Proof of the Holder lower bound on the growth factor of Fast Matrix Multiplication in Strassen's orbit

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> restart : with(LinearAlgebra) : with(Student[VectorCalculus]) : with(PolynomialTools) :
> HolderP := proc(A, p) local i, s; s := 0; for i from 1 to RowDimension(A) do s := s
      + (MatrixNorm(A[i], Frobenius, conjugate = false))^p; od; s^(1/p); end:
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L represents the left linear pre-additions performed by the original Strassen's algorithm on left-hand side of A*B
K represents all the transformation of any L, R or P matrix within Strassen's orbit

> $L := Matrix(7, 4, [[1, 0, 0, 1], [0, 1, 0, -1], [-1, 0, 1, 0], [1, 1, 0, 0], [1, 0, 0, 0], [0, 0, 0, 1], [0, 0, 1, 1]]) :$
> $W := \left\langle \langle r|x\rangle, \left\langle 0 \middle| \frac{1}{r} \right\rangle \right\rangle : V := \left\langle \langle s|y\rangle, \left\langle 0 \middle| \frac{1}{s} \right\rangle \right\rangle :$
> $K := KroneckerProduct(W, V) : L, W, V, K;$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} r & x \\ 0 & \frac{1}{r} \end{bmatrix}, \begin{bmatrix} s & y \\ 0 & \frac{1}{s} \end{bmatrix}, \begin{bmatrix} r\,s & r\,y & x\,s & x\,y \\ 0 & \frac{r}{s} & 0 & \frac{x}{s} \\ 0 & 0 & \frac{s}{r} & \frac{y}{r} \\ 0 & 0 & 0 & \frac{1}{r\,s} \end{bmatrix} \tag{1.1.1}$$

It is easier to study the Holder norm to the power -1/z, and use a nicer form

> $holderexp := -z :$
> $HNlk := HolderP\left(L \cdot K, \frac{1}{holderexp}\right) :$

> $$G := \left((r^2 + x^2)\cdot(s^2 + y^2) + \frac{\left(2\cdot x\cdot y + \frac{1}{r\cdot s}\right)}{r\cdot s} \right)^{-\frac{1}{2\cdot z}} + (r\cdot s)^{\frac{1}{z}} + \left(s^2 + \left(y + \frac{1}{s}\right)^2\right)^{-\frac{1}{2\cdot z}} \cdot \left((r^2 + x^2)^{-\frac{1}{2\cdot z}} + r^{\frac{1}{z}} \right) + \left(r^2 + \left(x - \frac{1}{r}\right)^2\right)^{-\frac{1}{2\cdot z}} \cdot \left((s^2 + y^2)^{-\frac{1}{2\cdot z}} + s^{\frac{1}{z}} \right)$$

$$+\left(\left(r^2+x^2\right)\cdot\left(s^2+y^2\right)\right)^{-\frac{1}{2\cdot z}};$$

$E := simplify\left(G^{holderexp}, symbolic\right):$

simplify(E − HNlk, symbolic);

$$G := \left(\left(r^2+x^2\right)\left(s^2+y^2\right)+\frac{2\,x\,y+\frac{1}{r\,s}}{r\,s}\right)^{-\frac{1}{2\,z}}+\left(r\,s\right)^{\frac{1}{z}}+\left(s^2+\left(y+\frac{1}{s}\right)^2\right)^{-\frac{1}{2\,z}}\left(\left(r^2+x^2\right)^{-\frac{1}{2\,z}}+r^{\frac{1}{z}}\right)+\left(r^2+\left(x-\frac{1}{r}\right)^2\right)^{-\frac{1}{2\,z}}\left(\left(s^2+y^2\right)^{-\frac{1}{2\,z}}+s^{\frac{1}{z}}\right)+\left(\left(r^2+x^2\right)\left(s^2+y^2\right)\right)^{-\frac{1}{2\,z}}$$

$$0 \tag{1.2.1}$$

We compute the gradient of the Holder norm, and check its root

> $fx := diff(E, x) : fy := diff(E, y) : fr := diff(E, r) : fs := diff(E, s) :$
> $gradE := [fx, fy, fr, fs] :$

