MATH1061 — Week 9 Tutorial (T5)

30 April 2013

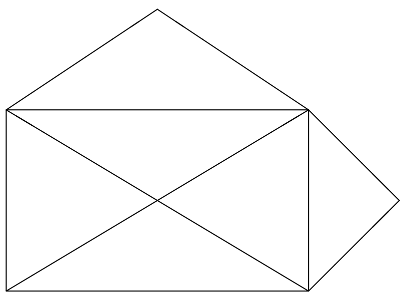
1. Do the following graphs exist? If so, draw one, if not, explain why.
   (a) A graph on 4 vertices with vertices of degree 1, 2, 3, 3.
   (b) A graph on 5 vertices with vertices of degree 1, 2, 3, 3, 5.
   (c) A simple graph on 6 vertices, all with degree 3.
   (d) A tree on 7 vertices with no leaves.
   (e) A tree on 5 vertices which contains a Hamiltonian path.
   (f) A connected graph with 8 vertices and 6 edges.

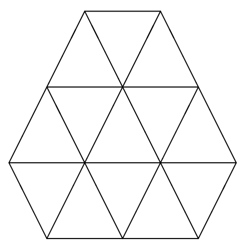
2. The complete bipartite graph $K_{1,n}$ is a special type of tree, called a *star*. Draw the star $K_{1,5}$, and write its adjacency matrix A and incidence matrix J.

3. In lectures we proved theorems about when an undirected graph has an Euler circuit or an Euler path (see workbook p.162). When does a **directed** graph have an Euler circuit or Euler path? (Note that a path in a directed graph must follow the directions of the arrows.) You may use the following definitions in your answer:

   In a directed graph, the *out degree* of a vertex v (denoted $\deg^+(v)$) is the number of arcs going out of v, and the *in degree* of v (denoted $\deg^-(v)$) is the number of arcs going into v.

4. Can the following figures be drawn in a single pen stroke? Explain why/why not, without trying to draw the figures. If not, what's the minimum number of pen lifts required?





Solutions:

1. (a) Not possible: total degree is 9, which is odd.

   (b) Possible, but not with simple graphs: An example would be to have a vertex of degree 5 with a loop, and a vertex of degree 3 with a loop.

   (c) Possible: vertex set $\{a, b, c, d, e, f\}$ and edge set $\{ab, bc, cd, de, ef, fa, ad, bf, ce\}$.

   (d) Not possible: Each vertex would have degree at least 2, for a total degree of at least 14, but a tree on 7 vertices needs to have total degree 12.

   (e) Possible: A line, with vertex set $\{a, b, c, d, e\}$ and edge set $\{ab, bc, cd, de\}$. The Hamiltonian path follows the vertices in the order a, b, c, d, e.

   (f) Not possible: Using 6 edges, we can only connect at most 7 vertices.

2. (Picture not shown) Adjacency matrix A and incidence matrix J below:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad J = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

3. Suppose G is a directed connected graph. G has an Eulerian cycle if and only if every vertex u has $\deg^+(u) = \deg^-(u)$.

   G has an Eulerian path from v to w if and only if vertex $\deg^+(v) = \deg^-(v) + 1$, $\deg^-(w) = \deg^+(w) + 1$, and every other vertex u has $\deg^+(u) = \deg^-(v)$.

4. Note that a figure can be drawn in one pen stroke if and only if a graph with that drawing, and vertices where lines meet, has an Eulerian cycle/path.

   The first figure has two 'vertices' of odd degree (the bottom-left has degree 3, and the top-right of the rectangle has degree 5), so there is an Eulerian path, so it is possible. Such a drawing would have to start at one of those two corners, and finish at the other.

   The second figure has six 'vertices' of odd degree (the six 'corners' all have degree 3), so there is no Eulerian path or cycle, so it is not possible. We can consider pen-lifts to be 'adding in an extra edge' between where the pen is lifted and where it is put back down. Since there are 6 vertices of odd degree, we can add 2 such edges between 4 of them, and there will be 2 vertices of odd degree remaining. So, two pen lifts are required.