Chapter 6
SPECIAL FUNCTIONS

Friedrich Wilhelm Bessel
(1784-1846)
Math/Astronomy
Discovery of Neptune
Bessel Functions

Lecture 22

6.3 Bessel Functions

E6.1 Electromagnetic Cylindrical Waveguide

Hermann Hankel
(1839-1873)
Math
Hankel Functions
Hankel Transform

E6.1 Electromagnetic Cylindrical Waveguide

In electrical engineering, metal cavity problems are very important. Here we consider a cylindrical metal hollow cavity in which is filled by vacuum or air.*

$\mathbb{D} = \varepsilon_0 \mathbb{E} + \mathbb{P}$

$\mathbb{B} = \mu_0 \mathbb{H} + \mu_0 \mathbb{M} \rightarrow \mathbb{H} = \frac{1}{\mu_0}\mathbb{B} - \mathbb{M}$

The EM waves in the cavity are governed by the **Vectorial Helmholtz equation**, which can be derived from Maxwell's equations:

$$\nabla^2 \mathbf{E}(\mathbf{r}) + k^2 \mathbf{E}(\mathbf{r}) = 0 \qquad \text{(E6.1)}$$

where the wave number is given by

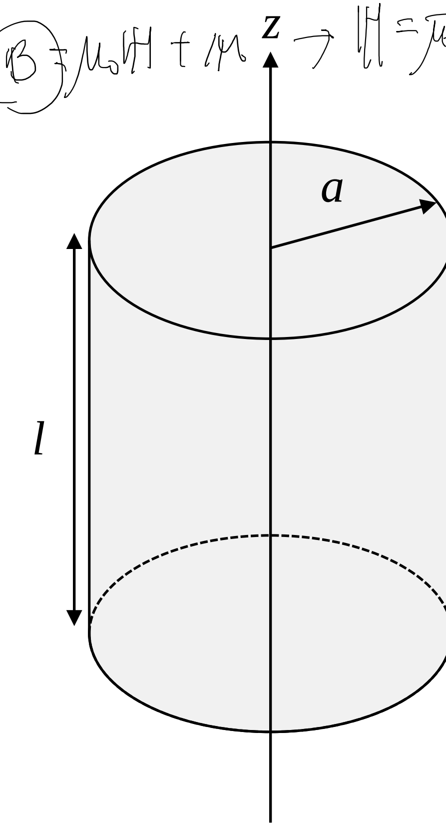
$$k = \frac{\omega}{c} \qquad \text{(E6.2)}$$

For E_z, we can use the **Scalar Helmholtz equation**:

$$\nabla^2 E_z(\mathbf{r}) + k^2 E_z(\mathbf{r}) = 0 \qquad \text{(E6.3)}$$

Using the **Method of Separation of Variables (MSE)**,

$$\boxed{E_z(\mathbf{r}) = R(\mathbf{r}_t)Z(z) = R(\rho,\phi)Z(z)} \qquad \text{(E6.4)}$$



*Typically, for EM waves at relatively low frequencies (rf and microwaves).
Then how about THz waves at ~ 10^{12} Hz?

Substituting (E6.4) into (E6.3),

$$\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+k^2\right]R(\rho,\phi)Z(z)+R(\rho,\phi)\frac{d^2Z(z)}{dz^2}=0 \tag{E6.5}$$

Dividing by $R(\rho,\phi)Z(z)$, we find

$$\underbrace{\frac{1}{R(\rho,\phi)}\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}+k^2\right]R(\rho,\phi)}_{\text{Function of }(\rho,\phi)\text{ only}}=\underbrace{-\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}}_{\text{Function of }z\text{ only}} \tag{E6.6}$$

→ different variables ⇒ constant.

and we see that we must have **some constant β** such that

$$\frac{1}{R(\rho,\phi)}\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right]R(\rho,\phi)+k^2=-\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}=\beta^2 \tag{E6.7}$$

and we find

$$\boxed{\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right]R(\rho,\phi)+(k^2-\beta^2)R(\rho,\phi)=0} \tag{E6.8}$$

$$\boxed{\frac{d^2Z(z)}{dz^2}+\beta^2Z(z)=0} \tag{E6.9}$$

From (E6.9), we first obtain

$$\boxed{Z(z) = A\cos\beta z + B\cos\beta z = A' e^{i\beta z} + B' e^{-i\beta z}} \qquad \text{(E6.10)}$$

Separation Variable.

