

Skew stickiness
oooooooooooo

Diamond trees and forests
oooooooooooo

Computing the SSR
oooooooooooo

AFV SSR
oooooooooooo

Computing skew-stickiness

Jim Gatheral



Bloomberg Quant Seminar
November 28, 2023

Outline of this talk

- The skew-stickiness ratio (SSR)
- The diamond product
- Trees and forests
- The \mathbb{F} -expansion and stochastic volatility
 - The Bergomi-Guyon smile expansion to all orders
- The SSR in terms of diamonds for very short times
- The time series of SSR from Vola Dynamics
- Properties of the SSR

Implied volatility

According to the definition of implied volatility $\sigma_{\text{BS}}(k, T)$, the market price of an option is given by

$$C(S, K, T) = C_{\text{BS}}(S, K, T, \sigma_{\text{BS}}(k, T))$$

where C_{BS} denotes the Black-Scholes formula and $k = \log K/S$ is the log-strike.

Hedging European options

To hedge options using the Black-Scholes formula, market makers need to hedge two effects:

- The explicit spot effect

$$\frac{\partial C}{\partial S} \delta S$$

and

- The change in implied volatility conditional on a change in the spot

$$\frac{\partial C}{\partial \sigma} \mathbb{E} [\delta \sigma | \delta S].$$

Estimating $\mathbb{E} [\delta\sigma(T) | \delta X]$

- ATM implied volatilities $\sigma(T) = \sigma_{BS}(0, T)$ and stock prices are both observable.
- Market makers can estimate the second component using a simple regression:

$$\delta\sigma(T) = \beta(T) \frac{\delta S}{S} + \text{noise} =: \beta(T) \delta X + \text{noise}. \quad (1)$$

- Then

$$\beta(T) = \frac{\mathbb{E} [\delta\sigma(T) | \delta X]}{\delta X}.$$

The skew-stickiness ratio

- For a given time to expiration T , we define the ATM volatility skew

$$\psi(T) = \frac{\partial}{\partial k} \sigma_{\text{BS}}(k, T) \Big|_{k=0}.$$

- Bergomi [Ber09, Ber16] calls

$$\mathcal{R}(T) = \frac{\beta(T)}{\psi(T)}$$

the *skew-stickiness ratio* or *SSR*.

Sticky delta and sticky strike

In the old days, traders would typically make one of two assumptions:

- *Sticky delta* where the ATM volatility is fixed.
 - In this case, when S increases to $S + \delta S$, $\delta\sigma(T) = 0$ so $\mathcal{R}(T) = 0$.
- or
- *Sticky strike* where the implied volatility is fixed for a given strike independent of the stock price.
 - In this case, when S increases to $S + \delta S$,
$$\delta\sigma(T) = \sigma_{\text{BS}}(S + \delta S, T) - \sigma_{\text{BS}}(S, T) \approx \psi(T) \delta S$$

so $\beta(T) = \psi(T)$ and $\mathcal{R} = 1$.
- Listed options were thought of as sticky strike and OTC options as sticky delta.

Empirical SSR

- Let's check some skew-stickiness ratios over the period June 1, 2010 to June 1, 2011, reproducing figures from an article I wrote with Mike Kamal [[KG09](#)] in the Encyclopedia of Quantitative Finance.

1-month SSR

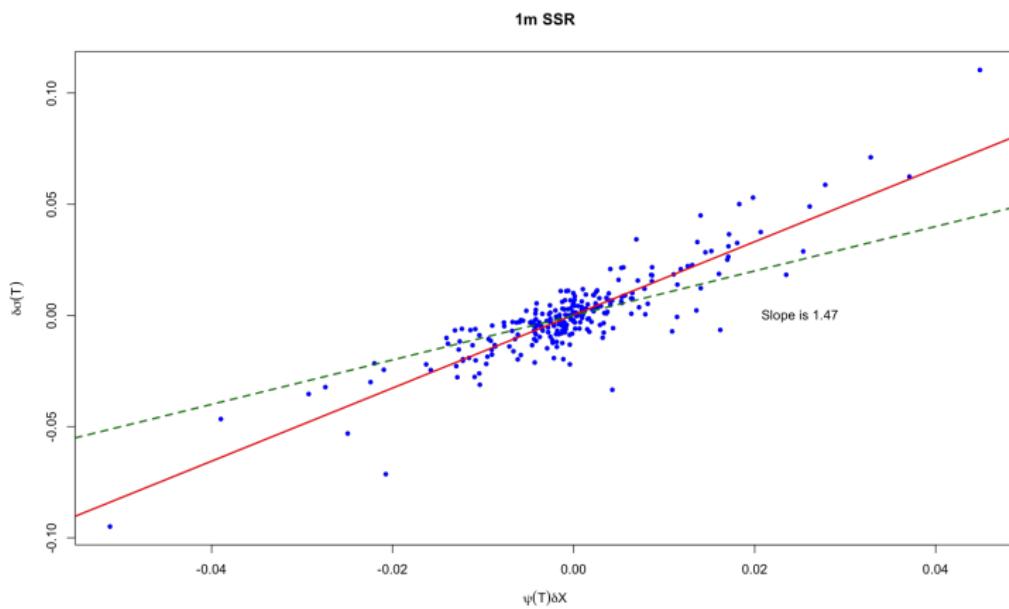


Figure 1: The 1-month skew-stickiness ratio (SSR). The "sticky strike" green line with slope 1 clearly doesn't fit.

