My Ideas

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***Abstract*—Just me talkin**

I. Introduction

This document aims to inform the reader about my current ideas on the relationship between network theory and the connections between rigid bodies and their states. It has been observed that the mass matrix M in the Euler-Lagrange equation resembles that of the graph Laplacian of an undirected graph. My research focuses on the network interpretation of a single robotic system which may lead to further research in modeling the connections of rigid bodies algebraically.

II. Setup

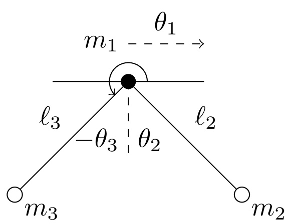

Fig. 1. Mechanical diagram for two single pendulums on a cart

Let the following equation be a simplified for of the Euler-Lagrange equation which represents the dynamics of a general robotic system:

$$M(\theta)\ddot{\theta} = f(\dot{\theta}, \theta)$$

where $M(\theta)$ is the mass matrix and $f(\dot{\theta}, \theta)$ is the sum of the coriolis and gravitational terms. It can be shown that the mass matrix is always symmetric and positive definite for any system described in this way, thus $M(\theta)$ is invertible:

$$\ddot{\theta} = M(\theta)^{-1} f(\dot{\theta}, \theta) \tag{1}$$

Now consider the consensus equation expressed with the graph Laplacian:

$$\dot{x} = -L(\mathcal{D})x$$

This is the form that I wish to compare to (1) to understand a graph theoretical meaning of the mass and inertial relationships between rigid bodies. The idea is that the network connections represented in the graph Laplacian are encoded in the mass matrix somehow. Let's examine $M(\theta)$ and $M(\theta)^{-1}$ for the system of two pendulums on a cart as shown in Figure 1:

$$M(\theta) = \begin{bmatrix} m_1 + m_2 + m_3 & l_2 m_2 \cos\theta_2 & l_3 m_3 \cos\theta_3 \\ l_2 m_2 \cos\theta_2 & l_2^2 m_2 & 0 \\ l_3 m_3 \cos\theta_3 & 0 & l_3^2 m_3 \end{bmatrix}$$

$$M(\theta)^{-1} = \frac{1}{d} \begin{bmatrix} 1 & -\frac{\cos\theta_2}{l_2} & -\frac{\cos\theta_3}{l_3} \\ -\frac{\cos\theta_2}{l_2} & \frac{m_1+m_2+m_3\sin^2\theta_2}{l_2^2 m_2} & \frac{\cos\theta_2\cos\theta_3}{l_2 l_3} \\ -\frac{\cos\theta_3}{l_3} & \frac{\cos\theta_2\cos\theta_3}{l_2 l_3} & \frac{m_1+m_2\sin^2\theta_2+m_3}{l_3^2 m_3} \end{bmatrix}$$

$$d = m_1 + m_2\sin^2\theta_2 + m_3\sin^2\theta_3$$

where d is a common denominator within $M(\theta)^{-1}$. Although, matrices are symmetric, this doesn't show that there is a mathematical link between rigid bodies. In the following sections, we will examine the decomposition of $M(\theta)^{-1}$.

Finally, let us formulate a graph which we expect to represent the two pendulum system. Let nodes (v_1, v_2, v_3) correspond to bodies 1, 2, and 3 in Figure 1. We expect an edge to exist between the pairs (v_1, v_2) and (v_1, v_3) because they are physically connected, and no edge between (v_2, v_3) because they are not. Also, the graph will be defined as undirected, because we do not yet know the direction of information flow within the graph. Furthermore, the Laplacian of an undirected graph is symmetric, which it will need to be to compare with $M(\theta)$ or $M(\theta)^{-1}$.

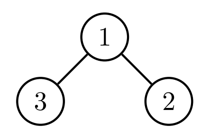

Fig. 2. Estimated graph for two pendulum system

$$L(\mathcal{G}) = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$L(\mathcal{G})$ is the graph Laplacian for the undirected graph in Figure 1. It is important to note that a zero entry in the (i, j) position indicates the nonexistence of an edge between nodes i and j.

III. Definitions

Definition 1. For a digraph $\mathcal{D}$, the incidence matrix $D(\mathcal{D}) \in \mathbb{R}^{n \times m}$ is a matrix defined as follows:

$$D(\mathcal{D}) = [d_{ij}]\,, \text{where } d_{ij} = \begin{cases} -1 & \text{if } v_i \text{ is the tail of } e_j \\ 1 & \text{if } v_i \text{ is the head of } e_j \\ 0 & otherwise \end{cases}$$

where v_i is the ith vertex of $\mathcal{G}$ and e_j is the jth edge. The incidence matrix captures the features of the edges of a graph and the vertices they connect to. One important feature of the incidence matrix is that each column describes one edge, so each column has one -1 and one 1, and $\mathbf{1} \in \mathcal{N}(D(\mathcal{D})^T)$.

Definition 2. The graph Laplacian of $\mathcal{G}$, the undirected version of $\mathcal{D}$, can be defined as:

$$L(\mathcal{G}) = D(\mathcal{D})D(\mathcal{D})^T$$

which describes the connections between nodes in the graph.
Definition 3. The edge Laplacian of $\mathcal{G}$ is defined as:

$$L_e(\mathcal{G}) = D(\mathcal{D})^T D(\mathcal{D})$$

which describes the connections between edges in the graph.
Definition 4. The Cholesky decomposition of a symmetric positive-definite matrix is written as:

$$A = LL^T$$

where L is a real lower triangular matrix with positive diagonal entries. This decomposition can also be done with upper triangular matrices such that $A = UU^T$.

