# Proof of the Holder lower bound on the growth factor of Fast Matrix Multiplication in Strassen's orbit

> $restart : with(LinearAlgebra) : with(Student[VectorCalculus]) : with(PolynomialTools) :$

> $HolderP := \mathbf{proc}(A, p)\ \mathbf{local}\ i, s;\ s := 0;\mathbf{for}\ i\ \mathbf{from}\ 1\ \mathbf{to}\ RowDimension(A)\ \mathbf{do}\ s := s + (MatrixNorm(A[i], Frobenius, conjugate = false))^{p};\ \mathbf{od};\ s^{\frac{1}{p}};\ \mathbf{end}:$

## L represents the left linear pre-additions performed by the original Strassen's algorithm on left-hand side of A*B
## K represents all the transformation of any L, R or P matrix within Strassen's orbit

> $L := Matrix(7, 4, [[1, 0, 0, 1], [0, 1, 0, -1], [-1, 0, 1, 0], [1, 1, 0, 0], [1, 0, 0, 0], [0, 0, 0, 1], [0, 0, 1, 1]]) :$
> $\mathrm{W} := \left\langle \langle r|x\rangle, \left\langle 0 \middle| \frac{1}{r} \right\rangle \right\rangle : \mathrm{V} := \left\langle \langle s|y\rangle, \left\langle 0 \middle| \frac{1}{s} \right\rangle \right\rangle :$
> $K := KroneckerProduct(W, V) : \mathrm{L, W, V, K};$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} r & x \\ 0 & \frac{1}{r} \end{bmatrix}, \begin{bmatrix} s & y \\ 0 & \frac{1}{s} \end{bmatrix}, \begin{bmatrix} r\,s & r\,y & x\,s & x\,y \\ 0 & \frac{r}{s} & 0 & \frac{x}{s} \\ 0 & 0 & \frac{s}{r} & \frac{y}{r} \\ 0 & 0 & 0 & \frac{1}{r\,s} \end{bmatrix} \tag{1.1.1}$$

## It is easier to study the Holder norm to the power -1/z, and use a nicer form

> $holderexp := -z : \mathrm{HNlk} := \mathrm{HolderP}\left(L \cdot K, \frac{1}{holderexp}\right) :$

> $\mathrm{G} := \left( (r^2+x^2)\cdot(s^2+y^2) + \frac{\left(2\cdot x\cdot y + \frac{1}{r\cdot s}\right)}{r\cdot s} \right)^{-\frac{1}{2\cdot z}} + (r\cdot s)^{\frac{1}{z}} + \left(s^2 + \left(y+\frac{1}{s}\right)^2\right)^{-\frac{1}{2\cdot z}} \cdot \left( (r^2+x^2)^{-\frac{1}{2\cdot z}} + r^{\frac{1}{z}} \right) + \left(r^2 + \left(x - \frac{1}{r}\right)^2\right)^{-\frac{1}{2\cdot z}} \cdot \left( (s^2+y^2)^{-\frac{1}{2\cdot z}} + s^{\frac{1}{z}} \right) + \left( (r^2+x^2)\cdot(s^2+y^2) \right)^{-\frac{1}{2\cdot z}};$

$E := simplify\left(G^{holderexp}, symbolic\right):$

simplify(E − HNlk, symbolic);

$$G := \left(\left(r^2+x^2\right)\left(s^2+y^2\right)+\frac{2\,x\,y+\frac{1}{r\,s}}{r\,s}\right)^{-\frac{1}{2z}}+(r\,s)^{\frac{1}{z}}+\left(s^2+\left(y+\frac{1}{s}\right)^2\right)^{-\frac{1}{2z}}\left(\left(r^2+x^2\right)^{-\frac{1}{2z}}+r^{\frac{1}{z}}\right)+\left(r^2+\left(x-\frac{1}{r}\right)^2\right)^{-\frac{1}{2z}}\left(\left(s^2+y^2\right)^{-\frac{1}{2z}}+s^{\frac{1}{z}}\right)+\left(\left(r^2+x^2\right)\left(s^2+y^2\right)\right)^{-\frac{1}{2z}}$$

$$0 \tag{1.2.1}$$

## We compute the gradient of the Holder norm, and check its root

> $fx := diff(E, x) : fy := diff(E, y) : fr := diff(E, r) : fs := diff(E, s) :$
> $gradE := [fx, fy, fr, fs] :$