3-month SSR

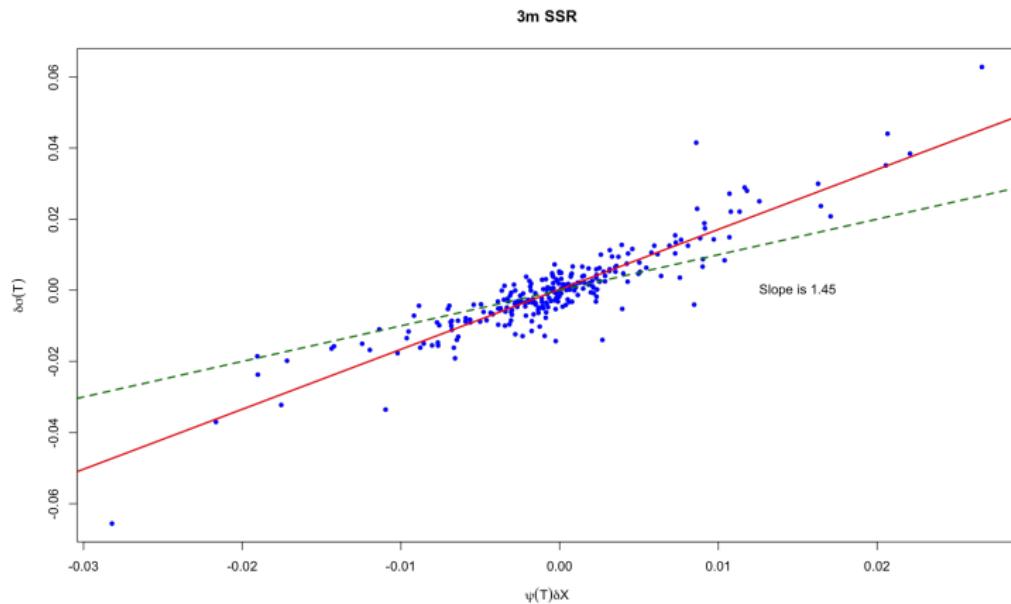


Figure 2: The 3-month skew-stickiness ratio (SSR). The "sticky strike" green line with slope 1 clearly doesn't fit.

6-month SSR

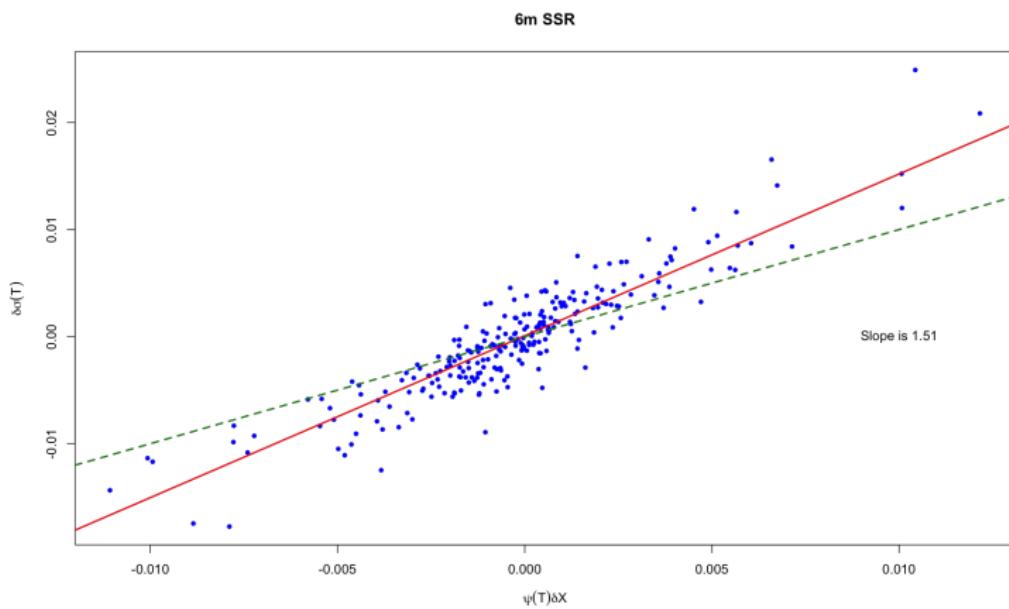


Figure 3: The 6-month skew-stickiness ratio (SSR). The "sticky strike" green line with slope 1 clearly doesn't fit.

