Homework 2 - INDENG 222 Spring 24.

Jupyter notebook only, individual homework

Your report must:

- have a clear structure,
- include the numerical simulations of your results in Pyhton with interpretation together with the code.
- Jupyter notebook file .ipynb
- **clarity of the code (comment) and interpretation of results (2pt)**

In this project, N is the number of time steps, M is the number of of path generated for the Brownian motion. For example, you could take $N = 100$ (or $N = 200$ to have a better accuracy if it runs not too slowly) and $M = 20,000$.

We refer to the Appendix to compute an expectation numerically.

The number of points is given for information and may be subjected to small changes.

1 Brownian motion 11pt

1. (2pt) By using the rescaled random walk, build a program which give one path of a Brownian motion on $[0, 1]$ with a time subdivision $t_i = \frac{i}{N}$, $0 \leq i \leq N$ and give an example of the value taken by W_1;
2. (3pt) Generate M path of a Brownian motion and prove numerically that $\mathbb{E}[W_t] = 0$ and $\text{Var}(W_t) = t$ for different values of t.
3. (2pt) Generate M path of a Brownian motion and prove numerically that $\text{Cov}(W_t, W_s) = \inf(t, s)$ for some value of s, t.
4. (4pt) Compute numerically $[W, W]_t$ and observe that $[W, W]_t = t$

   for $t = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$ and $t = 1$.

2 Geometric Brownian motion and Black-Scholes 10pt

In this section, we assume that the price at any time t of a risky asset is given by a geometric Brownian motion:

$$S_t = S_0 e^{\sigma W_t + (r - \frac{\sigma^2}{2} t)}.$$

1. (1pt) Compute the price of a Call option at time 0 by using the probabilistic formula and Monte Carlo simulations (see the appendix):

$$C_0 = \mathbb{E}[e^{-rT}(S_T - K)^+],$$

by generating $M = 10,000$ simulation of S_T for $T = 1$ with $\sigma = 0.1, S_0 = 100, r = 0.1, K = 100$.

2. (1pt) Create a function `CallBS` (resp. `PutBS`) with input r, σ, S_0, T, K which returns the Black-Scholes price of a European call (resp. put) option at time 0 with the analytic formula:

$$\texttt{CallBS}_0(r, \sigma, S_0, T, K) = S_0 N(d_+(T, S_0)) - Ke^{-rT} N(d_-(T, S_0)),$$

with

$$d_\pm(T, S_0) := \frac{\ln(S_0/K) + (r \pm \frac{\sigma^2}{2})T}{\sigma\sqrt{T}},$$

where N is the CDF of a standard normal distribution given by

$$N(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$

Compare this value with the probabilistic value found in Question 1.

3. (2pt) Fix the initial price $S_0 = 50, \sigma = 0.1, r = 0.1$ and $T = 1$. Observe that the price at time 0 of a call option is convex with respect to the strike (You may show a graph of $\texttt{CallBS}_0$ as a function of K).

4. (1pt) In the Black-Scholes model, the price of a put option is similarly given by

$$\texttt{PutBS}_0(r, \sigma, S_0, T, K) = Ke^{-rT} N(-d_-(T, S_0)) - S_0 N(-d_+(T, S_0)).$$

Check numerically the call-put parity relation in this model for initial price $S_0 = 50, \sigma = 0.1, r = 0.1, T = 1$ and $K = 55$.

5. (2pt) The Delta of a call option at time 0 denoted by Δ corresponds to the sensitivity (derivative) of the Call price at time 0 with respect to the underlying asset S, that is

$$\Delta = \frac{\partial \texttt{CallBS}(r, \sigma, S_0, T, K)}{\partial S_0} = N(d_+(T, S_0)).$$

Give the graph of the Delta of a Call option with strikes $K = 100$, $r = 0$, volatility $\sigma = 0.1$ and maturity $T = 1$ as a function of the underlying asset S_0 and interpret your results.

6. (3pt) The Vega of a call option at time 0 denoted by $\mathcal{V}$ corresponds to the sensitivity (derivative) of the Call price at time 0 with respect to the volatility σ, that is

$$\mathcal{V} = \frac{\partial \texttt{CallBS}(r, \sigma, S_0, T, K)}{\partial \sigma} = S_0 N'(d_+(T, S_0))\sqrt{T}.$$

   (a) Give the graph of the Velta of a Call option with strikes $K = 100$, $r = 0$, volatility $\sigma = 0.1$ and maturity $T = 1$ as a function of the underlying asset S_0 and propose an interpretation of your results.

   (b) Show numerically that $\texttt{CallBS}(r, \tilde{\sigma}, S_0, T, K) \approx \texttt{CallBS}(r, \sigma, S_0, T, K) + \mathcal{V}(\tilde{\sigma} - \sigma)$ for values $r, \sigma, S_0, T, K, \tilde{\sigma}$ of your own choice when $\tilde{\sigma} \approx \sigma$ ($\tilde{\sigma}$ and σ close enough but different).

A Monte Carlo method

Assume that we are able to generate M independent realizations of a random variable X. We denote by $X^{(j)}$ the jth realization of the random variable with $j = 1, \dots, M$. In Monte Carlo methods, we approach the expectation of X by the empirical mean, that is

$$\mathbb{E}[X] \approx \frac{1}{M} \sum_{j=1}^{M} X^{(j)}.$$