Stability, Fairness and the Pursuit of Happiness in Recommender Systems

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Top-k personalized recommendations are ubiquitous, but are they stable? We study whether, given complete information, buyers and sellers prefer to participate in matches formed by top-k recommendations rather than pursuing offline matches among themselves. When there are no constraints on the number of times an item is recommended, we observe that top-k recommendations are stable. When exposures are constrained, e.g., due to limited inventory or exposure opportunities, stable recommendations need not exist. We show that maximizing total buyer welfare under unit exposure constraints is stable, Pareto optimal and swap-envy free for orthogonal buyers, identical buyers, and buyers with dichotomous valuations. Most of these properties are retained under arbitrary exposure constraints. Finally, we consider variants of common recommendation strategies and find that they lead to substantial instability and envy in three real-world datasets. Among them, maximizing buyer welfare leads to the most stable outcomes and near-zero swap-envy.

1 INTRODUCTION

Recommender systems play an important role in market making by matching buyers to products (and their sellers) in large online platforms. They do so by learning buyers' preferences from past ratings and recommending to each buyer a subset of products she would like, from which the buyer typically chooses one. Traditionally, recommender systems focused on satisfying individual buyers, with the implicit assumption that matching buyers to products they like also benefits sellers by increasing sales and attracting more buyers to the market.

Recent research, however, questions this assumption. Buyer-focused recommender systems can concentrate sales on popular sellers and increase inequities [Fleder and Hosanagar, 2009]. This poses a risk to online marketplaces, since disgruntled sellers may withdraw inventory and target buyers through off-platform channels. As a result there are calls to design recommender systems that serve all stakeholders — buyers, sellers, and the platform [Abdollahpouri et al., 2020].

Such systems not only need to exhibit multi-sided fairness, but must also respect different stakeholders' constraints. For example, recommending a physical good to more buyers than the number of available copies can result in a costly stock-out. For platforms like ad-networks, exposure is a limited resource. This, along with contracted guarantees on exposures, limits how many times a seller can be recommended to potential buyers. Sellers, in turn, would like scarce exposures to target buyers who give them the best chance of making a sale.

Against this backdrop, we take inspiration from the matching literature and study the stability of recommender systems. Would buyers and sellers, given complete information on preferences and constraints of participants on both side, participate in a platform that uses a personalized recommender system? Or, might some prefer to pursue off-platform transactions among themselves?

This is an important question because even in offline settings with a high cost of discovery where a lack of complete information can let the market operate for some time, unstable markets unravel over time due to persistent incentives.[1] Discovery is easier with digital platforms. Furthermore, competition among digital platforms increasingly facilitates multi-homing and other forms of off-platform transactions [Bakos and Halaburda, 2020, Bryan and Gans, 2019]. A platform relying on unstable matches risks its participants easily migrating to alternative platforms.

1.1 Our Contributions

We start with McFadden [1973]'s choice model for the behaviour of a buyer who is recommended a set of items (or choice set). We investigate three properties towards robust recommendations. *Stability* requires that buyers and sellers would prefer to be matched via the recommendations made by the system rather than make side-deals [Roth and Sotomayor, 1992]. *Pareto optimality* (of buyer utilities) ensures efficiency: increasing the utility of any buyer comes at the cost of another. *Envy-freeness* guarantees that no buyer prefers to receive the recommendations made to another over her own. We propose a relaxation of envy-freeness suitable for the context of top-k recommendation called swap-envy-freeness up to one good (SEF1), which allows for envy but only to the extent that it can be eliminated by exchanging a pair of recommended items.

We study the existence of stable recommendations in Section 3. In the absence of exposure constraints, when each item can be recommended an arbitrary number of times, we observe that top-k recommendations are stable, envy free, and Pareto optimal. Limiting the number of times an item is recommended, however, may make it impossible to find a stable recommendation (Theorem 3). We show that the potential gain from deviating from the platform's recommendations, for both buyers and sellers, follows from the disparity in buyers valuations of items and the values

[1]For example, when matching interns to hospitals and matching children to schools [Roth, 1984, 2015].

of their choice sets. In fact, the gain can be (tightly) upper bound by the product of two factors that measure these disparities.

Next, we study restricted preference domains under a strong exposure constraint in which each item may be recommended only once. Specifically, we consider identical preferences, as might be the case when the quality of an item is publicly known and agreed upon, in Section 3.1 and dichotomous values, where each buyer views each item as either high value or low value, in Section 3.2. In both cases, somewhat surprisingly, maximizing the total expected buyer utility leads to stable, Pareto optimal and SEF1 recommendations. Notably, maximizing expected buyer utility under our choice model is equivalent to maximizing the *Nash welfare*, with respect to the exponentiated utilities, which is known to have several attractive fairness properties [Caragiannis et al., 2019].

These results are extended to arbitrary upper bounds on the number of times an item is recommended, with some caveats. Stability is not guaranteed for identical buyers in general (Example 8), though maximizing the expected buyer welfare remains PO and SEF1 (Theorem 9). In the dichotomous value setting, swap-envy-freeness is not guaranteed (Example 11), but maximizing buyer welfare still guarantees stability and Pareto optimality (Theorem 12).

Absent the guarantee of stable recommendations, we propose practical recommendation algorithms under exposure constraints in Section 4. The first algorithm draws on the theoretical results above and maximizes expected buyer welfare (Section 4.1), though this is not guaranteed to find a stable recommendation even when one exists. Round robin recommendations are considered in Section 4.2 and shown to be SEF1 under unit exposure constraints (Theorem 14). A greedy form of top-k recommendations in Section 4.3 better mimics decision-making in an online variant of the problem but is not guaranteed to return a recommendation that is stable, PO or SEF1. Finally we outline how any batch recommendation algorithm can be deployed in an online manner for users who arrive one at a time and must be recommended items as they arrive.

We conclude with a computational study using datasets from Amazon and Rent-the-runway. The algorithms all result in unstable recommendations in which a large fraction of participants have some incentive to deviate from the platform's recommendations. The magnitude of this incentive, as an improvement over the participant's current utility, range from 150% for greedy top-k to 8% for the welfare maximizing recommendations. The online variant of round robin, though worse than the batch version, performs significantly better than greedy top-k.

Our theoretical and computational results are in agreement: Platforms worried about participants pursuing off-platform transactions should maximize buyer welfare. It is provably stable in some settings and, even when stable recommendations can not be guaranteed, leads to the lowest incentive to deviate in our computational experiments.

1.2 Related Work

This paper combines ideas from work on recommender systems, matching, and fair division. We briefly discuss some of the most closely related work in each stream.

Multi-stakeholder recommendations, which consider multiple buyer, seller and platform preferences, have become increasingly popular [Abdollahpouri et al., 2020, Burke et al., 2016, Nguyen et al., 2017]. The study of fairness in recommender systems often takes place in the context of multi-objective recommendations [Abdollahpouri and Burke, 2019, Ekstrand et al., 2022, Patro et al., 2020]. Chakraborty et al. [2017] recognize the difficulty of achieving fair allocations under static recommendation. Drawing from research in processor scheduling, they point out that proportional fairness can be achieved by considering recommendations over time. Bateni et al. [2022] propose a stochastic approximation scheme based on the Eisenberg-Gale convex program for an online advertising system which maximizes platform revenue while being approximately fair towards buyers. Patro et al. [2020] propose a greedy version of round robin which is shown to be envy-free up to one good (EF1) for buyers and guarantees sellers some exposure. In the context of recommending packages/bundles to groups that consume them together, Serbos et al. [2017] show that maximizing fairness towards *all members of a group* is NP-hard and offer greedy approximation algorithms.

Questions of fairness become salient when buyers or sellers face constraints. Recommendation under capacity or exposure constraint has been studied, generally to maximize sales [Makhijani et al., 2019] or user utility [Sürer et al., 2018], rarely to unpack participation incentives. Elsewhere, Tennenholtz and Kurland [2019] point out that the standard relevance based ranking by content recommender systems can create incentives to produce homogeneous content. Related work offers Shapely value based probabilistic recommendations as a solution [Ben-Porat and Tennenholtz, 2018]. Eskandanian and Mobasher [2020] is one of the few using stable matching in a recommender system. They show that a Deferred Acceptance based algorithm can diversify recommendations.

We add to this growing body of literature by going beyond fairness, by asking if the participants will have the incentive to participate in a match produced by recommender system, under a one-shot matching scenario. Drawing inspiration from the Eisenberg-Gale program, we find preference domains where the interests of buyers and sellers can be aligned.

There is a long literature on stable matchings dating back to the 1950's [Gale and Shapley, 1962, Stalnaker, 1953]. Roth and Sotomayor [1990] and Abdulkadiroglu and Sönmez [2013] offer thorough treatments of the topic. Recommending k items to a buyer reminds of worker-firm [Kelso and Crawford, 1982] or college admissions [Gale and Shapley, 1962] matching programs where workers (students) are matched to firms (colleges), sometimes subject to quotas on the number of matches. Kelso and Crawford [1982] show that when preferences satisfy a substitutability condition and when workers' preferences depend only on the firm they apply to, not their co-workers, then stable many to one matchings exist. Our setting does not satisfy this condition: the probability an item is selected depends on the other items recommended to the buyer. Indeed, items (and their sellers) have preference orders not over buyers but over (buyer, choice set) pairs. This leads to complications even with unit exposure constraints. For example, in traditional deferred acceptance schemes, one side of the market proposes matches to the other in order of their preferences, and matches are tentatively accepted until a better proposal comes along. In our setting, the seller of an item can not accurately judge the attractiveness of being recommended to a buyer until the other $k-1$ items in that buyer's choice set are fixed.

Envy-free allocations [Foley, 1967] and relaxations thereof [Lipton et al., 2004] have been studied for divisible [Brams and Taylor, 1995, Procaccia, 2016] and indivisible goods [Alkan et al., 1991, Caragiannis et al., 2019, Lipton et al., 2004] in both static and dynamic settings [Benadè et al., 2018, 2022, Zeng and Psomas, 2020]. The concept of Nash welfare, or the product of agent utilities, originated in John Nash's solution to a bargaining problem [Nash, 1950]. Maximizing Nash welfare when allocating indivisible goods among agents with additive utilities is known to be EF1 and Pareto optimal (PO) [Caragiannis et al., 2019]. Maximizing Nash welfare is NP-hard for several bidding languages [Ramezani and Endriss, 2010]; Caragiannis et al. [2019] propose a computational approach which scales to reasonably sized instances. The notions of balance and impartiality we use in Section 3 have appeared in Huang et al. [2022].

In a similar spirit to our work, two-sided fairness has recently received attention in the fair division literature [Freeman et al., 2021, Gollapudi et al., 2020, Igarashi et al., 2022]. Caragiannis and Narang [2022] independently propose envy-freeness up to a single exchange of items in a setting where goods and chores are repeatedly matched to agents and find that a variation of round robin allocation adapted for repeated matchings is both EF1 and SEF1. We make a similar observation for round robin allocations in Section 4.2 and further show that maximizing expected buyer welfare is also SEF1. Igarashi et al. [2022] study the fair allocation of players to teams where teams have additive values for players and players have a weak preference order over teams and find that maximizing Nash welfare is EF1 and PO. As highlighted before, sellers in our setting have no such (weak) preference order over buyers. Igarashi et al. [2022] also consider two stability notions. An allocation is said to be *swap stable* when there is no pair of teams and players on those teams so that swapping the players makes at least one of the four parties better off while leaving none worse off. An allocation is *individually stable* when no player can deviate to another team without making one of the teams involved worse off. In keeping with the original stable marriage problem [Gale and Shapley, 1962], our notion of stability requires only that the deviating buyer and seller are strictly better off, not that the other parties involved are no worse off.

2 MODEL

We study a setting in which each of a batch of buyers is simultaneously recommended k items, and each item is subject to constraints on the number of times it is recommended. Let $\mathcal{B}$ denote a set of n buyers and $\mathcal{I}$ a set of m items. We assume every item is sold by a different seller and occasionally blur the distinction between recommending an item and recommending a seller to buyers.

A *(k−)recommendation* to buyer $b \in \mathcal{B}$ is a set of k unique items $\bar{A}_b \subseteq \mathcal{I}$, $|\bar{A}_b| = k$. Additionally, buyer b has (fixed) outside option ω_b, which represents not selecting any of the recommended items and instead sticking with the status quo or pursuing an off-platform transaction. Let $A_b = \bar{A}_b \cup \{\omega_b\}$ be the choice set of buyer b. We call the vector of recommendations $A = (A_b)_{b \in \mathcal{B}}$ a *recommendation profile*. The buyers who are recommended item i are denoted $A_i^{-1} = \{b \in \mathcal{B} : i \in A_b\}$.

