# Translation Association Schemes

A brief introduction

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# 1 Introduction

This document attemps to provide an introduction to the topic of translation association schemes. During my bachelor's thesis, which dealt with rank-metric codes, I had to learn from scratch all of the main identities and theorems of translation association schemes, and I found that there was not much content available on the internet. Furthermore, some of the main sources regarding them (which are cited throughout this document) were very hard to come by, and I could only get access to them researchers at my university who happened to have them. The content presented here is mostly self-contained, only requiring basic concepts of group theory and linear algebra which anyone studying a bachelor's in mathematics will be familiar with; hopefully it will be useful to the reader who is not yet familiar with association schemes and wants a quick but detailed overview of the properties of translation association schemes, with all the necessary proofs.

Association schemes are one of the more relevant objects of algebraic combinatorics. They are no more than special partitions of finite sets, but their particular structure has been studied thoroughly and a there is a very rich set of results and identities.

We start by defining association schemes.

**Definition 1.** Let $X$ be a finite set. An **association scheme** with $d$ classes is a pair $(X, \mathcal{R})$ such that

(i) $\mathcal{R} = \{R_0, R_1, \dots, R_d\}$ is a partition of $X \times X$;

(ii) $R_0 = \Delta := \{(x,x) \mid x \in X\}$;

(iii) For any $i \in \{0, 1, \dots, d\}$, $R_i^T := \{(y,x) \mid (x,y) \in R_i\}$ is a class in $\mathcal{R}$;

(iv) For any $i, j, k \in \{0, 1, \dots, d\}$ there is a constant, $p_{ij}^k$, such that, for any $(x,y) \in R_k$, $p_{ij}^k$ counts the number of $z \in X$ such that $(x,z) \in R_i$ and $(z,y) \in R_j$;

(v) For all $i, j, k \in \{0, 1, \dots, d\}$, $p_{ij}^k = p_{ji}^k$.

Some authors do not assume property (v), refer association schemes with this property as commutative association schemes. In this section we will assume all association schemes to be commutative.

In coding theory, we are usually working with association schemes where the relation is symmetric:

**Definition 2.** We say that an association scheme is **symmetric** if, instead of (iii), it holds that

$$R_i = R_i^T, \forall i \in \{0, 1, \dots, d\}$$

Unless specified otherwise, any association schemes mentioned from now on will be symmetric[1]. Note that for symmetric association schemes, (v) is not an axiom but rather a consequence of all the other axioms.

# 2 The theory of association schemes

In this section, we will go over the existence of the basis of idempotent matrices in the Bose-Mesner algebra. To do this, we will be mostly following Godsil's [2] which, instead of relying on know results of linear algebra, gives a more combinatorial construction of the basis of idempotents. Brouwer et al.'s [3] and and Van Lint and Wilson's [4] were also consulted to obtain some intermediate results.

In symmetric association schemes, the number $n_i := p_{ii}^0$, called the **valency** of $R_i$, is the number of $z \in X$ such that $(x, z) \in R_i$. Therefore, it holds that

$$n :=\mid X \mid= \sum_{i=0}^{d} n_{ii}^0$$

Note that $n_0 = 1$.

We now state some useful properties of the coefficients of association schemes.

**Lemma 3** ([3, Lemma 2.1.1]). *The parameters $n_i$ and $p_{ij}^k$ of an association scheme with $d$ classes satisfy the following:*

*(i)* $p_{0j}^k = \delta_{jk}$,

*(ii)* $p_{ij}^0 = \delta_{ij} n_j$,

*(iii)* $p_{ij}^k = p_{ji}^k$,

*Proof.* (i) Note that $(x, z) \in R_0 \iff x = z$. If $j = k$, then it is clear that $p_{0j}^k = 0$. If $j \neq 0$, then there is one $(x, y) \in R_k - R_j$, so for that $(x, y)$ there are no $z \in X$ such that $(x, z) = (x, x) \in R_0$ and $(z, y) = (x, y) \in R_j$, therefore $p_{0j}^k$.

(ii) If $i = j$, by definition $p_{jj}^0 = n_j$. Suppose $i \neq j$ and $p_{ij}^k >= 1$. Then, for every $x \in X$, $(x, x) \in R_0$ and there is at least one $z \in X$ such that $(x, z) \in R_i \iff (z, x) \in R_i$ and $(z, x) \in R_j$, so $(z, x) \in R_i \cap R_j$, which is a contradiction with the fact that $\mathcal{R}$ is a partition of $X \times X$.

[1] For a very thorough introduction to non-symmetric association schemes and related algebraic concepts, see University of Waterloo's notes from Godsil [1]

(iii) Given $i, j, k \in \{0, 1, \ldots, d\}$, for any $(x, y) \in R_k \iff (y, x) \in R_k$, $(x, z) \in R_i \iff (z, x) \in R_i$ and $(z, y) \in R_j \iff (y, z) \in R_j$, so it is clear that $p_{ij}^k = p_{ji}^k$.

■

One can view an association scheme as a complete graph with labeled edge: the vertices are the elements in $X$ and there is an edge with label $i$ between two vertices $x, y \in X$ if and only if $(x, y) \in R_i$. The subgraph obtained by only selecting the edges labeled $i$ is usually denoted by $X_i$.

We define the **adjancecy matrices** of an association scheme as the association matrices of each subgraph $X_i$, that is,

$$\begin{cases} (A_i)_{xy} = 1 & \text{if } (x, y) \in R_i \\ 0 & \text{otherwise} \end{cases}$$

These are clearly $n \times n$ symmetric matrices, and

$$A_0 = I$$

$$A_0 + A_1 + \cdots + A_d = J$$

where $I$ is the $n \times n$ identity matix and $J$ is the $n \times n$ matrix with all coefficients 1. This induces an equivalent definition of association schemes in terms of matrices, which is also commonly used in the literature ([2, Chapter 3]).

**Definition 4.** A (commutative, not necessarily symmetric) association scheme is a set of $n \times n$ $\{0, 1\}$-matrices $\mathcal{A} = \{A_0, A_1, \ldots, A_d\}$ with the following properties:

(i) $\sum_{i=1}^k = J$;

(ii) $A_0 = I$;

(iii) $A_i^T \in \mathcal{A}$ for all $i \in \{0, 1, \ldots, d\}$;

(iv) for all $i, j \in \{0, 1, \ldots, d\}$, the matrix product $A_i A_j$ is in the span of $\mathcal{A}$;

(v) $A_i A_j = A_j A_i$ for all $i$ and $j$.

It is immediate that the first three axioms of this definition are equivalent to the first three in the definition based on relations. The last two are also equivalent:

*Proof.* We define $p_{ij}^k$ as the coefficient of $A_k$ when we express $A_i A_j$ as an element in the span of $\mathcal{A}$, that is,

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$$

We now prove that this definition is equivalent to the one given previously. Note that $(A_iA_j)_{xy}$ is the number of $z \in X$ such that $(A_i)_{xz} = 1$ and $(A_j)_{zy} = 1$. From (iii), we deduce that, for every $x, y$ and every $i, j \in \{0, 1, \ldots, d\}$, either $(A_i)_{xy} = 0$ or $(A_j)_{xy} = 0$. Therefore, fixing $k, i, j \in \{0, 1, \ldots, d\}$, it is clear that for any $(A_k)_{xy}$ the number of $z \in X$ such that $(A_i)_{xz}$ and $(A_j)_{zy}$ is precisely the coefficient $p_{ij}^k$ and that this coefficient is independent of the choice of $x, y$. ∎

The definition of symmetric association schemes based on matrices is analogous. The set $\mathcal{A}$ clearly forms an algebra with respect to the matrix multiplication. Furthermore, $\mathcal{A} \cup \{0\}$ forms an algebra with respect to the **Hadamard** or **Schur** product, which is defined as

$$(A \circ B)_{xy} = A_{xy}B_{xy}$$

This is immediate from the fact that all matrices in the association scheme are Schur orthogonal and Schur idempotent, that is,

$$A_i \circ A_j = \begin{cases} A_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

It is also immediate that the matrices are linearly independent.

