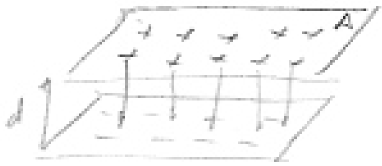
HW # 3 Jesus Gil Gil

3.1.1. Gauss

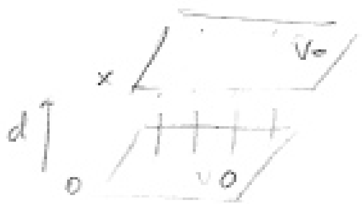


$$\Delta V = -\int_a^b \vec{E}\, dl$$

$$E\int_0^d dl = Ed. \quad \text{for } E = \frac{\sigma}{\varepsilon} \Rightarrow \Delta V = \frac{\sigma d}{\varepsilon}$$

$$Q = \sigma A \Rightarrow C = \frac{|Q|}{\Delta V} = \frac{\cancel{\sigma} A}{\frac{\cancel{\sigma} d}{\varepsilon}} = \frac{A\varepsilon}{d}$$

Laplace



$$\nabla^2 V(x) = 0 \Rightarrow \frac{\partial^2 V}{\partial x^2} = \frac{d^2 V}{dx^2} \Rightarrow \hat{y} \text{ and } \hat{z} \perp \hat{?}$$

$$\text{and } \frac{\partial^2 V}{\partial y^2} = 0, \ \frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{d^2 V}{dx^2} = 0 \Rightarrow V(x) = Ax + B$$

Boundary Conditions:

Top plate → V_0 ; $x = d$

Bottom plate 0 ; $x = 0$

$$V(d) = V_0 = Ad + B \Rightarrow A = \frac{V_0 - B}{d}$$

$$V(0) = 0 = \cancel{A}x + B \Rightarrow B = 0$$

$$A = \frac{V_0}{d}$$

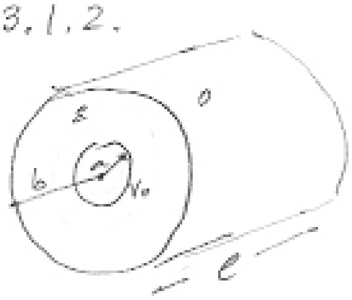
$$V(x) = \frac{V_0}{d} x$$

$$\vec{E} = -\nabla V \quad \vec{E} = -\frac{dV}{dx}\hat{x} = -\frac{V_0}{d}\hat{x}$$

$$\vec{\sigma}_x = \varepsilon \vec{E} = -\varepsilon \frac{V_0}{d}\hat{x} \Rightarrow Q = \sigma A = -\frac{\varepsilon V_0}{d} A$$

$$C = \frac{|Q|}{\Delta V} = \varepsilon \frac{\cancel{V_0}}{d} A \frac{1}{\cancel{V_0}} = \frac{\varepsilon A}{d}$$

3.1.2.



Gauss

$$\Delta V = -\int_a^b \vec{E}\, dA_s \Rightarrow A_s = 2\pi r \ell$$

$$Q = \sigma A = 2\pi r \ell \sigma \Rightarrow \sigma = \frac{Q}{2\pi r \ell}$$

$$E = \frac{\sigma}{\varepsilon} = \frac{Q}{2\pi \varepsilon r \ell}$$

$$\Delta V = \int_a^b \frac{Q}{2\pi\varepsilon r \ell}\, dr = \frac{Q \ln(b/a)}{2\pi\varepsilon\ell}$$

$$C = \frac{Q}{\Delta V} = \frac{\cancel{Q}}{\frac{\cancel{Q}\ln(b/a)}{2\pi\varepsilon\ell}} = \frac{2\pi\varepsilon\ell}{\ln(b/a)}$$

for $\frac{C}{\ell} = \mathcal{C}$ capacitance per unit ℓ $\Rightarrow$ $\mathcal{C} = \frac{2\pi\varepsilon}{\ln(b/a)}$

Laplace in cylindrical coordinates:

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \Rightarrow \text{No variations in } z \text{ or } \phi$$

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0 \Rightarrow r\frac{\partial V}{\partial r} = A \Rightarrow r\,dV = A\,dr =$$

$$\Rightarrow dV = \frac{A\,dr}{r} \Rightarrow V = A\ln(r) + B$$

Boundary conditions

$$\left.\begin{array}{ll} V = V_0 & \text{at } r = a \\ V = 0 & \text{at } r = b \end{array}\right\} \quad 0 = A\ln b + B \Rightarrow B = -A\ln b$$

$$V_0 = A\ln a + B = A\ln a - A\ln b = A\ln\left(\frac{a}{b}\right) \Rightarrow A = \frac{V_0}{\ln\frac{a}{b}}$$