> $explminpoint := simplify\left(subs\left(\left\{r = root[4]\left(\frac{3}{4}\right)\right\}, subs\left(\{s = r\}, subs\left(\left\{y = -\frac{2 \cdot s^3}{3}, x = \frac{2 \cdot r^3}{3}\right\}, [r, s, x, y]\right)\right)\right)\right) :$
$subminpoint := solve([r, s, x, y] - explminpoint);$
$map(simplify, subs(subminpoint, gradE));$

$$subminpoint := \left\{r = \frac{3^{1/4}\sqrt{2}}{2}, s = \frac{3^{1/4}\sqrt{2}}{2}, x = \frac{3^{3/4}\sqrt{2}}{6}, y = -\frac{3^{3/4}\sqrt{2}}{6}\right\}$$

$$[0, 0, 0, 0] \tag{1.3.1}$$

## We now compute the Hessian at that point, then the associated characteristic polynomial and eigenvalues

> $H := map(x \rightarrow simplify(radnormal(x, rationalized)), Hessian(E, [r, s, x, y] = explminpoint));$
$charP := FromCoefficientVector(map(x \rightarrow simplify(x, symbolic), CoefficientVector(simplify(expand(CharacteristicPolynomial(H, X)), radical), X)), X);$
$solutions := map(x \rightarrow simplify(x, symbolic), [solve(charP)]);$

$$H := \begin{bmatrix} \dfrac{13\sqrt{3}\,\Big(2^{-\frac{1}{z}}\Big(2^{\frac{1}{2z}}+6\,3^{\frac{1}{2z}}\Big)\Big)^{-z}\Big(18\,3^{\frac{1}{2z}}\,2^{\frac{1}{2z}}\,z+}{18\,z\Big(2^{\frac{1}{2z}}+6\,} \cdots \\[3em] -\dfrac{23\sqrt{3}\,\Big(2^{-\frac{1}{z}}\Big(2^{\frac{1}{2z}}+6\,3^{\frac{1}{2z}}\Big)\Big)^{-z}\Big(-24\,3^{\frac{1}{2z}}\,2^{\frac{3}{2z}}}{72\,z\Big(2^{\frac{1}{2z}}+6} \cdots \\[3em] \dfrac{\Big(9\,3^{\frac{1}{2z}}\Big(z-\frac{1}{6}\Big)2^{\frac{1}{2z}}+(18\,z-9)\,3^{\frac{1}{z}}+}{3\,z\Big(2^{\frac{1}{2z}}+6\,3} \cdots \\[3em] \dfrac{5\Big(2^{-\frac{1}{z}}\Big(2^{\frac{1}{2z}}+6\,3^{\frac{1}{2z}}\Big)\Big)^{-z}\Big(-24\,3^{\frac{1}{2z}}\,2^{\frac{3}{2z}}\,z+}{24\,z\Big(2^{\frac{1}{2z}}+6\,} \cdots \end{bmatrix}$$

$$charP := \frac{1}{z^4}\Bigg(9216\Bigg(\Bigg(\bigg(-\frac{2^{-\frac{2z-3}{2z}}}{18}-\frac{2^{\frac{3}{2z}}}{72}\bigg)3^{\frac{2z+1}{2z}}-\frac{3\,2^{\frac{1-z}{z}}\,3^{\frac{1}{z}}}{2}+z\Big(3^{\frac{1}{z}}\,2^{\frac{1}{z}}$$

$$+\frac{5\,2^{\frac{3}{2z}}\,3^{\frac{1}{2z}}}{6}+\frac{4^{\frac{1}{z}}}{9}\Big)\Bigg)\Bigg)^2\Big(2^{\frac{1}{2z}}+6\,3^{\frac{1}{2z}}\Big)^{-8-4z}\Bigg(\bigg(-2^{-\frac{2z-3}{2z}}+\Big(z$$

$$+\frac{1}{4}\Big)2^{\frac{3}{2z}}\bigg)3^{\frac{2z+1}{2z}}+18\bigg(2^{\frac{1}{z}}\,z-\frac{2^{\frac{1-z}{z}}}{2}\bigg)3^{\frac{1}{z}}\Bigg)^2 16^{-\frac{1}{z}}\Bigg)-\frac{1}{z^3}\Bigg(6144\Bigg($$

$$-\frac{3^{\frac{2z+1}{2z}}\,2^{\frac{3}{2z}}}{2}+4^{\frac{z+1}{z}}\,z+(30\,z-3)\,3^{\frac{1}{2z}}\,2^{\frac{3}{2z}}+36\,3^{\frac{1}{z}}\Big(z-\frac{3}{4}\Big)2^{\frac{1}{z}}\Bigg)$$

