HW-1 Solutions

Assign: 2017-03-21 (Tue) Due : 2017-03-27 (Mon 5pm)

1. Find an algebraic expression of the Levi-Civita symbol ε_{ijk} in terms of i, i, and k. [30]

$$\varepsilon_{ijk} = \frac{(i-j)(j-k)(k-i)}{2} = \begin{bmatrix} 1, & \text{even permutation}: (i,j,k) = (1,2,3),(2,3,1),(3,1,2) \\ -1, & \text{odd permutation}: (i,j,k) = (2,1,3),(3,2,1),(1,3,2) \\ 0, & \text{repeated index}: i=j,\ j=k,\ k=i \end{bmatrix}$$

2. Prove the following identities, if necessary, using special symbols.

a) $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{A}\times\mathbf{B}) = (AB)^2 - (\mathbf{A}\cdot\mathbf{B})^2$ [10]

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{A}\times\mathbf{B}) = (AB)^2\sin^2\theta = (AB)^2(1-\cos^2\theta) = (AB)^2 - (\mathbf{A}\cdot\mathbf{B})^2$$

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{A}\times\mathbf{B}) = \varepsilon_{ijk}A_jB_k\varepsilon_{imn}A_mB_n = \varepsilon_{ijk}\varepsilon_{imn}A_jB_kA_mB_n = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})A_jB_kA_mB_n$$

$$= A_jA_jB_kB_k - A_jB_jA_kB_k = (AB)^2 - (\mathbf{A}\cdot\mathbf{B})^2$$

b) $(\mathbf{A}\times\mathbf{B})\times\overline{\mathbf{I}} = \overline{\mathbf{I}}\times(\mathbf{A}\times\mathbf{B}) = \mathbf{BA} - \mathbf{AB}$ [10]

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{I} = (\mathbf{A}\times\mathbf{B})\times\mathbf{u}_k\mathbf{u}_k = (\mathbf{BA}-\mathbf{AB})\cdot\mathbf{u}_k\mathbf{u}_k = (\mathbf{BA}-\mathbf{AB})\cdot\mathbf{I} = \mathbf{BA}-\mathbf{AB}$$

$$\mathbf{I}\times(\mathbf{A}\times\mathbf{B}) = \mathbf{u}_k\mathbf{u}_k\times(\mathbf{A}\times\mathbf{B}) = \mathbf{u}_k\mathbf{u}_k\cdot(\mathbf{BA}-\mathbf{AB}) = \mathbf{I}\cdot(\mathbf{BA}-\mathbf{AB}) = \mathbf{BA}-\mathbf{AB}$$

You can try special symbols too.

c) $\nabla\times[\mathbf{a}\times\mathbf{b}\,f(\mathbf{r})] = (\mathbf{ab}-\mathbf{ba})\cdot\nabla f(\mathbf{r})$ ($\mathbf{a}$ and $\mathbf{b}$ are constant vectors.)

$$\nabla\times[\mathbf{a}\times\mathbf{b}\,f(\mathbf{r})] = \nabla f(\mathbf{r})\times(\mathbf{a}\times\mathbf{b}) + f(\mathbf{r})\cancel{\nabla\times(\mathbf{a}\times\mathbf{b})} = \nabla f(\mathbf{r})\times(\mathbf{a}\times\mathbf{b})$$

$$= \nabla f(\mathbf{r})\cdot(\mathbf{ba}-\mathbf{ab}) = (\mathbf{ab}-\mathbf{ba})\cdot\nabla f(\mathbf{r})$$

3 A point charge q at the origin is given by a charge density using Dirac delta function, $\rho(\mathbf{r}) = q\delta(\mathbf{r})$.

$$\rho(\mathbf{r}) = q\delta(\mathbf{r})$$

Consider a point dipole with two point charges, $+q$ and $-q$ at $\mathbf{r} = \pm(a/2)\hat{z}$. Then in the limit of $a\to 0$, what is the dipole density? [20]

The charge density $\rho_d(\mathbf{r})$ of the dipole is given by

$$\rho_d(\mathbf{r}) = \lim_{a\to 0} q\left[\delta\left(\mathbf{r}-\frac{a}{2}\hat{z}\right) - \delta\left(\mathbf{r}+\frac{a}{2}\hat{z}\right)\right] = -\frac{qa}{2}\hat{z}\cdot\nabla\delta(\mathbf{r}) - \frac{qa}{2}\hat{z}\cdot\nabla\delta(\mathbf{r})$$

$$= -qa\,\hat{z}\cdot\nabla\delta(\mathbf{r}) = \mathbf{d}\cdot\nabla\delta(\mathbf{r})$$

