Logic, Mathematics, Physics and Philosophy

Jonathan Ginsburg

September 22, 2014

Prologue

The title of the present text was thought to convey the order of increasing complexity of the mentioned concepts, which will be discussed throughout. However, we recognize that philosophy is more fundamental than the other three, but we placed it at the end due to the scope of book and the specificity of its orientation.

This treatise has two objectives, which correspond to personal thought clarification and note taking. These are notation and diction reference in logic, mathematics, physics, and philosophy; and the elaboration of selected topics related to the mentioned fields of study.

Throughout this text, when logic is mentioned it refers to Classical Logic, which dates back to Aristotle as perceived from his Organon. This formal discipline differs from other formal logics (e.g. Fuzzy Logic) in its foundational laws. They are called The Three Classical Laws of Thought. In Latin they are: Omne quod est, est; Nihil potest esse et non esse; and, Aut est aut non omne. The first is the Identity Law, from which we understand that everything is logically equivalent to itself. The second is the Law of Noncontradiction, which states that nothing can be (true) and not be (true) under the same circumstances. The last one is The Law of the Excluded Middle, which says that everything either is (true) or is not (true) [Boole, 1854].

In the first chapter a few terms are clarified using natural language.

Begun on October 6, 2014.
J.G.

Contents

Chapter 1

Glossary

This chapter must serve as a convention establishment of language between the text and the reader. It is in the form of a glossary and divided in four parts: General, Logic, Mathematics and Physics. The order of terms does not correspond to the alphabet. Instead it follows that of increasing complexity, namely, basic terms come first. Let us agree that fundamental words cannot be explained, but come naturally to the human thought.

1.1 General

- Definition: human assignment of meaning to a statement, concept or expression.
- Formal: that allows no ambiguity.
- Logic: formalization of human understanding and reasoning.
- Abstract: human artifice to explain, from within, the external; an approximation to Plato's Theory of Ideas.
- Theory: abstract thought for explaining.
- Fundamental: the starting point for building a theory.
- Primitive notion: fundamental abstraction, which cannot be defined through other concepts.
- Mathematics: the logic of measurements, count, sets, groups, et. al.
- Symbol: glyph used to externalize and record knowledge, understanding and reasoning.
- Valid: follows established rules.

- Operator: tool that builds an instance, called the return value, out of other instances, called operands.
- Unary operator: operator that uses one instance to build a new one.
- Binary operator: operator that uses two instances to build a new one.
- Order of precedence: a defined sequential hierarchy of operator applications to an expression containing more than one operator.

1.2 Logic

- Truth value: true or false.
- Proposition: statement with truth value.
- Predicate (Propositional function): statement that has unspecified variables, to which a truth value can be assigned if the unspecified becomes specified.
- Logical operator: tool that returns a new proposition, often called molecular proposition, out of a given number of propositions, often called atomic propositions, on which it operates.
- Truth table: table that indicates the truth value of the proposition obtained through the application of a logical operator, or a combination of these, over all the possible permutations of the truth values of the operated propositions.
- Tautology: (1) proposition that always has a truth value of true; (2) molecular proposition that has a truth table of true values for all the possible permutations of truth values of the given atomic propositions that are operated on.
- Absurd: (1) proposition that always has a truth value of false; (2) molecular proposition that has a truth table of false values for all the possible permutations of truth values of the given atomic propositions that are operated on.
- Conditional proposition: a proposition obtained through the application of the conditional logical binary operator.
- Rule of Inference: elaborated, and often named, conditional proposition that is a tautology, which always establishes true consequences out of true premises.
- Reasoning: logically formal and correct procedure of the application of rules of inference to a collection of premises.
- Argumentation: explicit reasoning; usually written.