We will denote the linear span of $\mathcal{A}$ over the complex numbers by $\mathbb{C}(\mathcal{A})$. In the literature, $\mathbb{C}(\mathcal{A})$ is often referred to as the **Bose-Mesner** algebra of $\mathcal{A}$.

All of this is summed up in the following lemma:

**Lemma 5.** *The Bose-Mesner algebra of an association scheme $\mathcal{A}$, $\mathbb{C}(\mathcal{A})$, is a commutative matrix algebra with identity, which is closed under transposition and contains the all-ones matrix.*

The Bose-Mesner algebra is interesting because a basis of orthogonal idempotent matrices exists for it. In the literature, this is commonly deduced using known results of linear algebra, since the fact that the $A_i$ commute implies that they can be simultaneously diagonalized ([3], [4]). However, a combinatorial construction of this basis is provided in [2], which we will develop in the next couple of pages.

**Theorem 6** ([2, Theorem 3.4.1])**.** *Let $M$ be a commutative matrix algebra with identity over $\mathbb{C}$. Assume that, for all $N \in M$, it holds that $N^2 = 0$ only if $N = 0$. Then, each matrix in $M$ can be expressed as a linear combination of pairwise othogonal independents.*

*Proof.* Suppose $A \in M$. Call $m_A$ the minimal polynomial of $A$, with

$$m_A(x) = \prod_{i=1}^{k} (x - \alpha_i)^{\lambda_i}$$

We now define

$$m_{A,i}(x) = \frac{m_A(x)}{(x-\alpha_i)^{\lambda_i}}$$

Note that $m_{A,1},\dots,m_{A,k}$ are coprime by definition and that they are univariate polynomials over a field. Therefore, Bezout's identity states that there are $f_1(x),\dots,f_k(x) \in \mathbb{C}[X]$ such that

$$f_1(x)m_{A,1}(x) + \dots + f_k(x)m_{A,k}(x) = 1$$

If we replace by $A$, we find that

$$f_1(A)m_{A,1}(A) + \dots + f_k(A)m_{A,k}(A) = I \tag{1}$$

We will use this identity to find the idempotent matrices. Define $E_i = f_i(A)m_{A,i}(A)$. First of all, we see that they are pairwise orthogonal. If $i \neq j$, then it is clear that $m_{A,i}m_{A_j}$ is a multiple of $m_A$, so

$$m_{A,i}m_{A,j} = 0$$

Therefore, $E_iE_j = 0$, i.e., they are pairwise othogonal.

We can use this to see that the $E_i$ are idempotent. If we multiply 1 by $E_i$ for a given $i \in \{1,\dots,k\}$, we get that:

$$E_i(E_1 + \dots E_k) = E_iI \iff E_i^2 = E_i$$

The only thing left to show is that $A$ is a linear combination of these $E_i$. It is immediate that $(x-\alpha_i)^{\lambda_i}f_i(x)$ is a multiple of $m_A(X)$, so it holds that

$$(A-\alpha_iI)^{\lambda_i}E_i = 0$$

Using the fact that $E_i$ is idempotent,

$$((A-\alpha_iI)E_i)^{\lambda_i} = 0$$

Note that, by hypothesis, $N^2 = 0 \Rightarrow N = 0$ for every $N \in M$. This also implies that $N^n = 0 \Rightarrow N = 0$, which we can prove by induction: if we suppose that $N^k = 0 \iff N = 0$ for all $1 <= k <= n-1$

$$N^n = 0 \Rightarrow N^{n+(n \pmod 2)} = 0 \Rightarrow (N^{\frac{n+(n \pmod 2)}{2}})^2 = 0 \Rightarrow N^{\frac{n+(n \pmod 2)}{2}} = 0 \Rightarrow N = 0$$

Therefore, from the previous identity we deduce that $(A-\alpha_iI)E_i = 0 \iff AE_i = \alpha_iE_i$. Multiplying 1 by $A$, we now get that

$$A = AE_i + \dots + AE_k = \alpha_1E_1 + \dots + \alpha_kE_k$$

so $A$ can be expressed as a linear combination of pairwise othogonal idempotents. Note that from $AE_i = \alpha_iE_i$ we get that the columns of $E_i$ are eigenvectors of A with eigenvalue $\alpha_i$.

■

We will now prove that this theorem's hypothesis holds for the Bose-Mesner algebra of any association scheme.

**Lemma 7** ([2, Lemma 3.4.2])**.** *If $N$ and $N^*$ commute, then $N^2 = 0 \Rightarrow N = 0$.*

*Proof.* If $N$ and $N^2$ commute and $N^2 = 0$, then it holds that

$$0 = (N^*)^2 N^2 = (N^* N)^2$$

which implies that

$$0 = tr((N^* N)^2) = tr((N^* N)^*(N^* N))$$

It is easy to show that $tr(H * H) = 0 \Rightarrow H = 0$ for any matrix $H$ with coefficients in $\mathbb{C}$. Note that

$$(H^* H)_{ii} = \sum_{j=1}^{n} \overline{H_{ji}} H_{ji}$$

which is a positive real number. Therefore,

$$tr(H^* H) = 0 \Rightarrow \sum_{j=1}^{n} \overline{H_{ji}} H_{ji} = 0, \forall i \Rightarrow \overline{H_{ji}} H_{ji} = 0, \forall i, j \Rightarrow H_{ji} = 0, \forall i, j$$

We now use this to deduce that $N^* N = 0$. Applying the same argument, we conclude that $N = 0$. ∎

One can define a partial ordering on the idempotent matrices of a commutative algebra. For any two idempotent $E$ and $F$, we write $E \leq F$ if $FE = E$. This relation is clearly a partial order:

1. Reflexive: $E^2 = E$ because $E$ is idempotent.
2. Transitive: $FE = E$ and $GF = F$ implies $GE = GFE = FE = E$.
3. Antisymmetric: if $FE = E$ and $EF = F$, since they commute we have that $E = FE = EF = F$.

Note that the fact that $FE = E$ implies that the column space of $E$ is a subspace of the colunm space of $F$:

*Proof.* If $c = (c_1, \ldots, c_n)$ is a column in $E$, and $f_1, \ldots, f_n$ are the columns of $F$, then

$$Fc = c \iff c_1 f_1 + \cdots + c_k f_k = c$$

which implies that $c \in \langle f_1, \ldots, f_k \rangle$. ∎

We will call a **minimal idempotent** a minimal element of the set of nonzero idempotent matrices with respect to this order, i.e., an idempotent matrix $A$ is minimal if there is no other non-zero idempotent matrix $F$, $F \neq E$, such that $F \leq E$. This concept will be useful in that there is a basis of idempotent matrices for the Bose-Mesner algebra of any association scheme.

**Lemma 8** ([2, Lemma 3.4.3])**.** *Any set of idempotent matrices in a commutative algebra contains a subset of minimal idempotent matrices. Furthermore, these minimal idempotent matrices are pairwise orthogonal.*

*Proof.* Suppose that $E$ and $F$ are distinct idempotents in a commutative algebra, $E \leq F \iff FE = E$. Then,

$$F - E \neq 0 \iff F - FE \neq 0 \iff F(I - E) \neq 0$$

Since $E$ is idempotent, we have that

$$E(I - E) = 0$$

From this and the previous inequality we can deduce that the span of $E$'s columns cannot be the same as the span of $F$'s columns, and since $E \leq F$ we get that the former is strictly contained in the latter. This implies that if $E \leq F$, then the dimension of the vector subspace generated by the columns of $E$ is strictly smaller that the dimension of the vector subspace generated by the columns of $F$. Now, suppose that $E_1, \dots, E_m$ are distint non zero idempotent matrices of a commutative algebra, such that

$$E_1 \leq \cdots \leq E_m$$

hen, $m$ must be bounded by the number of columns in these matrices, i.e., all chains in the partial order contain a finite number of elements and so they must have a minimal element. This proves that there is a set of minimal idempotent matrices.

