2010 Qualifying Exam: Mathematics

Caution!!!
Use separate answer books for Problems 1-5 (Math.-A) and 6-8 (Math.-B).

Problem 1. (10 points) Find the solution by utilizing Laplace transformation:

$$y^{(2)}(t) + 4y(t) = \cos(3t),$$

where $y(0) = 2$ and $y^{(1)}(0) = 0$.

Problem 2. (10 points) Green's Formula says that

$$\oint_C \{Ldx + Mdy\} = \int\int_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dxdy,$$

where C indicates the curve enclosing R, oriented counterclockwise. Let R be the region bounded by the triangle with vertices at $(0,0)$, $(2,0)$, and $(0,3)$. If we orient C in the counterclockwise direction, solve the following.

$$\oint_C \{(3x^2 + 4xy + y)dx + (5x + 2x^2)dy\}$$

Problem 3. (10 points) Find the limits.

(a)

$$\lim_{x\to 1} \frac{x^{1/4} - 1}{x^{1/3} - 1}$$

(b)

$$\lim_{x\to 0} \frac{\sin(5x)}{3x}$$

Problem 4. (10 points) Differentiate

$$y = (\sin x)^{x^3}.$$

Problem 5. (10 points) Find

$$\int \frac{3x+6}{x^2+5x+4}\,dx.$$

Caution!!!
Use separate answer books for Problems 1-5 (Math.-A) and 6-8 (Math.-B).

Problem 6. (10 points) Say True or False to each sub-problem. You don't need to justify your answer. (correct answer +2, no answer 0, incorrect answer −1)

(a) Given an M-by-N rectangular matrix, the sum of the dimension of the null space and the dimension of the row space is always equal to N.

(b) Given a square matrix, there always exist orthonormal eigenvectors.

(c) If a square matrix is symmetric, then its eigenvalues are always real.

(d) A summation of two periodic functions is always periodic.

(e) The Fourier transform of a periodic signal always consists of Dirac delta functions.

Problem 7. (20 points) Suppose that a square matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 1+\sin^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & 1+\cos^2\theta \end{pmatrix},$$

where $0 \le \theta < 2\pi$ is a constant. Answer the following questions.

(a) (5 points) Find the characteristic polynomial as a function only of λ, i.e., find a, b, and c such that

$$\det(\lambda\mathbf{I} - \mathbf{A}) = a\lambda^2 + b\lambda + c,$$

where $\mathbf{I}$ is the identity matrix.

(b) (5 points) Find the orthonormal eigenvectors of $\mathbf{A}$.

(c) (10 points) Find d and e such that

$$\mathbf{A}^3 + d\mathbf{A} + e\mathbf{I} = \mathbf{0},$$

where $\mathbf{0}$ is the zero matrix.

Problem 8. (20 points) Suppose that $x(t)$ is defined as a time-limited signal given by

$$x(t) \triangleq \begin{cases} -1, & \text{for } -T/2 \le x < 0 \\ 1, & \text{for } 0 \le x < T/2 \\ 0, & \text{elsewhere,} \end{cases}$$

for some $T > 0$. Answer the following questions.

(a) (5 points) Find the Fourier transform of $x(t)$.

(b) (5 points) Find the Fourier series representation of

$$\sum_{n=-\infty}^{\infty} x(t - nT).$$

(c) (10 points) Using the result in (b), find the limit of the series

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

Math. QE

2009.8

Caution!!!
Use separate answer books for Problems 1-5 (Math.-A) and 6-8 (Math.-B).

Problem 1.(10pt) Consider a linear system with transfer function $H(s) = \frac{1}{(s+2)^2(s+8)}$. Find the output when $u(t) = e^{-2t}u_s(t)$ is applied as an input, where $u_s(t)$ is a unit step function. Assume that the initial condition, $x(0_-) = 0$.

Problem 2.(10pt) Calculate

$$\int_C \frac{ydx - xdy}{x^2 + y^2},$$

where $C\ :\ (x-3)^2 + (y-5)^2 = 4$.
(Hint: Use Green's theorem.)

Problem 3.(10pt) Using the residue theorem, find

$$\int_C \frac{1 - 4z + 6z^2}{(z^2 + \frac{1}{4})(2 - z)} dz.$$

Problem 4.(10pt) Find

$$\lim_{x\to\infty} \sqrt{x}(\sqrt{x+1} - \sqrt{x}).$$

Problem 5.(10pt) Find

$$\int \frac{dx}{x^2\sqrt{x^2+4}}.$$

Caution!!!
Use a separate answer book (Math.-B) for Problems 6, 7, and 8.

6. (15 points) Say True of False for each sub-problem. You don't need to justify your answer. (Correct answer +1, no answer 0, incorrect answer -1)

   (a) If the columns of a square matrix are linearly independent, then the rows are also linearly independent.

   (b) The projection of a vector $\underline{a}$ in the direction of $\underline{b}(\neq \underline{0})$ is $\underline{a}^T\underline{b}$.

   (c) If two linearly independent vectors $\underline{a}$ and $\underline{b}$ are on the same plane, then their cross/vector product $\underline{a} \times \underline{b}$ is a normal vector to the plane.

   (d) If $a_{i,j}$ and $b_{i,j}$ are (i,j)th entries of the $N \times N$ square matrices $\mathbf{A}$ and $\mathbf{B}$, respectively, then the (i,j)th entry of the product $\mathbf{AB}$ is given by $\sum_{k=1}^{N} a_{i,k}b_{k,j}$.

   (e) If $\mathbf{A}$ and $\mathbf{B}$ are both $N \times N$ square matrices, then $\mathbf{AB} = \mathbf{BA}$.

   (f) If $\mathbf{A}$ and $\mathbf{B}$ are both $N \times N$ square matrices, then $\mathbf{AB} = 0$ implies $\mathbf{A} = 0$ or $\mathbf{B} = 0$.

   (g) The row space and the null space of a matrix $\mathbf{A}$ have the same dimension.

   (h) For any $N \times N$ matrices $\mathbf{A}$ and $\mathbf{B}$, $\det(\mathbf{AB})=\det(\mathbf{A})\det(\mathbf{B})$.

   (i) The eigenspace of an $N \times N$ matrix $\mathbf{A}$ is the same as the column space of $\mathbf{A}$.

   (j) The eigenvalue λ of an $N \times N$ matrix $\mathbf{A}$ is defined as a non-zero constant such that $\mathbf{A}\underline{x} = \lambda\underline{x}$ for some $\underline{x}$.

   (k) If an $N \times N$ matrix $\mathbf{A}$ is unitary, then $\mathbf{AA}^T = \mathbf{I}$.

   (l) The determinant of a unitary matrix is 1.

   (m) The eigenvalues of a Hermitian symmetric matrix are real.

   (n) If λ is an eigenvalue of $\mathbf{A}$, then it is also an eigenvalue of $\mathbf{T}^{-1}\mathbf{AT}$, where $\mathbf{T}$ is any non-singular matrix.

   (o) If $\mathbf{A}$ is a real square matrix, then $\underline{x}^T\mathbf{A}\underline{x} > 0$ for any non-zero real vector $\underline{x}$.

7. (15 points) When 2×2 matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix},$$

answer the following questions.

(a) (5 points) Find the eigenvalues and corresponding eigenvectors of $\mathbf{A}$.

(b) (10 points) Find the minimum and the maximum values of

$$f(\underline{x}) = \frac{\underline{x}^T \mathbf{A} \underline{x}}{\underline{x}^T \underline{x}},$$

where $\underline{x} \neq \underline{0}$.

8. (15 points) When $x(t)$ is defined as

$$x(t) = \begin{cases} 1, & \text{for } t = 0 \\ \frac{\sin \pi t}{\pi t}, & \text{elsewhere,} \end{cases}$$

and $y(t) = (x(t))^2$, answer the following questions.