- Proof: argumentation that ends in a specific, desired proposition.
- Conclusion: proposition obtained through a proof.
- Logical indicator: a symbol that alludes to logical operators and/or proofs.
- Logical equivalence: (1) the characteristic of two propositions that refer to the same thing (i.e. two names for the same abstraction); (2) property of two propositions that always coincide in their truth value.
- Logical definition: (1) human assignment of logical equivalence to a pair of propositions; (2) human assignment of name to a variable or variables of a predicate, if such predicate results true.
- Logical definition of truth value: human assignment of truth value to a given proposition.
- Axiom: human postulated proposition that generally appeals to the intuition.
- Theorem: often named, notable conclusion based on axioms.
- Corollary: conclusion based on theorems.
- Lemma: a logically indicated proposition used in an argument, as part of the premises.

1.3 Mathematics

1.3.1 Generalities

- Numerical value: a primitive notion of measurement or count.
- Number: a symbol that has numerical value.
- Scalar: a number.
- Mathematical object: number, set, vector, scalar, etc.
- Mathematical value: (1) numerical value; (2) primitive notion of essence of a mathematical object.
- Numerical definition: (1) human assignment of numerical value to a number; (2) human assignment of mathematical value to a mathematical object.
- Numerical equivalence: the characteristic of two numbers that have the same numerical value (i.e. two symbols for the same count or measurement).

- Arithmetical operator: tool that returns a number out of a given count of numbers.
- Logical-numerical operator: tool that returns a proposition out of a given count of numbers.

1.3.2 Set Theory

- Set Theory: the branch of mathematics that studies sets.
- Set: a primitive notion of a collection of distinct mathematical objects, called elements [Herstein, 2006, cfr. Preliminary Notions].
- Set operator: tool that returns a set out of a given number of sets.
- Logical-set operator: tool that returns a proposition out of a given number of sets.
- Numerical-set operator: tool that returns a number out of a given number of sets.
- Logical-mathematical operator: logical-set or logical-numerical operator.
- Universe: a primitive notion of a set containing all the instances in nature.

1.3.3 Number Classification

Missing...§ 1.1

1.3.4 Geometry

- Geometry: branch of mathematics that studies figures' shape, relative position and size.
- Coordinate: a number used to specify the place of a point in a space.
- Point: a primitive notion of a dimensionless geometrical object, of which a place in a space can be given with the use of coordinates.
- Dimension [of a space or object]: the minimum number of coordinates needed to specify a point within it.
- Origin: the point where normal axes for coordinate description join.
- Line: a collection of continuous points that builds a single-dimensional space.
- Length: the magnitude of the difference between the two coordinates of the end-points of a figure in single-dimensional space.
- Plane: two-dimensional space.

- Angle (Planar angle): the amount of opening of two lines or planes that intersect; measured with radians.
- Radian: the length of an arc, delimited by the intersection of two lines or planes of which the angular measure is to be given, of a unitary circle with center positioned at a point where the two lines or planes join.
- Solid Angle: the amount of opening of two lines that join at a reference point, of which each comes from an extremity of a certain object in three-dimensional space; measured with steradians.
- Steradian: the surface area of a unitary sphere, centered at a reference point, described by a cone's, built with vertex at the reference point and the lines to which a solid angular measurement is to be given, intersection with it.
- Magnitude: a positive number.
- Direction: a planar angle.
- Sense: positive or negative sign.
- Vector: a geometrical object that is described through two properties, namely, a sensed magnitude and a collection of directions.

1.3.5 Maps

- Map: rule that assigns mathematical objects to a given number of other mathematical objects.
- Function: a map that assigns a unique mathematical object to a given number of other mathematical objects.

1.4 Physics

1.4.1 Generalities

- Phenomenon: any event in nature.
- Mass: property of instances of nature that oppose acceleration and allow gravitational interaction.
- Space: the, hitherto assumed infinite, three-dimensional scenario where phenomena take place.
- Reference frame: an arbitrary selection of a spatial point used for geometrical descriptions of phenomena.

