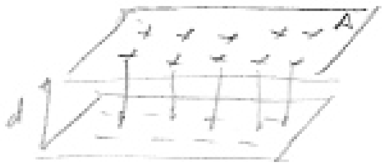
HW # 3 Jesus Gil Gil

3.1.1. <u>Gauss</u>

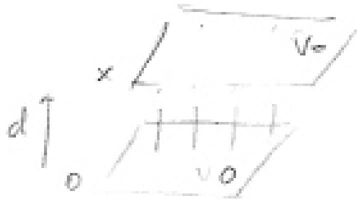


$$\Delta V = -\int_a^b \vec{E}\, dl$$

$$E\int_0^d dl = Ed. \quad \text{for } E = \frac{\sigma}{\varepsilon} \Rightarrow \Delta V = \frac{\sigma d}{\varepsilon}$$

$$Q = \sigma A \Rightarrow C = \frac{|Q|}{\Delta V} = \frac{\cancel{\sigma} A}{\frac{\cancel{\sigma} d}{\varepsilon}} = \frac{A\varepsilon}{d}$$

<u>Laplace</u>



$$\nabla^2 V(x) = 0 \Rightarrow \frac{\partial^2 V}{\partial x^2} = \frac{d^2 V}{dx^2} \Rightarrow \hat{y} \text{ and } \hat{z} \perp \hat{x}$$

$$\text{and } \frac{\partial^2 V}{\partial y^2} = 0, \ \frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{d^2 V}{dx^2} = 0 \Rightarrow V(x) = Ax + B$$

Boundary Conditions:

Top plate $\rightarrow V_0 \ ; \ x = d$

Bottom plate $0 \ ; \ x = 0$

$$V(d) = V_0 = Ad + B \Rightarrow A = \frac{V_0 - B}{d} \qquad A = \frac{V_0}{d}$$

$$V(0) = 0 = \cancel{A}x + B \Rightarrow B = 0$$

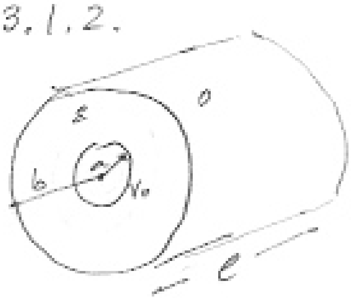
$$V(x) = \frac{V_0}{d} x$$

$$\vec{E} = -\nabla V \qquad \vec{E} = -\frac{dV}{dx}\hat{x} = -\frac{V_0}{d}\hat{x}$$

$$\vec{\sigma}_x = \varepsilon \vec{E} = -\varepsilon \frac{V_0}{d}\hat{x} \Rightarrow Q = \sigma A = -\frac{\varepsilon V_0}{d} A$$

$$C = \frac{|Q|}{\Delta V} = \frac{\varepsilon \cancel{V_0}}{d} A \frac{1}{\cancel{V_0}} = \frac{\varepsilon A}{d}$$

3.1.2.



Gauss

$$\Delta V = -\int_a^b \vec{E}\, dA_s \Rightarrow A_s = 2\pi r \ell$$

$$Q = \sigma A = 2\pi r \ell \sigma \Rightarrow \sigma = \frac{Q}{2\pi r \ell}$$

$$E = \frac{\sigma}{\varepsilon} = \frac{Q}{2\pi \varepsilon r \ell}$$

$$\Delta V = \int_a^b \frac{Q}{2\pi\varepsilon r \ell}\, dr = \frac{Q \ln(b/a)}{2\pi \varepsilon \ell}$$

$$C = \frac{Q}{\Delta V} = \frac{\cancel{Q}}{\frac{\cancel{Q} \ln(b/a)}{2\pi\varepsilon \ell}} = \frac{2\pi\varepsilon \ell}{\ln(b/a)}$$

for $\frac{C}{\ell} = \mathcal{C}$ capacitance per unit ℓ $\Rightarrow$ $\mathcal{C} = \frac{2\pi\varepsilon}{\ln(b/a)}$

Laplace in cylindrical coordinates:

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$ $\Rightarrow$ No variations in z or ϕ

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0 \Rightarrow r\frac{\partial V}{\partial r} = A \Rightarrow r\,dV = A\,dr. =$$

$$\Rightarrow dV = \frac{A\,dr}{r} \Rightarrow V = A\ln(r) + B$$

Boundary conditions

$V = V_0$ at $r = a$

$V = 0$ at $r = b$

$$0 = A\ln b + B \Rightarrow B = -A\ln b$$

$$V_0 = A\ln a + B = A\ln a - A\ln b = A\ln\left(\frac{a}{b}\right) \Rightarrow A = \frac{V_0}{\ln\frac{a}{b}}$$

