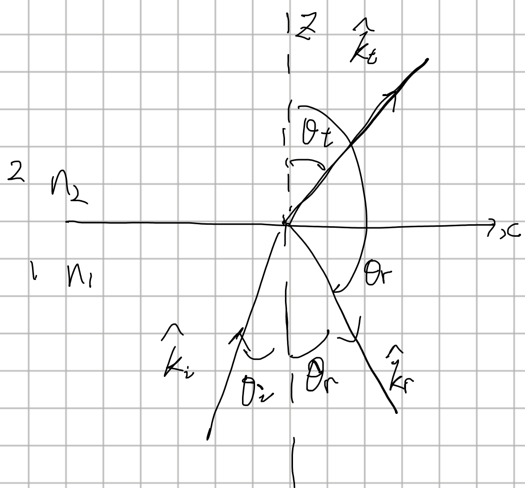
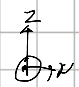
④ Reflection & Refraction at an interface

a) Snell's law





Plane wave

$$\psi = \psi(\omega t - \vec{k}\cdot\vec{r})$$
$$= \psi\left(\omega\left(t - \frac{\vec{k}\cdot\vec{r}}{\omega}\right)\right)$$
$$= \psi\left(t - \frac{\hat{k}\cdot\vec{r}}{v}\right).$$

cf. $v = \frac{\omega}{k} = \frac{c}{n}$

At the interface, the phase vector form of the secondary waves are the same as the primary (incident wave)

$$t - \frac{\vec{r}\cdot\hat{k}_i}{v_1} = t - \frac{\vec{r}\cdot\hat{k}_r}{v_1} = t - \frac{\vec{r}\cdot\hat{k}_t}{v_2}$$

On the boundary plane

$$\frac{x\hat{k}_{xi} + y\hat{k}_{yi}}{v_1} = \frac{x\hat{k}_{xr} + y\hat{k}_{yr}}{v_1} = \frac{x\hat{k}_{xt} + y\hat{k}_{yt}}{v_2} \Rightarrow \frac{\sin\theta_i}{v_1} = \frac{\sin\theta_r}{v_1} = \frac{\sin\theta_t}{v_2}$$

holds for every x & y

$$\hat{k}_{xi} = \sin\theta_i,\quad \hat{k}_{yi} = 0,\quad \hat{k}_{iz} = \cos\theta_i$$
$$\hat{k}_{xr} = \sin\theta_r,\quad \hat{k}_{yr} = 0,\quad \hat{k}_{rz} = \cos\theta_r$$
$$\hat{k}_{xt} = \sin\theta_t,\quad \hat{k}_{yt} = 0,\quad \hat{k}_{tz} = \cos\theta_t$$

$$\frac{\sin\theta_i}{\frac{c}{n_1}} = \frac{\sin\theta_t}{c/n_2} \Rightarrow \boxed{n_1\sin\theta_i = n_2\sin\theta_t}$$

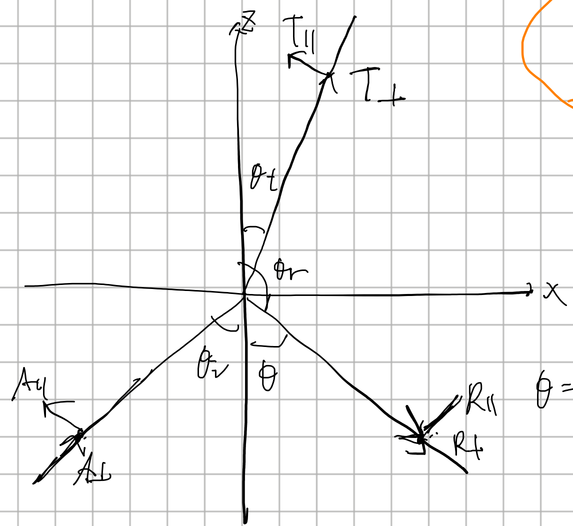
Snell's law
(law of refraction)

$\sin\theta_i = \sin\theta_r$, $\cos\theta_r = -\cos\theta_i$

$\theta_r = \pi - \theta_i$, $\theta_i = \theta_r$

law of reflection.

b) Fresnel Coefficients

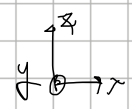


$\theta = \pi - \theta_r$

∥ TM, p-pol

⊥ TE, s-pol

→ depends on geometry



Incident wave

$E_x^i = -A_\parallel \cos\theta_i \, e^{-i\tau_i}$, $\tau_i = \omega t - \vec{k}_i \cdot \vec{r}$

