Chapter 6
SPECIAL FUNCTIONS

Friedrich Wilhelm Bessel
(1784-1846)
Math/Astronomy
Discovery of Neptune
Bessel Functions

Lecture 23

6.3 Bessel Functions

Hermann Hankel
(1839-1873)
Math
Hankel Functions
Hankel Transform

6.3.2 Generating Function of Integral Order

The Bessel functions can also be defined by a generating function without solving the Bessel's differential equation, which is more convenient for many practical problems.

Consider a function with two variables,

$$g(x,t)=\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right] \qquad (6.17)$$

$\exp\left(\frac{x}{2}\left(t-\frac{1}{t}\right)\right)$

Expanded in a Maclaurin series series, we can define Bessel functions of the first kind:

$$\boxed{\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right]=\sum_{-\infty}^{\infty}J_n(x)t^n} \qquad (6.18)$$

$= \exp\left(\frac{xt}{2}\right)\cdot\exp\left(-\frac{x}{2t}\right)$

$= \sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{xt}{2}\right)^r \sum_{s=0}^{\infty}\frac{1}{s!}\left(-\frac{x}{2t}\right)^s$

Proof) Expanding the argument of the exponential function, we have

$$\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right]=\exp\left(\frac{xt}{2}\right)\cdot\exp\left(-\frac{x}{2t}\right)=\sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{xt}{2}\right)^r\sum_{s=0}^{\infty}\frac{1}{s!}\left(-\frac{x}{2t}\right)^s$$

$= \sum_{n=0}^{\infty}\left[\sum_{r-s\geq 0,\ r,s\geq 0}\frac{(-1)^s}{r!\,s!}\left(\frac{x}{2}\right)^{r+s}\right]t^n$

$= \sum_{n=0}^{\infty}\left[\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s}\right]$

Introducing a new summation index $n=r-s\geq 0$, we find

$$\boxed{J_n(x)=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s}} \qquad (6.19)$$

$J_n(x) = \sum_{s=0}^{\infty}\frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s}$

which is the same as one in (6.10) obtained from the Bessel's differential equation.

6.3.3 Recurrence Relations

The recurrence relation of Bessel functions can be obtained from the generating function, which is much easier than from the form of series expansion.

Differentiating (6.19) with respect to t, we find

$$\frac{\partial}{\partial t}g(x,t)=\frac{1}{2}x\left(1+\frac{1}{t^2}\right)\exp\left[\frac{x}{2}\left(t-\frac{1}{t}\right)\right]=\sum_{n=-\infty}^{\infty}nJ_n(x)t^{n-1} \tag{6.20}$$

Substituting (6.18) into (6.20), we have

$$\boxed{J_{n-1}(x)+J_{n+1}=\frac{2n}{x}J_n(x)} \tag{6.21}$$

In a similar manner, differentiating (6.19) with respect to x, we find

$$\boxed{J_{n-1}(x)-J_{n+1}=2J_n'(x)} \tag{6.22}$$

also as a useful special case

$$\boxed{J_0'(x)=J_1(x)} \tag{6.23}$$

Multiplying $x^{\pm n}$ to (6.22)

$$\boxed{\begin{aligned}\frac{d}{dx}\left[x^nJ_n(x)\right]&=x^nJ_{n-1}(x)\\ \frac{d}{dx}\left[x^{-n}J_n(x)\right]&=-x^{-n}J_{n+1}(x)\end{aligned}} \tag{6.24}$$

Adding (6.21) and (6.22), we have

$$\boxed{J_{n-1}(x) = \frac{n}{x} J_{n\cancel{+1}}(x) + J_n'(x)} \qquad J_{n-1}(x) = \frac{n}{x} J_n(x) + J_n'(x) \tag{6.25}$$

