MATH1061 — Challenge Questions

27 September 2013

1 Sets

1. For a finite set X, show that $|\mathcal{P}(X)| = 2^{|X|}$ (hint: use induction).

2 Graphs

1. Show that a tree on $n \geq 2$ vertices has at least one leaf (hint: pick a vertex, and start drawing a path away from it).
2. Repeat the process in the hint to show that a tree on $n \geq 2$ vertices has at least two leaves.
3. Prove by induction that a tree on n vertices has $n-1$ edges. (Hint: if you remove a leaf from a tree, the result is also a tree)
4. Knowing that a tree on n vertices has $n-1$ edges, give an alternate proof that a tree on $n \geq 2$ vertices has at least two leaves.

3 Relations

For a relation ρ on a set A, we say that ρ is *irreflexive* if for all $x \in A$, it does **not** hold that $x\rho x$. We say that ρ is *asymmetric* if, whenever $x\rho y$, it does not hold that $y\rho x$.

1. Let $A = \{1, 2, 3\}$.
   (a) State a relation on A which is both symmetric and antisymmetric.
   (b) State a relation on A which is neither symmetric nor antisymmetric.
   (c) State a relation on A which is neither reflexive nor irreflexive.
   (d) State a relation on A which is neither symmetric nor asymmetric.
2. For any set A, find a relation which is both an equivalence relation and a partial order.
3. We say that a relation is a "strict partial order" if it is irreflexive, asymmetric and transitive.
   (a) Let A be a set, and let $\preceq$ be a partial order on A. Define a relation $\prec$ on A by: $a \prec b$ iff $a \preceq b$ and $a \neq b$. Show that $\prec$ is a strict partial order.
   (b) Let $\prec$ be a strict partial order on A, and define a relation $\preceq$ on A by: $a \preceq b$ iff $a \prec b$ or $a = b$. Show that $\preceq$ is a partial order.
4. Let ρ be a relation on a set A which is symmetric and transitive. Show that ρ is an equivalence relation.

4 Functions and Cardinality

In the following questions, our aim is to find a sensible way to talk about the 'size' of an infinite set, by using properties about the size of finite sets.

1. Let $X = \{1, 2, 3\}$, and $Y = \{1, 2, 3, 4\}$.
   (a) Construct a one-to-one function from X to Y, and an onto function from Y to X.
   (b) Explain why there are no surjections from X to Y.
   (c) Explain why there are no injections from Y to X.

2. Let A and B be finite sets.
   (a) Prove that $|A| \leq |B|$ iff there exists an injective function from A to B.
   (b) Prove that $|A| \geq |B|$ iff there exists a surjective function from A to B.
   (c) Prove that $|A| = |B|$ iff there exists a bijective function from A to B.

3. (a) Let $\iota_{\mathcal{X}}$ be the identity function on a set $\mathcal{X}$. Prove that $\iota_{\mathcal{X}}$ is a bijection.
   (b) If $f : \mathcal{X} \to \mathcal{Y}$ is a bijection, prove that f^{-1} is a bijection.
   (c) If $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ are both one-to-one, prove that $g \circ f$ is one-to-one. (This is a theorem from workbook chapter H.4.)
   (d) If $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ are both onto functions, prove that $g \circ f$ is onto. (This is a theorem from workbook chapter H.4.)
   (e) Define a relation $\sim$ on sets such that for any sets A and B, $A \sim B$ if and only if there exists a bijection from A to B. Show that $\sim$ satisfies the three axioms for being an equivalence relation.
   (f) Conclude from 2(c) and 3(e) that if A and B are finite, then $A \sim B$ iff $|A| = |B|$.

4. For any sets A and B (possibly infinite), we say that A and B have the same 'cardinality' if $A \sim B$, and we write $|A| = |B|$. Note that 'cardinality' can be thought of as 'size', but also allowing infinite sets. Read Chapter H.5 of the workbook before proceeding.
   (a) Let $\mathbb{Z}$ be the set of integers and let $\mathbb{Z}_{even}$ be the set of even integers. Show that $|\mathbb{Z}| = |\mathbb{Z}_{even}|$, i.e. show that they have the same cardinality, even though $\mathbb{Z}$ contains seemingly 'more' elements than $\mathbb{Z}_{even}$.
   (b) Let $\mathbb{Z}^+$ be the set of positive integers, also denoted $\mathbb{N}$. Research Cantor's diagonal argument, which shows that $|\mathbb{R}| \neq |\mathbb{Z}^+|$, hence $\mathbb{R}$ is uncountable.
   (c) Show that $\mathbb{Z}$ is countable, i.e. $|\mathbb{Z}| = |\mathbb{Z}^+|$.
   (d) Show that $\mathbb{Q}$ is countable.
   (e) Let $(-\frac{\pi}{2}, \frac{\pi}{2}) = \{x \in \mathbb{R} \mid -\frac{\pi}{2} < x < \frac{\pi}{2}\}$, i.e. the interval of real numbers between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Show that this interval has the same cardinality as $\mathbb{R}$.
   (f) Show that the interval $(0, 1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$ has the same cardinality as $(-\frac{\pi}{2}, \frac{\pi}{2})$.
   (g) Conclude that $(0, 1)$ is uncountable.

5. Now that we have different 'sizes' of infinity, we would like to compare them, and talk about when one cardinality is 'bigger' than another.

(a) For sets A, B, define a relation $\preceq$ such that $A \preceq B$ iff there exists an injection from A to B.

Let A, B, C be sets where A has the same cardinality as B. Show that if $A \preceq C$, then $B \preceq C$; and if $C \preceq A$, then $C \preceq B$. Hence conclude that this relation can be instead thought of as a relation on cardinalities, instead of a relation on sets.

(b) For sets A, B with cardinalities $|A|$ and $|B|$, we say that $|A| \leq |B|$ if there exists an injection from A to B. (Note that for the size of finite sets, this definition agrees with the result from 2(a).) Use the results from 3(a) and 3(c) to conclude that this relation is reflexive and transitive.

(c) The Cantor-Bernstein-Schroeder theorem shows that if there is an injection from A to B, and an injection from B to A, then there is a bijection from A to B. Conclude from this that the relation $\leq$ is a partial order on cardinalities.

(d) If $|A| \leq |B|$ and $|A| \neq |B|$, we say that $|A| < |B|$. Show that $|\mathbb{Z}^+| < |\mathbb{R}|$.

(e) Show that $|\mathbb{Q}| < |(0, 1)|$, hence conclude that there are more numbers between 0 and 1 than there are rational numbers in the entire number line.

(f) If A is finite and B is infinite, show that $|A| < |B|$.

(g) Using the theorem from page 244 of the workbook (known as Cantor's theorem), show that $|X| < |\mathcal{P}(X)|$ for any set X. Hence, show that there are infinitely many types of infinite cardinalities (hint: consider $\mathbb{Z}$, $\mathcal{P}(\mathbb{Z})$, $\mathcal{P}(\mathcal{P}(\mathbb{Z}))$, $\ldots$).