2

Time, uncertainty, and liquidity

Financial economics deals with the allocation of resources over time and in the face of uncertainty. Although we use terms like "present values," "states of nature," and "contingent commodities" to analyze resource allocation in these settings, the basic ideas are identical to those used in the analysis of consumer and producer behavior in ordinary microeconomic theory. In this chapter we review familiar concepts such as preferences, budget constraints, and production technologies in a new setting, where we use them to study the intertemporal allocation of resources and the allocation of risk. We use simple examples to explain these ideas and later show how the ideas can be extended and generalized.

2.1 EFFICIENT ALLOCATION OVER TIME

We begin with the allocation of resources over time. Although we introduce some new terminology, the key concepts are the same as concepts familiar from the study of efficient allocation in a "timeless" environment. We assume that time is divided into two periods, which we can think of as representing the "present" and the "future." We call these periods **dates** and index them by $t = 0, 1$, where date 0 is the present and date 1 is the future.

2.1.1 Consumption and saving

Suppose a consumer has an **income stream** consisting of Y_0 units of a homogeneous consumption good at date 0 and Y_1 units of the consumption good at date 1. The consumer's utility $U(C_0, C_1)$ is a function of his **consumption stream** (C_0, C_1), where C_0 is consumption at date 0 and C_1 is consumption at date 1. The consumer wants to maximize his utility but first has to decide which consumption streams (C_0, C_1) belong to his **budget set**, that is, which streams are feasible for him. There are several ways of looking at this question. They all lead to the same answer, but it is worth considering each one in turn.

Borrowing and lending

One way of posing the question (of which consumption streams the consumer can afford) is to ask whether the income stream (Y_0, Y_1) can be transformed into a consumption stream (C_0, C_1) by borrowing and lending. For simplicity, we suppose there is a bank that is willing to lend any amount at the fixed interest rate $i > 0$ per period, that is, the bank will lend one unit of present consumption today in exchange for repayment of $(1 + i)$ units in the future. Suppose the consumer decided to spend $C_0 > Y_0$ today. Then he would have to borrow $B = C_0 - Y_0$ in order to balance his budget today, and this borrowing would have to be repaid with interest iB in the future. The consumer could afford to do this if and only if his future income exceeds his future consumption by the amount of the principal and interest, that is,

$$(1 + i)B \leq Y_1 - C_1.$$

We can rewrite this inequality in terms of the consumption and income streams as follows:

$$C_0 - Y_0 \leq \frac{1}{1+i}(Y_1 - C_1).$$

Conversely, if the consumer decided to consume $C_0 \leq Y_0$ in the present, he could save the difference $S = Y_0 - C_0$ and deposit it with the bank. We suppose that the bank is willing to pay the same interest rate $i > 0$ on deposits that it earns on loans, that is, one unit of present consumption deposited with the bank today will be worth $(1 + i)$ units in the future. The consumer will receive his savings with interest in the future, so his future consumption could exceed his income by $(1 + i)S$, that is,

$$C_1 - Y_1 \leq (1 + i)S.$$

We can rewrite this inequality in terms of the consumption and income streams as follows:

$$C_0 - Y_0 \leq \frac{1}{1+i}(Y_1 - C_1).$$

Notice that this is the same inequality as we derived before. Thus, any feasible consumption stream, whether it involves saving or borrowing, must satisfy the same constraint. We call this constraint the **intertemporal budget constraint** and write it for future reference in a slightly different form:

$$C_0 + \frac{1}{1+i}C_1 \leq Y_0 + \frac{1}{1+i}Y_1. \tag{2.1}$$

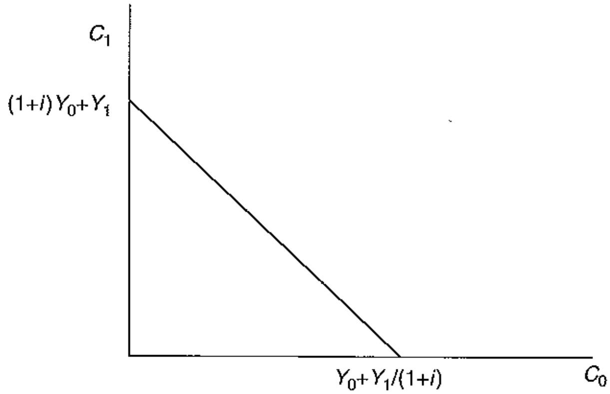

Figure 2.1. Intertemporal budget constraint.

