C) Methods of solving BV problem in magnetostatics.

$\nabla \cdot \vec{B} = 0$

$\nabla \times \vec{H} = \frac{4\pi}{c} J_{free}$

$\vec{B} = \mu \vec{H}$ linear substance.

A. General Method

$B = \nabla \times A$, $\nabla \times \left(\frac{1}{\mu} \nabla \times A\right) = \frac{4\pi}{c} J$, μ is uniform

$\nabla(\nabla \cdot \vec{A}) - \nabla^2 A = \frac{4\pi}{c} \mu J$

$\nabla^2 A = -\frac{4\pi}{c} \mu J$ with Coulomb gauge $\nabla \cdot A = 0$

$\vec{A} = \frac{1}{c} \int d^3x' \frac{\mu J}{|x - x'|}$

B. $J_{free} = 0$

$\nabla \cdot \vec{B} = 0$, $\nabla \times \vec{H} = 0 \Rightarrow H = -\nabla \Phi_m$

$0 = \nabla \cdot (\mu H) = -\mu \nabla^2 \Phi_m$

$\nabla^2 \Phi_m = 0$ [Laplace equation]

B.C. are used to match Φ_m's for differential

C. Ferromagnet, $M \neq 0$, $J = 0$

$0 = \nabla \cdot \vec{B} = \nabla \cdot (\vec{H} + 4\pi \vec{M})$, $\nabla \times \vec{H} = 0$, $H = -\nabla \Phi_m$

$\nabla \cdot \vec{H} = -4\pi \nabla \cdot M$

$\nabla^2 \Phi_m = 4\pi \nabla \cdot M = 4\pi \rho_M$, $\rho_M = \nabla \cdot M$

$$\Phi_m = -\int \frac{\nabla'\cdot \vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x' + \int da' \frac{\vec{M}\cdot\hat{n}}{|\vec{x}-\vec{x}'|}$$

contribution from boundary surfaces through which M is discontinuous

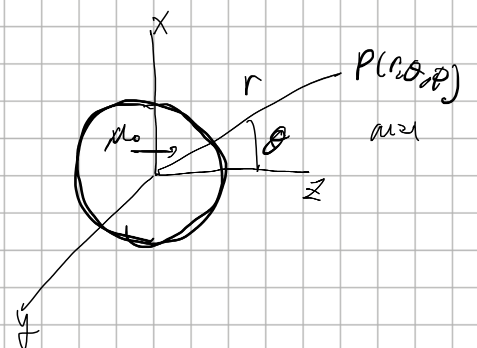
$$0 = \nabla\times H = \nabla\times(\vec{B} - 4\pi\vec{M}) \rightarrow \nabla\times\vec{B} = 4\pi\nabla\times\vec{M}$$

$$\vec{B} = \nabla\times\vec{A} \quad \text{with } \nabla\cdot\vec{A} = 0$$

$$\nabla^2\vec{A} = -\frac{4\pi}{c}\vec{J}_M, \quad \vec{J}_M = c\,\nabla\times\vec{M}$$

$$\vec{A} = \int \frac{\nabla'\times\vec{M}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x' + \int \frac{\vec{M}(\vec{x}')\times\hat{n}'}{|\quad|}\, da'$$

Ex) uniformly magnetized sphere



$a<r$

M_0 constant

$\vec{J}_{free} = 0$

$$\nabla\times\vec{H} = 0 \rightarrow H = -\nabla\Phi_m$$

$$\Phi_m = \cancel{\int \frac{-\nabla\cdot M}{|\quad|}\, d^3x'} + \int da' \frac{\vec{M}\cdot\hat{n}'}{|\quad|}$$

$0 \quad \nabla\cdot M = 0$

$$= M_0 a^2 \int d\Omega' \frac{\cos\theta'}{|\vec{x}-\vec{x}'|}, \quad \cos\theta' = P_1(\cos\theta')$$

$$= M_0 a^2 \int d(\cos\theta')\, d(\phi')\, P_1(\cos\theta')\, 4\pi \sum_l \sum_m \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi)$$