> $explminpoint := simplify\left(subs\left(\left\{r = root[4]\left(\frac{3}{4}\right)\right\}, subs\left(\{s = r\}, subs\left(\left\{y = -\frac{2\cdot s^3}{3}, x = \frac{2\cdot r^3}{3}\right\}, [r, s, x, y]\right)\right)\right)\right):$
> $subminpoint := solve([r, s, x, y] - explminpoint);$
> $map(simplify, subs(subminpoint, gradE));$

$$subminpoint := \left\{r = \frac{3^{1/4}\sqrt{2}}{2}, s = \frac{3^{1/4}\sqrt{2}}{2}, x = \frac{3^{3/4}\sqrt{2}}{6}, y = -\frac{3^{3/4}\sqrt{2}}{6}\right\}$$

$$[0, 0, 0, 0] \tag{1.3.1}$$

We now compute the Hessian at that point, then the associated characteristic polynomial and eigenvalues

> $H := map\left(x \rightarrow simplify(radnormal(x, rationalized)), Hessian(E, [r, s, x, y] = explminpoint)\right);$
> $charP := FromCoefficientVector\left(map\left(x \rightarrow simplify(x, symbolic), CoefficientVector(simplify(expand(CharacteristicPolynomial(H, X)), radical), X)\right), X\right);$
> $solutions := map\left(x \rightarrow simplify(x, symbolic), [solve(charP)]\right);$

$$H := \begin{bmatrix} \dfrac{13\sqrt{3}\,\Bigl(2^{-\frac{1}{z}}\Bigl(2^{\frac{1}{2z}}+6\,3^{\frac{1}{2z}}\Bigr)\Bigr)^{-z}\Bigl(18\,3^{\frac{1}{2z}}\,2^{\frac{1}{2z}}\,z+36}{18\,z\,\Bigl(2^{\frac{1}{2z}}+6\,3^{\frac{1}{2}}} \cdots \\[3ex] -\dfrac{23\sqrt{3}\,\Bigl(2^{-\frac{1}{z}}\Bigl(2^{\frac{1}{2z}}+6\,3^{\frac{1}{2z}}\Bigr)\Bigr)^{-z}\Bigl(-24\,3^{\frac{1}{2z}}\,2^{\frac{3}{2z}}\,z+}{72\,z\,\Bigl(2^{\frac{1}{2z}}+6\,3} \cdots \\[3ex] \dfrac{\Bigl(2^{-\frac{1}{z}}\Bigl(2^{\frac{1}{2z}}+6\,3^{\frac{1}{2z}}\Bigr)\Bigr)^{-z}\Bigl(9\,3^{\frac{1}{2z}}\Bigl(z-\frac{1}{6}\Bigr)}{3\,z\,\Bigl(2^{\frac{1}{2z}}+6\,3^{\frac{1}{2z}}} \cdots \\[3ex] \dfrac{5\,\Bigl(2^{-\frac{1}{z}}\Bigl(2^{\frac{1}{2z}}+6\,3^{\frac{1}{2z}}\Bigr)\Bigr)^{-z}\Bigl(-24\,3^{\frac{1}{2z}}\,2^{\frac{3}{2z}}\,z+18}{24\,z\,\Bigl(2^{\frac{1}{2z}}+6\,3^{\frac{1}{2}}} \cdots \end{bmatrix}$$

$$charP := \frac{1}{z^4}\Biggl(9216\Biggl(\Biggl(\Bigl(-2^{-\frac{2z-3}{2z}}+\Bigl(z+\frac{1}{4}\Bigr)2^{\frac{3}{2z}}\Bigr)3^{\frac{2z+1}{2z}}+18\,3^{\frac{1}{z}}\Bigl(2^{\frac{1}{z}}\,z$$

