Chapter 6
SPECIAL FUNCTIONS

Friedrich Wilhelm Bessel
(1784-1846)
Math/Astronomy
Discovery of Neptune
Bessel Functions

Lecture 20

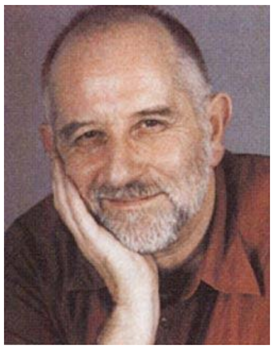
6.1 Introduction to Special Functions
6.2 Gamma Function
6.3 Bessel Functions

Hermann Hankel
(1839-1873)
Math
Hankel Functions
Hankel Transform

6.1 Introduction to Special Functions

"Special function, any of a class of mathematical functions that arise in the solution of various classical (and quantum) problems of physics." **- Encyclopedia Britannica -**

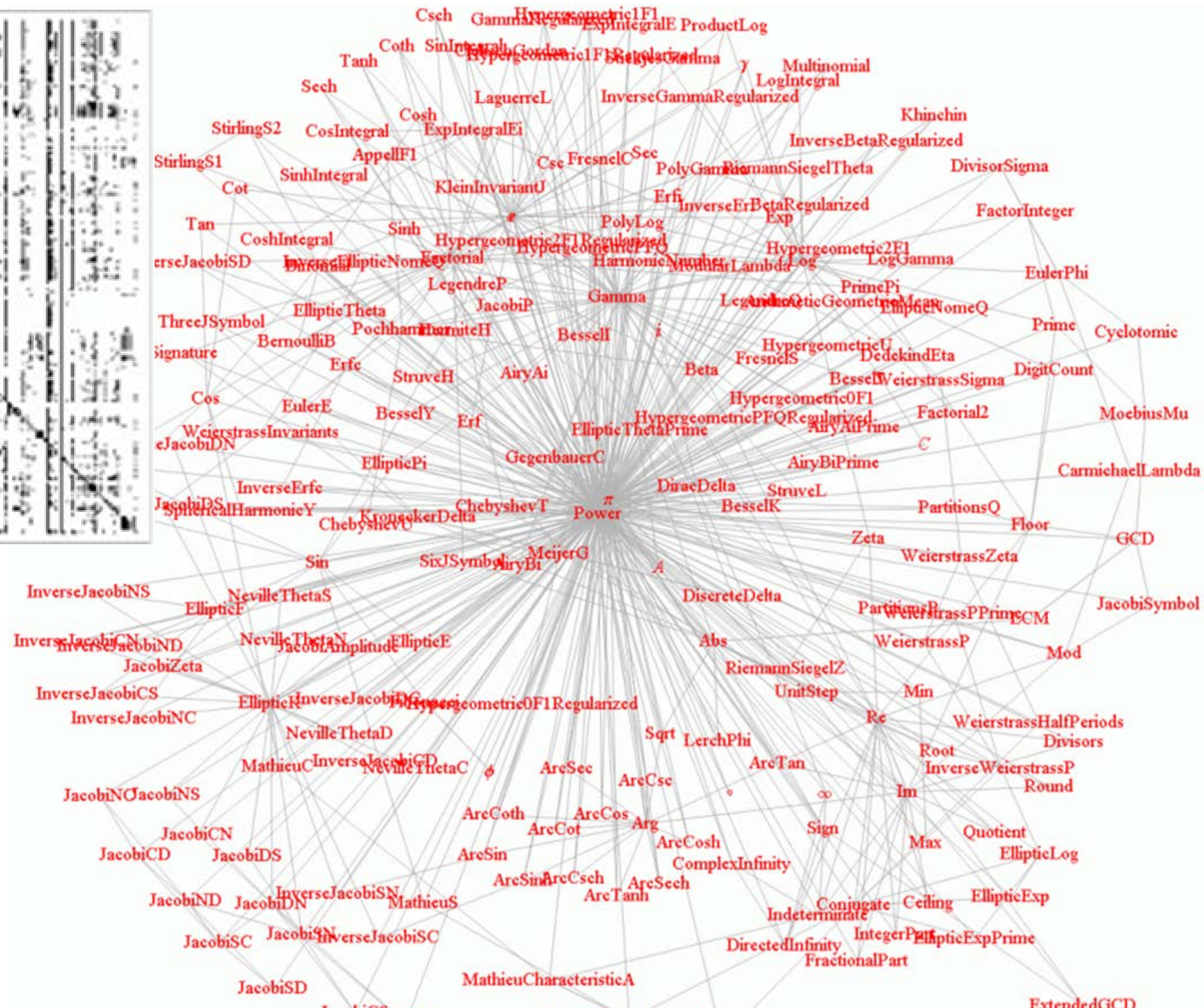
Why are Special Functions Special?



