1a) $f_{Y|\lambda} \sim \text{Poisson}(\lambda)$ $\lambda \sim \text{Gamma}(\mu, \tau)$ Jordan Garrett

$$f(y|\lambda) = \frac{\lambda^{y} \exp(-\lambda)}{y!}$$

$$f_\lambda(\lambda;\mu,\tau) = \frac{1}{\Gamma(\tau)}\left(\frac{\tau}{\mu}\right)^{\tau} \exp\left(-\frac{\tau\lambda}{\mu}\right)\lambda^{\tau-1}$$

$$\frac{\lambda^{y}\exp(-\lambda)}{y!}\left[\frac{1}{\Gamma(\tau)}\left(\frac{\tau}{\mu}\right)^{\tau}\exp\left(-\frac{\tau\lambda}{\mu}\right)\lambda^{\tau-1}\right]$$

$$= \int_0^\infty \frac{\lambda^{y}\exp(-\lambda)}{y!\,\Gamma(\tau)}\left(\frac{\tau}{\mu}\right)^{\tau}\exp\left(-\frac{\tau\lambda}{\mu}\right)\lambda^{\tau-1}\,d\lambda$$

$$= \int_0^\infty \frac{\lambda^{y+\tau-1}}{y!\,\Gamma(\tau)}\left(\frac{\tau}{\mu}\right)^{\tau}\exp-\left(\frac{\tau}{\mu}+1\right)\lambda \;\; d\lambda$$

$$= \int_0^\infty \frac{(\tau/\mu)^{\tau}}{y!\,\Gamma(\tau)}\lambda^{y+\tau-1}\exp-\left(\frac{\tau}{\mu}+1\right)\lambda\, d\lambda$$

$$= \frac{(\tau/\mu)^{\tau}}{y!\,\Gamma(\tau)}\int_0^\infty \lambda^{y+\tau-1}\exp-\left(\frac{\tau}{\mu}+1\right)\lambda\, d\lambda$$

$$= \frac{(\tau/\mu)^{\tau}}{y!\,\Gamma(\tau)}\frac{\Gamma(\tau+y)}{\left(\frac{\tau}{\mu}+1\right)^{y+\tau}}\int_0^\infty \frac{\left(\frac{\tau}{\mu}+1\right)^{y+\tau}}{\Gamma(\tau+y)}\lambda^{y+\tau-1}\exp-\left(\frac{\tau}{\mu}+1\right)\lambda\;\;d\lambda$$

$$= \frac{(\tau/\mu)^{\tau}}{y!\,\Gamma(\tau)}\frac{\Gamma(\tau+y)}{\left(\frac{\tau}{\mu}+1\right)^{y+\tau}}$$

$$= \frac{(\tau/\mu)^{\tau}}{\left(\frac{\tau}{\mu}+1\right)^{y+\tau}}\frac{\Gamma(\tau+y)}{y!\,\Gamma(\tau)} = \frac{\tau^{\tau}}{\mu^{\tau}}\cdot\frac{1}{\left(\frac{\tau}{\mu}+1\right)^{y+\tau}}\;\frac{\Gamma(\tau+y)}{y!\,\Gamma(\tau)}$$

$$= \frac{\tau^\tau}{\mu^\tau} \frac{\mu^{y+\tau}}{(\tau+\mu)^{y+\tau}} \frac{\Gamma(\tau+y)}{y!\,\Gamma(\tau)}$$

$$= \frac{\tau^\tau}{\cancel{\mu^\tau}} \frac{\mu^y + \cancel{\mu^\tau}}{(\tau+\mu)^{y+\tau}} \frac{\Gamma(\tau+y)}{y!\,\Gamma(\tau)}$$

$$= \frac{\tau^\tau \mu^y}{(\tau+\mu)^{y+\tau}} \frac{\Gamma(\tau+y)}{y!\,\Gamma(\tau)}$$

$$\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)$$

1 b)

$$= \exp\left\{ \ln\left[\frac{\tau^\tau \mu^y}{(\tau+\mu)^{y+\tau}} \frac{\Gamma(\tau+y)}{y!\,\Gamma(\tau)} \right] \right\}$$

$$= \exp\left\{ \ln\left[\frac{\tau^\tau \mu^y}{(\tau+\mu)^{y+\tau}} \right] + \ln\left[\frac{\Gamma(\tau+y)}{y!\,\Gamma(\tau)} \right] \right\}$$

$$= \exp\left\{ \tau\ln(\tau) + y\ln(\mu) - (y+\tau)\ln(\tau+\mu) + \ln\left[\frac{\Gamma(\tau+y)}{y!\,\Gamma(\tau)} \right] \right\}$$

