Political Districting without Geography

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Abstract

An analysis of political districting without geographical constraints yields insights into how to design an appropriate objective function for an optimization model. In particular, it reveals serious weaknesses of the popular efficiency gap criterion, as well as the sharp conflict between proportionality and competitiveness and how it might be overcome.

Keywords: Political districting, gerrymandering, efficiency gap

1. Introduction

Optimization models have been devised for political districting for more than half a century (e.g., [1, 2]). As observed in [3], these models have been almost exclusively concerned with the geographical layout of districts, aside from ensuring that districts have roughly equal populations. The very term *gerrymandering* refers to the salamander-like shape of districts that are contrived to benefit a certain party. Yet the fundamental problem with gerrymandering is not the shape of the districts, but the unfair representation that results. The "packing and cracking" strategies used in gerrymandering are based on the political demographics of districts, not their geography. Districts that look reasonable can be highly gerrymandered, while distorted and serpentine districts can provide fair representation.

In recent years, a rise in political polarization has led to concern about the competitiveness of districts as well as gerrymandering [4, 5, 6, 7]. When individual districts are dominated by a single political party, their representatives may be less inclined to negotiate compromise, possibly resulting in a

more partisan legislature. Yet competitiveness is no more tied to geography than gerrymandering is. Districts that concentrate a single point of view can be either compact or serpentine.

The nongeographical essence of the fair districting problem suggests that it can be usefully analyzed without the distraction of geographical constraints. We find that such an analysis, even though it relies solely on elementary algebra, reveals basic properties of the problem that, to our knowledge, have not been observed in the literature. For example, it deduces the theoretical potential of gerrymandering and clarifies the conflict between competitiveness and proportional representation. More importantly, it can lead to optimization models that better incorporate the fundamental goal of fair representation without sacrifice of competitiveness.

A geography-free analysis also reveals serious weaknesses in the recently much-discussed *efficiency gap* criterion for fair districting [6, 8, 9, 10]. The efficiency gap measures the extent to which the political parties differ in how many of their votes are "wasted." We show that minimizing the efficiency gap is consistent with highly nonproportional representation and extreme noncompetitiveness. It is therefore unsuitable, we argue, as an objective.

We do not oppose the use of geographical constraints, as they can serve a legitimate purpose. If nothing else, highly distended districts raise public suspicions of gerrymandering even if it does not exist. Yet we suggest that the districting problem can be better understood, and more satisfactory optimization models obtained, by viewing geography as a side constraint rather than a central element of the model.

2. The Basic Model

To simplify discussion we assume two political parties, A and B, although our analysis can be readily extended to multiple parties or interest groups. We let α and β be the fraction of the voting population aligned with parties A and B, respectively, where $\alpha + \beta = 1$. The legislature contains n seats, corresponding to n districts. We suppose that A is the majority party ($\alpha > \beta$), that all districts have the same population, and that every eligible voter votes. Our task is to decide, for each district i, the fraction α_i of its voters aligned with party A, and thereby the fraction $\beta_i = 1 - \alpha_i$ aligned with party B. Obviously, if all districts reflect the political composition of the population as a whole, then party A will take all n seats. The notation is summarized in Table 1.

We first investigate how to design districts so that a given number m of the districts are majority A. Let $\mathcal{A}$ be the index set of majority-A districts,

Table 1: List of Symbols

n	number of seats in the legislature
m	number of seats won by party A
α, β	fraction of total population that votes for party A, B
α_i, β_i	fraction of population of district i that votes for party A, B
$\bar{\alpha}, \bar{\beta}$	average of α_i, β_i across majority-A, majority-B districts
$\mathcal{A}, \mathcal{B}$	index set of majority-A, majority-B districts
ρ	proportionality ratio for party B: $(1 - m/n)/\beta$
Δ	voting margin $\alpha - \beta$ in the population as a whole
δ	district-level competitiveness margin
δ'	margin in noncompetitive districts
Δ_{eff}	efficiency gap

and similarly for $\mathcal{B}$. Then since all districts contain the same number of voters, we have

$$\frac{1}{n}\left(\sum_{i \in \mathcal{A}} \alpha_i + \sum_{i \in \mathcal{B}}(1 - \beta_i)\right) = \alpha \tag{1}$$

