The design of a distributed framework for the enumeration of sets of mutually orthogonal Latin squares

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Declaration

By submitting this thesis electronically, I declare that the entirety of the work contained therein is my own, original work, that I am the sole author thereof (save to the extent explicitly otherwise stated), that reproduction and publication thereof by Stellenbosch University will not infringe any third party rights and that I have not previously in its entirety or in part submitted it for obtaining any qualification.

Date: December 1, 2014

Abstract

Write your English abstract here.

Uittreksel

Skryf jou Afrikaanse uittreksel hier.

Acknowledgements

The author wishes to acknowledge the following people for their various contributions towards the completion of this work:

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Glossary

Something Description of that something.

Something Description of that something.

Something Description of that something.

List of Reserved Symbols

Symbols in this thesis conform to the following font conventions:		
A	Symbol denoting a **some general thing**	(Roman capitals)
$\mathcal{A}$	Symbol denoting a **some general thing**	(Calligraphic capitals)

Symbol	Meaning
$\times$	Symbol used to denote the multiplication operator
$\times$	Symbol used to denote the multiplication operator
$\times$	Symbol used to denote the multiplication operator
$\times$	Symbol used to denote the multiplication operator
$\times$	Symbol used to denote the multiplication operator

List of Acronyms

WISF: What It Stands For

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CHAPTER 1

Introduction

Contents

Chapter 1 will introduce the topic of Latin squares in a gripping way so that the readers interest is piqued enough to continue reading. It will formally define the problem tackled in this thesis, set clear objectives for the thesis and delimit the scope of the research in a reasonable and sensible way. Finally, this chapter will also give an overview of what the reader may expect from the rest of the thesis and briefly outline the content of the succeeding chapters.

1.1 Historical background

1.2 Problem statement

1.3 Scope and objectives

Objectives:

- Survey the literature on Latin squares and distributed computing projects
- Study the state-of-the-art techniques used for enumerating k-MOLS
- Design a fast algorithm for enumerating main classes of k-MOLS
- Implement said algorithm and verify results by comparing it to published findings
- Design and launch a distributed computing project
- Obtain (novel) results from the distributed enumeration
- Contribute towards answering the celebrated existence question of a 3-MOLS of order 10 by estimating the effectiveness of a distributed enumeration approach

Scope: Combinatorial designs other that Latin squareare considered to be beyond the scope of this thesis, except in cases where the use of these designs may contribute towards a clearer understanding of some topic. The thesis will focus on enumerating main classes of k-MOLS, any other equivalence classes are considered to be beyond the scope.

1.4 Thesis organisation

CHAPTER 2

Background

Contents

Chapter 2 introduces the basic theory of Latin squares. Its will start out with basic definitions of a Latin square, show the relationship between Latin squares and quasigroups and define properties such as symmetric, idempotent and unipotent. The notions of a universal and transversal will also be introduced. It will continue by exploring relationships between Latin squares, specifically the notion of orthogonality and k-MOLS and introducing alternative representations of k-MOLS, for example orthogonal arrays. It will also examine the different ways in which permutations may be applied to the symbol set and row or column indexing sets of a Latin square or a MOLS and give examples of conjugate operations. The chapter will conclude by reviewing some ways of recursively constructing Latin squares from smaller Latin squares by means of, amongst other techniques, elongation and taking direct products.

2.1 Basic definitions

A *Latin square* of order n is commonly defined (see, amongst others, Colbourn and Dinitz [7, Definition 1.1]) to be an $n \times n$ array in which every cell contains a single symbol from an n-set S, such that each symbol occurs exactly once in each row and column.

If, for example, S contained the four suits of playing cards, in other words $S = \{\blacklozenge, \heartsuit, \clubsuit, \spadesuit\}$, then the 4×4 array

$$\begin{bmatrix} \blacklozenge & \heartsuit & \clubsuit & \spadesuit \\ \spadesuit & \blacklozenge & \heartsuit & \clubsuit \\ \clubsuit & \spadesuit & \blacklozenge & \heartsuit \\ \heartsuit & \clubsuit & \spadesuit & \blacklozenge \end{bmatrix}$$

would be an example of a Latin square of order 4.