$$\sqrt{3}\Bigg(\frac{3^{\frac{2z+1}{2z}}\,2^{\frac{3}{2z}}}{72}+\frac{\Big(z-\frac{1}{2}\Big)3^{\frac{1}{2z}}\,2^{\frac{3}{2z}}}{6}+3^{\frac{1}{z}}\Big(z-\frac{1}{4}\Big)2^{\frac{1}{z}}\Bigg)\Big(2^{\frac{1}{2z}}$$

$$+6\,3^{\frac{1}{2z}}\Big)^{-6-3z}\,4^{-\frac{1}{z}}\Bigg(-\frac{3^{\frac{2z+1}{2z}}\,2^{\frac{1}{2z}}}{36}+\frac{2^{\frac{z+1}{z}}\,z}{36}+\frac{3^{\frac{1}{2z}}\,2^{\frac{1}{2z}}\,z}{2}+3^{\frac{1}{z}}\Big(z$$

$$-\frac{1}{2}\Big)\Big)X\Big)+\frac{1}{9\,z^{2}}\Bigg(5120\Bigg(\frac{1107\;3^{\frac{3}{2\,z}}\left(z^{2}-\frac{95}{164}\,z+\frac{95}{1312}\right)2^{\frac{5}{2\,z}}}{40}$$

$$+\,3^{\frac{1}{2\,z}}\,z\left(z-\frac{121}{640}\right)2^{\frac{7}{2\,z}}+\frac{1341\;8^{\frac{1}{z}}\left(z^{2}-\frac{54}{149}\,z+\frac{95}{4768}\right)3^{\frac{1}{z}}}{160}$$

$$+\,\frac{1107\;9^{\frac{1}{z}}\left(z^{2}-z+\frac{285}{1312}\right)4^{\frac{1}{z}}}{40}+\frac{13\,z^{2}\,16^{\frac{1}{z}}}{320}\Bigg)4^{-\frac{1}{z}}\left(2^{\frac{1}{2\,z}}+6\;3^{\frac{1}{2\,z}}\right)^{-4}$$

$$^{-2\,z}X^{2}\Bigg)-\frac{1}{9\,z}\Bigg(32\sqrt{3}\left(2^{\frac{1}{2\,z}}+6\;3^{\frac{1}{2\,z}}\right)^{-2-z}\Bigg(18\;3^{\frac{1}{2\,z}}\;2^{\frac{1}{2\,z}}\,z+36\;3^{\frac{1}{z}}\,z$$

$$+\,2^{\frac{z+1}{z}}\,z-3^{\frac{2\,z+1}{2\,z}}\;2^{\frac{1}{2\,z}}-18\;3^{\frac{1}{z}}\Bigg)X^{3}\Bigg)+X^{4}$$

$$solutions:=\Bigg[\Bigg\{X=\frac{1}{6\,z}\Bigg(\Bigg(\Big(32\,z-11\Big)\,3^{\frac{z+1}{2\,z}}+4\,z\sqrt{3}\;2^{\frac{1}{2\,z}} \tag{1.4.1}$$

$$+\sqrt{-384\,z\left(z-\frac{1}{16}\right)3^{\frac{1}{2\,z}}\;2^{\frac{1}{2\,z}}+\left(1344\,z^{2}-384\,z+39\right)3^{\frac{1}{z}}+48\;2^{\frac{1}{z}}\,z^{2}}\Bigg)$$

$$\left(2^{\frac{1}{2\,z}}+6\;3^{\frac{1}{2\,z}}\right)^{-z-1}\Bigg),z=z\Bigg\},\Bigg\{X=-\frac{1}{6\,z}\Bigg(\Bigg(\Big(-32\,z+11\Big)\,3^{\frac{z+1}{2\,z}}$$

$$-\,4\,z\sqrt{3}\;2^{\frac{1}{2\,z}}$$

$$+\sqrt{-384\,z\left(z-\frac{1}{16}\right)3^{\frac{1}{2\,z}}\;2^{\frac{1}{2\,z}}+\left(1344\,z^{2}-384\,z+39\right)3^{\frac{1}{z}}+48\;2^{\frac{1}{z}}\,z^{2}}\Bigg)$$