(a) (5 points) Sketch $y(t)$ and its Fourier transform.

(b) (10 points) Using the Fourier transform of

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t-k),$$

find and sketch

$$\sum_{k=-\infty}^{\infty} y(t-k).$$

QE Math 08

1 (10pt)

Write out Taylors formula for $\cos x \cos y$ about $(0, \pi)$ to three terms.

2 (10pt)

Green's formula is written as

$$\oint_B L dx + M dy = \iint_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dA$$

Evaluate the integral by Green's theorem. The integral is to be taken in the counterclockwise direction:

$$\oint_C \frac{y dx - x dy}{x^2 + y^2}$$

where $C \ : \ (x-3)^2 + (y-5)^2 = 4$.

3 (15pt)

Find the solution by utilizing Laplace transformation:

$$y'' + y = -9 \sin 2t$$

Initial conditions: $y(0) = 1$ and $y'(0) = 0$.

4 (15pt)

Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

a) Show that $f'(0) = 0$.
b) Show that $f'(x)$ is not continuous at $x = 0$.

Caution!!! Use a different answer book for Problems 5 and 6 from that for Problems 1-4.

5. (25 points) Suppose that an N-by-N matrix A has the (m, n)th entry given by

$$[A]_{m,n} = \frac{1}{\sqrt{N}} e^{j\frac{2\pi mn}{N}}$$

for $m, n = 1, 2, ..., N$. Answer the following questions.

(a) (5 points) When $N = 2$, find A, A^H, and A^{-1}, respectively, where the superscript ^{H} denotes Hermitian transposition.

(a) (10 points) Compute AA^H.

(b) (10 points) Let λ_i be the ith eigenvalue of A. Find $|\lambda_i|$ for $i = 1, 2, ..., N$.

6. (25 points) Suppose that a continuous-time signal $x(t)$ has the Fourier transform pair $X(j\omega)$ given by

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt,$$

and that a continuous-time signal $y(t)$ is defined as

$$y(t) = \sum_{k=-\infty}^{\infty} x(t - kT)$$

for some $T > 0$. Answer the following questions.

(a) (5 points) Find the inverse Fourier transform of $2\pi\delta(\omega - \frac{2\pi k}{T})$, where $\delta(\omega)$ is the Dirac delta function.

(b) (5 points) Find the kth Fourier series coefficient a_k such that

$$\sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi kt}{T}}.$$

(c) (15 points) Find the Fourier series representation of $y(t)$ in terms of $X(j\omega)$. (Hint: Use the results in (a) and (b).)

QE Math. 2007

1. (20 points) Answer the following questions.

   (a) (5 points) Let x_m be the mth entry of an M-by-1 vector $\mathbf{x}$, $a_{m,n}$ be the (m,n)th entry of an M-by-N matrix A, and let y_n be the nth entry of an N-by-1 vector $\mathbf{y}$. Find $\mathbf{x}^T A\mathbf{y}$ in terms of x_m, $a_{m,n}$ and y_n, where the superscript ^{T} denotes transposition.

   (b) (5 points) Show that all eigenvalues of an N-by-N real symmetric matrix B are real. (Hint. Consider $(\mathbf{v}^T B\mathbf{v})^*$, where $\mathbf{v}$ is an eigenvector of B and the superscript * denotes complex conjugation.)

   (c) (5 points) Find all the eigenvalues of the matrix C given by

   $$C = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}.$$

   (d) (5 points) Find the null space of the matrix D given by

   $$D = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

2. (20 points) Answer the following questions.

   (a) (5 points) Find the necessary and sufficient condition on a real number ω for $\int_0^T e^{j\omega t} dt = 0$, where T is a positive real number.

   (b) (15 points) Find the numbers a_n, for $n = -N, -N+1, ..., N$, that minimize

   $$\int_0^T \left| x(t) - \sum_{n=-N}^{N} a_n e^{j\frac{2\pi n t}{T}} \right|^2 dt,$$

   where $x(t)$ is a periodic function of t with period T (> 0). (Fully justify your answer.)

3. **(7pt)** Find the derivative of

$$\sin^{-1}(x^2 - 1).$$

4.**(7pt)** Find

$$\int (\ln(x))^2 dx.$$

5.**(16pt)** (a) Expand the following to the $2n^{th}$ order utilizing the Taylor series expansion:

$$\frac{1}{1+x^2}$$

(b) Find a series expansion of $\tan^{-1}(\cdot)$ utilizing result (a);

The result (b) can be used for calculating $\frac{\pi}{4} = \tan^{-1}(1)$. Another way of calculating π would be the following:
(c) Prove that

$$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}.$$

and show by utilizing the previous equation that

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right). \qquad (5-c)$$

(d) The target is to calculate $\pi = 3.141592\cdots$. Give a good reason why (5-c) is better than the calculation based on $\frac{\pi}{4} = \tan^{-1}(1)$.

6.**(7pt)** Show that $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.
(Hint. Utilize the integral test.)

7.**(15pt)**Solve the following differential equation utilizing Laplace transform:

$$\begin{aligned} y_1' &= -y_2, \\ y_2' &= y_1, \\ \text{initial condition} &: \; y_1(0) = 1, \qquad y_2(0) = 0. \end{aligned}$$

7.**(8pt)** Evaluate

$$\frac{1}{2\pi j} \oint_C \frac{\sin 3z}{z + \frac{\pi}{2}} dz$$

if C is the circle $|z| = 5$.

2006 Qualifying Exam. Math

1. (20 points) When the square matrices A and B are given, respectively, as

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 0 & 1 & 5 \\ 1 & 2 & -1 & 2 \\ -2 & -4 & 1 & 5 \\ 1 & -2 & 0 & 3 \end{bmatrix},$$

   (a) (5 points) Compute $A^T A$.

   (b) (5 points) Find all the eigenvectors and corresponding eigenvalues of the matrix A.

   (c) (10 points) Compute $\det(AB)$.

2. (30 points) When a continuous-time signal $x(t)$ is defined as

$$x(t) = \frac{\sin \pi t}{\pi t},$$

   answer the following questions.

   (a) (10 points) Find the continuous-time Fourier transform

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

   of $x(t)$ and sketch it. Carefully mark all the important values on the graph.

   (b) (20 points) Find the continuous-time Fourier transform of

$$y(t) = \sum_{n=-\infty}^{\infty} |x(t-n)|^2$$

   and sketch it. Carefully mark all the important values on the graph. (You can use the Dirac delta functions in your answer.)

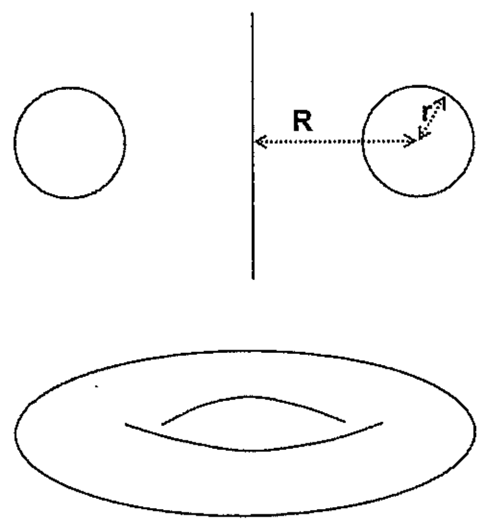
3. (10 points) Prove that $\ln(xy) = \ln x + \ln y$. (Hint: Use differentiation)

4. (10 points) Prove that $\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}$.

5. (10 points) Expand $\tan^{-1}(x)$ using Taylor series to the third order. (Hint: $\tan^{-1} x = \tan^{-1}(0 + x)$.)

6. (5 points) Using the previous result, find an approximate value of π. (Hint: $\frac{\pi}{4} = \tan^{-1}(1)$.)