$$= \exp\left\{ \tau\ln(\tau) + y\ln(\mu) - y\ln(\tau+\mu) - \tau(\ln(\tau+\mu)) + \ln\left[\frac{\Gamma(\tau+y)}{y!\,\Gamma(\tau)} \right] \right\}$$

$$= \exp\left\{ \tau\ln(\tau) + y\ln\left(\frac{\mu}{\tau+\mu}\right) - \tau(\ln(\tau+\mu)) + \ln\left[\frac{\Gamma(\tau+y)}{y!\,\Gamma(\tau)} \right] \right\}$$

i)

$$= \exp\left\{ y\ln\left(\frac{\mu}{\tau+\mu}\right) + \tau\left(\ln\left(\frac{\tau}{\tau+\mu}\right)\right) + \ln\left[\frac{\Gamma(\tau+y)}{y!\,\Gamma(\tau)} \right] \right\}$$

ii)

$\theta = \ln\left(\frac{\mu}{\tau+\mu}\right)$

$\mu = \frac{\tau e^\theta}{1-e^\theta}$

$\phi = 1$

$a(\phi) = 1$

$c = \ln\left[\frac{\Gamma(\tau+y)}{y!\,\Gamma(\tau)} \right]$

$$b(\theta) = -\tau\left(\ln\left(\frac{\tau}{\tau+\mu}\right)\right)$$

$$= -\tau\left[\ln(\tau) - \ln\left(\tau + \frac{\tau e^\theta}{1-e^\theta}\right)\right]$$

$$= -\tau\left[\ln(\tau) - \ln\left(\frac{\tau - \tau e^\theta + \tau e^\theta}{1-e^\theta}\right)\right]$$

$$= -\tau\left[\ln(\tau) - \ln\left(\frac{\tau}{1-e^\theta}\right)\right]$$

$$= -\tau\left[\ln(1-e^\theta)\right]$$

$$b'(\theta) = -\tau \frac{1}{1-e^\theta} - e^\theta$$

$$= \frac{\tau e^\theta}{1-e^\theta}$$

$$E(Y) = \mu$$

$$b''(\theta) = \frac{(1-e^\theta)\tau e^\theta - (-e^\theta)\tau e^\theta}{(1-e^\theta)^2}$$

$$= \frac{\tau e^\theta - \tau(e^\theta)^2 + \tau(e^\theta)^2}{(1-e^\theta)^2}$$

$$= \frac{\tau e^\theta}{(1-e^\theta)^2} = \frac{\tau e^\theta}{1-e^\theta} \frac{1}{1-e^\theta}$$

$$= \mu \frac{1}{1-e^\theta}$$

$$= \mu \left(1 + \frac{e^\theta}{1-e^\theta}\right)$$

$$= \mu \left(1 + \frac{1}{\mu\left(\frac{1-e^\theta}{e^\theta}\right)} \mu\right)$$

$$= \mu \left(1 + \frac{\mu}{\tau}\right)$$

$$V(\theta) = \mu + \frac{\mu^2}{\tau}$$

$$\phi = 1$$

$$Var(Y) = \mu + \frac{\mu^2}{\tau}$$

2. $D_M \triangleq 2 \sum_{i=1}^{n} w_i [y_i(\tilde{\theta}_i - \hat{\theta}_i) - b(\tilde{\theta}_i) + b(\hat{\theta}_i)]$

Saturated $\tilde{\mu}_i = y_i \quad a_i(\phi) = \frac{1}{w_i}$

- Binomial Proportion $\quad$ * Let $y_i = \frac{Z_i}{m_i}$ * $Z_i \overset{indep}{\sim} \text{Binomial}(m_i, \pi_i)$

$w_i = m_i \quad \hat{\theta}_i = \text{logit}(\hat{\pi}_i) \quad b(\hat{\theta}_i) = -\ln(1-\pi_i) \quad \phi = 1 \quad \tilde{\theta}_i = \text{logit}(y_i)$

$$l_M = \sum_{i=1}^{n} \{ m_i [y_i \ln(\frac{\hat{\pi}_i}{1-\hat{\pi}_i}) + \ln(1-\hat{\pi}_i)] \}$$

$$l_{saturated} = \sum_{i=1}^{n} \{ m_i [y_i (\ln(\frac{y_i}{1-y_i})) + \ln(1-y_i)] \}$$

$$D_M \triangleq 2 \sum_{i=1}^{n} m_i [y_i(\ln(\frac{y_i}{1-y_i}) - \ln(\frac{\hat{\pi}_i}{1-\hat{\pi}_i})) + \ln(1-y_i) - \ln(1-\hat{\pi}_i)]$$