$E_y^i = A_\perp e^{-i\tau_i}$

$E_z^i = A_\parallel \sin\theta_i \, e^{-i\tau_i}$

$$\vec{H}^i = \sqrt{\epsilon_r}\, \hat{k} \times \vec{E}^i$$

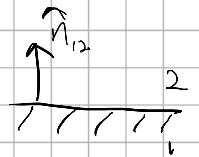
$$= \hat{x}\left(-A_\perp \cos\theta_i \sqrt{\epsilon_1}\, e^{-i\tau_i}\right) + \hat{y}\left(-A_\parallel \sqrt{\epsilon_1}\, e^{-i\tau_i}\right) + \hat{z}\left(A_\perp \sin\theta_i \sqrt{\epsilon_1}\, e^{-i\tau_i}\right)$$

Likewise,

$$\left[\begin{matrix} E_x^t = -T_\parallel \cos\theta_t \, e^{-i\tau_t}, & H_x^t = -T_\perp \cos\theta_t \sqrt{\epsilon_2}\, e^{-i\tau_t} \\ \vdots & \vdots \end{matrix}\right]$$

$$\left[\begin{matrix} E_x^r = -R_\parallel \cos\theta_r \, e^{-i\tau_r}, & H_x^r = -R_\perp \cos\theta_r \sqrt{\epsilon_1}\, e^{-i\tau_r} \\ \vdots & \vdots \end{matrix}\right]$$

Boundary condition from Maxwell's equation.



$$\hat{n}_{12}\times(\vec{E}_2-\vec{E}_1)=0, \quad \hat{n}_{12}\cdot(\vec{D}_2-\vec{D}_1)=0$$
$$\hat{n}_{12}\times(\vec{H}_2-\vec{H}_1)=0, \quad \hat{n}_{12}\cdot(\vec{B}_2-\vec{B}_1)=0$$

& Snell's law ($\vec{k}_i\cdot\vec{r}=\vec{k}_r\cdot\vec{r}=\vec{k}_t\cdot\vec{r}$ on $z=0$)

$$T_\parallel=\frac{2n_1\cos\theta_i}{n_2\cos\theta_i+n_1\cos\theta_t}A_\parallel \underset{\text{Snell's law}}{=} \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i+\theta_t)\cos(\theta_i-\theta_t)}A_\parallel$$

$$T_\perp = \qquad = \frac{2\sin\theta_t\cos\theta_i}{\sin(\theta_i+\theta_t)}A_\perp$$

$$R_\parallel = \qquad = \left[\tan(\theta_i-\theta_t)/\tan(\theta_i+\theta_t)\right]A_\parallel$$
$$R_\perp = \qquad = -\sin(\theta_i-\theta_t)/\sin(\theta_i+\theta_t)\ A_\parallel$$

Note 1. $0\le\theta_i+\theta_t\le\pi$, $\quad -\pi/2\le\theta_i-\theta_t\le\pi/2$

$\sin(\theta_i+\theta_t)\ge 0$, $\quad \cos(\theta_i-\theta_t)>0$

2. Reflected wave

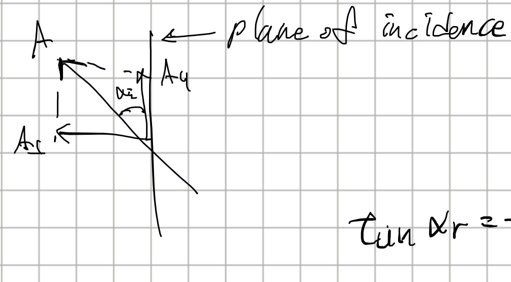
$n_2>n_1 \Rightarrow \theta_i>\theta_t \rightarrow R_\perp, A_\perp$: phase diff. π

$R_\parallel, A_\parallel$: various cases

$\theta_i+\theta_t>\pi/2$

$R_\parallel$ & $A_\parallel$: phase diff. π.

3. Transmited wave: along the same phase as the incident.

4. 