7. (15 points) Find the volume of the torus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2005.10.26

2005 Math Qualifying Exam.

1. (20pts) Mark `O' if the statement is right, mark `X' otherwise. There is a penalty of -4 points for each wrong answer. You may leave question unanswered.

a) Convolution of sinc function with itself reduces to a sinc function.( )
b) The limit of a sequence $\{f_n(t)\}$ of continuous functions is continuous.( )
c) Sum of two jointly Gaussian random variables are Gaussian ( )
d) An infinitely many differentiable function can be expressed as a Taylor series. ( )
e) Any continuous function $f:[0,1] \rightarrow [0,1]$ is integrable. ( )

2.(15pts) Determine second order Taylor formula for

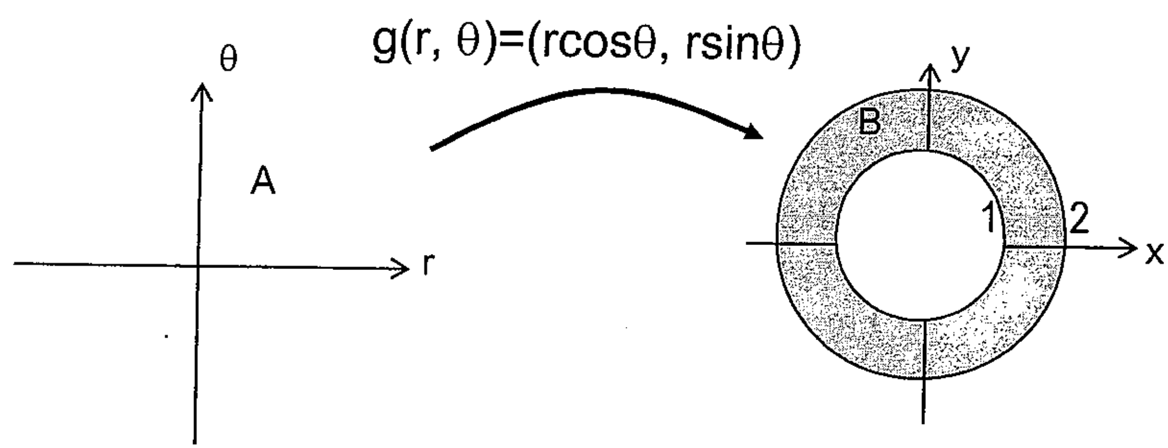
$$f(x, y) = \frac{1}{x^2 + y^2 + 1}, \qquad x_0 = 0,\ y_0 = 0.$$

3. (20pts) Find

$$\int_B (x^2+y^2)^{-\frac{3}{2}}\,dxdy$$

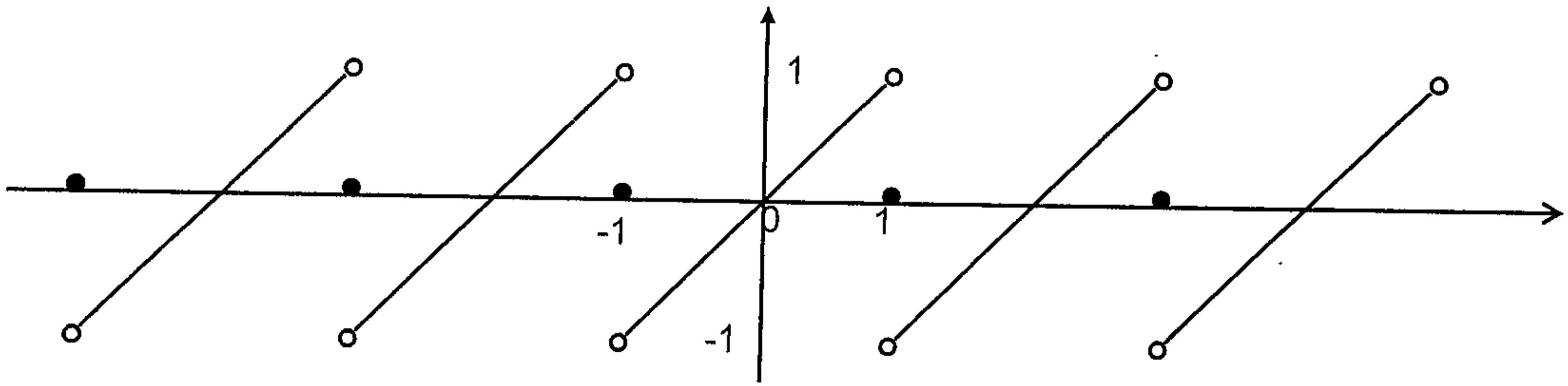
where B is given by the following annulus.

a) Let $(x, y)=g(r, \theta)$. Find A in (r, θ) plane such that $g(A)=B$.



b) Change the integral into the one in the polar coordinate and evaluate it.

4. (20pts) Find the Fourier transform of the following periodic function:



5. (10pts) Evaluate the following determinant:

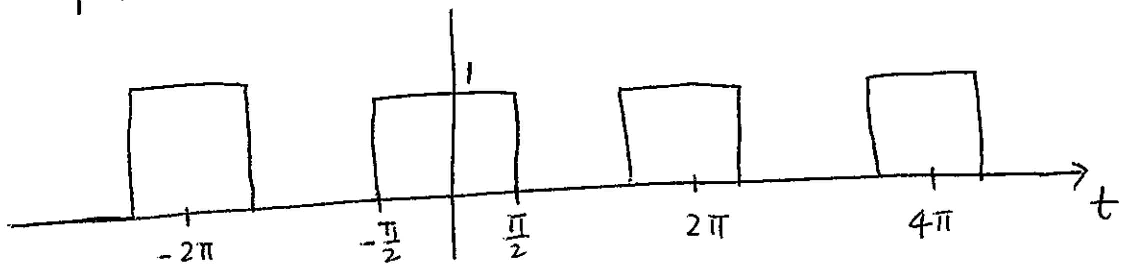
$$\begin{vmatrix} 1 & 2 & -1 & 2 \\ 3 & 0 & 1 & 5 \\ 1 & -2 & 0 & 3 \\ -2 & -4 & 1 & 5 \end{vmatrix}$$

6. (15pts) Solve the following differential equation using Laplace transform.
If you do not use Laplace transform, -7 point penalty will be applied.

$$y'' + y = 2\cos t, \qquad y(0) = 2, \qquad y'(0) = 0.$$

Math. Qualifying 2004. 11. 3

1. Find the Fourier Transform of the following periodic function. Sketch the solution (20)



2. Find the solution (20)

$$y''(t) + 2y'(t) + 5y(t) = 0, \qquad y(0) = 1, \quad y'(0) = 1$$

3. Find the convolution $y[n] = x[n] * h[n]$ (20)

$$x[n] = \left(-\tfrac{1}{2}\right)^n u[n-4]$$
$$h[n] = 4^n u[2-n]$$

4. Find the solution

$$\int_0^{2\pi} \frac{1}{5 - 4\cos\theta}\, d\theta$$

Hint: Let $z = e^{i\theta}$, then $\cos\theta = \frac{z + z^{-1}}{2}$, $dz = iz\,d\theta$.

