6.1 Faraday's Law and Ampere's Law.

$$E = E_{ox} \cos(k_z z - \omega t)\hat{x}$$

1. Find B using $\nabla \times E = -\frac{\partial B}{\partial t}$

Find the curl of E

$$\nabla \times E = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} \Rightarrow$$

$$\nabla \times E = \left(\cancel{\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}} \right)\hat{x} + \left(\frac{\partial E_x}{\partial z} - \cancel{\frac{\partial E_z}{\partial x}} \right)\hat{y} + \left(\cancel{\frac{\partial E_y}{\partial x}} - \cancel{\frac{\partial E_x}{\partial y}} \right)\hat{z} \Rightarrow$$

$$0 \qquad 0 \qquad 0 \qquad 0$$

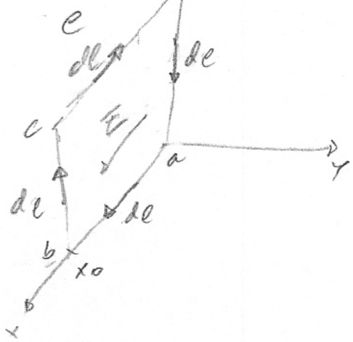
$$\nabla \times E = \frac{\partial E_x}{\partial z}\hat{y} = \frac{\partial}{\partial z} E_{ox} \cos(k_z z - \omega t)\hat{y} = -E_{ox} k_z \sin(k_z z - \omega t)\hat{y}$$

$$-\frac{\partial B}{\partial t} = -E_{ox} k_z \sin(k_z z - \omega t)\hat{y}$$

$$\int \frac{d}{dt} B\, dt = \int E_{ox} k_z \sin(k_z z - \omega t)\hat{y}\, dt$$

$$B = -\frac{E_{ox} k_z}{\omega} \cos(k_z z - \omega t)\hat{y}$$

2. Show that E satisfies Faraday's Law.



$$\oint E\, dl = -\frac{\partial \Phi_B}{\partial t}$$

$$\oint E\, dl = \int_a^b E\, dl + \int_b^c E\, \cancel{dl} + \int_c^d E\, dl + \int_d^a E\, \cancel{dl} =$$

$$0 \qquad 0$$

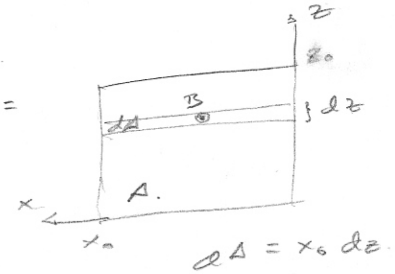
$$= \int_0^{x_0} E\, dx + \int_{x_0}^{0} E\, dx = \int_0^{x_0} E_{ox} \cos(k_z \cancel{z} - \omega t)\, dx + \int_{x_0}^{0} E_{ox} \cos(k_z z_0 - \omega t)\, dx$$

$\Rightarrow$

$$\oint E\,dl = x_0 E_{0x}\cos(\omega t) - x_0 E_{0x}\cos(\omega t - k_z z_0) =$$

$$= x_0 E_{0x}\left[\cos(\omega t) - \cos(\omega t - k_z z_0)\right] \checkmark$$

$$\Phi_B = \int B\cdot dA = \int_0^{z_0} -\frac{E_{0x}k_z}{\omega}\cos(k_z z - \omega t)\,x_0\,dz =$$



$$dA = x_0\,dz$$

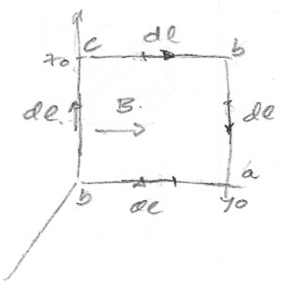
$$= \frac{E_{0x}x_0}{\omega}\left[\sin(\omega t - k_z z_0) - \sin(\omega t)\right]$$

