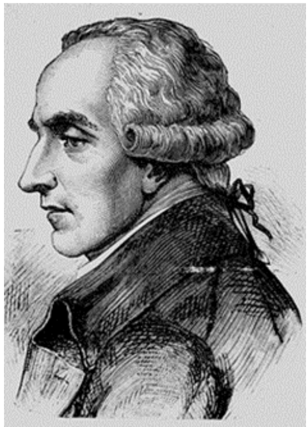
Chapter 4
INTEGRAL TRANSFORMS

Joseph Fourier
(1768-1830)
Math/Physics
Fourier Series/Transform

Lecture 13

4.2 Fourier Transform



Pierre-Simon Laplace
(1749-1827)
Math/Physic
Laplace Transform
Laplace Equation
(Scalar Potential Theory)

Heaviside Unit Step Function

Consider the FT of unit step function, and here we derive it more rigorously using the Dirac identity than in a course about “signal and processing.” The unit step function is usually defined as

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \tag{4.13}$$

and the FT is given by

$$\boxed{U(\omega) = \frac{i}{\omega + i0^+} = \pi\delta(\omega) + iPV\frac{1}{\omega}} \tag{4.14}$$

Proof) The FT is given by an improper integral

$$U(\omega) = \int_{-\infty}^{\infty} d\omega\, u(t) e^{-i\omega t} = \int_0^{\infty} d\omega\, e^{-i\omega t} \tag{4.15}$$

Thus we need a limiting process for regularization:

$$u(t) = \lim_{\varepsilon \to 0^+} u_\varepsilon(t) \tag{4.16}$$

using a sequence function

$$u_\varepsilon(t) = \begin{cases} e^{-\varepsilon t}, & t > 0 \\ 0, & t < 0 \end{cases} \tag{4.17}$$

Substituting (4.17) into (4.15),

$$\begin{aligned} U(\omega) &= \lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty} d\omega\, u_\varepsilon(t) e^{-i\omega t} = \lim_{\varepsilon\to 0^+}\int_{0}^{\infty} d\omega\, e^{-(\varepsilon+i\omega)t} \\ &= \frac{i}{\omega+i0^+} = i\left(PV\frac{1}{\omega} - i\pi\delta(\omega)\right) = \pi\delta(\omega) + iPV\frac{1}{\omega} \end{aligned} \tag{4.18}$$

where we have used the Dirac identity.

For the existence (or convergence) of its FT at the discontinuity at $t=0$, we need to define $u(0)=[u(0^+)+u(0^-)]/2=1/2$ in principle, but this is not necessary for most of physics and engineering problems.

[Reminder] Go back to "Ch3 Complex Variables" and see the Dirac Identity

Differential Operators

In the FT, the differential operators can be replaced simply by

$$\boxed{\begin{aligned} \nabla &\xrightarrow{FT} i\mathbf{k} \\ \partial/\partial t &\xrightarrow{FT} -i\omega \end{aligned}} \tag{4.19}$$

The proof is straightforward. Do it by yourself.

Integral Operator

At first glance, an integral operator $I(t)$, an inversion of a differential operator, might be involved with the inverse of $-i\omega$. However, in a more precise description, we should have more than that:

$$\boxed{I(t) = \int_{-\infty}^{t} dt' \xrightarrow{FT} U(\omega) = \frac{i}{\omega + i0^{+}} = \pi\delta(\omega) + iPV\left(\frac{1}{\omega}\right)} \tag{4.19}$$

or more explicitly

$$\boxed{\begin{aligned} &I(t)\,f(t) = \int_{-\infty}^{t} dt'\, f(t') \\ &\qquad \xrightarrow{FT} U(\omega)F(\omega) = \frac{iF(\omega)}{\omega + i0^{+}} = PV\left[-\frac{F(\omega)}{i\omega}\right] + \pi F(0)\delta(\omega) \end{aligned}} \tag{4.20}$$

Proof) Consider an improper integral

$$\int_{-\infty}^{t} dt' f(t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, F(\omega)\int_{-\infty}^{t} dt'\, e^{-i\omega t'} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, F(\omega)\left.\frac{e^{-i\omega t'}}{-i\omega}\right|_{-\infty}^{t} \tag{4.21}$$

This integral has two singularities ($\omega = 0$ and $t' = -\infty$), and thus we need a regularization process by $\omega \to \omega + i0^{+}$ to remove the singularities and to guarantee the convergence.

