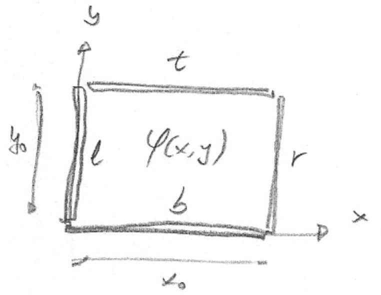
HW # 2

2.1. 'Laplace' Equation in two directions - Analytical.



$\varphi(x,y)$, $x_0, y_0, V_l, V_r, V_t, V_b$

$$\varphi = \varphi(x,y) \Rightarrow \nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$$

$\varphi(x,y) = X(x)\,Y(y) \Rightarrow$ Separation of Variables.

For $\nabla^2\varphi = 0 \Rightarrow$ $\dfrac{\partial^2\varphi}{\partial x^2} = \dfrac{d^2X}{dx^2}\,Y$ and $\dfrac{\partial^2\varphi}{\partial y^2} = \dfrac{d^2Y}{dy^2}\,X$

$$\frac{d^2X}{dx^2}\,Y + \frac{d^2Y}{dy^2}\,X = 0 = \frac{1}{X}\frac{d^2X}{dx} + \frac{1}{Y}\frac{d^2Y}{dy^2} \Rightarrow$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = -m^2 \Rightarrow m \text{ is separation constant}$$

$X(x) = A\cos mx + B\sin mx$ or $X(x) = Ae^{mx} + Be^{-mx}$

$Y(y) = Ce^{mx} + De^{-my}$ or $Y(y) = C\cosh mx + D\sinh my$

or any combination of "Boundary Value Problems" Notes:

I. $\varphi(x,y) = (A\cosh mx + B\sinh mx)(C\cos my + D\sin my)$

II. $\varphi(x,y) = (A\cos mx + B\sin mx)(C\cosh my + D\sinh my)$

III. $\varphi(x,y) = (Ae^{mx} + Be^{-mx})(C\sin my + D\cos my)$

IV. $\varphi(x,y) = (A\cos mx + B\sin mx)(Ce^{my} + De^{-my})$

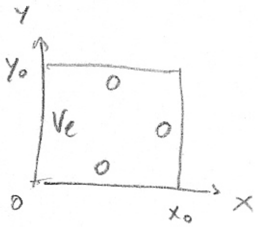
1. $\varphi_l(x,y) \Rightarrow V_t, V_r, V_b = 0$, $V_l = V$

2. $\varphi_b(x,y) \Rightarrow V_l, V_r, V_t = 0$, $V_b = V$

3. $\varphi_t(x,y) \Rightarrow V_l, V_r, V_b = 0$, $V_t = V$

4. $\varphi_r(x,y) \Rightarrow V_l, V_b, V_t = 0$, $V_r = V$

1. $\varphi_\ell(x,y) \Rightarrow V_b, V_t, V_r = 0 \quad V_\ell = V$



Boundary Conditions

1. $\varphi_\ell(0,y) = V_e$
2. $\varphi(x,0) = 0$
3. $\varphi(x,y_0) = 0$
4. $\varphi(x_0,y) = 0$

$$\varphi(x,y) = \left(Ae^{mx} + Be^{-mx}\right)\left(C\sin my + D\cos my\right)$$

2. $\varphi(x,0) \Rightarrow \varphi(x,y) = 0 \Rightarrow D\cos my = 0 \Rightarrow D = 0$

$$\varphi(x,y) = \left(Ae^{mx} + Be^{-mx}\right) C\sin my.$$

3. $\varphi(x,y_0) \Rightarrow C\sin my = 0 \Rightarrow m = \frac{n\pi}{y_0}$

$$\varphi(x,y) = \sum_{n=1}^{\infty}\left(Ae^{\frac{n\pi}{y_0}x} + Be^{-\frac{n\pi x}{y_0}}\right) C\sin\frac{n\pi y}{y_0}$$

4. $\varphi(x_0,y) = 0$

$$X(x_0) = Ae^{\frac{n\pi x_0}{y_0}} + Be^{-\frac{n\pi x_0}{y_0}} \Rightarrow A = -B\frac{e^{-\frac{n\pi x_0}{y_0}}}{e^{\frac{n\pi x_0}{y_0}}} \Rightarrow m = \frac{n\pi}{y_0}$$

$$X(x) = -B\frac{e^{-mx_0}}{e^{mx_0}}e^{mx} + Be^{-mx} = B\left(\frac{e^{-mx}e^{mx_0} - e^{-mx_0}e^{mx}}{e^{mx_0}}\right) =$$

$$= -2B\frac{\sinh(mx - mx_0)}{e^{mx_0}} = -2B\frac{\sinh\left(\frac{n\pi x}{y_0} - \frac{n\pi x_0}{y_0}\right)}{e^{\frac{n\pi x_0}{y_0}}}$$

$$B_n = -2BC\frac{1}{e^{\frac{n\pi x_0}{y_0}}} \Rightarrow \varphi(x,y) = \sum_{n=1}^{\infty} B_n\sinh\left(\frac{n\pi x}{y_0} - \frac{n\pi x_0}{y_0}\right)\sin\frac{n\pi y}{y_0}$$

