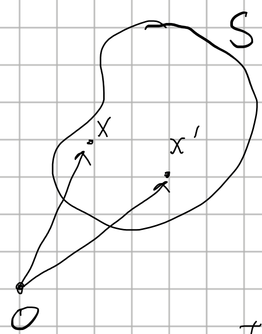
A Green's function Method

A Green's function is defined to be a solution of

$$\nabla'^2 G(\vec{x}, \vec{x}') = -4\pi\delta(\vec{x}-\vec{x}')$$

inside the surface S



G: potential at $\vec{x}'$ due to a unit charge at $\vec{x}$

In general $G(\vec{x}, \vec{x}') = \underbrace{\frac{1}{|\vec{x}-\vec{x}'|}}_{\text{particular solution}} + \vec{F}(\vec{x}, \vec{x}')$ with $\nabla'^2 F(\vec{x}, \vec{x}') = 0$

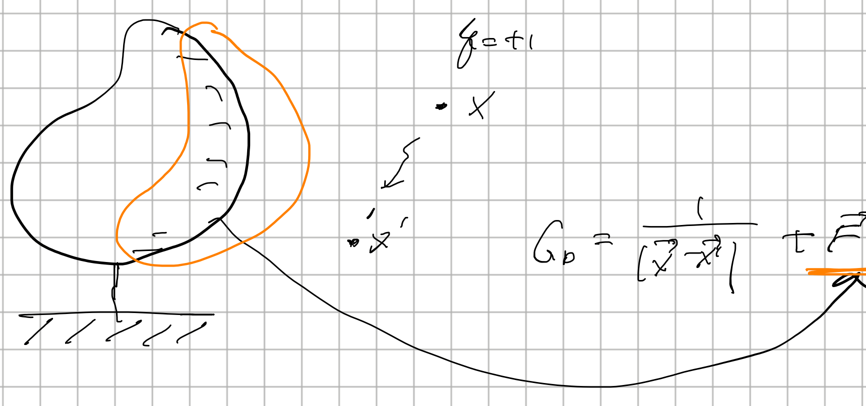
We have a function to choose F

For Dirichlet BVP, F is chosen in such a way that

$$G_D(\vec{x}, \vec{x}') = 0 \quad \text{for } \vec{x} \text{ on } S$$

G_D: solution for given geometrical arrangement of grounded conducting boundaries when only charge is a unit charge at $\vec{x}$

F: the potential due to the induced charge on the grounded S



$$G_D = \frac{1}{|\vec{x}-\vec{x}'|} + \vec{F}(\vec{x}, \vec{x}')$$

For Neumann BVP,

$$-\hat{n}\cdot\nabla' G_N(\vec{x},\vec{x}') = -\frac{4\pi}{A}, \quad \vec{x}' \text{ on } S$$

$$\underbrace{-\hat{n}\cdot\nabla'}_{\left(\equiv\frac{\partial}{\partial n'}\right)}$$

A: area of surface

If we specify $\frac{\partial n}{\partial n'} G=0$, then

$$\int d^3x \; \nabla'^2 G = -\int d^3x\,(-4\pi\delta(\vec{x}-\vec{x}'))$$

$$\int d\vec{a}\cdot\nabla' G = -4\pi$$

$$0 = -4\pi \quad \Rightarrow \frac{\partial}{\partial n'} G = -4\pi/A$$

$$\nabla'\cdot(\phi\nabla'\psi) = \nabla'\phi\cdot\nabla'\psi + \phi\nabla'^2\psi$$

$$\nabla'\cdot(\psi\nabla'\phi) = \nabla'\psi\cdot\nabla'\phi + \psi\nabla'^2\phi$$

$$\Rightarrow \nabla'\cdot(\phi\nabla'\psi - \psi\nabla'\phi) = \phi\nabla'^2\psi - \psi\nabla'^2\phi$$

$\int d^3x$ & Gauss' theorem.

$$\int d\vec{a}\cdot(\phi\nabla'\psi - \psi\nabla'\phi) = \int d^3x'(\phi\nabla'^2\psi - \psi\nabla'^2\phi)$$

a) Dirichlet BVP $\quad \psi = G_D, \; \nabla'^2 G_D = -4\pi\delta$

$G_D = 0, \; x'$ on S

$$\nabla'^2\phi = -4\pi\rho(\vec{x}')$$

$$\phi(\vec{x}) = \int d^3x'\,\rho(\vec{x}')\,G_D(\vec{x},\vec{x}') - \frac{1}{4\pi}\int da'\,\phi(\vec{x}')\frac{\partial}{\partial n'}G_D(\vec{x},\vec{x}')$$