$$-\frac{2^{\frac{1-z}{z}}}{2}\Bigr)\Biggr)^2\,16^{-\frac{1}{z}}\Biggl(\Biggl(\Bigl(-\frac{2^{-\frac{2z-3}{2z}}}{18}-\frac{2^{\frac{3}{2z}}}{72}\Bigr)3^{\frac{2z+1}{2z}}-\frac{3\,2^{\frac{1-z}{z}}\,3^{\frac{1}{z}}}{2}$$

$$+z\Bigl(3^{\frac{1}{z}}\,2^{\frac{1}{z}}+\frac{5\,2^{\frac{3}{2z}}\,3^{\frac{1}{2z}}}{6}+\frac{4^{\frac{1}{z}}}{9}\Bigr)\Biggr)\Biggr)^2\Bigl(2^{\frac{1}{2z}}+6\,3^{\frac{1}{2z}}\Bigr)^{-8-4z}\Biggr)$$

$$-\frac{1}{z^3}\Biggl(6144\sqrt{3}\Biggl(-\frac{3^{\frac{2z+1}{2z}}\,2^{\frac{1}{2z}}}{36}+\frac{2^{\frac{z+1}{z}}\,z}{36}+\frac{3^{\frac{1}{2z}}\,2^{\frac{1}{2z}}\,z}{2}+3^{\frac{1}{z}}\Bigl(z-\frac{1}{2}\Bigr)\Biggr)\Biggl($$

$$-\frac{3^{\frac{2z+1}{2z}}\,2^{\frac{3}{2z}}}{2}+4^{\frac{z+1}{z}}\,z+(30\,z-3)\,3^{\frac{1}{2z}}\,2^{\frac{3}{2z}}+36\,3^{\frac{1}{z}}\Bigl(z-\frac{3}{4}\Bigr)2^{\frac{1}{z}}\Biggr)$$

$$4^{-\frac{1}{z}}\Biggl(\frac{3^{\frac{2z+1}{2z}}\,2^{\frac{3}{2z}}}{72}+\frac{\Bigl(z-\frac{1}{2}\Bigr)3^{\frac{1}{2z}}\,2^{\frac{3}{2z}}}{6}+\Bigl(z-\frac{1}{4}\Bigr)3^{\frac{1}{z}}\,2^{\frac{1}{z}}\Biggr)\Bigl(2^{\frac{1}{2z}}$$

$$+ 6\, 3^{\frac{1}{2z}} \Big)^{-6-3z} X \Bigg)$$

$$+ \frac{1}{9 z^2} \Bigg(5120\, 4^{-\frac{1}{z}} \Bigg(\frac{1107 \left(z^2 - \frac{95}{164} z + \frac{95}{1312}\right) 3^{\frac{3}{2z}}\, 2^{\frac{5}{2z}}}{40} + 3^{\frac{1}{2z}} z \Big(z$$

$$- \frac{121}{640} \Big) 2^{\frac{7}{2z}} + \frac{1341\, 8^{\frac{1}{z}} \left(z^2 - \frac{54}{149} z + \frac{95}{4768}\right) 3^{\frac{1}{z}}}{160}$$

$$+ \frac{1107\, 9^{\frac{1}{z}} \left(z^2 - z + \frac{285}{1312}\right) 4^{\frac{1}{z}}}{40} + \frac{13 z^2\, 16^{\frac{1}{z}}}{320} \Bigg) \left(2^{\frac{1}{2z}} + 6\, 3^{\frac{1}{2z}}\right)^{-4}$$

$$^{-2z} X^2 \Bigg) - \frac{1}{9 z} \Bigg(32 \sqrt{3} \left(2^{\frac{1}{2z}} + 6\, 3^{\frac{1}{2z}}\right)^{-2-z} \Big(18\, 3^{\frac{1}{2z}}\, 2^{\frac{1}{2z}} z + 36\, 3^{\frac{1}{z}} z + 2^{\frac{z+1}{z}} z$$