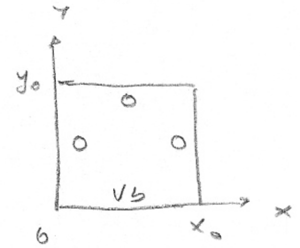
1. $\varphi(0,y) = V_e$

$$\varphi(0,y) = \sum_{n=1}^{\infty} -B_n\sinh\left(\frac{n\pi x_0}{y_0}\right)\sin\frac{n\pi y}{y_0} = V_e \Rightarrow \text{Fourier trick.}$$

$$-B_n\sinh\left(\frac{n\pi x_0}{y_0}\right) = \frac{2V_e}{y_0}\int_0^{y_0}\sin\frac{n\pi y}{y_0}\,dy = \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{2y_0}{n\pi} & \text{for } n \text{ odd.}\end{cases}$$

$$B_n = -\frac{4V_e}{n\pi}\frac{1}{\sinh\left(\frac{n\pi x_0}{y_0}\right)} \Rightarrow \varphi_\ell(x,y) = \sum_{n\ \mathrm{odd}} -\frac{4V_e}{n\pi}\frac{1}{\sinh\frac{n\pi x_0}{y_0}}\sin\frac{n\pi y}{y_0}\sinh\left(\frac{n\pi}{y_0}(x - x_0)\right)$$

2. $\varphi_b(x,y) \Rightarrow V_e, V_r, V_t = 0\ , \ V_b = 0$



1. $\varphi(0,y) = 0$
2. $\varphi(x,0) = V_b$
3. $\varphi(x,y_0) = 0$
4. $\varphi(x_0,y) = 0$

$$\varphi(x,y) = X(x)\ Y(y)$$

$$\varphi(x,y) = (A\cos mx + B\sin mx)(C\cosh my + D\sinh my)$$

1. $\varphi(0,y) = 0 \Rightarrow A\cos mx = 0 \Rightarrow A = 0$

$$\varphi(x,y) = B\sin mx\,(C\cosh my + D\sinh my)$$

4. $\varphi(x_0,y) = 0$

$$B\sin mx_0 = 0 \qquad \sin mx_0 = 0 \qquad m = \frac{n\pi}{x_0}$$

$$\varphi(x,y) = \sum_{n=1}^{\infty} B\sin\frac{n\pi x}{x_0}\left(C\cosh\frac{n\pi y}{x_0} + D\sinh\frac{n\pi y}{x_0}\right)$$

3. $\varphi(x,y_0) = 0$

$$C\cosh\frac{n\pi y_0}{x_0} + D\sinh\frac{n\pi y_0}{x_0} = 0 \Rightarrow C = -D\frac{\sinh\frac{n\pi y_0}{x_0}}{\cosh\frac{n\pi y_0}{x_0}}$$

$$Y(y) = C\cosh my + D\sinh my = -D\frac{\sinh my_0}{\cos my_0}\cosh my + D\sinh my =$$

$$= D\frac{\sinh my\cosh my_0 - \sinh my_0\cos my}{\cos my_0} = D\frac{\sinh(my - my_0)}{\cos my_0} \Rightarrow$$

$$\varphi(x,y) = \sum_{n=1}^{\infty} B\sin\frac{n\pi x}{x_0}\,D\,\frac{\sinh\left(\frac{n\pi y}{x_0} - \frac{n\pi y_0}{x_0}\right)}{\cosh\frac{n\pi y_0}{x_0}} \Rightarrow BD\frac{1}{\cosh\frac{n\pi y_0}{x_0}} = B_n$$

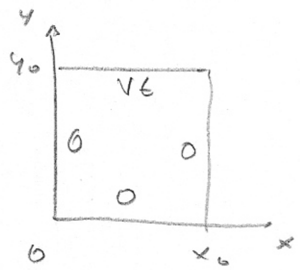
2. $\varphi(x,0) = V_b$

$$V_b = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{x_0}\sinh\left(\frac{n\pi}{x_0}(y-y_0)\right) = \sum_{n=1}^{\infty} -B_n\frac{\sin n\pi x}{x_0}\sinh\frac{n\pi y_0}{x_0}$$