- Matter: anything in the universe that occupies space and has mass.
- Body: finite-sized matter.
- Position: a vector that goes from a reference frame to a point particle or a body's center of mass.
- Center of mass: the mass weighted vectorial mean of positions of point particles that build a body.
- Velocity: the change of position in time.
- Acceleration: the change of velocity in time.
- Inertial reference frame: a reference frame that has no change in its velocity.
- Momentum: the amount of movement that a body has; mathematically defined as mass times velocity.
- Force: the change of momentum in time.
- Fundamental Interactions: the most fundamental forces known (i.e. gravitational, electromagnetic, weak nuclear and strong nuclear).

Chapter 2

Notation Conventions

Although any symbol can be used for unspesified instances of logic, mathematics and physics, there are certain conventions that help with avoiding confusion. In this book, this notation conventions will be followed, unless specified otherwise.

2.1 Logic

2.1.1 General Instances

- Proposition: represented with lower-case, Greek letters (e.g. χ).
- Predicate or propositional function: represented with a lower-case, Greek letter followed by a parenthesis containing a list (separated by commas) of Roman, lower-case letters that represent the unspecified variables (e.g. $\chi(x, y, z)$).

2.1.2 Specific Instances

- Tautology: T_0.
- Absurd: F_0.
- True: $\top$ (LaTeX's code is `\top`) or T.
- False: $\bot$ (LaTeX's code is `\bot`) or F.

2.2 Mathematics

2.2.1 General Instances

- Set: represented with capitalized, bold Roman letters (e.g. $\mathbf{A}$).
- Number: represented with a Roman, lower-case, stylized letter (e.g. x).

- Angle: represented with lower-case, Greek letters (e.g. α).
- Solid Angle: represented with upper-case, Greek letters (e.g. Ω).
- Vector: represented with Greco-Roman letters with a right-pointing arrow above (e.g. $\vec{v}$); LaTeX's code is `\vec{v}`.
- Function: represented with Roman, stylized letters, usually beginning with f and forward (e.g. f, g, h, etc.).

2.2.2 Constants and Specific Instances

Geometry and Algebra

- Ratio of a circle's circumference to its diameter: $\pi \approx 3.141592$.
- Imaginary unit: $i := \sqrt{-1}$.

Set Theory

- Universe: $\mathbb{U}$.

Number Classification

- Natural Numbers: $\mathbb{N}$.
- Integers: $\mathbb{Z}$, due to their name in German (i.e. Zahlen).
- Real Numbers: $\mathbb{R}$.
- Prime Numbers: $\mathbb{P}$.
- Rational Numbers: $\mathbb{Q}$.
- Irrational Numbers: $\mathbb{Q}'$.

2.3 Physics

2.3.1 Constants and Specific Instances

Mechanics

- Position: $\vec{r}$.
- Velocity: $\vec{v}$.
- Acceleration: $\vec{a}$.
- Angular velocity: $\vec{\omega}$.

Physical Constants

- Gravitational constant: $G \approx 6.67 \times 10^{-11}[\frac{\mathrm{N\cdot m^2}}{\mathrm{kg^2}}]$.

Chapter 3

Symbology and Notation

3.1 Logic

3.1.1 Logical Definition

Symbols	Definition	LaTeX Code
$\chi :\iff \psi$	Logical definition for propositions χ and ψ (i.e. χ and ψ are names for the same thing, ergo $\chi \longleftrightarrow \psi$ is a tautology)	`:\iff`

3.1.2 Propositional Logical Operators

Logical operators, listed in order of precedence, of which each has a defined truth table.