$$-\frac{d\Phi_B}{dt} = -\frac{d}{dt}\left[\frac{E_{0x}x_0}{\omega}\left[\sin(\omega t - k_z z_0) - \sin(\omega t)\right]\right] =$$

$$= -x_0 E_{0x}\left[\cos(\omega t - k_z - z_0) - \cos(\omega t)\right] = x_0 E_{0x}\left[\cos(\omega t) - \cos(\omega t - k_z z_0)\right] \checkmark$$

$$\oint E\,dl = -\frac{d\Phi_B}{dt} \checkmark$$

3. Show that B satisfies Ampere's Law for $J=0$ $\quad \oint B\cdot dl = \frac{1}{c^2}\frac{d\Phi_E}{dt}$



Same as before.

$$\oint B\cdot dl = \int_a^b B\,dl + \int_b^c \overset{0}{\cancel{B\,dl}} + \int_c^d B\,dl + \int_d^a \overset{0}{\cancel{B\,dl}} =$$

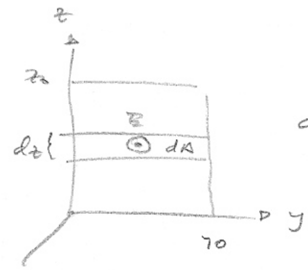
$$= \int_{y_0}^{0} B\,dy + \int_0^{y_0} B\,dy = \int_{y_0}^{0} -\frac{E_{0x}k_z}{\omega}\cos(k_z\cdot 0 - \omega t)\,dy +$$

$$+ \int_0^{y_0} -\frac{E_{0x}k_z}{\omega}\cos(k_z z_0 - \omega t)\,dy =$$

$$= E_{0x}y_0\frac{k}{\omega}\cos(\omega t) - E_{0x}y_0\frac{k}{\omega}\cos(\omega t - k_z z_0) =$$

$$= E_{0x}y_0\frac{k}{\omega}\left[\cos(\omega t) - \cos(\omega t - k_z z_0)\right] \qquad \text{Note that } \frac{k}{\omega} = \frac{1}{c}$$

$$\Phi_E = \int E\, dA = \int_0^{z_0} E\, y_0\, dz = \int_0^{z_0} E_{0x} \cos(k_z z - \omega t)\, y_0\, dz =$$



$dA = y_0\, dz$

$$= -\frac{E_{0x}\, y_0}{k}\left[\sin(\omega t - k_z z_0) - \sin(\omega t)\right]$$

$$\frac{\partial \Phi_E}{\partial t} = \frac{\partial}{\partial t}\left[-\frac{E_{0x}\, y_0}{k}\left[\sin(\omega t - k_z z_0) - \sin(\omega t)\right]\right] =$$

$$= -E_{0x}\, y_0\, \frac{\omega}{k}\left[\cos(\omega t - k_z z_0) - \cos(\omega t)\right] =$$

$$= E_{0x}\, y_0\, \frac{\omega}{k}\left[\cos(\omega t) - \cos(\omega t - k_z z_0)\right] \rightarrow \text{note that } \frac{\omega}{k} = c$$

$$\frac{1}{c} \cdot \frac{\partial \Phi_E}{dt} = E_{0x}\, y_0 \left[\cos(\omega t) - \cos(\omega t - k_z z_0)\right].$$

$$c \cdot \oint B\, dl = E_{0x}\, y_0 \left[\cos(\omega t) - \cos(\omega t - k_z z_0)\right]$$

$$c \oint B\, dl = \frac{1}{c}\, \frac{\partial \Phi_E}{\partial t} \Rightarrow \oint B\, dl = \frac{1}{c^2}\, \frac{\partial \Phi_E}{\partial t} \quad \checkmark$$