Let $\mathcal{A}$ denote the set of feasible recommendation profiles. A recommendation profile is feasible if it satisfies constraints on the number of exposures received by each item, encoded as $|A_i^{-1}| \le c_i$ for all $i \in \mathcal{I}$. Unless explicitly stated otherwise, we assume unit exposure constraints, i.e. $c_i = 1$ for all $i \in \mathcal{I}$. Under unit exposures $|A_i^{-1}| = 1$, we overload notation to let A_i^{-1} also refer to the (unique) buyer recommended item $i \in \mathcal{I}$. We use i interchangeably with the singleton set $\{i\}$.

Buyer behavior is assumed to follow a standard choice model [McFadden, 1973]. Choice models have been extensively used in the recommender systems literature to learn users-preferences from their selections [Moins et al., 2020, Song et al., 2022, 2019] and to predict user-choices when presented with a set of recommendations [Carroll et al., 2021, Chen et al., 2019, Fleder and Hosanagar, 2007]. They have also been used to study choices in many other disciplines [Berry, 1994, Chintagunta et al., 2002, Talluri and Van Ryzin, 2004].

Under this model buyer $b \in \mathcal{B}$ has utility $U_{bi} = u_{bi} + \epsilon_{bi}$ for item $i \in \mathcal{I}$, where u_{bi} is the expected utility that b has for i and ϵ_{bi}, drawn independently and identically from a Gumbel distribution, is an unknown random utility component which captures unobserved determinants of item utility. We take u_{bi} as arbitrary and known, for example, it may have been estimated using a collaborative filter, and call $v_{bi} = e^{u_{bi}}$ buyer b's *virtual value* for item i. Outside options are modeled similarly, the known component of buyer b's outside option ω_b is $u_{b\omega}$ and we assume all $b \in \mathcal{B}$ have $u_{b\omega} = u_\omega$. This is largely without loss of generality: when buyers have outside options with different values, we can create normalized utilities $u'_{bi} = u_{bi} - u_{b\omega}$ and all our results remain true when phrased in terms of the normalized utilities. Most examples further set $u_\omega = 0$ ($v_\omega = 1$) for concreteness, though any constant will do.[2]

Per choice model theory, buyer $b \in B$ considering choice set S realizes the previously unknown random components of item utilities, then (deterministically) selects the option $i \in S$ that provides greatest utility. McFadden [1973] shows the resulting probability that option i will be selected

[2]We follow the convention that $v_{bi} = 0$ when $u_{bi} \to -\infty$.

from choice set S is $\mathbb{P}(b,i,S) = e^{u_{bi}}/\sum_{j\in S} e^{u_{bj}} = v_{bi}/\sum_{j\in S} v_{bj}$.[3] Notice that a buyer's probability of selecting the outside option decreases in the quality of their k-recommendation.

The welfare of (the seller of) item i is assumed only to be increasing in the expected number of times i is selected, denoted $\mathbb{E}_i(A)$. This is flexible enough to capture, for example, sellers having different profit margins, where expected profit increases in the expected number of sales. The expected number of selections $\mathbb{E}_i(A) = \sum_{b\in A_i^{-1}} \mathbb{P}(b,i,A_b)$ which, under unit exposures, becomes $\mathbb{E}_i(A) = \mathbb{P}(i,A) \triangleq \mathbb{P}(A_i^{-1}, i, A_{A_i^{-1}})$.

At the time of recommendation, a buyer's *expected utility* from the choice set S is given by $u_b(S) = \mathbb{E}\left(\max_{i\in S}(U_{bi})\right) = \log\left(\sum_{i\in S} e^{u_{bi}}\right)$ [Williams, 1977]. The *utility profile* associated with recommendation A is $u(A) = (u_b(A_b))_{b\in\mathcal{B}}$. Accordingly, $v_b(S) = e^{u_b(S)} = \sum_{i\in S} v_{bi}$ and we call $v(A) = (v_b(A_b))_{b\in\mathcal{B}}$ the *virtual value profile.*

Notice that each buyer has a fixed preference order over $\mathcal{I} \cup \{\omega\}$, as determined by $\{u_{bi} : \forall i \in \mathcal{I} \cup \{\omega\}\}$. However, sellers of items do not have a fixed preference order over buyers: the probability of being selected depends both on the item's utility and on how much competition the item faces in a given choice set (the sum of virtual values of all other options in the choice set).

2.1 Measuring The Quality of a Recommendation

Efficiency. A natural requirement is that the recommendation profile is *Pareto optimal* (PO) with respect to buyers' welfare. Let $[n] = \{1,\ldots,n\}$. A vector $x \in \mathbb{R}^n$ *strictly dominates* $y \in \mathbb{R}^n$ when $x_i \geq y_i$ for all $i \in [n]$ and there exists $j \in [n]$ where $x_j > y_j$. A recommendation $A \in \mathcal{A}$ is Pareto optimal if there does not exist another recommendation $A' \in \mathcal{A}$ such that $u(A')$ strictly dominates $u(A)$. Notice that $u(A)$ is undominated exactly when $v(A)$ is undominated.

Fairness. A standard notion of fairness is *envy-freeness*, which we consider from the buyers' perspectives. Envy-freeness requires that each buyer prefers their choice set over the choice set of any other buyer. Formally, recommendation profile A is envy-free when $u_b(A_b) \geq u_b(A_{b'})$ (equivalently, $v_b(A_b) \geq v_b(A_{b'})$) for all $b, b' \in \mathcal{B}$. Envy-freeness is often impossible with indivisible objects (consider allocating a single valuable item to two agents) and is commonly relaxed to envy-freeness up to one item (EF1), which allows envy to exist but only to the extent that it can be eliminated by removing a single item from the envied agent's allocation. In our setting, each buyer must be recommended exactly k items and simply removing an item from a buyer's choice set is not an option. We propose a relaxation of envy-freeness, called *swap-envy-freeness*, to accommodate this. A recommendation profile is *swap-envy-free up to 1 item* (SEF1) when any pairwise envy between buyers $b \neq b' \in \mathcal{B}$ can be eliminated by exchanging a single pair of items between them. Formally, A is SEF1 if, for all $b, b' \in \mathcal{B}$ where b envies b', there exist a pair of items $i \in A_b, j \in A_{b'}$ so that $u_b(A_b \cup j \setminus i) \geq u_b(A_{b'} \cup i \setminus j)$.

Stability. Pareto optimality and (swap-)envy-freeness both focus on buyer satisfaction. These properties do not guarantee that buyers and sellers will continue to participate in a platform's recommendations. A buyer-item pair $(b,i) \in \mathcal{B} \times \mathcal{I}$ is called a *blocking pair* in recommendation profile A if both the buyer and the item's seller strictly benefit from i replacing some item j in A_b. Formally, for unit exposure constraints, (b,i) blocks when $i \notin A_b$ and there exists a item $j \in A_b$ so that both $u_{bi} > u_{bj}$ and $\mathbb{P}(i,A) < \mathbb{P}(b,i,A_b \cup i \setminus j)$. A recommendation profile is *stable* in the absence of a blocking pair. To handle general exposure constraints and the possibility that $j \in A_c$, we further require that the recommendation profile resulting from the deviation of i to A_b (ejecting j) can be feasibly extended to a full k-recommendation profile. This condition has no

[3] This expression for probability as a function of u_{bi}, incidentally, is the softmax function that is widely used during training of neural networks with discrete outcome variables [Goodfellow et al., 2016].

bite for the unconstrained and unit exposure settings: in the former there is nothing preventing simultaneously recommending i to both b and c; in the latter $j \in A_b$ implies $j \notin A_c$ since an item is recommended at most once. Elsewhere this is equivalent to assuming that dummy items are available to be recommended for which every buyer has the lowest possible value.

We illustrate some differences between these concepts with an example.

Example 1. *Consider an instance with buyers {1,2}, four items $\{a, b, c, d\}$, outside option ω with $u_\omega = 0$ and $k = 2$. Table 1 shows the virtual values for an instance with identical buyers and two good and two bad items. Suppose each item can only be recommended once.*

Table 1. Buyers' virtual values.

	a	b	c	d	ω
Buyer 1	2	2	1	1	1
Buyer 2	2	2	1	1	1

The outside option appears in every buyer's choice set, we omit explicitly including it except where an omission could create confusion. The recommendation profile $(\{a, b\}, \{c, d\})$ is Pareto optimal even though it recommends both good items to the same buyer. It is also SEF1: buyer 2 envies buyer 1 but swapping b with c removes the envy. Furthermore, it is unstable with blocking pair $(2, b)$: 2 prefers b over both c and d, and b has a larger probability of being purchased by 2 in the choice set $\{b, c\}$ than by 1 in the choice set $\{a, b\}$. Recommendation profile $(\{a, c\}, \{b, d\})$ is Pareto optimal, envy free and stable.

3 ON THE EXISTENCE OF STABLE RECOMMENDATIONS

We investigate the conditions under which stable recommendations exist. When the exposure constraints do not prevent any buyer from being recommended exactly their k most preferred items, top-k recommendations are stable. Omitted proofs appear in the appendix.

Observation 2. *In any instance where the unconstrained top-k recommendation profile is feasible, for example, when $\vec{c} = \infty$ or when $\vec{c} = 1$ and all buyers have disjoint sets of top-k items, recommending each buyer their k most liked items is feasible, stable, welfare optimal and envy-free.*

Stability, here, follows from the fact that a buyer recommended her k highest value items will be unwilling to participate in a blocking pair. Recommendation systems are often deployed to recommend digital products where exposure constraints are less likely to exists, for example, a streaming platform may not have a limit on the number of times a particular movie is streamed. It is reassuring to know that personalized top-k recommendation is stable in this setting. To the best of our knowledge, this property of top-k recommendation has not been discussed before.

Unfortunately, once the exposure constraints make it impossible to recommend to every buyer their most preferred k items, stable recommendations need no longer exist.

Theorem 3. *Under unit exposures, there exist instances where no stable recommendation exists.*

Proof. Consider the instance in Table 2 with buyers $\{1, 2\}$ and sellers $\{a, b, c, d\}$. We argue that this instance with $k = 2$ does not permit a stable recommendation under unit exposure constraints.

Since $A_2 = S \setminus A_1$, we need only check all possible A_1. For $A_1 \in \{\{ac\}, \{ad\}\}$, $(2, a)$ is a blocking pair as buyer 2 prefers a over all their recommended sellers and would be willing to eject seller b from their current choice set. For $A_1 \in \{\{ab\}, \{bc\}, \{bd\}\}$, $(2, b)$ blocks, since 2 will always be willing to accept b (it is one of 2's top-2 items), and b's purchase probability, which is at most $1/8$

Table 2. Buyers' virtual values.

	a	b	c	d	ω
Buyer 1	10	1	7	6	1
Buyer 2	10	8	4	5	1

when it is recommended to buyer 1, increases. Finally, for $A_1 = \{c, d\}$, $(1, a)$ blocks since a is 1's most liked item and the purchase probability of a is higher when competing with c or d in 1's choice set than when competing with b in 2's choice set. □

That stability cannot be guaranteed is foreshadowed by a more general result by Pycia [2012], which shows that stable coalitions can only be guaranteed when, for any pair of coalitions, the common members of the coalitions prefer the same coalition. In our setting, recommendations are coalitions consisting of a buyer and k sellers. If the buyer obtains higher welfare by changing some items of the choice set, the purchase probability of the remaining items must reduce due to increased competition. As a result, buyers and sellers prefer opposing choice sets.

A closer look at the instance in Theorem 3 reveals two situations which facilitate blocking pairs. First, whenever b was recommended to buyer 1, the massive discrepancy in the buyers' values for item b created the potential for b to increase its purchase probability by deviating from the recommendations. Second, the allocations with $b \in A_2$ lead to a large disparity in the bundle values, specifically, $|v_1(A_1) - v_2(A_2)| \geq 4$. When choice sets have very different values it leads to unequal levels of competition, and even an item valued identically by the buyers may benefit from deviating to seller where it will face less competition, as shown by the blocking pair $(2, a)$ in allocation $A_1 = \{a, c\}, A_2 = \{b, d\}$. We can formalize this intuition that the incentive to participate in a blocking pair depends on the degree to which buyers value the same item similarly and the difference in competition across choice sets in terms of *balance* and *impartiality* [Huang et al., 2022].