$$- 3^{\frac{2z+1}{2z}}\, 2^{\frac{1}{2z}} - 18\, 3^{\frac{1}{z}} \Big) X^3 \Bigg) + X^4$$

$$solutions := \Bigg[\Bigg\{ X = \frac{1}{6 z} \Bigg(\left(2^{\frac{1}{2z}} + 6\, 3^{\frac{1}{2z}}\right)^{-z-1} \Bigg((32 z - 11)\, 3^{\frac{z+1}{2z}} + 4 z \sqrt{3}\, 2^{\frac{1}{2z}} \qquad \textbf{(1.4.1)}$$

$$+ \sqrt{-384 \left(z - \frac{1}{16}\right) 3^{\frac{1}{2z}} z\, 2^{\frac{1}{2z}} + \left(1344 z^2 - 384 z + 39\right) 3^{\frac{1}{z}} + 48\, 2^{\frac{1}{z}} z^2} \Bigg) \Bigg), z$$

$$= z \Bigg\}, \Bigg\{ X = -\frac{1}{6 z} \Bigg(\Bigg((-32 z + 11)\, 3^{\frac{z+1}{2z}} - 4 z \sqrt{3}\, 2^{\frac{1}{2z}}$$

$$+ \sqrt{-384 \left(z - \frac{1}{16}\right) 3^{\frac{1}{2z}} z\, 2^{\frac{1}{2z}} + \left(1344 z^2 - 384 z + 39\right) 3^{\frac{1}{z}} + 48\, 2^{\frac{1}{z}} z^2} \Bigg)$$

$$\left(2^{\frac{1}{2z}} + 6\, 3^{\frac{1}{2z}}\right)^{-z-1} \Bigg), z = z \Bigg\}, \Bigg\{ X = \frac{1}{18 z} \Bigg(\Bigg((96 z - 63)\, 3^{\frac{z+1}{2z}} + 52 z \sqrt{3}\, 2^{\frac{1}{2z}}$$

$$+$$

$$\left(19584\left(z-\frac{237}{272}\right)3^{\frac{1}{2z}}\,z\,2^{\frac{1}{2z}}+\left(12096\,z^2-20736\,z+8991\right)3^{\frac{1}{z}}\right.$$

$$\left.\left.\left.+\,8112\,2^{\frac{1}{z}}\,z^2\right)^{1/2}\right)\left(2^{\frac{1}{2z}}+6\,3^{\frac{1}{2z}}\right)^{-z-1}\right],\ z=z\Big\},\ \Big\{X=-\frac{1}{18\,z}\left(\left(2^{\frac{1}{2z}}\right.\right.$$

$$\left.+\,6\,3^{\frac{1}{2z}}\right)^{-z-1}\left(\left(-96\,z+63\right)3^{\frac{z+1}{2z}}-52\,z\sqrt{3}\,2^{\frac{1}{2z}}\right.$$

$$+$$

$$\left(19584\left(z-\frac{237}{272}\right)3^{\frac{1}{2z}}\,z\,2^{\frac{1}{2z}}+\left(12096\,z^2-20736\,z+8991\right)3^{\frac{1}{z}}\right.$$

$$\left.\left.\left.+\,8112\,2^{\frac{1}{z}}\,z^2\right)^{1/2}\right)\right),\ z=z\Big\}\Big]$$

We now look for a nicer form of the eigenvalues for z>0

> tau := $3^{\frac{1}{2\cdot z}}$: nu := $3^{\frac{1+z}{2\cdot z}}$: lambda := $2^{\frac{1}{2\cdot z}}$:

> $d1 := 24(1-16\cdot z)\cdot \mathrm{tau}\cdot z\cdot \mathrm{lambda}+\left(1344\,z^2-384\,z+39\right)\cdot\tau^2+48\cdot\lambda^2\cdot z^2;$

