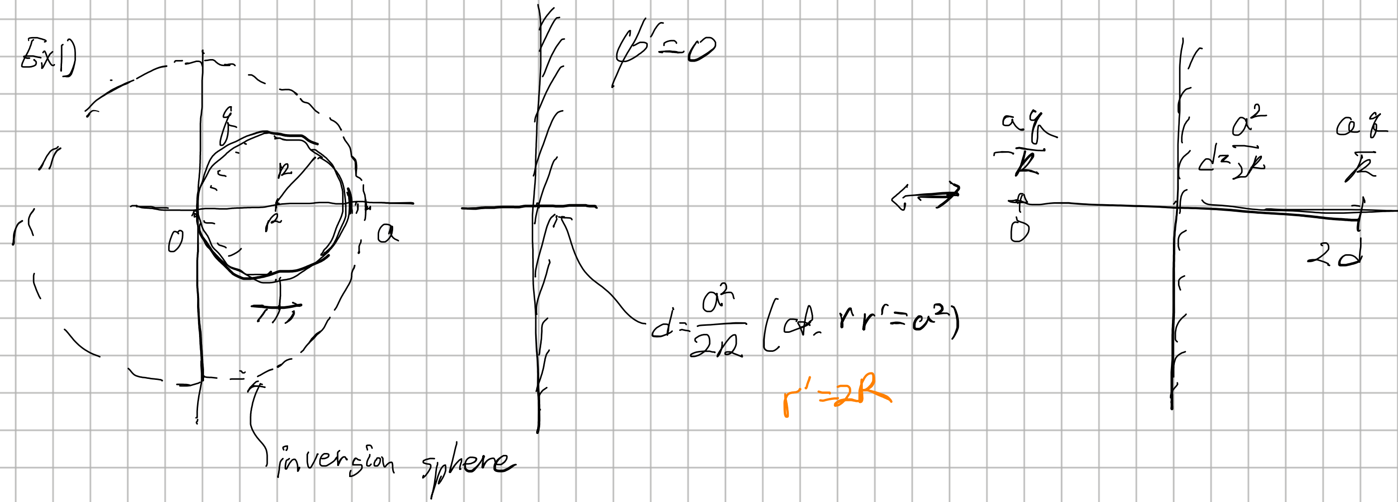
Ex1)



i) $\phi' = 0$

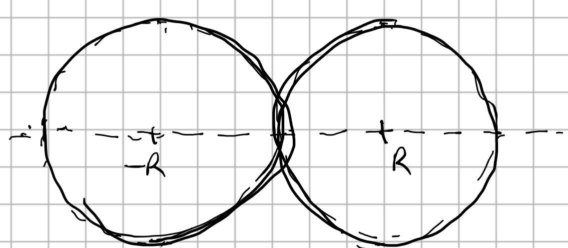
ii) outside the sphere

$$\phi' = \frac{a}{r}\,\phi\left(\frac{a^2}{r}, \theta, \varphi\right)$$

$$\frac{q}{-R} \rightarrow \frac{a}{r}\left(\frac{-q}{R}\right) = \frac{\left(-\frac{aq}{R}\right)}{r} \quad : \text{charge } \left(\frac{-aq}{R}\right) \text{ origin}$$

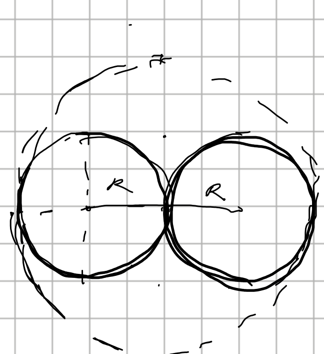
$$\frac{q}{|\vec{x} - \hat{x}R|} \rightarrow \frac{a}{r}\,\frac{q}{\left|\frac{a}{r}\hat{r} - \hat{x}R\right|} = \frac{aq/R}{\left|\vec{r} - \hat{x}\frac{a^2}{R}\right|} \quad : \text{charge } \left(\frac{aq}{R}\right) \text{ at } \frac{a^2}{R} = 2d$$

Ex2)

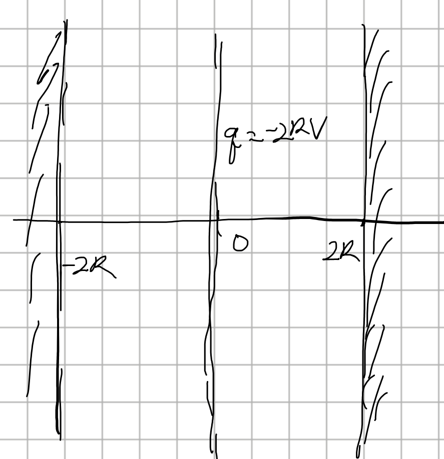


two conducting spheres touching each other

Capacitance of the system?