12-month SSR

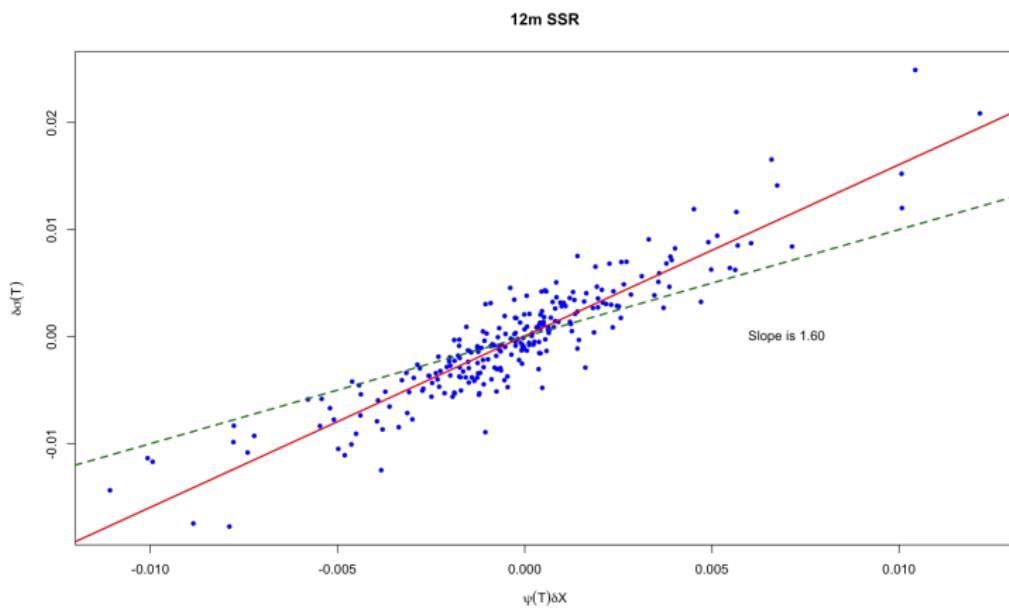


Figure 4: The 12-month skew-stickiness ratio (SSR). The "sticky strike" green line with slope 1 clearly doesn't fit.

The diamond product

Definition

Given two continuous semimartingales A, B with integrable covariation process $\langle A, B \rangle$, the diamond product of A and B is another continuous semimartingale given by

$$(A \diamond B)_t(T) := \mathbb{E} [\langle A, B \rangle_{t,T} | \mathcal{F}_t] = \mathbb{E} [\langle A, B \rangle_T | \mathcal{F}_t] - \langle A, B \rangle_t,$$

where $\langle A, B \rangle_{t,T} = \langle A, B \rangle_T - \langle A, B \rangle_t$.

Properties of the diamond product

- Commutative: $A \diamond B = B \diamond A$.
- Non-associative: $(A \diamond B) \diamond C \neq A \diamond (B \diamond C)$.
- $A \diamond B$ depends only on the respective martingale parts of A and B .
- $A \diamond B$ is in general not a martingale.

The \mathbb{G} -forest expansion

Theorem 1 (Theorem 1.1 of [FGR22])

Let Y_T be a real-valued, \mathcal{F}_T -measurable random variable with associated martingale $Y_t = \mathbb{E}_t[Y_T]$. Under natural integrability conditions, with a, b small enough, there is a.s. convergence of

$$\log \mathbb{E} \left[e^{aY_T + b\langle Y \rangle_T} \middle| \mathcal{F}_t \right] = aY_t + b\langle Y \rangle_t + \sum_{k \geq 2} \mathbb{G}_t^k(T), \quad (2)$$

where

$$\mathbb{G}^2 = \left(\frac{1}{2}a^2 + b \right) (Y \diamond Y)_t(T),$$

$$\mathbb{G}^k = \frac{1}{2} \sum_{j=2}^{k-2} \mathbb{G}^{k-j} \diamond \mathbb{G}^j + (a Y \diamond \mathbb{G}^{k-1}) \text{ for } k > 2. \quad (3)$$

Trees and forests

- The general term $\mathbb{G}_t^n(T)$ in (3) is naturally written as a linear combination of binary diamond trees¹.
- Hence the terminology *\mathbb{G} -forest expansion* for (2).
- Specifically, writing \bullet as a short-hand for Y , interpreted as single leaf, we have

$$\mathbb{G}^2 = \left(\frac{1}{2}a^2 + b\right) \bullet \text{v}\bullet$$

$$\mathbb{G}^3 = a \left(\frac{1}{2}a^2 + b\right) \bullet \text{v} \bullet \text{v} \bullet$$

$$\mathbb{G}^4 = \frac{1}{2} \left(\frac{1}{2}a^2 + b\right)^2 \bullet \text{v} \bullet \text{v} \bullet \text{v} \bullet + a^2 \left(\frac{1}{2}a^2 + b\right) \bullet \text{v} \bullet \text{v} \bullet \text{v} \bullet$$

$$\mathbb{G}^5 = a \left(\frac{1}{2}a^2 + b\right)^2 \bullet \text{v} \bullet \text{v} \bullet \text{v} \bullet \text{v} \bullet + \frac{1}{2}a \left(\frac{1}{2}a^2 + b\right)^2 \bullet \text{v} \bullet \text{v} \bullet \text{v} \bullet \text{v} \bullet$$

$$+ a^3 \left(\frac{1}{2}a^2 + b\right) \bullet \text{v} \bullet \text{v} \bullet \text{v} \bullet \text{v} \bullet \quad (4)$$

¹Trees stolen from [Hai13]!

Forward variance models

- Let S be a strictly positive continuous martingale.
 - Then $X := \log S$ is a semimartingale with quadratic variation process $\langle X \rangle$.
- Defining $V_t dt := d \langle X \rangle_t$, forward variances are given by $\xi_t(u) := \mathbb{E}[V_u | \mathcal{F}_t]$, $u > t$.
 - Forward variances are tradable assets (unlike spot variance).
 - We get a family of martingales indexed by their individual time horizons u .
- Following [BG12], it is natural to specify the dynamics of $\xi_t(u)$ for each $u > t$.
 - all conventional finite-dimensional Markovian stochastic volatility models may be cast as forward variance models.

Trees with colored leaves

- Denote $X \equiv \circ$.
- We could define $(X \diamond X) = M$, or $\circ\textcolor{brown}{\circ} \equiv \bullet$, resulting in trees with leaves of two colors.
 - In a forward variance model, X_t represents the log-stock price and $M_t(T)$, the expected total variance $\int_t^T \xi_t(u) du$.
- In general, we can always identify subtrees in this way and assign them a new variable name (and leaf color).
 - For example, we could define $\textcolor{brown}{\circ}\textcolor{brown}{\circ} \equiv \bullet$ to get

$$\textcolor{brown}{\circ} \textcolor{brown}{\circ} \textcolor{brown}{\circ} \textcolor{brown}{\circ} = \textcolor{brown}{\circ} \textcolor{brown}{\circ} \textcolor{brown}{\circ} = \bullet \textcolor{brown}{\circ},$$

and so on.