So for a point dipole, we can define the following quantities :

$\mathbf{d} = -qa\,\hat{z}$: Dipole Moment

$q_d(\mathbf{r}) = \mathbf{d}\cdot\nabla$: *Effective* Charge (Operator)

$\rho_d(\mathbf{r}) = q_d\delta(\mathbf{r}) = \mathbf{d}\cdot\nabla\delta(\mathbf{r})$: Charge Density

where for a non-zero dipole moment $\mathbf{d}$, we take $qa = d \neq 0$ as $q\to\infty$ and $a\to 0$.

HW-1 Solutions

Assign: 2017-03-21 (Tue) Due : 2017-03-27 (Mon 5pm)

4. The charge density of a moving point charge q is given by $\rho(\mathbf{r},t) = q\delta(\mathbf{r}-\mathbf{v}t)$ with constant velocity $\mathbf{v}$.

a) What is the current density? [10]

$\mathbf{j}(\mathbf{r},t) = q\mathbf{v}\delta(\mathbf{r}-\mathbf{v}t)$

b) Derive the current continuity equation directly from the charge density. [10]

Using a differential property of Dirac delta function:

$$\frac{\partial\delta(\mathbf{r}-\mathbf{r}')}{\partial\mathbf{r}} = \nabla\delta(\mathbf{r}-\mathbf{r}') = -\nabla'\delta(\mathbf{r}-\mathbf{r}') = -\frac{\partial\delta(\mathbf{r}-\mathbf{r}')}{\partial\mathbf{r}'}$$

then we have

$$\frac{\partial\rho(\mathbf{r},t)}{\partial t} = q\frac{d(\mathbf{v}t)}{dt}\frac{\partial}{\partial(\mathbf{v}t)}\delta(\mathbf{r}-\mathbf{v}t) = -q\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{r}}\delta(\mathbf{r}-\mathbf{v}t) = -\nabla\cdot[q\mathbf{v}\delta(\mathbf{r}-\mathbf{v}t)] = -\nabla\cdot\mathbf{j}(\mathbf{r},t)$$

which gives the current continuity or charge conservation law:

$$\frac{\partial\rho(\mathbf{r},t)}{\partial t} + \nabla\cdot\mathbf{j}(\mathbf{r},t) = 0$$

5. Prove the Helmholtz theorem:

$$\mathbf{F}(\mathbf{r}) = -\nabla\left(\int d^3\mathbf{r}'\frac{\nabla'\cdot\mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) + \nabla\times\left(\int d^3\mathbf{r}'\frac{\nabla'\times\mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) = -\nabla S(\mathbf{r}) + \nabla\times\mathbf{V}(\mathbf{r})$$

which is subject to the three infinite boundary conditions (see the Lecture slides). [20]

[Hint] Consider a vector identity, $\nabla^2\mathbf{F} = \nabla\nabla\cdot\mathbf{F} - \nabla\times\nabla\times\mathbf{F}$ and note that for a vectorial Poisson's equation $\nabla^2\mathbf{F}(\mathbf{r}) = -\mathbf{G}(\mathbf{r})$ with a source $\mathbf{G}(\mathbf{r})$, the solution is given by

$$\mathbf{F}(\mathbf{r}) = -\int d^3\mathbf{r}'\frac{\mathbf{G}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} = \int d^3\mathbf{r}'\frac{\nabla'^2\mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} = -\int d^3\mathbf{r}'\frac{\nabla'\nabla'\cdot\mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} + \int d^3\mathbf{r}'\frac{\nabla'\times\nabla'\times\mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