Definition 4 (α-balance). *The valuations of an item i is called α_i-balanced if $v_{bi} \leq \alpha_i \cdot v_{ci}$ for all $b, c \in \mathcal{B}$. When each $i \in \mathcal{I}$ is α_i-balanced, the instance is called α-balanced, with $\alpha = \max_i \alpha_i$.*

When buyers have identical values the instance is 1-balanced. Next, we parameterize the disparity across virtual values of choice sets in a given allocation.

Definition 5 (β-impartiality). *An allocation A is called β-impartial when β is the smallest value such that $v_b(A_b) \leq \beta \cdot v_c(A_c)$ for all $b, c \in \mathcal{B}$.*[4]

For an arbitrary allocation A with blocking pair $(b, i) \notin A$, let A' denote an allocation with $A'_b = A_b \cup i \setminus j$ for some $j \in A_b$. For example, under unit constraints if $i \in A_c$ then A' can be identical to A except that $A'_b = A_b \cup i \setminus j$ and $A'_c = A_c \cup j \setminus i$, i.e., the allocation that results from A when b deviates with i and the item ejected from A_b is recommended to c. We can upper bound the benefit from participating in a blocking pair in terms of the balancedness of the instance and the impartiality of A.

Theorem 6. *Consider an α-balanced instance with β-impartial allocation A with $i \in A_c$. For any blocking pair (b, i) of A, the multiplicative gain of buyer b and seller i when deviating from A to A' (as defined above) can be upper bound as*

$$\frac{\mathbb{P}(i, A')}{\mathbb{P}(i, A)} \leq \alpha_i \beta \leq \alpha\beta, \quad \textit{and} \quad \frac{v_b(A'_b)}{v_b(A_b)} < \alpha_i \beta \leq \alpha\beta.$$

[4] β_b-impartiality can be defined with $\beta_b = \max_{c \in \mathcal{B}}\{v_c(A_c)/v_b(A_b)\}$ and will lead to slightly stronger bounds.

Proof. We first bound the gain seller i can get from deviating. Observe that $v_{bj} \leq v_{bi}$ since j is ejected from A_b in favor of i when deviating. Now

$$\begin{aligned}\frac{\mathbb{P}(i,A')}{\mathbb{P}(i,A)} &= \frac{v_{bi}}{v_b(A_b)+v_{bi}-v_{bj}} \Big/ \frac{v_{ci}}{v_c(A_c)}\\ &= \frac{v_{bi}}{v_{ci}} \cdot \frac{v_c(A_c)}{v_b(A_b)+v_{bi}-v_{bj}}\\ &\leq \frac{v_{bi}}{v_{ci}} \cdot \frac{v_c(A_c)}{v_b(A_b)} \;\leq\; \alpha_i\beta.\end{aligned}$$

We now bound the welfare increase of buyer b. Since (b,i) is a blocking pair the purchase probability of i increases after deviation, so

$$\mathbb{P}(i,A) < \mathbb{P}(i,A') \iff \frac{v_{ci}}{v_c(A_c)} < \frac{v_{bi}}{v(A_b)+v_{bi}-v_{bj}} \iff v_b(A_b)+v_{bi}-v_{bj} < \frac{v_{bi}\cdot v_c(A_c)}{v_{ci}}. \quad (1)$$

We can now bound buyer b's multiplicative increase in welfare as

$$\frac{v_b(A'_b)}{v_b(A_b)} = \frac{v_b(A_b)+v_{bi}-v_{bj}}{v_b(A_b)} < \frac{v_{bi}}{v_{ci}} \cdot \frac{v_c(A_c)}{v_b(A_b)} \leq \alpha_i\beta,$$

where the second transition uses Equation (1).

□

As an example, consider allocation $A_1 = \{a,c,\omega\}, A_2 = \{b,d,\omega\}$ in the instance of Theorem 3 which has $\alpha_a = 1$ and $\beta = \frac{18}{14}$, implying a maximum benefit of $\frac{18}{14} \approx 1.29$. Participating in the blocking pair $(2,a)$ increases seller a's purchase probability by a factor of $\frac{10/16}{10/18} \approx 1.11$ when ejecting b. Buyer 2's welfare increases by a factor of $\frac{16}{14} \approx 1.15$. We show in the Appendix A.1 that there exist instances where both the above bounds are tight.

When the unconstrained top-k recommendation profile is feasible (as in Observation 2), top-k recommendations maximize buyer welfare. In the remainder of this section we show that maximizing buyer welfare guarantees stability under two other restricted preference domains: buyers with identical values and buyers with dichotomous values.

3.1 Identical Buyers

Suppose the buyers agree on a common evaluation u_i of each item $i \in \mathcal{I}$, so $u_i = u_{bi}$ for all $b \in \mathcal{B}$. Let A^* be a recommendation profile which maximizes total buyer welfare, i.e., A^* solves

$$\max_{A\in\mathcal{A}} \left\{\sum_{b\in\mathcal{B}} \log\left(\sum_{i\in A_b} e^{u_{bi}}\right)\right\} \equiv \max_{A\in\mathcal{A}} \left\{\sum_{b\in\mathcal{B}} \log(v_b(A_b))\right\} \equiv \max_{A\in\mathcal{A}} \left\{\prod_{b\in\mathcal{B}} v(A_b)\right\}.$$

Notice that maximizing buyer welfare equivalently maximizes the *Nash welfare* with respect to the virtual values. In the fair division literature, maximizing Nash welfare (on item values) is known to be EF1 and PO [Caragiannis et al., 2019]. It also has attractive properties here in the case of unit exposure constraints.

Theorem 7. *For unit exposure constraints and identical preferences, A^* is stable, PO and SEF1.*

Proof of Theorem 7. We first show A^* is stable. Assume for contradiction it is not, then there exists a blocking pair $(b,i) \in \mathcal{B} \times \mathcal{I}$. Let c be the buyer currently recommended i. By definition, a blocking pair implies $i \notin A^*_b$ and $\exists j \in A^*_b$ so that $v_{bj} < v_{bi}$ and $\mathbb{P}(i,A^*) < \mathbb{P}(i,A')$, where A' is constructed from A^* by exchanging i and j, i.e. $A'_b = A^*_b \cup i \setminus j$ and $A'_c = A^*_c \setminus i \cup j$ and $A^*_d = A'_d \,\forall d \notin \{b,c\}$.

Since A^* maximizes the product of virtual values $v_b(A_b^*) \cdot v_c(A_c^*) \geq v_b(A_b') \cdot v_c(A_c')$. It follows that

$$\frac{v_b(A_b^*)}{v_b(A_b')} \geq \frac{v_c(A_c')}{v_c(A_c^*)}$$
$$1 - \frac{v_b(A_b^*)}{v_b(A_b')} \leq 1 - \frac{v_c(A_c')}{v_c(A_c^*)}$$
$$\frac{(v_b(A_b^*) + v_i - v_j) - v_b(A_b^*)}{v_b(A_b')} \leq \frac{v_c(A_c^*) - (v_c(A_c^*) - v_i + v_j)}{v_c(A_c^*)}$$
$$\frac{v_i - v_j}{v_b(A_b')} \leq \frac{v_i - v_j}{v_c(A_c^*)}$$
$$v_b(A_b') \geq v_c(A_c^*).$$

As a result, $\mathbb{P}(i, A^*) = v_i/v_c(A_c^*) \geq v_i/v_b(A_b') = \mathbb{P}(i, A')$, contradicting the fact that i strictly benefited from participating in the blocking pair. We conclude no blocking pair exists and A^* is stable.

Finally, we show that A^* is SEF1 with respect to the virtual values, which implies SEF1 for the true utilities by the monotonicity of the logarithm. Assume for contradiction that A^* is not SEF1 for virtual values and suppose that buyer $b \in \mathcal{B}$ envies $c \in \mathcal{B}$. By definition, $v(A_b^*) < v(A_c^*)$ and, for every $i \in A_c^*, j \in A_b^*, v(A_b^* \setminus j \cup i) < v(A_c^* \setminus i \cup j)$.

Suppose for contradiction that $v_j \geq v_i$ for all $j \in A_b^*, i \in A_c^*$. Since $|A_b^*| = |A_c^*|$, it follows that $v(A_b^*) \geq v(A_c^*)$, which is not the case. We conclude that there exists at least one pair of sellers $(j, i) \in (A_b^*, A_c^*)$ such that $v_j < v_i$. Let A' be the recommendation that results from swapping i and j, in other words, $A_b' = A_b^* \cup i \setminus j, A_c' = A_c^* \setminus i \cup j$ and $A_d' = A_d^*$ for $d \in \mathcal{B} \setminus \{b, c\}$.

Set $\delta = v_i - v_j > 0$, so $v(A_b') = v(A_b^*) + \delta$ and $v(A_c') = v(A_c^*) - \delta$. Now

$$v(A_b') \cdot v(A_c') = (v(A_b^*) + \delta)(v(A_c^*) - \delta) = v(A_b^*) \cdot v(A_c^*) + \delta(v(A_c^*) - (v(A_b^*) + \delta)) \geq v(A_b^*) \cdot v(A_c^*),$$

where the final transition follows from

$$v(A_c^*) - (v(A_b^*) + \delta) = v(A_c') - v(A_b') > 0$$

since A^* is not SEF1 by assumption. This contradicts the fact that A^* maximized the product of virtual values. We conclude that A^* is SEF1.

Pareto optimality follows directly from A^* maximizing the product of buyers' virtual values. □

If it were possible to guarantee exactly equal values across all choice sets (i.e. find a perfectly impartial recommendation profile with $\beta = 1$), then Theorem 6 would imply that the allocation is stable, since $\alpha = 1$ for agents with identical preferences. Perfectly balancing virtual values is too much to hope for. The previous result shows that maximizing the product of buyer virtual values, which (at least in the identical preference case) leads to choice sets with very similar utilities, gets close enough to ensure stability. When items can be recommended more than once, carefully constructed upper bounds may enforce less impartial recommendation profiles where stability can no longer be guaranteed. The following example illustrates this.

Example 8. *We study an instance with $n = 2, k = 3$ and six unique items and values and upper bounds on the number of exposures as in Table 3. Without loss of generality, assume buyer 1 receives the most valuable item (the other case is symmetric). The resulting recommendation which maximizes the product of buyers' virtual values is indicated with checkmarks.*

This welfare maximizing recommendation profile is not stable: consider transferring a to 2*, displacing b from 2's current choice set. The only resulting recommendation is boxed (1 can not be recommended* *a second copy of b). Buyer 2's total virtual value increased from 20 to 22. Item a's probability of being selected also increased, from* $\frac{12}{26}$ *to* $\frac{12}{22}$*. We conclude that* $(2, a)$ *is a blocking pair.*

Notice that in the corresponding instance with seven items (two distinct items with value 10) and unit constraints the product maximizing choice sets are valued more similarly, with virtual values 22 and 24 ($\beta = \frac{24}{22} \approx 1.1$*, rather than 20 and 26 (*$\beta = \frac{26}{20} = 1.3$*).*

Table 3. Buyers' virtual values.

	a	*b*	*c*	*d*	*e*	*f*	ω
Values	*12*	*10*	*5*	*4*	*3*	*3*	*1*
Exposure limit	*1*	*2*	*1*	*1*	*1*	*1*	–
Buyer 1	$\checkmark$	$\boxed{\checkmark}$			$\boxed{\checkmark}$	$\square$	$\boxed{\checkmark}$
Buyer 2	$\square$	$\checkmark$	$\boxed{\checkmark}$	$\boxed{\checkmark}$			$\boxed{\checkmark}$

Despite stability breaking down, A^* remains PO (trivially) and SEF1. The argument that SEF1 holds is similar to the one presented in the proof of Theorem 7.

Theorem 9. *A^* is PO and SEF1 for buyers with identical preferences.*

3.2 Dichotomous Values

We now consider the case where buyers have dichotomous preferences with either high or low value for every item (and are indifferent among items of each type). Formally, assume $v_{bi} \in \{\ell, h\}$ for all $b \in \mathcal{B}$ and $i \in \mathcal{I}$, for some real-valued $\ell < h$. Buyers need not agree on the evaluation of an item: it may be of high value to some and of low value to others. Here, maximizing the total buyer welfare (or maximizing Nash welfare with respect to the virtual values) again leads to stability.