The $\tilde{\mathbb{F}}$ -forest expansion

A corollary of the \mathbb{G} -expansion is the $\tilde{\mathbb{F}}$ -expansion of [AGR2020]:

Corollary

The cumulant generating function (CGF) is given by

$$\psi_t(T; a) = \log \mathbb{E}_t \left[e^{iaX_T} \right] = iaX_t - \frac{1}{2}a(a+i)M_t(T) + \sum_{\ell=1}^{\infty} \tilde{\mathbb{F}}_{\ell}(a). \quad (5)$$

where the $\tilde{\mathbb{F}}_{\ell}$ satisfy the recursion

$$\tilde{\mathbb{F}}_0 = -\frac{1}{2}a(a+i)M_t = -\frac{1}{2}a(a+i) \bullet \text{ and for } k > 0,$$

$$\tilde{\mathbb{F}}_{\ell} = \frac{1}{2} \sum_{j=0}^{\ell-2} \left(\tilde{\mathbb{F}}_{\ell-2-j} \diamond \tilde{\mathbb{F}}_j \right) + ia \left(X \diamond \tilde{\mathbb{F}}_{\ell-1} \right). \quad (6)$$

Applying the recursion (6), the first few $\tilde{\mathbb{F}}$ forests are given by

$$\tilde{\mathbb{F}}_0 = -\frac{1}{2}a(a+i)$$

$$\tilde{\mathbb{F}}_1 = -\frac{i}{2}a^2(a+i)$$

$$\tilde{\mathbb{F}}_2 = \frac{1}{2^3}a^2(a+i)^2 + \frac{1}{2}a^3(a+i)$$

$$\tilde{\mathbb{F}}_3 = (\tilde{\mathbb{F}}_0 \diamond \tilde{\mathbb{F}}_1) + ia \diamond \tilde{\mathbb{F}}_2$$

$$= \frac{i}{2^2}a^3(a+i)^2 + \frac{i}{2^3}a^3(a+i)^2 + \frac{i}{2}a^4(a+i)$$

- Note that the total probability and martingale constraints are satisfied for each tree.
 - That is $\psi_t^T(0) = \psi_t^T(-i) = 0$.

The Bergomi-Guyon smile expansion

- The Bergomi-Guyon (BG) smile expansion (Equation (14) of [BG12]) reads

$$\sigma_{\text{BS}}(k, T) = \hat{\sigma}_T + \mathcal{S}_T k + \mathcal{C}_T k^2 + \mathcal{O}(\epsilon^3)$$

where the coefficients $\hat{\sigma}_T$, \mathcal{S}_T and \mathcal{C}_T are complicated combinations of trees such as .

- The beauty of the BG expansion is that in some sense, it yields direct relationships between the smile and autocovariance functionals.

A formal expansion

- Regarding the forest expansion (5) as a formal power series in ϵ whose power counts the forest index ℓ , the characteristic function of the log stock price may be written in the form

$$\varphi_t(T; a) = \exp \left\{ i a X_t - \frac{1}{2} a(a+i) M_t(T) + \sum_{\ell=1}^{\infty} \epsilon^\ell \tilde{\mathbb{F}}_\ell(a) \right\}.$$

- On the other hand, from for example equation (5.7) of [Gat06], with $X_t = 0$,

$$\int_0^\infty \frac{du}{u^2 + \frac{1}{4}} \operatorname{Re} \left[e^{-iuk} \left(\varphi_t^T(u - i/2) - e^{-\frac{1}{2}(u^2 + \frac{1}{4})\Sigma(k)} \right) \right] = 0 \quad (7)$$

where, fixing T , $\Sigma(k) = \sigma_{BS}^2(k, T) T$ is the implied total variance smile.

- Formally expand $\Sigma(k)$ as

$$\Sigma(k) = \sum_{\ell=0}^{\infty} \epsilon^\ell a_\ell(k).$$

- The power of ϵ counts the order of the forest expansion.
- We set $\epsilon = 1$ at the end of the computation.
- Equation (7) may then be rewritten in the form

$$\begin{aligned}
 & \int_0^\infty \frac{du}{u^2 + \frac{1}{4}} \operatorname{Re} \left[e^{-iuk} \exp \left\{ -\frac{1}{2} \left(u^2 + \frac{1}{4} \right) \sum_{\ell=0}^{\infty} \epsilon^\ell a_\ell(k) \right\} \right] \\
 &= \int_0^\infty \frac{du}{u^2 + \frac{1}{4}} \operatorname{Re} \left[e^{-iuk} e^{-\frac{1}{2} \left(u^2 + \frac{1}{4} \right) M_t(T)} \exp \left\{ \sum_{\ell=1}^{\infty} \epsilon^\ell \tilde{\mathbb{F}}_\ell(u - i/2) \right\} \right]. \tag{8}
 \end{aligned}$$

- Matching powers of ϵ on each side of (8) gives the coefficients $a_\ell(k)$ in terms of diamond trees, for any $\ell \in \mathbb{Z}^+$.