$d2 := 72\,(272\cdot z-237)\cdot\mathrm{tau}\cdot z\,\mathrm{lambda}+\left(12096\,z^2-20736\,z+8991\right)\cdot\tau^2$
$\quad +\,8112\,\lambda^2\,z^2;$

$b1 := (32\,z-11)\cdot\mathrm{nu}+4\sqrt{3}\,z\,2^{\frac{1}{2z}};$

$b2 := (96\,z-63)\cdot\mathrm{nu}+52\sqrt{3}\,z\,\mathrm{lambda};$

$n := \dfrac{\left(2^{\frac{1}{2z}}+6\,3^{\frac{1}{2z}}\right)^{-1-z}}{6\cdot z};$

$$d1 := (24-384\,z)\,3^{\frac{1}{2z}}\,z\,2^{\frac{1}{2z}}+\left(1344\,z^2-384\,z+39\right)\left(3^{\frac{1}{2z}}\right)^2+48\left(2^{\frac{1}{2z}}\right)^2 z^2$$

$$d2 := (19584\,z-17064)\,3^{\frac{1}{2z}}\,z\,2^{\frac{1}{2z}}+\left(12096\,z^2-20736\,z+8991\right)\left(3^{\frac{1}{2z}}\right)^2$$
$$+\,8112\left(2^{\frac{1}{2z}}\right)^2 z^2$$

$$b1 := (32\,z-11)\,3^{\frac{z+1}{2z}}+4\,z\sqrt{3}\,2^{\frac{1}{2z}}$$

$$b2 := (96\,z-63)\,3^{\frac{z+1}{2z}}+52\,z\sqrt{3}\,2^{\frac{1}{2z}}$$

(1.5.1)

$$n := \frac{\left(2^{\frac{1}{2z}} + 6\,3^{\frac{1}{2z}}\right)^{-z-1}}{6\,z} \tag{1.5.1}$$

> $eigs := \left[n\cdot(b1+\mathrm{sqrt}(d1)),\ n\cdot(b1-\mathrm{sqrt}(d1)),\ \frac{n}{3}\cdot(b2+\mathrm{sqrt}(d2)),\ \frac{n}{3}\cdot(b2-\mathrm{sqrt}(d2))\right];$
$sols := [subs(solutions[1], X), subs(solutions[2], X), subs(solutions[3], X), subs(solutions[4], X)]:$
$map(y \rightarrow simplify(y, symbolic), eigs - sols);$

$$eigs := \left[\frac{1}{6\,z}\left(\left(2^{\frac{1}{2z}} + 6\,3^{\frac{1}{2z}}\right)^{-z-1}\left((32\,z-11)\,3^{\frac{z+1}{2z}} + 4\,z\sqrt{3}\,2^{\frac{1}{2z}}\right.\right.\right.$$
$$+$$
$$\left.\left.\sqrt{(24-384\,z)\,3^{\frac{1}{2z}}\,z\,2^{\frac{1}{2z}} + \left(1344\,z^2-384\,z+39\right)\left(3^{\frac{1}{2z}}\right)^2 + 48\left(2^{\frac{1}{2z}}\right)^2 z^2}\right)\right)$$
$$,\ \frac{1}{6\,z}\left(\left(2^{\frac{1}{2z}} + 6\,3^{\frac{1}{2z}}\right)^{-z-1}\left((32\,z-11)\,3^{\frac{z+1}{2z}} + 4\,z\sqrt{3}\,2^{\frac{1}{2z}}\right.\right.$$
$$-$$
$$\left.\left.\sqrt{(24-384\,z)\,3^{\frac{1}{2z}}\,z\,2^{\frac{1}{2z}} + \left(1344\,z^2-384\,z+39\right)\left(3^{\frac{1}{2z}}\right)^2 + 48\left(2^{\frac{1}{2z}}\right)^2 z^2}\right)\right)$$
$$,\ \frac{1}{18\,z}\left(\left(2^{\frac{1}{2z}} + 6\,3^{\frac{1}{2z}}\right)^{-z-1}\left((96\,z-63)\,3^{\frac{z+1}{2z}} + 52\,z\sqrt{3}\,2^{\frac{1}{2z}}\right.\right.$$
$$+\left((19584\,z-17064)\,3^{\frac{1}{2z}}\,z\,2^{\frac{1}{2z}} + \left(12096\,z^2-20736\,z+8991\right)\left(3^{\frac{1}{2z}}\right)^2\right.$$
$$\left.\left.\left.+\,8112\left(2^{\frac{1}{2z}}\right)^2 z^2\right)^{1/2}\right)\right),\ \frac{1}{18\,z}\left(\left(2^{\frac{1}{2z}} + 6\,3^{\frac{1}{2z}}\right)^{-z-1}\left((96\,z-63)\,3^{\frac{z+1}{2z}}\right.\right.$$
$$+\,52\,z\sqrt{3}\,2^{\frac{1}{2z}}$$