Figure 2.1 illustrates the set of consumption streams (C_0, C_1) that satisfy the intertemporal budget constraint. It is easy to see that the income stream (Y_0, Y_1) must satisfy the intertemporal budget constraint. If there is neither borrowing nor lending in the first period then $C_0 = Y_0$ and $C_1 = Y_1$. The endpoints of the line represent the levels of consumption that would be possible if the individual were to consume as much as possible in the present and future, respectively. For example, if he wants to consume as much as possible in the present, he has Y_0 units of income today and he can borrow $B = Y_1/(1+i)$ units of the good against his future income. This is the maximum he can borrow because in the future he will have to repay the principal B plus the interest iB, for a total of $(1+i)B = Y_1$. So the maximum amount he can spend today is given by

$$C_0' = Y_0 + B = Y_0 + \frac{Y_1}{1+i}.$$

Conversely, if he wants to consume as much as possible in the future, he will save his entire income in the present. In the future, he will get his savings with interest $(1+i)Y_0$ plus his future income Y_1. So the maximum amount he can spend in the future is

$$C_1 = (1+i)Y_0 + Y_1.$$

Suppose now that consumption in the first period is increased by ΔC_0. By how much must future consumption be reduced? Every unit borrowed in the first period will cost $(1+i)$ in the second because interest must be paid. So the decrease in second period consumption is $\Delta C_1 = -(1+i)\Delta C_0$. This shows that he can afford any consumption stream on the line between the two endpoints with constant slope $= -(1+i)$. (See Figure 2.1.)

We have shown that any consumption stream that can be achieved by borrowing and lending must satisfy the intertemporal budget constraint. Conversely, we can show that any consumption stream (C_0, C_1) that satisfies the intertemporal budget constraint can be achieved by some feasible pattern of borrowing or lending (saving). To see this, suppose that the intertemporal budget constraint is satisfied by the consumption stream (C_0, C_1). If $C_0 > Y_0$ we assume the consumer borrows $B = C_0 - Y_0$. In the future he has to repay his loan with interest, so he only has $Y_1 - (1+i)B$ left to spend on consumption. However, the intertemporal budget constraint ensures that his planned future consumption C_1 satisfies

$$\begin{aligned} C_1 &\leq (1+i)(Y_0 - C_0) + Y_1 \\ &= Y_1 - (1+i)(C_0 - Y_0). \end{aligned}$$

So the consumer can borrow B units today and repay it with interest tomorrow and still afford his planned future consumption. The other case where $C_0 \leq Y_0$ is handled similarly. Thus, we have seen that the income stream (Y_0, Y_1) can be transformed into the consumption stream (C_0, C_1) through borrowing or lending at the interest rate i if and only if it satisfies the intertemporal budget constraint (2.1).

Wealth and present values

Another way of thinking about the set of affordable consumption streams makes use of the concept of present values. The present value of any good is the amount of present consumption that someone would give for it. The present value of one unit of present consumption is 1. The present value of one unit of future consumption is $1/(1+i)$, because one unit of present consumption can be converted into $1+i$ units of future consumption, and vice versa, through borrowing and lending. Thus, the present value of the income stream (Y_0, Y_1), that is, the value of (Y_0, Y_1) in terms of present consumption is

$$PV(Y_0, Y_1) \equiv Y_0 + \frac{1}{1+i} Y_1$$

and the present value of the consumption stream (C_0, C_1) is

$$PV(C_0, C_1) \equiv C_0 + \frac{1}{1+i} C_1.$$

The intertemporal budget constraint says that the present value of the consumption stream (C_0, C_1) must be less than or equal to the present value of the consumer's income stream.

