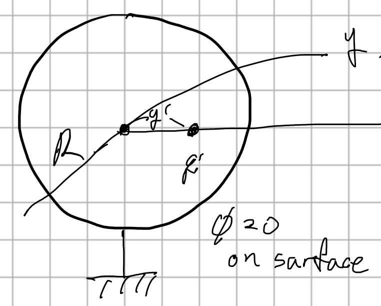
b) Grounded conducting sphere with a point charge



effective image charge q' & y' ?

(magnitude & position)

cf. Consider the following case.

$\cdot \vec{x}$

O q $\qquad$ $q'(d,0,0)$

$qq' < 0$

$|q| > |q'|$

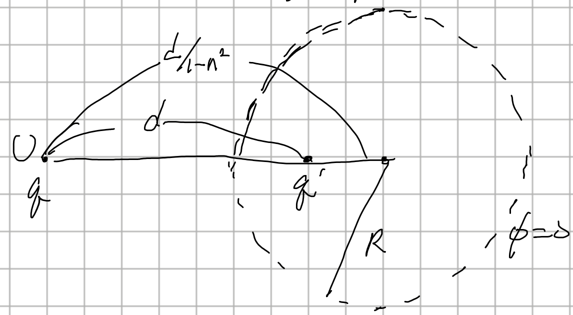
$$\phi(\vec{x}) = \frac{q}{\sqrt{x^2+y^2+z^2}} + \frac{q'}{\sqrt{(x-d)^2+y^2+z^2}}$$

For $\phi = 0$ surface, $0 = \frac{q}{\sqrt{\quad}} + \frac{q'}{\sqrt{\quad}}$

$$(x-d)^2+y^2+z^2 = \left(\frac{q'}{q}\right)^2 (x^2+y^2+z^2)$$

$q' = -nq, \quad 0 < n < 1$

$$\Rightarrow \left(x - \frac{d}{1-n^2}\right)^2 + y^2 + z^2 = \left(\frac{nd}{1-n^2}\right)^2$$



ϕ is zero on the surface

Centered at $\left(\frac{d}{1-n^2}, 0, 0\right)$

with a Radius of $R = \frac{n}{1-n^2} d$

$d' = \frac{d}{1-n^2} - d = \frac{n^2}{1-n^2} d$

Return to our problem

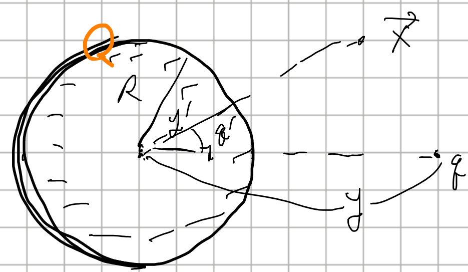
$$y = \frac{d}{1-n^2} = \frac{R}{n} = -\frac{q}{q'}R$$

$$y' = d' = \frac{n^2}{1-n^2}d = nR = -\frac{q'}{q}R = \frac{R^2}{y}$$

$$\Rightarrow q' = -q\frac{R}{y}, \quad y' = R\frac{R}{y}$$

$$\phi = \frac{q}{|\vec{x} - y\hat{y}|} + \frac{q'}{|\vec{x} - y'\hat{y}|}$$

cf. Insulated conducting sphere with a charge Q



$$\phi = \underbrace{\frac{q}{|\vec{x} - y\hat{y}|} + \frac{q'}{|\vec{x} - y'\hat{y}|}}_{\phi_1} + \underbrace{\frac{Q-q'}{|\vec{x}|}}_{\phi_2}$$

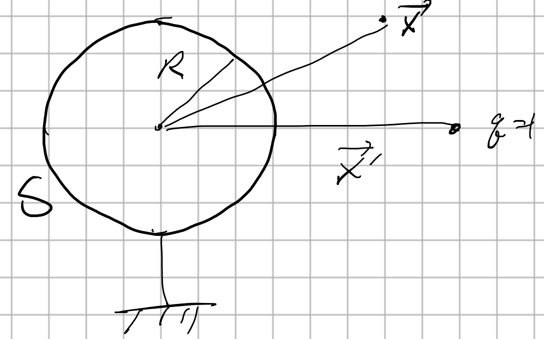
① grounded conducting sphere $\Rightarrow \phi_1$

