Exercises

1. Define a relation σ on $\mathbb{R}$ such that $x\sigma y$ if and only if $x - y \in \mathbb{Z}$. Show that σ is an equivalence relation. Describe the equivalence classes.

2. Let $f : \mathbb{Z} \to \mathbb{Z}$ be given by $f(x) = x^2$. Is f one-to-one? Is it onto?

3. Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = x + 7$. Is g one-to-one? Is it onto?

4. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \lfloor x \rfloor$. Is f onto?

   Similarly, let $g : \mathbb{R} \to \mathbb{Z}$ be given by $g(x) = \lfloor x \rfloor$. Is g onto?

Solutions

1. To show that σ is an equivalence relation, we must show that it is reflexive, symmetric, and transitive:

   (a) For any $x \in \mathbb{R}$, $x - x = 0 \in \mathbb{Z}$, so $x\sigma x$; therefore σ is reflexive.

   (b) For any $x, y \in \mathbb{R}$, if $x\sigma y$, then let $m = x - y, m \in \mathbb{Z}$. Then, $y - x = -m \in \mathbb{Z}$, so $y\sigma x$; therefore σ is symmetric.

   (c) For $x, y, z \in \mathbb{R}$, if $x - y \in \mathbb{Z}$ and $y - z \in \mathbb{Z}$, let $m = x - y$, $n = y - z$. Then, $x - z = (x - y) + (y - z) = m + n \in \mathbb{Z}$, so $x\sigma z$. Therefore, σ is transitive.

   Note that we could have avoided using the label m and just said "if $x - y \in \mathbb{Z}$, then $y - x = -(x - y) \in \mathbb{Z}$." Similarly we could have skipped the labels m and n in transitivity.

   Now, for any $x \in \mathbb{R}$, the equivalence class containing x consists of all real numbers which differ from x by an integer amount, i.e.

   $$[x] = \{\ldots, x - 2, x - 1, x, x + 1, x + 2, \ldots\}.$$

   The set of equivalence classes $[x]$ for each real number $0 \leq x < 1$ gives all equivalence classes. $\square$

2. f is not 1-to-1, a counterexample[1] is: $f(-1) = 1 = f(1)$, but $-1 \neq 1$.

   f is not onto: for instance, there is no integer x for which $x^2 = -1$.

3. g is one-to-one[2]: Suppose $x_1 + 7 = x_2 + 7$ for $x_1, x_2 \in \mathbb{R}$; subtracting 7 from each side, we have $x_1 = x_2$.

   g is onto[3]: For any $y \in \mathbb{R}$, the value $y - 7$ has $f(y - 7) = (y - 7) + 7 = y$.

4. f is not onto, since there is no $x \in \mathbb{R}$ for which $\lfloor x \rfloor = \frac{3}{2}$ (or any other non-integer value).

   g is onto, since for any $y \in \mathbb{Z}$, we have $g(y) = y$.

   Note that the choice of input to g is not unique, we could have also said $g\left(y + \frac{1}{2}\right) = y$, or $g(y + 0.013) = y$. To show that g is onto, we only need some value that works, we don't care how many such values there are.

---

[1] i.e. an example of $x_1, x_2 \in \mathbb{Z}$ such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$.

[2] To show g is one-to-one, we have to prove either "$x_1 \neq x_2 \implies g(x_1) \neq g(x_2)$" or "$g(x_1) = g(x_2) \implies x_1 = x_2$".

[3] We have to prove that for any $y \in \mathbb{R}$ (the codomain), there is some $x \in \mathbb{R}$ (the domain) such that $y = f(x)$ (informally, 'If I give you any y, how can you choose a value that when put into the function, gives y?').