MATH1061 — Week 10 Tutorial (T5)

7 May 2013

1. 5 cards are dealt out from a standard 52-card deck.
   (a) What's the probability that all cards are of the same suit?
   (b) What's the probability that there is a pair of cards with the same face value?
   (c) What's the probability that all cards are of the same suit, or that there is a pair of cards with the same face value?

2. We are painting the six faces of a cube (top, bottom, front, back, left, right) with 6 colours (red, orange, yellow, green, blue, white), each with a different colour. Note that two colourings of the cube are the same if one is a rotation of the other. How many different ways can we colour the cube?

3. Computers can store a set of data in a *hash table*, by storing the data into a number of *buckets*, then the data can be accessed efficiently by knowing which bucket to look in. Hash tables work most efficiently when the values are spread evenly across the buckets.

   If a hash table has buckets labelled from 0 to 15, and contains 137 values spread across the buckets, what is the largest number k such that we can guarantee there is a bucket of size at least k?

4. How many integers between 1 and 90 have a common factor with 90?

   (Hint: $\gcd(n, 90) \neq 1$ if and only if n is divisible by 2, 3, or 5. Use the inclusion-exclusion principle - workbook p.298)

5. Use the binomial theorem to prove that, for any nonnegative integer n,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

Solutions

1. (a) The probability is
$$1 \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{9}{48} = \frac{33}{16660} \approx 0.002.$$
(b) The complement of the event is that all cards have different face value, which has probability
$$1 \cdot \frac{48}{51} \cdot \frac{44}{50} \cdot \frac{40}{49} \cdot \frac{36}{48} = \frac{2112}{4165} \approx 0.507,$$
so the desired probability is $\frac{2053}{4165} \approx 0.493$.
(c) First note that the two events are mutually exclusive, since all 52 cards are distinct. Then, the probability is
$$\frac{33}{16660} + \frac{2053}{4165} = \frac{97}{196} \approx 0.495.$$

2. WLOG, the top face is red, and there is an axis of rotation perpendicular to the top face. Then, there are 5 choices for the bottom face. WLOG, the front face is fixed, then there are $3! = 6$ choices for the remaining 3 faces, for a total of 30 colourings.

   Alternative solution is to observe that there are $6! = 720$ colourings without excluding rotations, and the square has 24 symmetries, giving $720/24 = 30$ colourings.

3. There are a total of 16 buckets available to put values in. Since $137 > 8 \cdot 16$, by the pigeonhole principle, there is a group of at least 9 people who got the same mark. Since $137 < 9 \cdot 16$, this is the largest such number.

4. An integer has a common factor with 90 if and only if it is divisible by 2, 3, or 5. Let A_2 be the set of integers between 1 and 90 which are divisible by 2, A_3 be those divisible by 3, and A_5 be those divisible by 5. Then,
$$\begin{aligned}|A_2 \cup A_3 \cup A_5| &= |A_2| + |A_3| + |A_5| - |A_2 \cap A_3| - |A_2 \cap A_5| \\ &\quad - |A_3 \cap A_5| + |A_2 \cap A_3 \cap A_5| \\ &= 45 + 30 + 18 - 15 - 9 - 6 + 3 = 66\end{aligned}$$

5. $$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n = 2^n.$$