Using vector integro-differential theorems,

$$-\int d^3\mathbf{r}'\frac{\nabla'\nabla'\cdot\mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} = -\int d^3\mathbf{r}'\,\nabla'\left(\frac{\nabla'\cdot\mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) + \int d^3\mathbf{r}'\left(\nabla'\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right)\nabla'\cdot\mathbf{F}(\mathbf{r}')$$

$$= \cancel{-\oint d^2\mathbf{r}'\frac{\nabla'\cdot\mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}} - \int d^3\mathbf{r}'\left(\nabla\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right)\nabla'\cdot\mathbf{F}(\mathbf{r}') = -\nabla\int d^3\mathbf{r}'\frac{\nabla'\cdot\mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

$$\int d^3\mathbf{r}'\frac{\nabla'\times\nabla'\times\mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} = \int d^3\mathbf{r}'\,\nabla'\times\left(\frac{\nabla'\times\mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) - \int d^3\mathbf{r}'\left(\nabla'\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right)\times\nabla'\times\mathbf{F}(\mathbf{r}')$$

$$= \cancel{\oint d^2\mathbf{r}'\times\left(\frac{\nabla'\times\mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}\right)} + \int d^3\mathbf{r}'\left(\nabla\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right)\times\nabla'\times\mathbf{F}(\mathbf{r}') = \nabla\times\int d^3\mathbf{r}'\frac{\nabla'\times\mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

HW-1 Solutions

Assign: 2017-03-21 (Tue) Due : 2017-03-27 (Mon 5pm)

6. The Helmholtz theorem tells us that an arbitrary field is decomposed to the longitudinal and transverse components, $\mathbf{F}(\mathbf{r}) = \mathbf{F}_L(\mathbf{r}) + \mathbf{F}_T(\mathbf{r})$. Then what is the physical meaning of "Longitudinal," and "Transverse"? [10]
   [Hint] To answer the question, consider the definition of the Fourier transform:
$$\mathbf{F}(\mathbf{r}) = \frac{1}{(2\pi)^3}\int d^3\mathbf{k}\ \mathbf{F}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}$$

From the Helmholtz theorem in Prob. 5, an arbitrary field can be decomposed into longitudinal and transverse fields, $\mathbf{F}_\mathrm{L}(\mathbf{r})$ and $\mathbf{F}_\mathrm{T}(\mathbf{r})$. Taking Fourier transforms of these two fields,
$$\mathbf{F}_\mathrm{L}(\mathbf{k}) = \mathrm{FT}\left[-\nabla S(\mathbf{r})\right] = -i\mathbf{k}S(\mathbf{k})$$
$$\mathbf{F}_\mathrm{T}(\mathbf{k}) = \mathrm{FT}\left[\ \nabla\times\mathbf{V}(\mathbf{r})\right] = i\mathbf{k}\times\mathbf{V}(\mathbf{k})$$

Now we see their physical meanings:
"$\mathbf{F}_\mathrm{L}(\mathbf{k})$ is **longitudinal (or parallel)** to $\mathbf{k}$"
"$\mathbf{F}_\mathrm{T}(\mathbf{k})$ is **transverse (or perpendicular)** to $\mathbf{k}$"

7. Prove the Green's Theorem: [10]
$$\int_V dv\left[F(\mathbf{r})\nabla^2 G(\mathbf{r}) - G(\mathbf{r})\nabla^2 F(\mathbf{r})\right] = \oint_S d\mathbf{s}\cdot\left[F(\mathbf{r})\nabla G(\mathbf{r}) - G(\mathbf{r})\nabla F(\mathbf{r})\right]$$

Using a vector identity, $F(\mathbf{r})\nabla^2 G(\mathbf{r}) - G(\mathbf{r})\nabla^2 F(\mathbf{r}) = \nabla\cdot[F(\mathbf{r})\nabla G(\mathbf{r}) - G(\mathbf{r})\nabla F(\mathbf{r})]$,
$$\int_V dv\left[F(\mathbf{r})\nabla^2 G(\mathbf{r}) - G(\mathbf{r})\nabla^2 F(\mathbf{r})\right] = \int_V dv\nabla\cdot\left[F(\mathbf{r})\nabla G(\mathbf{r}) - G(\mathbf{r})\nabla F(\mathbf{r})\right] = \oint_S d\mathbf{s}\cdot\left[F(\mathbf{r})\nabla G(\mathbf{r}) - G(\mathbf{r})\nabla F(\mathbf{r})\right]$$

8. What are the SI (MKSA) units of electric and magnetic multipoles (monopole, dipole, and quadrupole)? [10]

Therefore, we see their physical meanings:

Electriconopole	[C]	Magnetic Monopole*	[A]
Electric Dipole	[C·m]	Magnetic Dipole	[A·m^2]
Electric Quadrupole	[C·m^2]	Magnetic Quadrupole	[A·m^3]

*The real magnetic monopole (charge) has not been observed, we can define and use it as a fictitious charge for theoretical convenience.

$\int_V dv\,\nabla\cdot A = \oint_S ds\cdot A$