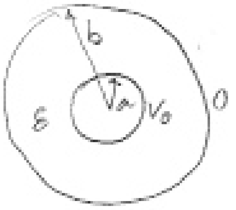
$$B = -\frac{V_0}{\ln\frac{a}{b}}\ln b \Rightarrow V = \frac{V_0}{\ln\frac{a}{b}}\ln(r) - \frac{V_0}{\ln\frac{a}{b}}\ln b = \frac{V_0}{\ln\frac{a}{b}}\ln\frac{r}{b}$$

$$V = \frac{V_0 \ln b/r}{\ln\frac{b}{a}} \Rightarrow \vec{E} = -\nabla V = \frac{-V_0}{\ln\frac{b}{a}}\frac{d(\ln b - \ln r)}{dr} = \frac{V_0}{r\ln\left(\frac{b}{a}\right)}$$

$$\sigma = \varepsilon E = \frac{\varepsilon V_0}{r\ln\left(\frac{b}{a}\right)} \Rightarrow Q = \sigma A = \frac{\varepsilon V_0 2\pi \cancel{r}\ell}{\cancel{r}\ln\left(\frac{b}{a}\right)} \Rightarrow C = \frac{Q}{\Delta V} = \frac{2\pi\varepsilon\ell \cancel{V_0}}{\cancel{V_0}\ln\left(\frac{b}{a}\right)} \Rightarrow \frac{C}{\ell} = \frac{2\pi\varepsilon}{\ln(b/a)}$$

3.1.3.

Gauss



$$\sigma = \frac{Q}{\varepsilon} \quad \sigma = \oint E\,ds = \int E\,ds\,\overset{1}{\cancel{\cos\theta}} = E\int ds = E4\pi r^2$$

$$E4\pi r^2 = \frac{Q}{\varepsilon} \Rightarrow E = \frac{Q}{4\pi r^2 \varepsilon}$$

$$\Delta V = \int_a^b E\,dr = \int_a^b \frac{Q}{4\pi r^2 \varepsilon}\,dr = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$C = \frac{Q}{\Delta V} = \frac{\cancel{Q}}{\frac{\cancel{Q}}{4\pi\varepsilon}\left(\frac{1}{a}-\frac{1}{b}\right)} = \frac{4\pi\varepsilon}{\frac{1}{a}-\frac{1}{b}} = \underline{\underline{\frac{4\pi\varepsilon ab}{a-b}}}$$

Laplace in spherical coordinates V is only function of r so

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0 \Rightarrow r \neq 0 \Rightarrow \frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0$$

$$r^2\frac{dV}{dr} = A \Rightarrow dV = \frac{A}{r^2}dr \Rightarrow V = -\frac{A}{r} + B$$

Boundary Conditions

$$\left.\begin{array}{ll} r=a & V=V_0 \\ r=b & V=0 \end{array}\right\} -\frac{A}{b} + B = 0 \Rightarrow B = \frac{A}{b} \text{ or } A = Bb$$

$$V_0 = -\frac{A}{a} + B = -\frac{A}{a} + \frac{A}{b} = A\left(\frac{1}{b} - \frac{1}{a}\right)$$

$$A = \frac{V_0}{\left(\frac{1}{b}-\frac{1}{a}\right)} \Rightarrow V(r) = \frac{V_0}{r\left(\frac{1}{a}-\frac{1}{b}\right)} - \frac{V_0}{b\left(\frac{1}{a}-\frac{1}{b}\right)} = \frac{V_0}{\left(\frac{1}{a}-\frac{1}{b}\right)}\left(\frac{1}{r} - \frac{1}{b}\right)$$

$$\bar{E} = -\nabla V = -\frac{dV}{dr}\hat{r} = -\frac{d}{dr}\left[\frac{V_0}{\frac{1}{a}-\frac{1}{b}}\left(\frac{1}{r}-\frac{1}{b}\right)\right] = -\frac{V_0}{\frac{1}{a}-\frac{1}{b}}\left(-\frac{1}{r^2}\right) = \frac{V_0}{r^2\left(\frac{1}{a}-\frac{1}{b}\right)}$$

$$\sigma = \varepsilon E = \frac{V_0\varepsilon}{r^2\left(\frac{1}{a}-\frac{1}{b}\right)} \Rightarrow Q = \int \sigma\,dA = \frac{V_0\varepsilon}{\left(\frac{1}{a}-\frac{1}{b}\right)}\int_0^{2\pi}\int_0^{\pi}\frac{\cancel{r^2}\sin\theta\,d\theta\,d\phi}{\cancel{r^2}} =$$

$$Q = \frac{V_0\varepsilon}{\frac{1}{a}-\frac{1}{b}}\underbrace{\left(-\cos\theta\right)\Big|_0^{\pi}}_{2}\;\underbrace{\phi\Big|_0^{2\pi}}_{2\pi} = \frac{4\pi\varepsilon V_0}{\frac{1}{a}-\frac{1}{b}} \Rightarrow C = \frac{Q}{\Delta V} = \frac{4\pi\varepsilon \cancel{V_0}}{\cancel{V_0}\left(\frac{1}{a}-\frac{1}{b}\right)} = \underline{\underline{\frac{4\pi\varepsilon}{\frac{1}{a}-\frac{1}{b}}}}$$

3.1.4.1 Equal and Opposite.

Yes it is true. The charge density may differ depending on their shape, but the total charge must be equal. If not, current (charges) will be drawn from the battery.

