HW-1

Assign: 2017-03-21 (Tue)

Due : 2017-03-27 (Mon)

1. Find an algebraic expression of the Levi-Civita symbol ε_{ijk} in terms of i, j, and k. [30]

$\varepsilon_{ijk} = e_i \cdot (e_j \times e_k)$

2. Prove the following identities, if necessary, using special symbols.

(a) $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{A}\times\mathbf{B}) = (AB)^2 - (\mathbf{A}\cdot\mathbf{B})^2$ $(A\times B) = (\varepsilon_{ijk} u_i A_j B_k)\cdot(\varepsilon_{ijk} u_i A_j B_k)$ [10]

(b) $(\mathbf{A}\times\mathbf{B})\times\bar{\mathbf{I}} = \bar{\mathbf{I}}\times(\mathbf{A}\times\mathbf{B}) = \mathbf{BA} - \mathbf{AB}$ [10]

(c) $\nabla\times[\mathbf{a}\times\mathbf{b}\,f(\mathbf{r})] = (\mathbf{ab} - \mathbf{ba})\cdot\nabla f(\mathbf{r})$ ($\mathbf{a}$ and $\mathbf{b}$ are constant vectors) [10]

3. A point charge q at the origin is given by a charge density using Dirac delta function:

$$\rho(\mathbf{r}) = q\delta(\mathbf{r})$$

Consider a **point dipole** with two opposite charges, $+q$ and $-q$ at $\mathbf{r} = \pm\frac{1}{2}a\hat{z}$ $(a \to 0)$.
What is the **point dipole density**? [20]

4. The charge density of a moving point charge q is given by $\rho(\mathbf{r},t) = q\delta(\mathbf{r} - \mathbf{v}t)$.

(a) What is the current density? [10]

(b) Derive the current continuity equation directly from the charge density. [10]

5. Prove the Helmholtz theorem:

$$\mathbf{F}(\mathbf{r}) = -\nabla\left(\int d^3\mathbf{r}' \frac{\nabla'\cdot\mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) + \nabla\times\left(\int d^3\mathbf{r}' \frac{\nabla'\times\mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}\right)$$

which is subject to the three infinite boundary conditions (see the Lecture slides). [20]

[Hint] Consider a vector identity $\nabla^2\mathbf{F} = \nabla\nabla\cdot\mathbf{F} - \nabla\times\nabla\times\mathbf{F}$, and note that for a vectorial Poisson's equation $\nabla^2\mathbf{F}(\mathbf{r}) = -\mathbf{G}(\mathbf{r})$ with a source $\mathbf{G}(\mathbf{r})$, the solution is given by

$$\mathbf{F}(\mathbf{r}) = -\int d^3\mathbf{r}' \frac{\mathbf{G}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

6. The Helmholtz theorem tells us that an arbitrary field is decomposed to the longitudinal and transverse components, $\mathbf{F}(\mathbf{r}) = \mathbf{F}_L(\mathbf{r}) + \mathbf{F}_T(\mathbf{r})$. Then what is the physical meaning of "Longitudinal," and "Transverse"? [10]

[Hint] To answer the question, consider the definition of the Fourier transform:

$$\mathbf{F}(\mathbf{r}) = \frac{1}{(2\pi)^3}\int d^3\mathbf{k}\ \mathbf{F}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}$$

7. Prove the Green's Theorem. [10]

$$\int_V dv[F(\mathbf{r})\nabla^2 G(\mathbf{r}) - G(\mathbf{r})\nabla^2 F(\mathbf{r})] = \oint_S d\mathbf{s}\cdot[F(\mathbf{r})\nabla G(\mathbf{r}) - G(\mathbf{r})\nabla F(\mathbf{r})]$$

8. What are the SI (MKSA) units of electric and magnetic multipoles (monopole, dipole, and quadrupole)? [10]