Theorem 10. *For buyers with dichotomous values, A^* is stable, PO and SEF1 under unit exposure constraints.*

Proof. *Stability:* Assume, for contradiction, A^* is not stable. Then there exists a blocking pair $(b, i) \in \mathcal{B} \times \mathcal{I}$. Let c be the buyer to whom i is currently recommended.

Observe that b's choice set contains at least one low-valued seller, otherwise she can not benefit from participating in a blocking pair. Let $j \in A^*_b$ be such an item with $v_{bj} = \ell$. Construct A' from A^* by exchanging i and j. So $A'_b = A^*_b \cup i \setminus j, A'_c = A^*_c \cup j \setminus i$ and $A'_d = A^*_d$ for all $d \in \mathcal{B} \setminus \{b, c\}$.

We now study the possible values of (v_{bi}, v_{ci}):

(1) (ℓ, ℓ): Participating in a blocking pair requires displacing a buyer with strictly lower value, which is not possible since $v_{bi} = \ell$. This contradicts (b, i) being a blocking pair.
(2) (ℓ, h): As in the previous case, $v_{bi} = \ell$, which contradicts (b, i) being a blocking pair.
(3) (h, ℓ): In this case i is assigned to c despite c having low and b having high value for it. It follows that $v_b(A') = v_b(A^*) - \ell + h > v_b(A^*)$ and $v_c(A') = v_c(A^*) - \ell + v_c(j) \geq v_c(A^*)$. However, this implies $v_b(A') \cdot v_c(A') > v_b(A^*) \cdot v_c(A^*)$, contradicting the fact that A^* maximizes the product of virtual values.
(4) (h, h): Suppose that $v_{cj} = h$. Now $v_b(A') > v_b(A^*)$ and $v_c(A') = v_c(A^*)$, contradicting that A^* maximizes the product of virtual values. We conclude that $v_{cj} = \ell$, so b and c value the

swapped items identically. We can now argue as in Theorem 7 that

$$\frac{v_b(A_b^*)\cdot v_c(A_c^*)}{v_b(A_b')\cdot v_c(A_c')} \geq 1$$
$$1-\frac{v_b(A_b^*)}{v_b(A_b')} \leq 1-\frac{v_c(A_c')}{v_c(A_c^*)}$$
$$\frac{(v_b(A_b^*)+v_{bi}-v_{bj})-v_b(A_b^*)}{v_b(A_b')} \leq \frac{v_c(A_c^*)-(v_c(A_c^*)-v_{ci}+v_{cj})}{v_c(A_c^*)}$$
$$\frac{h-\ell}{v_b(A_b')} \leq \frac{h-\ell}{v_c(A_c^*)}$$
$$v_b(A_b') \geq v_c(A_c^*).$$

As a result, $\mathbb{P}(i,A^*) = v_i/v_c(A_c^*) \geq v_i/v_b(A_b') = \mathbb{P}(i,A')$, contradicting the fact that i strictly benefited from participating in the blocking pair.

We conclude there exists no blocking pair (b,i), and thus that A^* is stable.

SEF1: Suppose, for contradiction, that A^* is not SEF1 with respect to the virtual values. This means there exists buyers b,c so that b envies c after every pairwise swap of items, i.e.

$$v_b(A_b^*)+v_{bi}-v_{bj} < v_b(A_c^*)-v_{bi}+v_{bj} \text{ for all } j\in A_b^*, i\in A_c^*.$$

Consider arbitrary $j\in A_b^*$ and $i\in A_c^*$ so that $v_{bj}=\ell$ and $v_{bi}=h$ (such items exist since b envies c). If it were the case that $v_{ci}=\ell$, then it would be possible to increase the product of utilities by exchanging i and j (b is better off and c is no worse off). The maximimality of A^* thus implies that $v_{ci}=h$. We similarly conclude that $v_{cj}=\ell$. This means both buyers agree that every item $j\in A_b^*$ with $v_{bj}=\ell$ is a low value item, and similarly every $i\in A_c^*$ with $v_{bj}=h$ is a high value item.

Let $H_b=\{k\in\mathcal{I} : v_{bk}=h\}$ and $L_b=\{k\in\mathcal{I} : v_{bk}=\ell\}$, and define H_c, L_c analogously. We've established that $H_b\cap A_c^*\subseteq H_c\cap A_c^*$ and $L_b\cap A_b^*\subseteq L_c\cap A_b^*$.

Construct A' from A^* by exchanging i and j. So $A_b'=A_b^*\cup i\setminus j$, $A_c'=A_c^*\cup j\setminus i$ and $A_d'=A_d^*$ for all $d\in\mathcal{B}\setminus\{b,c\}$. Let $\delta=h-\ell$.

Since A^* is not SEF1, $v_b(A_b')<v_b(A_c')$. Buyer c receives an ℓ item in exchange for an h one, so $v_c(A_c')=v_c(A_c^*)-\delta<v_c(A_c^*)$, similarly $v_b(A_b')=v_b(A_b^*)+\delta>v_b(A_b^*)$. Because c also has high value for those items in A_c' that b has high value for,

$$\begin{aligned}
v_b(A_c') &= v_b(A_c'\cap H_b)+v_b(A_c'\cap L_b)\\
&= h\cdot|A_c'\cap H_b|+\ell\cdot|A_c'\cap L_b|\\
&= v_c(A_c'\cap H_b)+\ell\cdot|A_c'\cap L_b|\\
&\leq v_c(A_c'\cap H_b)+v_c(A_c'\cap L_b)=v_c(A_c').
\end{aligned}$$

Putting it all together shows

$$v_b(A_b^*)<v_b(A_b')<v_b(A_c')\leq v_c(A_c')<v_c(A_c^*). \tag{2}$$

Finally,

$$\begin{aligned}
v_b(A_b')\cdot v_c(A_c') = (v_b(A_b^*)+\delta)(v_c(A_c^*)-\delta) &= v_b(A_b^*)\cdot v_c(A_c^*)+\delta(v_c(A_c^*)-\delta-v_b(A_b^*))\\
&= v_b(A_b^*)\cdot v_c(A_c^*)+\delta(v_c(A_c')-v_b(A_b^*))\\
&> v_b(A_b^*)\cdot v_c(A_c^*),
\end{aligned}$$

where the final transition is from Equation (2). This contradicts the fact that A^* maximized the product of virtual values.

We conclude that A^* is SEF1 in terms of virtual values. SEF1 in terms of utility follows from the fact that a buyer's utility is monotone in their virtual values.

Pareto optimality: This follows directly from A^* being welfare maximizing. □

As was the case for identical buyers, not all these properties generalize to arbitrary exposure constraints. Carefully chosen exposure constraints can force a situation where the choice sets of two buyers have to overlap. Suppose, when this happens, buyers disagree on the value of the commonly recommended items and agree on which of the other items have low and high values. Now maximizing buyer welfare will result in allocating the items values highly by both to whoever values the common items least, and the low value items to the other. This may create more envy than what can be eliminated by a single exchange of items. The following example illustrates this.

Example 11. *Consider an instance with $n = 2, k = 15$ and 20 unique items of three types. There are ten unique items of type a, each with exposure limit two and $v_{1a} = 2, v_{2a} = 1$. There are five unique items of type b, each with exposure limit one and $v_{1b} = 2, v_{2b} = 2$. There are five items of type c, each with exposure limit one, which both buyers value at one. Table 4 summarizes the instance.*

The unique recommendation which maximizes the product of buyers' virtual values (equivalently, expected welfare) is boxed in Table 4: Buyer 1 is recommended all items of type a and c, and buyer 2 all items of types a and b. Buyer 1 has virtual value 26 for her own choice set and 31 for buyer 2's set, so buyer 1 envies buyer 2. Since virtual values are at most two, no single exchange of items can eliminate this envy and the recommendation is not SEF1 with respect to virtual values. By monotonicity of the logarithm, the swap envy will remain when converting the virtual values into expected utilities.

Table 4. Buyers' virtual values.

Item type	*a*	*b*	*c*	*ω*
# of items	*10*	*5*	*5*	–
Exposure limit	*2*	*1*	*1*	–
Buyer 1	$\boxed{2}$	*2*	$\boxed{1}$	$\boxed{1}$
Buyer 2	$\boxed{1}$	$\boxed{2}$	*1*	$\boxed{1}$

Stability and Pareto optimality, however, can still be guaranteed. The proof that stability holds is largely similar to the unit exposure case, where the existence of a blocking pair contradicted A^* being welfare maximizing. The main wrinkle is that the seller ejected from the deviating buyer's choice set may already be recommended to the buyer who suffered from the deviation, so care must be taken when constructing the alternative recommendation A'.

Theorem 12. *A^* is stable and PO for buyers with dichotomous values.*

3.3 Other Restricted Preference Domains

We briefly remark on two other common structured preference domains. First, suppose buyers have factored utility structure: each buyer b is associated with a characteristic vector β_b, each item i with similar vector γ_i, and $u_{bi} = \langle \beta_b, \gamma_i \rangle$. A stable recommendation need not exist in setting: the instance in the proof of Theorem 3 can be (approximately) factorized. Second, when buyers have identical preference orders over the items (but potentially different values) a stable recommendation always exists under unit exposures for instances with $n = 2, k = 2$, but existence is not guaranteed for larger instances. Full details appear in the Appendix A.4.1.

4 RECOMMENDATION STRATEGIES UNDER EXPOSURE CONSTRAINTS

Despite it not being possible to guarantee stability for general instances, one my still attempt to find a stable recommendation on those instances that permit it. In Appendix A.5 we present a polynomially-sized integer program which finds stable recommendations when they exist and otherwise minimizes the largest incentive any seller has for participating in a blocking pair. Unfortunately, this approach does not appear to scale to realistic instances.

In this section we propose alternative practical recommendation strategies that can accommodate constraints on the number of exposures per product. These will help us measure the level of instability that can be expected in real world recommender systems. We focus on the unit exposure case, but all strategies are simple to modify to arbitrary exposure constraints.

4.1 Maximizing Total Buyer Welfare

A common platform objective is to maximize the total buyer welfare, which was shown to have the additional benefit of leading to stable, fair and efficient recommendations in specific preference domains. The resulting problem is

$$\begin{aligned} \text{maximize} \quad & \sum_{b \in \mathcal{B}} \log \left(\sum_{i \in \mathcal{I}} e^{\hat{u}_{bi}} x_{bi} + e^{u_\omega} \right) \\ \text{s.t.} \quad & \sum_{b \in \mathcal{B}} x_{bi} = 1, \ \text{for all } i \in \mathcal{I} && \text{(unit capacities)} \\ & \sum_{i} x_{bi} = k, \ \text{for all } b \in \mathcal{B}, \text{and} && (k\text{-recommendations}) \\ & x_{bi} \in \{0, 1\}, \ \text{for all } b \in \mathcal{B}, i \in \mathcal{I}, \end{aligned}$$

where $x_{bi} = 1$ when i appears in b's choice set. This is equivalent to maximizing the Nash welfare with respect to virtual values $v_{bi} = e^{\hat{u}_{bi}}$, which is NP-hard [Nguyen et al., 2013].

Because the total buyer welfare is directly maximized, buyers may be less willing to deviate from the platform's recommendations. However, even for unit capacities maximizing buyer welfare is not guaranteed to find a stable recommendation in instances where one exists, as the following example shows.

Example 13. *Consider the instance with virtual values as shown in Table 5. The recommendation that maximizes buyer welfare is* $A_1 = \{a, d\}, A_2 = \{b, c\}$ *(boxed in Table 5), however,* $(1, c)$ *is a blocking pair. At the same time, the underlined recommendation* $A_1 = \{c, d\}, A_2 = \{a, d\}$ *is stable.*

Table 5. Maximizing buyers' welfare need not result in a stable recommendation even when one exists.

	a	b	c	d	ω
Buyer 1	$\boxed{10}$	$\underline{6}$	$\underline{3}$	$\boxed{1}$	$\boxed{\underline{1}}$
Buyer 2	$\underline{10}$	$\boxed{9.5}$	$\boxed{0.5}$	$\underline{0.25}$	$\boxed{\underline{1}}$

4.2 Round Robin Recommendations

This strategy fixes a permutation of buyers, then cycles through the buyers k times. At each step, the active buyer is assigned her highest value item from the remaining items.

In the instance of Example 1, round robin results in each buyer being recommended one high value item and one low value item. An advantage of round robin is that common high value items are shared among buyers. This may increase the probability of high value items being selected by reducing the relative competition they face. We show that round robin, which is known to be EF1 in traditional indivisible goods settings, is SEF1.

