MATH1061 — Week 5 Tutorial (T5)

26 March 2013

1. Prove that there is no integer which is both even and odd.
2. Consider the statement "$\forall n \in \mathbb{Z}, k \in \mathbb{Z}^+$, if $5 \nmid n^k$, then $5 \nmid n$."
   (a) Prove this statement using a proof by contraposition.
   (b) Prove this statement using a proof by contradiction.
3. For all positive integers a, b, prove that $a \mid b$ if and only if $\gcd(a, b) = a$. (Note that to prove "X if and only if Y", you need to prove both "if X, then Y" and "if Y, then X"; because $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$.)
4. Find all possible values of $196x + 158y$ for integers x and y.
5. Find a solution to the equation $64x + 28y = 12$ where x, y are integers. What is another solution?

Solutions:

1. Proof by contradiction: suppose that there is an integer n which is both even and odd. Then, $n = 2a = 2b + 1$ for some integers a and b. Then, $a = b + \frac{1}{2}$, which is not an integer, a contradiction. So, our assumption is incorrect and the original claim is true.

2. (a) Let n be an integer and k be a positive integer. Suppose $5 \mid n$, i.e. $n = 5x$ for some $x \in \mathbb{Z}$. Then, $n^k = 5^k x^k = 5(5^{k-1}x^k)$, and since $5^{k-1}x^k \in \mathbb{Z}$, $5 \mid n^k$.
   (b) Suppose there is an integer n and k with $5 \nmid n^k$ and $5 \mid n$. Then, $n = 5x$ for $x \in \mathbb{Z}$, so $n^k = 5(5^{k-1}x^k)$, and since $5^{k-1}x^k \in \mathbb{Z}$, $5 \mid n^k$, which contradicts the fact that $5 \nmid n^k$, so our assumption is incorrect, and the original claim is true.

3. Let a, b be positive integers. First, we prove that "if $a \mid b$, then $\gcd(a, b) = a$": Suppose $a \mid b$. Then, since $a \mid a$, a is a common divisor of a and b. If c is a common divisor of a and b, then $c \mid a$, which means that $c \leq a$, so a is the greatest common divisor, of a and b.
   Next, we prove that "if $\gcd(a, b) = a$, then $a \mid b$": Suppose that $\gcd(a, b) = a$. Then, a is a common divisor of a and b; in particular, $a \mid b$.

4. Using the theorem on page 81 of the workbook, we see that all possible values of $196x + 158y$ are the integers c such that $\gcd(196, 158) \mid c$. By the Euclidean algorithm, $\gcd(196, 158) = \gcd(158, 38) = \gcd(38, 6) = \gcd(6, 2) = \gcd(2, 0) = 2$, so $196x + 158y = c$ if and only if $2 \mid c$. So, the even integers are all the possible values of $196x + 158y$.

5. By the Extended Euclidean algorithm, we find that $\gcd(64, 28) = 4$, and $4 = 64 \cdot -3 + 28 \cdot 7$. Multiplying by 3, we have $12 = 64 \cdot -9 + 28 \cdot 7$ (*), so one solution is $(x, y) = (-9, 7)$. Since $0 = 64 \cdot 28 + 28 \cdot -64$, we can add this on to (*) to obtain $12 = 64 \cdot 19 + 28 \cdot -43$, so another solution is $(x, y) = (19, -43)$.