CHAPTER 1
INTRODUCTION AND BACKGROUND

1.1 Introduction

Set theorists have devoted considerable energy toward the study of set-theoretic statements involving Aronszajn trees and Suslin trees. Variants of the generalized continuum hypothesis, forcing, additional axioms of set theory, and large cardinals, are among the techniques set theorists use in proving the consistency of such statements.

We prove results which assume particular cardinal arithmetic assumptions and consequences of the axiom of constructibility. In particular, we assume the existence of a regular cardinal λ, satisfying $\lambda^{<\lambda} = \lambda$, and the existence of two variations of diamond on a stationary subset of λ^+. The main results of our dissertation concern the existence under such hypotheses of large families of pairwise non-club isomorphic Aronszajn trees.

The history of this topic begins with a theorem of Gaifman and Specker [5] which asserts the existence from ZFC of many pairwise not isomorphic Aronszajn trees. Since that result was proven, the focus has turned to comparing Aronszajn trees with respect to isomorphisms on a club of levels, instead of on the entire tree. Abraham and Shelah [1] proved that the Proper Forcing Axiom implies that any two Aronszajn trees on the first uncountable cardinal are club isomorphic. This theorem was generalized to higher cardinals in recent work of Krueger [10]. Abraham and Shelah also proved that the opposite holds under diamond principles. In this dissertation we address the existence of pairwise not club isomorphic Aronszajn trees on higher cardinals from a variety of cardinal arithmetic and diamond principle assumptions. For example, on the successor of a regular cardinal, assuming GCH and the diamond principle on the critical cofinality, there exists a large collection of special Aronszajn trees such that any two of them do not contain club isomorphic subtrees.

Next we give an overview of the chapters and main theorems to be proven in this dissertation. As we do so, we discuss known results bearing connections to our theorems.

Throughout the remainder of Chapter 1 we explore necessary background for our results. Suslin trees, Aronszajn trees, club subsets of regular cardinals, club filters and non-stationary ideals, are among the fundamental concepts to be reviewed.

It is well known that variants of GCH entail the existence of Aronszajn trees. Specker [14] proved that assuming λ is a regular uncountable cardinal satisfying $\lambda^{<\lambda} = \lambda$, there exists a special λ^+-Aronszajn tree. In Chapter 2 we assume the existence of such a regular cardinal. We then present what may be considered a higher cardinal variant of the rationals, $\mathbb{Q}_\lambda$, per the formulation in [7]. Then properties of $\mathbb{Q}_\lambda$ demonstrating that (assuming $\lambda^{<\lambda} = \lambda$) $\mathbb{Q}_\lambda$ and the rationals share many powerful properties are proven. The reader may recall that it is properties on the rationals which witness that under ZF an ω_1-Aronszajn tree exists. These properties serve a similar role: allowing us to prove the existence of special λ^+-Aronszajn trees. Finally we define a type of Aronszajn tree predicated on $\mathbb{Q}_\lambda$, and show that such a tree of height λ^+ is special. However, it is not be until Chapter 4 that we prove the existence of such trees.

In Chapter 3 we explore diamond principles. Ronald Jensen [9] extracted the principle of diamond on ω_1, while proving that the axiom of constructibility implies the existence of an ω_1-Suslin tree. Assuming GCH, John Gregory proved in [6] that given a successor cardinal λ^+: for any $\kappa < \lambda$, diamond holds on the ordinals in λ^+ of cofinality κ. We formally introduce the diamond principle on the ordinals in λ^+ of cofinality λ where λ is regular and satisfies $\lambda^{<\lambda} = \lambda$. Then we develop auxiliary diamond sequences which suffice for constructing large collections of Aronszajn/Suslin trees in later chapters. Finally we explore the theory of weak diamond, introduced by Devlin and Shelah in [4]. Devlin and Shelah demonstrated that weak diamond on ω_1 is equivalent to $2^{\aleph_0} < 2^{\aleph_1}$. Moreover, the authors used their notion of weak diamond to extract a powerful ideal on ω_1, therein called the small ideal. The utility of the small ideal was demonstrated when the authors proved that the ideal satisfies normality: a crucial property in theory of ideals and filters.

