Chapter 3
COMPLEX VARIABLES

Leonard Euler
(1707-1783)
Math/Physics
Calculus, Euler-Lagrange Eq.
" *i*, *π*, *e*, *f*(*x*), Σ "
$\cos z, \sin z, \tan z$
$e^{i\pi} + 1 = 0$

Lecture 8

3.4 Power Series Expansion of Analytic Function
3.5 Cauchy Principal Value and Hilbert Transform

Augustin-Louis Cauchy
(1789-1857)
Math/Physics
Complex Analysis
Stress Tensor

3.4 Power Series Expansion of Analytic Function

The power series expansion of an analytic function is of practical use for many physics and engineering problems, which is **one of the most important applications of Cauchy's formula**.

Geometric Series: Radius of Convergence

For a geometric series with a real variable x, we can easily find the binomial expansion by performing simple divisions:

$$(1-x)^{-1} = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1 \tag{3.19}$$

Convergence condition.

where the series converges only for $|\cancel{z}| < 1$ (radius of convergence). In a similar manner, we can find the binomial expansion for a complex variable:

$$(1-z)^{-1} = \frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots = \sum_{n=0}^{\infty} z^n, \quad |z| < 1 \tag{3.20}$$

Next we consider the **Taylor and Laurent series** expansions of an analytic function, **inside circular and annular regions, respectively**, where the convergence condition of the geometric series in (3.20) will be used to prove the Taylor and Laurent theorems.

Taylor Expansion Theorem : Inside Circular Region Centered at z_0

If $f(z)$ is analytic inside and on a circular contour $C(z')$ centered at z_0 (Fig. 3-4), then $f(z)$ can be expanded by the Taylor series ~~inside~~ (not on) $C(z')$:

$$\boxed{f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n, \quad r<r'} \tag{3.21}$$

$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n, \; r<r'$

The Maclaurin series $(z_0=0)$ is just a special case of the Taylor series.

Proof)

$2\pi i f(z)=\oint_C dz'\frac{f(z')}{z'-z}$

$$2\pi i f(z)=\oint_C dz'\frac{f(z')}{z'-z} \quad \text{: Cauchy's Integral Formula}$$

$$=\oint_C dz'\frac{f(z')}{(z'-z_0)-(z-z_0)}$$

$$=\oint_C dz'\frac{f(z')}{z'-z_0}\left(1-\frac{z-z_0}{z'-z_0}\right)^{-1}=\oint_C dz'\frac{f(z')}{z'-z_0}\left(1-\frac{r}{r'}\right)^{-1} \Rightarrow \quad r<r'$$

$$=\oint_C dz'\frac{f(z')}{z'-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{z'-z_0}\right)^n=\sum_{n=0}^{\infty}(z-z_0)^n\underbrace{\oint_C dz'\frac{f(z')}{(z'-z_0)^{n+1}}}_{=2\pi i\frac{f^{(n)}(z_0)}{n!}}$$

$$=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

$r=|z-z_0|$

$r'=|z'-z_0|$

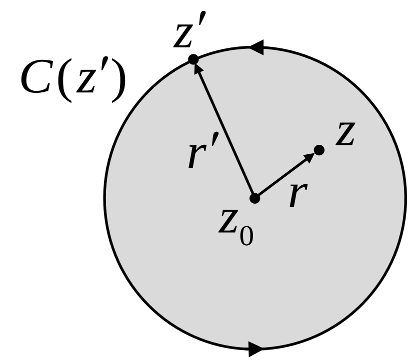

Fig. 3-4 Contour for Taylor Expansion

Laurent Expansion Theorem : Inside Annular Region Centered at $\mathbf{z_0}$

If $f(z)$ is analytic inside and on an annular contour of $\Gamma = C_1(z_1') + C_2(z_2')$ centered at z_0 (Fig. 3-5), then $f(z)$ can be expanded by the Laurent series inside (not on) Γ:

$$\boxed{f(z) = \sum_{n=-\infty}^{\infty} A_n(z_0)(z-z_0)^n} \tag{3.22}$$

$$r = |z - z_0|$$
$$r_1' = |z_1' - z_0|$$
$$r_2' = |z_2' - z_0|$$

with

$$\boxed{A_n(z_0) = \frac{1}{2\pi i}\oint_\Gamma dz' \frac{f(z')}{(z'-z_0)^{n+1}}, \quad n = 0, \pm 1, \pm 2, \cdots} \tag{3.23}$$

or more explicitly

$$\boxed{f(z) = \underbrace{\sum_{n=1}^{\infty} A_{-n}(z_0)(z-z_0)^{-n}}_{\text{Principal Part}} + \underbrace{\sum_{n=0}^{\infty} A_n(z_0)(z-z_0)^n}_{\text{Analytic Part}}} \tag{3.24}$$

with

$$\boxed{\begin{aligned} A_{-n}(z_0) &= \frac{1}{2\pi i}\oint_{C_1} dz_1' f(z_1')(z_1'-z_0)^{n-1}, \quad n = 1, 2, 3, \cdots \\ A_n(z_0) &= \frac{1}{2\pi i}\oint_{C_2} dz_2' \frac{f(z_2')}{(z_2'-z_0)^{n+1}}, \quad n = 0, 1, 2 \cdots \end{aligned}} \tag{3.25}$$

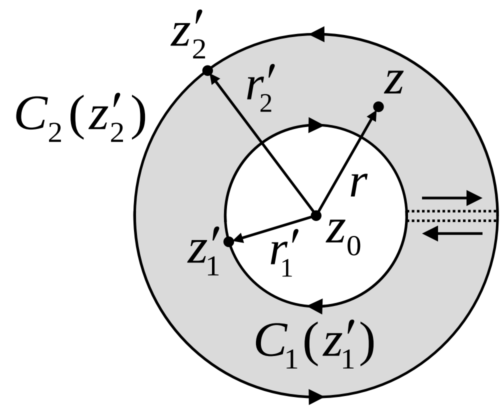

Fig. 3-5 Contour for Laurent Expansion

The inverse-power series is called the principal part while the normal-power series is called the analytic part. When $f(z)$ is also analytic inside C_1', including z_0, the integrand is also analytic inside, the principal part becomes zero and the Lorenz series reduces to the Taylor series. In other words, Laurent series is a generalization of Taylor series.

Proof)

$$2\pi i f(z) = \oint_\Gamma dz' \frac{f(z')}{z'-z} \quad : \text{Cauchy's Integral Formula}$$

$$= \oint_{-C_1} dz_1' \frac{f(z_1')}{(z_1'-z_0)-(z-z_0)} + \oint_{C_2} dz_2' \frac{f(z_2')}{(z_2'-z_0)-(z-z_0)}$$

$$= \frac{1}{z-z_0}\oint_{-C_1} dz_1' f(z_1')\left(\frac{z_1'-z_0}{z-z_0}-1\right)^{-1} + \oint_{C_2} dz_2' \frac{f(z_2')}{z_2'-z_0}\left(1-\frac{z-z_0}{z_2'-z_0}\right)^{-1} \Rightarrow r_1' < r < r_2'$$

$$= \sum_{n=0}^{\infty}\frac{1}{(z-z_0)^{n+1}}\oint_{C_1} dz_1' f(z_1')(z_1'-z_0)^n + \sum_{n=0}^{\infty}(z-z_0)^n\oint_{C_2} dz_2' \frac{f(z_2')}{(z_2'-z_0)^{n+1}}$$

$$= \sum_{n=1}^{\infty}\frac{1}{(z-z_0)^{n}}\oint_{C_1} dz_1' f(z_1')(z_1'-z_0)^{n-1} + \sum_{n=0}^{\infty}(z-z_0)^n\oint_{C_2} dz_2' \frac{f(z_2')}{(z_2'-z_0)^{n+1}}$$

$$= \sum_{n=-\infty}^{\infty}(z-z_0)^n\oint_\Gamma dz' \frac{f(z')}{(z'-z_0)^{n+1}}$$