Jordan Garrett

Homework 1

1\. a) $E[\eta] = E[K^{-1}\underline{\varepsilon}] = K^{-1}E[\underline{\varepsilon}] = K^{-1}\cdot 0 = 0$

Assumptions

$$\boxed{\begin{gathered} E[\varepsilon] = 0 \\ Var[\underline{\varepsilon}] = \sigma^2 V \end{gathered}}$$

$Var[\eta] = Var[K^{-1}\underline{\varepsilon}]$

$= K^{-1}\,Var[\varepsilon](K^{-1})^T$

$= K^{-1}\sigma^2 V(K^{-1})^T$

$= K^{-1}\sigma^2 KK^T(K^{-1})^T$

$= I\sigma^2 I$

$Var[\eta] = \sigma^2 I$

b) Model : $Z = D\beta + \eta$

i) $D = K^{-1}X \qquad V = KK^T \qquad Z = K^{-1}Y$

$\beta^* = (D^TD)^{-1}D^TZ$

$= [(K^{-1}X)^T(K^{-1}X)]^{-1}(K^{-1}X)^TK^{-1}Y$

$= [(K^{-1})^TX^TK^{-1}X]^{-1}(K^{-1})^TX^TK^{-1}Y$

$\beta^* = (X^TV^{-1}X)^{-1}X^TV^{-1}Y$

ii) $E[\beta^*] = E[(X^TV^{-1}X)^{-1}X^TV^{-1}Y]$

$= (X^TV^{-1}X)^{-1}X^TV^{-1}E[Y]$

$= \cancel{(X^T)^{-1}}\cancel{V}X^{-1}\cancel{X^T}\cancel{V^{-1}}E[Y]$

$= X^{-1}E[Y]$

$= \cancel{X^{-1}X}\beta$

$E[\beta^*] = \beta$

iii) $\text{Var}(\beta^*) = \text{Var}\left[(X^T V^{-1} X)^{-1} X^T V^{-1} Y\right]$

Let $A = (X^T V^{-1} X)^{-1} X^T V^{-1}$ and And $A^T = X V^{-1}(X^T V^{-1} X)^{-1}$

symmetric

then $\text{Var}(\beta^*) = \text{Var}[\,A Y\,]$

$$= A \,\text{Var}[Y]\, A^T \qquad * \text{Var}[Y] = \sigma^2 V$$

$$= (X^T V^{-1} X)^{-1} X^T V^{-1} \sigma^2 V \, X V^{-1} (X^T V^{-1} X)^{-1}$$

$$= \sigma^2 V (X^T V^{-1} X)^{-1} X^T V^{-1} X V^{-1} (X^T V^{-1} X)^{-1}$$

$$= \sigma^2 V (X^T V^{-1} X)^{-1} \cancel{X^T} V^{-1} \cancel{X} \cancel{V^{-1}} (X^T)^{-1} \cancel{V} X^{-1}$$

$$= \sigma^2 \cancel{V} (X^T V^{-1} X)^{-1} \cancel{V^{-1}}$$

$$= \sigma^2 (X^T V^{-1} X)^{-1}$$

iv) $RSS = (Z - D\beta^*)^T (Z - D\beta^*) \qquad V = KK^T$

$$= (K^{-1} Y - K^{-1} X \beta^*)^T (K^{-1} Y - K^{-1} X \beta^*)$$

$$= (Y - X\beta^*)^T (K^{-1})^T K^{-1} (Y - X\beta^*)$$

$$= (Y - X\beta^*)^T V^{-1} (Y - X\beta^*)$$

4. $Y \overset{indep}{\sim} N(\mu_i, \sigma^2) \quad i=1,\dots n \qquad N \sim \text{p.d.f} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$\mu_i = x_i^T \beta$

a) $P(Y_i = y_i \mid \mu_i, \sigma^2) = f_{Y_i}(y_i \mid x_i^T\beta, \sigma^2)$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y_i - x_i^T\beta}{\sigma}\right)^2}$$

$$= \frac{e^{-(y_i - x_i^T\beta)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}$$

$$L(\beta, \sigma^2 \mid y_i, x_i^T) = \prod_{i=1}^{n} f_{Y_i}(y_i \mid x_i\, \beta, \sigma^2)$$

$$= \prod_{i=1}^{n} \frac{e^{-(y_i - x_i^T\beta)^2/2\sigma^2}}{\sigma^2\sqrt{2\pi}}$$

$$L(\beta, \sigma^2 \mid y_i, x_i^T) = (2\pi\sigma^2)^{-n/2} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - x_i^T\beta)^2\right]$$

$$\ell(\beta, \sigma^2) = \ln\left[(2\pi\sigma^2)^{-n/2} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - x_i^T\beta)^2\right]\right]$$

$$\ell(\beta, \sigma^2) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - x_i^T\beta)^2$$

b) $\frac{\partial}{\partial\beta}\ell(\beta, \sigma^2) = 0$

$$\frac{\partial}{\partial\beta}\left(-\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - x_i^T\beta)^2\right)$$

$$0 = \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i(y_i - x_i^T\beta) \Rightarrow \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i x_i^T\beta$$

$$\sum_{i=1}^{n} x_i y_i = \beta \sum_{i=1}^{n} x_i x_i^T$$

$$\hat{\beta}_{MLE} = \left(\sum_{i=1}^{n} x_i x_i^T\right)^{-1} \sum_{i=1}^{n} x_i y_i \quad \text{or} \quad (X^T X)^{-1} X^T Y$$

c) $\hat{\beta}_{1_{MLE}} \pm 1.96 \cdot \sigma \sqrt{(X^T X)^{-1}}$

d) This confidence interval is exact.

e) $\hat{\beta}_{1\,MLE} \pm t_{n-p,\,0.975} \cdot \hat{\sigma} \sqrt{(X^T X)^{-1}}$

This confidence interval would be wider since we are estimating another parameter. If the true population distribution is normal, then the confidence interval is exact.

7. $$P(Y=y) = \begin{cases} \dfrac{p^y}{y(-\ln(1-p))} & \text{if } y \in 1, 2, \ldots \text{ where } 0 < p < 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $\mu = E(Y)$

$$= \exp\left[\ln\left(\frac{p^y}{y(-\ln(1-p))}\right)\right]$$

$$= \exp\left[y\ln(p) - \ln(y_i) - \ln(-\ln(1-p))\right]$$

$$= \exp\left[y\ln(p) - \ln(-\ln(1-p)) - \ln(y_i)\right]$$

$\theta_i = \ln(p)$

$p = e^{\theta_i}$

$b(\theta_i) = \ln(-\ln(1-p))$

$= \ln(-\ln(1-e^{\theta_i}))$

$a(\phi) = 1$

$\phi = 1$

$a(\phi) = \phi$

$c(y_i, \phi) = -\ln(y_i)$