The present value of the income stream (Y_0, Y_1) is also called the consumer's **wealth**, denoted by $W \equiv Y_0 + \frac{1}{1+i}Y_1$. The intertemporal budget constraint allows the consumer to choose any consumption stream (C_1, C_2) whose present value does not exceed his wealth, that is,

$$C_0 + \frac{C_1}{1+i} \leq W. \tag{2.2}$$

Dated commodities and forward markets

There is still a third way to interpret the intertemporal budget constraint (2.1). We are familiar with the budget constraint of a consumer who has to divide his income between two goods, beer and pizza, for example. There is a price at which each good can be purchased and the value of consumption is calculated by multiplying the quantity of each good by the price and adding the two expenditures. The consumer's budget constraint says that the value of his consumption must be less than or equal to his income. The intertemporal budget constraint (2.1) can be interpreted in this way too. Suppose we treat present consumption and future consumption as two different commodities and assume that there are markets on which the two commodities can be traded. We assume these markets are perfectly competitive, so the consumer can buy and sell as much as he likes of both commodities at the prevailing prices. The usual budget constraint requires the consumer to balance the value of his purchases and expenditures on the two commodities. If p_0 is the price of present consumption and p_1 is the price of future consumption, then the ordinary budget constraint can be written as

$$p_0 C_0 + p_1 C_1 \leq p_0 Y_0 + p_1 Y_1.$$

Suppose that we want to use the first-period consumption good as our **numeraire**, that is, measure the value of every commodity in terms of this good. Then the price of present consumption is $p_0 = 1$, since one unit of the good at date 0 is worth exactly one (unit of the good at date 0). How much is the good at date 1 worth in terms of the good at date 1? If it is possible to borrow and lend at the interest rate i, the price of the good at date 1 will be determined by **arbitrage**. If $p_1 > \frac{1}{1+i}$, then anyone can make a riskless arbitrage profit by selling one unit of future consumption for p_1, using the proceeds to buy $\frac{1}{1+i}$ units of present consumption, and investing the $\frac{1}{1+i}$ units at the interest rate i

to yield $(1+i)\frac{1}{1+i} = 1$ unit of future consumption. This strategy yields a profit of $p_1 - \frac{1}{1+i}$ at date 0 and has no cost since the unit of future consumption that is sold is provided by the investment at date 0. Such a risk free profit is incompatible with equilibrium, since anyone can use this arbitrage to generate unlimited wealth. Thus, in equilibrium we must have $p_1 \leq \frac{1}{1+i}$.

A similar argument can be used to show that if $p_1 < \frac{1}{1+i}$, it is possible to make a risk free profit by borrowing $\frac{1}{1+i}$ units of present consumption, buying one unit of future consumption at the price p_1, and using it to repay the loan at date 1. Thus, in equilibrium, we must have $p_1 \geq \frac{1}{1+i}$.

Putting these two arbitrage arguments together, we can see that, if it is possible to borrow and lend at the interest rate i and present consumption is the numeraire, the only prices consistent with equilibrium are $p_0 = 1$ and $p_1 = \frac{1}{1+i}$.

Substituting the prices $p_0 = 1$ and $p_1 = \frac{1}{1+i}$ into the budget constraint above, we see that it is exactly equivalent to the intertemporal budget constraint (2.1). Borrowing and lending at a constant interest rate is equivalent to having a market in which present and future consumption can be exchanged at the constant prices (p_0, p_1). The same good delivered at two different dates is two different commodities and present and future consumption are, in fact, simply two different commodities with two distinct prices. From this point of view, the intertemporal budget constraint is just a new interpretation of the familiar consumer's budget constraint.

Consumption and saving

Since the consumer's choices among consumption streams (C_0, C_1) are completely characterized by the intertemporal budget constraint, the consumer's decision problem is to maximize his utility $U(C_0, C_1)$ by choosing a consumption stream (C_0, C_1) that satisfies the budget constraint. We represent this decision problem schematically by

$$\begin{aligned} \max \quad & U(C_0, C_1) \\ \text{s.t.} \quad & C_0 + \frac{C_1}{1+i} = W. \end{aligned}$$

Note that here we assume the budget constraint is satisfied as an equality rather than an inequality. Since more consumption is preferred to less, there is no loss of generality in assuming that the consumer will always spend as much as he can on consumption. The solution to this maximization problem is illustrated in Figure 2.2, where the optimum occurs at the point on the budget constraint where the indifference curve is tangent to the budget constraint.

