1. Prove that for arbitrary sets A, B, C that
$$(A \setminus B) \setminus C = A \setminus (B \cup C).$$

2. Prove that for arbitrary sets A and B we have $A \subseteq B$ if and only if $A \cup B = B$.

3. Let $X = \{1, \{1\}, \{\emptyset\}\}$. Write down the elements of $\mathcal{P}(X) \setminus X$.

Challenge: If X is a finite set, prove by induction that $|\mathcal{P}(X)| = 2^{|X|}$.
Challenge: Prove that for arbitrary sets A and B, that $A \subseteq B$ iff $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Solution/outlines:

1. Element argument. $x \in (A \setminus B) \setminus C)$ iff $x \in A$ and $x \notin B$ and $x \notin C$, iff $x \in A$ and $\sim ((x \in B) \vee (x \in C))$, iff $x \in A \setminus (B \cup C)$.

2. Construct a correspondence with the predicate
$$(P(x) \to Q(x)) \leftrightarrow ((P(x) \vee Q(x)) \leftrightarrow Q(x)),$$
show that this is a tautology (truth table or otherwise). Or, show that if $A \subseteq B$ then $A \cup B \subseteq B \subseteq A \cup B$, and if $A \cup B = B$ then $A \subseteq B$.

3. $\mathcal{P}(X) \setminus X = \big\{\emptyset, \{\{1\}\}, \{\{\emptyset\}\}, \{1, \{1\}\}, \{1, \{\emptyset\}\}, \{\{1\}, \{\emptyset\}\}, \{1, \{1\}, \{\emptyset\}\}\big\}$.

4. (Challenge) Let $n = |X|$, induct on n. Base case: $\mathcal{P}(\emptyset) = \{\emptyset\}$. Inductive step: Pick $x \in X$, $X' = X \setminus \{x\}$,
$$\mathcal{P}(X) = \mathcal{P}(X) \cup \{A \cup \{x\} \mid A \in \mathcal{P}(X)\}.$$

5. (Challenge) ($\Rightarrow$) If $A \subseteq B$, then for any $X \in \mathcal{P}(A)$, we have $X \subseteq A \subseteq B$, so $X \subseteq B$, so $X \in \mathcal{P}(B)$. Therefore, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

   ($\Leftarrow$) If $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then in particular, $A \in \mathcal{P}(A) \subseteq \mathcal{P}(B)$, so $A \subseteq B$.