Theorem 14. *Round robin is SEF1.*

Proof. Let A be the round robin allocation. Suppose for contradiction A is not SEF1. Then there exists buyers $b, c \in \mathcal{B}$ so that b envies c after every exchange of items between their allocations. We may assume without loss of generality that b was after c in the permutation of buyers, otherwise b would not envy c. We ignore the other buyers, since their bundles do not affect b's envy towards c.

Label $A_b = \{b_1, b_2, \ldots, b_k\}$ and $A_c = \{c_1, c_2, \ldots, c_k\}$, where items are indexed in the order they are assigned. This implies, $v_b(b_i) > v_b(b_j)$ and similarly $v_c(c_i) > v_c(c_j)$ whenever $j > i$.

By the nature of round robin, when b was allocated b_i, all $c_j, j > i$ were still unassigned. This implies $v_b(b_i) > v_b(c_{i+1})$ for all $i \in [k-1]$. By assumption, b envies c, so $\sum_{i\in[k]} v_b(b_i) < \sum_{i\in[k]} v_b(c_i)$. Since $\sum_{i\in[k-1]} v_b(b_i) \geq \sum_{i\in[k-1]} v_b(c_{i+1})$, it follows that $v_b(c_1) > v_b(b_k)$.

Create a new allocation A' by swapping b_k and c_1 and leaving the other buyers unchanged, so $A'_b = A_b \setminus b_k \cup c_j$, $A'_c = A_c \setminus b_k \cup c_j$ and $A'_d = A_d$ for all $d \in \mathcal{B} \setminus \{b, c\}$. Now

$$v_b(A'_b) = v_b(c_1) + \sum_{i\in[k-1]} v_b(b_i) > v_b(b_k) + \sum_{i\in[k-1]} v_b(c_{i+1}) = v_b(A'_c),$$

contradicting that b has swap envy towards c. We conclude that round robin is SEF1. □

Because round robin leads to approximately equal welfare across buyers, choice sets are similarly competitive (or impartial). This should reduce sellers' incentive to deviate from the platforms recommendations, making stability more likely. Despite being SEF1, round robin need not be Pareto optimal or stable.

4.3 Greedy Top-k

This strategy iterates over buyers in a random order and recommends each buyer her top-k (remaining) items. This mimics how recommendations are often made in practice. Buyers arrive on a website one at a time, and the platform recommends the set of items that would provide the buyer the highest utility without considering future arrivals. Because greedy top-k considers only a single buyer at a time, its recommendations are based on much less information than those of the two preceding algorithms, both of which consider all buyers and their preferences when constructing a recommendation profile.

Despite the popularity of this approach, it is particularly susceptible to instability, especially when buyers agree that some items are superior to others. Consider the instance in Example 1. The greedy strategy will assign both high value items to first buyer, leaving the other buyer with both low value items. Sellers of both the high value items would get a higher purchase probability by displacing one of the two low value items in the second choice set. The second buyer will accept such displacement because it increases her welfare. As a result, the greedy recommendation is unstable. Identical values are not required for instability: any overlap in buyers' top-k items suffices.

Greedy top-k also fails to satisfy basic fairness and efficiency properties. Consider an instance with two buyers and $k \geq 4$ and assume that buyer 1 has value 1 for all items. Without loss of generality, suppose that items $[k]$ are included in buyer 1's choice set. Set buyer 2's values as follows: they value items $[k]$ highly and the remainder at some low value. Buyer 1 is considered first, without regard for other buyers, after which buyer 2 is recommended items $\{k+1, \ldots, 2k\}$, all of which they have low value for. This recommendation is neither Pareto optimal nor SEF1.

Proposition 15. *Greedy top-k recommendations need not be stable, SEF1 or Pareto optimal.*

4.4 Online variants of batch algorithms

Greedy top-k is better suited to certain online settings since it only requires a single buyer's preferences to make recommendations, while the preceding algorithms are more suited to batch recommendations. Round robin, for example, collapses to online top-k when directly deployed on repeated batches of size 1, which is contrary to the egalitarian nature of the algorithm. We outline a simple framework for converting a batch recommendation algorithm to an online algorithm by fixing past recommendations and simulating future arrivals using the empirical observations.

Let b_t denote the buyer arriving in time step t as well as her item preferences. By now, $t-1$ buyers have received recommendations, let $\mathcal{B}_{t-1} = \{b_s : s \leq t-1\}$ be those buyers and $A_1, \ldots, A_{t-1}$ their k-recommendations. Construct an instance $\mathcal{I}_t = (\hat{B}_t^n, c^t)$, where $\hat{B}_t^n$ consists of b_t together with with $n-t-1$ buyers drawn uniformly at random with replacement from $\mathcal{B}_t$, and remaining capacities $c_i^t = c_i - \sum_{j\in[t-1]} \mathbf{1}[i \in A_t]$. Notice c^t is exactly the capacities remaining after the first $t-1$ recommendations (the past is immutable), and $\hat{B}_t^n$ assumes future arrivals will have the same distribution as historical ones. If $\hat{A}_t, \ldots, \hat{A}_n$ are the batch recommendations on $\mathcal{I}_t$, recommend $\hat{A}_t$ to buyer b_t.

For illustration, we evaluate this online variant of round robin. Of course, the empirical online variant of an algorithm provides no more guarantees than the batch version.

5 MEASURING INSTABILITY IN REAL WORLD DATASETS

Theorem 3 shows that there are instances where stability is impossible, but we may hope that such instances are rare. Moreover, even in the absence of stability it is possible that there are very few sellers who participate in blocking pairs, or that they have so little to gain from deviating from the match that it is practically irrelevant. In this section we investigate whether common recommendation strategies lead to stable recommendations in real-world datasets and, if not, whether sellers have a significant incentive to deviate from the recommendation. We do this by first predicting ratings ($\hat{u}_{bi}$) for all buyer–item pairs using a collaborative filter and then making recommendations using the strategies described in Section 4. Throughout this section we assume unit exposure constraints.

5.1 Datasets

We use three datasets for our experiments: two with customers' ratings on products from Amazon (in the Automotive and Musical Instruments categories) and one with renters' ratings on clothing from Rent-the-runway (Table 6). These are classic physical goods markets with a natural capacity constraint dictated by inventory levels. Amazon is a platform that matches buyers to third-party sellers. Rent-the-runway is becoming a platform where multiple suppliers directly maintain their portfolio of garments [Chang, 2018]. As such, suppliers' incentive to participate in any recommender system used by these platforms become salient.

5.2 Experiments

We simulate the following scenario. On any given day, the platform decides to recommend products to a subset of buyers, either on its website or through promotional emails. The platform has estimated expected value of each item for each buyer from their past choices using a learning algorithm (e.g., a collaborative filter). The platform recommends $k = 5$ items to each buyer using one of the strategies outlined in Section 4, in a batch manner. For simplicity, we assume that the platform has only one copy of each item in its inventory and does not recommend any item to more than one buyer to avoid the risk of stockout.

Table 6. Summary of datasets used in the experiments.

				Rating summary			
Dataset	# users	# items	# ratings	Min	Median	Avg.	Max
Amazon Automotive	193651	79437	1711519	1	5	4.46	5
Amazon Musical Instrument	27530	10620	231392	1	5	4.47	5
Rent the Runway	105508	5850	192462	2	10	9.09	10

Note: All data are available at https://cseweb.ucsd.edu/~jmcauley/datasets.html. We use the full Rent-the-runway dataset. For Amazon, we use the small, dense subsets with buyers and items that occur at least five times.

We train SVD++, a matrix factorization based collaborative filter, on the entire dataset [Koren, 2008].[5] We randomly select $B \in \{50, 100, 200\}$ buyers and $k \times B$ items to form a pool of buyers and items to be matched. The collaborative filter's predicted ratings for each buyer-item pair is used as the buyer's utility for the item, we take $u_\omega = 0$.[6] We track five metrics for each recommendation strategy:

(1) **Welfare**: the average buyer welfare from their recommended choice sets. All else being equal, a platform would like to provide its buyers higher welfare.
(2) **Move**: the percentage of sellers who participate in a blocking pair, i.e. are able and prefer to pursue off-platform transactions. This measures how *widespread* the incentive to deviate is.
(3) **Gain**: the average percentage improvement in purchase probability of such sellers if they deviated to maximize their purchase probability. This measures how *strong* the incentive is.
(4) **Envy**: the percentage of buyers who envy another. Fewer buyers with envy may signal that the recommendation treats buyers more equally.
(5) **Swap Envy**: the percentage of buyers who envy another *even after their most preferred exchange of items*. Low swap envy implies that whatever envy exists is limited and can be removed by a single swap.

We draw 16 random buyer-item pools for each dataset and report average metrics in Table 7.

There is widespread and substantial incentive to deviate from the recommendations of the greedy top-k algorithm. Nearly all sellers participate in blocking pairs, and they often stand to improve their expected sales by more than 100% by deviating from the system's recommendations.[7] While most of the sellers would still like to deviate under the other strategies, their potential gain from doing so is substantially lower. Maximizing total buyer welfare appears to be the most stable: roughly half the sellers have incentive to deviate but they stand to gain only about 10-12%. Round robin performs slightly worse, but the simplicity of the algorithm may make it an attractive option when it is not computationally feasible to maximize utility on large datasets. The online variant of round robin compares poorly to the two batch recommendation strategies, but is comfortably best of the online algorithms with the average gain from deviating from the recommendations roughly half that of greedy top-k.

[5]We do not set aside a test set for the main experiment since out-of-sample prediction is not the goal. However, a separate evaluation using 20% data for testing shows a RMSE of 1.1 on a 10 point scale for Rent-the-runway and 0.64 − 0.7 on a 5 point scale on the two Amazon datasets. These suggest a reasonably accurate recommender system.
[6]So $v_\omega = 1$. The results were quantitatively similar and qualitatively identical without an outside option, i.e. $v_\omega = 0$.
[7]Some caution is warranted in interpreting these gains. We report the average incentive an *individual* seller has to deviate from the current matching. This is not necessarily the gains the sellers will realize if *all of them* deviate to maximize their expected sales. We expect that gains will be lower under under this setting, since particularly worse off buyers will attract multiple new sellers, thereby leading to increased competition.

Table 7. Welfare, instability and envy in top-5 recommendations under unit exposure constraints.

	Number of users/items									
	50/250					200/1000				
Automotive	Welfare⁺	Move⁻	Gain⁻	Envy⁻	Swap Envy⁻	Welfare⁺	Move⁻	Gain⁻	Envy⁻	Swap Envy⁻
Greedy top-5	6.27	96.8%	113.6%	76.0%	61.1%	6.36	99.2%	145.4%	71.7%	58.8%
	(0.01)	(0.23%)	(5.40%)	(1.36%)	(1.36%)	(0.01)	(0.05%)	(4.49%)	(0.75%)	(0.99%)
Round-robin	6.36	64.8%	11.5%	40.9%	0.0%	6.44	70.8%	11.7%	38.0%	0.0%
	(0.01)	(1.14%)	(0.24%)	(1.69%)	(0.00%)	(0.01)	(0.39%)	(0.13%)	(0.65%)	(0.00%)
Max Nash	6.39	50.1%	8.4%	48.5%	0.4%	6.47	56.6%	8.9%	44.2%	0.4%
	(0.01)	(1.15%)	(0.40%)	(2.67%)	(0.26%)	(0.01)	(0.67%)	(0.31%)	(1.75%)	(0.21%)
Online RR	6.27	92.8%	50.9%	64.4%	1.9%	6.36	97.7%	70.4%	51.8%	1.8%
	(0.01)	(0.53%)	(3.47%)	(1.90%)	(0.37%)	(0.01)	(0.26%)	(4.12%)	(0.66%)	(0.17%)
Musical Instr.										
Greedy top-5	6.26	97.0%	96.0%	75.4%	60.0%	6.33	99.1%	128.1%	73.8%	59.3%
	(0.01)	(0.18%)	(5.48%)	(1.45%)	(2.33%)	(0.01)	(0.07%)	(5.31%)	(0.64%)	(1.01%)
Round-robin	6.34	64.6%	10.3%	38.7%	0.0%	6.40	70.5%	10.8%	37.4%	0.0%
	(0.01)	(1.00%)	(0.26%)	(1.79%)	(0.00%)	(0.01)	(0.34%)	(0.14%)	(0.61%)	(0.00%)
Max Nash	6.37	48.7%	7.8%	46.8%	0.1%	6.43	55.2%	8.4%	50.1%	0.8%
	(0.01)	(1.64%)	(0.37%)	(2.33%)	(0.12%)	(0.01)	(0.72%)	(0.27%)	(1.81%)	(0.18%)
Online RR	6.26	93.6%	44.9%	63.2%	2.1%	6.32	97.7%	65.3%	50.2%	1.4%
	(0.01)	(0.49%)	(1.88%)	(2.18%)	(0.37%)	(0.00)	(0.20%)	(3.33%)	(0.97%)	(0.12%)
Rent-the-runway										
Greedy top-5	10.88	97.6%	143.0%	87.9%	79.4%	10.96	99.2%	169.0%	87.9%	77.1%
	(0.01)	(0.12%)	(4.77%)	(1.04%)	(1.48%)	(0.00)	(0.06%)	(4.72%)	(0.36%)	(0.57%)
Round-robin	10.99	70.8%	14.0%	55.6%	0.0%	11.07	73.3%	14.4%	51.3%	0.0%
	(0.01)	(0.95%)	(0.23%)	(2.22%)	(0.00%)	(0.00)	(0.47%)	(0.13%)	(1.25%)	(0.00%)
Max Nash	11.03	55.6%	10.9%	50.7%	0.2%	11.11	63.9%	13.5%	52.7%	0.3%
	(0.01)	(2.80%)	(0.98%)	(3.37%)	(0.17%)	(0.00)	(1.81%)	(0.92%)	(1.31%)	(0.14%)
Online RR	10.91	94.3%	52.8%	68.7%	1.1%	10.99	98.1%	84.5%	59.1%	1.6%
	(0.01)	(0.50%)	(2.75%)	(2.21%)	(0.30%)	(0.00)	(0.10%)	(1.81%)	(1.49%)	(0.12%)