$$\begin{aligned}
 a_0(k) &= M_t(T) = \bullet \\
 a_1(k) &= \left(\frac{k}{M} + \frac{1}{2} \right) \bullet \circlearrowleft \bullet \\
 a_2(k) &= \frac{1}{4} (\bullet \circlearrowleft \bullet)^2 \left\{ -\frac{5k^2}{M^3} - \frac{2k}{M^2} + \frac{3}{M^2} + \frac{1}{4M} \right\} \\
 &\quad + \frac{1}{4} (\bullet \circlearrowleft \bullet) \left\{ \frac{k^2}{M^2} - \frac{1}{M} - \frac{1}{4} \right\} \\
 &\quad + (\bullet \circlearrowleft \bullet) \left\{ \frac{k^2}{M^2} + \frac{k}{M} - \frac{1}{M} + \frac{1}{4} \right\}. \tag{9}
 \end{aligned}$$

- It is straightforward to verify that the resulting expansion coincides with that of Bergomi and Guyon up to second order in ϵ .

Computation of the ATM skew $\psi(T)$

- To first order in the forest expansion,

$$\Sigma(k) = M + \epsilon \left(\frac{k}{M} + \frac{1}{2} \right) \text{○} \text{▽} \text{○} + \text{ higher order.}$$

- Wlog, set $t = 0$ and drop the subscript.
- To lowest order in the forest expansion

$$\Sigma(0) = \text{○} = M(T) = \int_0^T \xi(u) du + \text{ higher order.}$$

and, by definition of ψ ,

$$2\sqrt{M}\sqrt{T}\psi(T) = \Sigma'(0) = \epsilon \frac{1}{M} \text{○} \text{▽} \text{○} + \text{ higher order.} \quad (10)$$

Computation of the regression coefficient $\beta(T)$

- We rewrite the regression (1) defining $\beta(T)$ as

$$\delta\sigma(0) = \frac{1}{2\sqrt{M}\sqrt{T}} \delta M_t(T) = \alpha + \beta(T) \frac{\delta S_t}{S_t} + \text{noise}.$$

- Thus, for fixed T , (reintroducing t),

$$\beta_t(T) = \frac{\mathbb{E}[d \langle M, X \rangle_t]}{2\sqrt{M}\sqrt{\tau} \mathbb{E}[d \langle X \rangle_t]} = \frac{-\frac{d}{dt}(X \diamond M)_t(T)}{2\sqrt{M}\sqrt{\tau} V_t} + \mathcal{O}(\epsilon), \quad (11)$$

where $X = \log S$, $\tau = T - t$.

- Recall that $(X \diamond M)_t(T) = \mathbb{E}_t \left[\int_t^T d \langle X, M(T) \rangle_s \right]$.

Computation of the SSR for very short times

- Putting (10) and (11) together, we get

$$\mathcal{R}_t(T) = \frac{\beta_t(T)}{\psi_t(T)} = -\frac{M_t(T) \frac{d}{dt}(X \diamond M)_t(T)}{V_t(X \diamond M)_t(T)} + \mathcal{O}(\epsilon^2).$$

- For very short times, with $\tau = T - t$, $M_t(T) \approx V_t \tau$, so

$$\mathcal{R}_t(T) \approx -\tau \frac{d}{dt} \log(X \diamond M)_t(T).$$

To first order in the forest expansion, for τ very small:

The SSR \mathcal{R} is given by the time derivative of the spot-volatility correlation functional $X \diamond M$.

A wild guess

- Suppose that something like

$$\mathcal{R}_t(T) \approx -\tau \frac{d}{dt} \log(X \diamond M)_t(T).$$

holds for all expirations, as implicitly argued in [Ber16].

- Then we have the following examples:

Some SSR examples

- The SABR model
 - $(X \diamond M)_t(T) \propto (T - t)^2$ so $\mathcal{R}_t(T) \approx 2$.
- The Heston model (with $V_t = \bar{V}$),

$$(X \diamond M)_t(T) = \rho \eta \bar{V} \int_0^\tau dt \int_t^\tau e^{-\kappa(u-t)} du$$

- For $\tau \ll 1/\kappa$, $(X \diamond M)_t(T) \sim \tau^2$ and $\mathcal{R}_t(T) \approx 2$.
- For $\tau \gg 1/\kappa$, $(X \diamond M)(\tau) \sim \tau$ and $\mathcal{R}_t(T) \approx 1$.

The n -factor Bergomi model

- Let κ_1 be the shortest timescale (largest) mean reversion coefficient and κ_n be the longest timescale (smallest) mean reversion coefficient.
 - For $\tau \ll 1/\kappa_1$, $(X \diamond M)_t(T) \sim \tau^2$ and $\mathcal{R}_t(T) \approx 2$.
 - For $\tau \gg 1/\kappa_n$, $(X \diamond M)_t(T) \sim \tau$ and $\mathcal{R}_t(T) \approx 1$.
- For classical stochastic volatility models in general, $\mathcal{R}_t(T) \approx 2$ for τ small and $\mathcal{R}_t(T) \approx 1$ for τ large.

The skew-stickiness ratio under rough volatility

- Under rough Heston

$$(X \diamond M)_t(T) = \frac{\rho \nu}{\Gamma(1 + \alpha)} \int_t^T \xi_t(s) (T - s)^\alpha ds \approx \frac{\rho \nu}{\Gamma(2 + \alpha)} V_t \tau^{\alpha+1}.$$

- Then, for very short times,

$$\mathcal{R}_t(T) \approx \tau \frac{d}{d\tau} \log(X \diamond M)(\tau) \approx \alpha + 1.$$

- Rough Heston is thus an explicit example of a model in which

$$\mathcal{R}_t(T) \approx 1 + \alpha = H + \frac{3}{2}.$$

- Fukasawa [Fuk21] derives the above result formally for generic rough volatility models in the limit $\tau \rightarrow 0$.

