b) Laplace Equation in a rectangular coordinate

$$\nabla^2 \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = 0$$

$\phi(x,y,z) = X(x)Y(y)Z(z)$ separation of variables

$$\underbrace{\frac{1}{X}\frac{d^2X(x)}{dx^2}}_{=-m^2} + \underbrace{\frac{1}{Y}\frac{d^2Y(y)}{dy^2}}_{=-n^2} + \underbrace{\frac{1}{Z}\frac{d^2Z(z)}{dz^2}}_{=m^2+n^2=l^2} = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -m^2 \Rightarrow X = e^{\pm imx}$$

Likewise $Y = e^{\pm iny}$, $Z = e^{\pm lz}$ $\quad l^2 = m^2+n^2$

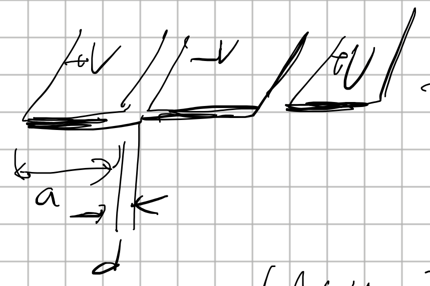
some of two solutions of Laplace equation is also a solution

Hence ~~the~~ most generation solution is

$$\phi(x,y,z) = \sum_{m,n}\left[A_m \sin(mx) + B_m \cos(mx)\right]$$
$$\times\left[C_n \sin(ny) + D_n \cos(ny)\right]$$
$$\times\left[E_l \sinh(lz) + F_l \sinh(lz)\right]$$

A, B, C, D, E, F to be determined by B.C.

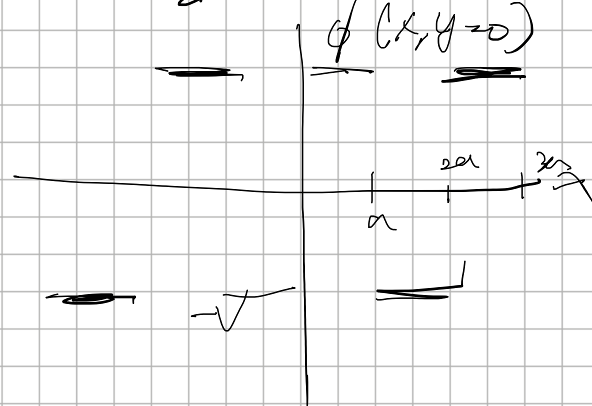
Ex)



---- Conducting plates

$d \ll a$

$\phi(x,y) \to ?$



$$\nabla^2 \phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 0$$

$$\phi = X(x)\,Y(y)$$

$$\phi(x,y) = \sum_{m=0}^{\infty}\left[A_m \sin(mx) + B_m \cos(mx)\right] \times \left[C_m e^{my} + D_m e^{-my}\right]$$

i) ϕ is finite, $C_m = 0$ otherwise $\phi \to \infty$, $y \to \infty$

ii) $y = 0$ $\quad \phi(x,y) = V(x) = \begin{cases} +V & 2an < x < 2an+a \\ -V & 2an+a < x < 2an+2a \end{cases}$

$$-\ \phi(x, y=0) = \phi(x+2a, y=0)$$

$$\sum\left[A_m' \sin(mx) + B_m' \cos(mx)\right]$$

$$= \sum\left[A_m' \sin(mx+2am) + B_m' \cos(mx+2am)\right]$$

$$2am = 2\pi l, \quad m = \frac{\pi l}{a}, \quad l = 0, 1, 2, 3 \cdots$$

$$-\ A_m' : \quad \underbrace{\int_0^{2a} dx\, V(x) \sin(mx)}_{\frac{4Va}{\pi l},\ l = 1, 3, 5} = A_m' \int_0^{2a} dx \sin^2(mx) = a A_m'$$

$$\Rightarrow A_m' = \frac{4V}{\pi l}$$

odd even

$$-B_m' = \int^{2a} dx\, \underbrace{V(x)\cos(mx)}_{\text{"odd"}} = a B_m'$$

$$\to B_m' = 0$$

$$\phi(x,y) = \sum_{l=1,3,5} \frac{4V}{\pi l} \sin\left(\frac{\pi l x}{a}\right) e^{-\frac{\pi l y}{a}}$$

$$= \frac{4V}{\pi} \operatorname{Im} \sum_{l=1,3,5} \frac{e^{i\frac{\pi l}{a}x}}{l} e^{-\frac{\pi l}{a}y}$$

$$= \frac{4V}{\pi} \operatorname{Im} \sum_{l=1,3,5} \frac{e^{\left(i\frac{\pi}{a}x - \frac{\pi}{a}y\right)l}}{l}$$