Symbols	Definition	LaTeX Code
$\neg\chi$	Operator for the negated proposition χ	`\neg`
$\chi \land \psi$	Operator for the conjunction of propositions χ and ψ	`\land`
$\chi \lor \psi$	Operator for the disjunction of propositions χ and ψ	`\lor`
$\chi \lor! \psi$	Operator for the exclusive disjunction of propositions χ and ψ	`\ \lor! \`
$\chi \to \psi$	Operator for the conditional of premises or antecedents χ to consequences or conclusions ψ	`\to`
$\chi \longleftrightarrow \psi$	Operator for the logical equivalence of propositions χ and ψ (i.e. $\chi \longleftrightarrow \psi :\iff \chi \to \psi \land \psi \to \chi$)	`\longleftrightarrow`

3.1.3 Logical Quantifiers

Symbols	Definition	LaTeX Code
$(\forall x)(\chi(x))$	Proposition that affirms that for every x in the universe, $\chi(x)$ is true	`\forall`
$(\exists x)(\chi(x))$	Proposition that affirms that there exists at least one x in the universe, for which $\chi(x)$ is true	`\exists`
$(\exists! x)(\chi(x))$	Proposition that affirms that there exists only one x in the universe, for which $\chi(x)$ is true	`\exists !`
$(\nexists x)(\chi(x))$	Proposition that affirms that there does not exist an x in the universe, for which $\chi(x)$ is true (i.e. $\neg(\exists x)(\chi(x))$)	`\nexists`

3.1.4 Nested Logical Quantifiers

Symbols	Definition	LaTeX Code
$(\forall x, y, \cdots)(\chi(x, y, \cdots))$	Proposition that affirms that for every combination of $x, y, \cdots$ in the universe, $\chi(x, y, \cdots)$ is true	`\forall`
$(\exists x, y, \cdots)(\chi(x, y, \cdots))$	Proposition that affirms that there is at least a combination $x, y, \cdots$ in the universe, for which $\chi(x, y, \cdots)$ is true	`\exists`
$(\exists! x, y, \cdots)(\chi(x, y, \cdots))$	Proposition that affirms that there is only one combination $x, y, \cdots$ in the universe, for which $\chi(x, y, \cdots)$ is true	`\exists !`

3.1.5 Logical Indicators

Symbols	Definition	LaTeX Code
$\chi \implies \psi$	Logical indicator that alludes to $\chi \to \psi$ being true, without showing explicitly how	`\implies`
$\chi \models \psi$	Logical indicator that alludes to $\chi \to \psi$ being true, without showing explicitly how	`\models`
$\chi \vdash_i \psi$	Logical indicator that alludes to $\chi \to \psi$ being true, which is obtained through proof i	`\vdash_i`
$\chi \iff \psi$	Logical indicator that alludes to $\chi \longleftrightarrow \psi$ being true, without showing explicitly how	`\iff`

3.2 Mathematics

3.2.1 Numerical Definition

Symbols	Definition	LaTeX Code
$x := y$	Numerical definition for symbol x and number y (i.e. x is defined to have the same numerical value as y)	`:=`

3.2.2 Arithmetical Operators

Symbols	Definition	LaTeX Code
$x + y$	Addition that returns $x+y$, the sum of numbers x, the augend, and y, the addend	`+`
$x - y$	Subtraction that returns $x-y$, the difference of numbers x, the minuend, and y, the subtrahend	`-`
$x \times y$	Multiplication that returns $x \times y$, the product of numbers x, the multiplicand, and y, the multiplier (i.e. $x \times y := \underbrace{y + y + \cdots + y}_{x \text{ times}}$)	`\times`
$x \div y$ or $\frac{x}{y}$	Division that returns $x \div y$, the quotient of numbers x, the numerator or dividend, and y, the denominator or divisor (i.e. $(x \times y) \div y = x :\iff \underbrace{y + y + \cdots + y}_{x \text{ times}} = x \times y$)	`\div` or `\frac{x}{y}`
x^y	Exponentiation that returns x^y, the power of number x, the base, to number y, the exponent (i.e. $x^y := \underbrace{x \times x \times \cdots \times x}_{y \text{ times}}$)	`^`
$\sqrt[y]{x}$ or $x^{\frac{1}{y}}$	y-th root that returns $\sqrt[y]{x}$, the y degree root of number x, the radicand (i.e. $\sqrt[y]{x^y} = x :\iff \underbrace{x \times x \times \cdots \times x}_{y \text{ times}} = x^y$)	`\sqrt[y]{x}`
$\sqrt{x}$	Square root of number x (i.e. $\sqrt[2]{x}$)	`\sqrt{}`
$\lvert x \rvert$	Absolute value of number x (i.e. $\sqrt{x^2}$)	`|x|`

