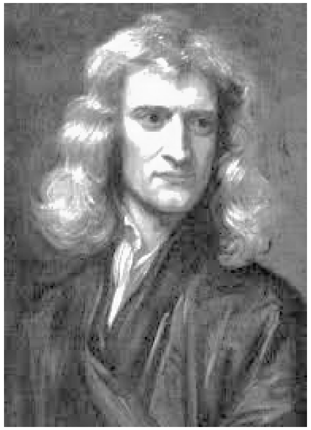
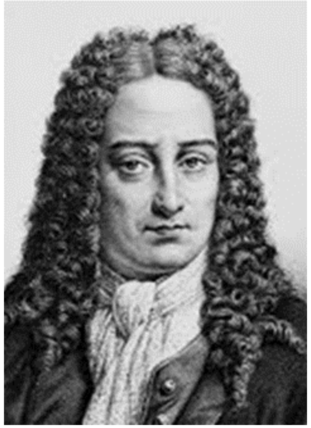
Chapter 5
PARTIAL DIFFERENTIAL EQUATIONS



Isaac Newton
(1642-1726)
Math/Physics
Universal Gravity
Newtonian Mechanics
***Differential* Calculus**

Lecture 17

5.1 Introduction to Differential Equations



Gottfried Wilhelm Leibniz
(1646-1716)
Math/Physic
***Integral* Calculus**
Leibniz Notation

5.1 Introduction to Partial Differential Equations

Most of the physics laws, such as Newton's law, Maxwell's equations, and Schrödinger equation of quantum mechanics, can be written as differential equations, typically in space and/or time derivatives. Depending on the number of variables, we have either an **ordinary differential equation (ODE) with a single variables** or a **partial differential equation (PDE) with n variables ($n \geq 2$)**.

Differential Notations

$$\frac{df(x)}{dx} = f'(x) = f^{(1)}(x) = f_x(x)$$

$$\frac{d^2 f(x)}{dx^2} = f''(x) = y^{(2)}(x) = f_{xx}(x), \;\cdots\;, \quad \frac{d^n y(x)}{dx^n} = y^{(n)}(x)$$

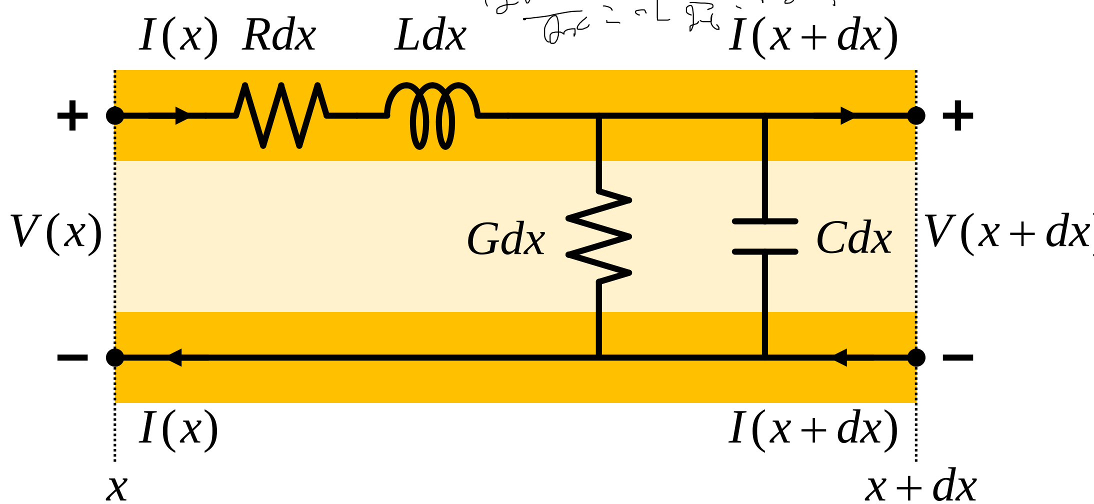
$$\frac{dg(x,y)}{dx} = g_x(x,y)$$

$$\frac{d^2 g(x,y)}{dxdy} = g_{xy}(x,y) = g_{yx}(x,y) = \frac{d^2 g(x,y)}{dydx}$$

Basic Concept for EE Students: Transmission Line (or Telegraph) Equation

The transmission line equation is a coupled partial differential equation:

$$\boxed{\begin{aligned}\frac{\partial V(x,t)}{\partial x} &= -L\frac{\partial I(x,t)}{\partial t} - RI(x,t)\\ \frac{\partial I(x,t)}{\partial x} &= -C\frac{\partial V(x,t)}{\partial t} - GV(x,t)\end{aligned}}$$



Decomposed into two second-order partial differential equations for voltage and current,

$$\boxed{\begin{aligned}\frac{\partial^2}{\partial x^2}V(x,t) &= LC\frac{\partial^2}{\partial t^2}V(x,t) + (RC+GL)\frac{\partial}{\partial t}V(x,t) + RGV(x,t)\\ \frac{\partial^2}{\partial x^2}I(x,t) &= LC\frac{\partial^2}{\partial t^2}I(x,t) + (RC+GL)\frac{\partial}{\partial t}V(x,t) + RGV(x,t)\end{aligned}}$$

we usually transform from time domain to frequency domain using harmonic time dependence, $e^{\pm i\omega t}$.

Types of Partial Differential Equations

The classification of PDEs is an important concept since the general theory and methods of solutions depends on the types of PDEs. There are three main classifications:

1) **Order** of the PDE: the order of the highest derivative.

$u_t = au_x$ (first order)

$u_t = bu_{xx}$ (second order)

2) **Dimensionality**: Number of Variables

$u_t = au_x$ (two variables, 1D in space and time parameter)

$u_t = b(u_{xx} + u_{yy})$ (two variables, 2D in space and time parameter)

3) **Linearity**: With/Without Superposition Principle

$u_{tt} = au_x$ (linear)

$u_x = bu^2$ (nonlinear)

The second-order linear PDEs with two variables are give by

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \quad (5.1)$$

4) **Homogeneity**: $G = 0$ or $G \neq 0$

$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$ (homogeneous)

$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \neq 0$ (nonhomogeneous)

Three Basic Types of Second-Order Linear PDEs

Considering the second-order linear PDEs in (5.1), $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we have thee basic types:

1) **Parabolic** PDEs: $B^2 - AC = 0$ (Diffusion or Heat Equation)
2) **Hyperbolic** PDEs: $B^2 - AC > 0$ (Wave Equation)
3) **Elliptic PDEs**: $B^2 - AC < 0$ (Laplace Equation)

Hyperbolic

Ex) $u_t = u_{xx} \rightarrow B^2 - AC = 0$ (Parabolic)

$u_{tt} = u_{xx} \rightarrow B^2 - AC = 0$ (Hyperbolic)

$u_{xx} + u_{yy} = 0 \rightarrow B^2 - 4AC = -4$ (Elliptic)

General Form of n^th-Order PDEs

$$F(x, u_x, u_{xx}, \cdots u^{(n)}) = 0 \quad \text{or} \quad F(x, u_x, u_{xx}, \cdots u^{(n-1)}) = u^{(n)}$$