6.2 Wave equation derivation.

$$E = E_{0x}(x,t)\hat{x} + E_{0y}(x,t)\hat{y} + E_{0z}(x,t)\hat{z}$$

$$B = B_{0x}(x,t)\hat{x} + B_{0y}(x,t)\hat{y} + B_{0z}(x,t)\hat{z}$$

1. $E_y(x,t)$, $E_z(x,t)$, $B_y(x,t)$ and $B_z(x,t)$ each obey.

$$\frac{\partial^2 f}{\partial u^2} = \frac{1}{c}\frac{\partial^2 f}{\partial t^2} \Rightarrow u = x, y, t \text{ and } f = E_y, E_z, B_x, B_z.$$

Faraday's law $\Rightarrow \nabla \times E = -\dfrac{\partial B}{\partial t}$ Ampere's Law $(J=0) \Rightarrow \nabla \times B = \dfrac{1}{c^2}\dfrac{\partial E}{\partial t}$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_{0x} & E_{0y} & E_{0z} \end{vmatrix} \qquad \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_{0x} & B_{0y} & B_{0z} \end{vmatrix}$$

$$\hat{x}(0) = -\frac{\partial B_{0x}}{\partial t} \quad (1)$$

$$\hat{y}\left(\frac{\partial E_{0z}}{\partial x}\right) = -\frac{\partial B_{0y}}{\partial t} \quad (2)$$

$$\hat{z}\left(\frac{\partial E_{0y}}{\partial x}\right) = -\frac{\partial B_{0z}}{\partial t} \quad (3)$$

$$\hat{x}(0) = \frac{1}{c^2}\frac{\partial E_{0x}}{\partial t} \quad (4)$$

$$\hat{y}\left(\frac{\partial B_{0z}}{\partial x}\right) = \frac{1}{c^2}\frac{\partial E_{0y}}{\partial t} \quad (5)$$

$$\hat{z}\left(\frac{\partial B_{0y}}{\partial x}\right) = \frac{1}{c^2}\frac{\partial E_{0z}}{\partial t}. \quad (6)$$

$$\frac{\partial}{\partial x}(3) \Rightarrow \frac{\partial^2 E_{0y}}{\partial x^2} = -\frac{\partial^2 B_{0z}}{\partial x\,\partial t} \quad ; \quad \frac{\partial}{\partial t}(5) \Rightarrow \frac{\partial^2 B_{0z}}{\partial t\,\partial x} = \frac{1}{c^2}\frac{\partial^2 E_{0y}}{\partial t^2}$$

$$\frac{\partial^2 E_{0y}}{\partial x^2} = -\frac{1}{c^2}\frac{\partial^2 E_{0y}}{\partial t^2} \quad \checkmark$$

$$\frac{\partial}{\partial x}(5) \Rightarrow \frac{\partial^2 B_{0z}}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E_{0y}}{\partial x\,\partial t} \quad ; \quad \frac{\partial}{\partial t}(3) \Rightarrow \frac{\partial^2 E_{0y}}{\partial x\,\partial t} = -\frac{\partial^2 B_{0z}}{\partial t^2}$$

$$\frac{\partial^2 B_{0z}}{\partial x^2} = -\frac{1}{c^2}\frac{\partial^2 B_{0z}}{\partial t^2} \quad \checkmark$$

$$\frac{\partial}{\partial x}(2) \Rightarrow \frac{\partial^2 E_{oz}}{\partial x^2} = -\frac{\partial^2 B_{oy}}{\partial x\, \partial t} \qquad \frac{\partial}{\partial t}(6) \Rightarrow \frac{\partial^2 B_{oy}}{\partial x\, \partial t} = \frac{1}{c^2}\frac{\partial^2 E_{oz}}{\partial t^2}$$

$$\frac{\partial^2 E_{oz}}{\partial x^2} = -\frac{1}{c^2}\frac{\partial^2 E_{oz}}{\partial t^2} \quad \checkmark$$

$$\frac{\partial}{\partial x}(6) \quad \frac{\partial^2 B_{oy}}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E_{oz}}{\partial x\, \partial t} \qquad \frac{\partial}{\partial t}(2) \Rightarrow \frac{\partial^2 E_{oz}}{\partial x\, \partial t} = -\frac{\partial^2 B_{oy}}{\partial t^2}$$

$$\frac{\partial^2 B_{oy}}{\partial x^2} = -\frac{1}{c}\frac{\partial^2 B_{oy}}{\partial t^2} \quad \checkmark$$

2. $E_x(x,t) = B_x(x,t) = 0$?