Integration over $\phi' \to m=0$

Integration over $\cos\theta' \to l=0$

$$\Phi_m = M_0 a^2\, 2\pi \frac{2}{2\cdot l+1} \frac{r_<}{r_>^2} P_l(\cos\theta) \qquad \text{cf. } Y_{lm} = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\; P_l^m(\cos\theta)\, e^{im\phi}$$

$$= \frac{4\pi}{3} M_0 a^2 \frac{r_<}{r_>^3}\cos\theta$$

Inside the sphere $r_< = r$, $r_> = a$

$$\Phi_m = \frac{4\pi}{3} M_0 \; r\cos\theta = \frac{4}{3}\pi M_0 z$$

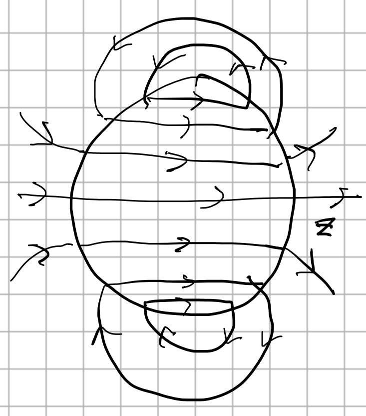
$$\vec{H}_{in} = -\nabla\Phi_m = -\frac{4\pi}{3} M_0 \hat{z}$$

$$\vec{B}_{in} = \vec{H}_{in} + 4\pi\vec{M}_0 = \frac{8\pi}{3} M_0 \hat{z}$$

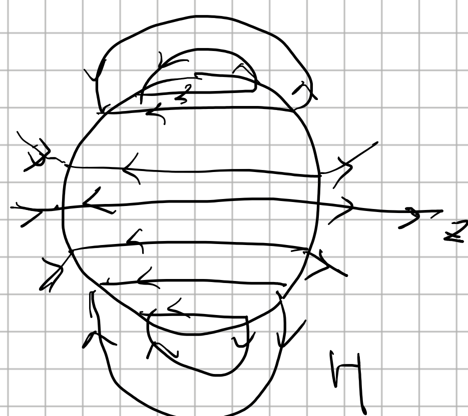
Outside the sphere $r_< = a$, $r_> = r$

$$\Phi_m = \frac{4\pi}{3} M_0 a^3 \frac{\cos\theta}{r^2} = \frac{\vec{m}\cdot\hat{r}}{r^3} \qquad \vec{m} = \frac{4\pi}{3} a^3 M_0 \hat{z}$$

$$\vec{H}_{out} = -\nabla\Phi_m = \vec{B}_{out} = \frac{4\pi}{3} M_0 a^3 \frac{1}{r^3}\left(2\cos\theta\,\hat{r} + \cos\theta\,\hat{\theta}\right)$$

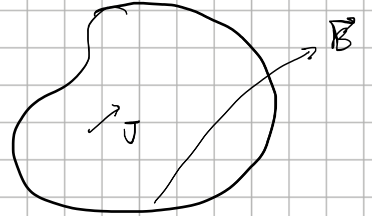


B



H

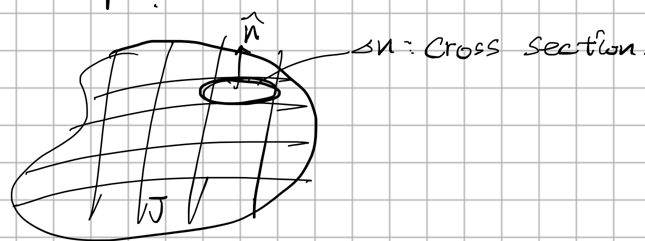
D) Energy stored in magnetic field

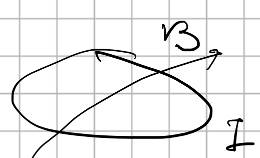


We would like to know how much work we need to do to establish the system, J & B shown in the left (a steady state distribution of currents and fields)

We build the system very slowly, steady-state current is assumed at each moment.