IV. Decomposition of the Inverse Mass Matrix

A. Two Pendulums on a Cart

Let the Cholesky decomposition of $M(\theta)^{-1}$ be defined as:

$$M(\theta)^{-1} = L(\theta)L(\theta)^T$$

For the two pendulum system described above, $L(\theta)$ is:

$$L(\theta) = \begin{bmatrix} \frac{1}{\sqrt{d}} & 0 & 0 \\ -\frac{\cos\theta_2}{l_2\sqrt{d}} & \frac{1}{l_2\sqrt{m_2}} & 0 \\ -\frac{\cos\theta_3}{l_3\sqrt{d}} & 0 & \frac{1}{l_3\sqrt{m_3}} \end{bmatrix}$$

Finally, something interesting happens if we evaluate $L(\theta)^T L(\theta)$:

$$L(\theta)^T L(\theta) = \begin{bmatrix} \frac{l_2^2 l_3^2 + l_2^2\cos^2\theta_3 + l_3^2\cos^2\theta_2}{l_2^2 l_3^2 d} & -\frac{\cos\theta_2}{l_2^2\sqrt{m_2 d}} & -\frac{\cos\theta_3}{l_3^2\sqrt{m_3 d}} \\ -\frac{\cos\theta_2}{l_2^2\sqrt{m_2 d}} & \frac{1}{l_2^2 m_2} & 0 \\ -\frac{\cos\theta_3}{l_3^2\sqrt{m_3 d}} & 0 & \frac{1}{l_3^2 m_3} \end{bmatrix}$$

We see that the structure is similar to the expected graph Laplacian $L(\mathcal{G})$! The idea here is that $L(\theta)$ is related to some incidence matrix $D(\mathcal{D})^T$, and $M(\theta)^{-1}$ is related to some edge Laplacian $L_e(\mathcal{G})$ with states θ. Therefore, the graph theoretical links between rigid bodies in a robotic system do not exist in terms of θ, but in terms of the edges of the underlying graph. However, none of this matters if it is not generalizable to all robotic systems, or at least those that satisfy some given conditions. Let's look at a couple more examples.

B. Double and Single Pendulums on a Cart

Figures 3 and 4 show the diagram and estimated graph for the system of two carts sliding on a platform connected to a pendulum. When we complete similar analysis as before we get $M(\theta)$ to be similar in structure to $L(\mathcal{G})$ like before (not shown, too big to fit).

$$L(\mathcal{G}) = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

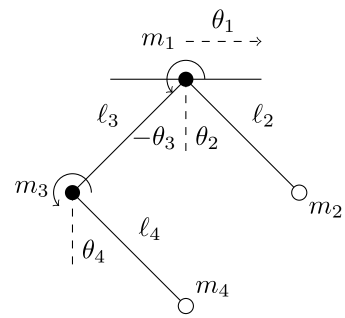

Fig. 3. Mechanical diagram for the double/single pendulum on a cart

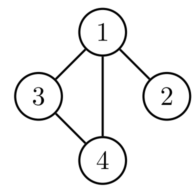

Fig. 4. Estimated graph for double/single pendulum system

$L(\mathcal{G})$ is the graph Laplacian for the undirected graph in Figure 4. $L(\theta)^T L(\theta)$ is a very complicated matrix, so only the structure is shown:

$$L(\theta)^T L(\theta) = \begin{bmatrix} * & * & * & * \\ * & * & 0 & 0 \\ * & 0 & * & * \\ * & 0 & * & * \end{bmatrix}$$

where $*$ represents a generally nonzero value. Again, the zero entries, which denote the nonexistence of an edge in the graph Laplacian, are consistent between $M(\theta)$, $L(\mathcal{G})$, and $L(\theta)^T L(\theta)$.

C. Two Carts on a Pendulum

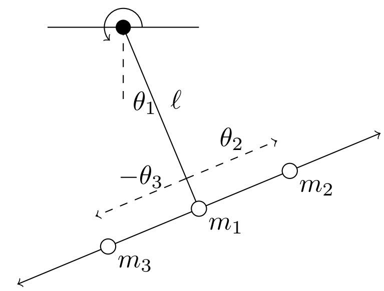

Fig. 5. Mechanical diagram for two carts on a pendulum

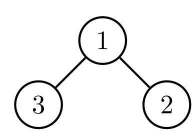

Fig. 6. Estimated graph for two carts on a pendulum system

Figures 5 and 6 show the diagram and estimated graph for the system of two carts sliding on a platform connected to a pendulum. When we complete similar analysis as before we get:

$$M(\theta)=\begin{bmatrix} l^2\left(m_1+m_2+m_3\right)+m_2\theta_2^2+m_3\theta_3^2 & lm_2 & lm_3 \\ lm_2 & m_2 & 0 \\ lm_3 & 0 & m_3 \end{bmatrix}$$

$$L(\theta)^T L(\theta)=\begin{bmatrix} * & * & * \\ * & * & 0 \\ * & 0 & * \end{bmatrix}$$

Now, what if we redefine the positions of the θs, masses, and nodes of the system? Let's see what happens if we switch nodes 1 and 3.

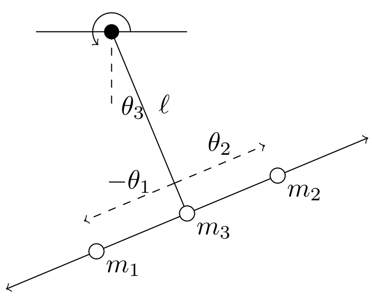

Fig. 7. Mechanical diagram for two carts on a pendulum

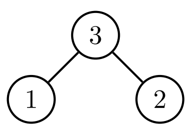

Fig. 8. Estimated graph for two carts on a pendulum system

We can see the modified diagram and graph in Figures 7 and 8. Now, let's recalculate the mass matrix and $L(\theta)^T L(\theta)$ for this redefined system:

$$M(\theta)=\begin{bmatrix} m_1 & 0 & lm_1 \\ 0 & m_2 & lm_2 \\ lm_1 & lm_2 & l^2\left(m_1+m_2+m_3\right)+m_1\theta_1^2(t)+m_2\theta_2^2(t) \end{bmatrix}$$

$$L(\theta)^T L(\theta)=\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

The mass matrix is what we expect, indicating no edge between 1 and 2, but the edge Laplacian does not have zero entries anywhere! This is due to the way the Cholesky decomposition is defined, there is no way (I think) to get zeros *only* in the $(1,2)$ and $(2,1)$ positions from the product of a lower and an upper triangular matrix (in that order). However, we can actually define the Cholesky decomposition as the product of an upper triangular matrix and its transpose by performing the algorithm starting from the bottom right of the matrix rather than the upper left. Let's define the upper triangular version as:

$$M(\theta)^{-1}=U(\theta)U(\theta)^T$$

Now, by doing this version of the decomposition, and switching the order, we get:

$$U(\theta)^T U(\theta)=\begin{bmatrix} * & * & * \\ * & * & 0 \\ * & 0 & * \end{bmatrix}$$

which is what we expect! The final permutation of defining this system is to make node 2 the pendulum that the carts are connected to, in other words, make node 2 the root of the graph. The algorithm yields matrices with all nonzero elements for both $L(\theta)^T L(\theta)$ and $U(\theta)^T U(\theta)$.