Yes, $\nabla \times E = 0\,\hat{x}$ and $\nabla \times B = 0\,\hat{x}$.

as the wave propagates in the $\hat{x}$ direction, the E and B fields are normal to $\hat{x}$ and therefore, B_x and E_x are 0.

3. $$\nabla^2 E = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} \quad \text{and} \quad \nabla^2 B = \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2}$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times \nabla \times E = -\frac{\partial}{\partial t}(\nabla \times B) \Rightarrow \nabla \times B = \frac{1}{c^2}\frac{\partial E}{\partial t} \text{ for } J = 0$$

$$-\nabla^2 E + \nabla(\underbrace{\cancel{\nabla \cdot E}}_{0}) = -\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial E}{\partial t}\right) \Rightarrow \nabla \cdot E = 0 \text{ for } \rho = 0$$

} No sources.

$$\nabla^2 E = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} \Rightarrow \nabla^2 \text{ is } \frac{\partial^2}{\partial x^2} \text{ (in our case).}$$

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} \quad \checkmark$$

$$\nabla \times B = \frac{1}{c^2}\frac{\partial E}{\partial t} \Rightarrow \nabla \times \nabla \times B = \frac{1}{c^2}\frac{\partial}{\partial t}(\nabla \times E) \Rightarrow \nabla \times E = -\frac{\partial B}{\partial t}.$$

$$\Rightarrow -\nabla^2 B + \nabla(\underbrace{\cancel{\nabla \cdot B}}_{0}) = \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\frac{\partial B}{\partial t}\right) \Rightarrow \nabla^2 B = \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2}$$

$$\frac{\partial^2 B}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2} \quad \checkmark$$

6.3 Wave equation Solutions

1. $E = E_{0x} \cos(k_z z - \omega t + \delta_x)\hat{x}$ satisfies $\nabla^2 E = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2}$

* $\nabla^2 E = \frac{\partial^2 E}{\partial z^2} = \frac{\partial^2}{\partial z^2} E_{0x} \cos(k_z z - \omega t + \delta_x)\hat{x} =$

$= -k_z^2 E_{0x} \cos(k_z z - \omega t + \delta_x)\hat{x}$

$\frac{\partial^2 E}{\partial t^2} = \frac{\partial^2}{\partial t^2} E_{0x} \cos(k_z z - \omega t + \delta_x)\hat{x} =$

$= -\omega^2 E_{0x} \cos(k_z z - \omega t + \delta_x)\hat{x}$

for $\frac{1}{c^2} = \frac{k^2}{\omega^2}$

$-k_z^2 E_{0x} \cos(k_z z - \omega t + \delta_x)\hat{x} = -\frac{k^2}{\cancel{\omega^2}} \cancel{\omega^2} E_{0x} \cos(k_z z - \omega t + \delta_x)\hat{x}$ ✓

* $B = B_{0x} \cos(k_z z - \omega t + \delta'_x)\hat{x} + B_{0y} \cos(k_z z - \omega t + \delta'_y)\hat{y}$

$\nabla^2 B = \frac{\partial^2 B}{\partial z^2} = \frac{\partial^2}{\partial z^2}\left[B_{0x} \cos(k_z z - \omega t + \delta'_x)\hat{x} + B_{0y} \cos(k_z z - \omega t + \delta'_y) y'\right] =$

$= -k^2 B_{0x} \cos(k_z z - \omega t + \delta'_x)\hat{x} - k^2 B_{0y} \cos(k_z z - \omega t + \delta'_y) y' =$

