2006 수학

1. (a) $A^T = A$ 이므로 $A^TA = A^2$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 4I.$$

(b) $A\underline{x} = \lambda\underline{x}$, $A^2\underline{x} = \lambda A\underline{x} = 4I\underline{x} = \lambda^2\underline{x}$

$\therefore \lambda^2 = 4$, $\lambda_1 = 2$, $\lambda_2 = -2$.

① $\lambda_1 = 2$

$$\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 3 & -1 \\ -1 & 1 & -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - x_2 - x_3 - x_4 &= 0 \\ -x_1 + x_2 + x_3 + x_4 &= 0 \\ -x_1 + x_2 + 3x_3 - x_4 &= 0 \\ -x_1 + x_2 - x_3 + 3x_4 &= 0 \end{aligned}$$

$$\begin{cases} x_3 = x_4, \\ x_1 - x_2 = 2x_3 = 2x_4. \end{cases} \quad \therefore p_1 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}, \quad p_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

② $\lambda_2 = -2$

$$\begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 + 3x_2 - x_3 - x_4 &= 0 \\ x_1 - x_2 + x_3 + x_4 &= 0 \end{aligned} \Rightarrow \begin{cases} x_1 = -x_2 \\ x_3 + x_4 = 2x_2 \end{cases} \quad \therefore p_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad p_4 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

(c) $\det(AB) = \det A \det B$ (∵ A, B are square matrices.)

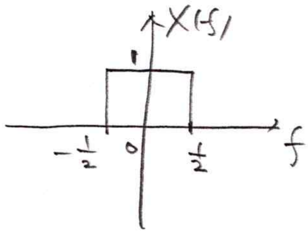
$$\det(A) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & 0 & -2 \end{vmatrix} \Rightarrow \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -2 & -2 \end{vmatrix} \Rightarrow \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -4 \end{vmatrix} = \underline{16}.$$

$$\det(B) = \begin{vmatrix} 3 & 0 & 1 & 5 \\ 0 & 2 & -\frac{4}{3} & \frac{1}{3} \\ 0 & -4 & \frac{5}{3} & \frac{25}{3} \\ 0 & -2 & -\frac{1}{3} & \frac{1}{3} \end{vmatrix} = \begin{vmatrix} 3 & 0 & 1 & 5 \\ 0 & 2 & -\frac{4}{3} & \frac{1}{3} \\ 0 & 0 & -1 & 9 \\ 0 & 0 & -\frac{5}{3} & \frac{2}{3} \end{vmatrix} = \begin{vmatrix} 3 & 0 & 1 & 5 \\ 0 & 2 & -\frac{4}{3} & \frac{1}{3} \\ 0 & 0 & -1 & 9 \\ 0 & 0 & 0 & -\frac{43}{3} \end{vmatrix} = \underline{86}.$$

$\therefore \underline{\det(AB) = 1376}$

2. $x(t) = \mathrm{sinc}\, t$.

(a) $X(f) = \int_{-\infty}^{\infty} \mathrm{sinc}\, t\, e^{-j2\pi f t}\, dt = \Pi(f) = \begin{cases} 1, & (-\frac{1}{2} \le f \le \frac{1}{2}) \\ 0, & \text{otherwise} \end{cases}$



(b) $y(t) = |x(t)|^2 * \sum_{n=-\infty}^{\infty} \delta(t-n)$

$Y(f) = \left(X(f) * X(f)\right) \cdot \sum_{k=-\infty}^{\infty} \delta(f-k)$

$= \underline{\delta(f)}$ //

3. $\frac{\partial \ln(xy)}{\partial y} = \frac{x}{xy} = \frac{1}{y}$,

$\ln(xy) = \ln y + f(x)$

$\frac{\partial \ln(xy)}{\partial x} = \frac{y}{xy} = \frac{1}{x} = f'(x)$, $f(x) = \ln x + C$

$\therefore \ln(xy) = \ln x + \ln y + C$.

for $x = y = 1$, we have $C = 0$.