$$- \left((19584 z - 17064)\, 3^{\frac{1}{2z}}\, z\, 2^{\frac{1}{2z}} + (12096 z^2 - 20736 z + 8991) \left(3^{\frac{1}{2z}}\right)^2 + 8112 \left(2^{\frac{1}{2z}}\right)^2 z^2 \right)^{1/2} \Bigg) \Bigg) \Bigg]$$

$$[0, 0, 0, 0] \qquad \textbf{(1.5.2)}$$

We check that the eigenvalues are real on the positive plane: the minimal values of d1 and d2 are positive

> $dd1 := diff(\mathrm{d1}, z) : \mathrm{dd2} := \mathrm{diff}(\mathrm{d2}, z) :$
> $\mathrm{min1} := solve(\mathrm{dd1}) : min2 := solve(\mathrm{dd2}) : evalf([min1, min2]);$

$$[0.3926777710, 0.5524430714] \qquad \textbf{(1.6.1)}$$

>

d1 is decreasing before min1, increasing afterwards ; d2 is decreasing before min1, increasing afterwards ;

$$\left[\left[evalf\left(subs\left(z = min1 - \frac{1}{10}, dd1\right)\right), evalf\left(subs\left(z = \mathrm{min1} + \frac{1}{10}, dd1\right)\right)\right], \left[evalf\left(subs\left(z = min2 - \frac{1}{10}, dd2\right)\right), evalf\left(subs\left(z = min2 + \frac{1}{10}, dd2\right)\right)\right]\right];$$

$$[[-4224.401642, 1059.281801], [-107526.9425, 35945.63615]] \qquad \textbf{(1.6.2)}$$

> # *Their minimal values are positive:*
> $evalf([subs(z = min1, d1), subs(z = min2, d2)]);$

$$[1121.901516, 184.766516] \qquad \textbf{(1.6.3)}$$

We end by checking that the eigenvalues are positive after 0.5171, hence the Hessian is definite positive and the extremal point is a local minimum

> # *we search for the minimal value of the derivative of the potentially smallest 2 eigenvalues*
> $\mathrm{sbd1} := \mathrm{b1}^2 - \mathrm{d1} : \mathrm{sbd2} := \mathrm{b2}^2 - \mathrm{d2} :$
> $bd1 := diff(\mathrm{sbd1}, z) : \mathrm{bd2} := \mathrm{diff}(\mathrm{sbd2}, z) :$
> $bin1 := solve(\mathrm{bd1}) : bin2 := solve(\mathrm{bd2}) : evalf([bin1, bin2]);$

$$[0.3123021197, 0.3123021197] \qquad \textbf{(1.7.1)}$$

>

sbd1 is decreasing before bin1, increasing afterwards ; sbd2 is decreasing before bin2, *increasing afterwards* ;

$$\left[\left[evalf\left(subs\left(z = bin1 - \frac{1}{10}, bd1\right)\right), evalf\left(subs\left(z = bin1 + \frac{1}{10}, bd1\right)\right)\right], \left[evalf\left(subs\left(z = bin2 - \frac{1}{10}, bd2\right)\right), evalf\left(subs\left(z = bin2 + \frac{1}{10}, bd2\right)\right)\right]\right];$$

$$[[-300500.8617, 6683.547019], [-2.704507802 \times 10^6, 60151.92294]] \qquad \textbf{(1.7.2)}$$