The time series of SSR from Vola Dynamics

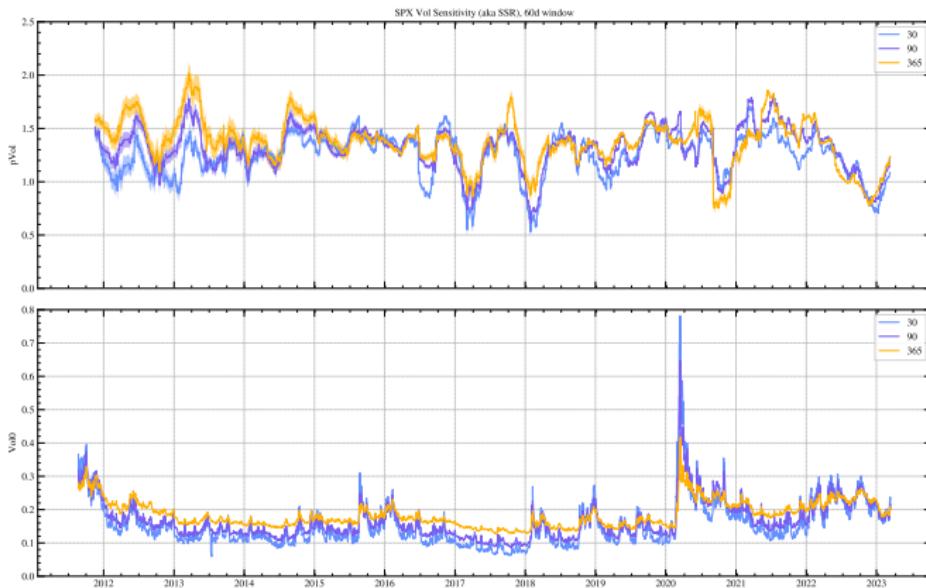


Figure 5: The 30 day, 90 day, and 1 year skew-stickiness ratios (SSR), with a trailing window of 60 days from Vola Dynamics.

The time series of SSR from Vola Dynamics

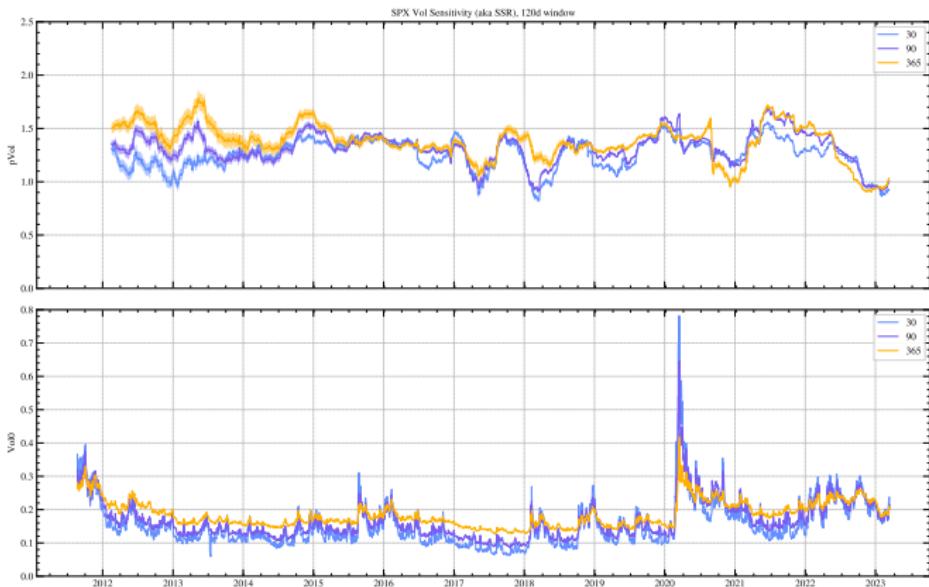


Figure 6: The 30 day, 90 day, and 1 year skew-stickiness ratios (SSR), with a trailing window of 120 days from Vola Dynamics.

Is rough volatility consistent with the SSR time series?

- To be consistent with rough volatility, we would need $\mathcal{R}(T) > \frac{3}{2}$.
 - We see that, empirically, $0.9 < \mathcal{R}(T) < 1.7$.
- Maybe volatility is super-rough?
- Moreover the previous analysis only applies in the limit $T - t = \tau \rightarrow 0$
- and, we haven't taken account of the shape of the forward variance curve.

More carefully: The SSR in AFV models

- In affine forward variance models [GKR19],

$$\frac{dS_t}{S_t} = \sqrt{V_t} dZ_t; \quad d\xi_t(u) = \kappa(u - t) \sqrt{V_t} dW_t.$$

- $M_t(T) = \int_t^T \xi_t(u) du$ so

$$dM_t(T) = \left(\int_t^T \kappa(u - t) du \right) \sqrt{V_t} dW_t =: \tilde{\kappa}(T - t) \sqrt{V_t} dW_t,$$

and

$$d\langle X \rangle_t = V_t dt; \quad d\langle X, M \rangle_t = \rho \tilde{\kappa}(T - t) V_t dt.$$

- Thus

$$(X \diamond M)_t(T) = \rho \int_t^T \xi_t(s) \tilde{\kappa}(T - s) ds.$$

Dependence of the SSR on ξ

- Recall that

$$\mathcal{R}_t(T) = \frac{\beta_t(T)}{\psi_t(T)} = \frac{M_t(T) d \langle X, M \rangle_t}{d \langle X \rangle_t (X \diamond M)_t(T)} + \mathcal{O}(\epsilon^2).$$

- Then, in AFV models,

$$\mathcal{R}_t(T) = \frac{\left(\int_t^T \xi_t(s) ds \right) \tilde{\kappa}(T-t)}{\int_t^T \xi_t(s) \tilde{\kappa}(T-s) ds}. \quad (12)$$

- $R_t(T)$ depends on the shape of $\xi_t(u)$!
 - A monotonic increasing ξ causes the SSR to increase, and vice versa.

