MATH1061 — Week 4 Tutorial (T5)

19 March 2013

Prove or disprove the following claims:

1. There is an integer n such that $n > 3$ and $2^n - 1$ is prime.
2. Every integer is either even or odd.
3. The sum of any two irrational numbers is irrational.
4. $\forall r, s \in \mathbb{Q}$, if $r < s$, then $\exists t \in \mathbb{Q}$ such that $r < t < s$.
5. $\forall a, b, c \in \mathbb{Z}$, if $a \mid c$ and $b \mid c$, then $(a + b) \mid c$.
6. There is an integer n such that $n^2 \equiv 2 \pmod 3$.
7. $\forall x \in \mathbb{R}$, $\lfloor x \rfloor \leq \lceil x \rceil$.

Hints:

1. The claim is true — Give an example.

2. The claim is true — Substitute $d = 2$ into the Quotient-Remainder Theorem (p. 58 workbook); we want to show that every integer satisfies the definition of 'even' or 'odd' (p. 43 workbook).

3. The claim is false — Find a counter-example.

4. The claim is true — The proof will start with something like "Let $r, s \in \mathbb{Q}$ such that $r < s$, then $r = \frac{a}{b}, s = \frac{c}{d}$ where a, b, c, d are …". Then use r and s to make a new rational number (call it t), and show that $r < t < s$ and t is rational.

5. The claim is false — Find a counter-example.

6. The claim is false — We need to prove that for any integer n, that $n^2 \not\equiv 2 \pmod 3$. We prove this by division into cases, using the fact that for an integer n, either $n \equiv 0$, or $n \equiv 1$, or $n \equiv 2 \pmod 3$. For each of these three cases, work out what n^2 is congruent to (mod 3).

7. The claim is true — Apply the definitions of $\lfloor x \rfloor$ and $\lceil x \rceil$ (p. 64 workbook), i.e. "$\lfloor x \rfloor$ is an integer n such that …, and $\lceil x \rceil$ is an integer m such that …. Then, …". Don't use the same letter for $\lfloor x \rfloor$ and $\lceil x \rceil$!

Solutions:

1. *True* — Example: For $n = 5$, $2^5 - 1 = 31$, which is prime.

2. *True* — For any integer n, by the Quotient-Remainder Theorem with $d = 2$, $n = 2q + r$ for some integer r with $0 \leq r < 2$. The only possibilities are $r = 0$, so $n = 2q$ which is even, or $r = 1$, so $n = 2q + 1$ which is odd.

3. *False* — Counter-example: $\sqrt{2}$ and $-\sqrt{2}$ are irrational, but $\sqrt{2} + (-\sqrt{2}) = 0$ is rational.

4. *True* — Let $r, s \in \mathbb{Q}$ such that $r < s$, then $r = \frac{a}{b}, s = \frac{c}{d}$ for some $a, b, c, d \in \mathbb{Z}$ where $b, d \neq 0$. Then, let $t = \frac{r+s}{2} = \frac{ad+bc}{2bd}$. Now, since $r < s$, then $\frac{r+r}{2} < \frac{r+s}{2} < \frac{s+s}{2}$, so $r < t < s$. Also, $ad + bc, 2bd \in \mathbb{Z}$, and $2bd \neq 0$, so t is rational.

5. *False* — Counter-example: $a = 2, b = 3, c = 6$. Observe that $2 \mid 6$ and $3 \mid 6$, but $(2 + 3) = 5 \nmid 6$.

6. *False* — Prove the negation, by division into cases modulo 3: if $n \equiv 0 \pmod 3$, then $n^2 \equiv 0^2 = 0 \pmod 3$; if $n \equiv 1 \pmod 3$, then $n^2 \equiv 1^2 = 1 \pmod 3$; if $n \equiv 2 \pmod 3$, then $n^2 \equiv 2^2 = 4 \equiv 1 \pmod 3$. In each of these cases, $n^2 \not\equiv 2 \pmod 3$.

7. *True* — Let $n = \lfloor x \rfloor$ and $m = \lceil x \rceil$, then $n \leq x < n+1$ and $m-1 < x \leq m$. Therefore, $n \leq x \leq m$, so $n \leq m$. Substituting in the definitions of n and m, $\lfloor x \rfloor \leq \lceil x \rceil$.