Michael Berry
(1941-)
Physics
Berry Phase

One of the principal applications of these (special) functions was in the **compact expression of approximations to physical problems** for which explicit analytical solutions could not be found. But since the 1960s, when **scientific computing** became widespread, ….. this would **make the special functions redundant**. Similar skepticism came from some pure mathematicians, whose ignorance about special functions, and lack of interest in them, …… in the 1970s, and I was seeking a graduate student to pursue these investigations, a mathematician recommended somebody as being very bright, very knowledgeable, and interested in applications. But **this student had never heard of Bessel functions** (nor could he carry out the simplest integrations). ….. There are mathematical theories in which **some classes of special functions appear naturally**. ….. by increasing complexity, starting with polynomials and algebraic functions and progressing through **the "elementary" or "lower" transcendental functions** (logarithms, exponentials, sines and cosines, and so on) to the **"higher" transcendental functions (Bessel, parabolic cylinder, and so on)**.

Relations between Functions (Data from Wolfram Function Site)



6.2 Gamma Functions

The gamma function is widely used in physics and engineering, and was first introduced by Euler to generalize the factorial function. There are several definitions of the gamma function. Note that the gamma function **cannot be defined by differential equations**.

Infinite Limit (Euler)

$\Gamma(z)=\lim_{n\to\infty}\frac{1\cdot2\cdot3\cdots n}{z(z+1)(z+2)\cdots(z+n)}n^{z}$

$$\Gamma(z)=\lim_{n\to\infty}\frac{1\cdot 2\cdot 3\cdots n}{z(z+1)(z+2)\cdots(z+n)}n^{z},\quad z\neq 0,-1,-2,-3,\cdots \tag{6.1}$$

$z\neq 0,\cdots$

$$\Gamma(z+1)=z\Gamma(z)$$

$\Gamma(z+1)=z\Gamma(z)$

$$\Gamma(n)=(n-1)!,\quad n:\text{integer}$$

$\Gamma(n)=(n-1)!$

Definite Integral (Euler)

$\Gamma(z)=\int_0^\infty dt\,e^{-t}t^{z-1}$

$$\Gamma(z)=\int_0^\infty dt\,e^{-t}t^{z-1}=\int_0^1 dt\left[\ln\left(\frac{1}{t}\right)\right]^{z-1},\quad \mathrm{Re}(z)>0 \tag{6.2}$$