Now we have

$$\left.\frac{e^{-i(\omega+i0^+)t'}}{-i(\omega+i0^+)}\right|_{-\infty}^{t} = \frac{ie^{-i\omega t}}{\omega+i0^+} = i\left[PV\left(\frac{1}{\omega}\right) - i\pi\delta(\omega)\right]e^{-i\omega t}$$
$$= PV\left(-\frac{1}{i\omega}\right)e^{-i\omega t} + \pi\delta(\omega) \tag{4.22}$$

Substituting (4.22) into (4.21), we have

$$\int_{-\infty}^{t} dt' f(t') = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega F(\omega)\left[PV\left(-\frac{1}{i\omega}\right)e^{-i\omega t} + \pi\delta(\omega)\right]$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\left[PV\left(-\frac{F(\omega)}{i\omega}\right) + \pi F(0)\delta(\omega)\right]e^{-i\omega t} \tag{4.23}$$

Fourier Series: Fourier Transform of Periodic Functions

If $f(x)$ is periodic, *i.e.*, subject to a **periodic boundary condition**, $f(x) = f(x+\Lambda)$ then it can be represented by a Fourier series with $K = 2\pi/\Lambda$:

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} F_n e^{inKx}} \tag{4.24}$$

This is just the Fourier transform of a Dirac comb

$$\boxed{F(k) = \sum_{n=-\infty}^{\infty} F_n \delta(k - nK)} \tag{4.25}$$

where the Fourier coefficient F_n and wave number (spatial frequency) K are given by

$$\boxed{F_n = F(nK) = \frac{2\pi}{\Lambda}\int_0^{\Lambda} dx\, f(x) e^{-inKx}} \tag{4.26}$$

Proof) With the periodic boundary condition,

$$F(k) = \int_{-\infty}^{\infty} dx\, f(x) e^{-ikx} = \int_{-\infty}^{\infty} dx\, f(x+\Lambda) e^{-ikx} = \int_{-\infty}^{\infty} dx'\, f(x') e^{-ikx'} e^{ik\Lambda} \tag{4.27}$$

So we obtain $e^{ik\Lambda} = 1$ which is the **quantization condition of momentum** of k:

$p = \hbar k$

$E = \hbar \omega$

$$k = \frac{2\pi n}{\Lambda} = nK, \quad n = 0, \pm 1, \pm 2, \cdots \tag{4.28}$$

This means that the FT is given by

$$F(k) = \sum_{n=-\infty}^{\infty} F(nK)\delta(k - nK) = \sum_{n=-\infty}^{\infty} F_n \delta(k - nK) \tag{4.29}$$

Now we consider the inverse FT and an overlap integral over $0 \le x \le \Lambda$,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, F(k) e^{ikx} = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} F_n e^{imKx} \tag{4.30}$$

$$\int_0^{\Lambda} dx\, f(x) e^{-inKx} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} F_n \int_0^{\Lambda} dx\, e^{i(m-n)Kx} = \frac{\Lambda}{2\pi} F_n \tag{4.31}$$

[ToDo] Poisson's Summation
Computing long-range interactions
such as Coulomb interactions in periodic systems (crystals)

excitation → casuality

Convolution and Correlation

response

With two functions $f(t)$ and $g(t)$ we define the convolution and correlation functions as

$$\boxed{\begin{aligned} cv_{fg}(t) &= f(t)*g(t)=\int_{-\infty}^{\infty}dt'\, f(t-t')g(t') \\ cr_{fg}(t) &= f(t)\circ g(t)=\int_{-\infty}^{\infty}dt'\, f(t'-t)g(t') \end{aligned}}$$

Convolution
Correlation (4.32)

Then we have the FTs:

$$\boxed{\begin{aligned} CV_{fg}(\omega) &= F(\omega)G(\omega) \\ CR_{fg}(\omega) &= F(\omega)\,G(-\omega)=F(-\omega)\,G(\omega) \end{aligned}}$$

Convolution
Correlation (4.33)

special case

For $f(t)=g(t)$, we define the autocorrelation function as

$$\boxed{\begin{aligned} cv_{ff}(t) &= f(t)*f(t)=\int_{-\infty}^{\infty}dt'\, f(t-t')g(t') \\ cr_{fg}(t) &= f(t)\circ g(t)=\int_{-\infty}^{\infty}dt'\, f(t'-t)g(t') \end{aligned}}$$

Autocorrelation (4.32)

and otherwise it is called cross-correlation function.

Proof)

$$CV_{fg}(\omega)=\int_{-\infty}^{\infty}d\omega\left[\int_{-\infty}^{\infty}dt'\, f(t-t')g(t')\right]e^{i\omega t}=\int_{-\infty}^{\infty}dt'\, f(t-t')e^{i\omega(t-t')}\int_{-\infty}^{\infty}d\omega\, g(t')e^{i\omega t'}=F(\omega)G(\omega)$$

$$CR_{fg}(\omega)=\int_{-\infty}^{\infty}d\omega\left[\int_{-\infty}^{\infty}dt'\, f(t'+t)g(t')\right]e^{i\omega t}=\int_{-\infty}^{\infty}dt'\, f(t'+t)e^{i\omega(t'+t)}\int_{-\infty}^{\infty}d\omega\, g(t')e^{-i\omega t'}=F(\omega)G(-\omega)$$

Convolution versus Correlation in Physics

- Convolution is related with the causality condition between excitation and response.
  → **Electrodynamics and Linear Response Theory** in Solid State Physics
- Correlation is a kind of measure of the **order and noise** of a physical system

[ToDo] Weiner-Khintchine Theorem

Correlation Theorem for Real-Valued Function

$f(\mathbb{r}, t)$: from real physics. $\Rightarrow$ $F(\mathbb{k}, \omega)$

$\Rightarrow$ real $\Rightarrow$ complex.