3.1.4.2 Justification of shape. $\nabla^2 \varphi(x) = \frac{\partial^2 \varphi}{\partial x^2} = \frac{d^2 \varphi}{dx^2}$

$\nabla^2 \varphi(x) = \nabla \cdot \nabla \varphi(x) = \left[\frac{\partial}{\partial x}\right] \cdot \left[\frac{\partial \varphi}{\partial x}\right] = \frac{d^2 \varphi}{dx^2}$ → partial derivates are used for multiple variables, and ordinary derivate for single variables. Because $\varphi(x)$ is only dependent on x $\frac{\partial^2 \varphi}{\partial x^2}$ can be written as $\frac{d^2 \varphi}{dx^2}$

3.1.4.3 Aproximation

- Only the Concentric sherical shells capacitance is exact.
- For the parallel plate, the capacitance is aproximated, as the fringing field are ignored. This capacitance is nearly exact, when the Area >>> d.
- Similarly, the long coxial cilinders are aproximated for the same reason (fringing fields), but C becomes nearly exact as the length of the Cylinder is much greater than the radious L >>> r

3.1.4.4

$$\Delta V_T = \Delta V_1 + \Delta V_2 = E \int_0^{d_1} dl + E \int_0^{d_2} dl = \frac{\sigma d_1}{\varepsilon} + \frac{\sigma d_2}{\varepsilon}$$

$$Q = \sigma A \Rightarrow C = \frac{Q}{\Delta V} \Rightarrow \Delta V = \frac{Q}{C_T} \Rightarrow$$

$$\Delta V_T = \frac{Q}{C_T} \; ; \; \Delta V_1 = \frac{\sigma A}{C_1} \; ; \; \Delta V_2 = \frac{\sigma A}{C_2} \Rightarrow \frac{1}{C_1} = \frac{\cancel{\sigma} d_1}{\varepsilon \cancel{\sigma} A} \; ; \; \frac{1}{C_2} = \frac{\sigma d_2}{\varepsilon \sigma A}$$

$$\frac{1}{C_T} = \frac{\Delta V_T}{Q} = \frac{\cancel{\sigma}(d_1 + d_2)}{\varepsilon \cancel{\sigma} A} = \frac{d_1}{\varepsilon A} + \frac{d_2}{\varepsilon A} = \frac{1}{C_1} + \frac{1}{C_2}$$

3.2. Potential due to Point charges.

I was worried that someone would catch the infinity issue, and I am glad you did.

Technically, if q2 is positive it takes and infinite amount of work to move q1 from +infinity to x2. This “large” positive work is canceled by negative work as you move from x2 to x1. One can work around the infinity by assuming the charge q2 is uniformly distributed on a small sphere. The field will not be infinite for an arbitrarily small sphere.

3.2.1 W_1 to move q_1 from ∞ to x_1

$U = \infty \Rightarrow$ as q_1 moves closer to q_3 the work increases to overcome the charge repulsion force between q_1 and q_3

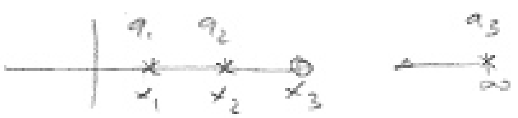
For example let's look at q_1 as it approaches q_3 with a separation as $d = [pq_1 - x_3]$

d is positive in this equation (ideally you would have labeled it on your sketch). As a result, the sign of U only depends on the signs of q1 and q3.

$$U = -\int_{\infty}^{d} \frac{q_1 q_3}{4\pi\varepsilon r^2}\,dr = \frac{q_1 q_3}{4\pi\varepsilon d} \Rightarrow \text{and } U \propto \frac{1}{d} \Rightarrow U \to \infty \text{ as } d \to 0$$

3.2.2 $\Rightarrow U = \infty$, same as before

3.2.3 $\Rightarrow$



$$\Rightarrow U = \frac{q_3 q_1}{4\pi\varepsilon(x_1 - x_3)} + \frac{q_3 q_2}{4\pi\varepsilon(x_2 - x_3)}$$

Should be |x_1 - x_3| and |x_2 - x_3|.

For in class Q? |———|

$$\varphi_1 = \frac{q_2}{4\pi\varepsilon[x_2 - x_1]} + \frac{q_3}{4\pi\varepsilon[x_3 - x_1]}$$

$$\varphi_2 = \frac{q_1}{4\pi\varepsilon[x_1 - x_2]} + \frac{q_3}{4\pi\varepsilon[x_3 - x_1]}$$

$$\varphi_3 = \frac{q_1}{4\pi\varepsilon[x_1 - x_3]} + \frac{q_2}{4\pi\varepsilon[x_2 - x_3]}$$

$\varphi_\infty = 0$ as $[\infty - x] = \infty \Rightarrow \frac{q}{\infty} = 0$