Let $\bar{\alpha}$ be the average fraction of party A adherents in majority-A districts, and similarly for $\bar{\beta}$, so that

$$\bar{\alpha} = \frac{1}{m}\sum_{i \in \mathcal{A}} \alpha_i \qquad \bar{\beta} = \frac{1}{n-m}\sum_{i \in \mathcal{B}} \beta_i$$

Then (1) immediately implies

$$\frac{m}{n}\bar{\alpha} + \left(1 - \frac{m}{n}\right)(1 - \bar{\beta}) = \alpha$$

From this we have the following.

Proposition 1. If the districts have equal population, then

$$\frac{m}{n} = \frac{\alpha + \bar{\beta} - 1}{\bar{\alpha} + \bar{\beta} - 1} \tag{2}$$

Thus the number of seats allocated to party A is determined by the *average* fraction of A voters in majority-A districts and the average fraction of B voters in majority-B districts. The distribution of A and B voters across their respective majority districts has no effect.

We can also derive bounds on the fractions $\bar{\alpha}$ and $\bar{\beta}$. We first note that (2) implies

$$\bar{\alpha} = \frac{\alpha - \left(1 - \frac{m}{n}\right)(1 - \bar{\beta})}{\frac{m}{n}} \qquad \bar{\beta} = \frac{\beta - \frac{m}{n}(1 - \bar{\alpha})}{1 - \frac{m}{n}} \tag{3}$$

3

Due to the fact that $\frac{1}{2} < \bar{\beta} \leq 1$, the first equation in (3) implies

$$\tfrac{1}{2} + \frac{n}{m}(\alpha - \tfrac{1}{2}) < \ \bar{\alpha} \leq \frac{n}{m}\alpha \tag{4}$$

Since the upper bound in (4) may be greater than 1, we replace it with $\min\{1, (n/m)\alpha\}$. We substitute into (4), so modified, the expression for $\bar{\alpha}$ in (3) to obtain bounds on $\bar{\beta}$. This yields

Proposition 2. If the districts have equal population, then the fractions $\bar{\alpha}$ and $\bar{\beta}$ have the bounds

$$\tfrac{1}{2} + \frac{n}{m}(\alpha - \tfrac{1}{2}) < \bar{\alpha} \leq \min\left\{1, \ \frac{n}{m}\alpha\right\} \tag{5}$$

$$\tfrac{1}{2} < \bar{\beta} \leq \min\left\{1, \ \frac{n}{n-m}\beta\right\} \tag{6}$$

As a running example, suppose the electorate consists of 60% party A supporters ($\alpha = 0.6$). If we wish to allot 7 of 10 legislative seats to party A ($m/n = 7/10$), the average fraction $\bar{\alpha}$ of A voters in majority-A districts must be between 64% and 86%, from (5). The resulting average fraction $\bar{\beta}$ of B voters in majority-B districts can be anything between 50% and 100%, from (6).

3. Gerrymandering

The above simple model reveals the theoretical limits of gerrymandering. Suppose we want to gerrymander the districts so that party B will win $n - m > n/2$ of the seats and control the legislature, even though it is the minority party. This is accomplished by cracking and packing. We *crack* the B vote by letting the majority-B districts have a small average margin ϵ, so that $\bar{\beta} - (1 - \bar{\beta}) = \epsilon$, or $\bar{\beta} = \frac{1}{2}(1 + \epsilon)$. Substituting this into the expression for $\bar{\alpha}$ in (3), we have

$$\bar{\alpha} = \frac{n}{m}\alpha - \tfrac{1}{2}(1 - \epsilon)\left(\frac{n}{m} - 1\right) \tag{7}$$

This is the average fraction of A voters that must be *packed* into majority-A districts to ensure that $n - m$ districts vote B by an average margin of ϵ.