Current distribution is now broken up into the network of elementary current loops.





Change of flux
↓
electromotive force is induced
↓
Current changes

To keep the current constant the source of the current must do work.

$$\frac{dW}{dt} = \underbrace{-I\mathcal{E}}_{} = \frac{I}{c}\frac{dF}{dt} \quad \rightarrow \quad \mathcal{E} = -\frac{1}{c}\frac{dF}{dt}$$

work done by the electromotive force to the current I

$$\delta W = \frac{I}{c}\delta F$$

$$\Delta(\delta W) = \frac{J\Delta\sigma}{c}\int n\cdot\delta\vec{B}\,da = \frac{J\Delta\sigma}{c}\int n\cdot(\nabla\times\delta\vec{A})\,da$$

$$= \frac{J\Delta\sigma}{c}\oint_c \delta\vec{A}\cdot d\vec{l}$$

$$\delta W = \frac{1}{c}\int \delta A\cdot J\,d^3x = \frac{1}{c}\int \delta A\cdot\frac{c}{4\pi}\nabla\times\vec{H}\,d^3x$$

$$= \frac{1}{4\pi}\int d^3x\left[\vec{H}\cdot(\nabla\times\delta A) + \nabla\cdot\underbrace{\cancel{(\vec{H}\times\delta\vec{A})}}_{0}\right]$$

localized source & field

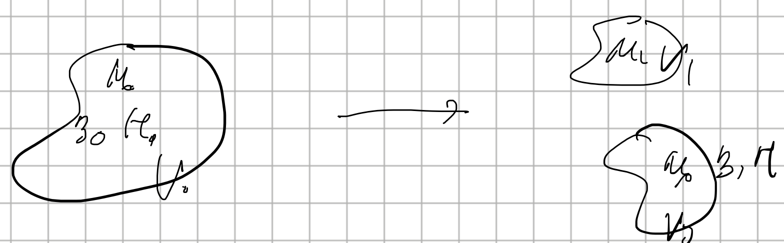
$$= \frac{1}{4\pi}\int d^3x\,\vec{H}\cdot\delta\vec{B}$$

linear medium: $\vec{H}\cdot\delta B = \frac{1}{2}\delta(\vec{H}\cdot\vec{B})$

$$W = \frac{1}{8\pi}\int d^3x\,\vec{H}\cdot\vec{B} = \frac{1}{2c}\int d^3x\,\vec{J}\cdot\vec{A}$$

cf. $W = \frac{1}{8\pi}\int \vec{E}\cdot\vec{D}\,d^3x$
$= \frac{1}{2}\int d^3x\,\rho\,\Phi$

* the change of energy when a permeable body is introduced.



$$\Delta W = \frac{1}{8\pi}\int d^3x\,(\vec{H}\cdot\vec{B} - \vec{H}_0\cdot\vec{B}_0)$$

$$= \frac{1}{8\pi}\int_{V_1} d^3x\,(\vec{B}\cdot\vec{H}_0 - \vec{H}\cdot\vec{B}_0) = \frac{1}{2}\int_{V_1} d^3x\,\vec{M}\cdot\vec{B}_0$$

battery connect

at fixed current this includes the work done by the source against the induced current.

cf. $\Delta W = -\frac{1}{2}\int_{V_1} d^3x\,\vec{P}\cdot\vec{E}_0$ at fixed charge.

cf. $F_{\xi} = \left(\frac{\partial W}{\partial \xi} \right)_{\text{current fixed}}$

force acting on the body for a displacement ξ

cf. $\Delta W = -\vec{m} \cdot \vec{B}_0$ ← no battery

permanent magnet in an external field.