Suppose now that $E$ and $F$ are distinct minimal idempotent matrices. It is immediate that $EF \leq E$ and $FE \leq F$. Therefore, $EF = FE = 0$, and they are pairwise orthogonal. ∎

**Theorem 9** ([2, Theorem 3.4.4])**.** *Let $\mathcal{A}$ be an association scheme and $\mathbb{C}(\mathcal{A})$ its its Bose-Mesner algebra. Then $\mathbb{C}(\mathcal{A})$ has a basis of pairwise othogonal idempotet matrices $\{E_0, \dots, E_d\}$. Furthermore, this basis satisfies the following properties:*

*(i)* $E_0 = \frac{1}{n}J$;

*(ii)* $\sum_{i=0}^{d} E_i = I$;

*(iii)* $E_i^T \in \{E_0, \dots, E_d\}$ *for all* $i \in \{0, 1, \dots, d\}$*;*

*(iv) for any* $i, j \in \{0, 1, \dots, d\}$*,* $E_i \circ E_j$ *is in* $\mathbb{C}(\mathcal{A})$*;*

*(v)* $E_i E_j = \delta_{ij}$ *for all* $i, j \in \{0, 1, \dots, d\}$*.*

*Proof.* We start by proving the first statement. If $A \in \mathbb{C}(\mathcal{A})$, then $A^* \in \mathbb{C}(\mathcal{A})$ because the matrices $A_0, A_1, \dots, A_d$ have coefficients in $\{0, 1\}$ and $A_i \in \mathcal{A}$ for all $i \in \{0, 1, \dots, d\}$. Then we can use 7 to prove that $\mathbb{C}(\mathcal{A})$ satisfies the conditions of 6. Therefore, for every matrix in $\mathbb{C}(\mathcal{A})$ there is a set $\{E_1^A, \dots, E_k^A\}$ idempotent, paiwise orthogonal matrices such that $A$ is a linear combination of them. Consider the set

$$S = \cup_{A \in \mathbb{C}(\mathcal{A})} \{E_1^A, \dots, E_k^A\}$$

Note that $E_i^A$ and $E_j^B$ are not necessarily orthogonal if $A \neq B$. Consider the subset of minimal idempotent matrices in $S$, $S'$, which exists due to 8. All elements in $S'$ are then pairwise orthogonal. To prove the statement we just have to show that $S'$ spans $\mathbb{C}(\mathcal{A})$. Fix any $F \in S$, and define $F_0$ as the sum of all $E \in S'$ such that $E \leq F$. Note that $F_0$ is idempotent because all matrices in $S'$ are pairwise othogonal. Moreover, $F_0 \leq F$, since

$$FF_0 = F(\sum_{E \in S' | FE = E} E) = \sum_{E \in S' | FE = E} FE = \sum_{E \in S' | FE = E} = E = F_0$$

Now suppose $F_0 \neq F$. Then, it holds that

$$(F - F_0)^2 = F^2 - 2FF_0 + F_0^2 = F - 2F_0 + F_0 = F - F_0$$

so $F - F_0$ is an idempotent matrix. Notice that

$$F(F - F_0) = F^2 - F_0 = F - F_0$$

so $(F - F_0) \leq F$. This implies the existence of a non-zero minimal idempotent matrix $G$ such that $G \leq F - F_0 \leq F$, but this contradicts the definition of $F_0$. So $F = F_0$, and we have proved that $\mathbb{C}(\mathcal{A})$ is spanned by $S'$. Note that the matrices in $S$ must be linearly independent due to being orthogonal, so $S' = \{E_1, \dots, E_k\}$ is, in fact, a basis for $\mathbb{C}(\mathcal{A})$.

We now prove the properties of these matrices $\{E_1, \dots E_k\}$:

(i) Note that $J = A_0 + A_1 + \cdots + A_d \in \mathbb{C}(\mathcal{A})$. Define $E_0 := f_1(J) m_{J,1}(J) \in S$ following the notation in 6. We know that it is idempotent, so we just have to show that it has rank 1, i.e., its column space has dimension 1. Suppose that $p(x) := p_0 + p_1 x + p_2 x^2 + \cdots + p_m x^m$. Note that $J^i$ has rank 1 for all $i \in \mathbb{N}$, so $p(J)$ is a matrix of rank 1 plus the matrix $p_0 I$. It is clear then that the dimension of the column space over $\mathbb{C}$ of $p(J)$ is 1. From this we get that $E_0$ is the product of two rank one matrices, so it has rank at most one.

(ii) We know that $I \in \mathbb{C}(\mathcal{A})$, so there exist $a_0, a_1, \dots, a_d \in \mathbb{C}$ such that

$$I = \sum_{i=0}^{d} a_i E_i$$

We just have to show that $a_i = 0, \forall i \in \{0, 1, \dots, d\}$. Note that, for any $i$, it holds that

$$E_i = IE_i = (\sum_{i=0}^{d} a_i E_i)E_i = a_i E_i^2 = a_i E_i$$

so $a_i = 1$.

(iii) For a given $i \in \{0, 1, \dots, d\}$, $E_i = f_i(A)m_{A,i}(A)$ for some $A \in \mathbb{C}(\mathcal{A})$. Then,

$$E_i^T = f_i(A)^T m_{A,i}(A)^T = f_i(A^T)m_{A,i}(A^T)$$

Clearly $A^T = (\sum_{i=0}^{d} \alpha_i A_i)^T = \sum_{i=0}^{d} \alpha_i A_i^T \in \mathbb{C}(\mathcal{A})$, since for any $i \in 0, 1, \dots, d$, $A_i^T \in \mathcal{A}$. Therefore, $E_i^T \in S$. Suppose $E_i^T$ is not minimal. Then, there exists an $F \in S$, $F \neq E_i^T$, such that $F \leq E_i^T \iff E_i^T F = F$. Since $\mathbb{C}(\mathcal{A})$ is a commutative algebra, it then holds that

$$(E_i^T F)^T = F^T \iff F^T E_i = F^T \iff E_i F^T = F^T$$

that is, $F^T \leq E_i$, which contradicts the fact that $E_i$ is minimal. We conclude that $E_i^T$ mut be minimal.

(iv) It is immediate from the fact that $\mathbb{C}(\mathcal{A})$ is closed under the *Schur* multiplication.

(v) It is immediate from the fact that any $E_i, E_j \in S'$ are pairwise othogonal and idempotent.

■

**Lemma 10.** *The idempotent matrices $\{E_0, E_1, \dots, E_d\}$ that form a basis of $\mathbb{C}(\mathcal{A})$ are also Hermitian.*

*Proof.* We have already shown that $\mathbb{C}(\mathcal{A})$ is closed under conjugation. Therefore, for any $i \in \{0, 1, \dots, d\}$, it holds that there exist $a_{i0}, a_{i1}, \dots, a_{id} \in \mathbb{C}$ such that

$$E_i^* = \sum_{i=0}^{d} a_{ij} E_j$$

Note that, for all $j \in \{0, 1, \ldots, d\}$, it holds that

$$E_i^* E_j = a_j E_j$$

Suppose that $E_i^*$ is not minimal. Then there is an idempotent matrix $F \in \mathbb{C}(\mathcal{A})$, $F \neq E_i^*$, such that $F \leq E_i^* \iff E_i^* F = F$. Since $\mathbb{C}(\mathcal{A})$ is commutative, this implies that

$$(E_i^* F)^* = F^* \iff F^* E_i = F^* \iff E_i F^* = F^*$$

which contradicts the fact that $E_i$ is minimal. Therefore, $E_i^*$ is minimal, and so for any $i \in \{0, 1, \ldots, d\}$ there is a $j \equiv j(i) \in \{0, 1, \ldots, d\}$ such that $E_i^* E_j(i) \neq 0$.