$=\int_0^1 dt\left[\ln\left(\frac{1}{t}\right)\right]^{z-1}$

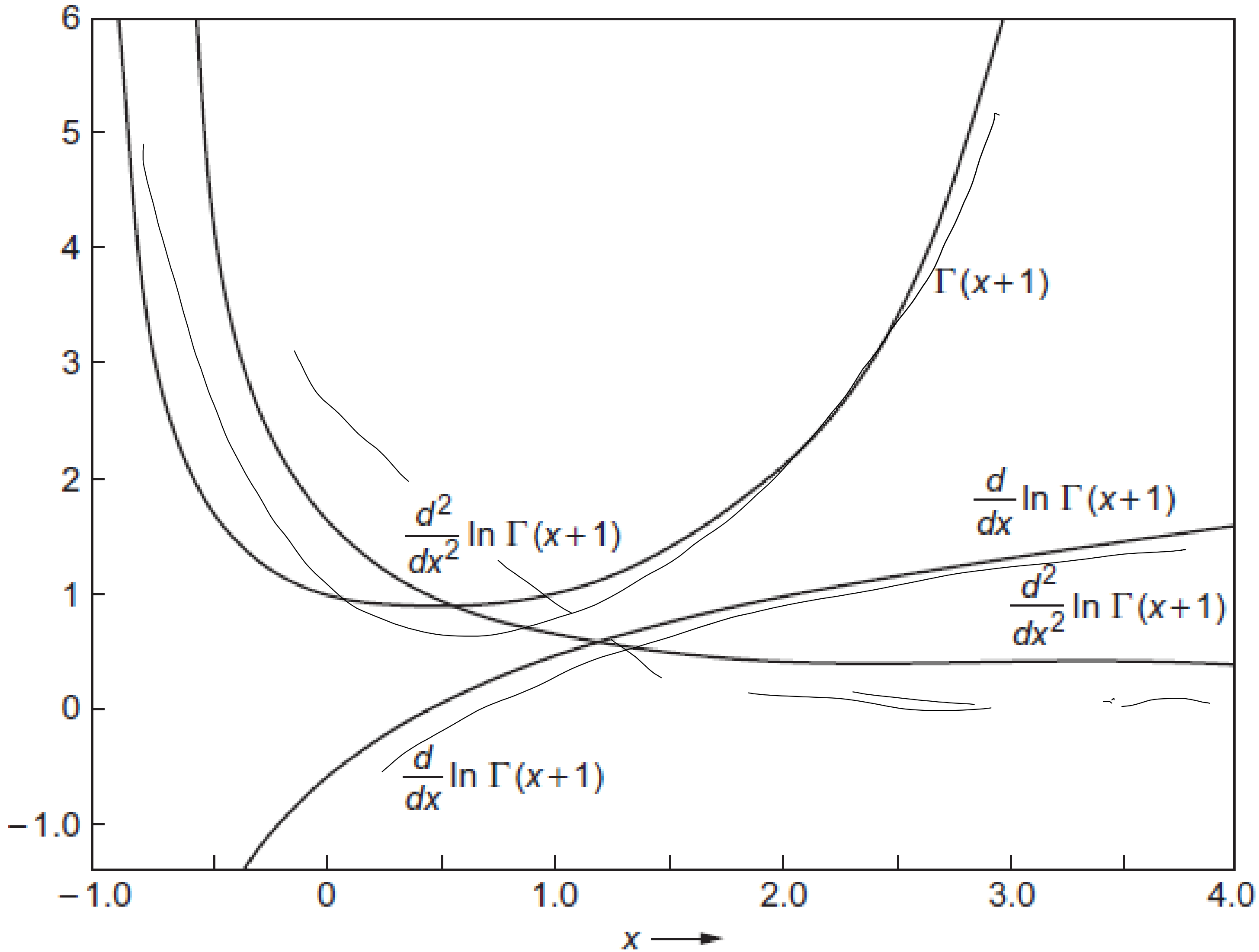
Definite Integral (Weierstrass)

$$\frac{1}{\Gamma(z)}=ze^{\gamma z}\prod_{n=1}^{\infty}\frac{1}{z}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}} \tag{6.3}$$

$\frac{1}{\Gamma(z)}=ze^{\gamma z}\prod_{n=1}^{\infty}\frac{1}{z}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$

$\gamma=0.5772156619\cdots$: Euler-Mascheroni Constant

Gamma Function and its Derivatives



6.3 Bessel Functions

The Bessel functions are essential for many physical equations in circular cylindrical coordinates such as Helmholtz or wave equations in classical and quantum physics. Some problems in physics involve integrals related with **Bessel functions, even when the original problems do not explicitly involve cylindrical or spherical geometry**.