Let $S(\boldsymbol{L})$ denote the symbol set of a Latin square $\boldsymbol{L}$ and let $R(\boldsymbol{L})$ and $C(\boldsymbol{L})$ denote its row and column indexing sets, respectively. For any $i \in R(\boldsymbol{L})$ and $j \in C(\boldsymbol{L})$ define $\boldsymbol{L}(i, j) \in S(\boldsymbol{L})$ as the element in the i-th row and the j-th column of $\boldsymbol{L}$. In the remainder of this thesis it is assumed that $R(\boldsymbol{L}) = C(\boldsymbol{L}) = S(\boldsymbol{L}) = \mathbb{Z}_n$, without any subsequent loss of generality.

The *transpose* of $\boldsymbol{L}$, denoted $\boldsymbol{L}^T$, is the Latin square for which $\boldsymbol{L}^T(j,i) = \boldsymbol{L}(i,j)$ for all $i \in R(\boldsymbol{L})$ and $j \in C(\boldsymbol{L})$. The *k-th diagonal* of $\boldsymbol{L}$ is the set of entries $\{((k+i) \mod n, i) \mid i \in \mathbb{Z}_n\}$ and the 0-th diagonal is simply referred to as the *main diagonal*. Any row or column in which all of the entries appear in numerical order, *i.e.* $0, 1, \ldots, n-1$, is said to be in *natural order*.

Let $\boldsymbol{L}(i)$ and $\boldsymbol{L}^T(j)$ denote the i-th row and the j-th column of the Latin square $\boldsymbol{L}$, respectively (note that the j-th column of $\boldsymbol{L}$ is, by definition, also the j-th row of $\boldsymbol{L}^T$). A Latin square may also be defined as an $n \times n$ array with the additional property that every row and column is a permutation of the elements of $S(\boldsymbol{L})$. Any individual row or column is therefore a permutation. For instance row i may be expressed as the permutation

$$\boldsymbol{L}(i) = \begin{pmatrix} 0 & 1 & \ldots & n-1 \\ \boldsymbol{L}(i,0) & \boldsymbol{L}(i,1) & \ldots & \boldsymbol{L}(i,n-1) \end{pmatrix}.$$

It is clear that every element $k \in \mathbb{Z}_n$ is mapped to a distinct element $\boldsymbol{L}(i,k) \in S(\boldsymbol{L})$ by every permutation in the set of row permutations $\{\boldsymbol{L}(i) \mid i \in \mathbb{Z}_n\}$ in order to prevent the repetition of symbols in column k. A similar condition may be imposed on the set of column permutations, $\{\boldsymbol{L}^T(j) \mid j \in \mathbb{Z}_n\}$.

Although Latin squares were studied by Leonard Euler in 1782, British mathematician Arthur Cayley was first to notice, nearly a century later, that the multiplication table (or *Cayley table*, see Appendix A) of a group is an appropriately bordered Latin square. When the abstract concept of a group was generalised to *quasigroups* and *loops* during the 1930s, Latin squares again emerged as the corresponding Cayley tables, as is evident from the following result which may be found in Dénes and Keedwell [9, Theorem 1.1.1].

Theorem 2.1 ([9]). *The Cayley table of a quasigroup is a Latin square.*

Given a Latin square $\boldsymbol{L}$, the *underlying quasigroup* of $\boldsymbol{L}$ may be defined as the group $(G, \circ)$ where $a \circ b = c$ if $\boldsymbol{L}(a,b) = c$. In the case where the first row and column of $\boldsymbol{L}$ both appear in natural order, $\boldsymbol{L}$ is said to be a *reduced Latin square* or *in standardised form*. The element 0 in the underlying quasigroup of a reduced Latin square $\boldsymbol{L}$ is therefore the identity element of the quasigroup $(G, \circ)$ so that $(G, \circ)$ may be referred to as the *underlying loop* of $\boldsymbol{L}$. The Cayley table of the group $(\mathbb{Z}_n, +)$ provides such a reduced form Latin square for all orders $n \in \mathbb{Z}$. For example, the reduced form Latin square

$$\boldsymbol{L}_{2.1} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 0 \\ 2 & 3 & 4 & 5 & 0 & 1 \\ 3 & 4 & 5 & 0 & 1 & 2 \\ 4 & 5 & 0 & 1 & 2 & 3 \\ 5 & 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

of order 6. The Cayley table of the group $(\mathbb{Z}_n, +)$ is also an example of a *symmetric* Latin square, that is, a Latin square such that $\boldsymbol{L}(i,j) = \boldsymbol{L}(j,i)$ for all $i \in R(\boldsymbol{L}), j \in C(\boldsymbol{L})$.