We only have to prove that this $j(i)$ is $i$ itself. We know that, for all $k \in \{0, 1, \ldots, d\} \setminus j(i)$,

$$E_i^* E_k = 0 \iff a_k = 0$$

But, since $tr(E_i^* E_i) > 0$, it must also hold that

$$E_i^* E_i \neq 0 \iff a_i \neq 0$$

so we conclude that $j(i) = i$ and $E_i^* = a_i E_i$. Using the fact that $tr(E_i) = tr(E_i^*)$, we conclude that $a_i = 1$ and $E_i^* = E_i$. ■

We can use the basis of idempotents, $\{E_0, E_1, \ldots, E_d\}$, to state some key identities which are fundamental to the study of association schemes.

**Definition 11.** Let $\mathcal{A} = \{A_0, A_1, \ldots, A_d\}$ be an association scheme. For any given $i$ in $\{0, 1, \ldots, d\}$, we define the $p_i(j)$ as the $j$-th coordinate of $A_i$ in the basis $\{E_0, E_1, \ldots, E_d\}$, that is,

$$A_i = \sum_{j=0}^{d} p_i(j) E_j$$

The values $p_i(j)$ are called the **eigenvalues** of $\mathcal{A}$.

The chosen name for the $p_i(j)$ is not coincidental. Since the $E_i$ are idempotent and pairwise orthogonal, we have that

$$A_i E_j = p_i(j) E_j$$

from which it is immediate that the columns of $E_j$ are eigenvectors of $A_i$, with eigenvalue $p_i(j)$. Note that $p_i(0)$ is the valency of the of the graph $X_i$, that is, the number of edges that are associated to each vertex:

*Proof.* Note that

$$A_i E_0 = p_i(0) E_0 \iff A_i J = p_i(0) J$$

By looking at the LHS it is clear that the $x$-th row of the resulting matrix is precisely the valency of $x$ in $X_i$. Then, the RHS tells us that this valency is constant for all $x \in X$ with value $p_i(0)$. ■

*Remark* 12. Note that, since we are working with commutative, symmetric association schemes, the $A_i$ matrices are symmetric matrices over $\mathbb{R}$, and so their eigenvalues are also in $\mathbb{R}$.

The valency is usally referred to as $v_i := p_i(0)$.

We can make analog constructions using $\{A_0, A_1, \ldots, A_d\}$ as the basis for $\mathbb{C}(\mathcal{A})$:

**Definition 13.** Let $\mathcal{A} = \{A_0, A_1, \ldots, A_d\}$ be an association scheme. For any given $j \in \{0, 1, \ldots, d\}$, we define the $q_j(i)$ as $n$ times the $i$-th coordinate of $E_j$ in the basis $\{A_0, A_1, \ldots, A_d\}$, that is,

$$E_j = \frac{1}{n} \sum_{i=0}^{d} q_j(i) A_i$$

The scalars $q_j(i)$ are called the dual eigenvalues of $\mathcal{A}$.

**Lemma 14.** *Let $\mathcal{A} = \{A_0, \ldots, A_d\}$ be an association scheme. Then, for any $j \in \{0, 1, \ldots, d\}$, it holds that*

$$q_j(0) = tr(E_j) = rk(E_j)$$

*Proof.* We start by noting that complex idempotent matrices can only have eigenvalues 0 or 1. Then, by changing to the eigenvector basis, it is easy to see that the trace of an idempotent matrix is precisely its rank.

By the definition of $q_j(i)$, it holds that

$$\mathrm{tr}(E_j) = \frac{1}{n} \sum_{i=0}^{d} q_j(i) \mathrm{tr}(A_i)$$

Since $tr(A_i) = 0$ if $i \neq 0$ and $A_0 = I$, it is clear that the right hand side is just $q_j(0)$. ■

The values $q_j(0)$, for $j \in \{0, 1, \ldots, d\}$, is often referred to as the **multiplicities** of the association scheme, denoted by $m_j$.

**Proposition 15.** *Let $\mathcal{A} = \{A_0, A_1, \ldots, A_d\}$ be an association scheme, let $\{E_0, E_1, \ldots, E_d\}$ its orthogonal idempotent matrix basis. Then, for any given $i, j \in \{0, 1, \ldots, d\}$ it holds that*

$$\sum_{k=0}^{d} p_k(i) q_j(k) = \begin{cases} 0, i \neq j \\ n, i = j \end{cases}$$

*Proof.* Fix some $i, j \in \{0, 1, \ldots, d\}$. Looking at the definitions of $q_j(i)$ and $p_i(j)$, we get that

$$E_j = \frac{1}{n} \sum_{k=0}^{d} q_j(k) A_k \iff E_j = \frac{1}{n} \sum_{k=0}^{d} q_j(k) \sum_{l=0}^{d} p_k(l) E_l$$

from which

$$E_j E_i = \frac{1}{n} \sum_{k=0}^{d} q_j(k) \sum_{l=0}^{d} p_k(l) E_l E_i \iff n\delta_{ij} = \sum_{k=0}^{d} q_j(k) p_k(i)$$

■

**Definition 16.** We define the **eigenmatrix** of the association scheme as the matrix $P$ whose coefficients are given by

$$P_{i,j} = p_j(i)$$

We define the **dual eigenmatrix** of the association scheme as the matrix $Q$ whose coefficients are given by

$$Q_{i,j} = q_j(i)$$

Note that, from 15, it is immediate that

$$PQ = nI \tag{2}$$

Similarly, it holds that

$$QP = nI \tag{3}$$

Note for a fixed $A_i$, the column space of each of the $E_j$ defines an eigenspace of $A_i$, with all of the column spaces beign pair-wise disjoint due to the $E_j$ being orthogonal. What we have not seen yet is that the $E_j$ define *all* of the eigenspaces of $A_i$. Since the sum of the dimensions of the complex eigenspaces of any complex $n \times n$ matrix must be equal to $n$, it is sufficient to show that the sum of all the multiplicities is equal to $|X|$.

**Proposition 17.** *Let $m_0, m_1, \ldots, m_d$ be the multiplicities of an associtiation scheme over the set $X$. Then, it holds that*

$$\sum_{k=0}^{d} m_k = |X|$$

*Proof.* From (3) we get that

$$|X| = (QP)_{00} = \sum_{k=0}^{d} Q_{0k} P_{k0} = \sum_{k=0}^{d} q_k(0) p_0(k)$$

Using the fact that $p_0(k) = 1$, we get that

$$|X| = \sum_{k=0}^{d} m_k$$

■

Therefore, we may conclude the following:

**Corollary 18.** *For any given $i \in \{0, 1, \dots, d\}$, the column spaces of the matrices $\{E_0, E_1, \dots, E_d\}$ are the eigenspaces of the matrix $A_i$.*

One can define an inner product in $\mathbb{C}(\mathcal{A})$. If $M, N \in \mathbb{C}(\mathcal{A})$, we define a bilinear form $\langle M, N \rangle$ as

$$\langle M, N \rangle := tr(M^* N)$$

This satisfies the properties necessary for it to be an inner product:

*Proof.*

- $\langle N, M \rangle = tr(N^* M) = tr(M N^*) = \overline{tr(M^* N)}$.
- $\langle M, aN_1 + bN_2 \rangle = tr(M^*(aN_1 + bN_2)) = tr(M^*(aN_1) + M^*(bN_2)) = atr(M^* N_1) + btr(M^* N_2) = a\langle M N_1 \rangle b \langle M N_2 \rangle$.
- $\langle M, M \rangle = tr(M^* M) = \sum_{i=0}^{n} \overline{M_{ii}} M_{ii} = \sum_{i=0}^{n} \mid M_{ii} \mid^2 > 0$.

