relation between latent and infectious period and generation time for compartmental models

For an SI^kR model, the time from becoming infectious to infection in a compartmental model with k stages of infection and a per stage rate q follows a distribution which has the following Laplace transform (see e.g. Svensson Math Biosci 2008)

In[1]:=
```
lapl[t_] := (1 - (1 + q t) ^ (-k)) / (q k t)
```

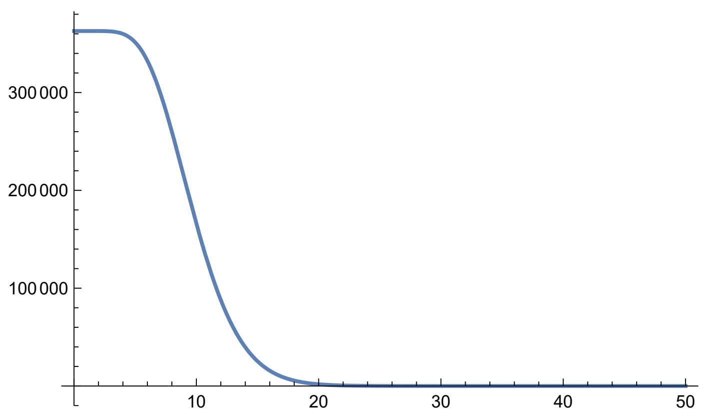
In[2]:=
```
InverseLaplaceTransform[lapl[t], t, x]
```

Out[2]=
$$\frac{\text{Gamma}\left[k, \frac{x}{q}\right]}{k\, q\, \text{Gamma}[k]}$$

example of the shape of this distribution

In[3]:=
```
Plot[Gamma[10, x], {x, 0, 50}]
```

Out[3]=



The moment generating function is the Laplace transform with the sign of the argument reversed

In[4]:=
```
momgenfun[s_] := lapl[-s]
```

In[5]:=
```
momgenfun[s]
```

Out[5]=
$$-\frac{1-(1-q\, s)^{-k}}{k\, q\, s}$$

The cumulant generating function is the logarithm of the moment generating function

In[6]:=
```
cumgenfun[s_] = Log[Simplify[lapl[-s]]]
```

Out[6]=
$$\mathrm{Log}\left[-\frac{1-(1-q\,s)^{-k}}{k\,q\,s}\right]$$

In[7]:=
```
cumgenfun2[s_] := Log[((1 - q s)^-k - 1)/(k q s)]
```

The mean and variance of the distribution are given by the first and second derivative of the cumulant generating function evaluated at zero

mean (confirming results of Svensson 2008)

In[8]:=
```
firstder[s_] := FullSimplify[D[cumgenfun2[s], {s, 1}]]
```

In[9]:=
```
Limit[firstder[s], s → 0]
```

Out[9]=
$$\frac{1}{2}(1+k)\,q$$

variance

In[10]:=
```
secondder[s_] := FullSimplify[D[cumgenfun2[s], {s, 2}]]
```

In[11]:=
```
Limit[secondder[s], s → 0]
```

Out[11]=
$$\frac{1}{12}\left(5+6\,k+k^2\right)q^2$$

For an SE^k1I^k2R model, the time from becoming infectious to infection in a compartmental model with k1 latent stages, k2 stages of infection, and a per stage rate q follows a distribution which has the following Laplace transform

In[12]:=
```
laplgentime[t_] := ((1 + q t)^(-k1)) (1 - (1 + q t)^(-k2)) / (q k2 t)
```

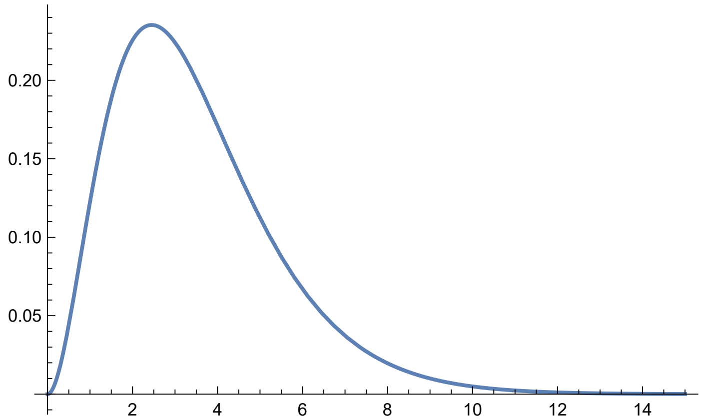
In[13]:=
```
InverseLaplaceTransform[laplgentime[t], t, x]
```

Out[13]=
$$\frac{-\frac{\mathrm{Gamma}\left[k1,\frac{x}{q}\right]}{\mathrm{Gamma}[k1]}+\frac{\mathrm{Gamma}\left[k1+k2,\frac{x}{q}\right]}{\mathrm{Gamma}[k1+k2]}}{k2\,q}$$

example of the shape of this distribution

In[14]:=
```
Plot[(Gamma[4, x] / Gamma[4] - Gamma[2, x] / Gamma[2]) / 2, {x, 0, 15}]
```

Out[14]=



The moment generating function is the Laplace transform with the sign of the argument reversed

