2017 Qualifying Exam: Mathematics

Use separate answer books for Problems 1-5 (Math.-A) and Problems 6-8 (Math.-B)

Problem 1. (5 points) Find the limit

$$\lim_{y\to 0}\frac{y+\sin(2y)}{y\sqrt{y^2+2\sin(y)+1}-\sqrt{\sin^2(y)-y+1}}$$

Problem 2. (10 points) Find the solution by using Laplace transformation:

$$y^{(2)}(t)+2y^{(1)}(t)+y(t)=e^{-t}$$

where $y^{(1)}(0)=4$ and $y(0)=2$.

Problem 3. (10 points) Express $f(x,y)=3x^2y^2\cos(y)$ in the Taylor series expansion up to second order around $(x,y)=(2,0)$. Note that the answer should be written in the following form:

$$f(x,y)=f(2,0)+\begin{bmatrix} \cdot & \cdot \end{bmatrix}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}+\frac{1}{2}\begin{bmatrix} \cdot & \cdot \end{bmatrix}\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$$

Problem 4. (10 points) Green's Formula says that

$$\oint_C \{Ldx + Mdy\} = \iint_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dxdy,$$

where *C* indicates the curve enclosing *R*, oriented counterclockwise. Let C be the circle $x^2 + y^2 = 1$, oriented counterclockwise. Evaluate the following integral:

$$\oint_C \{(\cos(x^2) - y^2)dx + x^3 dy\}$$

Problem 5. (15 points) There is an ellipse defined by $x = 4\cos(t)$ and $y = 3\sin(t)$.

(a) (5 points) Find the area enclosed by the ellipse.

(b) (10 points) Find the volume obtained by rotating the ellipse around the *x*-axis.