■

We can define this inner product in an equivalent way that is sometimes more convenient. First, for a matrix $A \in \mathbb{C}(\mathcal{A})$, define $sum(A)$ as the sum of all the entries in $A$. Then, we can rewrite

$$\langle M, N \rangle = sum(\overline{M} N)$$

Note that both $\{E_0, E_1, \dots, E_d\}$ and $\{A_0, A_1, \dots, A_k\}$ form an orthogonal basis with respect to this inner product.

It is particularly interesting to consider the inner products between $A_i$ and $E_j$ for all $i, j \in \{0, 1, \dots, d\}$. Following the first definition, we get that

$$\langle A_i, E_j \rangle = tr(A_i^T E_j) = \overline{tr(E_j^* A_i)} = \overline{tr(E_j A_i)} = \overline{tr(A_i E_j)} = tr(\overline{p_i(j) E_j}) = \overline{p_i(j)} m_j$$

while with the second definition we get that

$$\langle A_i, E_j \rangle = sum(E_j \circ A_i) = \sum_{k=0}^{d} \frac{1}{n} q_j(i) sum(A_k \circ A_i) = q_j(i) v_i$$

Thus, we have that

$$\frac{\overline{p_i(j)}}{v_i} = \frac{q_j(i)}{m_j}$$

We will use $\Delta_v$ and $\Delta_m$ to denote the diagonal matrix of valencies and multiplicities, respectively. Then, we can express the previous equality as

$$\Delta_m P = Q^* \Delta_v$$

As we will see in the next section, the action of a group $G$ on the set $X$ over which we have an association scheme can have a lot of interesting properties. For this reason, for any given $g \in G$ we will define $P_g$ as the permutation matrix of $g$ over $X$, that is, the matrix indexed by $X$ whose $(x, y - x)$ element is

$$(P_g)_{xy} = \begin{cases} 1 & \text{if y = gx,} \\ 0 & \text{otherwise} \end{cases}$$

This allows us to define the following:

**Definition 19.** The **centralizer algebra**, or **commutant**, of an association scheme $(X, \mathcal{R})$ is the algebra of all complex matrices that commute with $P_g$ for all $g \in G$. We denote it as $\mathcal{C}(X)$.

In [2, Lemma 3.2.1], a fundamental property of the centralizer algebra of association schemes is stated, but not proven. A proof is provided below.

**Lemma 20.** *The commutant of a set of $n \times n$ permutation matrices is a Schur-closed matrix algebra that contains $I$ and $J$ and is closed under transposition.*

*Proof.* The matrix $I$ is precisely the permutation matrix of the identity permutation, and is clearly commutative.

For any $g \in G$, the $x$-th row of the matrix $PJ$ is an array whose every entry is the number of elements $y \in X$ such that $x = gy$. Note that, for every such $y$, there is a $z := gy \in X$ such that $z = gx$, and that $JP$ is a matrix whose $x$-th column is an array where all the entries are the number of elements $z \in X$ such that $z = gx$. Therefore, it must hold that $JP = PJ$.

Supose now that $A, B$ are two matrices in the commutant, and consider any $\lambda \in \mathbb{C}$ and $g \in G$. Then it holds that

$$\begin{aligned}
&(A+B)P_g = AP_g + BP_g = P_gA + P_gB = P_g(A+B) \\
&(AB)P_g = A(BP_g) = (AP_g)B = P_g(AB) \\
&(\lambda A)P_g = \lambda(AP_g) = \lambda(P_gA) = P_g(\lambda A)
\end{aligned}$$

which proves that the commutant is a matrix algebra. Furthermore,

$$A^T P_g = (P_g^T A)^T = (P_{g^{-1}} A)^T = (A P_{g^{-1}})^T = P_g A^T$$

so it is closed under transposition.

We only have left to check wether it is closed under the Schur multiplication. Let $P$ be any permutation matrix of the form

$$P = \begin{pmatrix} — & e_{r_1}^T & — \\ & \vdots & \\ — & e_{r_n}^T & — \end{pmatrix}, \quad P = \begin{pmatrix} | & & | \\ e_{c_1} & \cdots & e_{c_n} \\ | & & | \end{pmatrix}$$

for some permutations $(c_1, \dots, c_n)$, $(r_1, \dots, r_n)$ of $(1, \dots, n)$, and where $e_i$ is the $i$-th canonical column vector. Then, $A$ belongs to the commutant if and only if, for all $i, j$,

$$(PA)_{ij} = (AP)_{ij} \iff a_{r_i,j} = a_{i,c_j}$$

Therefore, for any $A, B$ in the commutant, it holds that

$$(A \circ B)_{r_i,j} = a_{r_i,j} b_{r_i,j} = a_{i,c_j} b_{i,c_j} = (A \circ B)_{i,c_j}$$

and so $A \circ B$ is also in the commutant, proving that the commutant is closed under the Schur multiplication. ■

Another concept that is often talked about in the literature, and that we will utilize at the end of this section, is the inner and outer distributions.

**Definition 21.** Let $(X, \mathcal{R})$ be $d$-class association scheme, and let $Y$ be a nonempty subset of $X$. We define the **inner distribution** of $Y$ as the row vector $a = (a_0, a_1, \dots, a_d)$ where

$$a_i = \frac{1}{|Y|} |(Y \times Y) \cap R_i|$$

*Remark* 22. The inner distribution is sometimes found in the literature as

$$a_i = \frac{1}{|Y|} \boldsymbol{x_Y}^T A_i \boldsymbol{x_Y}$$

where $\boldsymbol{x_Y}$ is the characteristic vector of $Y$, that is,

$$\boldsymbol{x_Y}(t) = \begin{cases} 1 \text{ if } t \in Y \\ 0 \text{ otherwise} \end{cases}$$

One can easily check that this two definitions are equivalent.

**Lemma 23** ([4, Theorem 30.3])**.** *The distribution vector $a$ of a nonempty subset $Y$ of an association scheme $(X, \mathcal{R})$ satisfies*

$$aQ \geq 0$$

*where 0 is the $(d+1)$-dimensional zero row vector.*

*Proof.* Since $E_j$ is idempotent and symmetric, it holds that

$$0 \leq |\boldsymbol{x_Y}^T E_j|^2 = \boldsymbol{x_Y}^T E_j^T E_j \boldsymbol{x_Y} = \boldsymbol{x_Y}^T E_j \boldsymbol{x_Y} = \frac{1}{|X|}\boldsymbol{x_Y}^T (\sum_{i=0}^{d} Q_{ij} A_i)\boldsymbol{x_Y} =$$

$$= \frac{|Y|}{|X|}(\sum_{i=0}^{d} Q_{ij} \boldsymbol{x_Y}^T A_i \boldsymbol{x_Y})$$

■

The lemma above, when applied to the association scheme obtained over matrices, gives us an analog of the *MacWilliams inequalities* established for classical codes.

**Definition 24.** Let $(X, \mathcal{R})$ be $d$-class association scheme, and let $Y$ be a nonempty subset of $X$. We define the outer distribution of $Y$ as the $|X| \times (d+1)$ matrix B where

$$B_{xi} := |\{y \in Y \mid (x, y) \in R_i\}| = (A_i \chi)_x$$

**Lemma 25** ([3, Lemma 2.5.1])**.** *The inner distribution $a$ and outer distribution $B$ of a nonempty subset $Y$ of an association scheme $(X, \mathcal{R})$ satisfy*

$$(BQ)_{xj} = n\chi^T E_j \chi, \quad (|Y|aQ)_j = |X|\chi_y^T E_j \chi$$

*Proof.*

$$(BQ)_{xj} = \sum_i B_{xi} Q_{ij} = \sum_i (A_i \chi)_x Q_{ij} = |X|(E_j \chi)_x$$

$$(|Y|aQ)_j = (\chi^T BQ)_j = |X|\chi^T E_j \chi$$

■

# 3 Translation association schemes

In this section we will over some properties which are specific to a certain type of association schemes called translation association schemes.