Fourier trick $\Rightarrow$ $-B_n\sinh\frac{n\pi y_0}{x_0} = \frac{2}{x_0}\int_0^{x_0} V_b\sin\frac{n\pi x}{x_0}\,dx = \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{2x_0}{n\pi} & \text{for } n \text{ odd} \end{cases}$

$$B_n = -\frac{4V_b}{n\pi}\frac{1}{\sinh\frac{n\pi y_0}{x_0}} \Rightarrow \varphi_b(x,y) = \sum_{n\,odd} -\frac{4V_b}{n\pi}\frac{1}{\sinh\frac{n\pi y_0}{x_0}}\frac{\sin n\pi x}{x_0}\sinh\left(\frac{n\pi}{x_0}(y-y_0)\right)$$

3. $\varphi_t(x,y) \Rightarrow V_l, V_r, V_b = 0 \quad V_t = V$



1. $\varphi(0,y) = 0$
2. $\varphi(x,0) = 0$
3. $\varphi(x,y_0) = V_b$
4. $\varphi(x_0,y) = 0$

$$\varphi(x,y) = (A\cos mx + B\sin mx)(C\cosh my + D\sinh my)$$

1. $\varphi(0,y) = X(0)\,Y(y) \Rightarrow X(0) = 0 \Rightarrow A\cos mx = 0 \Rightarrow A = 0$

$$\varphi(x,y) = B\sin mx\,(C\cosh my + D\sinh my).$$

2. $\varphi(x,0) = X(x)\,Y(0) \Rightarrow Y(0) = 0 \Rightarrow \quad C\cosh my = 0 \Rightarrow C = 0$

$$\varphi(x,y) = B\sin mx \;\; D\sinh my$$

4. $\varphi(x_0,y) \Rightarrow \quad B\sin mx = 0 \Rightarrow m = \frac{n\pi}{x_0}$

$$\varphi(x,y) = \sum_{n=1}^{\infty} B\sin\frac{n\pi x}{x_0}\; D\sinh\frac{n\pi y}{x_0} \Rightarrow B_n = BD.$$

$$\varphi(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{x_0}\,\sinh\frac{n\pi y}{x_0}$$

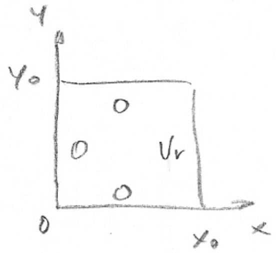
3. $\varphi(x,y_0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{x_0}\,\sinh\frac{n\pi y_0}{x_0} = V_b \quad \Rightarrow$ Fourier trick.

$$\sinh\frac{n\pi y_0}{x_0}\,B_n = \frac{2V_t}{x_0}\int_0^{x_0}\sin\frac{n\pi x}{x_0}\,dx = \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{4V_t}{n\pi} & \text{for } n \text{ odd} \end{cases}$$

$$B_n = \frac{4V_t}{n\pi}\,\frac{1}{\sinh\frac{n\pi y_0}{x_0}}$$

$$\varphi_t(x,y) = \sum_{n\ \text{odd}}^{\infty} \frac{4V_t}{n\pi}\,\frac{1}{\sinh\frac{n\pi y_0}{x_0}}\,\sin\frac{n\pi x}{x_0}\,\sinh\frac{n\pi y}{x_0}$$

4. $\varphi_r(x,y) \Rightarrow V_\ell, V_t, V_b = 0, V_r = V$



1. $\varphi(0,y) = 0$
2. $\varphi(x,0) = 0$
3. $\varphi(x, y_0) = 0$
4. $\varphi(x_0, y) = V_r$

$\varphi(x,y) = X(x)\, Y(x)$

$$\varphi(x,y) = (Ae^{mx} + Be^{-mx})(C \sin my + D \cos my)$$

1. $\varphi(0,y) = X(0)\, Y(y) \Rightarrow X(0) \quad A + B = 0 \Rightarrow A = -B$

$$\varphi(x,y) = A(e^{mx} - e^{-mx})(C \sin my + D \cos my)$$

2. $\varphi(x,0) = X(x)\, Y(0) \Rightarrow D \cos my = 0 \Rightarrow D = 0$

$$\varphi(x,y) = A(e^{mx} - e^{-mx})\, C \sin my$$

3. $\varphi(x, y_0) = X(x)\, Y(y_0) \Rightarrow C \sin my = 0 \Rightarrow m = \frac{n\pi}{y_0}$

$$\varphi(x,y) = \sum_{n=1}^{\infty} A\left(e^{\frac{n\pi x}{y_0}} - e^{-\frac{n\pi x}{y_0}}\right) C \sin \frac{n\pi y}{y_0} \Rightarrow$$

and $e^{\frac{n\pi x}{y_0}} - e^{-\frac{n\pi x}{y_0}} = 2 \sinh \frac{n\pi x}{y_0} \Rightarrow 2AC = C_n$

$$\varphi(x,y) = \sum_{n=1}^{\infty} C_n \sinh \frac{n\pi x}{y_0} \sin \frac{n\pi y}{y_0}$$