② Disconnect the wire & charge the sphere to $Q-q'$

$Q-q'$ will be uniformly distributed because the electrostatic force due to q is already balance by q'

$$\phi = \frac{Q-q'}{|\vec{x}|}$$

c) Green's function for Dirichlet BVP

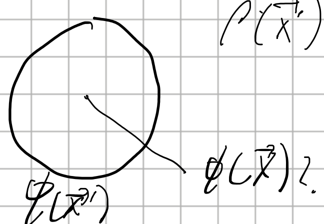


$$\nabla^2 G_D(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}')$$

potential at $\vec{x}$ due to $q=1$ at $\vec{x}'$

$G_D(\vec{x},\vec{x}') = 0$ on S

$$G_D(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + \frac{-\frac{R}{|\vec{x}'|}}{\left|\vec{x} - \frac{R}{|\vec{x}'|^2}\vec{x}'\right|} \quad , \quad |\vec{x}|>R, \ |\vec{x}'|>R$$



$\rho(\vec{x}')$ $$\phi = \int d^3x' \rho(\vec{x}')\left[\frac{1}{|\ \ |} - \frac{R}{|\vec{x}'|}\frac{1}{|\ \ |}\right]$$

$$+\frac{1}{4\pi}\int d \quad \phi(\vec{x}) \frac{x^2-R^2}{R(R^2-2R\hat{x}'\cdot\vec{x}+x^2)^{3/2}}$$

$$\hat{x}' = \vec{x}'/|\vec{x}'|$$

d) Inversion Method

From solved problems, others can be solved by an inversion process.

i) Basic theorem

If $\phi(r, \theta, \phi)$ is a solution of Poisson's equation in all space

$$\nabla^2 \phi(r, \theta, \phi) = -4\pi \rho(r, \theta, \phi)$$

for a charge density ρ, then

$$\phi'(r, \theta, \phi) = \frac{a}{r} \phi\left(\frac{a^2}{r}, \theta, \phi\right), \quad \rho' = \left(\frac{a}{r}\right)^2 \rho\left(\frac{a^2}{r}, \theta, \phi\right)$$ (a is constant (radius of inversion))

also satisfy Poisson's equation.

proof of this theorem is H.W.

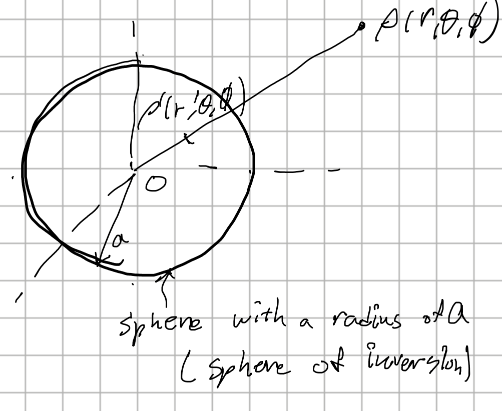
$$q \text{ at } r \longrightarrow q\frac{a}{r} \text{ at } \frac{a^2}{r}$$

$$\sigma \longrightarrow \left(\frac{a}{r}\right)^3 \sigma\left(\frac{a^2}{r}, \theta, \phi\right)$$

$$\rho \longrightarrow \left(\frac{a}{r}\right)^5 \sigma\left(\frac{a^2}{r}, \theta, \phi\right)$$

$$\phi \longrightarrow \phi' = \frac{a}{r} \phi\left(\frac{a^2}{r}, \theta, \phi\right)$$

ii) mapping by inversion.