**Definition 26** ([3, Section 2.10]). A **translation association scheme** is an association scheme $(X, \mathcal{R})$ in which the underlying set $X$ has the structure of an abelian group and, for all classes $R \in \mathcal{R}$, it holds that

$$(x, y) \in R \iff (x + z, y + z) \in R, \forall z \in X$$

It is clear that the association scheme obtained with the rank distance over the set of matrices with coefficients in $\mathbb{F}_q$ must also be a translation association scheme.

Working with a translation association scheme is useful because we can use the structure of the abelian group $X$ to give another meaning to some of the identities that hold for general association schemes. For this, an inner product on $X$ [2]. is necessary. Note that, since $X$ is an abelian finite group, the existence of an inner product is guaranteed: the fundamental theorem of finite abelian groups states that $X$ is a direct sum of cyclic subgroups of prime-power order, that is,

$$X = (\mathbb{Z}/e_1\mathbb{Z}) \oplus (\mathbb{Z}/e_2\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/e_m\mathbb{Z})$$

and if we denote the lowest common multiple of $e_1, \ldots, e_m$ by $e$, then we can define the inner product of $X$ as

$$\langle x, y \rangle = \sum_{i=1}^{m} \frac{e}{e_i} x_i y_i$$

where $x, y \in X$, $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m)$. Sometimes it might be useful to use a different inner product, if it exists.

We will now reintroduce the concept of the centralizer algebra. When dealing with translation association schemes, the set $X$ is now isomorphic to its group of permutations. Therefore, the permutation matrices can be expressed as $P_x$, where $x \in X$, and for every $a, b \in X$

$$(P_x)_{ab} = \begin{cases} 1 \text{ if b = a + x} \\ 0 \text{ otherwise} \end{cases}$$

It is easy to check that the properties hold for the permutation matrices of a finite abelian set:

$$P_x P_y = P_{x+y},\ P_x P_y = P_y P_x,\ (P_x)^s = P_{sx},\ P_x \circ P_y = \delta_{xy} P_x,\ P_x^T = P_{-x}$$

[2]This must not be confused with the inner product defined previously over the matrices in $\mathbb{C}(\mathcal{A})$. Even when $X$ is a set of squared matrices, the two are not related.

**Definition 27.** The **restricted centralizer algebra** of a translation association scheme $(X, \mathcal{R})$, $\mathcal{C}_r(X)$, is the real subalgebra of real, symmetric matrices in $\mathcal{C}(X)$. It is spanned by $P_x + P_{-x}$, for all $x \in X$.

This can analogously be defined for regular association schemes.

**Lemma 28.** *Every adjacency matrix $A_i$ of a translation association scheme on $X$ can be expressed as*

$$A_i = \sum_{x \in N_i} P_x$$

*where*

$$N_i = \{x \in X \mid (0, x) \in R_i\} \tag{4}$$

*Proof.* In a translation associan scheme, for any $a, b \in X$, it holds that

$$(a, b) \in N_i \iff b - a \in N_i$$

Note that, for every $z \in N_i$,

$$(P_z)_{ab} \begin{cases} 1 \text{ if } b - a = z \\ 0 \text{ otherwise} \end{cases}$$

Also, we observe that $P_x \circ P_y = \delta_{xy} P_x$, that is, for two different permutation matrices, there are no entries which are one in both. Therefore,

$$(\sum_{x \in N_i} P_x)_{ab} = \begin{cases} 1 \text{ if } b - a = x \text{ for some } x \in N_i \\ 0 \text{ otherwise} \end{cases}$$

which proves our statement. ■

From this, the following is immediately clear:

**Lemma 29.** *The Bose-Mesner algebra of a translation association scheme, $\mathbb{C}(\mathcal{A})$, is a subalgebra of $\mathcal{C}_r(X)$.*

**Definition 30.** We define the **character group** $X^*$ of a finite abelian group $X$ as the group of all functions $\chi : X \to S^1$ such that

$$\chi(x + y) = \chi(x)\chi(y), \; \forall x, y \in X$$

where the group operation is defined as

$$\chi_1\chi_2(x) = \chi_1(x)\chi_2(x)$$

*Remark* 31. Characters are often defined as functions from $X$ to $\mathbb{C}^*$, but note that for every $x \in X$ there is an $m_x \in \mathbb{N}$ such that $m_x x = 0$, and therefore

$$1 = \chi(0) = \chi(m_x x) = \chi(x)^{m_x} \implies |\chi(x)| = 1$$

for all $\chi \in X^*$.

For each character in $\chi$, we can construct an $|X|$-dimensional vector indexed by $x \in X$ whose $x$-th coordinate is $\chi(x)$, that is,

$$\chi_x = \chi(x) \tag{5}$$

**Proposition 32** (Proposition 2.10.7, [3]). *The character group $X^*$ of a finite abelian group $X$ is isomorphic to $X$. Moreover, there exists an isomorphism $f : X \to X^*$ such that $f(x)(y) = f(y)(x)$ for all $x, y \in X$.*

*Proof.* Note that $X$ can be expressed a direct product of cyclic groups due to it being finite and abelian: $X = X_1 \times \cdots \times X_r$. Let $\{x_1, \ldots, x_r\}$ be a basis of $X$, and let $m_i$ be the order of $x_i$, for $i \in \{0, \ldots, r\}$. Then, we may define a set of characters, $\{\xi_1, \ldots, \xi_r\}$, such that $\xi_i(x_j) = 1$ for all $j \neq i$ and $\xi_i(x_j) = e^{\frac{2\pi i}{m_i}}$. We will show that this set acts as a basis of $X^*$. Let $\chi \in X^*$. For any $x \in X$, there are $\lambda_1, \ldots, \lambda_r \in \mathbb{Z}$ such that $x = \lambda_1 x_1 + \cdots + \lambda_r x_r$, and so

$$\chi(x) = \chi(\lambda_1 x_1 + \cdots + \lambda_r x_r) = \chi(x_1)^{\lambda_1} \cdots \chi(x_r)^{\lambda_r}$$

Note that, for each $i \in \{1, \ldots, r\}$ it must hold that

$$\chi(x_i)^{m_i} = \chi(m_i x_i) = \chi(0) = 1$$

Therefore, for each $i \in \{1, \ldots, r\}$ there is an integer $k_i$ such that $\chi(x_i) = e^{\frac{2\pi i k_i}{m_i}} = \xi_i(x_i)^{k_i}$, and so

$$\chi = \xi_1^{\lambda_1 k_1} \cdots \xi_r^{\lambda_r k_r}$$

proving that $\{\xi_1, \ldots, \xi_r\}$ is a basis of $X^*$.

Let us now construct the isomophism $f$ by mapping one basis to the other, that is,

$$f(x_i) = \xi_i$$

Then, for any $x, y \in X$, $x = \lambda_1 x_1 + \cdots + \lambda_r x_r$ and $y = \nu_1 y_1 + \cdots + \nu_r y_r$

$$\begin{aligned} f(x)(y) = (\xi_1^{\lambda_1} \cdots \xi_r^{\lambda_r})(\xi_1 \nu_1 + \cdots + \xi_r \nu_r) &= \xi_1^{\lambda_1}(\nu_1 x_1 + \cdots + \nu_r x_r) \cdots \xi_r^{\lambda_r}(\nu_1 x_1 + \cdots + \nu_r x_r) = \\ &= (\xi_1^{\lambda_1})^{\nu_1} \cdots (\xi_r^{\lambda_r})^{\nu_r} = (\xi_1^{\nu_1})^{\lambda_1} \cdots (\xi_r^{\nu_r})^{\lambda_r} = \\ &= \xi_1^{\nu_1}(\lambda_1 x_1 + \cdots + \lambda_r x_r) \cdots \xi_r^{\nu_r}(\lambda_1 x_1 + \cdots + \lambda_r x_r) = \\ &= f(y)(x) \end{aligned}$$

■

Note that each $x \in X$ defines a character $x^{**}$ on $X^*$ by $x^{**}(\chi) = \chi(x)$. Furthermore, by 32, we have that $|X| = |X^*| = |X^{**}|$. Therefore, we can think of $X^{**}$ as being *cannonically isomorphic* to $X$, and we will use the two terms interchangeably.