V. Conclusions

- If all nonzero values exist along the diagonal, first row, and first column of $M(\theta)$, the matrix $L(\theta)^T L(\theta)$ derived from the Cholesky decomposition of $M(\theta)^{-1}$ resembles the structure of the mass matrix $M(\theta)$.
- To make the first row and column of $L(\theta)^T L(\theta)$ contain no zeros, θ_1 must define the position of the body that is "linked" to all other bodies, or in other words, node 1 must have edges that connect to all other nodes.
- There may exist another decomposition that can generalize the similarity between $M(\theta)$ and $L(\theta)^T L(\theta)$ or $U(\theta)^T U(\theta)$, which actually resembles an incidence matrix (currently working on this).
- There's a pretty good chance that none of this is actually leading anywhere and I'm just going crazy.

VI. Questions

- If what I have done so far is correct, and we can define an underlying graph for a robotic system, what can we do with it?
- Even if we can find a graph theoretical interpretation of $M(\theta)^{-1}$, it is still multiplied by the vector of nonlinear terms $f(\dot{\theta},\theta)$ in (1), so how is it useful?
- Say I can find a generalizable decomposition of $M(\theta)^{-1}$, does it even mean anything if the structure does not resemble that of an incidence matrix?

VII. More

For Kuramoto synced pendulums:

$$M(\theta)=\begin{bmatrix} \sum_{i=1}^{n} m_i & \dots & m_i\ell_i\cos\theta_i & \dots \\ \vdots & \ddots & & \\ m_i\ell_i\cos\theta_i & & m_i\ell_i^2 & \\ \vdots & & & \ddots \end{bmatrix}$$

$$f(\theta,\dot{\theta})=\begin{bmatrix} \sum_{i=2}^{n} m_i\ell_i\sin(\theta_i)\dot{\theta}_i^2 \\ \vdots \\ -gm_i\ell_i\sin\theta_i \\ \vdots \end{bmatrix}$$

$$[L(\theta)_{11}] = \frac{1}{\sqrt{m_1 + \sum\limits_{i=2}^{n} m_i \sin^2\theta_i}}$$

$$[L(\theta)_{i1}] = \frac{-\cos\theta_i}{\ell_i\sqrt{m_1 + \sum\limits_{i=2}^{n} m_i \sin^2\theta_i}}$$

$$[L(\theta)_{ii}] = \frac{1}{\ell_i\sqrt{m_i}}$$

$$L(\theta) = \begin{bmatrix} \frac{1}{\sqrt{m_1+\sum_{i=2}^{n} m_i\sin^2\theta_i}} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \frac{-\cos\theta_i}{\ell_i\sqrt{m_1+\sum_{i=2}^{n} m_i\sin^2\theta_i}} & 0 & \frac{1}{\ell_i\sqrt{m_i}} & 0 \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

$$L(\theta)^{-1} = \begin{bmatrix} \sqrt{m_1 + \sum\limits_{i=2}^{n} m_i\sin^2\theta_i} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \sqrt{m_i}\cos\theta_i & 0 & \ell_i\sqrt{m_i} & 0 \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

$$x = L(\theta)^{-1}\dot{\theta} = \begin{bmatrix} \left(\sqrt{m_1 + \sum\limits_{i=2}^{n} m_i\sin^2(\theta_i)}\right)\dot{\theta}_1 \\ \vdots \\ \sqrt{m_i}(\cos(\theta_i)\dot{\theta}_1 + \ell_i\dot{\theta}_i) \\ \vdots \end{bmatrix}$$

My way of thinking:

- $M(\theta)^{-1}$ is the edge Laplacian
- $L(\theta)^T$ is the incidence matrix
- $L(\theta)^T L(\theta)$ is the graph Laplacian
- $x_e = \dot{\theta}$
- Edge agreement equation: $\ddot{\theta} = M(\theta)^{-1}\dot{\theta}$ (kind of but not really)
- Node agreement equation: $\dot{x} = L(\theta)^T L(\theta)x$ where x is the vector of vertices in the underlying graph which I believe correspond to the bodies themselves rather than the θs
- $\dot{\theta} = L(\theta)x$

$$\ddot{\theta} = L(\theta)L(\theta)^T f(\dot{\theta}, \theta)$$

$$\dot{x} = L(\theta)^T f(\dot{\theta},\theta) = \begin{bmatrix} \frac{1}{\sqrt{m_1+\sum_{i=2}^{n} m_i\sin^2\theta_i}} & \dots & \frac{-\cos\theta_i}{\ell_i\sqrt{m_1+\sum_{i=2}^{n} m_i\sin^2\theta_i}} & \dots \\ 0 & \ddots & 0 & \dots \\ \vdots & \ddots & \frac{1}{\ell_i\sqrt{m_i}} & \ddots \\ 0 & \dots & 0 & \ddots \end{bmatrix} \begin{bmatrix} \sum\limits_{i=2}^{n} m_i\ell_i\sin(\theta_i)\dot{\theta}_i^{\,2} \\ \vdots \\ -gm_i\ell_i\sin\theta_i \\ \vdots \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} \frac{\sum_{i=2}^{n} m_i\ell_i\sin(\theta_i)\dot{\theta}_i^{\,2}}{\sqrt{m_1+\sum_{i=2}^{n} m_i\sin^2\theta_i}} + \sum\limits_{i=2}^{n} \frac{gm_i\sin\theta_i\cos\theta_i}{\sqrt{m_1+\sum_{i=2}^{n} m_i\sin^2\theta_i}} \\ \vdots \\ -g\sqrt{m_i}\sin\theta_i \\ \vdots \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} \frac{\sum_{i=2}^{n} m_i\sin\theta_i(\ell_i\dot{\theta}_i^{\,2} + g\cos\theta_i)}{\sqrt{m_1+\sum_{i=2}^{n} m_i\sin^2\theta_i}} \\ \vdots \\ -g\sqrt{m_i}\sin\theta_i \\ \vdots \end{bmatrix}$$

I think there's something here, but I'm going too deep into the graph theory analogies with systems that don't really fit. The problem is the nonlinear vector f and the Laplacians and incidence matrices are functions of time.