Express in contour integral (10 pt)

Apply Residue (10 pt)

5. Show that (20)

$$\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{vmatrix} = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$$

Math. Qualifying Exam. 2003

#1.(15pt) Solve the following via Laplace transform:

$$y'' - y' - 2y = 10\sin t, \qquad y(0) = 1, \qquad y'(0) = -3$$

#1.(20pt) Solve the following integral. (Hint: Use integration by part)

$$\int_{-\infty}^{\infty} e^{-x^2} x^2 dx$$

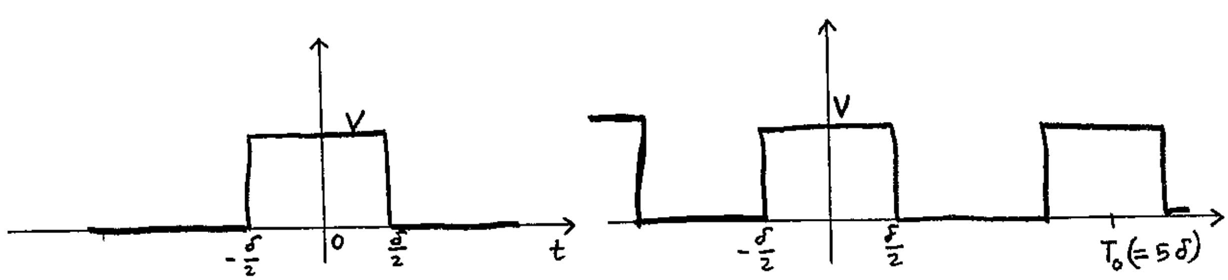
#1.(15pt) Evaluate the following integral where C is the unit circle(counterclockwise).

$$\int_C \frac{z^2 \sin z}{4z^2 - 1} dz$$

#1.(20pt) Find e^{At}, where

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

#1.(30pt) Find the Fourier transform $F(\omega)$ for the voltage pulse shown below:



Sketch the spectra $F(\omega)$ of the two. Need to specify the crossing points of x and y axes.

Find the relationship between two spectra.

2002 Qualifying Mathematics Examination

1. **(15 pt)** Compute $\cos(A)$, where
$$A = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

2. **(15 pt)** The characteristic equation of the following matrix is given by $(x-1)(x-3)^3$.
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -27 & 54 & -36 & 10 \end{bmatrix}.$$
Calculate A^{-1}. (Hint: Cayley-Hamilton theorem)

3. **(20 pt)** Solve the following differential equation.
$$y^{(2)} + 3y^{(1)} + 2y = 2,\ y(0) = 0,\ y^{(1)}(0) = 0.$$

4. **(10 pt)** Let $f(x) = x^{1/(x-1)}$ for $x \neq 1$. How should $f(1)$ be defined in order to make f be continuous at $x = 1$.

5. **(10 pt)** Evaluate the limit.
$$\lim_{x \to 0^+} (1 + \sin 2x)^{1/x}$$

6. **(10 pt)** Use appropriate theorems, consider the Fourier transform of
$$x(t) = e^{-a|t|} \cos 2\pi f_0 t.$$
Get the Fourier transform of $\cos 2\pi f_0 t$ in the limit as $a \to 0$.

7. **(10 pt)** Verify Green's theorem in the plane for
$$\oint x^2 y dx + (y^3 - xy^2) dy$$
where C is the boundary of the region enclosed by the circles $x^2 + y^2 = 4$, $x^2 + y^2 = 16$.

Hint: (Green's theorem)
$$\oint_C P dx + Q dy = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$
where C is the boundary of a bounded area R and P and Q are differentiable in R.

8. **(10 pt)** Find $\int \ln x dx$.

2002 Qualifying Mathematics Examination

1. (10 points) Compute $\sin(A)$, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}.$$

2. (10 points) Compute

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 2 & 2 & 2 & 2 & 3 \end{bmatrix}^{-1}.$$

hint: $A(I + BA)^{-1}$.

3. (10 points) Find the minimal polynomial of A, where

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

4. (20 points) Solve the following differential equation:

$$\ddot{y}(t) + 2\dot{y}(t) + y(t) = 1, \quad y(0) = 2, \; \dot{y}(0) = 0.$$

5. (35pt) a) One can express the following function as a sum of Fourier series:

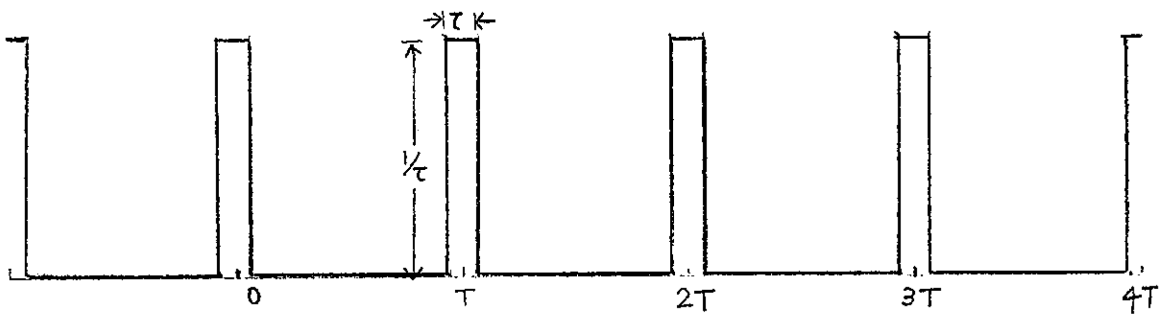
$$p(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn2\pi f_s t}$$

Obtain C_n by utilizing $\{e^{jn2\pi f_s t}\}_n$ as a set of orthonormal basis.

Hint: Note that

$$p(t) = \sum_{n=-\infty}^{\infty} < p(t), e^{jn2\pi f_s t} >$$

and C_n is the projection of $p(t)$ into $e^{jn2\pi f_s t}$.



b) Multiplying an arbitrary signal $x(t)$ by $p(t)$, we have

$$x_s(t) = \sum_{n=-\infty}^{\infty} C_n x(t) e^{jn2\pi f_s t}.$$

Take Fourier transform of $x_s(t)$.

Hint:

$$X_s(f) = \int_{-\infty}^{\infty} x_s(t) e^{-jn2\pi f_s t}.$$

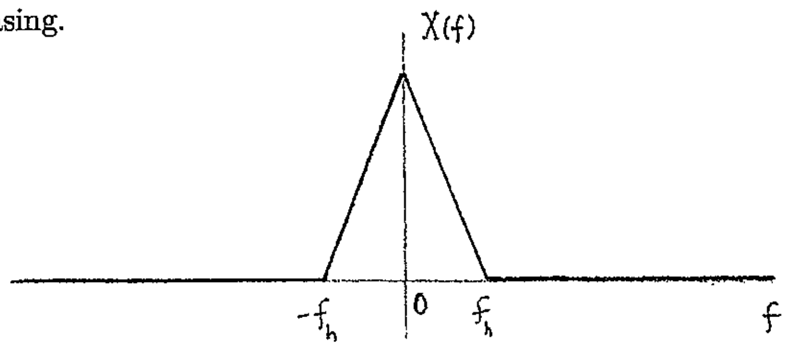
Find a relation between $X_s(f)$ and $X(f)$

Hint: Frequency shifting property needs to be used.

c) If $\tau \rightarrow 0$, then $p(t)$ converges to a pulse train. In such a case find C_n and rewrite the relation obtained in b).

d) Draw spectra of $X_s(f)$ for two cases: i) $f_s(=\frac{1}{T}) > 2f_h$, ii) $f_s(=\frac{1}{T}) > 2f_h$.

Illustrate aliasing.



e) We want to reconstruct the original signal $x(t)$ from the sampled signal $x_s(t) = x(t)\sum_{k=-\infty}^{\infty}\delta(t-kT)$ by applying an ideal filter whose spectrum is $H(f)$. Give the specification of $H(f)$ and obtain $h(t)$, where $h(t)$ is the impulse response of the filter.

f) Write $x(t)$ as the output of filtering $x_s(t)$ by the filter $h(t)$.

g) Illustrate why the ideal filter $h(t)$ is not realizable.