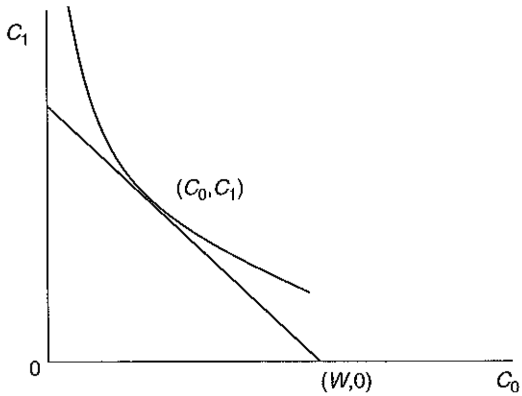

Figure 2.2. Consumption and saving.

The slope of the budget constraint is $-(1+i)$ and the slope of the indifference curve at the optimum point is

$$-\frac{\frac{\partial U}{\partial C_0}(C_0^*, C_1^*)}{\frac{\partial U}{\partial C_1}(C_0^*, C_1^*)},$$

so the tangency condition can be written as

$$\frac{\frac{\partial U}{\partial C_0}(C_0^*, C_1^*)}{\frac{\partial U}{\partial C_1}(C_0^*, C_1^*)} = (1+i).$$

It is easy to interpret the first-order condition by rewriting it as

$$\frac{\partial U}{\partial C_0}(C_0^*, C_1^*) = (1+i)\frac{\partial U}{\partial C_1}(C_0^*, C_1^*).$$

The left hand side is the marginal utility of consumption at date 0. The right hand side is the marginal utility of $(1+i)$ units of consumption at date 1. One unit of the good at date 0 can be saved to provide $1+i$ units of the good at date 1. So the first-order condition says that the consumer is indifferent between consuming one unit at date 0 and saving it until date 1 when it will be worth $(1+i)$ units and then consuming the $(1+i)$ units at date 1.

An alternative to the graphical method of finding the optimum is to use the method of Lagrange, which requires us to form the Lagrangean function

$$\mathcal{L}(C_0, C_1, \lambda) = U(C_0, C_1) - \lambda\left(C_0 + \frac{1}{1+i}C_1 - W\right)$$

and maximize the value of this function with respect to C_0, C_1, and the Lagrange multiplier λ. A necessary condition for a maximum at $(C_0^*, C_1^*, \lambda^*)$ is that the derivatives of $\mathcal{L}(C_0, C_1, \lambda)$ with respect to these variables should all be zero. Then

$$\frac{\partial \mathcal{L}}{\partial C_0}(C_0^*, C_1^*, \lambda^*) = \frac{\partial U}{\partial C_0}\left(C_0^*, C_1^*\right) - \lambda^* = 0,$$
$$\frac{\partial \mathcal{L}}{\partial C_1}(C_0^*, C_1^*, \lambda^*) = \frac{\partial U}{\partial C_0}\left(C_0^*, C_1^*\right) - \frac{\lambda^*}{1+i} = 0,$$
$$\frac{\partial \mathcal{L}}{\partial \lambda}(C_0^*, C_1^*, \lambda^*) = C_0^* + \frac{1}{1+i}C_1^* - W = 0.$$

The first two conditions are equivalent to the tangency condition derived earlier. To see this, eliminate λ^* from these equations to get

$$\frac{\partial U}{\partial C_0}\left(C_0^*, C_1^*\right) = \lambda^* = (1+i)\frac{\partial U}{\partial C_0}\left(C_0^*, C_1^*\right).$$

The last of the three conditions simply asserts that the budget constraint must be satisfied.

As before, the optimum (C_0^*, C_1^*) is determined by the tangency condition and the budget constraint.

Clearly, the optimal consumption stream (C_0^*, C_1^*) will be a function of the consumer's wealth W and the rate of interest i. If the pattern of income were $(W, 0)$ instead of (Y_1, Y_2) the value of wealth would be the same and hence the budget line would be the same. So the same point (C_1^*, C_2^*) would be chosen. In fact (Y_1, Y_2) could move to any other point on the budget line without affecting consumption. Only savings or borrowing would change.