3.2.3 Logical-Numerical Operators

Symbols	Definition	LaTeX Code
$x > y$	Number x is greater than number y, which is logically equivalent to $y < x$	`>`
$x < y$	Number x is smaller than number y, which is logically equivalent to $y > x$	`<`
$x = y$	Number x is equal to number y	`=`
$x \approx y$	Number x is relatively approximately equal to number y	`\approx`
$x \geq y$	$x > y \vee! \; x = y$	`\geq`
$x \leq y$	$x < y \vee! \; x = y$	`\leq`

3.2.4 Set Definition

Symbols	Definition	LaTeX Code
$\mathbf{A} := \mathbf{B}$	Set definition for sets $\mathbf{A}$ and $\mathbf{B}$ (i.e. $\mathbf{A}$ is a set built with the same elements as $\mathbf{B}$)	`:=`

3.2.5 Logical-Set Operators

Symbols	Definition	LaTeX Code
$x \in \mathbf{A}$	Element x belongs to $\mathbf{A}$	`\in`
$x, y, \cdots \in \mathbf{A}$	Elements $x, y, \cdots$ belong to $\mathbf{A}$	`\in`
$x \notin \mathbf{A}$	Element x does not belong to $\mathbf{A}$ (i.e. $\neg(x \in \mathbf{A})$)	`\notin`
$\mathbf{A} = \mathbf{B}$	Set $\mathbf{A}$ is equal to set $\mathbf{B}$ (i.e. $(\forall x)(x \in \mathbf{A} \longleftrightarrow x \in \mathbf{B})$ is true)	`=`
$\mathbf{A} \subset \mathbf{B}$	$\mathbf{A}$ is a subset of $\mathbf{B}$ (i.e. $(\forall x)(x \in \mathbf{A} \rightarrow x \in \mathbf{B})$)	`\subset`
$\mathbf{A} \supset \mathbf{B}$	$\mathbf{A}$ is a superset of $\mathbf{B}$ (i.e. $(\forall x)(x \in \mathbf{B} \rightarrow x \in \mathbf{A})$)	`\supset`
$\mathbf{A} \subseteq \mathbf{B}$	$\mathbf{A} \subset \mathbf{B} \vee \mathbf{A} = \mathbf{B}$	`\subseteq`
$\mathbf{A} \supseteq \mathbf{B}$	$\mathbf{A} \supset \mathbf{B} \vee \mathbf{A} = \mathbf{B}$	`\supseteq`

3.2.6 Set Operators

Listed in their order of precedence.

Symbols	Definition	LaTeX Code
$\{x \in \mathbf{A} \mid \chi(x)\}$	Set operator that builds a new set, which contains all elements x that give a truth value of true for the propositional function $\chi(x) \wedge x \in \mathbf{A}$. Note that if no $\mathbf{A}$ is specified, then it is assumed that $\mathbf{A} = \mathbb{U}$.	`{	}`
$\{x_1, x_2, \cdots, x_n\}$	Set operator that builds a new set, containing elements $x_1, x_2, \cdots, x_n$	`{ \cdots }`
$\mathbf{A} \setminus \mathbf{B}$	Set subtraction (i.e. $\{x \in \mathbf{A} \mid x \notin \mathbf{B}\}$)	`\setminus`
$\mathbf{A} \cup \mathbf{B}$	Set union (i.e. $\{x \mid x \in \mathbf{A} \vee x \in \mathbf{B}\}$)	`\cup`
$\mathbf{A} \cap \mathbf{B}$	Set intersection (i.e. $\{x \mid x \in \mathbf{A} \wedge x \in \mathbf{B}\}$)	`\cap`