$A_\parallel = A\cos\alpha_i$, $A_\perp = A\sin\alpha_i$

$\tan\alpha_i=\frac{A_\perp}{A_\parallel}$, $\quad \tan\alpha_r=\frac{R_\perp}{R_\parallel}$ $\quad \tan\alpha_t=\frac{T_\perp}{T_\parallel}$

$$\tan\alpha_r=-\frac{\cos(\theta_i-\theta_t)}{\cos(\theta_i+\theta_t)}\tan\alpha_i \qquad \tan\alpha_t=\cos(\theta_i-\theta_t)\tan\alpha_i$$

$0\le\theta_i\le\pi/2$, $0<\theta_t<\pi/2$

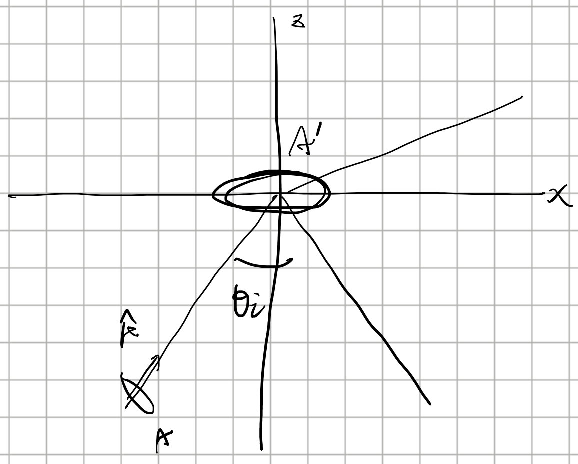
$|\tan\alpha_r|\ge|\tan\alpha_i|$

$|\tan\alpha_t|\le|\tan\alpha_i|$

c) reflectivity & transmitance (energy flow)

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{H} = \hat{k} \frac{c}{4\pi} \sqrt{\varepsilon}\, \vec{E}^2 = \hat{k} \frac{c}{4\pi} n \vec{E}^2$$

$$H = \sqrt{\varepsilon}\, \hat{k} \times \vec{E}, \quad n = \sqrt{\varepsilon}$$



$$J_i = \frac{E_{in}}{A'} = \frac{E_{in}}{\frac{A}{\cos\theta_i}}$$

$$= E_{in}/A \cos\theta_i = S_i \cos\theta_i$$

$$= \frac{c}{4\pi} n_1 |A|^2 \cos\theta_i$$

$$J_r = \frac{c}{4\pi} n_1 |R|^2 \cos\theta_i$$

$$J_t = \frac{c}{4\pi} n_2 |T|^2 \cos\theta_t$$

$$R = \frac{J_r}{J_i} = \frac{|R|^2}{|A|^2} \quad : \text{Reflectivity}$$

$$\mathcal{T} = \frac{J_t}{J_i} = \frac{n_2 \cos\theta_t}{n_1 \cos\theta_i} \frac{|T|^2}{|A|^2} \quad : \text{transmittance.}$$

$$R + \mathcal{T} = 1 \qquad (\text{energy conservation})$$

HW show this

$$A_{\parallel} = A \cos\alpha_i, \quad A_{\perp} = A \sin\alpha_i$$

$$R = R_{\parallel} \cos^2\alpha_r + R_{\perp} \sin^2\alpha_r \qquad R_{\parallel} + \mathcal{T}_{\parallel} = 1$$

$$\mathcal{T} = \mathcal{T}_{\parallel} \cos^2\alpha_t + \mathcal{T}_{\perp} \sin^2\alpha_t \qquad R_{\perp} + \mathcal{T}_{\perp} = 1$$

$$R_{\parallel,\perp} = J^r_{\parallel,\perp} / J^i_{\parallel,\perp}$$

$$\mathcal{T}_{\parallel,\perp} = J^t_{\parallel,\perp} / J^i_{\parallel,\perp}$$

show this too

$R_{\parallel} = \dfrac{\tan^2(\theta_i-\theta_t)}{\tan^2(\theta_i+\theta_t)}$, $\quad T_{\parallel} = \dfrac{\sin(2\theta_i)\sin(2\theta_t)}{\sin^2(\theta_i+\theta_t)\cos^2(\theta_i-\theta_t)}$

i) Brewster angle

$\theta_i+\theta_t = \pi/2$, $\quad \tan(\theta_i+\theta_t)=\infty$, $R_{\parallel}=0$, $T_{\parallel}=1$

$n_1\sin\theta_i = n_2\sin\theta_t \longrightarrow \tan\theta_B = n_2/n_1 \qquad \theta_B = 56.40^\circ$

for glass invisible.