$\therefore \underline{\ln(xy) = \ln x + \ln y}$

4. $\tan^{-1} x = y$, $\tan y = x$, $\sec^2 y\, dy = dx$, $\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y}$

$= \frac{1}{1+x^2}$

$\therefore \underline{\frac{dy}{dx} = \frac{d \tan^{-1} x}{dx} = \frac{1}{1+x^2}}$

5. $\tan^{-1}(x) = \tan^{-1}(0) + \frac{1}{1!}(\tan^{-1}(x))'\Big|_{x=0}(x-0) + \frac{1}{2!}(\tan^{-1}(x))''\Big|_{x=0}(x-0)^2$

$+ \frac{1}{3!}(\tan^{-1}(x))'''\Big|_{x=0}(x-0)^3$

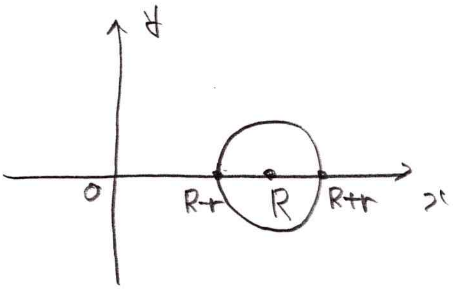
$= 0 + \frac{1}{1}(x) + \frac{1}{2!}\left(\frac{-2x}{(1+x^2)^2}\Big|_{x=0}\right)(x-0)^2 + \frac{1}{3!}\left(\frac{-2(1+x^2)^2}{(1+x^2)^4}\Big|_{x=0}\right)(x-0)^3$

$= \underline{x - \frac{1}{3}x^3}$ //

6. $\tan^{-1}x \approx x - \frac{1}{3}x^3$ 이므로 $\frac{\pi}{4} = \tan^{-1}(1) \approx 1 - \frac{1}{3} = \frac{2}{3}$

$\therefore \pi \approx \frac{8}{3}$

7.



$(x-R)^2 + y^2 = r^2$, $(x-R)^2 = r^2 - y^2$, $x - R = \pm\sqrt{r^2-y^2}$, $x = R \pm \sqrt{r^2-y^2}$

$\therefore V = \pi\int_{-r}^{r}\left(R+\sqrt{r^2-y^2}\right)^2 dy - \pi\int_{-r}^{r}\left(R-\sqrt{r^2-y^2}\right)^2 dy$

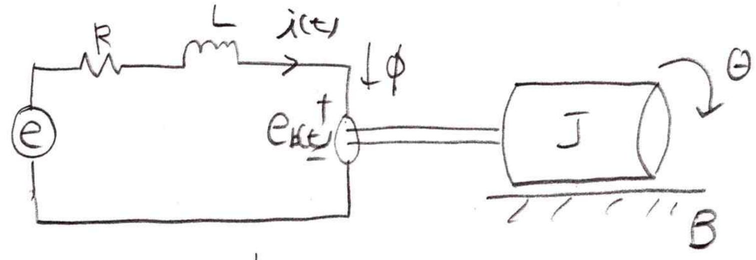
$= 4\pi R\int_{-r}^{r}\sqrt{r^2-y^2}\,dy$, Let $y = r\sin\theta$, $dy = r\cos\theta\, d\theta$.

$= 4\pi R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2\cos^2\theta\, d\theta = 4\pi r^2 R\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\, d\theta = \underline{2\pi^2 r^2 R}$.

2006 제어

필수

1. a)