$$= 2 \sum_{i=1}^{n} m_i [y_i(\ln(\frac{y_i}{1-y_i}) - \ln(\frac{\hat{\pi}_i}{1-\hat{\pi}_i})) + \ln(\frac{1-y_i}{1-\hat{\pi}_i})]$$

$$= 2 \sum_{i=1}^{n} m_i [y_i(\ln(y_i) - \ln(1-y_i) - \ln(\hat{\pi}_i) - \ln(1-\hat{\pi}_i)) + \ln(\frac{1-y_i}{1-\hat{\pi}_i})]$$

$$= 2 \sum_{i=1}^{n} m_i [y_i(\ln(\frac{y_i}{\hat{\pi}_i}) - \ln(\frac{1-y_i}{1-\hat{\pi}_i})) + \ln(\frac{1-y_i}{1-\hat{\pi}_i})]$$

$$= 2 \sum_{i=1}^{n} m_i [y_i \ln(\frac{y_i}{\hat{\pi}_i}) - y_i \ln(\frac{1-y_i}{1-\hat{\pi}_i}) + \ln(\frac{1-y_i}{1-\hat{\pi}_i})]$$

$$= 2 \sum_{i=1}^{n} m_i [y_i \ln(\frac{y_i}{\hat{\pi}_i}) + (-y_i + 1) \ln(\frac{1-y_i}{1-\hat{\pi}_i})]$$

$$= 2 \sum_{i=1}^{n} m_i [y_i \ln(\frac{y_i}{\hat{\pi}_i}) + (1-y_i) \ln(\frac{1-y_i}{1-\hat{\pi}_i})]$$

$$= 2 \sum_{i=1}^{n} [m_i y_i \ln(\frac{y_i}{\hat{\pi}_i}) + (m_i - m_i y_i) \ln(\frac{1-y_i}{1-\hat{\pi}_i})]$$

+ Poisson

$\hat{\theta}_i = \ln(\mu_i) \quad \tilde{\theta}_i = \ln(y_i) \quad b(\hat{\theta}_i) = \hat{\mu}_i \quad b(\tilde{\theta}) = y_i$

$a_i(\phi) = 1 = w_i$

$\phi = 1$

$$D_M \triangleq 2 \sum_{i=1}^{n} \left[y_i \left(\ln(y_i) - \ln(\hat{\mu}_i) \right) - y_i + \hat{\mu}_i \right]$$

$$= 2 \sum_{i=1}^{n} \left[y_i \left(\ln\left(\frac{y_i}{\hat{\mu}_i} \right) \right) - (y_i - \hat{\mu}_i) \right]$$

+ Gaussian

$\hat{\theta}_i = \hat{\mu}_i \quad b(\hat{\theta}_i) = \frac{\hat{\mu}_i^2}{2} \quad \tilde{\theta}_i = y_i \quad b(\tilde{\theta}_i) = \frac{y_i^2}{2}$

$a_i(\phi) = \frac{\phi}{w_i} = \sigma^2$

$\phi = \sigma^2$

$w_i = 1$

$$D_M \triangleq 2 \sum_{i=1}^{n} \left[y_i (y_i - \hat{\mu}_i) - \frac{y_i^2}{2} + \frac{\hat{\mu}_i^2}{2} \right]$$

$$= 2 \sum_{i=1}^{n} \left[y_i (y_i - \hat{\mu}_i) - \frac{y_i^2 + \hat{\mu}_i^2}{2} \right]$$

$$= 2 \sum_{i=1}^{n} \left[y_i^2 - y_i \hat{\mu}_i - \frac{y_i^2 + \hat{\mu}_i^2}{2} \right]$$

$$= 2 \sum_{i=1}^{n} \left[\frac{2y_i^2 - 2y_i \hat{\mu}_i - y_i^2 + \hat{\mu}_i^2}{2} \right]$$

$$= 2 \sum_{i=1}^{n} \left[\frac{y_i^2 - 2y_i \hat{\mu}_i + \hat{\mu}_i^2}{2} \right]$$

$$= 2 \sum_{i=1}^{n} \left[\frac{(y_i - \hat{\mu}_i)^2}{2} \right] = \sum_{i=1}^{n} (y_i - \hat{\mu}_i)^2$$