On the other hand, an increase in W to W', say, will shift the budget line out and increase consumption. The case illustrated in Figure 2.3 has the special property that the marginal rate of substitution is constant along a straight line OA through the origin. The slope of the budget line does not change so in this case the point of tangency moves along the line OA as W changes. In this special case, C_1 is proportional to W.

Problems

1. An individual consumer has an income stream (Y_0, Y_1) and can borrow and lend at the interest rate i. For each of the following data points, determine whether the consumption stream (C_0, C_1) lies within the consumer's budget

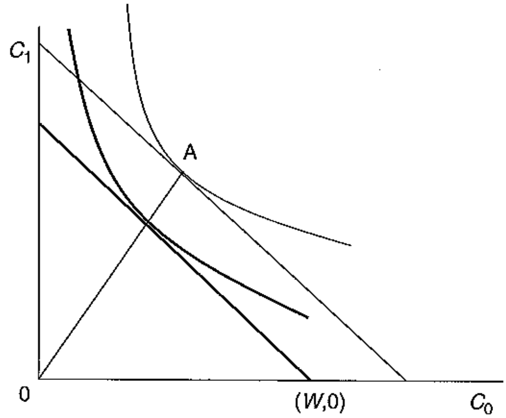

Figure 2.3. The effect of an increase in wealth.

set (i.e. whether it satisfies the intertemporal budget constraint).

(C_0, C_1)	(Y_0, Y_1)	$(1+i)$
(10, 25)	(15, 15)	2
(18, 11)	(15, 15)	1.1
(18, 11)	(15, 15)	1.5
(10, 25)	(15, 15)	1.8

Draw a graph to illustrate your answer in each case.

2. An individual consumer has an income stream $(Y_0, Y_1) = (100, 50)$ and can borrow and lend at the interest rate $i = 0.1\dot{1}$. His preferences are represented by the additively separable utility function

$$U(C_0, C_1) = \log C_0 + 0.9 \log C_1.$$

The marginal utility of consumption in period t is

$$\frac{d \log C_t}{dC_t} = \frac{1}{C_t}.$$

Write down the consumer's intertemporal budget constraint and the first-order condition that must be satisfied by the optimal consumption stream. Use the first-order condition and the consumer's intertemporal budget constraint to find the consumption stream (C_0^*, C_1^*) that maximizes utility. How much will the consumer save in the first period? How much will his savings be worth in the second period? Check that he can afford the optimal consumption C_1^* in the second period.

2.1.2 Production

Just as we can cast the consumer's intertemporal decision into the familiar framework of maximizing utility subject to a budget constraint, we can cast the firm's intertemporal decision into the form of a profit- or value-maximization problem.

Imagine a firm that can produce outputs of a homogeneous good in either period subject to a production technology with decreasing returns. Output at date 0 is denoted by Y_0 and output at date 1 is denoted by Y_1. The possible combinations of Y_0 and Y_1 are described by the production possibility curve illustrated in Figure 2.4.

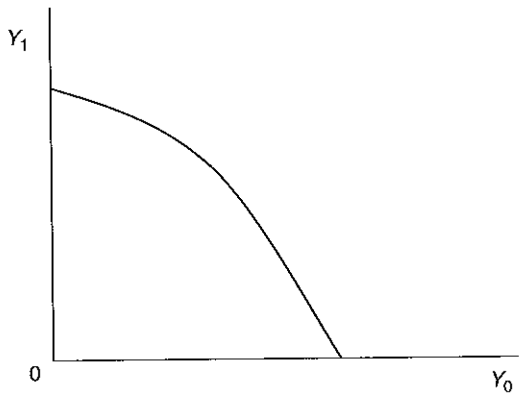

Figure 2.4. The production possibility curve.

Note the following properties of the production possibility curve:

- the curve is downward sloping to the right because the firm must reduce output tomorrow in order to increase the output today;
- the curve is convex upward because of the diminishing returns – as the firm decreases output today, each additional unit of present output foregone adds less to output tomorrow.