$$\left(2^{\frac{1}{2\,z}}+6\;3^{\frac{1}{2\,z}}\right)^{-z-1}\Bigg),z=z\Bigg\},\Bigg\{X=\frac{1}{18\,z}\Bigg(\left(2^{\frac{1}{2\,z}}+6\;3^{\frac{1}{2\,z}}\right)^{-z-1}\Bigg(\Big(96\,z$$

$$-\,63\Big)\,3^{\frac{z+1}{2\,z}}+52\,z\sqrt{3}\;2^{\frac{1}{2\,z}}$$

$+$

$\left(19584\left(z-\frac{237}{272}\right)z\,3^{\frac{1}{2z}}\,2^{\frac{1}{2z}}+\left(12096\,z^2-20736\,z+8991\right)3^{\frac{1}{z}}\right.$

$\left.\left.\left.+8112\,2^{\frac{1}{z}}\,z^2\right)^{1/2}\right)\right),z=z\Big\},\ \Big\{X=-\frac{1}{18\,z}\Big(\Big(\big(-96\,z+63\big)\,3^{\frac{z+1}{2z}}$

$-52\,z\sqrt{3}\,2^{\frac{1}{2z}}$

$+$

$\left(19584\left(z-\frac{237}{272}\right)z\,3^{\frac{1}{2z}}\,2^{\frac{1}{2z}}+\left(12096\,z^2-20736\,z+8991\right)3^{\frac{1}{z}}\right.$

$\left.+8112\,2^{\frac{1}{z}}\,z^2\right)^{1/2}\Big)\left(2^{\frac{1}{2z}}+6\,3^{\frac{1}{2z}}\right)^{-z-1}\Big),z=z\Big\}\Big]$

## We now look for a nicer form of the eigenvalues for z>0

> tau := $3^{\frac{1}{2\cdot z}}$ : nu := $3^{\frac{1+z}{2\cdot z}}$ : lambda := $2^{\frac{1}{2\cdot z}}$ :

> $d1 := 24(1-16\cdot z)\cdot\mathrm{tau}\cdot z\cdot\mathrm{lambda}+\left(1344\,z^2-384\,z+39\right)\cdot\tau^2+48\cdot\lambda^2\cdot z^2;$
$d2 := 72\,(272\cdot z-237)\cdot\mathrm{tau}\cdot z\,\mathrm{lambda}+\left(12096\,z^2-20736\,z+8991\right)\cdot\tau^2$
$+8112\,\lambda^2\,z^2;$

$b1 := (32\,z-11)\cdot\mathrm{nu}+4\sqrt{3}\,z\,2^{\frac{1}{2z}};\ b2 := (96\,z-63)\cdot\mathrm{nu}+52\sqrt{3}\,z\,\mathrm{lambda};$

$n := \frac{\left(2^{\frac{1}{2z}}+6\,3^{\frac{1}{2z}}\right)^{-1-z}}{6\cdot z};$

$$d1 := (24-384\,z)\,3^{\frac{1}{2z}}\,z\,2^{\frac{1}{2z}}+\left(1344\,z^2-384\,z+39\right)\left(3^{\frac{1}{2z}}\right)^2+48\left(2^{\frac{1}{2z}}\right)^2z^2$$

$$d2 := (19584\,z-17064)\,3^{\frac{1}{2z}}\,z\,2^{\frac{1}{2z}}+\left(12096\,z^2-20736\,z+8991\right)\left(3^{\frac{1}{2z}}\right)^2+8112\left(2^{\frac{1}{2z}}\right)^2z^2$$

$$b1 := (32\,z-11)\,3^{\frac{z+1}{2z}}+4\,z\sqrt{3}\,2^{\frac{1}{2z}}$$

$$b2 := (96\,z - 63)\,3^{\frac{z+1}{2\,z}} + 52\,z\sqrt{3}\,2^{\frac{1}{2\,z}}$$

$$n := \frac{\left(2^{\frac{1}{2\,z}} + 6\,3^{\frac{1}{2\,z}}\right)^{-z-1}}{6\,z} \tag{1.5.1}$$

