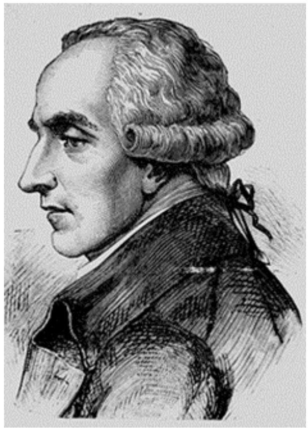
Chapter 4
INTEGRAL TRANSFORMS

Joseph Fourier
(1768-1830)
Math/Physics
Fourier Series/Transform

Lecture 16

4.3 Laplace Transform



Pierre-Simon Laplace
(1749-1827)
Math/Physic
Laplace Transform
Laplace Equation
(Scalar Potential Theory)

Scaling $$\mathfrak{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right) \tag{4.43}$$

Shifting $$\mathfrak{L}[f(t-a)u(t-a)] = e^{-as}F(s), \quad a > 0 \tag{4.44}$$

Attenuation $$\mathfrak{L}[e^{-at} f(t)] = F(s+a), \quad a > 0 \tag{4.45}$$

Derivative
$$\begin{gathered} \mathfrak{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0) \\ \mathfrak{L}\left[f^{(n)}(t)\right] = s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0) \end{gathered} \tag{4.46}$$

Integral $$\mathfrak{L}\left[\int_0^t d\tau\, f(\tau)\right] = \frac{F(s)}{s} \tag{4.47}$$

Time Product $$\mathfrak{L}\left[t^n f(t)\right] = (-1)^n F^{(n)}(s) \tag{4.48}$$

Time Division $$\mathfrak{L}\left[\frac{1}{t} f(t)\right] = \int_s^\infty ds' F(s') \tag{4.49}$$

Periodic Function $$\mathfrak{L}\left[f(t) = f(t+T)\right] = \frac{1}{1-e^{-Ts}} \int_0^T dt\, f(t) \tag{4.50}$$

Initial/Final Values
$$\begin{gathered} f(0^+) = \lim_{s\to\infty} sF(s) \\ f(\infty) = \lim_{s\to 0} sF(s) \end{gathered} \tag{4.51}$$

Inverse Laplace Transform: Partial Fraction Decomposition

For many problems, the LT is given by a ratio of two polynomials

$$F(s) = \frac{P_m(s)}{Q_n(s)} = \frac{p_0 + p_1 s + p_2 s^2 + \cdots + p_m s^m}{q_0 + q_1 s + q_2 s^2 + \cdots + q_n s^n}, \quad m < n \tag{4.52}$$

where without loss of generality we may assume that the degree of $P(s)$ is lower than that of $Q(s)$. By factoring $Q(s)$, we have in general

$$Q_n(s) = q_n (s - s_1)^{r_1} (s - s_2)^{r_2} \cdots (s - s_i)^{r_i} \cdots (s - s_n)^{r_n} \tag{4.53}$$

Here, s_i is the c_i^{th}-th order zeros (poles) of $Q(s)$ ($F(s)$),* and we have

$$\begin{aligned} F(s) = &\left[\frac{S_1^{(1)}}{(s-s_1)} + \frac{S_1^{(2)}}{(s-s_1)^2} + \cdots + \frac{S_1^{(r_1)}}{(s-s_1)^{r_1}} \right] + \left[\frac{S_2^{(1)}}{(s-s_2)} + \frac{S_2^{(2)}}{(s-s_2)^2} + \cdots + \frac{S_2^{(r_2)}}{(s-s_2)^{r_2}} \right] \\ &+ \cdots + \left[\frac{S_n^{(1)}}{(s-s_n)} + \frac{S_n^{(2)}}{(s-s_n)^2} + \cdots + \frac{S_n^{(r_n)}}{(s-s_n)^{r_n}} \right] \end{aligned} \tag{4.54}$$

* It should be noted that unfortunately there is no general zero-finding algorithm for n^{th}-order polynomials except for $n \le 4$.

which is given in a compact form by

$$F(s)=\sum_{i=1}^{n}\left[\sum_{j=1}^{r_n}\frac{S_i^{(j)}}{(s-s_i)^j}\right] \tag{4.55}$$

include negative

Analytic part

Laurent expansion ⊂ Taylor expansion

Inverse Laplace Transform: Residue Method

Now it should be noted that the summation in the parenthesis in (4.55) corresponds to a **Principal part** of **Laurent series expansion** for s_i (Lecture-8) and therefore we can use the Residue theorem.