In addition to symmetry, a Latin square $\boldsymbol{L}$ may also have various other structural properties. It may, for example, contain an $s \times s$ subarray which is itself also a Latin square, called a *subsquare* of side s. If $R' \subset R(\boldsymbol{L})$ and $C' \subset C(\boldsymbol{L})$ are subsets of the row and column indexing sets, both with cardinality s, then a subsquare is formally defined as the set of entries $\{(i,j) \mid i \in R', j \in C'\}$ in $\boldsymbol{L}$. It is easy to see that, as a subsquare is embedded in a Latin square, a necessary and sufficient condition for the existence of a subsquare is that it contains exactly s different symbols. For example, the Latin square

$$\boldsymbol{L}_{2.2} = \begin{bmatrix} \mathbf{0} & \mathbf{3} & 6 & \mathbf{1} & 5 & 4 & 2 \\ \mathbf{3} & \mathbf{1} & 4 & \mathbf{0} & 2 & 6 & 5 \\ 6 & 4 & 2 & 5 & 1 & 3 & 0 \\ \mathbf{1} & \mathbf{0} & 5 & \mathbf{3} & 6 & 2 & 4 \\ 5 & \underline{2} & 1 & \underline{6} & 4 & 0 & 3 \\ 4 & \underline{6} & 3 & \underline{2} & 0 & 5 & 1 \\ 2 & 5 & 0 & 4 & 3 & 1 & 6 \end{bmatrix}$$

contains at least two disjoint subsquares, a 3×3 subsquare (shown in boldface), defined by $R' = \{0, 1, 3\}$ and $C' = \{0, 1, 3\}$, and a 2×2 subsquare (underlined), defined by $R'' = \{4, 5\}$ and $C'' = \{1, 3\}$. Such a 2×2 subsquare of a Latin square $\boldsymbol{L}$ is also sometimes called an *intercalate* of $\boldsymbol{L}$.

The relationship between the Cayley tables of quasigroups and Latin squares extend naturally to subquasigroups and subsquares. More specifically, the Cayley table of a subquasigroup G' of a quasigroup G is a Latin subsquare of the Latin square defined by the Cayley table of G. Conversely, an appropriately bordered Latin subsquare in the Latin square defined by the Cayley table of G is the Cayley table of a subquasigroup.

Interestingly, due to a group theoretic result by HB Mann and WA McWorter (see [22]), the largest possible subsquare of an $n \times n$ Latin square has sides $s \leq \lfloor n/2 + 1 \rfloor$.

Two important notions when dealing with Latin squares are those of transversals and universals. Leonard Euler introduced the notion of a *transversal* of a Latin square under the name *formule directrix* in [11] and it has also merely been called a *directrix*, notably by HW Norton [25]. A transversal V of a Latin square $\boldsymbol{L}$ of order n, is a set of n distinct, ordered pairs (i, j), one from each row and column, containing all of the n symbols exactly once [7, Definition 1.27]. Transversals are important for many constructions of Latin squares and have close ties to complete mappings in quasigroups (see Appendix), as is highlighted by the following result, which may be found in [7, Definition 6.5].

Theorem 2.2 ([7]). *There is a one-to-one correspondence between the transversals of a Latin square $\boldsymbol{L}$ and the complete mappings of a quasigroup $(G, \circ)$ with $\boldsymbol{L}$ as Cayley table.*

A *universal*, U, of a Latin square $\boldsymbol{L}$ is a set of n distinct, ordered pairs (i, j), one from each row and column, containing only one symbol. A universal is therefore the set of all the entries containing a single symbol in $\boldsymbol{L}$, a particularly useful concept introduced by Kidd, Burger and van Vuuren in 2012 to facilitate the enumeration of specific classes of Latin squares [19].