Given an inner product $\langle \cdot, \cdot \rangle$ over $X \times X$ and a character $\chi$ on $X$, we define a **multiplicative inner product** over $X \times X$ as

$$A \cdot B = \chi \langle A, B \rangle$$

where it is clear that

$$A \cdot B = B \cdot A; \quad A \cdot B = 1 \iff A = 0 \text{ or } B = 0; \quad A \cdot (B_1 + B_2) = (A \cdot B_1)(A \cdot B_2)$$

The character group is sometimes given a diferent definition in the literature. In [5], given any multiplicative inner product $\cdot$, the $X^*$ is defined as the set of $\chi : X \to S^1$ such that

$$\chi(y) = y \cdot x$$

for some $x \in X$. Note that every such $\chi$ is clearly a character under our previous definition. From 32, we deduce that this set is not only well defined (that is, it does not depend on the choice of the multiplicative inner product), but it also must be exactly the same as the previouly defined $X^*$, since one is included in the other and they have the same cardinality, $|X|$.

**Proposition 33** (Proposition 2.10.8, [3])**.** *Let $Y$ be a subgroup of $|X|$. Then, for each $\chi \in X^*$, we have that*

$$\sum_{y \in Y} \chi(y) = \begin{cases} |Y| \text{ if } \chi(y) = 1 \text{ for all } y \in Y, \\ 0 \text{ otherwise} \end{cases}$$

*Proof.* Suppose there is a $y_0 \in Y$ such that $\chi(y_0) \neq 1$. Then

$$\chi(y_0) \sum_{y \in Y} \chi(y) = \sum_{y \in Y} \chi(y + y_0) = \sum_{y \in Y} \chi(y)$$

which implies that $\sum_{y \in Y} \chi(y) = 0$. ∎

This proposition allows us to see further into the relationship between the translation association scheme over $X$ and $X^*$.

**Proposition 34.** *Let $(X, \mathcal{R})$ be a $d$-class translation association scheme. For every association matrix $A_j$, each character $\chi \in X^*$ is an eigenvector of $A_j$, with eigenvalue $\sum_{x \in N_j} \chi(x)$. Furthermore, iterating over all $\chi \in X^*$ we get all the eigenvalues of $A_j$.*

*Proof.* Fix any $\chi \in X^*$. Since $X$ is a translation is an association scheme, we have that $(x, y) \in R_j \iff y - x \in N_j$. Also, we have that

$$\chi(y) = \chi(y - x + x) = \chi(y - x)\chi(x)$$

We now look at the value of the $x$-th element of the vector $A_j\chi$:

$$(A_j\chi)_x = \sum_{(x,y)\in R_j} \chi(y) = \sum_{z\in N_j} \chi(z)\chi(x) = \left(\sum_{z\in X_j} \chi(z)\right) \chi(x)$$

It is therefore clear that for any $j$, any $\chi \in X^*$ is an eigenvector of $A_j$, with the desired eigenvalue.

Note that, any distinct $\chi_1, \ldots, \chi_k$ in $X^*$ are linearly independent when taken as vectors indexed by $X$. This can be proved by induction. Therefore, iterating over $\chi \in X^*$ must give all the eigenvalues of $A_j$. ∎

**Definition 35.** The **dual scheme** $(X^*, \mathcal{R}^*)$ of a translation association scheme $(X, \mathcal{R})$ is the character group $X^*$ of $X$ with $\mathcal{R}^* = \{R_0^*, R_1^*, \ldots, R_d^*\}$, where

$$R_i^* = \{(\chi, \psi) \in X^* \times X^* \mid E_i(\chi^{-1}\psi) = \chi^{-1}\psi\}$$

We will write $N_i^* = \{\eta \in X^* \mid (0, \eta) \in X \times X\}$.

*Remark* 36. For any given $i, j \in \{0, 1, \ldots, d\}$, for all $\eta \in N_i^*$ it holds that

$$P_{ij} = \sum_{x\in N_j} \eta(x) \tag{6}$$

*Proof.* Directly from 34. ∎

One of the most important properties of translation association schemes, as we will see in this section, it that their corresponding dual schemes are also translation association schemes.

**Definition 37.** We define the **Fourier transform** $\tau : \mathcal{C}(X) \to \mathcal{C}(X^*)$ as the semilinear map defined by

$$(P_x)^\tau := \sum_{\chi\in X^*} \chi(x)P_\chi$$

$$(\sum_{x\in X} \alpha_x P_x)^\tau := \sum_{x\in X} \overline{\alpha}_x (P_x)^\tau$$

Note that, as we saw in 29, $\mathbb{C}(\mathcal{A})$ is a subset of $\mathcal{C}(x)$, so we may apply $\tau$ to matrices in $\mathbb{C}(\mathcal{A})$.

**Proposition 38.** *For all $A, B \in \mathcal{C}(X)$ and all $\alpha \in \mathbb{C}$, it holds that*

*(i)* $(A+B)^\tau = A^\tau + B^\tau$, $(\alpha A)^\tau = \overline{\alpha} A^\tau$,

*(ii)* $(AB)^\tau = A^\tau \circ B^\tau$, $(A \circ B)^\tau = |X|^{-1} A^\tau B^\tau$,

*(iii)* $A^{\tau\tau} = |X| A$,

*(iv)* $I^\tau = J$, $J^\tau = |X| I$.

*Proof.* The first property is obvious from the definition of the Fourier transform. We will suppose $A = \sum_{x \in X} \alpha_x P_x$ and $B = \sum_{y \in X} \beta_y P_y$.

(ii) We first prove that $(P_x P_y)^\tau = P_x^\tau \circ P_y^\tau$.

$$(P_x P_y)^\tau = P_{x+y}^\tau = \sum_{\chi \in X^*} \chi(x+y) P_\chi = \sum_{\chi \in X^*} \chi(x)\chi(y) P_\chi = P_x \circ P_y$$

From this it follows that

$$(AB)^\tau = \sum_{x \in X} \sum_{y \in X} \overline{\alpha_x} \overline{\beta_y} (P_x P_y)^\tau = \sum_{x \in X} \sum_{y \in X} \overline{\alpha_x} \overline{\beta_y} P_x^\tau \circ P_y^\tau = \left( \sum_{x \in X} \alpha_x P_x \right)^\tau \circ \left( \sum_{y \in X} \beta_y P_y \right)^\tau$$
$$= A^\tau \circ B^\tau$$

We now show that $(A \circ B)^\tau = |X|^{-1} A^\tau B^\tau$. We first prove that $(P_x \circ P_y)^\tau = |X|^{-1} P_x^\tau P_y^\tau$.

Suppose $x = y$. Since $P_x \circ P_x = P_x$, we have that

$$P_x^\tau P_x^\tau = \sum_{\chi \in X^*} \sum_{\eta \in X^*} \chi(x)\eta(x) P_\chi P_\eta = \sum_{\chi \in X^*} \sum_{\eta \in X^*} (\chi\eta)(x) P_{\chi\eta} = |X| \sum_{\chi \in X^*} \chi(x) P_\chi = |X| P_x \circ P_x$$

Now suppose $x \neq y$. Then, since $P_x \circ P_y = 0$

$$P_x^\tau P_y^\tau = \sum_{\chi \in X^*} \sum_{\eta \in X^*} \chi(x)\eta(y) P_\chi P_\eta = \sum_{\chi \in X^*} \sum_{\eta \in X^*} (\chi\eta)(x) P_{\chi\eta} = |X| \sum_{\chi \in X^*} \chi(x) P_\chi = |X| P_x \circ P_x$$

■

We can now prove that the dual scheme is a translation association scheme.