$$e(t) = R\,i(t) + L\frac{di(t)}{dt} + e_b(t)$$

$$T_m = K_i\, i(t)$$

$$e_b(t) = K_b\,\omega_m(t) = K_b\frac{d\theta_m(t)}{dt}$$

$$J\frac{d^2\theta_m(t)}{dt} + B\frac{d\theta_m(t)}{dt} + T_L = T_m$$

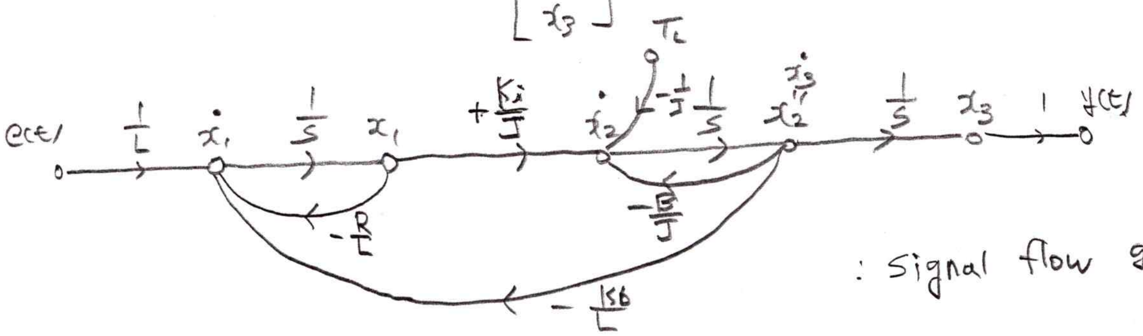
∴ differential equation : $\begin{cases} \dfrac{di(t)}{dt} = -\dfrac{R}{L}i(t) - \dfrac{1}{L}e_b(t) + \dfrac{1}{L}e(t) \\ T_M = K_i\, i(t) \\ e_b(t) = K_b\,\omega_m(t) = K_b\dfrac{d\theta_m(t)}{dt} \\ \dfrac{d^2\theta_m(t)}{dt^2} = -\dfrac{B}{J}\dfrac{d\theta_m(t)}{dt} + \dfrac{1}{J}T_m - \dfrac{1}{J}T_L \end{cases}$

b), c), d) 한꺼번에

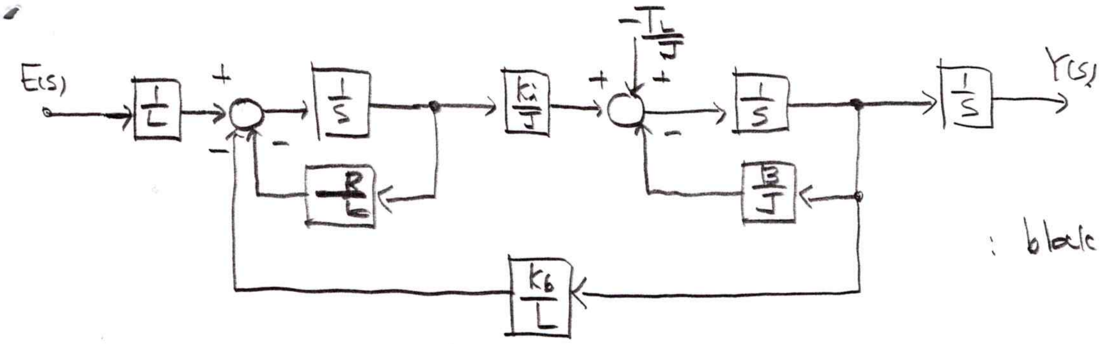
$\begin{cases} x_1(t) = i(t) \\ x_2(t) = \omega_m(t) = \dot{x}_3(t) \\ x_3(t) = \theta_m(t) \end{cases}$ $\quad y(t) = \theta_m(t) = x_3(t)$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{K_b}{L} & 0 \\ +\frac{K_i}{J} & -\frac{B}{J} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{J} \\ 0 \end{bmatrix} T_L + \begin{bmatrix} \frac{1}{L} \\ 0 \\ 0 \end{bmatrix} e(t)$$

$$y(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$: state equation.



: signal flow graph



: block diagram.

By Mason's gain formula, we have

$$H(s) = \frac{Y(s)}{E(s)} = \frac{\frac{1}{s^3}\cdot\frac{k_i}{LJ}}{1+\frac{R}{Ls}+\frac{B}{Js}+\frac{k_i k_b}{s^2 JL}+\frac{RB}{s^2 JL}} = \frac{k_i}{JLs^3+(JR+BL)s^2+(k_b k_i+RB)s}$$

: transfer function.

e) $E(s)=1$ 이므로 $Y(s) = H(s)$, $L=0$ 이므로

$$Y(s) = \frac{k_i}{JRs^2+(k_b k_i+RB)s} = \frac{k_i}{s(s+A)}, \quad \text{for } A = \frac{k_b k_i + RB}{JR}.$$

$$\frac{k_i}{s(s+A)} = \frac{p}{s}+\frac{q}{s+A}. \qquad \begin{array}{l} p+q=0 \\ Ap+q=k_i \end{array} \qquad p=\frac{k_i}{A-1},\ q=\frac{-k_i}{A-1}.$$

$$Y(s) = \frac{k_i}{A-1}\cdot\frac{1}{s} - \frac{k_i}{A-1}\,\frac{1}{s+A}$$

$$\therefore\ y(t) = \frac{k_i}{A-1}\left(1-e^{-At}\right)u_s(t) \quad (u_s(t)\text{: unit step function}).$$

제어선택

1. $[B \quad AB \quad A^2B \cdots]$ matrix가 full rank 이면 controllable.