VIII. Overparamaterization

A. Revoulte Joint

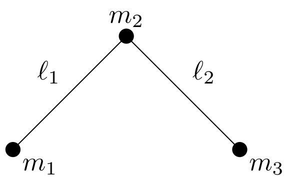

Fig. 9. Figure

$$q = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix}$$

$$D(\mathcal{G}) = \begin{bmatrix} I_{2\times 2} & 0 \\ -I_{2\times 2} & I_{2\times 2} \\ 0 & -I_{2\times 2} \end{bmatrix}$$

$$L(\mathcal{G}) = D(\mathcal{G})D(\mathcal{G})^T = \begin{bmatrix} I_{2\times 2} & -I_{2\times 2} & 0 \\ -I_{2\times 2} & 2I_{2\times 2} & -I_{2\times 2} \\ 0 & -I_{2\times 2} & I_{2\times 2} \end{bmatrix}$$

$$q^T L(\mathcal{G})q = \ell_1^2 + \ell_2^2$$

$$h(q) = q^T L(\mathcal{G})q - \ell_1^2 - \ell_2^2 = 0$$

$$\frac{dh}{dt} = 2q^T L(\mathcal{G})\dot{q} = 0$$

This is the constraint we will use in the form:

$$\alpha^T \dot{q} = 0, \ \alpha = L(\mathcal{G})q$$

And the Euler-Lagrange equation:

$$M\ddot{q} + Mg_6 = f + \alpha\lambda$$

where $g_6 = \begin{bmatrix} 0 & g & 0 & g & 0 & g \end{bmatrix}^T$, f is the externally applied force at the generalized coordinates, and $\alpha\lambda$ is the constraint force.

$$\ddot{q} = -g_6 + M^{-1}(f + \alpha\lambda) = -g_6 + M^{-1}f + M^{-1}L(\mathcal{G})q\lambda$$

$$\alpha^T\ddot{q} + \dot{\alpha}^T\dot{q} = 0$$

$$q^T L(\mathcal{G})\ddot{q} + \dot{q}^T L(\mathcal{G})\dot{q} = 0$$

$$q^T L(\mathcal{G})(-g_6 + M^{-1}f + M^{-1}L(\mathcal{G})q\lambda) + \dot{q}^T L(\mathcal{G})\dot{q} = 0$$

$$M\ddot{q} + Mg_6 + \left(\frac{\dot{q}^T L(\mathcal{G})\dot{q} - q^T L(\mathcal{G})g_6}{q^T L(\mathcal{G})M^{-1}L(\mathcal{G})q}\right)L(\mathcal{G})q = \left(I - \frac{L(\mathcal{G})qq^T L(\mathcal{G})M^{-1}}{q^T L(\mathcal{G})M^{-1}L(\mathcal{G})q}\right)f$$

The denominator kind of has an interesting structure:

$$q^T L(\mathcal{G})M^{-1}L(\mathcal{G})q$$

$$= q^T D(\mathcal{G})D(\mathcal{G})^T M^{-1}D(\mathcal{G})D(\mathcal{G})^T q$$

Now, let $q_e = D(\mathcal{G})^T q$ be the edge states associated with $\mathcal{G}$. The expression becomes:

$$q_e^T D(\mathcal{G})^T M^{-1}D(\mathcal{G})q_e$$

which kind of looks like the quadratic form of the weighted edge Laplacian with the inverse of the masses as weights.

Separate idea: can we use notions of leader-follower formation control to control a robotic system in which one or more bodies are grounded?

$$f = B(q)T = \begin{bmatrix} R_{90}\frac{r_1 - r_2}{\|r_1 - r_2\|} & 0_{2\times 1} \\ 0_{2\times 1} & 0_{2\times 1} \\ 0_{2\times 1} & R_{90}\frac{r_3 - r_2}{\|r_3 - r_2\|} \end{bmatrix} \begin{bmatrix} T_1 \\ T_3 \end{bmatrix}$$

$$R_{90} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

B. Prismatic Joint

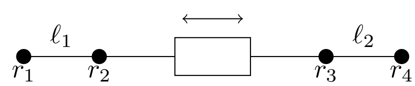

Fig. 10. Figure

$r_2 - r_1$ is parallel to $r_4 - r_3$ and pointing in the same direction, so:

$$(r_2 - r_1)^T(r_4 - r_3) = \|r_2 - r_1\|\|r_4 - r_3\| = \ell_1\ell_2$$

which can be rewritten as:

$$q^T L_1(\mathcal{G})q = 2\ell_1\ell_2$$

where:

$$L_1(\mathcal{G}) = \begin{bmatrix} 0 & 0 & I_{2\times 2} & -I_{2\times 2} \\ 0 & 0 & -I_{2\times 2} & I_{2\times 2} \\ I_{2\times 2} & -I_{2\times 2} & 0 & 0 \\ -I_{2\times 2} & I_{2\times 2} & 0 & 0 \end{bmatrix}$$

Also, length must be preserved:

$$(r_2 - r_1)^T(r_2 - r_1) = \ell_1^2$$

$$(r_4 - r_3)^T(r_4 - r_3) = \ell_2^2$$

$$q^T L_2(\mathcal{G})q = \ell_1^2 + \ell_2^2$$

where:

$$L_2(\mathcal{G}) = \begin{bmatrix} I_{2\times 2} & -I_{2\times 2} & 0 & 0 \\ -I_{2\times 2} & I_{2\times 2} & 0 & 0 \\ 0 & 0 & I_{2\times 2} & -I_{2\times 2} \\ 0 & 0 & -I_{2\times 2} & I_{2\times 2} \end{bmatrix}$$