If G_D & $\phi(S)$ are known, ϕ can be found

b) Neumann BVP

$$\psi = G_N, \; \nabla'^2 G_N = -4\pi\delta(\vec{x}-\vec{x}')$$

$$\frac{\partial}{\partial n'} G_N = -4\pi/A$$

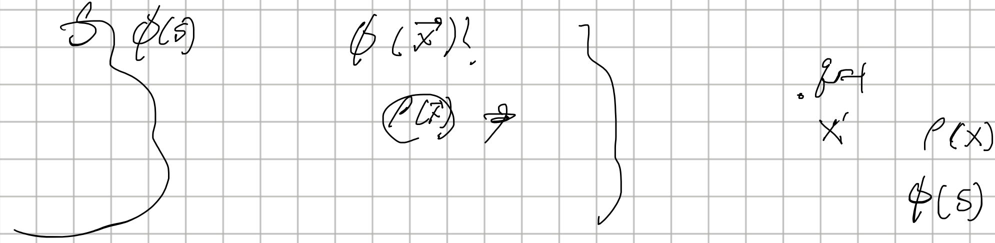
$$\nabla'^2\phi = -4\pi\rho(\vec{x}')$$

$$\phi(x) = \langle\phi\rangle + \int d^3x' \rho(\vec{x}')\, G_N + \frac{1}{4\pi}\int da' \left(\frac{\partial}{\partial n'}\phi(\vec{x}')\right) G_N$$

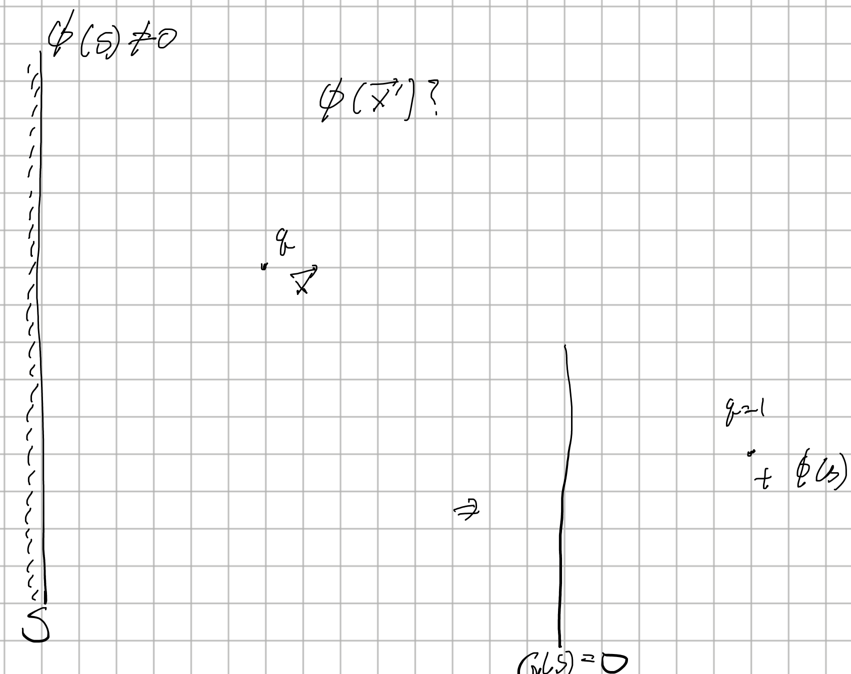
$$\langle\phi\rangle = \int da'\, \phi(\vec{x}')/A$$

Green's function is only determined by a geometry

Green's function separates out the geometry and charge distribution.



Ex)



$$G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} + F$$

F is chosen in such a way that

$$\nabla'^2 F = 0 \qquad \text{with } G(S) = 0$$

$\Rightarrow$ By intuition or your study

$$F = \frac{1}{|\vec{x} - \vec{X}_{Im}|}, \quad \vec{X}_{Im} : \text{the position of image charge}$$

$$\vec{X}_{im} = -\vec{x}$$

$$G_D = \frac{1}{|\vec{x'} - \vec{x}|} - \frac{1}{|\vec{x'} + \vec{x}|}$$

$$\phi(\vec{x'}) = \int d^3x'' \rho(\vec{x''}) G_D(\vec{x'}, \vec{x''}) - \frac{1}{4\pi} \int da'' \phi(\vec{x''}) \frac{\partial}{\partial n''} G_D(\vec{x'}, \vec{x''})$$