Multiplying (4.55) by $(s-s_i)^{j-1}$, we have

$$(s-s_i)^{j-1}F(s)=\sum_{i=1}^{n}\left[\sum_{j=0}^{r_n}\frac{S_i^{(j)}}{s-s_i}\right] \tag{4.56}$$

Therefore we find that $A_i^{(j)}$ is just the **residue** of $(s-s_i)^{j-1}F(s)$ at each **simple pole** at s_i :

$$\boxed{S_i^{(j)}=\text{Res}\left[(s-s_i)^{j-1}F(s)\right]_{s=s_i}=\frac{1}{(r_n-j)!}\lim_{s\to s_i}\frac{d^{r_n-j}}{ds^{r_n-j}}\left[(s-s_i)^{r_n}F(s)\right]} \tag{4.57}$$

Ex) Simple poles

$$F(s)=\frac{1}{(s-2)(s-3)}=\frac{A}{s-2}+\frac{B}{s-3}$$

$$A=(s-2)F(s)\Big|_{s=2}=-1 \qquad \rightarrow \qquad f(t)=-e^{2t}+e^{3t}$$

$$B=(s-3)F(s)\Big|_{s=3}=1$$

Ex) Simple and Double poles

two terms

$$F(s)=\frac{s^2+s+1}{s^2(s-1)^2}=\underbrace{\frac{A_1}{s}+\frac{A_2}{s^2}}+\underbrace{\frac{B_1}{(s-1)}+\frac{B_2}{(s-1)^2}}$$

$$A_1=\frac{1}{1!}\frac{d}{ds}\frac{s^2+s+1}{(s-1)^2}\Bigg|_{s=0}=3, \quad A_2=\frac{1}{0!}\frac{s^2+s+1}{(s-1)^2}\Bigg|_{s=0}=1 \qquad \rightarrow \qquad f(t)=3+t-3e^t+3te^t$$

$$B_1=\frac{1}{1!}\frac{d}{ds}\frac{s^2+s+1}{s^2}\Bigg|_{s=0}=-3, \qquad B_2=\frac{1}{0!}\frac{s^2+s+1}{s^2}\Bigg|_{s=1}=3$$

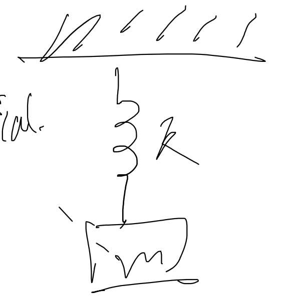
friction: dissipate Energy to air molecule

Classical 1-D Damped Oscillator

Damped oscillator is a model system in classical physics, and the its dymanics is given by a simple Newton's equation with a frictional force,

Hook's law.

$$m\frac{d^2x(t)}{dt^2}+\gamma\frac{dx(t)}{dt}+kx(t)=0$$



kinetic ⟷ potential.

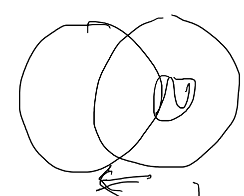
with the $\boxed{\text{initial conditions}}$ of $x(0)=x_0$ and $v(0)=0$.

↳ initial velocity.

Solution) Taking LT, for a small damping ($\gamma^2<4km$), we have

$$m\left[s^2X(s)-sx_0\right]+\gamma\left[sX(s)-x_0\right]+kX(s)=0$$

$$\rightarrow \quad X(s)=\frac{ms+\gamma}{ms^2+\gamma s+k}=\frac{s+\gamma/m}{\left(s+\dfrac{\gamma}{2m}\right)^2+\left(\dfrac{k}{m}-\dfrac{\gamma^2}{4m^2}\right)}$$



polarization.

and we find

$$x(t)=x_0\exp\left(-\frac{\gamma}{2m}t\right)\left[\cos\omega_1 t-\phi\right]$$

Energy dissipated to thermal Energy.

FT, LT important theorem.

$$\text{with}\quad \omega_0^2=\frac{k}{m},\quad \omega_1^2=\frac{k}{m}-\frac{\gamma^2}{4m^2}>0,\quad \phi=\tan^{-1}\left(\frac{\gamma}{2m\omega_1}\right)$$