From (E6.8), we use the MSE one more time by trying

$$\boxed{R(\rho,\phi) = U(\rho)\Psi(\phi)} \qquad \text{(E6.11)}$$

β: propagation constant

Substituting (E6.11) into (E6.8),

$$\boxed{\begin{aligned}&\Psi(\phi)\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right]U(\rho) + U(\rho)\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\Psi(\phi)\\ &= -(k^2 - \beta^2)U(\rho)\Psi(\phi) = \gamma^2 U(\rho)\Psi(\phi)\end{aligned}} \qquad \text{(E6.12)}$$

Similarly, dividing by $U(\rho)\Psi(\phi)$,

$$\frac{1}{U(\rho)}\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho}\right]U(\rho) + \frac{1}{\rho^2}\frac{1}{\Psi(\phi)}\frac{d^2}{d\phi^2}\Psi(\phi) = -\gamma^2 \qquad \text{(E6.13)}$$

and then multiplying ρ^2,

$$\underbrace{\frac{\rho^2}{U(\rho)}\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho}\right]U(\rho) + \gamma^2\rho^2}_{\text{Function of } \rho \text{ only}} = \underbrace{-\frac{1}{\Psi(\phi)}\frac{d^2}{d\phi^2}\Psi(\phi)}_{\text{Function of } \phi \text{ only}} = \nu^2 \qquad \text{(E6.14)}$$

Different Variables

Now we have two ODEs:

$$\frac{d^2}{d\phi^2}\Psi(\phi) = v^2\Psi(\phi) \tag{E6.14}$$

$$\frac{d^2U(\rho)}{d\rho^2} + \frac{1}{\rho}\frac{dU(\rho)}{d\rho} + \left(\gamma^2 - \frac{v^2}{\rho^2}\right)U(\rho) = 0 \tag{E6.15}$$

Noting that the **azimuthal function** must be **periodic**, we have

$$\boxed{v = m = 0, \pm 1, \pm 2, \pm 3, \cdots} \tag{E6.16}$$

we find the azimuthal ODE:

$$\frac{d^2}{d\phi^2}\Psi(\phi) = m^2\Psi(\phi) \tag{E6.17}$$

and the azimuthal function is given by

$$\boxed{\Psi_m(\phi) = C\cos m\phi + D\sin m\phi = C'e^{im\phi} + D'e^{-im\phi}} \tag{E6.18}$$

Substituting (E5.16) into (E6.15) we find the **Bessel's ODE**:

$$\frac{d^2U(\rho)}{d\rho^2} + \frac{1}{\rho}\frac{dU(\rho)}{d\rho} + \left(\gamma^2 - \frac{m^2}{\rho^2}\right)U(\rho) = 0 \tag{E6.19}$$

Now we obtain the cylindrical functions are given by the **Bessel Functions**:

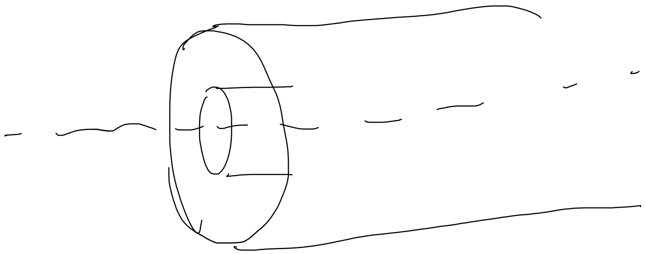
$$\boxed{U_m(\rho) = P_m J_m(\gamma\rho) + Q_m Y_m(\gamma\rho)} \tag{E6.20}$$

After very long derivation, from (E6.10), (E6.18), and (E6.20) substituting we finally obtain the **general** guided modes of the cylindrical waveguide,

$$\begin{aligned} E_z(\rho,\phi,z) = \sum_{m=0}^{\infty} \Big\{ & \left[P_m J_m(\gamma\rho) + Q_m Y_m(\gamma\rho)\right]\left[C_m \cos m\phi + D_m \sin m\phi\right] \\ & \left[A' \exp(i\beta_m z) + B' \exp(-i\beta_m z)\right]\Big\} \end{aligned} \tag{E6.21}$$

One more thing! For a real physics problem, there is **no infinite field!**

$$\boxed{\begin{aligned} E_z(\rho,\phi,z) = \sum_{m=0}^{\infty} \Big\{ & P_m J_m(\gamma\rho)\left[C_m \cos m\phi + D_m \sin m\phi\right] \\ & \left[A' \exp(i\beta_m z) + B' \exp(-i\beta_m z)\right]\Big\} \end{aligned}} \tag{E6.22}$$