Note: 1) Welfare: The welfare of the buyers from their choice sets. 2) Move: The % of all the sellers who can and would like to move to a different buyer. 3) Gain: The % they would gain in probability of purchase by doing so (over a baseline average of ≈ 0.2). 4) Envy: The % of buyers who envy the allocation of another. 5) Swap Envy: The % of buyers who envy another even after their most preferred swap. Reported numbers are averages over 16 random draws. The numbers in parenthesis are standard errors. The experiments with 100 users and 500 items produce qualitatively similar results and are omitted due to space constraints. ⁻: smaller the better; ⁺: larger the better.

Regarding envy, we see that 70-90% of the buyers have envy under the greedy strategy; moreover, the fraction of the envious buyers does not reduce significantly even when they are allowed to swap an item with the envied buyers. Round robin (both versions) and max welfare do much better. Though roughly 50% of buyers still have envy, the recommendations are very nearly SEF1, so whatever envy exists can be eliminated by exchanging a single pair of items between choice sets.

Unsurprisingly, maximizing total welfare leads to the highest buyer utility. Round robin yields utilities within 1% of optimal. The online algorithms, which only see a single buyer at a time, perform 1-2% worse. Greedy top-k's myopic decisions are punished by the fact that there are decreasing marginal utility to adding items to the choice set.

It is worth noting that the online version of round robin comfortably outperformed greedy top-k despite its simplicity and not explicitly optimizing for any of the metrics on which it was evaluated. Further investigation into online strategies that outperform greedy top-k is warranted.

In all, maximizing buyer welfare stands out as with the highest welfare, smallest incentives to deviate from the recommendations and near-zero swap envy.

6 DISCUSSION

Personalized recommendations play an important role in matching buyers to sellers in large marketplaces. We initiate a study of the stability of such recommendations under a standard choice model and, somewhat surprisingly, find that maximizing buyer welfare is not only provably stable in restricted preference domains but also leads to relatively low incentives to pursue off-platform transactions in real data sets.

Does stability matter? We cite well-known examples of offline matching markets unraveling in the absence of stability, and there is every reason to believe that, because of easier search and discovery, online markets are even more susceptible to this. Of course, deviating from a platform's recommendation may not look exactly as we modelled it, for example, it may take the form of disgruntled sellers multi-homing on multiple marketplaces or unsatisfied buyers browsing multiple retailers before making a purchase. These still have the effect of driving transactions off-platform, which is what we are interested in.

Relationship to multi-stakeholder recommendations. Our findings suggest that maximizing only buyer welfare is reasonable when the objective is a stable marketplace. But stability of one marketplace may not be enough in an ecosystem where multiple marketplaces/platforms compete. There is a growing interest in fairness towards certain groups at a marketplace (e.g., exposure for minority owned or small businesses). An explicit consideration of buyer, seller, and platform objectives (emphasized in recent recommender systems literature) may be necessary to make a platform successful in this environment.

Incentive compatibility and strategyproofness. We assume buyer preferences are estimated from their past online behaviour with, for example, a collaborative filtering algorithm. As such, we expect these values to be reasonably resistant to manipulation, since manipulation would require buyers modifying their browsing, search or purchase history for a period of time. An alternative setting may be one in which agents directly report preferences: now questions around strategyproofness and incentive compatibility may be interesting.

Future work. Several interesting questions remain open: Are there scalable methods that find stable recommendations when they exist and otherwise minimize the incentive to deviate from the recommendations? The integer prgram in Appendix A.5 is a step in this direction but scalability remains an issue. We ignored issues of competition but it may be important to understand, for example, how a platform's recommendation algorithm should change when a rival actively attempts to lure away users. Finally, is there a clean characterization of exactly which instances or preference structures permit stable recommendations? Do real-world instances satisfy these conditions?

We believe the stability of personalized recommendations is a topic of enormous practical relevance which, thusfar, has escaped the attention it deserves.

REFERENCES

H. Abdollahpouri, G. Adomavicius, R. Burke, I. Guy, D. Jannach, T. Kamishima, J. Krasnodebski, and L. Pizzato. 2020. Multistakeholder recommendation: Survey and research directions. *User Modeling and User-Adapted Interaction* 30, 1 (2020), 127–158.

H. Abdollahpouri and R. Burke. 2019. Multi-Stakeholder Recommendation and its Connection to Multi-Sided Fairness. *arXiv preprint arXiv:1907.13158* (2019).

A. Abdulkadiroglu and T. Sönmez. 2013. Matching markets: Theory and practice. *Advances in Economics and Econometrics* 1 (2013), 3–47.

A. Alkan, G. Demange, and D. Gale. 1991. Fair allocation of indivisible goods and criteria of justice. *Econometrica* 59, 4 (1991), 1023–1039.

Y. Bakos and H. Halaburda. 2020. Platform Competition with Multihoming on Both Sides: Subsidize or Not? *Management Science* 66, 12 (2020), 5599–5607.

M. Bateni, Y. Chen, D. F. Ciocan, and V. Mirrokni. 2022. Fair resource allocation in a volatile marketplace. *Operations Research* 70, 1 (2022), 288–308.

O. Ben-Porat and M. Tennenholtz. 2018. A game-theoretic approach to recommendation systems with strategic content providers. *Proceedings of the 32nd Annual Conference on Neural Information Processing Systems (NIPS)* 31 (2018).

G. Benadè, A. Kazachkov, C.-A. Psomas, and A. D. Procaccia. 2018. How to Make Envy Vanish Over Time. In *Proceedings of the 19th ACM Conference on Economics and Computation (EC)*. 593–610.

G. Benadè, A. Kazachkov, C.-A. Psomas, A. D. Procaccia, and D. Zeng. 2022. Fair and Efficient Online Allocations. Manuscript.

S. T. Berry. 1994. Estimating discrete-choice models of product differentiation. *The RAND Journal of Economics* (1994), 242–262.

S. J. Brams and A. D. Taylor. 1995. An Envy-Free Cake Division Protocol. *The American Mathematical Monthly* 102, 1 (1995), 9–18.

K. A. Bryan and J. S. Gans. 2019. A theory of multihoming in rideshare competition. *Journal of Economics & Management Strategy* 28, 1 (2019), 89–96.

R. D. Burke, H. Abdollahpouri, B. Mobasher, and T. Gupta. 2016. Towards Multi-Stakeholder Utility Evaluation of Recommender Systems. *UMAP (Extended Proceedings)* 750 (2016).

I. Caragiannis, D. Kurokawa, H. Moulin, A. D. Procaccia, N. Shah, and J. Wang. 2019. The unreasonable fairness of maximum Nash welfare. *ACM Transactions on Economics and Computation (TEAC)* 7, 3 (2019), 1–32.

I. Caragiannis and S. Narang. 2022. Repeatedly Matching Items to Agents Fairly and Efficiently. *arXiv preprint arXiv:2207.01589* (2022).

M. Carroll, D. Hadfield-Menell, S. Russell, and A. Dragan. 2021. Estimating and penalizing preference shift in recommender systems. In *Proceedings of the 15th ACM Conference on Recommender Systems*. 661–667.

A. Chakraborty, A. Hannak, A. J. Biega, and K. Gummadi. 2017. Fair sharing for sharing economy platforms. In *Fairness, accountability and transparency in recommender systems — Workshop on responsible recommendation.*

L. Chang. 2018. Rent the Runway is launching a new way for designers to partner with the company — and it's great news for subscribers.

X. Chen, S. Li, H. Li, S. Jiang, Y. Qi, and L. Song. 2019. Generative adversarial user model for reinforcement learning based recommendation system. In *Proceedings of the 16th International Conference on Machine Learning (ICML)*. 1052–1061.

P. Chintagunta, A. Petrin, B. Bronnenberg, R. Goettler, P. Seetharaman, K Sudhir, R. Thomadsen, Y. Zhao, et al. 2002. Structural applications of the discrete choice model. *Marketing letters* 13, 3 (2002), 207–220.

M. D. Ekstrand, A. Das, R. Burke, and F. Diaz. 2022. Fairness in Recommender Systems. In *Recommender Systems Handbook*. Springer-Verlag, 679–707.

F. Eskandanian and B. Mobasher. 2020. Using stable matching to optimize the balance between accuracy and diversity in recommendation. In *Proceedings of the 28th ACM Conference on User Modeling, Adaptation and Personalization*. 71–79.

D. Fleder and K. Hosanagar. 2009. Blockbuster culture's next rise or fall: The impact of recommender systems on sales diversity. *Management science* 55, 5 (2009), 697–712.

D. M. Fleder and K. Hosanagar. 2007. Recommender systems and their impact on sales diversity. In *Proceedings of the 8th ACM Conference on Economics and Computation (EC)*. 192–199.

D. Foley. 1967. Resource allocation and the public sector. *Yale Economics Essays* 7 (1967), 45–98.

R. Freeman, E. Micha, and N. Shah. 2021. Two-sided Matching Meets Fair Division. *arXiv preprint arXiv:2107.07404* (2021).

D. Gale and L. S. Shapley. 1962. College Admissions and the Stability of Marriage. *Americal Mathematical Monthly* 69, 1 (1962), 9–15.

S. Gollapudi, K. Kollias, and B. Plaut. 2020. Almost Envy-Free Repeated Matching in Two-sided Markets. In *International Conference on Web and Internet Economics*. 3–16.

I. Goodfellow, I. Bengio, and I. Courville. 2016. *Deep Learning*. MIT Press. http://www.deeplearningbook.org.

Z. Huang, M. Li, X. Shu, and T. Wei. 2022. Online Nash Welfare Maximization Without Predictions. arXiv preprint arXiv:2211.03077. (2022).

A. Igarashi, Y. Kawase, W. Suksompong, and H. Sumita. 2022. Fair Division with Two-Sided Preferences. *arXiv preprint arXiv:2206.05879* (2022).

A. S. Kelso and V. P. Crawford. 1982. Job Matching, Coalition Formation, and Gross Substitutes. *Econometrica: Journal of the Econometric Society* (1982), 1483–1504.

Y. Koren. 2008. Factorization meets the neighborhood: a multifaceted collaborative filtering model. In *Proceedings of the 14th International Conference on Knowledge Discovery and Data Mining (KDD)*. 426–434.

R. J. Lipton, E. Markakis, E. Mossel, and A. Saberi. 2004. On Approximately Fair Allocations of Indivisible Goods. In *Proceedings of the 6th ACM Conference on Economics and Computation (EC)*. 125–131.

R. Makhijani, S. Chakrabarti, D. Struble, and Y. Liu. 2019. LORE: A large-scale offer recommendation engine with eligibility and capacity constraints. In *Proceedings of the 13th ACM Conference on Recommender Systems*. 160–168.

D McFadden. 1973. Conditional logit analysis of qualitative choice behavior. *Frontiers in Econometrics* (1973).