6. (15pt) Evaluate

$$\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}.$$

Math. Qualifying Exam

(October. 27, 2000)

1.(15) We want to show that the Fourier transform of the impulse train gives us another impulse train in the frequency domain. Specifically, we want to derive the Fourier transform of $h(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$. But, the answer must be also expressed in the form of impulse train in the frequency domain. Note that we define the Fourier transform of $x(t)$ by $\mathcal{F}(x(t)) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$. Take the following step:

a) Show that if $\mathcal{F}(x(t)) = X(f)$, then $\mathcal{F}(e^{j2\pi f_0 t}x(t)) = X(f - f_0)$.

b) Take the Fourier series expansion of $h(t)$ using the basis function $e^{j2\pi n \frac{t}{T}}$. (Hint : Integrating Interval $[-\frac{T}{2}, \frac{T}{2}]$)

c) Now take the Fourier transform of the (Fourier) series expression obtained in b) using the relation in a). Write down the desired result.

2.(10) Note that $\mathcal{F}[\Pi(t)] = \text{sinc}(f)$ where

$$\Pi(t) = \begin{cases} 1, & -1/2 \le t \le 1/2 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \text{sinc}(f) = \frac{\sin \pi f}{\pi f}.$$

Utilizing the above fact, show that

$$\text{sinc}(t) * \text{sinc}(t) = \text{sinc}(t),$$

where $*$ denotes the convolution.

3.(7) Evaluate

$$\lim_{t\to\infty} \left[\frac{m+1}{m-1}\right]^m.$$

4.(8) Using the Taylor series, expand x^2y at $(x, y) = (1, -1)$. The answer must be expressed as a polynomial of $(x - 1)$ and $(y + 1)$.

5.(10) Given the causal transfer function $H(s) = \frac{s+1}{s^2+5s+6}$ calculate the output when the input signal is $x(t) = \cos 2t$.

6.(25) Evaluate the integral using contour integral. (Do not forget to specify the contour of integration and the pole within the contour.)

$$I = \int_0^{2\pi} \frac{d\theta}{A + B\cos\theta}, \qquad (A > |B|)$$

7.(25) a) Write

$$f(x,y,z) = 3x^2 - y^2 - 2z^2 - 8xy + 12yz + 4zx$$

in the quadratic form, i.e., find the matrix A such that $f(x,y,z) = \mathbf{x}^T A\mathbf{x}$, where $\mathbf{x} = [x,\ y,\ z]^T$.

b) Using $\mathbf{X} = P\mathbf{x}$, rotate the coordinates to obtain

$$f(X,Y,Z) = aX^2 + bY^2 + cZ^2,$$

where $\mathbf{X} = [X,\ Y,\ Z]^T$ and P is a 3×3 constant matrix. Find a, b, c.

c) Find the min. and max. of $f(x,y,z)$ when $x^2 + y^2 + z^2 = 4$.

1999. 10. 20

수학

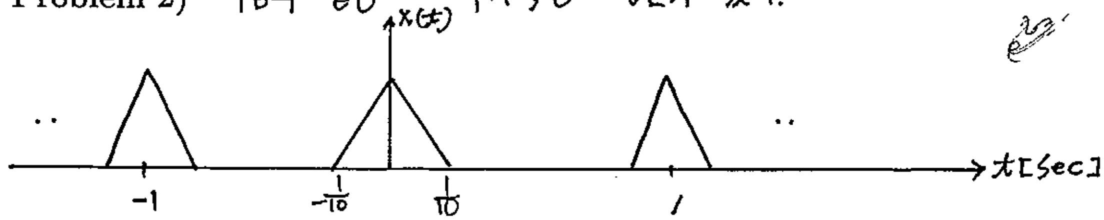
(15) **Problem 1)**

1. Using contour Integral, evaluate

$$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{e^x + 1}\, dx \qquad (0 < a < 1)$$

Don't forget to specify the contour for the integral

(25) **Problem 2)** 다음과 같은 주기적인 신호가 있다.



a. $x(t)$의 Fourier Transform $x(\omega)$를 구하고 $|x(\omega)|$를 sketch하라

b. $x(t)$를 $\Delta t = \frac{1}{100}[sec]$로 sampling한 신호 $x_s(t)$의 Fourier Transform $X_s(\omega)$를 구하고 $|x_s(\omega)|$를 sketch 하라.

c. $x_s(\omega)$는 periodic한가? periodic이라면 그 주기는 얼마인가?

(15) **Problem 3)**

Find the critical points of $f(x, y) = x^3 - 3x^2 + y^2$ and determine whether f has local maximum, local minimum, or saddle at each of these critical points.

(10) **Problem 4)**

Prove that

$$\begin{matrix} lim \\ n \to \infty \end{matrix} \quad \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) = \log 2$$

(15) **Problem 5)**

Prove that

$$y(t) = e^{\int_0^t p(\tau)d\tau} y(0) + \int_0^t e^{\int_\tau^t p(\sigma)d\sigma} q(\tau)d\tau$$

$$\frac{d}{dt}\int^t f(t,\tau)\,d\tau = f(t,t) \cdot \frac{d}{dt} f(t,\tau)$$

is the solution of

$$\frac{d}{dt}y(t) = p(t)y(t) + q(t),$$

where $p(t), q(t)$ are continuous scalar functions of t.

(20) Problem 6)

Suppose that $A \in R^{n\times n}$ has n distinct eignenvalues $\lambda_1, \cdots, \lambda_n$. Prove that there exist matrix $T \in R^{n\times n}$ such that

$$TAT^{-1} = \begin{bmatrix} \lambda_1 & & 0 \\ \cdot & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Further, show that

$$\det A = \prod_{i=1}^{n} \lambda_i$$

$$\frac{dy}{p(t)y(t)} = q(t)$$

Qualifying Exam (수학) 98.10.23

1. (20pt)

a) $f(t) = \cos 2\pi f_0 t$ 의 Fourier Transform $F(f)$를 구하고 sketch 하라

b) $f(t)$를 sampling 함수

$$S(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

로 sampling 한 $f_s(t)$의 Fourier transform $F_s(f)$를 구하고 sketch하라. 단 $\frac{1}{T} = 4f_0$ 이고 δ는 delta 함수를 의미한다.

c) $f_s(t)$를 linear interpolation 으로 신호를 복원 하고자 한다. 복원된 신호 $f_R(t)$의 Fourier transform $F(f_R)$를 구하고 sketch 하라.

[hint] $f_R(t) = f_s(t) * h(t)$로 나타낼수있다.

(10pt)
2. Matrix A의 eigenvalue 와 eigenvector를 구하라.

$$A = \begin{bmatrix} 7 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

(20pt)
3. 아래적분을 Contour Integral 을 이용하여 구하시오
(Contour of Integration 은 반드시 표시할것)

$$\int_0^{\infty} \frac{\sin^2 x}{x^2}\, dx$$

[Hint] $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

(10pt)
4. Compute $\lim_{n\to\infty} n \log_e\left(1 + \frac{1}{n}\right)$

(10pt)
5. 다음함수는 (x_0, y_0)에서 Taylor Series (2차 까지)로 전개하시오

$$f(x, y) = e^{-x^2-y^2} \cos(xy)$$

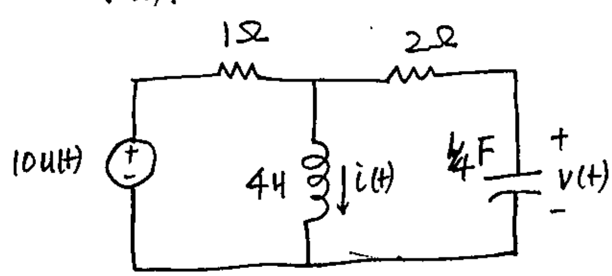
$x_0 = 0,\ y_0 = 0$

(10pt)
6. Show

$$\int_0^{\infty} \frac{\sin x}{x^2+1}\, dx$$

is convergent.

(20pt)
7. Determine the capacitor voltage $v(t)$.



$i(0) = 2$ A

$V(0) = 20$V

$u(t)$: unit step function.

* Use Laplace transform

Qualifying Examination 97

MATHEMATICS

Oct.16, 1996

Linear Algebra

1. (10 point) Compute e^A, where $A = \begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}$.