Dependence of the SSR on ξ

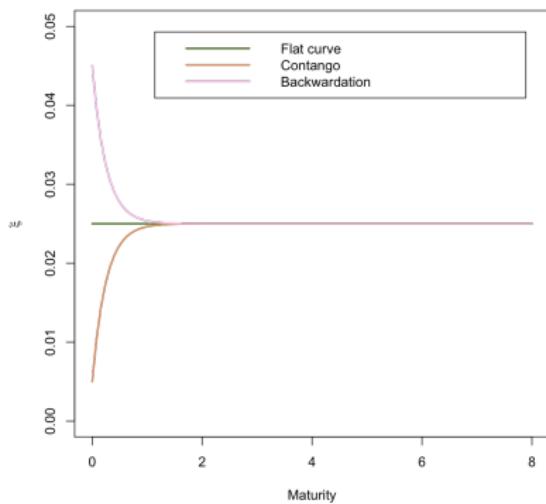
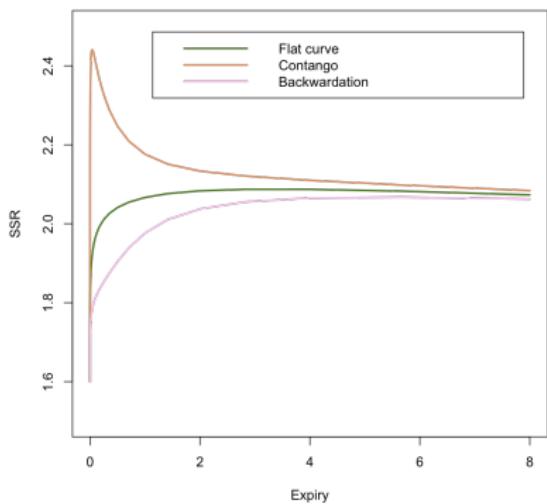


Figure 7: The SSR depends on the shape of the forward variance curve.

Path-dependence of the SSR

- With $dW_t = \rho dZ_t + \sqrt{1 - \rho^2} dZ_t^\perp$,

$$\begin{aligned}\xi_t(u) &= \bar{\xi} + \int_{-\infty}^t \kappa(u - r) \sqrt{V_r} dW_r \\ &= \bar{\xi} + \rho \int_{-\infty}^t \kappa(u - r) \frac{dS_r}{S_r} + \text{independent noise.}\end{aligned}$$

- The forward variance curve depends on a weighted average of historical stock returns.
- We deduce that $\mathcal{R}_t(T)$ depends on weighted average historical stock returns.
 - Historical negative returns should cause the SSR to increase, and vice versa.
 - From Figure 7, the SSR seems pretty sensitive to the shape of the forward variance curve.

How good is the skew approximation?

- Under rough Heston, with reasonable parameters:

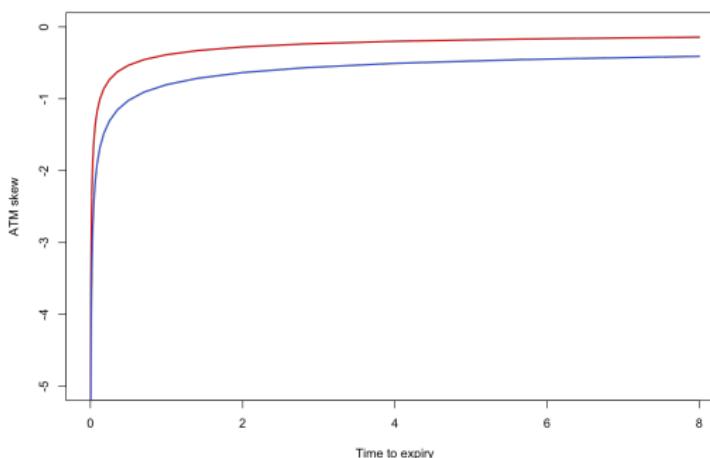


Figure 8: The rough Heston skew is in red and the first order approximation in blue.

- The first order approximation gets the shape correct but not the level.

How short is short?

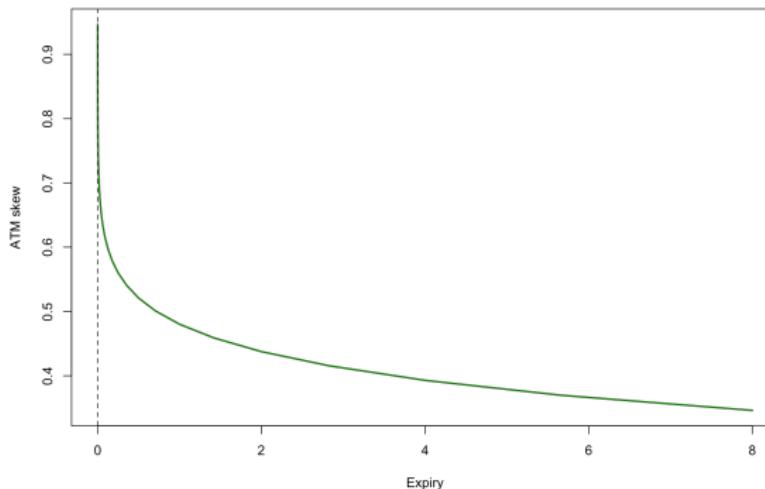


Figure 9: The ratio of the rough Heston skew to the BG first order approximation.