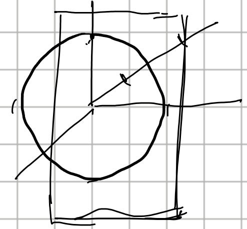
$P(r, \theta, \phi)$ & $P'(r' = \frac{a^2}{r}, \theta, \phi)$

are inverse to each other with respect to the sphere

$$rr' = a^2$$

* the inversion of the inverse point is the original point

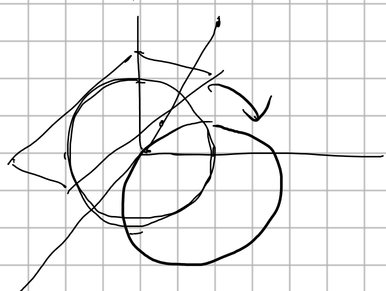
* a plane passing through the point O



a point & its inverse point are on the same line

$\Rightarrow$ a plane passing through O $\rightarrow$ inverse to itself.

* a plane not passing through O



plane equation.

$$lx + my + nz + q = 0 \quad (q \neq 0)$$

$$\frac{x}{r} = \frac{x'}{r'}, \quad \frac{y}{r} = \frac{y'}{r'}, \quad \frac{z}{r} = \frac{z'}{r'}$$

$rr' = a$

$x = \frac{x'}{r'}$, $r = \frac{a^2}{r'^2} x'$, $y = \cdots$, $z = \cdots$

$$x'^2 + y'^2 + z'^2 + \frac{a^2}{q}(lx' + my' + nz') = 0$$

$\vec{x}' = 0 = (0, 0, 0)$. satisfies this equation)

a plane not passing throug the point O

$\rightleftarrows$ Sphere passing througe O

* a sphere does not pass through the point O

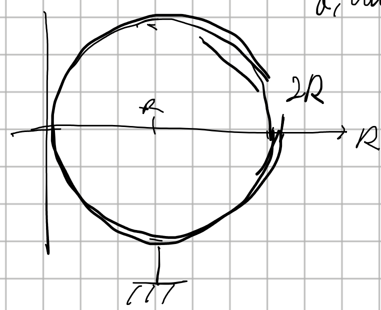
$$x'^2 + y'^2 + z'^2 + \frac{a^2}{q}(lx' + my' + nz') + \frac{a^4}{q} = 0$$

* sphere that does not pass through "O"

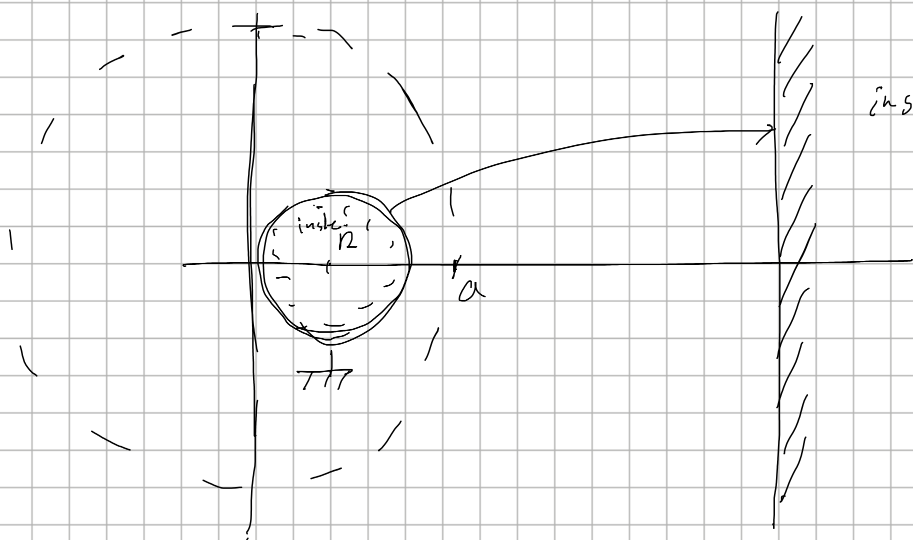
$\leftrightarrows$ another sphere not passing through the point O

Ex

q, uniformly distributed.



$$\phi = \begin{cases} 0 & \text{inside the sphere} \\ \dfrac{q}{|\vec{x}-2R|} - \dfrac{q}{R} & \text{outside sphere} \end{cases}$$



inside.