2. (10 point) Find the characteristic polynomial and the minimal polynomial of

$$\begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Complex Variables

3. (10 point) Compute

$$\int_{-\infty}^{\infty} \frac{\sin 2x}{x^2 + x + 1} dx.$$

4. (20 point) A function f of two real variables is defined for each point (x, y) in the unit square $0 \le x \le 1,\ \ 0 \le y \le 1$ as follows

$$f(x, y) \triangleq \begin{cases} 1 & \text{if} \quad x \text{ is rational.} \\ 2y & \text{if} \quad x \text{ is irrational.} \end{cases}$$

   (a) Compute $\bar{\int}_0^1 dx$ and $\underline{\int}_0^1 dx$ in terms of y.

   (b) Show that $\int_0^1 f(x, y)dy$ exists for each x and compute $\int_0^t f(x, y)dy$ in terms of x and t for $0 \le x \le 1,\ \ 0 \le t \le 1$.

Here,

$$\begin{aligned}
\overline{\int_a^b} f d\alpha &\triangleq \inf\{\mathcal{U}(p, f, \alpha) | p \in \mathcal{P}[a,b]\}, \\
\underline{\int_a^b} f d\alpha &\triangleq \sup\{\mathcal{L}(p, f, \alpha) | p \in \mathcal{P}[a,b]\}, \\
\mathcal{U}(p, f, \alpha) &\triangleq \sum_{k=1}^{n} \sup_{x_{k-1} \le x < x_k} \{f(x)\}, \\
\mathcal{L}(p, f, \alpha) &\triangleq \sum_{k=1}^{n} \inf_{x_{k-1} \le x < x_k} \{f(x)\},
\end{aligned}$$

where $\mathcal{P}$ is a partition of $[a, b]$.

Fourier Transformation

Let $x(t)$ be a given signal with Fourier transform $X(w)$. Define the signal

$$f(t) \triangleq \frac{d^2}{dt^2} x(t).$$

5. (10 point) Suppose that

$$X(w) = \begin{cases} 1 & |w| < 1 \\ 0 & |w| > 1 \end{cases}$$

Evaluate

$$\int_{-\infty}^{\infty} |f(t)|^2 dt.$$

6. (10 point) What is the inverse Fourier transform of $f(w/4)$?

Miscellanies

7. (20 point) Solve

$$\frac{d^2}{dt^2} y + 2\frac{d}{dt} y + y = \sin 10t. \quad y(0) = 1, \quad \frac{d}{dt} y(0) = 2.$$

8. (10 point) When $a > 1$, compute the maximum of

$$y = \frac{\log_a x}{x}.$$

95학년도 박사과정 수학 입학시험

1994. 10. 19

1. Evaluate the limit of
(10pt)

$$\left(\frac{x^n+y^n}{2}\right)^{\frac{1}{n}}, \quad x \geq y > 0$$

2. Find
(10pt)

$$\iint_A xy \sin(x^2-y^2)\, dx\, dy$$

where $A = \{(x,y) \mid 0 < y < 1,\ x > y \text{ and } x^2-y^2 < 1\}$

3. Let
(10pt)

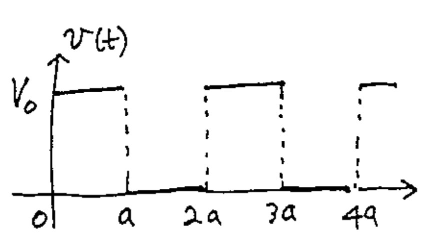
$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is irrational} \\ 0 & \text{if } x \text{ is rational} \end{cases}$$

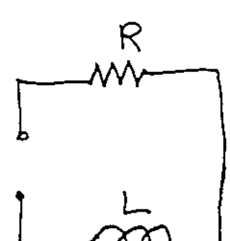
Show $f'(0)$ exists. Is f continuous at 0 ?

4 Find a matrix $P \in \mathbb{R}^{3\times n}$ such that PAP^{-1} is diagonal :
(15pt)

$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

5 Find the steady-state current in the circuit :
(15pt)





6 (15 pt) Evaluate the integral below using contour integral.
(Don't forget to specify the contour of integration.)

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx$$

7 (25 pt) Given a function $x(t) = \cos 2\pi f_0 t$,

a. Calculate its Fourier transform $X(f)$ and sketch it.

b. Calculate the Fourier transform $X_s(f)$ of its sampled version $x_s(t)$ with sampling interval t_s and sketch it. (Assume $t_s << 1/f_0$)

c. The function, $x_r(t)$, is reconstructed by linearly interpolating $x_s(t)$. Calculate its Fourier transform $X_r(f)$ and sketch it. Indicate the reconstruction error in the Fourier domain.

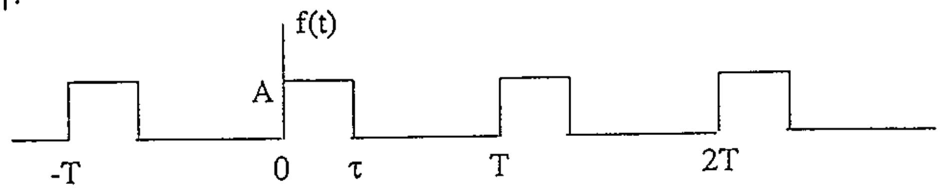
QE'93 : 93/11/7

Mathematics

박사과정 재학생은 #5문제까지, 신입생은 끝까지 풀 것.

Problem #1 [20점]

a. convolution theorem을 이용하여 아래 주기적파형 f(t)의 Fourier Transform F(ω)를 수식으로 구하고 magnitude를 sketch하시오. (T = 4τ 라 가정하시오.) 단 rectangular pulse 의 Fourier Transform이 sinc함수이며, impulse train 의 Fourier Transform이 간격이 다른 impulse train이라는 사실과 Fourier Transform의 shift property를 이용해도 된다.



b. 위 파형 f(t)를 $\Delta t = \tau/4$ 의 간격으로 sampling 한 파형을 $f_s(t)$라 할 때 이의 Fourier Transform $F_s(\omega)$를 구하고 magnitude를 sketch하시오.

c. 만일 위에서 sampling function이 delta function이 아닌 폭이 τ_a인 ($\tau_a << \tau$) rectangular function일 때 이의 $|F_s(\omega)|$ 에의 영향을 sketch하시오.

Problem #2 [20점]

a. $f(x,y,z) = 3x^2 - y^2 - 2z^2 - 8xy + 12yz + 4zx$ 를 quadratic form $\underline{x}^t A \underline{x}$ 로 표시할 때 matrix A 를 구하라. 단 $\underline{x} = (x\ y\ z)^t$ 이다.

b. matrix A 가 Hermitian 임을 보여라.

c. $\underline{X} = P\underline{x}$ (단 $\underline{X} = (X\ Y\ Z)^t$ 임)로 좌표회전하여 $f(X,Y,Z) = aX^2 + bY^2 + cZ^2$ 의 형태로 만들 때 a, b, c와 matrix P를 구하라.

d. matrix P는 unitary임을 보여라.

e. $x^2 + y^2 + z^2 = 1$일 때 f(x,y,z)의 최대값과 최소값을 구하라.

Problem #3 [20점]

contour integral 을 이용하여 아래의 적분을 i) $t < 0$ 와 ii) $t > 0$ 로 나누어 구하시오. contour를 정확히 표시할 것.

$$I(t) = \frac{A}{2\pi}\int_{-\infty}^{\infty} \frac{e^{j\omega t}\, d\omega}{R + j\omega L}$$

[20pt]**Prob.** #4: Compute the second order Taylor formular for $f(x,y) = e^x \cos y$ around (0,0).