In the example, suppose party B wishes to win 6 of the 10 seats even though it has only 40% of the vote. We need only give the majority-B districts a slight majority of B voters and pack the majority-A districts with slightly more than 75% A voters on the average.

To find the largest number of B districts we can engineer (i.e., the largest value of $n - m$), we note that when $\bar{\beta} = \frac{1}{2}(1 + \epsilon)$, (2) implies

$$\frac{m}{n} = \frac{\alpha - \frac{1}{2}(1 - \epsilon)}{\bar{\alpha} - \frac{1}{2}(1 - \epsilon)} \tag{8}$$

To maximize $n - m$ for a given n, we note from (8) that the smallest integer m such that $\bar{\alpha} \leq 1$ is

$$m = \left\lceil n \frac{\alpha - \frac{1}{2}(1 - \epsilon)}{1 - \frac{1}{2}(1 - \epsilon)} \right\rceil = n - \left\lfloor \frac{2n\beta}{1 + \epsilon} \right\rfloor$$

Thus the largest value of $n - m$ we can obtain is

$$n - m = \left\lfloor \frac{2n\beta}{1 + \epsilon} \right\rfloor = \begin{cases} \lfloor 2n\beta \rfloor, & \text{if } \lfloor 2n\beta \rfloor < 2n\beta \\ \lfloor 2n\beta \rfloor - 1, & \text{if } \lfloor 2n\beta \rfloor = 2n\beta \end{cases}$$

where the second equality holds for sufficiently small $\epsilon > 0$. Also we have $n - m > n/2$ when $\beta > \frac{1}{4}$. Thus

Proposition 3. If the districts have equal population, gerrymandering can yield at least as many as $\lfloor 2\beta n \rfloor - 1$ seats for the minority party when $2n\beta$ is integral and $\lfloor 2\beta n \rfloor$ seats otherwise. In particular, the minority party can control the legislature if it accounts for more than a quarter of the population.

For example, if only 41% of the population votes for party B, we can gerrymander the districts so that party B wins 8 of the 10 seats. In fact, gerrymandering can give party B control of the legislature if it accounts for only 26% of the population. Gerrymandering can therefore be very powerful.

4. Proportionality and Competitiveness

Proportionality, or proportional representation, means that the fraction of districts that favor a given party is roughly the fraction of people who belong to that party. Competitiveness means that the minority party in a district has some chance of winning future elections, which can occur when the fraction of people who belong to it is not too much less than 50%. We will see that competitiveness in all districts is sharply at odds with proportionality.

We define a proportionality ratio ρ to be the ratio of party B's representation in the legislature to its representation in the population, so that

5

$\rho = (1 - m/n)/\beta$. A ratio $\rho = 1$ is ideal, while $\rho = 0$ means that the minority party wins no seats at all, and $\rho > 1$ means it is overrepresented. Maximizing proportionality corresponds to minimizing $|1 - \rho|$.

We measure competitiveness in a district by the margin of that district's majority party over its minority party. Thus if we require a margin of δ in every district, we have $\delta = \alpha_i - \beta_i$ in majority-A districts and $\delta = \beta_i - \alpha_i$ in majority-B districts. This implies

$$\beta = \tfrac{1}{2} - \left(\frac{m}{n} - \tfrac{1}{2}\right)\delta, \ \ \text{or} \ \ \frac{m}{n} = \tfrac{1}{2} + \frac{\tfrac{1}{2} - \beta}{\delta} \tag{9}$$

The tradeoff between proportionality and competitiveness is more intuitive when the competitiveness margin δ is compared to the overall margin Δ between the parties. Thus we let $\Delta = \alpha - \beta = 1 - 2\beta$, so that $\beta = \tfrac{1}{2}(1 - \Delta)$. Using this, (9), and the definition of ρ, we obtain the following, which holds with or without geographical constraints:

Proposition 4. If all the districts have the same population, and Δ is the voting margin in the population as a whole, then a margin of δ in each district results in a proportionality ratio

$$\rho = \frac{1 - \Delta/\delta}{1 - \Delta} \tag{10}$$

This result implies a severe incompatibility between proportionality and general competitiveness. We first note that $\rho \leq 1$ because $\delta \leq 1$. Furthermore, we can see as follows that greater competitiveness in all districts (smaller δ) implies much less proportionality (smaller ρ). Since necessarily $\rho \geq 0$, (10) implies $\delta \geq \Delta$. Formula (10) also reminds us that the minority party wins no seats at all when $\delta = \Delta$. Now suppose, for example, that the minority party represents 48% of the voters, so that $\Delta = 4\%$. Then achieving a district margin as small as $\delta = 8\%$ already requires the party to settle for only 25% of the seats (since $\rho = 52\%$ and $\rho\beta = 25\%$). Thus even a modest degree of competitiveness excludes any semblance of proportionality.

5. Efficiency Gap

The efficiency gap is a much-discussed measure of gerrymandering. When the gap is small, gerrymandering is presumably less severe, which suggests that a reasonable objective is to minimize the efficiency gap. However, we will see that there are three problems with minimizing the efficiency gap.

- The efficiency gap is fully determined by the total population of districts won by the majority party. It is insensitive to any other characteristics of the districting plan.

- Minimizing the efficiency gap is consistent with a substantial lack of proportionality, except when the the two parties have roughly equal support in the population.

- Minimizing the efficiency gap is consistent with a complete absence of competitiveness.

It therefore seems desirable to strive for proportionality and competitiveness directly, rather than use the efficiency gap as a measure of fairness.

5.1. Computing the Efficiency Gap

The efficiency gap is defined as the absolute difference between the number of votes "wasted" by party A and the number wasted by party B, divided by the total number of votes. The number of votes wasted by party A in a given district is the number of votes cast for A minus the number necessary to win, or if A loses in the district, the total number of votes cast for A in the district; and similarly for B.

We no longer assume that all districts have equal size, and so the treatment to follow is fully general with respect to the calculation of the efficiency gap. Let p_i be the population (number of voters) in district i. Let P be the total population, p_A the total population of majority-A districts, and similarly for p_B, so that

$$P = \sum_{i=1}^{n} p_i \qquad p_A = \sum_{i \in \mathcal{A}} p_i \qquad p_B = \sum_{i \in \mathcal{B}} p_i$$

The number of votes wasted by parties A and B, respectively, is given by

$$\sum_{i \in \mathcal{A}} (\alpha_i - \tfrac{1}{2}) p_i + \sum_{i \in \mathcal{B}} \alpha_i p_i \ \text{ and } \ \sum_{i \in \mathcal{B}} (\beta_i - \tfrac{1}{2}) p_i + \sum_{i \in \mathcal{A}} \beta_i p_i$$

The absolute difference is

$$\left| \sum_{i \in \mathcal{A}} (\alpha_i - \beta_i - \tfrac{1}{2}) p_i - \sum_{i \in \mathcal{B}} (\beta_i - \alpha_i - \tfrac{1}{2}) p_i \right| = \left| \sum_{i=1}^{n} \alpha_i p_i - \sum_{i=1}^{n} \beta_i p_i + p_B - \tfrac{1}{2} P \right|$$

Dividing by P, we obtain the following, which holds with or without geographical constraints or equal district populations:

Proposition 5. The efficiency gap is given by

$$\Delta_{\text{eff}} = \left| \alpha - \beta + \frac{p_B}{P} - \tfrac{1}{2} \right| = \left| \frac{p_B}{P} - 2\beta + \tfrac{1}{2} \right|$$

Thus for a given β, the efficiency gap depends only on the fraction of the population that lives in majority-B (or majority-A) districts. The distribution of A and B voters across individual districts has no influence.

