2007 수학

1. (a) $X^T A = [x_1 \cdots x_M] \begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & & \vdots \\ a_{M1} & & a_{MN} \end{bmatrix} = \left[\sum_{k_1=1}^{M} x_{k_1} a_{k_1 1} \; \sum_{k_2=1}^{M} x_{k_2} a_{k_2 2} \cdots \sum_{k_n=1}^{M} x_{k_n} a_{k_n N} \right]$

$$X^T A Y = X^T A \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} = \sum_{l=1}^{N} y_l \sum_{k=1}^{M} x_k a_{kl} = \underline{\sum_{l=1}^{N} \sum_{k=1}^{M} x_k a_{kl} y_l.}$$

(b) $B\underline{x} = \lambda \underline{x} \cdots ①$

$B^* \underline{x}^* = \lambda^* \underline{x}^* \Rightarrow B\underline{x}^* = \lambda^* \underline{x}^*.$

$(\underline{x}^*)^T \lambda \underline{x} = (\underline{x}^*)^T B \underline{x} = (B^T \underline{x}^*)^T \underline{x} = (B\underline{x}^*)^T \underline{x} = \lambda^* (\underline{x}^*)^T \underline{x}$

$= (\underline{x}^*)^T \lambda^* \underline{x}$

$\therefore \lambda = \lambda^*$, so all λ's are real when B is real symmetric.

(c) $|\lambda I - C| = \begin{vmatrix} \lambda-1 & -3 \\ -4 & \lambda-5 \end{vmatrix} = \lambda^2 - 6\lambda + 5 - 12 = \lambda^2 - 6\lambda - 7 = (\lambda+1)(\lambda-7) = 0.$

$\therefore \underline{\lambda_1 = -1, \quad \lambda_2 = 7}$

(d) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 + 2x_2 = 0 \quad \therefore \underline{\begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}}.$

2. (a) $\int_0^T e^{j\omega t} dt = \frac{1}{j\omega} [e^{j\omega T} - 1] = 0, \quad e^{j\omega T} = 1 = e^{j2\pi k}, \quad \forall k \in \mathbb{Z}.$

$\therefore \omega = \frac{2\pi k}{T}, \quad \forall k \in \mathbb{Z}.$

(b) Let $x(t) = \sum_{k=-\infty}^{\infty} b_k e^{j\frac{2\pi}{T}kt}$, then $\int_0^T \left| \sum_{k=-\infty}^{\infty} b_k e^{j\frac{2\pi}{T}kt} - \sum_{n=-N}^{N} a_n e^{j\frac{2\pi}{T}nt} \right|^2 dt$

$= \int_0^T \left| \sum_{k=-\infty}^{-N-1} b_k e^{j\frac{2\pi}{T}kt} - \sum_{k=N+1}^{\infty} b_k e^{j\frac{2\pi}{T}kt} + \sum_{n=-N}^{N} (b_n - a_n) e^{j\frac{2\pi}{T}nt} \right|^2 dt = g(t)$, then,

$\min\{g(t)\} = \min\left\{ \sum_{n=-N}^{N} (b_n - a_n) e^{j\frac{2\pi}{T}nt} \right\} \Rightarrow a_n = b_n.$

$\therefore \underline{b_n = \frac{1}{T} \int_0^T x(t) e^{-j\frac{2\pi}{T}nt} dt = a_n}$

3. $\frac{d}{dx}\left(\sin^{-1}(x^2-1)\right)' = \frac{2x}{\sqrt{1-(x^2-1)^2}} = \frac{2x}{\sqrt{2x^2-x^4}} = \frac{2}{\sqrt{2-x^2}}$

4. $\int (\ln x)^2 dx = x(\ln x)^2 - \int 2(\ln x)\frac{1}{x}\cdot x\, dx = x(\ln x)^2 - 2\int \ln x\, dx + C_2$
$+C_1$