[20pt]**Prob.** #5 Solve the following differential equation using the Laplace transform:

$$y''(t) + 2y'(t) + 2y(t) = 4; \quad y(0) = 0, \; y'(0) = 0$$

____________ Students in the Ph. D. course stop at this point.____________

[20pt]**Prob.** #6 Prove that

$$\ln 2 = \lim_{n\to\infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right].$$

(Hint: Use Riemann sums)

[20pt]**Prob.** #7 Use Green's theorem to show that the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is πab.

[20pt]**Prob.** #8 Mark 'T' for *True Statement* or 'F' for *False Statement*. There is a penalty of -2 points for each wrong answer. You do not need to state the jestification.

(1) If f is a continuous function on [0,1], then f is bounded on [0,1].

(2) An integrable function on [0,1] must be continuous on [0,1].

(3) If U and V are open subsets of $\mathbb{R}$, then $U \times V = \{(x,y) | x \in U, \; y \in V\}$ is open.

(4) If f and g are integrable functions on $[a,b]$, then $f - 2g$ is integrable on $[a,b]$.

(5) Any bounded sequence in $\mathbb{R}^n$ must have a convergent subsequence.

(6) If f is a continuous real-valued function on [0,1] such that $f(x) \geq 0$ for all $x \in [0,1]$ and $\int_0^1 f(x)dx = 0$, then $f(x) = 0$ for all $x \in [0,1]$.

(7) If f is an infinitely differentiable real-valued function on $\mathbb{R}$, then f must have a power series expansion about each point of $\mathbb{R}$.

(8) If f_1,f_2,f_3, $\cdots$ are all continuous real-valued functions on (0,1), then $f(x) = \lim_{n\to\infty} f_n(x)$ is also continuous on (0,1).

(9) Assume F is a smooth vector (field) in $\mathbb{R}^3$. If $\nabla \times F = 0$, locally there exists $f : \mathbb{R}^3 \to \mathbb{R}$ such that $F = \text{grad} f$.

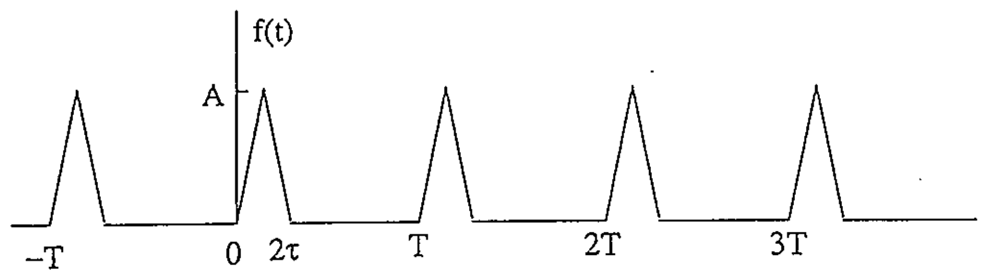
(10) If f is a differentiable real-valued function on $(0,1)$ and $f(1/2) \geq f(x)$ for all $x \in (0,1)$, then $f'(1/2) = 0$

QE'93 : 93/3/30

Mathematics

Problem #1 [25점]

a. convolution theorem을 이용하여 아래 주기적파형 f(t)의 Fourier Transform F(ω)를 수식으로 구하고 magnitude를 sketch하시오. 단 rectangular pulse 의 Fourier Transform이 sinc함수이며, impulse train 의 Fourier Transform이 간격이 다른 impulse train이라는 사실과 Fourier Transform의 shift property를 이용해도 된다.



b. Gibbs phenomenon에 대해 설명하시오.(발생이유, discontinuity 에서 두드러지는 이유 등 포함)

Problem #2 [25점]

2차원 random vector $x = (x_1,x_2)$의 sample 이 다음과 같이 7개가 주어졌다.

$(x_1,x_2) = \{ (0,0)\ (1,0)\ (0,1)\ (1,1)\ (2,1)\ (2,2)\ (1,2) \}$

a. mean vector m_x 과 covariance matrix C를 구하시오.

b. covariance matrix C의 eigenvalue와 normalized eigenvector를 구하시오.

c. $ACA^T = D$ (D는 diagonal matrix) 로 하는 matrix A를 구하고 이때의 D를 구하시오.

d. matrix A가 unitary matrix 가 됨을 보이시오.

e. random vector x 를 unitary matrix P 로 $y = (y_1,y_2) = P\,(x-m_x)$와 같이 변환시킬 때 y_1의 variance 를 최대로 하는 P와 A 사이의 관계를 구하고 이때의 $E\{y_1^2\}$, $E\{y_2^2\}$, $E\{y_1y_2\}$를 구하시오. 단 E{ }는 expectation operator임.

문제 #3, #4, #5 중 택 2.

Problem #3 [15점]

set $f(x) = \begin{bmatrix} x, & x: \text{rational} \\ 1-x, & x: \text{irrational} \end{bmatrix} \quad \{ 0 \le x \le 1 \}$

show that f is continuous at $x = 1/2$

Problem #4 [15점]

$\int (\ln x)^n \, dx$ 을 부정적분하시오.

Problem #5 [15점]

Evaluate $\int_0^\infty dx / (x^6+1)$

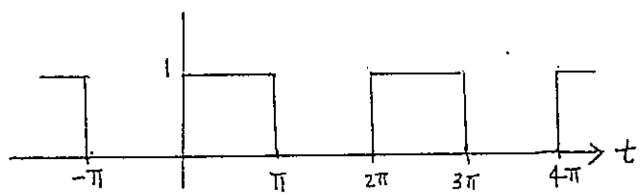
Problem #6 [20점]

Solve $y'' + y = 3\cos(2t)$, $y(0) = 3$, $y'(0) = 0$

Math. Qualifying Exam.

March 1992

Prob. 1.(20pt) Find the Fourier series of the following function:



Prob. 2.(15pt) Let A be a $n \times n$ nonsingular matrix. The trace of A is defined to be the sum of the diagonal elements, i.e., $\mathrm{tr}(A)=\sum_{i=1}^{n} a_{ii}$. Show that $\mathrm{tr}(A)=\mathrm{tr}(TAT^{-1})$, where T is a $n \times n$ nonsingular matrix.

Prob. 3.(15pt) Find the mass of a solid body S determined by the inequalities of spherical coordinates:

$$0 \le \theta \le \frac{\pi}{2}, \qquad \frac{\pi}{4} \le \varphi \le \arctan 2, \qquad 0 \le \rho \le \sqrt{6}.$$

The density, given as a function of the spherical coordinate (θ, φ, ρ), is equal to $\dfrac{1}{\rho}$.

Prob. 4.(15pt) Using $(\arctan x)' = \dfrac{1}{1+x^2}$, show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots.$$

Prob. 5.(15pt) Evaluate

$$\int_C \frac{\sinh z}{4z^2+1}\,dz,$$

where C is the unit circle.