3.2.7 Logical Map Notation

Symbols	Definition	LaTeX Code
$f : x \mapsto y$	There is a map f, which assigns mathematical objects y to mathematical objects x, the parameters	`f: x \mapsto y`
$f : \mathbf{A} \to \mathbf{B}$	There is a map f, which assigns elements,mathematical objects, of $\mathbf{A}$, the domain, to elements,mathematical objects, of $\mathbf{B}$, the codomain or image	`\to`
$f(x)$	There is a map f, which assigns mathematical objects $f(x)$ to mathematical objects x, the parameters	`f(x)`

Chapter 4

Axioms

4.1 Mathematics

4.1.1 Properties of $\mathbb{R}$

Identity

- I) Identity $(\forall a)(a \in \mathbb{R} \rightarrow a = a)$.

Addition

- A0) Symmetry of the Addition $(\forall a, b, c)(a, b, c \in \mathbb{R} \wedge a = b \rightarrow a + c = b + c)$.
- A1) Stability or Closure of the Addition over $\mathbb{R}$ $(\forall a, b)(a, b \in \mathbb{R} \rightarrow (\exists!(a+b))(a + b \in \mathbb{R}))$.
- A2) Associativity $(\forall a, b, c)(a, b, c \in \mathbb{R} \rightarrow (a + b) + c = a + (b + c))$.
- A3) Addition Neuter $(\forall a)(a \in \mathbb{R} \rightarrow (\exists!0)(0 \in \mathbb{R} \wedge a + 0 = a))$.
- A4) Addition Inverse $(\forall a)(a \in \mathbb{R} \rightarrow (\exists!(-a))((-a) \in \mathbb{R} \wedge a + (-a) = 0))$.
- A5) Commutativity $(\forall a, b)(a, b \in \mathbb{R} \rightarrow a + b = b + a)$.

Multiplication

- M0) Symmetry of the Multiplication $(\forall a, b, c)(a, b, c \in \mathbb{R} \wedge a = b \rightarrow ac = bc)$.
- M1) Stability or Closure of the Multiplication over $\mathbb{R}$ $(\forall a, b)(a, b \in \mathbb{R} \rightarrow (\exists!(ab))(ab \in \mathbb{R}))$.
- M2) Associativity $(\forall a, b, c)(a, b, c \in \mathbb{R} \rightarrow (ab)c = a(bc))$.
- M3) Multiplication Neuter $(\forall a)(a \in \mathbb{R} \rightarrow (\exists!1)(1 \in \mathbb{R} \wedge a1 = a))$.

- M4) Multiplication Inverse $(\forall a)(a \in \mathbb{R} \wedge \neg(a = 0) \rightarrow (\exists!(a^{-1}))((a^{-1}) \in \mathbb{R} \wedge a(a^{-1}) = 1))$.
- M5) Commutativity $(\forall a, b)(a, b \in \mathbb{R} \rightarrow ab = ba)$.

Distributivity

- AM) Distributivity $(\forall a, b, c)(a, b, c \in \mathbb{R} \rightarrow a(b + c) = ab + ac)$.

Ordering

Note: $(\forall x)(x \in \mathbb{P} :\iff x \in \mathbb{R} \wedge x > 0)$. We call $\mathbb{P}$ the real positives in this sub subsection.

- Or1) Trichotomy $(\forall a)(a \in \mathbb{R} \rightarrow a = 0 \vee! \; a > 0 \vee! \; a < 0)$.
- Or2) Closure of the Addition over $\mathbb{P}$ $(\forall a, b)(a, b \in \mathbb{P} \rightarrow a + b \in \mathbb{P})$.
- Or3) Closure of the Multiplication over $\mathbb{P}$ $(\forall a, b)(a, b \in \mathbb{P} \rightarrow ab \in \mathbb{P})$.

Chapter 5

Definitions

5.1 Mathematics

Definition 5.1.1. $(\forall a, -b)(a, -b \in \mathbb{R} \rightarrow a - b := a + (-b))$

Chapter 6

Proof Compendium

6.1 Mathematics

As we progress in the number of proofs of this section, more rules of inference, definitions and axioms will be considered implicit.