Let $L(\mathcal{G}) = L_1(\mathcal{G}) + L_2(\mathcal{G})$

$$h(q) = q^T L(\mathcal{G})q - 2\ell_1\ell_2 - \ell_1^2 - \ell_2^2 = 0$$

$$\frac{dh}{dt} = 2q^T L(\mathcal{G})\dot{q} = 0$$

So,

$$L(\mathcal{G}) = \begin{bmatrix} I_{2\times2} & -I_{2\times2} & I_{2\times2} & -I_{2\times2} \\ -I_{2\times2} & I_{2\times2} & -I_{2\times2} & I_{2\times2} \\ I_{2\times2} & -I_{2\times2} & I_{2\times2} & -I_{2\times2} \\ -I_{2\times2} & I_{2\times2} & -I_{2\times2} & I_{2\times2} \end{bmatrix}$$

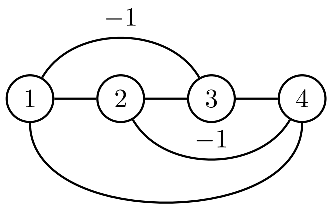

Fig. 11. Graph

C. Learning the network

Given n measurements of position and velocity, if we minimize the following function, we can find the best estimate of the graph Laplacian for a system.

$$\underset{L(\mathcal{G})}{\operatorname{argmin}} \sum_{i=1}^{n} (q_i^T L(\mathcal{G}) \dot{q}_i)$$

D. Trying again

- n sets of coordinates (nodes)
- k constraints (edges)

$$Q = \begin{bmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \in \mathbb{R}^{n\times2},\ G = \begin{bmatrix} 0 & g \\ \vdots & \vdots \\ 0 & g \end{bmatrix} \in \mathbb{R}^{n\times2}$$

$$M = \begin{bmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{bmatrix} \in \mathbb{R}^{n\times n}$$

$$F = \begin{bmatrix} f_{1,x} & f_{1,y} \\ \vdots & \vdots \\ f_{n,x} & f_{n,y} \end{bmatrix}$$

$$M\ddot{Q} + MG + \Gamma = F$$

Constraints in terms of the graph

$$D(G) = \begin{bmatrix} | & & | \\ d_1 & \dots & d_k \\ | & & | \end{bmatrix} \in \mathbb{R}^{n\times k}$$

$$L_i(\mathcal{G}) = d_i d_i^T \in \mathbb{R}^{n\times n}$$

$$h_i(Q) = \operatorname{Tr}(Q^T L_i(\mathcal{G}) Q) + c_i = 0$$

$$\frac{dh_i}{dQ} = 2Q^T L_i(\mathcal{G})$$

$$A(Q) \triangleq \frac{1}{2}\frac{dh}{dQ} = \begin{bmatrix} Q^T L_1(\mathcal{G}) \\ \vdots \\ Q^T L_k(\mathcal{G}) \end{bmatrix} \in \mathbb{R}^{2k\times n}$$

$$\lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{bmatrix} \in \mathbb{R}^{k\times1}$$

$$\begin{aligned} \Gamma &= A(Q)^T(\lambda \otimes I_{2\times2}) \\ &= \begin{bmatrix} Q^T d_1 d_1^T \\ \vdots \\ Q^T d_k d_k^T \end{bmatrix}^T \begin{bmatrix} \lambda_1 I_{2\times2} \\ \vdots \\ \lambda_k I_{2\times2} \end{bmatrix} \\ &= \begin{bmatrix} d_1 d_1^T Q & \dots & d_k d_k^T Q \end{bmatrix} \begin{bmatrix} \lambda_1 I_{2\times2} \\ \vdots \\ \lambda_k I_{2\times2} \end{bmatrix} \\ &= \sum_{i=1}^{k} d_i \lambda_i d_i^T Q \\ &= D(\mathcal{G}) \Lambda D(\mathcal{G})^T Q \\ &= L_w(\mathcal{G}) Q \end{aligned}$$

$$M\ddot{Q} + MG + D(\mathcal{G})\Lambda D(\mathcal{G})^T Q = F$$

$$\ddot{Q} = -G + M^{-1}(F - D(\mathcal{G})\Lambda D(\mathcal{G})^T Q)$$

The equivalent to the velocity constraint $A(q)\dot{q} = 0$ in this case is:

$$\begin{bmatrix} \operatorname{Tr}(Q^T L(\mathcal{G})_1 \dot{Q}) \\ \vdots \\ \operatorname{Tr}(Q^T L(\mathcal{G})_k \dot{Q}) \end{bmatrix} = 0$$

Take the derivative:

$$\begin{bmatrix} \operatorname{Tr}(Q^T L_1(\mathcal{G}) \ddot{Q} + \dot{Q}^T L(\mathcal{G})_1 \dot{Q}) \\ \vdots \\ \operatorname{Tr}(Q^T L_k(\mathcal{G}) \ddot{Q} + \dot{Q}^T L(\mathcal{G})_k \dot{Q}) \end{bmatrix} = 0$$

$$\begin{bmatrix} \operatorname{Tr}(Q^T L_1(\mathcal{G}) \ddot{Q}) + \|\dot{Q}^T d_1\|^2 \\ \vdots \\ \operatorname{Tr}(Q^T L_k(\mathcal{G}) \ddot{Q}) + \|\dot{Q}^T d_k\|^2 \end{bmatrix} = 0$$

$$\begin{bmatrix} \operatorname{Tr}(Q^T L_1(\mathcal{G}) M^{-1}(F - D(\mathcal{G})\Lambda D(\mathcal{G})^T Q)) + \|\dot{Q}^T d_1\|^2 \\ \vdots \\ \operatorname{Tr}(Q^T L_k(\mathcal{G}) M^{-1}(F - D(\mathcal{G})\Lambda D(\mathcal{G})^T Q)) + \|\dot{Q}^T d_k\|^2 \end{bmatrix} = 0$$

$$\begin{bmatrix} \operatorname{Tr}(Q^T L_1(\mathcal{G}) M^{-1} D(\mathcal{G})\Lambda D(\mathcal{G})^T Q) \\ \vdots \\ \operatorname{Tr}(Q^T L_k(\mathcal{G}) M^{-1} D(\mathcal{G})\Lambda D(\mathcal{G})^T Q) \end{bmatrix}$$