$= x(\ln x)^2 - 2\left[x\ln x - \int dx\right] = x(\ln x)^2 - 2x\ln x + 2x + C$

5. (a) $\frac{1}{1+x^2} = 1 + \frac{1}{1!}\left(\frac{-2x}{(1+x^2)^2}\right)\Big|_{x=0}\cdot(x-0) + \frac{1}{2!}\,\frac{-2(1+x^2)^2+8x(1+x^2)\cdot 2x}{(1+x^2)^4}\Big|_{x=a}(x-0)^2 + \dots$

$= 1 - \frac{2}{2!}x^2 + \frac{24}{4!}x^4 + \dots + \frac{(2n)!}{(2n)!}x^{2n}(-1)^n$

$= 1 - x^2 + x^4 - x^6 + \dots + x^{2n}(-1)^n$

(b) $\tan^{-1}x = y$, $\tan y = x$, $\sec^2 y\, dy = dx.$, $\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y}$
$= \frac{1}{1+x^2}.$

$\therefore (\tan^{-1}x)' = \frac{1}{1+x^2} = 1 - x^2 + x^4 \dots + x^{2n}(-1)^n$

$\Rightarrow \tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots + \frac{1}{2n+1}x^{2n+1}(-1)^n$

(c) $\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x\cos y + \cos x\sin y}{\cos x\cos y - \sin x\sin y} = \frac{\frac{\sin x\cos y}{\cos x\cos y} + \frac{\cos x\sin y}{\cos x\cos y}}{1 - \frac{\sin x\sin y}{\cos x\cos y}}$

$= \frac{\tan x + \tan y}{1 - \tan x\tan y}$

Let $\tan^{-1}\left(\frac{1}{2}\right) = a$, $\tan^{-1}\left(\frac{1}{3}\right) = b$,
$\tan\left(\frac{\pi}{4}\right) = \tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a\tan b} = \frac{\tan(\tan^{-1}\frac{1}{2}) + \tan(\tan^{-1}\frac{1}{3})}{1 - \tan(\tan^{-1}\frac{1}{2})\tan(\tan^{-1}\frac{1}{3})}$

$= \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}} = \frac{\frac{5}{6}}{1-\frac{1}{6}} = 1 = \tan\frac{\pi}{4}$

$\therefore \frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$

(d) $\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{n}x^n + \cdots + \frac{1}{2n+1}x^{2n+1}(-1)^n$ 에서

$$\frac{\pi}{4} \approx \tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{n} + \cdots \quad ①$$

$$\frac{\pi}{4} \approx \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{1}{2} - \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5}\left(\frac{1}{2}\right)^5 - \frac{1}{n}\left(\frac{1}{2}\right)^n + \cdots$$
$$+ \frac{1}{3} - \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 - \frac{1}{n}\left(\frac{1}{3}\right)^n + \cdots \quad ②$$

①의 경우보다 ②의 경우가 $\frac{\pi}{4}$를 계산하는 데에 있어서 더 정확하다

6. Integral test : $\int_2^\infty \frac{1}{n\ln n}dn$, $\ln n = t$, $\frac{1}{n}dn = dt$, $\int_2^\infty \frac{1}{n\ln n}dn = \int_{\ln 2}^\infty \frac{1}{t}dt$.

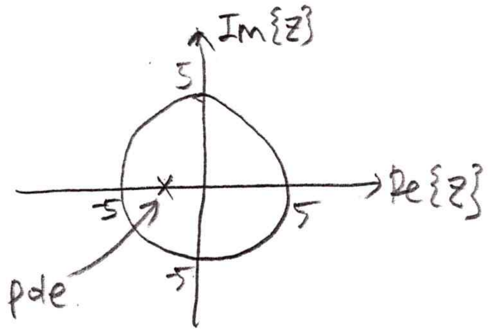
$\int_{\ln 2}^\infty \frac{1}{t}dt = \ln t \Big|_{\ln 2}^\infty \to \infty$. $\therefore$ $\underline{\sum_{n=2}^\infty \frac{1}{n\ln n} \text{ diverges}}$.