> $eigs := \left[n\cdot(b1 + \mathrm{sqrt}(d1)),\ n\cdot(b1 - \mathrm{sqrt}(d1)),\ \frac{n}{3}\cdot(b2 + \mathrm{sqrt}(d2)),\ \frac{n}{3}\cdot(b2 - \mathrm{sqrt}(d2))\right];$
>
> $sols := [subs(solutions[1], X), subs(solutions[2], X), subs(solutions[3], X), subs(solutions[4], X)]:$
>
> $map(y \rightarrow simplify(y, symbolic), eigs - sols);$

$$eigs := \left[\frac{1}{6\,z}\left(\left(2^{\frac{1}{2\,z}} + 6\,3^{\frac{1}{2\,z}}\right)^{-z-1}\left((32\,z - 11)\,3^{\frac{z+1}{2\,z}} + 4\,z\sqrt{3}\,2^{\frac{1}{2\,z}} + \sqrt{(24 - 384\,z)\,3^{\frac{1}{2\,z}}\,z\,2^{\frac{1}{2\,z}} + (1344\,z^2 - 384\,z + 39)\left(3^{\frac{1}{2\,z}}\right)^2 + 48\left(2^{\frac{1}{2\,z}}\right)^2 z^2}\right)\right)\right.$$

$$, \frac{1}{6\,z}\left(\left(2^{\frac{1}{2\,z}} + 6\,3^{\frac{1}{2\,z}}\right)^{-z-1}\left((32\,z - 11)\,3^{\frac{z+1}{2\,z}} + 4\,z\sqrt{3}\,2^{\frac{1}{2\,z}} - \sqrt{(24 - 384\,z)\,3^{\frac{1}{2\,z}}\,z\,2^{\frac{1}{2\,z}} + (1344\,z^2 - 384\,z + 39)\left(3^{\frac{1}{2\,z}}\right)^2 + 48\left(2^{\frac{1}{2\,z}}\right)^2 z^2}\right)\right)$$

$$, \frac{1}{18\,z}\left(\left(2^{\frac{1}{2\,z}} + 6\,3^{\frac{1}{2\,z}}\right)^{-z-1}\left((96\,z - 63)\,3^{\frac{z+1}{2\,z}} + 52\,z\sqrt{3}\,2^{\frac{1}{2\,z}} + \left((19584\,z - 17064)\,3^{\frac{1}{2\,z}}\,z\,2^{\frac{1}{2\,z}} + (12096\,z^2 - 20736\,z + 8991)\left(3^{\frac{1}{2\,z}}\right)^2 + 8112\left(2^{\frac{1}{2\,z}}\right)^2 z^2\right)^{1/2}\right)\right), \frac{1}{18\,z}\left(\left(2^{\frac{1}{2\,z}} + 6\,3^{\frac{1}{2\,z}}\right)^{-z-1}\left((96\,z - 63)\,3^{\frac{z+1}{2\,z}}\right.\right.$$

$$+ 52 z \sqrt{3}\, 2^{\frac{1}{2z}}$$

$$- \left( \left(19584 z - 17064\right) 3^{\frac{1}{2z}} z\, 2^{\frac{1}{2z}} + \left(12096 z^2 - 20736 z + 8991\right) \left(3^{\frac{1}{2z}}\right)^2 + 8112 \left(2^{\frac{1}{2z}}\right)^2 z^2 \right)^{1/2} \Bigg) \Bigg) \Bigg]$$

$$[0, 0, 0, 0] \quad \textbf{(1.5.2)}$$

## We check that the eigenvalues are real on the positive plane: the minimal values of d1 and d2 are positive

> $dd1 := diff(d1, z) : dd2 := diff(d2, z) :$
> $min1 := solve(dd1) : min2 := solve(dd2) : evalf([min1, min2]);$

$$[0.3926777710, 0.5524430714] \quad \textbf{(1.6.1)}$$

> *# d1 is decreasing before in1, increasing afterwards; d2 is decreasing before min2, increasing afterwards*
> $\left[\left[evalf\left(subs\left(z = min1 - \frac{1}{10}, dd1\right)\right), evalf\left(subs\left(z = min1 + \frac{1}{10}, dd1\right)\right)\right], \left[evalf\left(subs\left(z = min2 - \frac{1}{10}, dd2\right)\right), evalf\left(subs\left(z = min2 + \frac{1}{10}, dd2\right)\right)\right]\right];$

$$[[-4224.401642, 1059.281801], [-107526.9425, 35945.63615]] \quad \textbf{(1.6.2)}$$

> *# Their minimal values are positive:*
> $evalf([subs(z = min1, d1), subs(z = min2, d2)]);$

$$[1121.901516, 184.766516] \quad \textbf{(1.6.3)}$$

## We end by checking that the eigenvalues are positive after 0.5171, hence the Hessian is definite positive and the extremal point is a local minimum