Prob. 6.(20pt) Solve

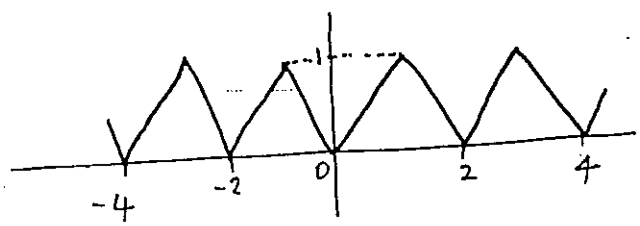
$$y'' + 4y = 4(\cos 2t - \sin 2t), \qquad y(0) = 1, \qquad y'(0) = 2.$$

1992 Math. Qualifying Exam.

1. Find $\lim_{x \to 0} (\cos x)^{\frac{1}{x^2}}$
(5pt)

2. Investigate the continuity of $f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$
(15pt) at $(0,0)$.

3. Expand the following function in a Fourier series.
(15pt)



4. Evaluate $\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}$
(15pt)

5. 어떤 causal system 의 impulse response 가 $h(t)$ 일 때 입력으로 $e^{j\omega t}$
(20pt) 를 가하면 해당 출력이 $H(j\omega)e^{j\omega t}$ 가 됨을 증명하라. 여기서

$$\int h(\tau) \frac{e^{j\omega(t-\tau)}}{e^{j\omega t}e^{-j\omega\tau}} d\tau = H \cancel{(s)} e^{j\omega t}$$

$H(s) = \int_0^{\infty} h(\tau) e^{-s\tau} d\tau$ (impulse response 의 Laplace transform) 이다.

$$e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow H(j\omega)e^{j\omega t}$$

6. Solve the following differential equation by means of Laplace transform.
(20pt)

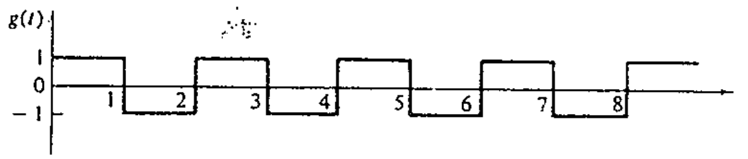
$y'(0) = -2$

1991 Qualifying Mathematics Examination

1. (15pt) Find the Laplace trasforms of the following functions.

   a) $g(t) = t\cos(5t)$,

   b) $\longrightarrow$



2. (15pt) Solve the following differential equation.

$$y'' + y = -9\sin 2t, \qquad y(0) = 1, \quad y'(0) = 0\,.$$

3. (15pt) Find the determinant of the following matrix.

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 3 & 0 & 1 & 5 \\ 1 & -2 & 0 & 3 \\ -2 & -4 & 1 & 6 \end{bmatrix}$$

4. (15pt) Evaluate the following integral:

$$\int_0^\infty \frac{1}{1+x^4}dx$$

5. (20pt) The bivariate random vector (x, y) has the joint density function.

$$f(x,y) = \begin{cases} c\cdot xy, & \text{if } 0 \le x \le y \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

Find constant c and the expectation of y, i.e., $E[y]$.

6.(20pt) Mark "O" if the statement is right, or "X" otherwise. There is penalty of -5 points for each wrong answer.

   a. $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \cdots$ diverges. ( )

   b. The following function is not continuous at all points in (0,1). ( )

$$f(x) = \begin{cases} 1, & \text{rational number} \\ 0, & \text{irrational number} \end{cases}.$$

   c. For some vector $V \in \mathbb{R}^3$,

$$\nabla \times (\nabla \times V) = \nabla\nabla\cdot V - \nabla\cdot\nabla V,$$

   where $\nabla\cdot$, ∇, and $\nabla\times$ denote divergence, gradient, and curl, respectively. ( )

   d. Consider the sequence of continuous functions, $\{f_n(x) : [0,1] \to [0,1]\}$. Then, the limit $f(x) := \lim_{n\to\infty} f_n(x)$ is also continuous over (0,1). ( )

Mathematics Qualifying Exam.

Problem 1.(10pt) A method of calculating $\pi = 3.14159\cdots$ is to utilize the relation, $\arctan(1) = \frac{\pi}{4}$. Illustrate how to calculate π. In the Taylor series expansion you should obtain at least 3 terms.

Problem 2.(10pt) Show that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges, where the base of log is e.

Problem 3.(15pt) Show that

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx = \sqrt{\pi}.$$

Hint :
$$\left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\, dx dy.$$

Problem 4.(15pt) Let A and P be $n \times n$ matrices. Assume P is nonsingular. Show that the eigenvalues of A and PAP^{-1} are the same.

Problem 5.(15pt) Let C be the unit circle in $\mathbb{C}$ centered at the origin. Evaluate

$$\int_C \frac{z^2 \sin z}{4z^2 - 1}\, dx,$$

Problem 6.(15pt) Solve the following differential equation:

$$y'' + y = 2\cos t, \qquad y(0) = 2, \quad y'(0) = 0.$$

Problem 7.(20pt) Mark O if the statement is right, or X otherwise. There is a penalty of -5 points for each wrong answer. If you are not sure, you may leave ( ) blank. (Don't explain.)

a) If $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist and are continuous in an open neighborhood of (x_0, y_0), then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ in a neighborhood of (x_0, y_0). ( )

b) If the Fourier integral (transform) of a real function $f(t)$ is real, then $f(t)$ is even. ( )

c) If an eigenvalue of a square matrix A, then the determinant of A is zero. ( )

d)
$$\nabla \cdot (E \times H) = -E \cdot \nabla \times H + H \cdot \nabla \times E \qquad (\ )$$

(8) If $f_1, f_2, f_3, \cdots$ are all continuous real-valued functions on (0,1), then $f(x) = \lim_{n\to\infty} f_n(x)$ is also continuous on (0,1). T

(9) Assume F is a smooth vector (field) in $\mathbb{R}^3$. If $\nabla \times F = 0$, locally there exists $f : \mathbb{R}^3 \to \mathbb{R}$ such that $F = \text{grad} f$. T

(10) If f is a differentiable real-valued function on $(0,1)$ and $f(1/2) \geq f(x)$ for all $x \in (0,1)$, then $f'(1/2) = 0$ T

Mathematics Problems

Problem M1: (25점)

a. Evaluate

$$\frac{1}{2\pi i}\oint_C \frac{4}{4z^2-1}dz,$$

where C is the unit circle.

b. We want to evaluate the following by using the contour integral:

$$I = \int_0^\infty \frac{dx}{1+x^2}$$

Specify the integration contour and obtain I by applying the residue theorem.

Problem M2: (25점)

a. Find the eigenvalues and the eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

and obtain matrix P such that $P^{-1}AP$ is a diagonal matrix.

b. i) The characteristic equation of the following matrices are given by $(t+2)^2(t-4)$.

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}.$$

For each matrix, obtain the basis of the eigenspace.

ii) Which matrix can be diagonalized? Explain why.

Problem M3: Suppose that the probabilty density function of the random variable X is given by $f(x)$ and let $Y = 2X$. What is the probabilty density funtion of Y? (10점)

Problem M4:

$$\lim_{x \to 0} x^x = \qquad (5점)$$

Problem M5: Solve the following ordinary differential eqaution:

$$y'' + y = -15 \sin 4x, \qquad y(0) = y'(0) = 0 \qquad (10점)$$

O-X Problems: Mark "O" or "X ". There is a penalty of -5 points for each wrong answer. (25점)

M6.1 : $\sum_{n=0}^{\infty} \frac{|cos x|}{n^2}$ converges as $n \to \infty$ for all real number x. ( )

M6.2 : If $\mathtt{curl} F = 0$, then (locally) there exists a scalar potential function f such that $F = \mathtt{grad} f$. ( )

M6.3 : According to the definition of the integration in the high school textbooks or in the freshman calculus, the funtion

$$f(x) = \begin{cases} x^2, & \text{if } x \text{ is rational number,} \\ 0, & \text{if } x \text{ is irrational number} \end{cases}$$

is integrable for the closed interval $[0, 1]$. ( )

M6.4 : The funtion

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is differentiable at $x = 0$. ( )

M6.5 : If X_1 and X_2 are Gaussian random variables, then $X_1 + X_2$ is also a Gaussian random variable. ( )

[1]

(a) Prove that the rank of a matrix (not necessarily square) is not changed by multiplication by a nonsingular matrix.

(b) Let A and B be n x n nonsingular square matrices. Prove that AB has an inverse, namely

$$(AB)^{-1} = B^{-1} A^{-1}.$$

Use the definition of an inverse matrix.

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2. (1) Find the general solution of the following differential equation

$$\frac{d^3y}{dx^3} - \frac{dy}{dx} = 2x + 1 - 4\cos x + 2e^x .$$

(2) Evaluate an integral $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\, dx.$