7. $\begin{cases} sY_1 - y_1(0) = -Y_2 \\ sY_2 - y_2(0) = Y_1 \end{cases}$ $\quad \begin{cases} sY_1 - 1 = -Y_2 \\ sY_2 = Y_1 \end{cases}$ $\Rightarrow$ $s^2Y_2 - 1 = -Y_2$, $Y_2(s^2+1) = 1$, $Y_2 = \frac{1}{s^2+1}$

$\therefore$ $Y_1(s) = \frac{s}{s^2+1}$, $Y_2(s) = \frac{1}{s^2+1}$.

$\therefore$ $y_1(t) = (\cos t)u_s(t)$, $y_2(t) = (\sin t)u_s(t)$, where $u_s(t)$: unit step f.t

8.



$$\therefore \oint_c \frac{\sin 3z}{z + \frac{\pi}{2}} dz = 2\pi j \sin\left(-\frac{3}{2}\pi\right) = +2\pi j$$

$$\therefore \frac{1}{2\pi j}\oint_c \frac{\sin 3z}{z + \frac{\pi}{2}} dz = +1$$

2007 제어

제어 필수

1. Differential equation : $r\cos\theta - Mg\sin\theta = ML\ddot{\theta}$

a) $\therefore \ddot{\theta}(t) = -\frac{g}{L}\sin\theta(t) + \frac{r(t)}{ML}\cos\theta(t)$

b) Equilibrium state $\Rightarrow$ $\ddot{\theta}(t)=0$, $\frac{g}{L}\sin\theta_0(t) = \frac{\cos\theta_0(t)}{ML} r_0$,

$r_0 = Mg\tan\theta_0 = \frac{9.8}{\sqrt{3}}$ (nominal value of $r(t)$)

c) $\dot{x}_1(t) = x_2(t)$,

$\dot{x}_2(t) = -9.8\sin x_1(t) + r(t)\cos x_1(t)$

$\Delta\dot{x}_1(t) = \Delta x_2(t)$

$\Delta\dot{x}_2(t) = -9.8\cos x_{o1}\cdot\Delta x_1(t) + \cos x_{o1}\cdot\Delta r(t) - r_0\sin x_{o1}\cdot\Delta x_1(t)$

$= -4.9\sqrt{3}\,\Delta x_1(t) - \frac{4.9}{\sqrt{3}}\Delta x_1(t) + \frac{\sqrt{3}}{2}\Delta r(t)$

$= -\frac{19.6}{\sqrt{3}}\Delta x_1(t) + \frac{\sqrt{3}}{2}\Delta r(t)$

$$\therefore \begin{bmatrix} \Delta\dot{x}_1(t) \\ \Delta\dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{19.6}{\sqrt{3}} & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\sqrt{3}}{2} \end{bmatrix} \Delta r(t)$$

$$\Delta y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix}$$

∴ 제어 선택

1. ① relaxed system at t_0

: System의 output $y[t_0, \infty]$이 input $u[t_0, \infty]$에 의해서만 결정될 때 t_0에서 relaxed system이다.

② causal system

: $t=t_0$에서의 output이 $t>t_0$에서의 input의 영향을 받지 않으면 causal system

③ Stable in the sense of Lyapunov

: Finite 한 Any initial state $X(t_0)$가 bounded response를 excite한 때

④ Asymptotically stable

: 일단 Lyapunov stable 하면서, $\|x(0)\| < \delta$, $\lim_{t\to\infty}\|x(t)\| = 0$ $(\delta>0)$ 이면 asymptotically stable.

2. $G_p(s) = \dfrac{s+1}{s^3+4s^2+5s+2}$

i) $\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x$

ii) $\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & -5 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

full rank가 아니므로 NOT observable!

iii) 뒷장.