T. Moins, D. Aloise, and S. J. Blanchard. 2020. RecSeats: A hybrid convolutional neural network choice model for seat recommendations at reserved seating venues. In *Proceedings of the 14th ACM Conference on Recommender Systems*.

309–317.

J. Nash. 1950. The bargaining problem. *Econometrica* 18, 2 (1950), 155–162.

P. Nguyen, J. Dines, and J. Krasnodebski. 2017. A Multi-Objective Learning to Re-rank Approach to Optimize Online Marketplaces for Multiple Stakeholders. *arXiv preprint arXiv:1708.00651* (2017).

T. T. Nguyen, M. Roos, and J. Rothe. 2013. A survey of approximability and inapproximability results for social welfare optimization in multiagent resource allocation. *Annals of Mathematics and Artificial Intelligence* 68, 1 (2013), 65–90.

G. K. Patro, A. Biswas, N. Ganguly, K. P. Gummadi, and A. Chakraborty. 2020. Fairrec: Two-sided Fairness for Personalized Recommendations in Two-sided Platforms. In *Proceedings of The Web Conference 2020*. 1194–1204.

A. D. Procaccia. 2016. Cake Cutting Algorithms. In *Handbook of Computational Social Choice*, F. Brandt, V. Conitzer, U. Endress, J. Lang, and A. D. Procaccia (Eds.). Cambridge University Press, Chapter 13.

M. Pycia. 2012. Stability and preference alignment in matching and coalition formation. *Econometrica* 80, 1 (2012), 323–362.

S. Ramezani and U. Endriss. 2010. Nash social welfare in multiagent resource allocation. In *Proceedings of the 12th International Workshop on Agent-Mediated Electronic Commerce (AMEC)*. 117—131.

A. E. Roth. 1984. The evolution of the labor market for medical interns and residents: a case study in game theory. *Journal of political Economy* 92, 6 (1984), 991–1016.

A. E. Roth. 2015. *Who gets what–and why: the new economics of matchmaking and market design*. Houghton Mifflin Harcourt.

A. E. Roth and M. Sotomayor. 1990. *Two-Sided Mathing: A Study in Game-Theoretic Modelling and Analysis*. Cambridge University Press.

A. E. Roth and M. Sotomayor. 1992. Two-sided matching. *Handbook of game theory with economic applications* 1 (1992), 485–541.

D. Serbos, S. Qi, N. Mamoulis, E. Pitoura, and P. Tsaparas. 2017. Fairness in package-to-group recommendations. In *Proceedings of the 26th International World Wide Web Conference (WWW)*. 371–379.

Y. Song, Z.n Li, and N. Sahoo. 2022. Matching returning donors to projects on philanthropic crowdfunding platforms. *Management Science* 68, 1 (2022), 355–375.

Y. Song, N. Sahoo, and E. Ofek. 2019. When and how to diversify—a multicategory utility model for personalized content recommendation. *Management Science* 65, 8 (2019), 3737–3757.

J. M. Stalnaker. 1953. The Matching Program for Intern Placement. *Journal of Medical Education* 28, 11 (1953), 13–19.

Ö. Sürer, R. Burke, and E. C. Malthouse. 2018. Multistakeholder recommendation with provider constraints. In *Proceedings of the 12th ACM Conference on Recommender Systems*. 54–62.

K. Talluri and G. Van Ryzin. 2004. Revenue management under a general discrete choice model of consumer behavior. *Management Science* 50, 1 (2004), 15–33.

M. Tennenholtz and O. Kurland. 2019. Rethinking search engines and recommendation systems: a game theoretic perspective. *Commun. ACM* 62, 12 (2019), 66–75.

H. C. W. L. Williams. 1977. On the formation of travel demand models and economic evaluation measures of user benefit. *Environment and planning A* 9, 3 (1977), 285–344.

D. Zeng and A. Psomas. 2020. Fairness-efficiency tradeoffs in dynamic fair division. In *Proceedings of the 21st ACM Conference on Economics and Computation*. 911–912.

A PROOFS OMITTED FROM THE MAIN BODY

A.1 Incentive to Deviate

Example 16. *Consider the instance in Table 8 with exposure constraints equal to 1. Note the the instance is 2-balanced. The recommendation $A_1 = \{c, d\}$ is β-impartial, with $\beta = \frac{2}{2-\epsilon}$, and (1, a) is a blocking pair, The purchase probability of item a is 1/2 in A_1. After deviating to $A_1' = \{a, d\}$, the*

Table 8. Buyers' virtual values.

	a	*b*	*c*	*d*
Buyer 1	*2*	*1*	$2-\epsilon$	*0*
Buyer 2	*1*	*1*	*1*	*0*

purchase probability of a is 1. The difference between seller a's multiplicative improvement and the bound of $\alpha\beta$ in Theorem 6 is

$$\alpha\beta - \frac{P(a, A')}{P(a, A)} = 2 \cdot \frac{2}{2-\epsilon} - \frac{1}{1/2} = \frac{4-4+2\epsilon}{2-\epsilon} = \frac{2\epsilon}{2-\epsilon} \longrightarrow 0 \text{ as } \epsilon \longrightarrow 0.$$

By considering the deviation to $A_1'' = \{a, c\}$, we similarly find that the upper bound on buyer improvement is tight. Specifically,

$$\alpha\beta - \frac{v_1(A'')}{v_1(A)} = 2 \cdot \frac{2}{2-\epsilon} - \frac{4-\epsilon}{2-\epsilon} = \frac{\epsilon}{2-\epsilon} \longrightarrow 0 \text{ as } \epsilon \longrightarrow 0.$$

A.2 Identical Buyers

THEOREM 9. *A^* is PO and SEF1 for buyers with identical preferences.*

PROOF. Pareto optimality again follows from maximizing expected buyer welfare.

Suppose for contradiction that A^* is not SEF1. Then there exists buyers b, c, so that b envies c even any feasible exchange of items between A_b^* and A_c^*. Let $X = A_b^* \cap A_c^*$ denote the items recommended to both b and c. An exchange of items $i \in X$ and $j \in A_c^*$ can only be feasible if $j = i$, otherwise c ends up being recommended i twice. Such an exchange does not change buyer bundles or utility, so we may safely ignore them.

Let $S = \{(i, j) : i \in A_b^* \setminus X, j \in A_c^* \setminus X, v_i < v_j\}$ denote the set of feasible exchanges that are (strictly) improving for b. Suppose $S = \emptyset$. If $|X| = k$, then b and c are recommended identical choice sets, and there is no envy. We may conclude that $|X| < k$. Since $k = |A_b^*| = |A_c^*|$, it follows that $|A_b^* \setminus X| = |A_c^* \setminus X| > 0$. Let $i^- = \text{argmin}_i\{v_i : i \in A_b^* \setminus X\}$ and $j^+ = \text{argmax}_j\{v_j : j \in A_c^* \setminus X\}$. If $S = \emptyset$, then in particular $(i^-, j^+) \notin S$ and, since this is a feasible exchange, it follows that $v_{i^-} > v_{j^+}$. Then

$$\begin{aligned} v(A_b^*) \geq v(X) + v_{i^-} \cdot |A_b^* \setminus X| &> v(X) + v_{j^+} \cdot |A_b^* \setminus X| \\ &= v(X) + v_{j^+} \cdot |A_c^* \setminus X| = v(A_c^*), \end{aligned}$$

contradicting that b envies c. It follows that $S \neq \emptyset$.

Select arbitrary $(i, j) \in S$. By assumption, $v(A_b^*) + v_j - v_i < v(A_c^*) + v_i - v_j$. Construct A' by exchanging i and j and keeping the rest of the recommendation unchanged, so $A_b' = A_b^* \cup \{j\} \setminus \{i\}$ and $A_c' = A_c^* \cup \{i\} \setminus \{j\}$ $A_d' = A_d^*$ for all $d \in \mathcal{B} \setminus \{b, c\}$.

Set $\delta = v_j - v_i > 0$, so $v(A_b') = v(A_b^*) + \delta$ and $v(A_c') = v(A_c^*) - \delta$. Now

$$v(A_b') \cdot v(A_c') = (v(A_b^*) + \delta)(v(A_c^*) - \delta) = v(A_b^*) \cdot v(A_c^*) + \delta(v(A_c^*) - (v(A_b^*) + \delta)) \geq v(A_b^*) \cdot v(A_c^*),$$

where the final transition follows from

$$v(A_c^*) - (v(A_b^*) + \delta) \geq v(A_c') - v(A_b') > 0$$

since A^* is not SEF1 by assumption. This contradicts the fact that A^* maximized the product of virtual values. We conclude that A^* is SEF1. □

A.3 Dichotomous buyers

Theorem 12. *A^* is stable and PO for buyers with dichotomous values.*

Proof. As before, set $\ell = e^a$ and $h = e^{a'}$. We show that maximizing the product of buyer virtual values, which is equivalent to maximizing buyer welfare, is stable (Pareto optimality follows from maximizing buyer welfare).

Assume, for contradition, A^* is not stable. Then there exists a blocking pair $(b, i) \in \mathcal{B} \times \mathcal{I}$. Let c be the buyer to whom (the relevant copy of) i is currently recommended.

Since b is willing to participate in the blocking pair, $v_{bi} = h$ and her choice set contains at least one low-valued item. Let $j \in A_b^*$ be such an item with $v_{bj} = \ell$. Construct A' from A^* by transferring i from c to b's choice sets, and completing c's choice set by recommending some item j' that is below capacity after the transfer. Note that j' must exist, by the definition of a blocking pair, and j' need not be j, in particular, when $j \in A_c^*$ using $j' = j$ is infeasible. Now $A_b' = A_b^* + i - j, A_c' = A_c^* + j' - i$ and $A_d' = A_d^*$ for all $d \in \mathcal{B} \setminus \{b, c\}$.

We now consider the possible values of $(v_{bi}, v_{cj'})$:

(1) $(\ell, \cdot)$: Now $v_b(A_b') \leq v_b(A_b^*)$. This contradicts (b, i) being a blocking pair, since b must strictly gain from participating in a blocking pair and can not do so if $v_{bi} = \ell = v_{bj}$.

(2) (h, h): Now $v_b(A_b') > v_b(A_b^*)$ since $v_{bi} > v_{bj}$ and $v_c(A_c') \geq v_c(A_c^*)$ since $v_{cj'} = h \geq v_{ci}$. This contradicts that A^* maximizes the product of virtual values.

(3) (h, ℓ): Now $v_b(A_b') = v_b(A_b^*) - \ell + h > v_b(A_b^*)$. We will handle the cases of $v_{ci} = \ell$ and $v_{ci} = h$ seperately.
First, suppose $v_{ci} = \ell$. Then $v_c(A_c') = v_c(A_c^*)$, implying $v_c(A_c') \cdot v_b(A_b') > v_c(A_c^*) \cdot v_b(A_b^*)$, contradicting A^* maximizing the product of virtual values.
Suppose instead $v_{ci} = h$. Now $v_c(A_c') = v_c(A_c^*) + \ell - h < v_c(A_c^*)$.
We know that $v_b(A_b^*) < v_b(A_b^*) + h - \ell = v_b(A_b') < v_c(A_c^*)$, because both b and i benefit from participating in the blocing pair and $v_{bi} = h = v_{ci}$ by assumption. As a result, $v_c(A_c^*) - v_b(A_b^*) > h + \ell$. In contrast, $v_c(A_c^*) - v_c(A_c') = h + \ell$. It follows that $v_b(A_b^*) < v_c(A_c')$, and the resulting product of virtual values of

$$\begin{aligned} v_b(A_b') \cdot v_c(A_c') &= (v_b(A_b^*) + h - \ell)(v_c(A_c^*) - h + \ell) \\ &= v_b(A_b^*) \cdot v_c(A_c^*) + (h - \ell)[v_c(A_c^*) - h + \ell - v_b(A_b^*)] \\ &= v_b(A_b^*) \cdot v_c(A_c^*) + (h - \ell)[v_c(A_c') - v_b(A_b^*)] \\ &> v_b(A_b^*) \cdot v_c(A_c^*) \end{aligned}$$

contradicts the fact that A^* was welfare maximizing.

We conclude there exists no blocking pair (b, i). Hence, A^* is stable. □

A.4 Other Preference Domains

A.4.1 Latent factor models. First, we show that the instance of Theorem 3 (Table 2) can be factorised, which implies that stability can not be guaranteed when buyer values come from a latent factor model.