The production technology can be represented by a transformation function $F(Y_0, Y_1)$. A pair of outputs (Y_0, Y_1) is feasible if and only if it satisfies the inequality

$$F(Y_0, Y_1) \leq 0.$$

The function F is said to be **increasing** if an increase in Y_0 or Y_1 increases the value $F(Y_0, Y_1)$. The function F is said to be **convex** if, for any output streams

(Y_0, Y_1) and (Y_0', Y_1') and any number $0 < t < 1$,

$$F\left(t\left(Y_0, Y_1\right) + (1-t)\left(Y_0', Y_1'\right)\right) \leq tF\left(Y_0, Y_1\right) + (1-t)\,F\left(Y_0', Y_1'\right).$$

If F is increasing, then the production possibility curve is downward sloping. If the function F is convex, the production possibility curve is convex upward. In other words, if (Y_0, Y_1) and (Y_0', Y_1') are feasible, then any point on the line segment between them is feasible.

To illustrate the meaning of the transformation curve, suppose that the firm's past investments produce an output of $\bar{Y}_0$ at the beginning of period 0. The firm can re-invest K_0 units and sell the remaining $Y_0 = \bar{Y}_0 - K_0$. The investment of K_0 units today produces $Y_1 = G(K_0)$ in the future (assume there is no investment in the future because the firm is being wound up). Then the firm can produce any combination of present and future goods (Y_0, Y_1) for sale that satisfies $Y_0 \leq \bar{Y}_0$ and $Y_1 = G(\bar{Y}_0 - Y_0)$. Then the transformation curve $F(Y_0, Y_1)$ can be defined by

$$F(Y_0, Y_1) = Y_1 - G(\bar{Y}_0 - Y_0).$$

Which combination of outputs Y_0 and Y_1 should the firm choose? In general, there may be many factors that will guide the firm's decision, but under certain circumstances the firm can ignore all these factors and consider only the market value of the firm. To see this, we need only recall our discussion of the consumer's decision. Suppose that the firm is owned by a single shareholder who receives the firm's outputs as income. If the consumer can borrow and lend as much as he wants at the rate i, all he cares about is his wealth, the present value of the income stream (Y_0, Y_1). The exact time-profile of income (Y_0, Y_1) does not matter. So if the firm wants to maximize its shareholder's welfare, it should maximize the shareholder's wealth. To make these ideas more precise, suppose that the firm has a single owner-manager who chooses both the firm's production plan (Y_0, Y_1) and his consumption stream (C_0, C_1) to maximize his utility subject to his intertemporal budget constraint. Formally, we can write this decision problem as follows:

$$\begin{aligned} \max \quad & U(C_0, C_1) \\ \text{s.t.} \quad & F(Y_0, Y_1) \leq 0 \\ & C_0 + \tfrac{1}{1+i} C_1 \leq W \equiv Y_0 + \frac{1}{1+i} Y_1. \end{aligned}$$

Then it is clear that the choice of (Y_0, Y_1) affects utility only through the intertemporal budget constraint and that anything that increases the present value of the firm's output stream will allow the consumer to reach a more desirable consumption stream. Thus, the joint consumption and production decision above is equivalent to the following two-stage procedure. First, have the firm maximize the present value of outputs:

$$\begin{aligned} \max \quad & W \equiv Y_0 + \frac{1}{1+i} Y_1 \\ \text{s.t.} \quad & F(Y_0, Y_1) \leq 0. \end{aligned}$$

The present value of outputs is also known as the **market value of the firm,** so this operational rule can be rephrased to say that firms should always maximize market value. Then have the consumer maximize his utility taking the firm's market value as given:

$$\begin{aligned} \max \quad & U(C_0, C_1) \\ \text{s.t.} \quad & C_0 + \frac{1}{1+i} C_1 \leq W, \end{aligned}$$

where $W = Y_0 + \frac{1}{1+i} Y_1$. Figure 2.5 illustrates this principle for the case of a single shareholder.

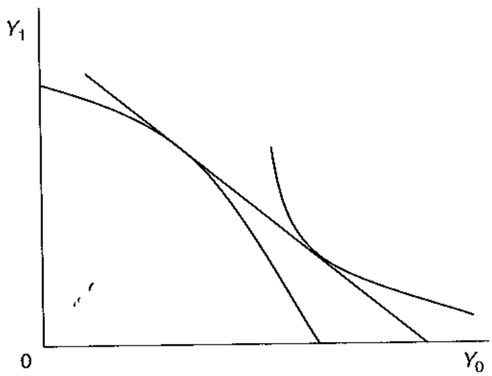

Figure 2.5. Value maximization and utility maximization.