$$= \begin{bmatrix} \operatorname{Tr}(Q^T L_1(\mathcal{G}) M^{-1} F) + \|\dot{Q}^T d_1\|^2 \\ \vdots \\ \operatorname{Tr}(Q^T L_k(\mathcal{G}) M^{-1} F) + \|\dot{Q}^T d_k\|^2 \end{bmatrix}$$

$$\begin{bmatrix} \operatorname{Tr}(D(\mathcal{G})^T Q Q^T L_1(\mathcal{G}) M^{-1} D(\mathcal{G})\Lambda) \\ \vdots \\ \operatorname{Tr}(D(\mathcal{G})^T Q Q^T L_k(\mathcal{G}) M^{-1} D(\mathcal{G})\Lambda) \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{Tr}(Q^T L_1(\mathcal{G}) M^{-1} F) + \|\dot{Q}^T d_1\|^2 \\ \vdots \\ \mathrm{Tr}(Q^T L_k(\mathcal{G}) M^{-1} F) + \|\dot{Q}^T d_k\|^2 \end{bmatrix}$$

Define the transformation $\mathrm{T}_{\mathrm{diag}} : \mathbb{R}^{k \times k} \to \mathbb{R}^k$ by

$$\mathrm{T}_{\mathrm{diag}}(A) = \mathrm{T}_{\mathrm{diag}}\left(\begin{bmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kk} \end{bmatrix}\right) = \begin{bmatrix} a_{11} \\ \vdots \\ a_{kk} \end{bmatrix}$$

$$\begin{bmatrix} \mathrm{T}_{\mathrm{diag}}(D(\mathcal{G})^T Q Q^T L_1(\mathcal{G}) M^{-1} D(\mathcal{G}))^T \\ \vdots \\ \mathrm{T}_{\mathrm{diag}}(D(\mathcal{G})^T Q Q^T L_k(\mathcal{G}) M^{-1} D(\mathcal{G}))^T \end{bmatrix} \lambda$$

$$= \begin{bmatrix} \mathrm{Tr}(Q^T L_1(\mathcal{G}) M^{-1} F) + \|\dot{Q}^T d_1\|^2 \\ \vdots \\ \mathrm{Tr}(Q^T L_k(\mathcal{G}) M^{-1} F) + \|\dot{Q}^T d_k\|^2 \end{bmatrix}$$

If the constraints are independent, the matrix should be invertible:

$$\lambda = \begin{bmatrix} \mathrm{T}_{\mathrm{diag}}(D(\mathcal{G})^T Q Q^T L_1(\mathcal{G}) M^{-1} D(\mathcal{G}))^T \\ \vdots \\ \mathrm{T}_{\mathrm{diag}}(D(\mathcal{G})^T Q Q^T L_k(\mathcal{G}) M^{-1} D(\mathcal{G}))^T \end{bmatrix}^{-1}$$

$$* \begin{bmatrix} d_1^T M^{-1} F Q^T d_1 + \|\dot{Q}^T d_1\|^2 \\ \vdots \\ d_k^T M^{-1} F Q^T d_k + \|\dot{Q}^T d_k\|^2 \end{bmatrix}$$

$$\lambda = \left(D(\mathcal{G})^T M^{-1} D(\mathcal{G}) \odot Q_e Q_e^T\right)^{-1} \begin{bmatrix} f_{1e}^T r_{1e} + \|\dot{r}_{1e}\|^2 \\ \vdots \\ f_{ke}^T r_{ke} + \|\dot{r}_{ke}\|^2 \end{bmatrix}$$

If we can cancel out the force terms within λ, we only need information of each bar's angular speed and the angle between adjacent bars to solve.

E. Example

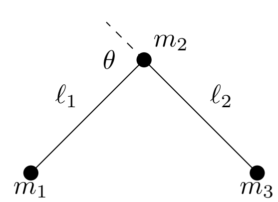

Fig. 12. Figure

$$n = 3,\ k = 2$$

$$Q = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix},\ G = \begin{bmatrix} 0 & g \\ 0 & g \\ 0 & g \end{bmatrix}$$

$$M = \begin{bmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{bmatrix},\ F = \begin{bmatrix} f_1^T \\ f_2^T \\ f_3^T \end{bmatrix}$$

$$M\ddot{Q} + MG + \Gamma = F$$

$$D(\mathcal{G}) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$L_1(\mathcal{G}) = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$L_2(\mathcal{G}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Solve for λ and substitute with edge variables:

$$Q_e = D(\mathcal{G})^T Q = \begin{bmatrix} (r_1 - r_2)^T \\ (r_2 - r_3)^T \end{bmatrix} = \begin{bmatrix} r_{1e}^T \\ r_{2e}^T \end{bmatrix}$$

$$\lambda = \begin{bmatrix} \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\ell_1^2 & -\frac{1}{m_2} r_{1e}^T r_{2e} \\ -\frac{1}{m_2} r_{1e}^T r_{2e} & \left(\frac{1}{m_2} + \frac{1}{m_3}\right)\ell_2^2 \end{bmatrix}^{-1}$$

$$* \begin{bmatrix} r_{1e}^T \left(\frac{f_1}{m_1} - \frac{f_2}{m_2}\right) + \|\dot{r}_{1e}\|^2 \\ r_{2e}^T \left(\frac{f_2}{m_2} - \frac{f_3}{m_3}\right) + \|\dot{r}_{2e}\|^2 \end{bmatrix}$$

$$= \frac{1}{\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\ell_1^2 \left(\frac{1}{m_2} + \frac{1}{m_3}\right)\ell_2^2 - \left(\frac{1}{m_2} r_{1e}^T r_{2e}\right)^2}$$

$$* \begin{bmatrix} \left(\frac{1}{m_2} + \frac{1}{m_3}\right)\ell_2^2 & \frac{1}{m_2} r_{1e}^T r_{2e} \\ \frac{1}{m_2} r_{1e}^T r_{2e} & \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\ell_1^2 \end{bmatrix}$$

$$* \begin{bmatrix} r_{1e}^T \left(\frac{f_1}{m_1} - \frac{f_2}{m_2}\right) + \|\dot{r}_{1e}\|^2 \\ r_{2e}^T \left(\frac{f_2}{m_2} - \frac{f_3}{m_3}\right) + \|\dot{r}_{2e}\|^2 \end{bmatrix}$$