$= -k^2 \left[B_{0x} \cos(k_z z - \omega t + \delta'_x)\hat{x} + B_{0y} \cos(k_z z - \omega t + \delta'_y)\hat{y}\right]$

$\frac{\partial^2 B}{\partial t^2} = \frac{\partial^2}{\partial t^2}\left[B_{0x} \cos(k_z z - \omega t + \delta'_x)\hat{x} + B_{0y} \cos(k_z z - \omega t + \delta'_y)\hat{y}\right] =$

$= -\omega^2 \left[B_{0x} \cos(k_z z - \omega t + \delta'_x)\hat{x} + B_{0y} \cos(k_z z - \omega t + \delta'_y)\hat{y}\right]$

for $\frac{1}{c^2} = \frac{k^2}{\omega^2}$. same as before:

$-k^2 \left[B_{0x} \cos(k_z z - \omega t + \delta'_x)\hat{x} + B_{0y} \cos(k_z z - \omega t + \delta'_y)\hat{y} = -\frac{k^2}{\cancel{\omega^2}} \cancel{\omega^2} \left[B_{0x} \ldots\right]\right.$ ✓

- k_z and w are related by $c = \frac{w}{k}$
- E is linearly polarized & B is circularly polarized. In the case of E & B both having the same polarization, their magnitudes (E_{ox} & B_{ox}) are related by:

  $E_{ox} = B_{oy} \cdot c$ according to the solution in 6.1 using Faraday's law.

  In our case E & B are different traveling waves

- In the case of B circular polarized, the relation between B_{ox} and B_{oy} will give a circular polarized is $B_{ox} = B_{oy}$, and eliptical polarized if $B_{ox} \neq B_{oy}$. and the difference between δ'_x & δ'_y will give us left-hand or right-hand polarization and the angle of polarization.

2. For Both B & E be consistent with Maxwell equations,

$B_{ox} = 0 \Rightarrow B = B_{oy} \cos(k_z z - wt + \delta'_y)\hat{y}$

Then $E_{ox} = B_{oy} c$, $c = \frac{w}{k}$ and $\delta_x = \delta'_y$

3. Redo $E = E_{oy} \cos(k_z z - wt + \delta_y)\hat{y}$

$B_{oy} = 0 \Rightarrow B = B_{ox} \cos(k_z z - wt + \delta'_x)\hat{x}$

Same as before $E_{oy} = c B_{ox}$, $k = \frac{w}{c}$ and $\delta_y = \delta'_x$

4. $k = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$ $\Rightarrow B = \frac{1}{c}\hat{k} \times E$

$$\hat{k} \times E = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ k_x & k_y & k_z \\ E_x & 0 & 0 \end{vmatrix} = \hat{y}(k_z E_x) + \hat{z}\left(\cancelto{0}{-k_y E_x}\right)$$

(2) $\hat{k} \times \bar{E} = k_z E_{ox} \cos(k_z z - wt + \delta_x)\hat{y}$

From Faraday's law $\nabla \times E = -\frac{\partial B}{\partial t}$

$\nabla \times E = \hat{y} \frac{\partial E_x}{\partial z} = -k_z E_{ox} \sin(k_z z - \omega t + \delta_x)\hat{y}$ integrate to get B

$B = -\int -k_z E_{ox} \sin(k_z z - \omega t + \delta_x)\hat{y}\, dt = \frac{1}{\omega} \underbrace{k_z E_{ox} \cos(k_z z - \omega t + \delta_x)\hat{y}}_{\hat{k} \times E}$

for $k = |k|\hat{k}_z$

$B = \frac{1}{c} \hat{k} \times E$

(3) Same for $E = E_{oy} \cos(k_z z - \omega t + \delta_y)\hat{y}$

$$k \times E = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ k_x & k_y & k_z \\ 0 & E_{oy} & 0 \end{vmatrix} = -\hat{x}(k_z E_{oy}) + \hat{z}\cancelto{0}{(k_x E_{oy})}$$

From Faraday's law $\nabla \times E \Rightarrow k \times E = -\frac{\partial B}{\partial t}$

$\nabla \times E = -\hat{x} \frac{\partial E_x}{\partial z} = k_z E_{oy} \sin(kz - \omega t + \delta_y)\hat{x}$ integrate to get B.

