, what are all the values of n and m which make the arrangement possible?

3. Challenge: Let G be a graph and let A be its adjacency matrix. Make a conjecture about how the entry in row i column j of A^2 relates to the vertices v_i and v_j of the graph. What about the entries of the matrix A^n?

   (Remark: Computers are efficient at doing matrix calculations, so if we want to write a computer program that stores and analyses graphs, we can store the adjacency matrix of the graph in the computer).

Solutions:

1. (a) Not possible: total degree is 9, which is odd.

   (b) Possible, but not with simple graphs: An example would be to have a vertex of degree 5 with a loop, and a vertex of degree 3 with a loop.

   (c) Possible: vertex set $\{1, 2, 3, 4, 5, 6\}$ and edge set $\{12, 23, 34, 45, 56, 61, 14, 26, 35\}$.

   (d) Not possible: Each vertex would have degree at least 2, for a total degree of at least 10, but a tree on 5 vertices needs to have total degree 8.

   (e) Not possible: Using 6 edges, we can only connect at most 7 vertices.

   (f) Not possible: Using 9 edges and 10 vertices, we can only construct a tree, but no cycles. We could construct a cycle of length n and a tree of order $10 - n$, but they would not be connected.

2. (a) Possible: The problem reduces to finding an Euler path or Euler circuit of K_7. Each vertex in K_7 has degree 6, so they're all even.

   (b) Not possible: The problem reduces to finding an Euler path or Euler circuit of $K_{4,3}$. The vertices in the order-4 partite set have degree 3.

   (c) Not possible: The problem reduces to finding a Hamiltonian circuit of $K_{4,3}$, and we can check that $K_{4,3}$ has no Hamiltonian circuit.

   (d) a. is possible iff $n + m$ is odd.

   b. is possible iff n and m are both even (which gives an Eulerian circuit), or $m = 2$ and n is odd, or $n = 2$ and m is odd (which gives an Eulerian path).

   c. is possible iff $n = m$.

3. The entry in row i column j of the matrix A^n represents the number of walks of length n from v_i to v_j.