Exercises:

1. For a positive integer n, use $\sum$ notation to write an expression for the first n terms of the sum
$$2 + 6 + 18 + 54 + 162 + \ldots$$
Show that the first n terms of this sum is equal to $3^n - 1$.

2. For what positive integers n is it true that $n! > 2^n$? Use induction to prove that it is true for these values of n.

3. The *Fibonacci sequence* is the sequence $\{F_n\}$ defined by
$$F_0 = 0; \quad F_1 = 1; \quad \forall n \geq 2,\ F_n = F_{n-1} + F_{n-2}.$$
Let $\phi = \frac{1+\sqrt{5}}{2}$. It can be shown that $1 < \phi < 2$ and that $\phi^2 = \phi + 1$. Use these facts to prove that $F_n < \phi^n$ for all $n \geq 0$.

4. [1] There must be something wrong with the following proof. What is it?

Theorem 1. *Let a be any positive number. For all integers $n \geq 0$, we have $a^n = 1$.*

Proof. If $n = 0$, $a^n = a^0 = 1$, so the base case holds. If the theorem is true for all of $0, 1, \ldots, n-1, n$ (i.e. $a^0 = 1, a^1 = 1, \ldots, a^{n-1} = 1, a^n = 1$), then we have
$$a^{n+1} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1,$$
so the theorem is true for $n+1$ as well. By the principle of strong induction, $a^n = 1$ for all integers $n \geq 0$. $\square$

5. Let d be a positive integer. For any integer $n \geq 0$, prove that there exist integers q, r with $0 \leq r < d$ such that $n = dq + r$.

Hint: Use the well-ordering principle in a proof by contradiction.

Remark: This is a part of a familiar theorem, which is stated on p.58 of the workbook.

6. We can define a set of arithmetic expressions over the real numbers as follows:

   I. BASE: Each real number r is an arithmetic expression.

   II. RECURSION: If u and v are arithmetic expressions, then $(u + v)$ and $(u - v)$ are arithmetic expressions.

   III. RESTRICTION: There are no arithmetic expressions over the real numbers other than those obtained from I and II.

Show: (a) that $((0.1 + (3.2 - 0)) + (-0.2 - \pi))$ is an arithmetic expression, and (b) that $(2 + (1.414 - 12.1) + 13.2)$ is not an arithmetic expression.

[1]This question is from Donald Knuth's The Art of Computer Programming (3rd ed.), §1.2.1, Q.2

Challenge[2] exercises:

C1. Show that

$$\sum_{k=1}^{n} 3 \cdot 4^{k-1} = 4^n - 1.$$

Compare this equation to Exercise 1 above, and Example D.2.4 (p.96 of workbook), and suggest a generalisation. Prove this generalisation is true. Use this to factorise the polynomial $x^n - 1$.

C2. Prove that the sum of the cubes of the first n positive integers is equal to the square of the sum of the first n positive integers. That is, for all integers $n \geq 1$, prove that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$

C3. The following algorithm takes a non-negative integer n as a parameter. Prove that the algorithm returns the value of n^2. (Related reading: Epp, 4th ed. §5.5)

function SQUARE(n)
Require: $n \in \mathbb{Z}$, $n \geq 0$.
Ensure: Return value is n^2.
    $m \leftarrow 0$
    $k \leftarrow 0$
    while $k < n$ **do**
        $m \leftarrow m + 2k + 1$
        $k \leftarrow k + 1$
    end while
    return m
end function

C4. Prove the result in Exercise 5 holds also holds for negative n. (Hint: if n is a negative integer, then $-n$ is a positive integer, and we've already proved it for the positive integers.) Why couldn't we easily use the well-ordering principle to prove it for all integers n, instead of just the non-negative ones? There's one more part of the Quotient-Remainder Theorem we haven't yet proved, what is it? Can you prove it?

C5. Design and implement a computer program which takes as input an arithmetic expression as defined in Exercise 6, and computes the result of applying the additions and subtractions. Extend your program to also detect for erroneous input (i.e. input which is not an arithmetic expression).[3]

[2] i.e., if you can do any of these questions, then you're going *way* above the course expectations. But hey, maybe you like this stuff, or maybe you want to see some applications to computer science.

[3] If you're interested in this sort of thing, consider taking COMP3506 in the future.

Hints:

1. Here's a pattern in the first few elements: $2 = 2 \cdot 3^0$, $6 = 2 \cdot 3^1$, $18 = 2 \cdot 3^2$, $54 = 2 \cdot 3^3$. Use an induction proof, and remember that $3^n + 2 \cdot 3^n = 3^{n+1}$.

2. Check what $n!$ and 2^n are for small integers n. Use a proof by induction to show that it's true for all n above a certain point.

3. Use strong induction, and remember that $\phi^n + \phi^{n-1} = \phi^{n-1}(\phi + 1)$.

4. What values of n does the inductive step work on?

5. Use the well-ordering principle in a proof by contradiction. Let S be the set of all non-negative integers n for which we *cannot* write $n = dq + r$ for any suitable q, r. Then find a contradiction involving the smallest element of S.

6. (a) Show that "3.2", "0", and "(3.2 - 0)" are arithmetic expressions, and so on.

   (b) If it were an arithmetic expression, explain why at least one of the following would have to be true:

   - The whole expression is a real number.
   - "$2 + (1.414 - 12.1)$" and "13.2" are both arithmetic expressions.
   - "2" and "$(1.414 - 12.1) + 13.2$" are both arithmetic expressions.
   - "$2 + (1.414$" and "$12.1) + 13.2$" are both arithmetic expressions.

No hints for the challenge tasks, otherwise they wouldn't be challenges.