6.3.4 Integral Representation

The integral representation of Bessel functions provides powerful tools for many physical problems. Considering a sinusoidal exponential functions for t, we have

$$\begin{aligned}\exp\left(ix\sin\theta\right) &= J_0 + 2\left[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \cdots\right] \\ &\quad + i2\left[J_1(x)\cos\theta + J_3(x)\cos 3\theta + \cdots\right]\end{aligned} \tag{6.26}$$

where we used $i\sin\theta$

$$\begin{aligned}J_1(x)e^{i\theta} + J_{-1}(x)e^{-i\theta} &= i2J_1(x)\sin\theta \\ J_2(x)e^{i2\theta} + J_{-2}(x)e^{-i2\theta} &= \cancel{i}2J_2(x)\cos 2\theta\end{aligned} \tag{6.27}$$

and so on. Therefore, directly from (6.26) we obtain

$$\begin{aligned}\cos(x\sin\theta) &= J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)\cos(2n\theta) \\ \sin(x\sin\theta) &= 2\sum_{n=1}^{\infty} J_{2n-1}(x)\cos\left[(2n-1)\theta\right]\end{aligned} \tag{6.28}$$

By using the orthogonality of sinusoidal functions with $m,n>0$,

$$\int_0^{\pi} d\theta \cos m\theta \cos n\theta = \frac{\pi}{2}\delta_{mn}$$
$$\int_0^{\pi} d\theta \sin m\theta \sin n\theta = \frac{\pi}{2}\delta_{mn} \tag{6.29}$$

we find

$$\int_0^{\pi} d\theta \cos(x\sin\theta)\cos n\theta = \begin{cases} J_n(x), & n:\text{even} \\ 0, & n:\text{odd} \end{cases}$$
$$\int_0^{\pi} d\theta \sin(x\sin\theta)\sin n\theta = \begin{cases} 0, & n:\text{even} \\ J_n(x), & n:\text{odd} \end{cases} \tag{6.30}$$

Adding these two equations, we have

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} d\theta \cos(n\theta - x\sin\theta)\cos n\theta \tag{6.31}$$

with a special case

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} d\theta \cos(x\sin\theta)\cos n\theta = \frac{1}{2\pi}\int_0^{2\pi} d\theta \cos(x\sin\theta)\cos n\theta \tag{6.32}$$