$B = -\int k_z E_{oy} \sin(kz - \omega t + \delta_y)\hat{x}\, dt = \frac{1}{\omega}\underbrace{(-k_z E_{oy} \cos(kz - \omega t + \delta_y)\hat{x})}_{\hat{k} \times E}$

for $k = |k|\hat{k}_z$

$B = \frac{1}{c} \hat{k} \times E$

6.4. Complex Form.

$E \equiv Re[\hat{E} e^{-i(\omega t - k\cdot r)}]$ $\qquad k = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}$

$B = Re[\hat{B} e^{-i(\omega t - k\cdot r)}]$ $\qquad r = x\hat{x} + y\hat{y} + z\hat{z}$

where $\tilde{E}$ & $\tilde{B}$ are complex constants as components.

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times E = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) +$$

$$+ \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) =$$

$$= \tilde{E} e^{-i(\omega t - kr)}[(k_y - k_z)\hat{x} + (k_z - k_x)\hat{y} + (k_x - k_y)\hat{z}]$$

Integrate over time to get B.

$$B = -\int \tilde{E} e^{-i(\omega t - kr)} [(k_y - k_z)\hat{x} + (k_z - k_x)\hat{y} + (k_x - k_y)\hat{z}]\, dt =$$

$$= \frac{1}{\omega} \tilde{E} e^{-i\left(\frac{2\omega t - 2kr + \pi}{2}\right)}[(k_y - k_z)\hat{x} + (k_z - k_x)\hat{y} + (k_x - k_y)\hat{z}]$$

$$\tilde{E} = E e^{i\delta} \Rightarrow \text{absorbe } \frac{\pi}{2} \Rightarrow \hat{E} = E e^{i(\delta - \frac{\pi}{2})}$$

$$= \frac{1}{\omega} \tilde{E} e^{-i(\omega t - kr)}[(k_y - k_z)\hat{x} + (k_z - k_x)\hat{y} + (k_x - k_y)\hat{z}]$$

$$\hat{k} \times \hat{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ k_x & k_y & k_z \\ E_x & E_y & E_z \end{vmatrix} = \hat{x}(k_y E_z - k_z E_y) + \hat{y}(k_z E_x - k_x E_z) + \hat{z}(k_x E_y - k_y E_x)$$

for $E_x, E_y, E_z = Re[\tilde{E} e^{-i(\omega t - (k_x + k_y + k_z) r)}]$

$$= \tilde{E} e^{-i(\omega t - kr)} [(k_y - k_z)\hat{x} + (k_z - k_x)\hat{y} + (k_x - k_y)\hat{z}$$

from $\tilde{B} = \frac{1}{\omega} \underbrace{\tilde{E} e^{-i(\omega t - kr)} [(k_y - k_z)\hat{x} + (k_z - k_x)\hat{y} + (k_x - k_y)\hat{z}]}_{\hat{k} \times \tilde{E}}$

using $k = |k| \hat{k}$

$$\tilde{B} = \frac{k}{\omega} \hat{k} \times \tilde{E} = \frac{1}{c} \hat{k} \times \tilde{E}$$

for $\tilde{E} = E e^{i\delta}$ and $\tilde{B} = B e^{i\delta}$ and substituting above.

$$B \cancel{e^{i\delta}} = \frac{1}{c} \hat{k} \times E \cancel{e^{i\delta}} \Rightarrow B = \frac{1}{c} \hat{k} \times E$$

meaning that the magnitudes of B and E are related by

$$B = \frac{1}{c} E$$