MATH1061 — Week 9 Tutorial (T5)

30 April 2013

1. Do graphs with the following descriptions exist? If so, draw one, if not, explain why.
   (a) A graph on 4 vertices with vertices of degree 1, 2, 3, 3.
   (b) A graph on 5 vertices with vertices of degree 1, 2, 3, 3, 5.
   (c) A simple graph on 6 vertices, all with degree 3.
   (d) A tree on 7 vertices with no leaves.
   (e) A tree on 5 vertices which contains a Hamiltonian path.
   (f) A connected graph with 8 vertices and 6 edges.

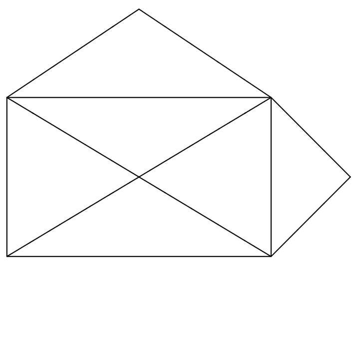
2. The complete bipartite graph $K_{1,n}$ is a special type of tree, called a *star*. Draw the star $K_{1,5}$, and write its adjacency matrix and incidence matrix. Describe the adjacency matrix of $K_{1,n}$.

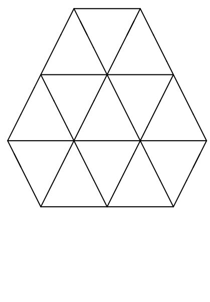
3. Recall from lectures that a connected graph has an Euler circuit if and only if every vertex of has even degree, and that it has an Euler path from v to w if and only if v and w have odd degree and all other vertices have even degree.

   In a directed graph, the *out degree* of a vertex v (denoted $\deg^+(v)$) is the number of arcs going out of v, and the *in degree* of v (denoted $\deg^-(v)$) is the number of arcs going into v.

   Let G be a directed, connected graph (i.e. there is a path from each vertex to every other vertex that follows the directions of the edges). Make a statement about when G has an Euler circuit. Make another statement about when G has an Euler path from v to w.

4. Can the following figures be drawn in a single pen stroke? Explain why/why not. If not, what's the minimum number of pen lifts required?





Solutions:

1. (a) Not possible: total degree is 9, which is odd.
   (b) Possible, but not with simple graphs: An example would be to have a vertex of degree 5 with a loop, and a vertex of degree 3 with a loop.
   (c) Possible: vertex set $\{1, 2, 3, 4, 5, 6\}$ and edge set $\{12, 23, 34, 45, 56, 61, 14, 26, 35\}$.
   (d) Not possible: Each vertex would have degree at least 2, for a total degree of at least 14, but a tree on 7 vertices needs to have total degree 12.
   (e) Possible: A line, with edge set $\{12, 23, 34, 45\}$.
   (f) Not possible: Using 6 edges, we can only connect at most 7 vertices.

2. (TODO Picture) Adjacency matrix A and incidence matrix J below:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad J = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

3. Suppose G is adirected connected graph. G has an Eulerian cycle if and only if every vertex u has $\deg^+(u) = \deg^-(u)$.

   G has an Eulerian path from v to w if and only if vertex $\deg^+(v) = \deg^-(v) + 1$, $\deg^-(w) = \deg^+(w) + 1$, and every other vertex u has $\deg^+(u) = \deg^-(v)$.

4. Note that a figure can be drawn in one pen stroke if and only if a graph with that drawing, and vertices where lines meet, has an Eulerian cycle/path.

   The first figure has two 'vertices' of odd degree (the bottom-left has degree 3, and the top-right of the rectangle has degree 5), so there is an Eulerian path, so it is possible. Such a drawing would have to start at one of those two corners, and finish at the other.

   The second figure has six 'vertices' of odd degree (the six 'corners' all have degree 3), so there is no Eulerian path or cycle, so it is not possible.