- The approximate formula overestimates the skew.
 - The ratio hits 0.95 for expirations later than 15 seconds!

Cancelation of errors?

- Maybe errors in the skew $\psi(T)$ are balanced to some extent by similar errors in $\beta(T)$.
- Figure 10 confirms this guess.
 - However, the skew stickiness ratio remains far above $H + \frac{3}{2}$ in the rough Heston model, for any expiration of practical interest.
- The short term limiting expression for the SSR in terms of $X \diamond M$ has no practical relevance!

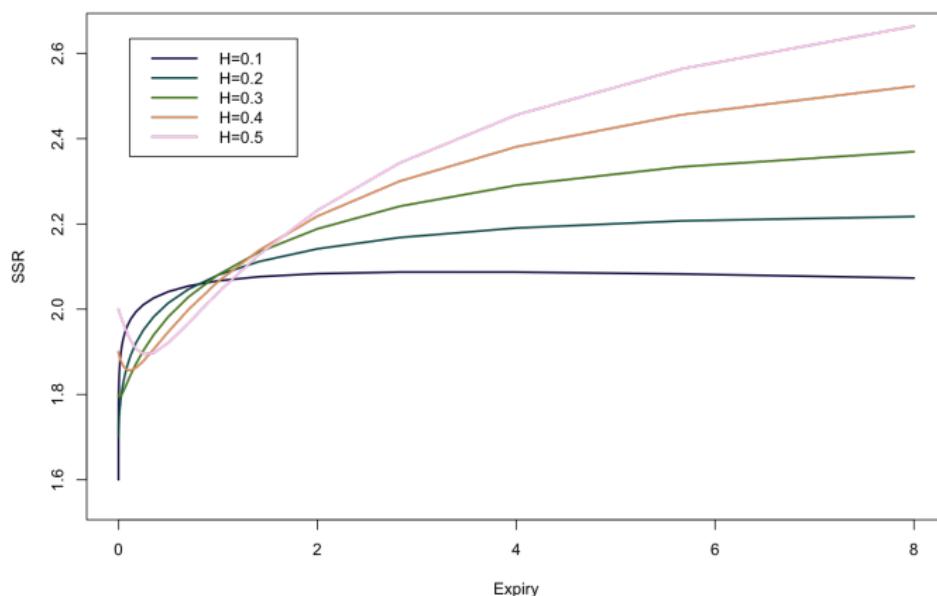
SSR under rough Heston ($\lambda = 0$) for various H 

Figure 10: Theoretical SSR under rough Heston with $\xi = 0.025$ and the Padé (4,4) approximation [GR19] of the characteristic function.

Summary

- Empirically, implied volatility moves proportionally to the implied volatility skew.
 - The constant of proportionality, the SSR, is historically between 1 and 1.5.
- Under stochastic volatility, the SSR for very short expirations is given by the time derivative of $\log X \diamond M$.
 - $\mathcal{R}_t(T) \approx H + \frac{3}{2}$ where H is the Hurst exponent of V_t .
 - The SSR depends on the recent history of the underlying stock return process.
- For longer expirations, under rough Heston at least, it seems that $\mathcal{R}_t(T) \gtrapprox 2$.
 - We need to check if this remains true under other modeling assumptions.
- For the moment, the empirical behavior of the SSR seems to be inconsistent with (classical or rough) stochastic volatility!

References

-  **Elisa Alòs, Jim Gatheral, and Radoš Radoičić.**
Exponentiation of conditional expectations under stochastic volatility.
Quantitative Finance, 20(1):13–27, 2020.
-  **Lorenzo Bergomi**
Smile dynamics IV.
Risk December, pages 94–100, 2009.
-  **Lorenzo Bergomi and Julien Guyon.**
Stochastic volatility's orderly smiles.
Risk May, pages 60–66, 2012.
-  **Lorenzo Bergomi.**
Stochastic Volatility Modeling.
Chapman and Hall/CRC, 2016.
-  **Peter K Friz, Jim Gatheral and Radoš Radoičić.**
Forests, cumulants, martingales.
The Annals of Probability, 50(4):1418–1445, 2022.

-  Masaaki Fukasawa.
Volatility has to be rough.
Quantitative Finance, 21(01):1–8, 2021.
-  Jim Gatheral.
The volatility surface: A practitioner's guide.
John Wiley & Sons, 2006.
-  Jim Gatheral and Martin Keller-Ressel.
Affine forward variance models.
Finance and Stochastics, 23(3):501–533, 2019.
-  Jim Gatheral and Radoš Radoičić.
Rational approximation of the rough Heston solution.
International Journal of Theoretical and Applied Finance, 22(3):1950010–19, 2019.
-  Martin Hairer.
Solving the KPZ equation,
Annals of Mathematics, 178:559–664, 2013.
-  Michael Kamal and Jim Gatheral
Implied Volatility Surface,
Encyclopedia of Quantitative Finance, Wiley, 2009.