**Theorem 39** (Theorem 2.2.10, [3])**.** *For any $d$-class translation association scheme $(X, \mathcal{R})$, with eigenmatrices $P, Q$, the dual scheme $(X^*, \mathcal{R})$ is a $d$-class translation associaion scheme with eigenmatrices $P^* = Q$, $Q^* = P$.*

*Furthermore, it holds that*

$$Q_{ij} = \sum_{\chi \in N_j^*} \chi(x) \text{ for any } x \in N_i \tag{7}$$

*and that*

$$E_j = |X|^{-1} \sum_{x \in X} \sum_{\chi \in N_j^*} \chi(x) P_x \tag{8}$$

*Proof.* We start by showing that $A_i^* = E_i^\tau$ for all $i \in \{0, 1, \ldots, d\}$. Let us fix any $(\chi, \psi) \in R_k^*$ and call $\nu = \chi^{-1}\psi \in N_k^*$. Then, applying first 28 and then (6), we get that

$$\begin{aligned}(E_i^\tau)_{\chi\psi} &= |X|^{-1} \sum_{j=0}^{d} \overline{q_i(j)} (A_j^\tau)_{\chi\psi} = |X|^{-1} \sum_{j=0}^{d} \overline{q_i(j)} \sum_{x \in N_j} (P_x^\tau)_{\chi\psi} = \\ &= |X|^{-1} \sum_{j=0}^{d} \overline{q_i(j)} \sum_{x \in N_j} (P_x^\tau)_{\chi\psi} = |X|^{-1} \sum_{j=0}^{d} \overline{q_i(j)} \sum_{x \in N_j} \sum_{\eta \in X^*} \overline{\eta(x)} (P_\eta)_{\chi\psi} = \\ &= |X|^{-1} \sum_{j=0}^{d} \overline{q_i(j)} \sum_{l=0}^{d} \sum_{\eta \in X_l^*} (P_\eta)_{\chi\psi} \sum_{x \in N_j} \overline{\eta(x)} = |X|^{-1} \sum_{j=0}^{d} \overline{q_i(j)} \sum_{l=0}^{d} \sum_{\eta \in X_l^*} \overline{p_j(l)} (P_\eta)_{\chi\psi}\end{aligned}$$

Note that $(P_\nu)_{\chi\psi}$ is 1 since $\nu = \chi^{-1}\psi = \psi\chi^{-1} \iff \psi = \nu\chi$. Applying this and 15, we get that

$$(E_i^\tau)_{\chi\psi} = |X|^{-1} \overline{\sum_{j=0}^{d} q_i(j) p_j(k)} = \delta_{ik}$$

Therefore, $(E_i^\tau)_{\chi,\psi}$ is 1 if $(\chi, \psi) \in R_i^*$ and 0 otherwise, which implies $E_i^\tau = A_i^*$.

We have that $(\mathbb{C}(\mathcal{A}))^\tau = \langle A_0^*, A_1^*, \ldots, A_d^* \rangle_{\mathbb{C}}$. We will now show that $X^*$ holds the properties of a translation association scheme:

(i) To show that $R^* = \{R_0, R_1, \ldots, R_d\}$ partitions $X^* \times X^*$, we note that every $\nu \in X^*$ is an eigenvector of $A_j$, which implies that it must be in the column space of some $E_i$. It is easy to check that the column spaces of idempotent matrices give eigenvectors of eigenvalue 1.

(ii) Since $E_0 = |X|^{-1}J$, it is clear that $E_0 v = v$ if and only if $v = 0$, hence $R_0^* = \{(\chi, \chi) \mid \chi \in X^*\}$.

(iii) Note that $\psi^{-1}\chi = \overline{\chi^{-1}\psi}$. Therefore, if $(\chi,\psi) \in R_i^*$, that is, $\chi^{-1}\psi$ is in the column space of $E_i$, it is clear that $\psi^{-1}\chi$ must also be in the column space of $E_i$, and thus $(\psi,\chi) \in R_i^*$.

(iv) From 38, we get that

$$A_i^* A_j^* = E_i^\tau E_j^\tau = |X|(E_i \circ E_j)^\tau$$

which is in $\mathbb{C}(\mathcal{A}^*) = \mathbb{C}(\mathcal{A})^\tau$ since, by 9, $E_j \circ E_i$ is in $\mathbb{C}(\mathcal{A})$.

(v) Let us fix $(\chi,\psi) \in R_i^*$. For any $\eta \in X^*$,

$$E_i((\chi\eta)^{-1}\psi\eta) = E_i(\chi^{-1}\psi) = \chi^{-1}\psi = \eta^{-1}\chi^{-1}\psi\eta = (\eta\chi)^{-1}(\psi\eta)$$

and so $(\chi\eta,\psi\eta) \in R_i^*$.

This proves that $X^*$ is, in fact, a translation association scheme.

Since we can identify $X^{**}$ with $X$, it must now also hold that

$$A_i = A_i^{**} = (E_i^*)^\tau$$

Thus,

$$|X|E_i^* = ((E_i^*)^\tau)^\tau = A_i^\tau \left(\sum_{j=0}^{d} p_i(j)E_j\right)^\tau = \sum_{j=0}^{d} \overline{p_i(j)}E_j^\tau = \sum_{j=0}^{d} p_i(j)A_i^*$$

and so $Q^* = P$. Furthermore, $Q = Q^{**} = P^*$.

It is now clear that equation (7) follows from (6). We may concude that

$$\begin{aligned} E_j = |X|^{-1}\sum_{i=0}^{d} q_j(i)A_i = |X|^{-1}|X|^{-1}\sum_{i=0}^{d} Q_{ij}A_i = |X|^{-1}\sum_{i=0}^{d}\left(\sum_{\chi\in N_j^*}\chi(x)\right)\left(\sum_{x\in N_i} P_x\right) = \\ = |X|^{-1}\sum_{x\in X} P_x\left(\sum_{\chi\in N_j^*}\chi(x)\right) \end{aligned}$$

■

In the context of additive codes in translation association schemes, the concept of duality takes on a different meaning:

**Definition 40.** Let $Y$ be an additive code of the translation association scheme $(X,\mathcal{R})$ . We define the **dual code** $Y'$ of $Y$ as the subgroup

$$Y' = \{\chi \in X^* \mid \chi(y) = 1 \text{ for all } y \in Y\} = \{y' \in X \mid y \cdot y' = 1 \text{ for all } y \in Y\}$$

The dual code of $Y$ is defined in such a way that their inner distribution is determined by the inner distribution of their dual, namely:

**Theorem 41** ([3, Theorem 2.10.12])**.** *Let $Y$ be an additive code $Y$ of a translation association scheme $(X, \mathcal{R})$. Then, the inner distributions $a, a^*$ of $Y$ and its dual code $Y'$ are related by the formulae*

$$a' = \frac{1}{|Y|} aQ, \quad a = \frac{|Y|}{|X|} a'P$$

*Proof.* We start by proving that for translation association schemes the inner distribution can actually be expressed as $a_i = |N_i^* \cap Y|$: for every $y \in N_i^* \cap Y$, every $z \in Y$ gives a different $(z, y+z)$ in $(Y \times Y) \cap R_i$.

Now, to prove the formulae above, we fix a $k \in \{0, 1, \ldots, d\}$ and develop

$$|Y|^2 a_k = |Y|^2 |Y' \cap N_k^*|$$

Note that from 33 we get that the previous expression above is equal to

$$\sum_{\nu \in N_k^*} \sum_{y \in Y} |Y| \nu(y) = \sum_{\nu \in N_k^*} \sum_{y \in Y} \nu(Y) (\sum_{z \in Y} \nu(-z)) = \sum_{\nu \in N_k^*} \sum_{y,z \in Y} \nu(Y - z)$$

From 39, we get that this is equal to

$$|X| \boldsymbol{x_Y}^T E_k \boldsymbol{x_Y}$$

and, from 25, this equals

$$|Y|(aQ)_k$$

■

This last proposition, when applied to the translation association scheme obtained over matrices with the rank distance, gives an analog of the *MacWilliams identities* established for classical codes.

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