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SDS 383 D - Statistical Modeling 2
Homework

Exercise 1.1

$$X_1, \ldots X_n \sim Bernoulli(p), \quad h(p) \sim Beta(\alpha, \beta)$$

$$
\begin{aligned}
p|X_1, \ldots X_N \propto & f(x_1, \ldots x_n|p) \cdot h(p) \\
\propto & p^{\sum x_i}(1-p)^{n-\sum x_i} \cdot p^{\alpha-1}(1-p)^{\beta-1} \\
\propto & p^{\sum x_i + \alpha - 1}(1-p)^{\beta + n - \sum x_i - 1}
\end{aligned}
$$

$$p|X_1, \ldots X_N \sim Beta(\sum x_i + \alpha, \beta + n - \sum x_i)$$

Exercise 1.2

$$X_1, \ldots X_N \overset{iid}{\sim} Cat(p), \quad p = (p_1, \ldots p_K), \quad f(x|p) = \prod_{i=1}^{n} \prod_{k=1}^{K} p_k^{I(x_i=k)}$$
$$p \sim Dirichlet(\alpha_1, \ldots \alpha_K)$$

$$\pi(p) = \frac{\Gamma(\sum_k \alpha_k)}{\prod_K \Gamma(\alpha_k)} \prod_{k=1}^{K} p^{\alpha_k - 1} I(\sum_{k=1}^{K-1} p_k \leq 1, p_k \geq 0)$$

$$
\begin{aligned}
p|X_{1:n} \propto & f(x_{1:n}|p) \cdot \pi(p) \\
\propto & \prod_{i=1}^{n} \prod_{k=1}^{K} p_k^{I(x_i=k)} \cdot \prod_{k=1}^{K} p_k^{\alpha_k - 1} \\
\propto & \prod_{k=1}^{K} p_K^{n_k + \alpha_k - 1}, \quad \text{where } n_k = \sum_{i=1}^{N} I(x_i = k) \\
p|X_{1:n} \sim & Dir(n_1 + \alpha_1, \ldots n_k + \alpha_k)
\end{aligned}
$$

Exercise 1.3

$X \sim Gam(\alpha, b)$ and $Y \sim Gam(\beta, b)$ and X is independent of Y.

1.3.a

Let $W = X+Y$ and $Z = X/(X+Y)$. Show that W and Z are independent with $W \sim Gam(\alpha + \beta, b)$ and $Z \sim Beta(\alpha, \beta)$.

This is a bivariate random variable transformation:

Change of variables equation: $f_{w,z} = f_{x,y}(h_1(w, z), h_2(w, z))|J|$

$g_1(X, Y) = W = X + Y$

$g_2(X, Y) = Z = X/(X + Y)$

$h_1(Z, W) = X = ZW$

$h_2(Z, W) = Y = W(1 - Z)$

$|J| = abs\left(\begin{vmatrix} \frac{\partial X}{\partial W} & \frac{\partial X}{\partial Z} \\ \frac{\partial Y}{\partial W} & \frac{\partial Y}{\partial Z} \end{vmatrix}\right) = \begin{vmatrix} Z & W \\ 1 - Z & -W \end{vmatrix} = abs(-w) = w$

$f_{w,z} = f_x(zw)f_y(w - zw)w$

$f_{w,z} = \dfrac{b^\alpha(zw)^{\alpha-1}e^{-bzw} \cdot b^\beta(w(1 - z))^{\beta-1}e^{-bw(1-z)}w}{\Gamma(\alpha)\Gamma(\beta)} \cdot I(zw \geq 0) \cdot I(w - zw \geq 0)$

$f_{w,z} = \dfrac{w^{\alpha+\beta-1}b^{\alpha+\beta}e^{-bw} \cdot z^{\alpha-1}(1 - z)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \cdot I(0 \leq z \leq 1) \cdot I(w \geq 0)$

$f_{w,z} = \dfrac{b^{\alpha+\beta}w^{\alpha+\beta-1}e^{-bw}I(w \geq 0)}{\Gamma(\alpha + \beta)} \cdot \dfrac{\Gamma(\alpha + \beta)z^{\alpha-1}(1 - z)^{\beta-1}I(0 \leq z \leq 1)}{\Gamma(\alpha)\Gamma(\beta)}$

So W and Z are independent by factorization theorem and $W \sim Gam(\alpha + \beta, b)$ and $Z \sim Beta(\alpha, \beta)$

1.3.b

Explain how to use a random gamma sampler to sample from a $Beta(\alpha, \beta)$ random variable using R:

We can use the R function (rgamma(n, shape, rate)) to sample two gamma random variables with shape parameters $\alpha = 10, \beta = 2$ and rate parameters equal $b = 5, b = 5$ and then use the transformation from the previous step to get a sample from a $Beta(\alpha = 10, \beta = 2)$ distribution.

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x <- rgamma(1, 10, 5)
y <- rgamma(1, 2, 5)
z = x / (x + y)
```

Exercise 1.4