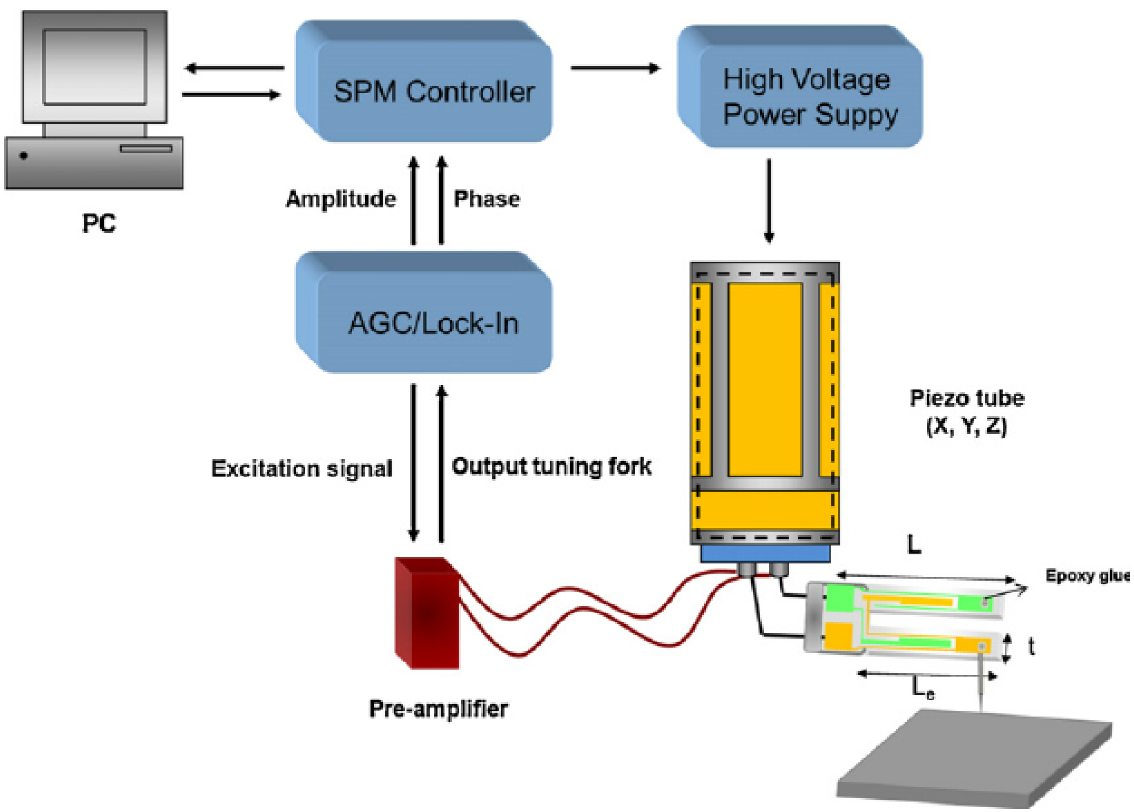
Also using

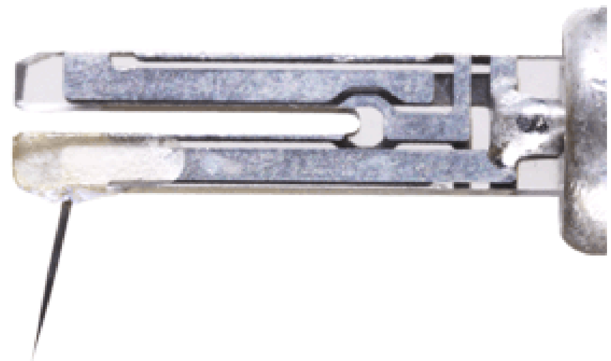
$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \cos(x\sin\theta)\cos n\theta = 0 \tag{6.33}$$

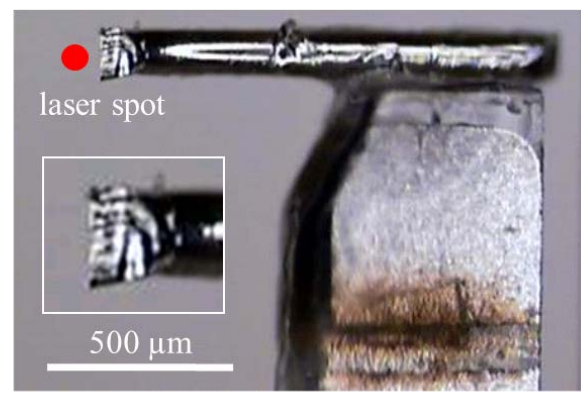
we finally have a useful integral representation

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \exp(x\sin\theta) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \exp(x\cos\theta) \tag{6.34}$$

Application Example: Direct Measurement of the Oscillation Amplitude of AFM Tip







[Quiz-3] Fourier Series of Generating Function

$$g(x,t) = \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right]$$

Find the Fourier series expansion of the generating function for $t = e^{\pm i\theta}$.