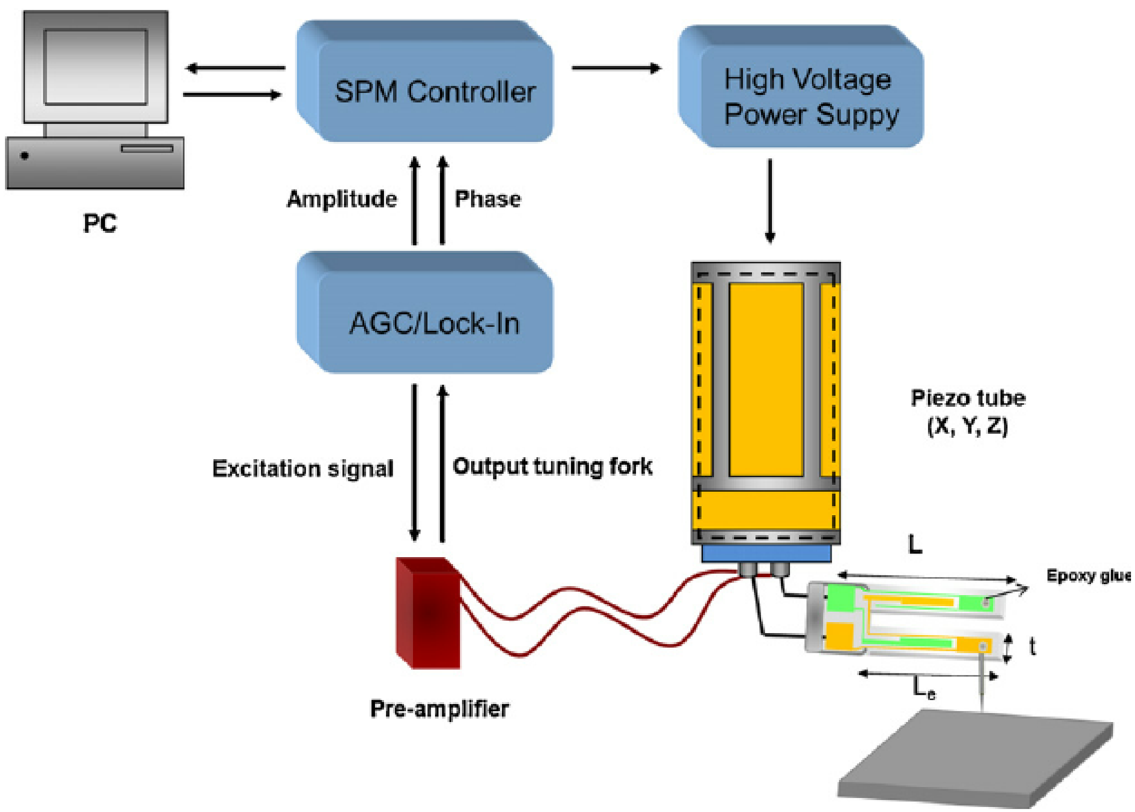
Also using

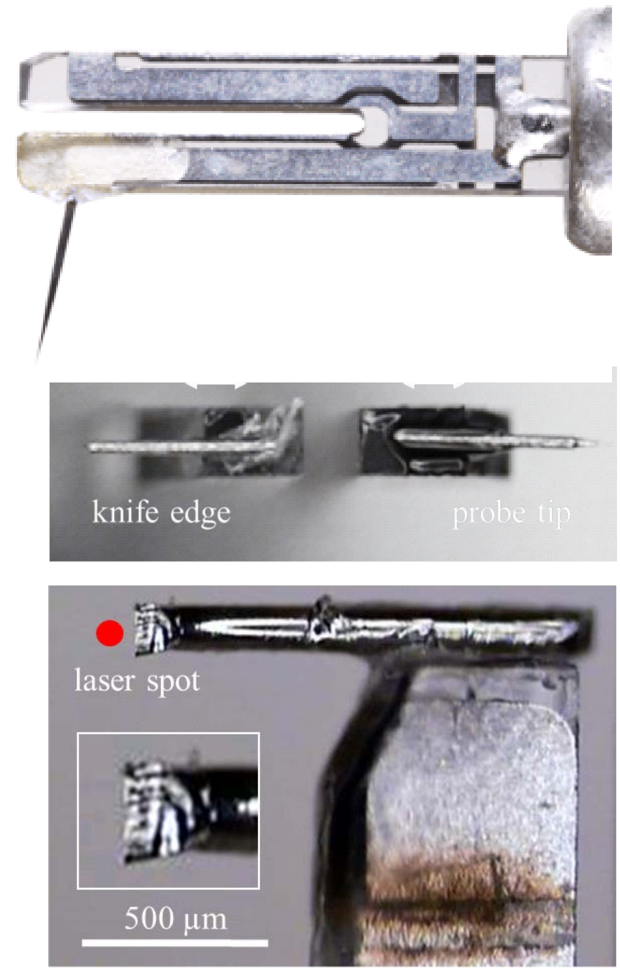
$$\frac{1}{2\pi}\int_0^{2\pi} d\theta \cos(x\sin\theta)\cos n\theta = 0 \tag{6.33}$$

we finally have a useful integral representation

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \exp(x\sin\theta) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \exp(x\cos\theta) \tag{6.34}$$

Application Example: Direct Measurement of the Oscillation Amplitude of AFM Tip





[Quiz-3] Fourier Series of Generating Function

$$g(x,t) = \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right]$$

Find the Fourier series expansion of the generating function for $t = e^{\pm i\theta}$.