4. $\varphi(x_0, y) = V_r \Rightarrow \sum_{n=1}^{\infty} C_n\, 2 \sinh \frac{n\pi x_0}{y_0} \sin \frac{n\pi y}{y_0} = V_r$.

$$\sinh \frac{n\pi x_0}{y_0} \frac{y_0}{2} C_n = \int_0^{y_0} V_r \sin \frac{n\pi y}{y_0}\, dx = \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{2y_0}{n\pi} & \text{for } n \text{ odd.} \end{cases}$$

$$C_n = \frac{4V_r \cancel{y_0}}{\cancel{y_0}\, n\pi} \frac{1}{\sinh \frac{n\pi x_0}{y_0}}$$

$$\varphi_r(x,y) = \sum_{n \text{ odd}}^{\infty} \frac{4V_r}{n\pi} \frac{1}{\sinh \frac{n\pi x_0}{y_0}} \sinh \frac{n\pi x}{y_0} \sin \frac{n\pi y}{y_0}$$

5.

$$\varphi(x,y) = \varphi_e(x,y) + \varphi_b(x,y) + \varphi_t(x,y) + \varphi_r(x,y)$$

$$\varphi_e(x,y) = \sum_{n\,odd} -\frac{4V_e}{n\pi}\frac{1}{\sinh\frac{n\pi x_0}{y_0}} \sin\frac{n\pi y}{y_0} \sinh\left(\frac{n\pi}{y_0}(x-x_0)\right)$$

$$\varphi_b(x,y) = \sum_{n\,odd} -\frac{4V_b}{n\pi}\frac{1}{\sinh\frac{n\pi y_0}{x_0}} \sin\frac{n\pi x}{x_0} \sinh\left(\frac{n\pi}{x_0}(y-y_0)\right)$$

$$\varphi_t(x,y) = \sum_{n\,odd} \frac{4V_t}{n\pi}\frac{1}{\sinh\frac{n\pi y_0}{x_0}} \sin\frac{n\pi x}{x_0} \sinh\frac{n\pi y}{x_0}$$

$$\varphi_r(x,y) = \sum_{n\,odd} \frac{4V_r}{n\pi}\frac{1}{\sinh\frac{n\pi x_0}{y_0}} \sin\frac{n\pi y}{y_0} \sinh\frac{n\pi x}{y_0}$$

Each equation satisfies Laplace equation, and so by linearity, the sum also satisfies Laplace equation with the same boundaries.

To validate this calculate each independent $\varphi(x,t)$ at $x, y = 0.5$

$x_0, y_0 = 1$ and $V_e, V_b, V_t, V_r = 1$

$$\varphi_e(0.5, 0.5) = \sum_{n=1}^{10} -\frac{4}{n\pi}\frac{1}{\sinh n\pi} \sin(\tfrac{1}{2}n\pi) \sinh(n\pi(0.5-1)) = 0.25$$

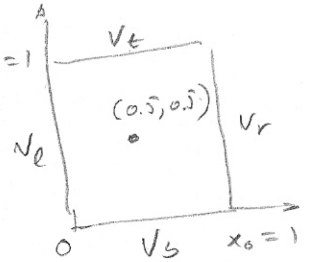
$$\varphi_b(0.5, 0.5) = \sum_{n=1}^{10} -\frac{4}{n\pi}\frac{1}{\sinh n\pi} \sin(\tfrac{1}{2}n\pi) \sinh(n\pi(0.5-1)) = 0.25$$

$$\varphi_t(0.5, 0.5) = \sum_{n=1}^{10} \frac{4}{n\pi}\frac{1}{\sinh n\pi} \sin(\tfrac{1}{2}n\pi) \sinh(n\pi\tfrac{1}{2}) = 0.25$$

$$\varphi_r(0.5, 0.5) = \sum_{n=1}^{10} \frac{4}{n\pi}\frac{1}{\sinh n\pi} \sin(\tfrac{1}{2}n\pi) \sinh(n\pi\tfrac{1}{2}) = 0.25$$

$$\varphi(x,y) = 0.25 + 0.25 + 0.25 + 0.25 = 1$$

$$\varphi(0.5, 0.5) = \frac{V_e + V_b + V_t + V_r}{4} = 1$$



6. I could not find any books to help with this. However, as we saw in HW1, Gauss law applies only to "infinity" long lines or surfaces, so it will be difficult to use Gauss law in this situation, without applying techniques like the taylor Series of HW1.