We lift up the principle of weak diamond and its associated small ideal to regular cardinals above ω_1. We then present a reformulation of each. Said formulation is suited to use in our main theorem in Chapter 5 wherein we prove that assuming our variation of weak diamond and $\lambda^{<\lambda} = \lambda$, there exists a collection of 2^{λ^+}-many pairwise non-club isomorphic special Aronszajn trees.

In Chapter 4 we prove our first main theorem. Abraham and Shelah proved in [1] that assuming weak diamond on ω_1, and hence the existence of the small ideal, there exist $2^{\aleph_1}$-many special Aronszajn trees such that no two trees are club isomorphic. We prove that assuming diamond on a particular stationary subset of λ^+, there exists a collection of 2^{λ^+}-many special Aronszajn trees such that no two members of the collection contain club isomorphic subtrees, a property which implies that such trees are not club isomorphic and called non-near by Abraham and Shelah. In particular we define such a collection of λ^+-trees predicated on $\mathbb{Q}_\lambda$ from Chapter 2, as we prove in Chapter 2, which are special and Aronszajn. To meet these goals we need to develop various branch set lemmas and tree extension lemmas which align with the diamond technology developed in Chapter 3. Before moving onto our discussion of Chapters 5 and 6, let us further discuss results related to the main theorem of Chapter 4. These results help to in fact frame all three main theorems of this dissertation.

In the same paper where Abraham and Shelah proved the non-club isomorphic result stated above, the authors proved the consistency of the statement that every pair of normal ω_1 Aronszajn trees are club isomorphic. Laver and Shelah [12] then went on to use iterated forcing to prove: assuming the existence of a weakly compact cardinal, it is consistent that there do not exist any ω_2-Suslin trees. Subsequently, in [2] Baumgartner went on to use iterated forcing to prove that under the proper forcing axiom no ω_2-Aronszajn tree exists. Finally, Krueger [10] went on to expand on the work of Abraham and Shelah. Krueger proved that assuming the the existence of an ineffable cardinal, a stronger assumption than that of Laver and Shelah, CH is consistent with the statement that all countably closed, ω_2-Aronszajn trees are club isomorphic, a condition which implies that no ω_2-Suslin tree exists. Thus his work lifted up the result of Abraham and Shelah on club isomorphic Aronszjan trees to ω_2, while also reaching the consistency statement with respect to ω_2-Suslin trees attained by Laver and Shelah.

In Chapter 5 prove that if one assumes $\lambda^{<\lambda} = \lambda$ and our formulation of weak diamond from Chapter 3, then there exists 2^{λ^+}-many non-club isomorphic special λ^+-Aronszajn trees. It is worth noting that we make much use of the technology developed in both Chapters 3 and 4. In particular, the special λ^+-trees we construct are also standard. It is also worth noting that while we weaken our inductive hypotheses to weak diamond we are weakening our theorem to pairwise non-club isomorphic as well. Finally we note that in [1] Abraham and Shelah outline a proof of the consistency that CH holds and any two ω_1-Aronszajn trees are near. Therefore, our result in Chapter 4 required the stronger assumption of diamond and it is unlikely one can hope to strengthen the result in Chapter 5 to pairwise non-near.

In Chapter 6 we maintain our assumption of $\lambda^{<\lambda} = \lambda$, assume $\diamondsuit(S^{\lambda^+}_{\lambda})$, and address the question of whether or not these assumptions are enough to produce large collections of non-near λ^+-Suslin trees. We use our diamond assumptions to define a type of α-tree predicated on $\diamondsuit(S^{\lambda^+}_{\lambda})$. Then we verify that such trees of height λ^+ are Suslin. Finally we prove that under stated assumptions there exist 2^{λ^+}-many λ^+-Suslin trees which are not near.

Note that our theorems amalgamate to suggest that models with fine structure have a variety of tree collections which are far, in the sense that either they are either pairwise non-club isomorphic, or pairwise non-near.

For easy reference, we state here the main results of the dissertation. Note that in the following $S^{\lambda^+}_{\lambda}$ denotes the set $\{\alpha \in \lambda^+ : \mathrm{cf}(\alpha) = \lambda\}$, where λ is a regular cardinal, and Φ denotes the weak diamond principle. Also the phrase "not near" means having no subtrees which are club isomorphic.