In[15]:=
```
momgenfungentime[s_] := laplgentime[-s]
```

In[16]:=
```
momgenfungentime[s]
```

Out[16]=

$$-\frac{(1-q\,s)^{-k1}\left(1-(1-q\,s)^{-k2}\right)}{k2\,q\,s}$$

The cumulant generating function is the logarithm of the moment generating function

In[17]:=
```
cumgenfungentime[s_] = Log[Simplify[laplgentime[-s]]]
```

Out[17]=

$$\mathrm{Log}\left[-\frac{(1-q\,s)^{-k1-k2}\left(-1+(1-q\,s)^{k2}\right)}{k2\,q\,s}\right]$$

In[18]:=
```
Limit[cumgenfungentime[s], s → 0]
```

Out[18]=

0

The mean and variance of the distribution are given by the first and second derivative of the cumulant generating function evaluated at zero

mean

In[19]:=
```
firstdergentime[s_] := FullSimplify[D[cumgenfungentime[s], {s, 1}]]
```

In[20]:=
```
Limit[firstdergentime[s], s → 0]
```

Out[20]=

$$\frac{1}{2}(1+2\,k1+k2)\,q$$

example

In[21]:= $\frac{1}{2}$ (1 + 2 k1 + k2) q /. {k1 → 2, k2 → 2, q → 1}

Out[21]=

$$\frac{7}{2}$$

variance

In[22]:=
```
seconddergentime[s_] := FullSimplify[D[cumgenfungentime[s], {s, 2}]]
```

In[23]:=
```
Limit[seconddergentime[s], s → 0]
```

Out[23]=

$$\frac{1}{12}\left(5 + 12\, k1 + 6\, k2 + k2^2\right) q^2$$

example

In[24]:= $\frac{1}{12}$ (5 + 12 k1 + 6 k2 + k2^2) q /. {k1 → 2, k2 → 2, q → 1}

Out[24]=

$$\frac{15}{4}$$

For an SE^k1I^k2R model, the time from becoming infectious to infection in a compartmental model with k1 latent stages and a latent stage rate q1, k2 stages of infection and an infectious per stage rate q2 follows a distribution which has the following Laplace transform

In[25]:=
```
laplgentime[t_] := ((1 + q1 t) ^ (-k1)) (1 - (1 + q2 t) ^ (-k2)) / (q2 k2 t)
```

In[26]:=
```
InverseLaplaceTransform[laplgentime[t], t, x]
```

Out[26]=

$$\frac{1 - \frac{\text{Gamma}\left[k1, \frac{x}{q1}\right]}{\text{Gamma}[k1]} - \text{InverseLaplaceTransform}\left[\frac{(1+q1\,t)^{-k1}\,(1+q2\,t)^{-k2}}{t}, t, x\right]}{k2\, q2}$$

The moment generating function is the Laplace transform with the sign of the argument reversed

In[27]:=
```
momgenfungentime[s_] := laplgentime[-s]
```

In[28]:=
```
momgenfungentime[s]
```

Out[28]=

$$-\frac{(1 - q1\, s)^{-k1}\left(1 - (1 - q2\, s)^{-k2}\right)}{k2\, q2\, s}$$

The cumulant generating function is the logarithm of the moment generating function

In[29]:=
```
cumgenfungentime[s_] = Log[Simplify[laplgentime[-s]]]
```

Out[29]=

$$\mathrm{Log}\left[-\frac{(1-q1\ s)^{-k1}\left(1-(1-q2\ s)^{-k2}\right)}{k2\ q2\ s}\right]$$

In[30]:=
```
Limit[cumgenfungentime[s], s → 0]
```

Out[30]=

0

The mean and variance of the distribution are given by the first and second derivative of the cumulant generating function evaluated at zero

mean

In[31]:=
```
firstdergentime[s_] := FullSimplify[D[cumgenfungentime[s], {s, 1}]]
```

In[32]:=
```
Limit[firstdergentime[s], s → 0]
```

Out[32]=

$$\frac{1}{2}\ (2\ k1\ q1 + q2 + k2\ q2)$$

variance

In[33]:=
```
seconddergentime[s_] := FullSimplify[D[cumgenfungentime[s], {s, 2}]]
```

In[34]:=
```
Limit[seconddergentime[s], s → 0]
```

Out[34]=

$$k1\ q1^2 + \frac{1}{12}\ \left(5 + 6\ k2 + k2^2\right)\ q2^2$$

higher derivatives are not very informative

In[35]:=
```
thirddergentime[s_] := FullSimplify[D[cumgenfungentime[s], {s, 3}]]
```

In[36]:=
```
Limit[seconddergentime[s], s → 0]
```

Out[36]=

$$k1\ q1^2 + \frac{1}{12}\ \left(5 + 6\ k2 + k2^2\right)\ q2^2$$

In[37]:=
```
fourthdergentime[s_] := FullSimplify[D[cumgenfungentime[s], {s, 4}]]
```

In[38]:=
```
Limit[seconddergentime[s], s → 0]
```

Out[38]=

$$k1\ q1^2 + \frac{1}{12}\ \left(5 + 6\ k2 + k2^2\right)\ q2^2$$