3. $(X_1 \ldots X_k) \sim$ Multinomial

n = total # trials

k = # of categories

$0 \le p_i < 1$ for $i = 1, \ldots k$ and $\sum_{i=1}^{k} p_i = 1$ ← lagrange constraint

a) $X_1, \ldots, X_k$ are not independent their respective probabilities $(p_1 \ldots p_k)$ sum to 1 (i.e. $\sum_{i=1}^{k} p_i = 1$)

b) $\mathcal{L}(p_i \mid x_i) = n! \prod_{i=1}^{k} \left(\frac{p_i^{x_i}}{x_i!} \right)$

$$\ell(\mathcal{L}(p_i \mid x_i)) = \ln \left\{ n! \prod_{i=1}^{k} \left(\frac{p_i^{x_i}}{x_i!} \right) \right\}$$

$$= \ln(n!) + \sum_{i=1}^{k} \ln \left(\frac{p_i^{x_i}}{x_i!} \right)$$

$$= \ln(n!) + \sum_{i=1}^{k} x_i \ln(p_i) - \sum_{i=1}^{k} \ln(x_i!)$$

$$\ell'(p, \lambda) = \ell(p) + \lambda\left(1 - \sum_{i=1}^{k} p_i\right)$$

$$\frac{\partial \ell'(p, \lambda)}{\partial p_i} = \frac{\partial \ell(p)}{\partial p_i} + \frac{\partial \lambda(1 - \sum_{i=1}^{k} p_i)}{\partial p_i}$$

$$= \frac{x_i}{p_i} - \frac{\partial \lambda - \lambda \sum_{i=1}^{k} p_i}{\partial p_i}$$

$$= \frac{x_i}{p_i} - \lambda \Rightarrow p_i = \frac{x_i}{\lambda}$$

* since $\sum_{i=1}^{k} p_i = \sum_{i=1}^{k} \frac{x_i}{\lambda}$,

then: $1 = \sum_{i=1}^{k} \frac{x_i}{\lambda} \Rightarrow 1 = \frac{1}{\lambda} \sum_{i=1}^{k} x_i \Rightarrow \lambda = n$

$$\hat{p}_i = \frac{x_i}{n}$$

For any given p_i

$$\hat{p}_{i_{MLE}} = \frac{x_i}{n}$$

Thus

$$p_{MLE} = \left(\frac{x_1}{n} \cdots \frac{x_k}{n}\right)$$

c) If $k = 2$ then

$$f_{x_1,x_2}(x_1, x_2 \mid p_1, p_2) = \frac{n!}{x_1!\, x_2!} p_1^{x_1} p_2^{x_2}$$

$$= \frac{n!}{x_1!\, x_2!} p_1^{x_1} (1 - p_1)^{x_2}$$

* x_1 = success
x_2 = failures
n = # trials
p_1 = prob of succes
p_2 = prob of falure *

$$= \frac{n!}{x_1!\,(n-x_1)!} p_1^{x}(1-p_1)^{n-x_1}$$

4a) $M_Y(t) = E[Y_j] = \sum_{j=1}^{k} e^{ty_j} P(Y = y)$

$$= \sum_{j=1}^{k} e^{ty_j} \frac{\lambda^{y_j} \exp(-\lambda)}{y_j!}$$

$$= \sum_{j=1}^{k} \frac{(e^t \lambda)^{y_j} \exp(-\lambda)}{y_j!}$$

$$= \exp(-\lambda) \sum_{j=1}^{k} \frac{(e^t \lambda)^{y_j}}{y_j!}$$

$$= \exp(-\lambda)(e^{\lambda e^t}) \sum_{j=1}^{k} \left(\frac{e^t \lambda)^{y_j}}{y_j!} e^{-\lambda e^t}$$

poisson pmf

$$= \exp(-\lambda(e^{\lambda e^t})$$

$$= e^{-\lambda + \lambda e^t}$$

$$= e^{\lambda(e^t - 1)}$$

$$\frac{\partial}{\partial t} = e^{\lambda(e^t-1)} \cdot \lambda \cdot e^t$$

$$\Big|_{t=0} = \lambda_j = E(Y_j)] = \mu_j$$

$$E\left[\sum_{j=1}^{k} Y_j\right] = \sum_{j=1}^{k} [E(Y_j)]$$

$$= \sum_{j=1}^{k} \mu_j$$

4b) $f(y_1, \ldots y_k) = \frac{e^{-\mu_1} \mu^{y_1}}{y_1!} \cdot \frac{e^{-\mu_2} \mu^{y_2}}{y_2!} \cdots \frac{e^{-\mu_k} \mu^{y_k}}{y_k!}$

$$= \frac{e^{\sum_{j=1}^{k} -\mu_j}}{y_1! \ldots y_k!} \mu^{y_1} \cdot \mu^{y_2} \ldots \mu^{y_k}$$