Both transversals and universals may be expressed in permutation form. A *transversal permutation* v sets $v(i) = j$ if $(i, j) \in V$, while the *universal permutation of* k sets $u_k(i) = j$ if $\boldsymbol{L}(i, j) = k$. In the Latin square $\boldsymbol{L}_{2.2}$, the main diagonal is clearly a transversal, say V, and the universal of 0 is given by $U_0 = \{(0, 0), (1, 3), (2, 6), (3, 1), (4, 5), (5, 4), (6, 2)\}$. The corresponding permutations are $v = \left(\begin{smallmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{smallmatrix}\right)$ and $u_0 = \left(\begin{smallmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 3 & 6 & 1 & 5 & 4 & 2 \end{smallmatrix}\right)$.

A Latin square which contains a transversal in natural order on its main diagonal, like $\boldsymbol{L}_{2.2}$, is said to be *idempotent*. Formally, an idempotent Latin square of order n has $\boldsymbol{L}(i, i) = i$ for all $i \in \mathbb{Z}_n$. A Latin square with a universal on the main diagonal is said to be *unipotent*.

2.2 Orthogonal Latin squares

According to Colbourn and Dinitz [7, Definition 3.1] two Latin squares of order n, $\boldsymbol{L}$ and $\boldsymbol{L}'$, are considered *orthogonal* if $\boldsymbol{L}(i,j) = \boldsymbol{L}(k,l)$ and $\boldsymbol{L}'(i,j) = \boldsymbol{L}'(k,l)$ implies that $i = k$ and $j = l$. Equivalently, orthogonality implies that every element of $\mathbb{Z}_n \times \mathbb{Z}_n$ appears exactly once among the ordered pairs $(\boldsymbol{L}(i,j), \boldsymbol{L}'(i,j))$ for $i, j \in \mathbb{Z}_n$.

Latin squares were first formally defined by Leonard Euler when he considered the so-called "36-Officers problem" asking whether it is possible to arrange thirty-six soldiers of six different ranks and from six different regiments in a square such that every row and column contained exactly one soldier of every rank, and one soldier from every regiment [10]. Labelling the ranks and regiments from the symbol set $\mathbb{Z}_n$, it is clear that Euler was attempting to find a pair of orthogonal Latin squares of order 6 where the entry in $\boldsymbol{L}(i,j)$ would indicate the rank of the soldier in position (i,j) and $\boldsymbol{L}'(i,j)$ his regiment. Euler was unable to find such an arrangement of soldiers and continued to propose what has become known as Euler's Conjecture, that no pair of orthogonal Latin squares order n exist when $n = 4m + 2$ for integer values of m [10].

Euler's hunch was lent some credence more than a century later when amateur French mathematician Gaston Tarry proved in two papers that a solution to the "36-Officers problem" (and hence to the special case of Euler's Conjecture where $n = 6$) does, indeed, not exist [30]. Sixty year later, however, pairs of orthogonal Latin squares were constructed of order 22 [5] and order 10 [28], thereby disproving Euler's Conjecture, before Bose, Shrikhande and Parker showed that it is possible to construct such pairs for all cases of Euler's Conjecture except when $n = 6$ [4].

It should be noted that orthogonality may also be expressed in terms of transversals and universals, specifically, it is necessary that the entries making up every transversal in $\boldsymbol{L}$ correspond to a universal in $\boldsymbol{L}'$ [**wallis**]. It follows that a Latin square $\boldsymbol{L}$ has an orthogonal mate $\boldsymbol{L}'$ if and only if $\boldsymbol{L}$ has n disjoint universals [**denes1**], as each of these transversals will correspond to a universal in $\boldsymbol{L}'$. The Latin square $\boldsymbol{L}$ of order $2k$ with $\boldsymbol{L}(i,j) = i + j \pmod{2k}$, in other words, the Latin square which is the the Cayley table of the the group $(\mathbb{Z}_{2k}, +)$, is an example of a Latin square without any transversals and therefore has no orthogonal mate.

According to Colbourn and Dinitz [7, Definition 3.3], the notion of orthogonality may be generalised a set of *mutually orthogonal* Latin squares $\boldsymbol{L}_1, \boldsymbol{L}_2, \ldots, \boldsymbol{L}_k$, or k-MOLS, where $\boldsymbol{L}_i$ and $\boldsymbol{L}_j$ are orthogonal for all $1 \leq i < j \leq k$. The set of Latin squares

$$\mathcal{M} = \left\{ \begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \\ 1 & 0 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix} \right\},$$

for example, form a 3-MOLS of order 4.