(Controllability : state $x(t_0)$ 에서 state $x(t_1)$ 로 transfer 하는 input $u[t_0,t_1]$ 이 존재. ($t_1 > t_0$, t_1 is finite value)).

$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix}$ matrix가 full rank 이면 observable.

(Observability : time interval input $u[t_0,t_1]$, output $y[t_0,t_1]$을 아는 것으로 state $x(t_0)$를 결정하기 충분할 때 observable, for finite $t_1 > t_0$).

2. ① $\lambda_1 = 1$ case

$$\begin{bmatrix} 1 & -6 & 5 \\ -1 & 1 & -2 \\ -3 & -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} x_1 - 6x_2 + 5x_3 = 0 \\ -x_1 + x_2 - 2x_3 = 0 \\ -3x_1 - 2x_2 - 3x_3 = 0 \end{cases}$$

$-5x_2 + 3x_3 = 0$, $5x_2 = 3x_3$

$-3x_1 - 7x_2 = 0$. $3x_1 = -7x_2$.

$$\therefore p_1 = \begin{bmatrix} 7 \\ -3 \\ -5 \end{bmatrix}$$

② $\lambda_2 = 1$ case

$$\begin{bmatrix} 1 & -6 & 5 \\ -1 & 1 & -2 \\ -3 & -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ -5 \end{bmatrix}$$

$$\begin{cases} x_1 - 6x_2 + 5x_3 = 7 \\ -x_1 + x_2 - 2x_3 = -3 \\ -3x_1 - 2x_2 - 3x_3 = -5 \end{cases}$$

$-5x_2 + 3x_3 = 4$

$5x_1 + 7x_3 = 11$

$$\therefore p_2 = \begin{bmatrix} \frac{11}{5} \\ -\frac{4}{5} \\ 0 \end{bmatrix}$$

③ $\lambda_3 = 2$ case

$$\begin{bmatrix} 2 & -6 & 5 \\ -1 & 2 & -2 \\ -3 & -2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} 2x_1 - 6x_2 + 5x_3 = 0 \\ -x_1 + 2x_2 - 2x_3 = 0 \\ -3x_1 - 2x_2 - 2x_3 = 0 \end{cases}$$

$2x_1 + 4x_2 = 0$, $x_1 = -2x_2$.

$x_3 = 2x_2$

$$\therefore p_3 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

3. $\dot{x} = Ax + Bu$ 에서 $x = P\bar{x}$, $\dot{x} = P\dot{\bar{x}}$ 이므로

$P\dot{\bar{x}} = AP\bar{x} + Bu \Rightarrow \dot{\bar{x}} = P^{-1}AP\bar{x} + P^{-1}Bu$. ($\because P$ is nonsingular).

Then $\bar{A} = P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

$\bar{B} = P^{-1}B = \begin{bmatrix} 7 & \frac{11}{5} & 2 \\ -3 & -\frac{4}{5} & -1 \\ -5 & 0 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

$[P^{-1}B \quad (P^{-1}AP)P^{-1}B \quad (P^{-1}AP)^2P^{-1}B] = P^{-1}[B \quad AB \quad A^2B]$,

P는 full rank 이므로 $[B \quad AB \quad A^2B]$가 full rank 이면 controllable.

$$[B \quad AB \quad A^2B] = \begin{bmatrix} 0 & 1 & -18 \\ 1 & 2 & 13 \\ 1 & 6 & 31 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 13 \\ 0 & 1 & -18 \\ 1 & 6 & 31 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 13 \\ 0 & 1 & -18 \\ 0 & 4 & 18 \end{bmatrix}$$

$$\downarrow$$

Full rank! $\longrightarrow \begin{bmatrix} 1 & 2 & 13 \\ 0 & 1 & -18 \\ 0 & 0 & 90 \end{bmatrix}$

$\therefore$ Controllable 하다