$\therefore$ iii) $\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2-k_1 & -5-k_2 & -4-k_3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix} u$

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 2+k_1 & 5+k_2 & \lambda+4+k_3 \end{vmatrix} = \lambda(\lambda^2 + (4+k_3)\lambda + (5+k_2)) + (2+k_1)$$

$$= \lambda^3 + (4+k_3)\lambda^2 + (5+k_2)\lambda + (2+k_1)$$

$$= (\lambda^2+8\lambda+25)(\lambda+5) = \lambda^3+5\lambda^2+8\lambda^2+40\lambda+25\lambda+125$$

$$= \lambda^3 + 13\lambda^2 + 65\lambda + 125$$

$$\therefore \underline{K = [123 \quad 60 \quad 9]}$$

Transfer function $H(s) = C(sI-A)^{-1}B$,

$$H(s) = [1 \quad 1 \quad 0] \underbrace{\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 125 & 65 & s+13 \end{bmatrix}^{-1}}_{\tilde{A}} \begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}$$

$$= \frac{(\tilde{A}_{1,3} + \tilde{A}_{2,3})P}{\det(sI-A)}, \quad \tilde{A}_{1,3} = 1, \quad \tilde{A}_{2,3} = S$$

$$\therefore H(s) = \frac{p(s+1)}{s^3+13s^2+65s+125}.$$

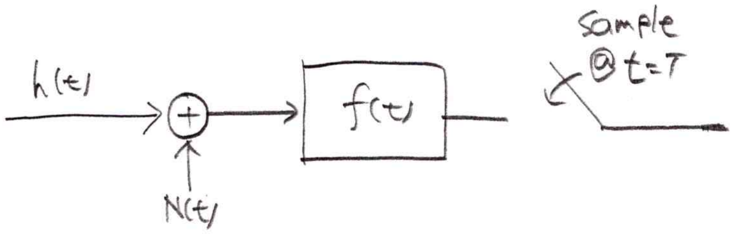
unit-step input

$$e_{ss} = \lim_{t\to\infty}(v(t)-y(t)) = \lim_{s\to 0} s(V(s)-Y(s)) = \lim_{s\to 0} sV_s(1-H(s))$$

$$= \lim_{s\to 0}(1-H(s)) = 0, \quad \lim_{s\to 0} H(s) = \frac{p}{125} = 1. \quad \therefore \underline{p=125}$$

∴ 2007 통신

1.



(a) $N = N(t) * f(t)\Big|_{t=T} = \int_{-\infty}^{\infty} N(\tau) f(t-\tau) d\tau \Big|_{t=T} = \int_{-\infty}^{\infty} N(\tau) f(T-\tau) d\tau.$

$E[N] = \int_{-\infty}^{\infty} E[N(\tau)] f(T-\tau) d\tau = 0.$

$Var(N) = E[N^2] = E\left[\int_{-\infty}^{\infty} N(\tau_1) f(T-\tau_1) d\tau_1 \int_{-\infty}^{\infty} N(\tau_2) f(T-\tau_2) d\tau_2\right]$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[N(\tau_1) N(\tau_2)] f(T-\tau_1) f(T-\tau_2) d\tau_1 d\tau_2$

$= \frac{N_0}{2} \int_{-\infty}^{\infty} f^2(T-\tau) d\tau = \frac{N_0}{2} \int_{-\infty}^{\infty} |F(f)|^2 df$ (by Parseval's theorem)

$\therefore \underline{N \sim N\left(0, \frac{N_0}{2} \int_{-\infty}^{\infty} |F(f)|^2 df\right)}$

(b) $A = h(t) * f(t)\Big|_{t=T} = \int_{-\infty}^{\infty} h(\tau) f(T-\tau) d\tau$, $h(\tau) = \int_{-\infty}^{\infty} H(f) e^{j2\pi f\tau} df$ 이므로

$A = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(f) e^{j2\pi f\tau} f(T-\tau) d\tau df = \int_{-\infty}^{\infty} H(f) \left(\int_{-\infty}^{\infty} f(T-\tau) e^{j2\pi f\tau} d\tau\right) df,$

$T-\tau = \tau', \left(\int_{-\infty}^{\infty} f(\tau') e^{-j2\pi f\tau'} d\tau'\right) e^{j2\pi fT} = F(f) e^{j2\pi fT}.$

$\therefore \underline{A = \int_{-\infty}^{\infty} H(f) F(f) e^{j2\pi fT} df}$

∴ (c) $SNR = \dfrac{A^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}|F(f)|^2 df} = \dfrac{\left|\int_{-\infty}^{\infty} H(f)F(f)e^{j2\pi fT}df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}|F(f)|^2 df} \le \dfrac{\int_{-\infty}^{\infty}|H(f)e^{j2\pi fT}|^2 df \int_{-\infty}^{\infty}|F(f)|^2 df}{\frac{N_0}{2}\int_{-\infty}^{\infty}|F(f)|^2 df}$

