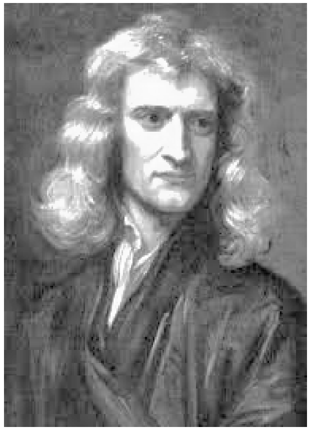
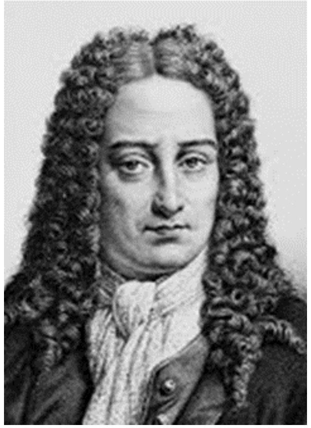
Chapter 5
DIFFERENTIAL EQUATIONS

Lecture 19

5.3 Series Solutions: Frobenius Method



Isaac Newton
(1642-1726)
Math/Physics
Universal Gravity
Newtonian Mechanics
***Differential* Calculus**



Gottfried Wilhelm Leibniz
(1646-1716)
Math/Physic
***Integral* Calculus**
Leibnitz Notation

5.2 Series Solutions: Frobenius Method

1D second-order inhomogeneous ODEs

For many problems in physics and engineering, we need to solve 1D second-order inhomogeneous ODEs:

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)=F(x) \tag{5.12}$$

for which we usually use the **method of separation of variables**, for example, in Cartesian, cylindrical, and spherical coordinates.

The most general solution is given by

$$\boxed{y(x)=y_h(x)+y_p(x)} \tag{5.13}$$

y_1 and y_2 is independent. (ex. Vector)

initial condition.

We define two kinds of solutions:

$$\boxed{y_h(x)=A_1y_1(x)+A_2y_2(x):}\ \textbf{Homogeneous Solutions}\ \text{for}\ F(x)=0$$

$$y_p(x):\ \textbf{Particular Solution}\ \text{for}\ F(x)\neq 0 \tag{5.14}$$

and the homogeneous solutions can be obtained in **a form of power series**:

$$\boxed{y_h(x)=x^k\left(a_0+a_1x+a_2x^2\cdots\right)=\sum_{n=0}^{\infty}a_nx^{k+n},\quad a_0\neq 0}$$

[Q] What determines A_1 and A_2?

Ex) Find the homogeneous solutions of a **classical linear oscillator**:

$$\frac{d^2y}{dx^2} + \omega^2 y = 0 \tag{5.15}$$

Let's try a series solution:

$$y(x) = x^k \sum_{n=0}^{\infty} a_n x^{n} = \sum_{n=0}^{\infty} a_n x^{k+n} \ (a_0 \neq 0) \tag{5.16}$$

from which we have

$a_0 k(k-1)$, $a_1(k+1)k$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} a_n (k+n)(k+n-1) x^{k+n-2} \tag{5.17}$$

$a_{n+2}(k+n+2)(k+n+1) = -\omega^2 a_n$

Substituting (5.16) and (5.17) into (5.15),

$$\sum_{n=0}^{\infty} a_n (k+n)(k+n-1) x^{k+n-2} = -\omega^2 \sum_{n=0}^{\infty} a_n x^{k+n}$$

or

$$a_0 k(k-1)x^{k-2} + a_1(k+1)kx^{k-1} + a_2(k+2)(k+1)x^k + \cdots$$
$$= -\omega^2 \left[a_0 x^k + a_1 x^{k+1} + a_2 x^{k+2} + \cdots \right] \tag{5.18}$$

$\frac{d^2y}{dx^2} + \omega^2 y = 0$

$\to y(x) = x^k \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{k+n}$

$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} a_n (k+n)(k+n-1) x^{k+n-2} = -\omega^2 \sum_{n=0}^{\infty} a_n x^{k+n}$

$a_{n+2}(k+n+2)(k+n+1) x^{k+n}$

So we have

$$a_{n+2} = -\frac{\omega^2}{(k+n+2)(k+n+1)} a_n \quad \text{Recurrence Relation} \tag{5.19}$$

Now we note that at least the first term of LHS in (5.18) should vanish

$$a_0 k(k-1) = 0 \tag{5.20}$$

$a_0 \neq 0$

Since we initially assumed that $a_0 \neq 0$, we should have two cases

$$k = 0 \quad \text{or} \quad k = 1 \tag{5.21}$$

1) $k = 0$: $a_{n+2} = -\dfrac{\omega^2}{(n+2)(n+1)} a_n \Rightarrow (-1)^n \dfrac{\omega^{2n}}{(2n)!} a_0$

$$\rightarrow a_2 = -\frac{\omega^2}{1\cdot 2} a_0 = -\frac{\omega^2}{2!} a_0, \quad a_4 = -\frac{\omega^2}{3\cdot 4} a_2 = \frac{\omega^2}{4!} a_0, \quad a_6 = \frac{\omega^2}{5\cdot 6} a_4 = -\frac{\omega^2}{4!} a_0, \quad \cdots$$

2) $k = 1$: $a_{n+2} = -\dfrac{\omega^2}{(n+3)(n+2)} a_n = (-1)^n \dfrac{\omega^{2n}}{(2n+1)!} a_0$

$$\rightarrow a_2 = -\frac{\omega^2}{2\cdot 3} a_0 = -\frac{\omega^2}{3!} a_0, \quad a_4 = -\frac{\omega^2}{4\cdot 5} a_2 = \frac{\omega^2}{5!} a_0, \quad a_6 = \frac{\omega^2}{6\cdot 7} a_4 = -\frac{\omega^2}{7!} a_0, \quad \cdots$$

Therefore we finally have

$$y_h(x) = \begin{bmatrix} y_1(x) = y(x)\big|_{k=0} = a_0\left[1 - \dfrac{(\omega x)^2}{2!} + \dfrac{(\omega x)^4}{2!} + \cdots\right] = a_0 \cos\omega x \\ y_2(x) = y(x)\big|_{k=1} = \dfrac{a_0}{\omega}\left[\omega x - \dfrac{(\omega x)^3}{3!} + \dfrac{(\omega x)^5}{5!} + \cdots\right] = \dfrac{a_0}{\omega}\sin\omega x \end{bmatrix} \tag{5.22}$$

around $x=0$.

5.3 Some Special Functions

Screening Effect..

Bessel Function

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \tag{5.23}$$

Legendre Function

$$(1-x^2)\frac{d^2 y}{dx^2} + 2x\frac{dy}{dx} + n(n+1)y = 0 \tag{5.24}$$

Associate Legendre Function

$$(1-x^2)\frac{d^2 y}{dx^2} + 2x\frac{dy}{dx} + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0 \tag{5.25}$$

Hermite Polynomial

$$\frac{d^2 y}{dx^2} + 2x\frac{dy}{dx} + 2ny = 0 \tag{5.26}$$

Laguerre Polynomial

$$x\frac{d^2 y}{dx^2} + (1-x)\frac{dy}{dx} + ny = 0 \tag{5.27}$$

Associate Laguerre Polynomial

$$x\frac{d^2 y}{dx^2} + (m+1-x)\frac{dy}{dx} + (n-m)y = 0 \tag{5.28}$$