In fact, this argument extends to the case where there are many shareholders with different time preferences. Some shareholders may be impatient and want to consume more in the present while others are more patient and are willing to postpone consumption, but all will agree that a change in production that increases the present value of output must be a good thing, because it increases the consumers' wealth. Figure 2.6 illustrates the case of two shareholders with equal shares in the firm. Then they have identical budget constraints with

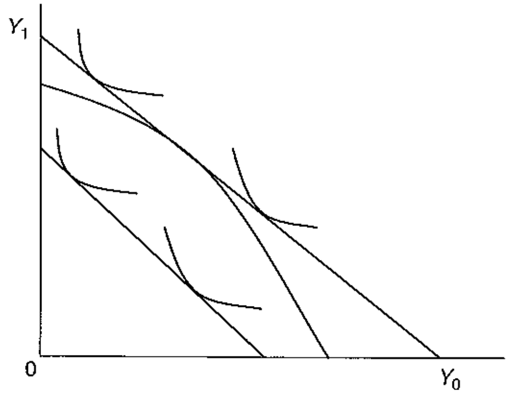

Figure 2.6. The separation theorem.

slope $-(1 + i)$, i.e. their budget constraint is parallel to the maximum value line that is tangent to the production possibility frontier. Each shareholder will choose to consume the bundle of goods (C_0, C_1) that maximizes his utility subject to the budget constraint. Because they have different time preferences, represented here by different indifference curves, each shareholder will choose a different bundle of goods, as indicated by the different points of tangency between indifference curves and budget constraint. Nonetheless, both agree that the firm should maximize its market value, because maximizing the value of the firm has the effect of putting both shareholders on the highest possible budget constraint. This is known as the **separation theorem** because the firm's decision to maximize its value is separate from shareholders' decisions to maximize their utility.

Problem

3. A firm has 100 units of output at the beginning of period 0. It has three projects that it can finance. Each project requires an input of I units of the good at date 0 and produces Y_1 units of the good at date 1. The projects are defined in the following table:

Project	Investment I	Output Y_1
1	20	30
2	30	48
3	50	70

Which projects should the firm undertake when the interest factor is

$$1 + i = 2, 1.5, 1.1?$$

Trace the firm's production possibility curve (the combinations of Y_0 and Y_1 that are technologically feasible) assuming that the firm can undertake a fraction of a project. Use this diagram to illustrate how changes in the discount factor change the firm's decision.

2.2 UNCERTAINTY

In the same way that we can extend the traditional analysis of consumption and production to study the allocation of resources over time, we can use the same ideas to study the allocation of risk bearing under uncertainty. Once again we shall use a simple example to illustrate the general principles.

2.2.1 Contingent commodities and risk sharing

We assume that time is divided into two periods or dates indexed by $t = 0, 1$. At date 0 (the present) there is some uncertainty about the future. For example, an individual may be uncertain about his future income. We can represent this uncertainty by saying that there are several possible **states of nature.** A state of nature is a complete description of all the uncertain, exogenous factors that may be relevant for the outcome of an individual's decision. For example, a farmer who is planting a crop may be uncertain about the weather. The size of his crop will depend on choices he makes about the time to plant, the use of fertilizers, etc., as well as the weather. In this case, we identify the weather with the state of nature. Each state would be a complete description of the weather – the amount of rainfall, the temperature, etc. – during the growing season. The outcome of the farmer's choices will depend on the parameters he determines – time to plant, etc. – and the state of nature. In other words, once we know the farmer's choices and the state of nature, we should know the size of the crop; but even after the farmer has determined all the parameters that he controls, the fact that the state is still unknown means that the size of the crop is uncertain.

In what follows we assume that the only uncertainty relates to an individual's income, so income is a function of the state of nature. The true state of nature is unknown at date 0 and will be revealed at date 1. For simplicity, suppose that there are two possible states indexed by $s = H, L$. The letters H and L stand for "high" and "low."

Commodities are distinguished by their physical characteristics, by the date at which they are delivered, and by the state in which they are delivered. Thus, the consumption good in state H is a different commodity from the consumption good in state L. We call a good whose delivery is contingent