$$= \frac{1}{\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\ell_1^2 \left(\frac{1}{m_2} + \frac{1}{m_3}\right)\ell_2^2 - \left(\frac{1}{m_2} r_{1e}^T r_{2e}\right)^2}$$

$$* \left[\begin{matrix} \left(\frac{1}{m_2} + \frac{1}{m_3}\right)\ell_2^2 \left(r_{1e}^T \left(\frac{f_1}{m_1} - \frac{f_2}{m_2}\right) + \|\dot{r}_{1e}\|^2\right) + \frac{1}{m_2} r_{1e}^T r_{2e} \left(r_{2e}^T \left(\frac{f_2}{m_2} - \right.\right. \\ \frac{1}{m_2} r_{1e}^T r_{2e} \left(r_{1e}^T \left(\frac{f_1}{m_1} - \frac{f_2}{m_2}\right) + \|\dot{r}_{1e}\|^2\right) + \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\ell_1^2 \left(r_{2e}^T \left(\frac{f_2}{m_2} - \right.\right. \end{matrix} \right.$$

Simplifications:

$$\|r_{ie}\| = \ell_i$$

$$r_{1e}^T r_{2e} = \ell_1 \ell_2 \cos\theta$$

For $\|\dot{r}_{1e}\|$, we consider the relative velocities of the masses. Let us write out: $\dot{r}_{1e} = \dot{r}_1 - \dot{r}_2$. If we choose a reference frame to be at m_2, we can say $\dot{r}'_{1e} = \dot{r}'_1$. The r_{1e} vector will always spin around the frame with constant radius, so there is a direct relationship between $\dot{r}_{1e}$ and the angular speed of the rotation of ℓ_1 given by:

$$\|\dot{r}_{1e}\|^2 = (\ell_1 \omega_1)^2$$

where ω_1 is the angular speed of the rotation of ℓ_1 with respect to a frame fixed on ℓ_1 (body frame).

F. Add restrictions

$$f_1 \perp r_{1e},\ f_2 = 0,\ f_3 \perp r_{2e}$$

$$\lambda = \frac{1}{\left(\frac{1}{m_1 m_2} + \frac{1}{m_1 m_3} + \frac{1}{m_2 m_3} + \frac{\sin^2\theta}{m_2^2}\right)\ell_1^2\ell_2^2} * \begin{bmatrix} \left(\frac{1}{m_2} + \frac{1}{m_3}\right)\ell_2^2\|\dot{r}_{1e}\|^2 + \frac{1}{m_2} r_{1e}^T r_{2e}\|\dot{r}_{2e}\|^2 \\ \frac{1}{m_2} r_{1e}^T r_{2e}\|\dot{r}_{1e}\|^2 + \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\ell_1^2\|\dot{r}_{2e}\|^2 \end{bmatrix} \quad (2)$$

$$\lambda = \frac{1}{\frac{1}{m_1 m_2} + \frac{1}{m_1 m_3} + \frac{1}{m_2 m_3} + \frac{\sin^2\theta}{m_2^2}} \begin{bmatrix} \left(\frac{1}{m_2} + \frac{1}{m_3}\right)\omega_1^2 + \frac{\ell_2}{m_2 \ell_1}\cos\theta\omega_2^2 \\ \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\omega_2^2 + \frac{\ell_1}{m_2 \ell_2}\cos\theta\omega_1^2 \end{bmatrix} \quad (3)$$

$$\begin{aligned} \Gamma &= D(\mathcal{G})\Lambda(\theta,\omega_1,\omega_2)D(\mathcal{G})^T Q \\ &= \begin{bmatrix} \lambda_1 r_{1e}^T \\ -\lambda_1 r_{1e}^T + \lambda_2 r_{2e}^T \\ -\lambda_2 r_{2e}^T \end{bmatrix} \end{aligned}$$

We see that Γ can be expressed entirely with edge variables. So, the only place where absolute variables are used in the full Euler-Lagrange equation is the $\ddot{Q}$ term, which means we can simplify the dynamics to $\ddot{Q} = \ldots$ with only knowledge of edge variables given a control F.

G. Control

Let's say the control goal is to cause $\theta \to 180°$ (bring the two rods together) and orient them in a specific direction, in this case make them flat. We specify the edge vectors to achieve: $r_{1e} \to \begin{bmatrix} \ell_1 \\ 0 \end{bmatrix}$ and $r_{2e} \to \begin{bmatrix} -\ell_2 \\ 0 \end{bmatrix}$ which is equivalent to:

$$Q_e \to Q_{e,d}$$

$$Q_{e,d} = \begin{bmatrix} \ell_1 & 0 \\ -\ell_2 & 0 \end{bmatrix} \quad (4)$$

If this is the goal, we can write the feedback linearization as:

$$\ddot{Q}_e = -K_1\left(Q_e - Q_{e,d}\right) - K_2\dot{Q}_e$$

The Euler-Lagrange equation is:

$$M\ddot{Q} + MG + \Gamma = F$$

$$\ddot{Q} + G + M^{-1}\Gamma = M^{-1}F$$

$$D(\mathcal{G})^T\ddot{Q} + D(\mathcal{G})^T G + D(\mathcal{G})^T M^{-1}\Gamma = D(\mathcal{G})^T M^{-1}F$$

Now convert to edge states using the edge transformation $Q_e = D(\mathcal{G})^T Q$:

$$\ddot{Q}_e + D(\mathcal{G})^T M^{-1}\Gamma(Q_e,\dot{Q}_e) = D(\mathcal{G})^T M^{-1}F$$

$$\begin{aligned} D(\mathcal{G})^T M^{-1}F = &-K_1\left(Q_e - Q_{e,d}\right) - K_2\dot{Q}_e \\ &+ D(\mathcal{G})^T M^{-1}\Gamma(Q_e,\dot{Q}_e) \end{aligned} \quad (5)$$

Now, if we have direct access to the weighted difference of force vectors on adjacent masses, we can control Q_e. We also have the freedom to add to the force vectors components that provide each mass with the same amount of acceleration without breaking this equation, meaning the robot can fight gravity while achieving relative consensus.