> *# we search for the minimal value of the derivative of the potentially smallest 2 eigenvalues*
> $sbd1 := b1^2 - d1 : sbd2 := b2^2 - d2 :$
> $bd1 := diff(sbd1, z) : bd2 := diff(sbd2, z) :$
> $bin1 := solve(bd1) : bin2 := solve(bd2) : evalf([bin1, bin2]);$

$$[0.3123021197, 0.3123021197] \quad \textbf{(1.7.1)}$$

> *# sdb1 is decreasing before bin1, increasing afterwards; sdb2 is decreasing before bin2, increasing afterwards*
> $\left[\left[evalf\left(subs\left(z = bin1 - \frac{1}{10}, bd1\right)\right), evalf\left(subs\left(z = bin1 + \frac{1}{10}, bd1\right)\right)\right], \left[evalf\left(subs\left(z = bin2 - \frac{1}{10}, bd2\right)\right), evalf\left(subs\left(z = bin2 + \frac{1}{10}, bd2\right)\right)\right]\right];$

$$[[-300500.8617, 6683.547019], [-2.704507802 \times 10^6, 60151.92294]] \quad \textbf{(1.7.2)}$$

> *# Their minimal values are not positive:*
> $evalf([subs(z = bin1, sbd1), subs(z = bin2, sbd2)]);$

$$[-1193.675072, -10743.07550] \tag{1.7.3}$$

> # *But sbd1 and sbd2 will be positive after their root:and b1>sqrt(d1) as well as b2>sqrt (d2)*

$evalf([[solve(b1^2 - d1)], [solve(b2^2 - d2)]]);$

$$[[0.5170849365], [0.5170849365]] \tag{1.7.4}$$

> # *b1 & b2 have no real roots and is positive and are thus positive after 0.5171*

$sin1 := solve(diff(b1, z)) : sin2 := solve(diff(b2, z)) : evalf([sin1, sin2]);$

$$[0.2724631658 + 0.3178238892\,I, 0.2696570864 + 0.4323870216\,I] \tag{1.7.5}$$

> # *b1 & b2 are thus increasing since their derivative is positive and are thus positive after 0.5171*

$evalf([subs(z=1, diff(b1, z)), subs(z=1, diff(b2, z))]);$

$$[67.79595805, 316.8478787] \tag{1.7.6}$$

> # *b1 & b2 are thus positive after 0.5171*

$evalf(\, subs(z=0.5171, [b1, b2]));$

$$[34.79848790, 24.10032976] \tag{1.7.7}$$

> # *Thereore b1+sqrt(d1) & b2+srqrt(d1) are positive as well as b1* $-\,\mathrm{sqrt}(d1)$ *& b2* $-\, srqrt(d1)$

# *since n is positive, all four eigenvalues are positive*

> $evalf(subs(z=0.5171 + rand(\,), eigs));$

$$[1.772443547 \times 10^{-742450478122}, 5.764470143 \times 10^{-742450478123}, 2.861834327 \times 10^{-742450478122}, 3.570157026 \times 10^{-742450478123}] \tag{1.7.8}$$

## The extremal point is a minimum, we search for the limit of the combined Holder norms at infinity

> $MinPoint := simplify(expand(simplify(subs(subminpoint, E)\,)\,)\,);$

$$MinPoint := \left(2^{-\frac{1}{2z}} + 6\, 3^{\frac{1}{2z}}\, 2^{-\frac{1}{z}}\right)^{-z} \tag{1.8.1}$$

> # *first Holder conjugates:* $\frac{1}{-\frac{1}{z}} + \frac{1}{-\frac{1}{z}} + \frac{1}{-\frac{1}{z}} + (1 + 3z) = 1$

$LowerBound1 := limit(MinPoint^3 \cdot 7^{1+3\cdot z}, z=\text{infinity});$

$$LowerBound1 := \frac{28\, 2^{11/14}\, 3^{5/7}}{9} \tag{1.8.2}$$

> # *second Holder conjugates:* $\frac{1}{-\frac{1}{z}} + \frac{1}{-\frac{1}{z}} + \frac{1}{\frac{1}{1+2z}} = 1$

$LowerBound2 := limit(MinPoint^2 \cdot subs(z=-(1 + 2\cdot z), MinPoint), z=\text{infinity});$

$$LowerBound2 := \frac{28\, 2^{11/14}\, 3^{5/7}}{9} \tag{1.8.3}$$

> $evalf([LowerBound1, LowerBound2]);$

$$[11.75546969, 11.75546969] \tag{1.8.4}$$