5.2. *Minimizing the Efficiency Gap*

We now consider how to minimize the efficiency gap for a given β. The gap is zero when $p_B/P = 2\beta - \tfrac{1}{2}$. Since $p_B/P \geq 0$ and $\beta \leq \tfrac{1}{2}$, this minimum can be achieved only when $\tfrac{1}{4} \leq \beta \leq \tfrac{1}{2}$. When $0 \leq \beta \leq \tfrac{1}{4}$, we must set $p_B/P = 0$ to obtain a minimum efficiency gap of $\tfrac{1}{2} - 2\beta$. Thus we have the following, which does not assume equal district populations:

Proposition 6. If there are no geographical constraints, the efficiency gap is minimized when

$$\frac{p_B}{P} = \max\left\{ 2\beta - \tfrac{1}{2}, 0 \right\} \tag{11}$$

and the resulting gap is

$$\Delta_{\text{eff}} = \begin{cases} \tfrac{1}{2} - 2\beta & \text{if } 0 \leq \beta \leq \tfrac{1}{4} \\ 0 & \text{if } \tfrac{1}{4} \leq \beta \leq \tfrac{1}{2} \end{cases} \tag{12}$$

This minimum may not be achievable in the presence of geographical constraints.[1]

If we assume the districts have equal population, $p_B/P = 1 - m/n$. Thus if $\tfrac{1}{4} \leq \beta \leq \tfrac{1}{2}$, we minimize the efficiency gap by choosing m so that $1 - m/n$ is as close as possible to $2\beta - \tfrac{1}{2}$. That is, we set $m = \lfloor (\tfrac{3}{2} - 2\beta)n + \tfrac{1}{2} \rfloor$. The resulting proportionality ratio is

$$\rho = \frac{1 - (1/n)\lfloor (\tfrac{3}{2} - 2\beta)n + \tfrac{1}{2} \rfloor}{\beta} \tag{13}$$

If $0 \leq \beta \leq \tfrac{1}{4}$, we set $m = n$, and the proportionality ratio is $\rho = 0$.

In the example with $\beta = 40\%$ and equally sized districts, the efficiency gap is minimized at zero when $m = 7$. We can achieve this gap with any districting plan in which party B wins $1 - m/n = 30\%$ of the districts. The resulting proportionality ratio is $\rho = 75\%$, from (13).

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[1]As a real-world example of the effect of geographical constraints, Massachusetts is known to have roughly 30% Republican voters spread fairly homogeneously throughout the state. This, together with state laws governing redistricting, makes it impossible to create any districts won by the Republican party [11] and, correspondingly, leads to a minimum efficiency gap much larger than the bound above.

Table 2: Effect of minimizing the efficiency gap

β range	Δ_{eff}	ρ	δ
$0 \leq \beta \leq \frac{1}{4}$	$\frac{1}{2} - 2\beta$	0	$1 - 2\beta$
$\frac{1}{4} \leq \beta \leq \frac{1}{2}$	0	$2 - \frac{1}{2\beta}$	$\frac{1}{2}$

5.3. Proportionality and Competitiveness

A minimized efficiency gap is consistent with a severe lack of proportionality and competitiveness. Supposing again that the districts have equal size, (13) implies that the proportionality ratio decreases rapidly with the minority party's share of the population. For example, if there are 10 districts, the minority party obtains 30% of the seats when its share is 40%, but it receives only 10% of the seats when its share is 30%, and no seats at all when its share is 25%.

A minimized efficiency gap also implies a very large competitiveness margin. Recall that $p_B/P = 1 - m/n$ when the districts have equal size. If we again suppose that δ is the competitiveness margin in every district, then we have from (9) that

$$\delta = \frac{\frac{1}{2} - \beta}{\frac{1}{2} - p_B/P}$$

Putting this together with Proposition 6, we conclude the following.