$\Rightarrow$



For a convenience

$\phi = 0$ on the spheres

$\phi(\infty) = -V$

The inversion process, potential at infinity

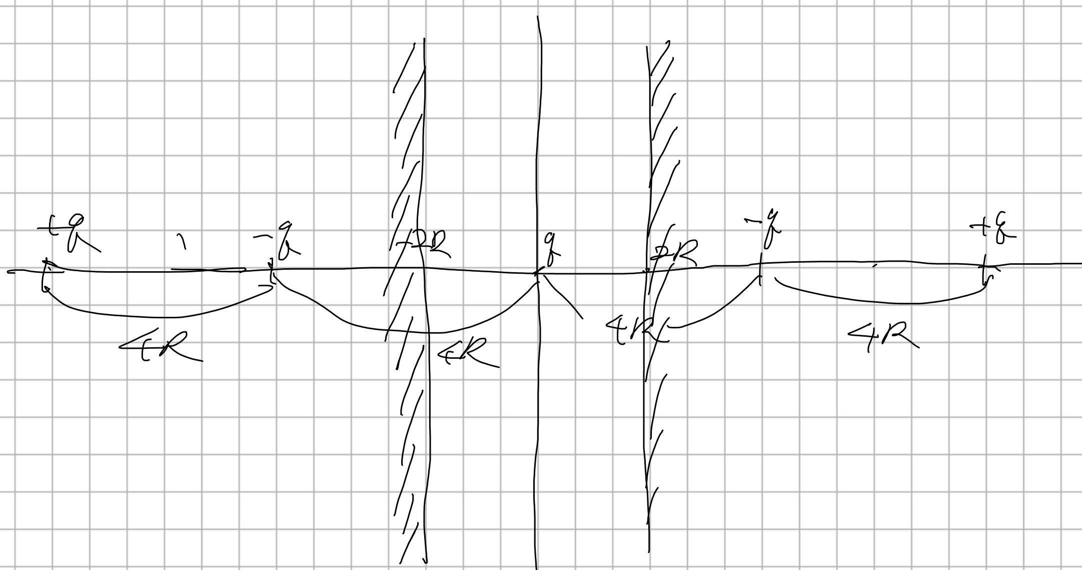
$\Rightarrow$ a point charge at origin

$\phi(\infty) = -V \rightarrow \phi = \frac{a}{r}\phi(\infty) = \frac{2R(-V)}{r}$

$\rightarrow q = -2RV$

the charge inside the conduction system

$= \sum$ inverted charges of image charges induced by $q = -2RV$



$$Q = \underset{\substack{\uparrow \\ \text{left\&right}}}{2} \sum_{n}^{\infty} \underbrace{(2R) \cdot \frac{1}{4Rn} \cdot \underset{\text{image charge}}{\underline{(-1)^n q}}}_{\text{inverse of image charges.}}$$

$$= q \sum_{n}^{\infty} \frac{(-1)^n}{n}$$

cf. $\ln(1+x) = \sum \frac{(-1)^{n+1}}{n} x^n$

$\ln 2 = \sum_n \frac{(-1)^n}{n}.$

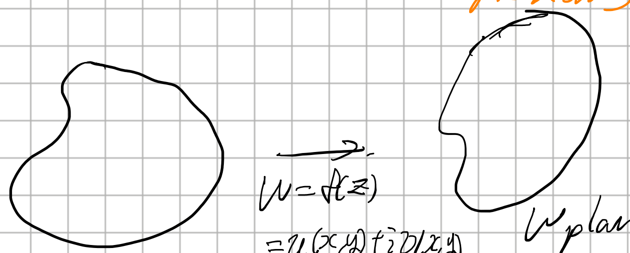
$= q(-\ln 2)$

$= 2RV \ln 2 \Rightarrow C = Q/V = 2R\ln 2.$

D) Complex Variable Method

a) Basics

- 2 dimensional problems.
-



simple geometry

solution is known.

The condition that $w = f(z)$ is analytic

(Cauchy - Riemann Condition)

$$\frac{dw}{dz} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta w}{\Delta z}$$ should be independent of a path theory

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{W(z+\Delta z, \bar{z})}{\Delta z} = \lim \left[\frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta x + i\Delta y} + i\frac{v(x+\Delta x, y+\Delta y) - v(x,y)}{\Delta x + i\Delta y} \right]$$

$$= \lim_{\Delta x, \Delta y \to 0} \frac{\left[\Delta x \frac{\partial u}{\partial x} + i\Delta y \frac{\partial v}{\partial y}\right] + i\left[\Delta x \frac{\partial v}{\partial x} - i\Delta y \frac{\partial u}{\partial y}\right]}{\Delta x + i\Delta y}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

* $W = u + iv = 2x + iy$

$$\frac{\partial u}{\partial x} = 2, \quad \frac{\partial v}{\partial y} = 1 \Rightarrow W = f(z) = 2x + iy \text{ is not analytic}$$

* $$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial}\right) = 0$$

u & v are satisfies of Laplace equation,
if $w = u + iv$ is analytic.