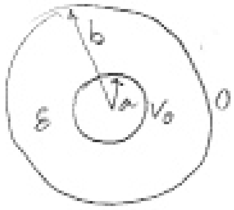
$$B = -\frac{V_0}{\ln\frac{a}{b}}\ln b \Rightarrow V = \frac{V_0}{\ln\frac{a}{b}}\ln(r) - \frac{V_0}{\ln\frac{a}{b}}\ln b = \frac{V_0}{\ln\frac{a}{b}}\ln\frac{r}{b}$$

$$V = \frac{V_0\ln b/r}{\ln\frac{b}{a}} \Rightarrow \vec{E} = -\nabla V = \frac{-V_0}{\ln\frac{b}{a}}\,\frac{d(\ln b - \ln r)}{dr} = \frac{V_0}{r\ln\left(\frac{b}{a}\right)}$$

$$\sigma = \varepsilon E = \frac{\varepsilon V_0}{r\ln\left(\frac{b}{a}\right)} \Rightarrow Q = \sigma A = \frac{\varepsilon V_0 2\pi\cancel{r}\ell}{\cancel{r}\ln\left(\frac{b}{a}\right)} \Rightarrow C = \frac{Q}{\Delta V} = \frac{2\pi\varepsilon\ell\cancel{V_0}}{\cancel{V_0}\ln\left(\frac{b}{a}\right)} \Rightarrow \frac{C}{\ell} = \frac{2\pi\varepsilon}{\ln(b/a)}$$

3.1.3.

Gauss



$$\sigma = \frac{Q}{\varepsilon} \quad \sigma = \oint E\,ds = \int E\,ds\,\overset{1}{\cancel{\cos\theta}} = E\int ds = E4\pi r^2$$

$$E4\pi r^2 = \frac{Q}{\varepsilon} \Rightarrow E = \frac{Q}{4\pi r^2\varepsilon}$$

$$\Delta V = \int_a^b E\,dr = \int_a^b \frac{Q}{4\pi r^2\varepsilon}dr = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{a}-\frac{1}{b}\right)$$

$$C = \frac{Q}{\Delta V} = \frac{\cancel{Q}}{\frac{\cancel{Q}}{4\pi\varepsilon}\left(\frac{1}{a}-\frac{1}{b}\right)} = \frac{4\pi\varepsilon}{\frac{1}{a}-\frac{1}{b}} = \frac{4\pi\varepsilon ab}{a-b}$$

Laplace in spherical coordinates V is only function of r so

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0 \Rightarrow r \neq 0 \Rightarrow \frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0$$

$$r^2\frac{dV}{dr} = A \Rightarrow dV = \frac{A}{r^2}dr \Rightarrow V = -\frac{A}{r} + B$$

Boundary Conditions

$$\left.\begin{matrix} r=a & V=V_0 \\ r=b & V=0 \end{matrix}\right\} \quad -\frac{A}{b} + B = 0 \Rightarrow B = \frac{A}{b} \text{ or } A = Bb$$

$$V_0 = -\frac{A}{a} + B = -\frac{A}{a} + \frac{A}{b} = A\left(\frac{1}{b}-\frac{1}{a}\right)$$

$$A = \frac{V_0}{\left(\frac{1}{b}-\frac{1}{a}\right)} \Rightarrow V(r) = \frac{V_0}{r\left(\frac{1}{a}-\frac{1}{b}\right)} - \frac{V_0}{b\left(\frac{1}{a}-\frac{1}{b}\right)} = \frac{V_0}{\left(\frac{1}{a}-\frac{1}{b}\right)}\left(\frac{1}{r}-\frac{1}{b}\right)$$

$$\bar{E} = -\nabla V = -\frac{dV}{dr}\hat{r} = -\frac{d}{dr}\left[\frac{V_0}{\frac{1}{a}-\frac{1}{b}}\left(\frac{1}{r}-\frac{1}{b}\right)\right] = -\frac{V_0}{\frac{1}{a}-\frac{1}{b}}\left(-\frac{1}{r^2}\right) = \frac{V_0}{r^2\left(\frac{1}{a}-\frac{1}{b}\right)}$$

$$\sigma = \varepsilon E = \frac{V_0\varepsilon}{r^2\left(\frac{1}{a}-\frac{1}{b}\right)} \Rightarrow Q = \int \sigma\,dA = \frac{V_0\varepsilon}{\left(\frac{1}{a}-\frac{1}{b}\right)}\int_0^{2\pi}\int_0^{\pi}\frac{\cancel{r^2}\sin\theta\,d\theta\,d\phi}{\cancel{r^2}} =$$

$$Q = \frac{V_0\varepsilon}{\frac{1}{a}-\frac{1}{b}}\underbrace{(-\cos\theta)\Big|_0^{\pi}}_{2}\;\underbrace{\phi\Big|_0^{2\pi}}_{2\pi} = \frac{4\pi\varepsilon V_0}{\frac{1}{a}-\frac{1}{b}} \Rightarrow C = \frac{Q}{\Delta V} = \frac{4\pi\varepsilon \cancel{V_0}}{\cancel{V_0}\left(\frac{1}{a}-\frac{1}{b}\right)} = \frac{4\pi\varepsilon}{\frac{1}{a}-\frac{1}{b}}$$

3.1.4.1 Equal and Opposite.

Yes it is true. The charge density may differ depending on their shape, but the total charge must be equal. If not, current (charges) will be drawn from the battery.