6.1.1 First Order Theorems for $\mathbb{R}$

Theorem 6.1.1.1. $(\forall x)(x \in \mathbb{R} \to 0x = 0)$

Proof.

1. $x \in \mathbb{R}$, premise.
   1.1. $(\forall a)(a \in \mathbb{R} \to (\exists!0)(0 \in \mathbb{R} \wedge a + 0 = a))$, A3.
   1.2. $x \in \mathbb{R} \to (\exists!0)(0 \in \mathbb{R} \wedge x + 0 = x)$, Universal Specification of 1.1 with (x/a).
   1.3. $(\exists!0)(0 \in \mathbb{R} \wedge x + 0 = x)$, Modus Ponendo Ponens (MP) of 1.2 and 1.
   1.4. $0 \in \mathbb{R} \wedge x + 0 = x$, Existential Specification of 1.3.
   1.5. $0 \in \mathbb{R}$, Simplification of 1.4.
   1.6. $x + 0 = x$, Simplification of 1.4.
   1.7. $x, 0 \in \mathbb{R}$, Conjunction of 1.5 and 1.
   1.8. $(\forall a, b)(a, b \in \mathbb{R} \to (\exists!(ab))(ab \in \mathbb{R}))$, M1.
   1.9. $x, 0 \in \mathbb{R} \to (\exists!(x0))(x0 \in \mathbb{R})$, Universal Specification of 1.8 with $(x/a, 0/b)$.
   1.10. $(\exists!(x0))(x0 \in \mathbb{R})$, Modus Ponendo Ponens (MP) of 1.7 and 1.9.
   1.11. $x0 \in \mathbb{R}$, Existential Specification of 1.10.
   1.12. $(\forall a)(a \in \mathbb{R} \to a = a)$, Identity (I).
   1.13. $x0 \in \mathbb{R} \to x0 = x0$, Universal Specification of 1.12 with $(x0/a)$.
   1.14. $x0 = x0$, Modus Ponendo Ponens (MP) of 1.11 and 1.13.
   1.15. $(\forall a, b, c)(a, b, c \in \mathbb{R} \wedge a = b \to a + c = b + c)$, A0.
   1.16. $x0 \in \mathbb{R} \wedge x0 = x0 \to x0 + x0 = x0 + x0$, Universal Specification of 1.15 with $(x0/a, x0/b, x0/c)$.