> # *Their minimal values are not positive:*
> $evalf([subs(z = bin1, sbd1), subs(z = bin2, sbd2)]);$

(1.7.3)

$$[-1193.675072, -10743.07550] \quad (1.7.3)$$

> # But sbd1 and sbd2 will be positive after their root:and b1>sqrt(d1) as well as b2>sqrt (d2)

$evalf([[solve(b1^2 - d1)], [solve(b2^2 - d2)]])$;

$$[[0.5170849365], [0.5170849365]] \quad (1.7.4)$$

> # b1 & b2 have no real roots and is positive and are thus positive after 0.5171

$sin1 := solve(diff(b1, z)) : sin2 := solve(diff(b2, z)) : evalf([sin1, sin2])$;

$$[0.2724631658 + 0.3178238892\,I, 0.2696570864 + 0.4323870216\,I] \quad (1.7.5)$$

> # b1 & b2 are thus increasing since their derivative is positive and are thus positive after 0.5171

$evalf([subs(z=1, diff(b1, z)), subs(z=1, diff(b2, z))])$;

$$[67.79595805, 316.8478787] \quad (1.7.6)$$

> # b1 & b2 are thus positive after 0.5171

$evalf(subs(z=0.5171, [b1, b2]))$;

$$[34.79848790, 24.10032976] \quad (1.7.7)$$

> # Thereore b1+sqrt(d1) & b2+srqrt(d1) are positive as well as b1 − sqrt(d1) & b2 − srqrt(d1)

since n is positive, all four eigenvalues are positive

> $evalf(subs(z=0.5171 + rand(\ ), eigs))$;

$$[3.606560116 \times 10^{-352849030632}, 1.172951779 \times 10^{-352849030632}, 5.823247547 \times 10^{-352849030632}, 7.264539357 \times 10^{-352849030633}] \quad (1.8.8)$$

The extremal point is a minimum, we search for the limit of the combined Holder norms at infinity

> $MinPoint := simplify(subs(subminpoint, E))$;

$$MinPoint := \left(2^{-\frac{1}{2z}} + 3^{\frac{2z+1}{2z}}\, 2^{-\frac{1}{z}} + 2^{\frac{z-1}{z}}\, 3^{\frac{1}{2z}} + 3^{\frac{1}{2z}}\, 2^{-\frac{1}{z}}\right)^{-z} \quad (1.8.1)$$

> # first Holder conjugates : $\frac{1}{-\frac{1}{z}} + \frac{1}{-\frac{1}{z}} + \frac{1}{-\frac{1}{z}} + (1 + 3z) = 1$

$LowerBound1 := limit(MinPoint^3 \cdot 7^{1+3 \cdot z}, z = \text{infinity})$;

$$LowerBound1 := \frac{28\, 2^{11/14}\, 3^{5/7}}{9} \quad (1.8.2)$$

> # second Holder conjugates: $\frac{1}{-\frac{1}{z}} + \frac{1}{-\frac{1}{z}} + \frac{1}{\frac{1}{1+2z}} = 1$

$LowerBound2 := limit(MinPoint^2 \cdot subs(z = -(1 + 2 \cdot z), MinPoint), z = \text{infinity})$;

$$LowerBound2 := \frac{28\, 2^{11/14}\, 3^{5/7}}{9} \quad (1.8.3)$$

> $evalf([LowerBound1, LowerBound2])$;

$$[11.75546969, 11.75546969] \quad (1.8.4)$$