H. Back to the example

Let's say the force thrusters on m_1 and m_3 must be perpendicular to the rods and the thruster on m_2 is turned off. The scalars $u_1,\ u_2 \in \mathbb{R}$ are the control inputs.

$$R_{90} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$f_1 = u_1 R_{90}\frac{r_{1e}}{\ell_1}$$

$$f_2 = 0$$

$$f_3 = u_2 R_{90}\frac{r_{2e}}{\ell_2}$$

The left side of equation 5 works out to:

$$D(\mathcal{G})^T M^{-1}F = \begin{bmatrix} u_1\frac{1}{m_1\ell_1}(R_{90}r_{1e})^T \\ -u_2\frac{1}{m_3\ell_2}(R_{90}r_{2e})^T \end{bmatrix}$$

We cannot choose this matrix arbitrarily given the constraints on the thruster directions, so we must project onto the workspace. Let the ith row of equation 5 take the form:

$$u_i a_i^T \approx b_i^T$$

The best solution for u_i is:

$$u_i = \frac{a_i^T b_i}{a_i^T a_i}$$

Using this as the control force, we get the following solutions of r_{1e} and r_{2e} over time.

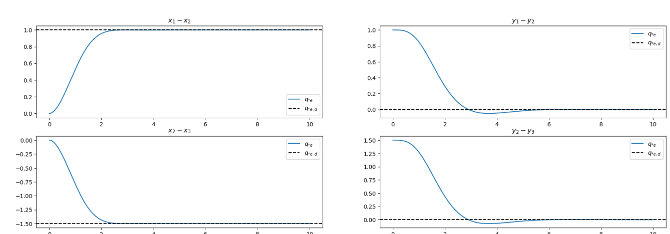

Fig. 13. Simulated edge states plot

In the simulation, $\ell_1 = 1$ and $\ell_2 = 1.5$, so the control objective from equation 4 is satisfied.

I. Closer look at the solution

The control objective has been achieved when $Q_e = Q_{e,d}$ and $\dot{Q}_e = 0$. Looking at equation 2, we see that if $\dot{Q}_e = 0$, then $\lambda = 0$. The error terms in equation 5 will go to zero and the Γ will also go to zero if $\lambda = 0$. Therefore, it makes sense for the forces to converge to zero as the objective is achieved as seen in Figure 15. If $\dot{Q}_e$ is desired to be something other

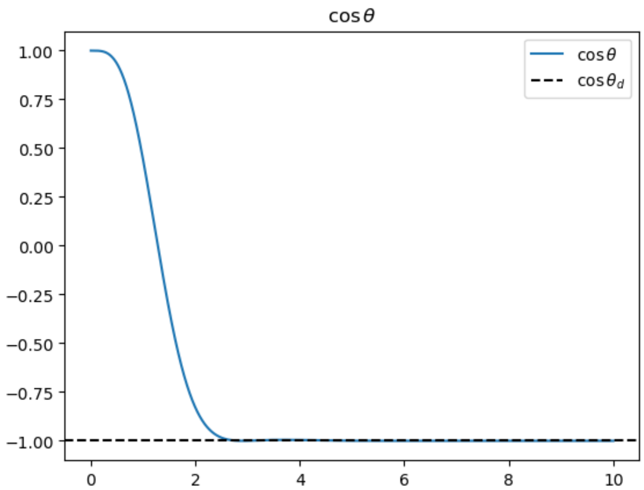

Fig. 14. Simulated angle between rods

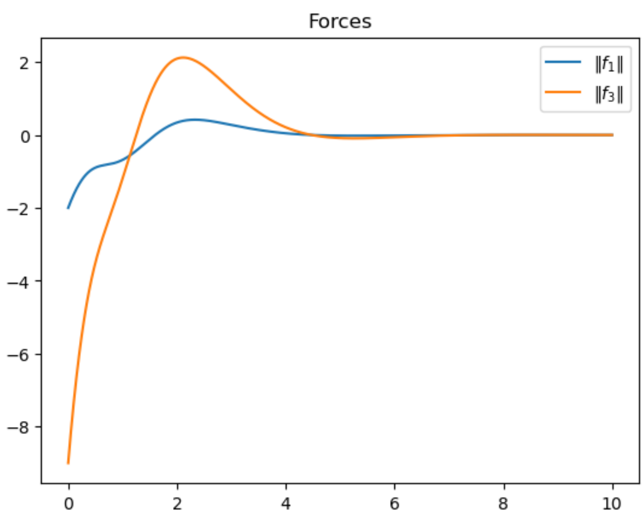

Fig. 15. Simulated control forces

than zero (rods keep spinning at a desired frequency) then the forces would need to keep going.

Understand the solution better

The solution works for every combination of initial conditions and control objectives that I tried including 90 degree configurations and nonzero initial velocities that are feasible.

Lyapunov function

What if we remove all restrictions from f_2, but then place restrictions on f_1 and f_3 so that the force terms go away from λ?

Leader-follower network,

Longer snake, is there a local network?

Mapping between difference of forces and a single force on the link.

J. Longer snake

$$D(\mathcal{G}) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

K. Node control

$$\begin{gathered} f_2 = m_2(-k_3(r_2 - r_{2,d}(t)) - k_4(\dot{r}_2 - \dot{r}_{2,d}(t)) + \ddot{r}_{2,d}(t)) \\ + m_2 \begin{bmatrix} 0 \\ g \end{bmatrix} \\ - \lambda_1(Q_{e,d}, \dot{Q}_{e,d}) r_{1e,d} + \lambda_2(Q_{e,d}, \dot{Q}_{e,d}) r_{2e,d} \end{gathered} \tag{6}$$

Note that equation 6 only contains state information about r_2 and $\dot{r}_2$.

Gravity seems to be a really big issue to overcome.

IX. Summary

- Matrix differential form of the Euler-Lagrange equation
- Derivation of Γ (constraint force vectors) and λ (constraint force magnitudes)
- Example of simple overparameterized system with distance constraints 2-bar
- What info is needed for λ
- PD control of edge states
- Adding control of global variables with edge control (leader-follower)
- More examples: jointed wing, star, X-wing