$$P\left(Y_1 = y_1, \ldots Y_k = y_k \,\middle|\, \sum_{i=1}^{k} Y_j = m\right) = \frac{P\left(\sum_{i=1}^{k} Y_j = m \,\middle|\, Y_1 = y_1, \ldots Y_k = y_k\right)}{P\left(\sum_{i=1}^{k} Y_j\right)}$$

$$P\left(\sum_{j=1}^{k} Y_j = m \mid Y_1 = y_1 \ldots Y_k = y_k\right) = \sum_{j=1}^{k} \frac{e^{-\sum_{j=1}^{k}\mu_j} \mu_1^{y_1} \cdots \mu_k^{y_k}}{y_1! \ldots y_k!}$$

$$P\left(\sum_{j=1}^{k} Y_j = m\right) = \frac{e^{\sum_{j=1}^{k}\mu_j} \left(\sum_{j=1}^{k} \mu_j\right)^m}{m!}$$

$$\frac{P\left(\sum_{j=1}^{k} Y_j = m \mid Y_1 = y_1 \ldots Y_k = y_k\right)}{P\left(\sum_{j=1}^{k} Y_j = m\right)} = \sum_{j=1}^{k} \frac{\cancel{e^{-\sum_{j=1}^{k}\mu_j}} \mu_1^{y_1} \cdots \mu_k^{y_k}}{y_1! \ldots y_k!} \frac{m!}{e^{-\sum_{j=1}^{k}\mu_j} \left(\sum_{j=1}^{k} \mu_j\right)^m}$$

$$= \frac{m! \; \mu_1^{y_1} \cdots \mu_k^{y_k}}{y_1! \ldots y_k! \sum_{j=1}^{k} \mu_j}$$

$$= \frac{m!}{y_1! \ldots y_k!} \cdot \frac{\mu_1^{y_1}}{\left(\sum_{j=1}^{k} \mu_j\right)^m} \cdots \frac{\mu_k^{y_k}}{\sum_{j=1}^{k} \mu_j}$$

Let $\pi_i = \dfrac{\mu_i}{\sum_{j=1}^{k} \mu_j}$

then

$$= \frac{m!}{y_1! \ldots y_k!} \pi_1^{y_1} \ldots \pi_k^{y_k}$$

5a

Random

$$Y_i \overset{indep}{\sim} Poisson(\mu_i) \quad i \in \{1, \dots 49\}$$

Y_i = the number of counted bicyclists at i^{th} location on Saturday

$$E(Y_i) = \mu_i$$

Systematic

$$\eta_i = \beta_0 + \beta_1 \text{ off-peak}_i + \beta_2 \text{ pm-peak}_i$$

Link

$$g(\mu_i) = \ln(\mu_i) = \eta_i$$

$$\mu_i = e^{\beta_0 + \beta_1 \text{ off-peak}_i + \beta_2 \text{ pm-peak}_i}$$

off-peak$_i$ = number of counted bicyclists at i^{th} location between 10 am - 12 noon

pm-peak$_i$ = number of counted bicyclists at i^{th} location between 5 - 7 pm

5b)

$$3.0874 + 0.0115(12) + 0.00144(14) = 3.25$$

$$e^{3.25} = 25.67$$

≈ 26 bicyclists are expected to appear at the second location on a Saturday

5c) i. $H_0 : \beta_2 = 0$

$H_1 : \beta_2 \neq 0$

If H_0 is true, $\dfrac{\hat{\beta}_2 - 0}{\widehat{se}(\hat{\beta}_2)} \overset{approx}{\sim} N(0,1)$

Reject H_0 when $\left| \dfrac{\hat{\beta}_{2\,observed} - 0}{\widehat{se}(\hat{\beta}_{2\,observed})} \right| > z_{\alpha/2} = 1.96$

$$\frac{0.00144}{0.0006363} = 2.27 > 1.96$$

* Reject null that the reduced model is the true model

ii. $z_{\alpha/2} = z_{0.01/2} = z_{0.005} = 2.58$

$2.27 < 2.58$

* Retain null that the reduced mode is the true model

5d) i $\hat{\phi} = \dfrac{\text{Residual Deviance}}{\text{Residual Df}}$

$= \dfrac{462.47}{46}$

$\hat{\phi} = 10$

There is evidence for over-dispersion since $\hat{\phi} = 10$, but the amount of over-dispersion is not large enough (eg. $\hat{\phi} > 30$) to suggest that the Poisson distribution is a poor assumption of the model's random compenent.

ii I would like to compare fit 2 with another fit in which we instead use a negative binomial, to see how much over dispersion there is. Further, I would include location as a covariate to determine if bicyclist activity is dependent on location.