Example 17. *A stable matching need not exist when values come from a latent factor model. Let* $n = 2, k = 2, f = 2$. *Consider* $\beta_1 = (0.6, 1.4), \beta_2 = (1.7, 0.3)$ *and* $\sigma_a = (1.1, 1.2), \sigma_b = (1.3, -0.5), \sigma_c = (0.6, 1.2), \sigma_d = (0.7, 1)$. *The resulting value and utility matrices are shown below.*

Table 9. Valuation matrix (left) and utility matrix (right).

	a	*b*	*c*	*d*
1	*2.34*	*0.08*	*2.04*	*1.82*
2	*2.23*	*2.06*	*1.38*	*1.49*

	a	*b*	*c*	*d*
1	*10.39*	*1.08*	*7.69*	*6.17*
2	*9.39*	*7.84*	*3.97*	*4.44*

It is straightforward to verify that this instance allows the same blocking pairs identified in Theorem 3.

A.4.2 Identical preference orders. Next, we consider the case where buyers have identical preference orders over the items, but not identical values.

The following example with two buyers, six items, unit exposure constraints and $k = 3$ shows that stability can not be guaranteed.

Table 10. Buyers' virtual values: identical preference orders do not guarantee stable recommendations existing.

	a	b	c	d	e	f
1	1	4	5	6	7	10
2	0.5	1.7	4.5	5	9	10

However, stability can be guaranteed in the restricted case of unit exposure constraints with $n = 2 = k, m = 4$.

Proposition 18. *Consider two buyers with identical preference orders and four items* $\{a, b, c, d\}$ *labeled in decreasing order of their virtual values. When* $k = 2$ *under unit exposure constraints, at least one of* $\{\{a, d\}, \{b, c\}\}$ *and* $\{\{b, c\}, \{a, d\}\}$ *is stable.*

Proof. Let $A_1 = \{\{a, d\}, \{b, c\}\}$ and $A_2 = \{\{b, c\}, \{a, d\}\}$.

First, we show that in either A_1 or A_2 seller a is unwilling to participate in a blocking pair. Suppose this is not the case, and a participates in a blocking pair in both A_1 and A_2. Then, from A_1, we conclude $\frac{v_{1a}}{v_{1a}+v_{1d}} < \frac{v_{2a}}{v_{2a}+v_{2c}}$. Similarly, from A_2, $\frac{v_{2a}}{v_{2a}+v_{2d}} < \frac{v_{1a}}{v_{1a}+v_{1c}}$. Together, it follows that

$$\frac{v_{2a}}{v_{2a} + v_{2d}} < \frac{v_{1a}}{v_{1a} + v_{1c}} < \frac{v_{1a}}{v_{1a} + v_{1d}} < \frac{v_{2a}}{v_{2a} + v_{2c}} < \frac{v_{2a}}{v_{2a} + v_{2d}},$$

a contradiction. We conclude a is unwilling to participate in a blocking pair in at least one of A_1, A_2; relabel the buyers so that this happens in A_1.

Consider $A_1 = \{\{a, d\}, \{b, c\}\}$. By construction, a does not participate in a blocking pair. Item b does not participate in a blocking pair, since it is currently selected with probability greater than 0.5 but this drops to less than 0.5 after deviating. Item d does not participate in a blocking pair, since it is the least liked item and can not displace another item from a choice set. Assume that c participates in a blocking pair, otherwise we have found a stable recommendation. It follows that

$$\frac{v_{2c}}{v_{2b} + v_{2c}} < \frac{v_{1c}}{v_{1a} + v_{1c}}. \tag{3}$$

Now consider $A_2 = \{\{b, c\}, \{a, d\}\}$. By the same reasoning as before, b and d do not participate in blocking pairs. We investigate whether a or c can participate ina blocking pair.

(1) Suppose a participates in a blocking pair. Then $\frac{v_{2a}}{v_{2a}+v_{2d}} < \frac{v_{1a}}{v_{1a}+v_{1c}}$. Equivalently,

$$\frac{v_{2d}}{v_{2a}+v_{2d}} > \frac{v_{1c}}{v_{1a}+v_{1c}} >^{\text{(By Eq. 3)}} \frac{v_{2c}}{v_{2b}+v_{2c}} > \frac{v_{2d}}{v_{2b}+v_{2d}} > \frac{v_{2c}}{v_{2a}+v_{2c}},$$

a contradiction.

(2) Suppose c participates in a blocking pair. Then $\frac{v_{1c}}{v_{1c}+v_{1b}} < \frac{v_{2c}}{v_{2a}+v_{2c}}$. It follows that

$$\frac{v_{1c}}{v_{1c}+v_{1b}} < \frac{v_{2c}}{v_{2a}+v_{2c}} < \frac{v_{2c}}{v_{2b}+v_{2c}} <^{\text{(By Eq. 3)}} \frac{v_{1c}}{v_{1a}+v_{1c}} < \frac{v_{1c}}{v_{1c}+v_{1b}},$$

a contradiction.

We conclude either A_1 is stable or, when c participates in a blocking pair in A_1, A_2 is stable. □

A.5 Finding Stable Recommendations, When They Exist

Example 1 shows that stability can not be guaranteed for general preferences. However, there may still be many instances that permit stable recommendations. We construct an integer program to find a stable match if it exists and, if not, returns recommendations in which the sellers' benefit from participating in a blocking pair is as small as possible.

There are some obstacles to overcome. Stability depends on items' purchase probabilities, which are inherently nonlinear. One option is to assign choice sets to buyers and precompute all the resulting purchase probabilities, however, this leads to an exponentially sized program. We present below a formulation with $O(|\mathcal{B}|^2 \cdot |\mathcal{I}|^2) = O(|\mathcal{B}|^4 \cdot k^2)$ variables and as many constraints.

Define binary variable x_{bi} which takes value 1 exactly when $i \in \mathcal{I}$ is recommended to $b \in \mathcal{B}$. The following constraints ensures k sellers are recommended to each buyer and that the recommendation satisfies capacity constraints

$$\sum_{b\in\mathcal{B}} x_{bi} = c_i, \forall i \in \mathcal{I}, \tag{4}$$

$$\sum_{i\in\mathcal{I}} x_{bi} = k, \forall b \in \mathcal{B}. \tag{5}$$

Define continuous variable $g \geq 0$ to capture the maximum multiplicative improvement any seller can get from deviating from any solution x. Consider arbitrary $(b, i) \in \mathcal{B} \times \mathcal{I}$ and let c the buyer that i is recommended to and j seller currently recommended to b who can be displaced by i (i.e. $v_{bi} > v_{bj}$).
Then

$$g \geq \frac{v_{bi}}{\sum_{l\in\mathcal{I}} x_{bl}v_{bl} - v_{bj} + v_{bi}} \Bigg/ \frac{v_{ci}}{\sum_{l\in\mathcal{I}} x_{cl}v_{cl}},$$

where the numerator is the purchase probability of i after transferring into b's bundle and the denominator is their current purchase probability with c. We rewrite this as

$$g \cdot \left(\sum_{l\in\mathcal{I}} x_{bl}v_{bl} - v_{bj} + v_{bi}\right) \geq \frac{v_{bi}}{v_{ci}} \cdot \left(\sum_{l\in\mathcal{I}} x_{cl}v_{cl}\right).$$

Define continuous variable $z_{bi} = g \cdot x_{bi}$ for all $b \in \mathcal{B}, i \in \mathcal{I}$. To ensure that z_{bi} takes the appropriate values, we require constraints

$$z_{bi} \geq 0, \tag{6}$$

$$z_{bi} \leq x_{bi} \cdot G, \tag{7}$$

$$z_{bi} \leq g, \tag{8}$$

$$z_{bi} \geq g + (x_{bi} - 1)G, \tag{9}$$

for all $b \in \mathcal{B}, i \in \mathcal{I}$ and some upper bound G on g. Substituting into the above we obtain

$$\sum_{l \in \mathcal{I}} z_{bl} v_{bl} - g v_{bj} + g v_{bi} \geq \frac{v_{bi}}{v_{ci}} \cdot \sum_{l \in \mathcal{I}} x_{cl} v_{cl}, \tag{10}$$

which should hold for all $b \neq c \in \mathcal{B}, i \neq j \in \mathcal{I}$ as long as $x_{bj} = 1 = x_{ci}$ and $v_{bi} > v_{bj}$. To enforce this we create a new indicator variable

$$\delta_{bj}^{ci} = \begin{cases} 1, & \text{when } x_{bj} = 1 = x_{ci} \text{ and } v_{bi} > v_{bj}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Now we can rewrite eq. (10) as

$$\sum_{l \in \mathcal{I}} z_{bl} v_{bl} - g v_{bj} + g v_{bi} \geq \frac{v_{bi}}{v_{ci}} \cdot \sum_{l \in \mathcal{I}} x_{cl} v_{cl} - (1 - \delta_{bj}^{ci}) M. \tag{11}$$

The following two constraints ensure that δ_{bj}^{ci} takes on the value 1 when expected,

$$(1 - \delta_{bj}^{ci}) M' \geq v_{bj} - v_{bi} - (x_{bj} + x_{bi} - 2) M, \tag{12}$$

$$-\delta_{bj}^{ci} M' \leq v_{bj} - v_{bi} - (x_{bj} + x_{bi} - 2) M, \tag{13}$$

for $M' > 2M$. Observe that when $x_{bj} = 1 = x_{ci}$ and $v_{bi} > v_{bj}$, eq. (12) does not bind and eq. (13) becomes $-\delta_{bj}^{ci} M' < 0$, implying $\delta_{bj}^{ci} = 1$. When $x_{ci} + x_{ct} < 2$, eq. (12) becomes $(1 - \delta_{bj}^{ci}) M' \geq 0$, ensuring $\delta_{bj}^{ci} = 0$, while (13) does not bind. Similarly $v_{bi} \leq v_{bj}$, implies $\delta_{bj}^{ci} = 0$.

This yields the mixed integer program

$$\begin{aligned} & \min g \\ \text{s.t.} \quad & \text{eqs. (4) to (5)} && \text{(assignment constraints)} \\ & \text{eqs. (6) to (9) } \forall b \in \mathcal{B}, i \in \mathcal{I} && \text{(linearization constraints)} \\ & \text{eqs. (11) to (13) } \forall b \neq c \in \mathcal{B}, i \neq j \in \mathcal{I} && \text{(stablity constraints)} \\ & g \geq 0, \; x \in \{0,1\}^{|\mathcal{B}| \times |\mathcal{I}|}, \; z \in \{0,1\}^{|\mathcal{B}| \times |\mathcal{I}|} \\ & \delta_{bj}^{ci} \in \{0,1\}^{|\mathcal{B}| \times |\mathcal{I}| \times |\mathcal{B}| \times |\mathcal{I}|} \end{aligned}$$

When $g \leq 1$, no seller can improve their purchase probability by participating in a blocking pair and the recommendation is stable.

A.5.1 Scalability. We compare the computational cost of the integer program with the three recommendation strategies in Section 4 by training SVD++, a matrix factorization based collaborative filter, on the datasets described in Section 5.1. An instance is created setting $k = 3$ and randomly selecting $B \in \{2, 2^2, \ldots, 2^8\}$ buyers and a corresponding number of random items and taking the collaborative filter's estimated buyer-item ratings as values. The time it takes each approach to yield a recommendation is visually represented in Figure 1. Clearly, the integer program does not scale to reasonable sizes; we exclude it from further experiments. Maximizing buyer welfare performs well on these tests, but may eventually present computational challenges. Round robin and greedy top-k is consistently extremely fast for all problem sizes.

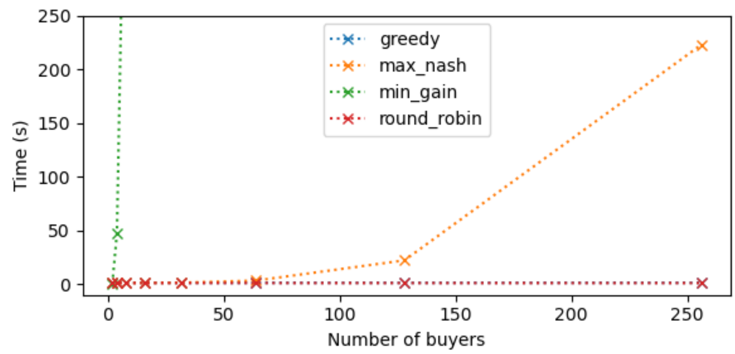

Fig. 1. Time to find recommendations as function of the number of buyers with $k = 5$.