$= \dfrac{2}{N_0}\int_{-\infty}^{\infty}|H(f)e^{j2\pi fT}|^2 df$

equality가 성립할 조건 : $F(f) = \left(H(f)e^{j2\pi fT}\right)^* = H^*(f)e^{-j2\pi fT}$

$\Rightarrow f(t) = h^*(-(t-T)) = \underline{h^*(T-t)}$ ← Matched filter.

(d) $p(t) = h(t) * f(t)$, For zero-ISI condition,

$\begin{cases} p(0) = \text{Constant} \\ p(nT) = 0, \text{ for } n \neq 0,\ n \in \mathbb{Z} \end{cases}$, $\sum_{k=-\infty}^{\infty} P\left(f - \frac{k}{T}\right) = \text{constant}.$

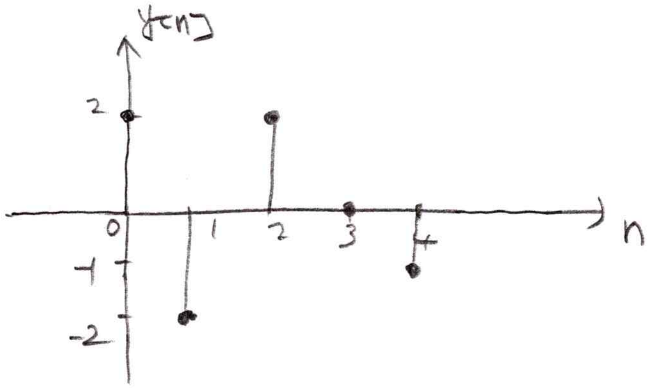
2. (a) $H[k] = \sum_{n=0}^{4} h[n] e^{-j\frac{2\pi}{5}nk} = 1 - e^{-j\frac{2\pi}{5}k} + e^{-j\frac{4\pi}{5}k}$

$X[k] = \sum_{n=0}^{4} x[n] e^{-j\frac{2\pi}{5}nk} = 1 - e^{-j\frac{2\pi}{5}k} + e^{-j\frac{6\pi}{5}k}.$

$Y[k] = X[k]H[k] = 1 - e^{-j\frac{2\pi}{5}k} + e^{-j\frac{6\pi}{5}k} - e^{-j\frac{2\pi}{5}k} + e^{-j\frac{4\pi}{5}k} - e^{-j\frac{8\pi}{5}k}$
$+ e^{-j\frac{4\pi}{5}k} - e^{-j\frac{6\pi}{5}k} + e^{-j\frac{10\pi}{5}k}$

$= 1 - 2e^{-j\frac{2\pi}{5}k} + 2e^{-j\frac{4\pi}{5}k} - e^{-j\frac{8\pi}{5}k} + \underbrace{e^{-j\frac{10\pi}{5}k}}_{1}$

∴ $y[n] = \{2, -2, 2, 0, -1\}$



(b) $\text{DFT}\{x[n]\} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk} = H[k]$

$$y[n] = \text{DFT}\{H[k]\} = \sum_{k=0}^{N-1} H[k] e^{-j\frac{2\pi}{N}nk} = \sum_{k=0}^{N-1} \left(\sum_{m=0}^{N-1} x[m] e^{-j\frac{2\pi}{N}mk} \right) e^{-j\frac{2\pi}{N}nk}$$

$$= \sum_{m=0}^{N-1} x[m] \underbrace{\sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}k(m+n)}}_{\rightarrow \frac{1-e^{-j2\pi(m+n)}}{1-e^{-j\frac{2\pi}{N}(m+n)}}}$$

$$\therefore y[n] = \sum_{m=0}^{N-1} x[m] \frac{1-e^{-j2\pi(m+n)}}{1-e^{-j\frac{2\pi}{N}(m+n)}}$$