cf. $\ln \frac{1+u}{1-u} = 2\sum_{l=1,3,5} \frac{u^l}{l} = \frac{2V}{\pi} \operatorname{Im} \ln \frac{1+u}{1-u}$

$$u = e^{i\frac{\pi}{a}x - \frac{\pi}{a}y}$$

$$\frac{1+u}{1-u} = \rho e^{i\theta} = \frac{2V}{\pi}\theta$$

$$\tan\theta = \frac{\operatorname{Im}\left(\frac{1+u}{1-u}\right)}{\operatorname{Re}\left(\frac{1+u}{1-u}\right)} = \frac{\sin\left(\frac{\pi}{a}x\right)}{\sinh\left(\frac{\pi y}{a}\right)}$$

$$\phi(x,y) = \frac{2V}{\pi} \tan^{-1}\left(\frac{\sin\left(\frac{\pi}{a}x\right)}{\sinh\left(\frac{\pi y}{a}\right)}\right)$$

Laplace equation in a spherical coordinate.

$$\nabla^2 \Phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\Phi\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial\phi^2}\Phi = 0$$

$$\Phi = \frac{u(r)}{r} P(\theta) Q(\phi)$$

$$\underbrace{r^2\sin^2\theta\left[\frac{1}{u}\frac{d^2}{dr^2}u + \frac{1}{P}\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)\right]}_{\text{function of } r \,\&\, \theta \text{ only.}} + \underbrace{\frac{1}{Q}\frac{d^2Q}{d\phi^2}}_{\text{function of } \phi \text{ only.}} = 0$$

① $\frac{1}{Q}\frac{d^2Q}{d\phi^2} = -m^2$, $Q_m = e^{\pm im\phi}$, $Q_m(\phi) = Q_m(\phi+2\pi)$

$\Rightarrow m = \text{integer}$.

$$\int_0^{2\pi} d\phi \, e^{im\phi} e^{-im'\phi} = 2\pi\delta_{m,m'}$$

② $$\underbrace{r^2\frac{1}{u}\frac{d^2}{dr^2}u}_{\substack{\text{function of } r \text{ only} \\ = l(l+1)}} + \underbrace{\frac{1}{P}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) - \frac{m^2}{\sin^2\theta}}_{\substack{\text{function of } \theta \text{ only} \\ = -l(l+1)}} = 0$$

this condition. is required by a finite solution for $P(\theta)$

$\Downarrow$

$$\frac{d^2}{dr^2}u = \frac{l(l+1)}{r^2}u$$

$\Rightarrow u \simeq r^{l+1}$ or r^{-l}

$$u(r) = A_l r^{l+1} + B_l r^{-l}$$

③ $$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + l(l+1)P - \frac{m^2}{\sin^2\theta}P = 0$$

for $x = \cos\theta$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P = 0$$

A finite solution P exists only

if l must be non-negative integer

& $-l \le m \le l$

$m = 0$: Legendre equation

$m \ne 0$: associated Legendre equation.

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\left(\frac{d}{dx}\right)^{l+m}(x^2-1)^l$$

For a freedom, $P_l^m(x)$ form a set of orthogonal functions

$$\int_{-1}^{1} dx\, P_l^m(x)\, P_{l'}^m(x) = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'}$$

$$-1 \le x \le 1$$

$\Rightarrow P_l^m(\cos\theta)\,Q_m(\phi)$ also form of orthogonal functions

$$-\;Y_{l,m}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)\,e^{im\phi}$$, spherical harmonics

The most general solution:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_l r^l + B_l r^{-(l+1)}\right] Y_{l,m}(\theta,\phi)$$

cf. azimuthal symmetry $\rightarrow m=0$

(no dependence on ϕ)

$$Y_{l,0}(\cos\theta) = P_l(\cos\theta)$$

Legendre polynomial

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + l(l+1)P_l(x) = 0, \quad x = \cos\theta$$

① $P_l(x) = \dfrac{1}{2^l\, l!}\left(\dfrac{d}{dx}\right)^l (x^2-1)^l$

② $P_0(x) = 1,\quad P_1(x) = x,\quad P_2(x) = \frac{1}{2}(3x^2-1) \cdots$

③ $\int_{-1}^{1} dx\, P_l(x) P_{l'}(x) = \frac{2}{2l+1}\delta_{ll'}$

$$f(x) = \sum_{l=0}^{\infty} A_l P_l(x), \quad -1 \le x \le 1$$

④ $\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + \dfrac{B_l}{r^{l+1}}\right] P_l(\cos\theta)$