1.17. $x0 \in \mathbb{R} \wedge x0 = x0$, Conjunction of 1.11 and 1.14.
1.18. $x0 + x0 = x0 + x0$, Modus Ponendo Ponens (MP) of 1.16 and 1.17.
1.19. $(\forall a, b, c)(a, b, c \in \mathbb{R} \rightarrow a(b + c) = ab + ac)$, AM.
1.20. $x, 0 \in \mathbb{R} \rightarrow x(0 + 0) = x0 + x0$, Universal Specification of 1.19 with $(x/a, 0/b, 0/c)$.
1.21. $x(0 + 0) = x0 + x0$, Modus Ponendo Ponens (MP) of 1.7 and 1.20.
1.22. $0 \in \mathbb{R} \rightarrow (\exists!0)(0 \in \mathbb{R} \wedge 0 + 0 = 0)$, Universal Specification of 1.1 with $(0/a)$.
1.23. $(\exists!0)(0 \in \mathbb{R} \wedge 0 + 0 = 0)$, Modus Ponendo Ponens (MP) of 1.5 and 1.22.
1.24. $0 \in \mathbb{R} \wedge 0 + 0 = 0$, Existential Specification of 1.23.
1.25. $0 + 0 = 0$, Simplification of 1.24.
1.26. $x0 = x0 + x0$, Identity of 1.21 and 1.25. ??????
1.27. $(\forall a, b)(a, b \in \mathbb{R} \rightarrow ab = ba)$, M5.
1.28. $x, 0 \in \mathbb{R} \rightarrow x0 = 0x$, Universal Specification of 1.27 with $(x/a, 0/b)$.
1.29. $x0 = 0x$, Modus Ponendo Ponens (MP) of 1.7 and 1.28.
1.30. $0x = 0x + 0x$, Identity of 1.26 and 1.29. ??????
1.31. $(\forall a)(a \in \mathbb{R} \rightarrow (\exists!(-a))((-a) \in \mathbb{R} \wedge a + (-a) = 0))$, A4.
1.32. $0x \in \mathbb{R} \rightarrow (\exists!(-0x))((-0x) \in \mathbb{R} \wedge 0x + (-0x) = 0)$, Universal Specification of 1.31 with $(0x/a)$.
1.33. $0x \in \mathbb{R}$, Identity 1.11 and 1.29. ??????
1.34. $(\exists!(-0x))((-0x) \in \mathbb{R} \wedge 0x + (-0x) = 0)$, Modus Ponendo Ponens (MP) of 1.32 and 1.33.
1.35. $(-0x) \in \mathbb{R} \wedge 0x + (-0x) = 0$, Existential Specification of 1.34.
1.36. $(-0x) \in \mathbb{R}$, Simplification of 1.35.
1.37. $0x + (-0x) = 0$, Simplification of 1.35.
1.38. $0x, 0x + 0x, -0x \in \mathbb{R} \wedge 0x = 0x + 0x \rightarrow 0x + (-0x) = 0x + 0x + (-0x)$, Universal Specification of 1.15 with $(0x/a, (0x + 0x)/b, -0x/c)$.
1.39. $(\forall a, b)(a, b \in \mathbb{R} \rightarrow (\exists!(a + b))(a + b \in \mathbb{R}))$, A1.
1.40. $0x \in \mathbb{R} \rightarrow (\exists!(0x + 0x))(0x + 0x \in \mathbb{R}))$, Universal Specification of 1.39 with $(0x/a, 0x/b)$.
1.41. $(\exists!(0x + 0x))(0x + 0x \in \mathbb{R}))$, Modus Ponendo Ponens (MP) of 1.40 and 1.33.
1.42. $0x + 0x \in \mathbb{R}$, Existential Specification of 1.41.
1.43. $0x, 0x + 0x, -0x \in \mathbb{R} \wedge 0x = 0x + 0x$, Conjunction of 1.33, 1.42, 1.36 and 1.30.
1.44. $0x + (-0x) = 0x + 0x + (-0x)$, Modus Ponendo Ponens (MP) of 1.43 and 1.38.
1.45. $0 = 0x + 0$, Identity of 1.37 and 1.44. ??????
1.46. $0x \in \mathbb{R} \rightarrow (\exists!0)(0 \in \mathbb{R} \wedge 0x + 0 = 0x))$, Universal Specification of 1.1 with $(0x/a)$.
1.47. $(\exists!0)(0 \in \mathbb{R} \wedge 0x + 0 = 0x))$, Modus Ponendo Ponens (MP) of 1.33 and 1.46.
1.48. $0 \in \mathbb{R} \wedge 0x + 0 = 0x$, Existential Specification of 1.47.
1.49. $0x + 0 = 0x$, Simplification of 1.48.

1.50. $0x = 0$, Identity of 1.45 and 1.49. ??????

2. $x \in \mathbb{R} \to 0x = 0$ Conditional Proposition from 1 to 1.50.
3. $(\forall x)(x \in \mathbb{R} \to 0x = 0)$ Universal Generalization of 2.

$\therefore (\forall x)(x \in \mathbb{R} \to 0x = 0)$ $\square$

Bibliography

[Boole, 1854] Boole, G. (1854). *An investigation of the laws of thought: on which are founded the mathematical theories of logic and probabilities*, volume 2. Walton and Maberly.

[Herstein, 2006] Herstein, I. N. (2006). *Topics in algebra*. John Wiley & Sons.