MOLS have been shown to have important applications to coding theory [20], subfields of statistics including experimental design (notably by RA Fisher in [12] and [13]) and the scheduling of sports tournaments (see, amongst many others, Kidd [18], Keedwell [17] and Robinson [29]).

It is natural to consider the number of Latin squares in the largest possible MOLS of order n, denoted by $N(n)$. It is possible to establish an upper bound for $N(n)$ by considering a MOLS with the property that every Latin square has been relabelled so that the first row appears in natural order. There are clearly exactly $n - 1$ possible symbols for the first element in the second row of the Latin square and, therefore, at most $n - 1$ Latin squares in the MOLS. This informal argument may be formalised (see, for example, Dénes and Keedwell [**denes1**]) to prove the well-known result that $N(n) \leq n - 1$ for all orders $n > 1$. Although such an $(n-1)$-MOLS, or *complete MOLS*, clearly exists for $n = 4$ in the example above and in general whenever n is a prime power [], RH Bruck and HJ Ryser showed in [**bruck**] that there is also an infinite set of orders for which $N(n) < n - 1$ [1].

2.3 Operations on Latin squares

permutations and conjugates
orthogonal array

standard form wallis p183

2.4 Constructions of Latin squares

Direct product etc

2.5 Chapter summary

[1]These results are due to the fact, first proven by RC Bose in [**bosepp**], that a complete MOLS of order n exists if and only if there exist a finite projective plane of order n, however, finite projective planes are considered outside the scope of this study. For further information regarding the equivalence of finite projective planes and complete MOLS, see Mann, Keedwell and Martin.[**mann**]. A proof of the non-existence of a finite projective plane of order 6, and hence a solution to the "36-Officers problem," may also be of interest and may be found in MacInnes [**McInnes**].

CHAPTER 3

Enumeration of MOLS

Contents

Chapter 3 will give an overview of the main equivalence classes of Latin squares and motivate why the decision was made to focus only on enumerating main classes. Some examples of existing enumeration techniques for MOLS will be described, after which an algorithm for the enumeration of main classes will be designed. The algorithm will be verified by comparing its results with previously published findings.

3.1 Historical overview

3.2 Main classes of MOLS

3.3 Enumeration methodology

3.4 Chapter summary

CHAPTER 4

The design of a distributed computing project

Contents

Chapter 4 introduces the concept of distributed computing by giving its history and a summary of some well-known projects, together with their results and the frameworks they were built on. The different distribution frameworks that are available today will then be surveyed. The working of Berkeley Open Infrastructure for Network Computing, in particular, will be examined in detail, as well as all the requirements for a BOINC project. The chapter will conclude in specifying the design choices neccessary to create a BOINC project for the enumeration of main classes of Latin squares, for example how to break the search tree up into workunits, how to assign workunits to a user based on system specifications etc.

4.1 A brief history of distributed computing

4.2 Using Berkeley Open Infrastructure for Network Computing

4.2.1 Basic workflow

4.2.2 The components of a distributed computing project

4.3 Designing a BOINC project for the enumeration of MOLS

4.4 Chapter summary

CHAPTER 5

Deployment of the distributed enumeration project

Contents

Chapter 5 will contain details pertaining to the method of deploying a small-scale distributed search based on the design set out in Chapter 4 as a proof of concept. It will include how awareness of the project was raised in the scientific community, the challenges involved in launching the project as well as the lessons learnt for the possible future deployment of larger scale projects. The chapter will conclude with the results of the distributed computing project designed in Chapter 4, most likely the enumeration of main classes of MOLS of order 8 or 9. It will also give a summary of the distributed enumeration in terms of number of volunteers, locality, how work was done by each of the volunteers etc.

5.1 Methodology

5.1.1 Technical specifications

5.1.2 Raising public awareness

5.1.3 Challenges

s

5.2 Distributed enumeration of MOLS

5.2.1 Enumeration summary

5.2.2 Results

5.3 Chapter summary

CHAPTER 6

Conclusions

Contents

The final chapter summarizes the work in this thesis, briefly recount the contents of each preceding chapter and proceeds to evaluate the contributions made. The thesis concludes with a reflection on possible continued studies arising from the contributions made by the thesis.

6.1 Overview of work

6.2 Summary of contributions

6.3 Possible further work

APPENDIX A

Group theory

group, quasi group, loop Cayley table complete mapping

[*]

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