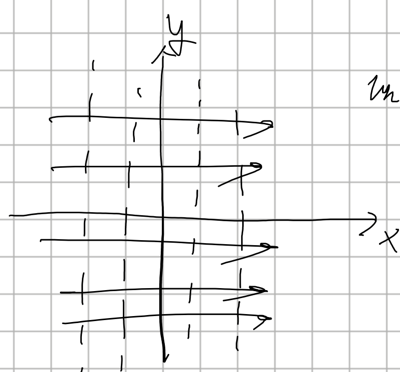
* $\nabla u \cdot \nabla v$

$$= \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial x} = 0$$

$\nabla u \perp \nabla v$

u: potential or field line

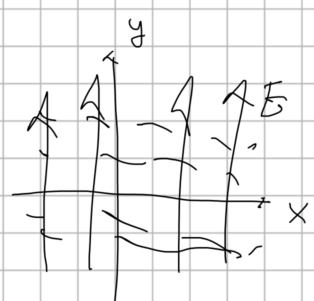
v: field line potential



uniform field E

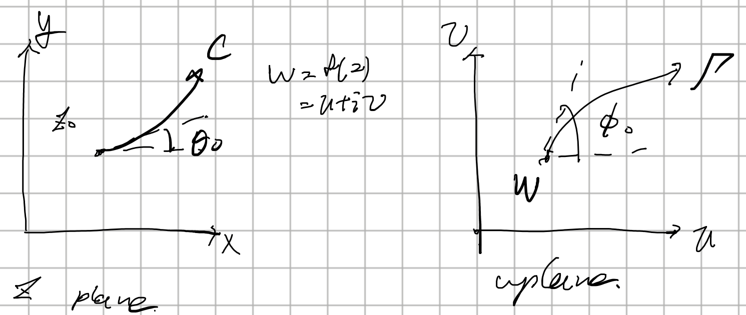
$W = -Ex$

$\Rightarrow W = -Ez$
$= \underbrace{-Ex}_{\text{potential}} \underbrace{- iEy}_{\text{field line}}$



$\rightarrow$ field line potential

b) Conformal Transfer motion.



$C : z(t) = x(t) + iy(t)$ $\qquad$ $\Gamma : w(t) = f(z(t))$

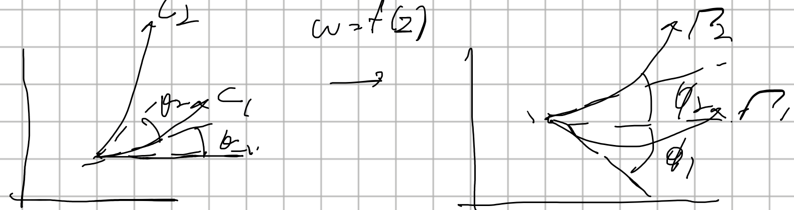
$a \le t \le b$

$$w'(t) = f'(z(t)) \cdot z'(t)$$

($f(z)$ should be analytic)

$$\arg[w'(t)] = \arg[z'(t)] + \arg[f'(z)]$$

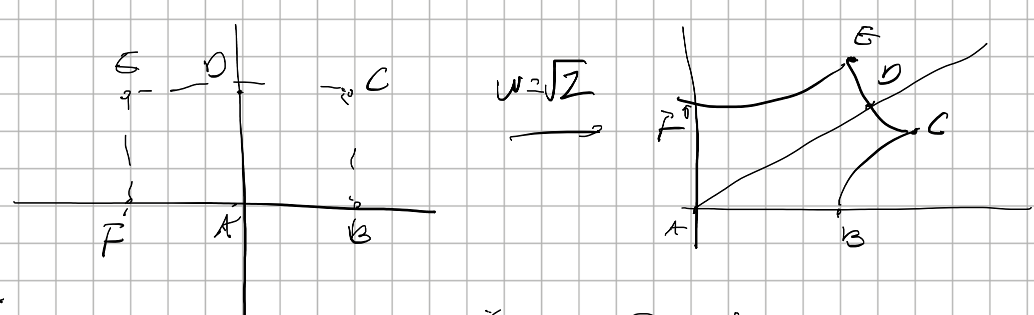
$z = z_0, \quad \phi_0 = \theta_0 + \psi_0$



$\therefore \phi_1 = \theta_1 + \psi_0 . \quad \psi_0 = \arg[f'(z)]_{z=z_0}$

$\phi_2 = \theta_2 + \psi_0$ $\qquad$ cf. $f'(z_0) = 0$, critical point

$\phi_2 - \phi_1 = \theta_2 - \theta_1$ $\quad$ $\therefore$ angle is preserved $\therefore$ Conformal



z-plane	w-plane
A (0,0)	(0,0)
B (1,0)	(1,0)
C (1,1)	$\left(\sqrt{\frac{1+\sqrt{2}}{2}}, \sqrt{\frac{\sqrt{2}-1}{2}}\right)$
D (0,1)	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
F (-1,0)	(0,1)

$w = z^{1/2}, \; z = w^2 = u^2 + 2iuv - v^2$

$u = \sqrt{\frac{x + \sqrt{x^2+y^2}}{2}}, \; v = \sqrt{u^2 - x}$