If $\phi(S) = 0$, $\phi(x) = \int d^3x'' \, q\delta(\vec{x''} - \vec{x}) \left[\frac{1}{|\vec{x'} - \vec{x''}|} - \frac{1}{|\vec{x'} + \vec{x''}|} \right]$

$$= \frac{q}{|\vec{x'} - \vec{x}|} + \frac{-q}{|\vec{x'} - (-\vec{x})|}$$

contribution due to the charges on S induced by q

induced charge

$$\sigma = \frac{1}{4\pi} E_{surface} = -\frac{1}{4\pi} (\nabla\phi) \cdot x \Big|_{x=0}$$

$$\vec{x} = (d, 0, 0)$$

$$= \frac{q}{4\pi} \frac{-2d}{(d^2+y^2+z^2)^{3/2}}$$

$$\int \sigma ds = \int_{-\infty}^{\infty} dy dz \frac{q}{4\pi} \cdot \frac{-2d}{(d^2+y^2+z^2)^{3/2}} = -q$$

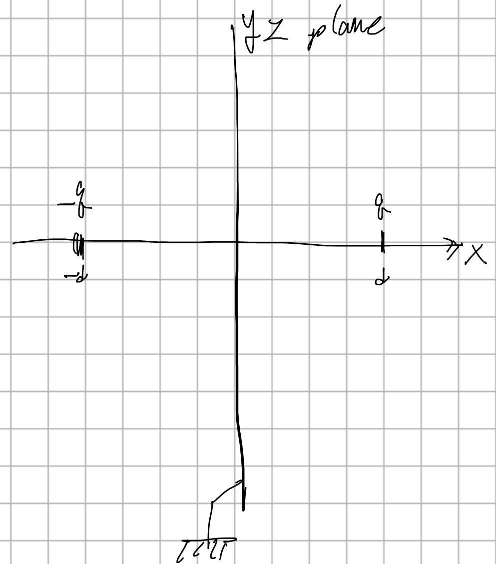
B. Image charge Method

More intuitive boundar are indirectly treated

$$\phi_{total} = \phi_{real\ charge} + \phi'_{\text{due to fictions image charges suitably placed}}$$

satisfies the boundary conditions ϕ_{total} is valid only in the physical region external to image charges.

a)



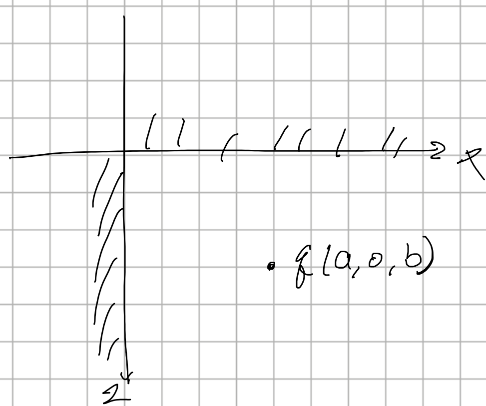
$$\nabla^2 \phi = -4\pi\delta(\vec{x}-d\hat{x})$$

$$\phi(x=0,y,z)=0$$

An Image charge at $\vec{x}=-d\hat{x}$

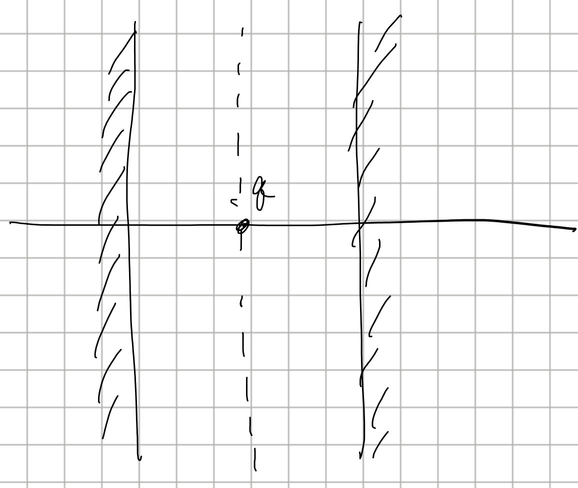
$$\phi = \frac{q}{|\vec{x}-d\hat{x}|} + \frac{-q}{|\vec{x}-(-d\hat{x})|} \quad \text{for } x>0$$

b)



$\phi = ?$

c)



$\phi(x) = ?$