3.1.4.2 Justification of shape. $\nabla^2 \varphi(x) = \frac{\partial^2 \varphi}{\partial x^2} = \frac{d^2 \varphi}{dx^2}$

$\nabla^2 \varphi(x) = \nabla \cdot \nabla \varphi(x) = \left[\frac{\partial}{\partial x}\right] \cdot \left[\frac{\partial \varphi}{\partial x}\right] = \frac{d^2 \varphi}{dx^2}$ $\longrightarrow$ partial derivatives are used for multiple variables, and ordinary derivative for single variables. Because $\varphi(x)$ is only dependent on x $\frac{\partial^2 \varphi}{\partial x^2}$ can be written as $\frac{d^2 \varphi}{dx^2}$

3.1.4.3 Aproximation

- Only the Concentric spherical shells capacitance is exact.
- For the parallel plate, the capacitance is aproximated, as the fringing field are ignored. This capacitance is nearly exact, when the Area $\gg\gg d$.
- Similarly, the long coxial cilinders are aproximated for the same reason (fringing fields), but C becomes nearly exact as the length of the cylinder is much greater than the radious $L \gg\gg r$

3.1.4.4

$$\Delta V_T = \Delta V_1 + \Delta V_2 = E\int_0^{d_1} de + E\int_0^{d_2} de = \frac{\sigma d_1}{\varepsilon} + \frac{\sigma d_2}{\varepsilon}$$

$$Q = \sigma A \Rightarrow C = \frac{Q}{\Delta V} \Rightarrow \Delta V = \frac{Q}{C_T} \Rightarrow$$

$$\Delta V_T = \frac{Q}{C_T} \;;\; \Delta V_1 = \frac{\sigma A}{C_1} \;;\; \Delta V_2 = \frac{\sigma A}{C_2} \Rightarrow \frac{1}{C_1} = \frac{\cancel{\sigma} d_1}{\varepsilon \cancel{\sigma} A} \;;\; \frac{1}{C_2} = \frac{\sigma d_2}{\varepsilon \sigma A}$$

$$\frac{1}{C_T} = \frac{\Delta V_T}{Q} = \frac{\cancel{\sigma}(d_1 + d_2)}{\varepsilon \cancel{\sigma} A} = \frac{d_1}{\varepsilon A} + \frac{d_2}{\varepsilon A} = \frac{1}{C_1} + \frac{1}{C_2}$$

3.2. Potential due to Point charge.

3.2.1 W_1 to move q_1 from ∞ to x_1

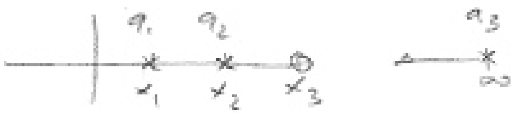
$U = \infty \Rightarrow$ as q_1 moves closer to q_3 the work increases to overcome the charge repulsion force between q_1 and q_3

For example let's look at q_1 as it approaches q_3 with a separation as $d = [pq_1 - x_3]$

$$U = -\int_{\infty}^{d} \frac{q_1 q_3}{4\pi\varepsilon r^2} dr = \frac{q_1 q_3}{4\pi\varepsilon d} \Rightarrow \text{ and } U \propto \frac{1}{d} \Rightarrow U \to \infty \text{ as } d \to 0$$

3.2.2 $\Rightarrow U = \infty$, same as before

3.2.3 $\Rightarrow$



$$\Rightarrow U = \frac{q_3 q_1}{4\pi\varepsilon(x_1 - x_3)} + \frac{q_3 q_2}{4\pi\varepsilon(x_2 - x_3)}$$

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$$\varphi_1 = \frac{q_2}{4\pi\varepsilon[x_2 - x_1]} + \frac{q_3}{4\pi\varepsilon[x_3 - x_1]}$$

$$\varphi_2 = \frac{q_1}{4\pi\varepsilon[x_1 - x_2]} + \frac{q_3}{4\pi\varepsilon[x_3 - x_1]}$$

$$\varphi_3 = \frac{q_1}{4\pi\varepsilon[x_1 - x_3]} + \frac{q_2}{4\pi\varepsilon[x_2 - x_3]}$$

$\varphi_\infty = 0$ as $[\infty - x] = \infty \Rightarrow \frac{q}{\infty} = 0$