Proposition 7. Suppose that all districts have equal population and competitiveness margin δ. Then the minimum efficiency gap Δ_{eff}, along with the resulting proportionality ratio ρ and competitiveness margin δ, are as given in Table 2.

A minimum efficiency gap of zero, which occurs whenever $\frac{1}{4} \leq \beta \leq \frac{1}{2}$, results in an extremely large competitiveness margin of 50%. Concurrently, the proportionality ratio can range anywhere between 0 and 1 when $\frac{1}{4} \leq \beta \leq \frac{1}{2}$. For example, $\beta = 30\%$ results in a proportionality ratio of $\frac{1}{3}$ and a competitiveness margin of 50%. Therefore minimizing efficiency gap can result in *simultaneously* poor proportionality and competitiveness.

6. Designing an Objective Function

We have seen that there is a sharp conflict between proportionality and general competitiveness, with or without geographical constraints. Minimizing the efficiency gap only exacerbates the problem. While it has the

virtue of a direct concern with gerrymandering rather than geographical features, it can result in substantial disproportionality *and* a total lack of competitiveness.

One possible escape from this dilemma is to aim for proportionality while achieving competitiveness in *some* districts, with possibly wider margins in the remaining districts. We will find that a satisfactory degree of proportionality is consistent with a surprisingly large number of highly competitive districts. This, in turn, suggests a practical objective for the districting problem.

We therefore require a small competitiveness margin of ϵ in only k majority-A districts and k majority-B districts, where $k \leq \min\{m, n-m\}$, and allow a larger margin of δ' in the remaining districts. This implies

$$\beta = \tfrac{1}{2} - \left(\frac{m}{n} - \tfrac{1}{2}\right)\delta', \ \text{ or } \ \frac{m}{n} = \tfrac{1}{2} + \frac{\tfrac{1}{2} - \beta}{\delta'} \tag{14}$$

Note that k and ϵ drop out of the formula, and we again obtain (9) except that δ' replaces δ. The margin ϵ in the competitive districts has no effect on the fraction β of B voters, because any change in the margin of majority-B competitive districts is balanced by an equal change in the same number of majority-A districts. For similar reasons, the number k of competitive districts has no effect, so long as $k \leq \min\{m, n-m\}$. We therefore obtain (10) with δ' in place of δ.

Proposition 8. If all the districts have the same population, and the same number of majority-A and majority-B districts are competitive (with equal margins), then a margin of δ' in each noncompetitive district results in a proportionality ratio

$$\rho = \frac{1 - \Delta/\delta'}{1 - \Delta} \tag{15}$$

Because ρ is independent of k and δ, we can suppose that all $n - m$ majority-B districts are competitive with an arbitrarily small margin. Returning to a previous example, we found that when the minority party represents $\beta = 48\%$ of voters, even a relatively large margin of 8% in all districts yields only 25% of seats for the minority party. However, if we allow a margin of $\delta' = 20\%$ in noncompetitive districts, the resulting proportionality ratio is $\rho = 5/6$, from (15). Thus the minority party wins $\rho\beta = 40\%$ of the seats, not much less than its 48% representation in the population. If there are 10 seats, the minority party wins 4 of them, and we can suppose that all of these districts, as well as 4 districts controlled by the majority party, are very competitive. Only two districts are relatively noncompetitive with a 20% margin for the majority party.

This is an ideal result that assumes an absence of geographical constraints. Yet it suggests that a reasonable objective is to maximize proportionality (by minimizing $|1 - \rho|$) subject to lower bounds on the number of competitive districts controlled by the two parties. An upper bound can be placed on the acceptable margin δ' in the remaining, possibly noncompetitive districts. This allows an optimization model to focus on two primary goals of political districting: fair representation without gerrymandering, and avoidance of excessive polarization. Neither of these goals is fundamentally geographical in nature, and the conflict between them can be kept to a minimum by avoiding geographical constraints whenever possible.

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