6.3.1 Bessel's Differential Equation: Series Method

The Bessel functions can be defined through an ordinary differential equation,

$$x^2 y'' + xy' + (x^2 - \nu^2) y = x(xy')' + (x^2 - \nu^2) y = 0 \tag{6.4}$$

By using a series method, we have

$$\boxed{y = \sum_{n=0}^{\infty} a_n x^{n+s}}$$

$$y' = \sum_{n=0}^{\infty} a_n (n+s) x^{n+s-1}$$

$$xy' = \sum_{n=0}^{\infty} a_n (n+s) x^{n+s} \tag{6.5}$$

$$(xy')' = \sum_{n=0}^{\infty} a_n (n+s)^2 x^{n+s-1}$$

$$x(xy')' = \sum_{n=0}^{\infty} a_n (n+s)^2 x^{n+s}$$

$y = \sum_{n=0}^{\infty} a_n x^{n+s}$

$y' = \sum_{n=0}^{\infty} a_n (n+s) x^{n+s-1}$

$xy' = \sum_{n=0}^{\infty} a_n (n+s) x^{n+s}$

$x(xy')' = \sum_{n=0}^{\infty} a_n (n+s)^2 x^{n+s}$ $((n+s)^2 - \nu^2)$

$\Rightarrow \sum_{n=0}^{\infty} a_n (n+s)^2 x^{n+s} + (x^2 - \nu^2) \sum_{n=0}^{\infty} a_n x^{n+s} = 0$

$\sum_{n=0}^{\infty} a_n x^{n+s+2} - \nu^2 \sum_{n=0}^{\infty} a_n x^{n+s}$

Substituting (6.5) into (6.4) and organizing the coefficients of the powers of x, we find the coefficient of x^s such that

$$s^2 - v^2 = 0 \quad \rightarrow \quad s = \pm v \qquad \text{Indicial Equation} \tag{6.6}$$

and for the coefficient of x^{s+n} we have

$$a_n = -\frac{a_{n-2}}{(n+s)^2 - v^2} \tag{6.7}$$

from which we find $a_1 = 0$, and consequently $a_n = 0$ for all odd-power terms, *i.e.*, leaving only even-power terms.

In case of $s = v$, we find

$$a_n = -\frac{a_{n-2}}{2^2 n(n+v)} \tag{6.8}$$

and using the definition of the gamma function,

$$a_{2n} = -a_0 \frac{\Gamma(1+v)}{n!2^{2n}\Gamma(n+1+v)} \tag{6.9}$$

we find a solution for $s = v$

$$\boxed{y = J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+v+1)} \left(\frac{x}{2}\right)^{2n+v}} \qquad \text{Bessel Function of the First Kind, Order } \mathbf{v} \tag{6.10}$$

A **second-order ODE** should have **two independent solutions**, and therefore we need to find the second solution of the Bessel equation.

Recalling that we did not used the second solution, $s=-v$ of the indicial equation in (6.6), we can try $J_{-v}(x)$ as a second solution,

$$\boxed{J_{-v}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+1)\Gamma(n-v+1)}\left(\frac{x}{2}\right)^{2n-v}} \tag{6.11}$$

However, this second solution has a problems with an integral n:

$$J_{-v}(x)=(-1)^n J_v(x) \qquad J_{-\nu}(x)=(-)^n J_\nu(x) \tag{6.12}$$

where the two solutions are not linearly independent.

This problem can be removed by using another combination:

$$\boxed{Y_v(x)=\frac{\cos v\pi J_v(x)-J_{-v}(x)}{\sin n\pi}} \qquad Y_\nu(x)=\frac{\cos\nu\pi J_\nu(x)-J_{-\nu}(x)}{\sin n\pi} \tag{6.13}$$

$$\Rightarrow \quad \boxed{y=Y_v(x)=\frac{\cos v\pi J_v(x)-J_{-v}(x)}{\sin n\pi}} \tag{6.14}$$

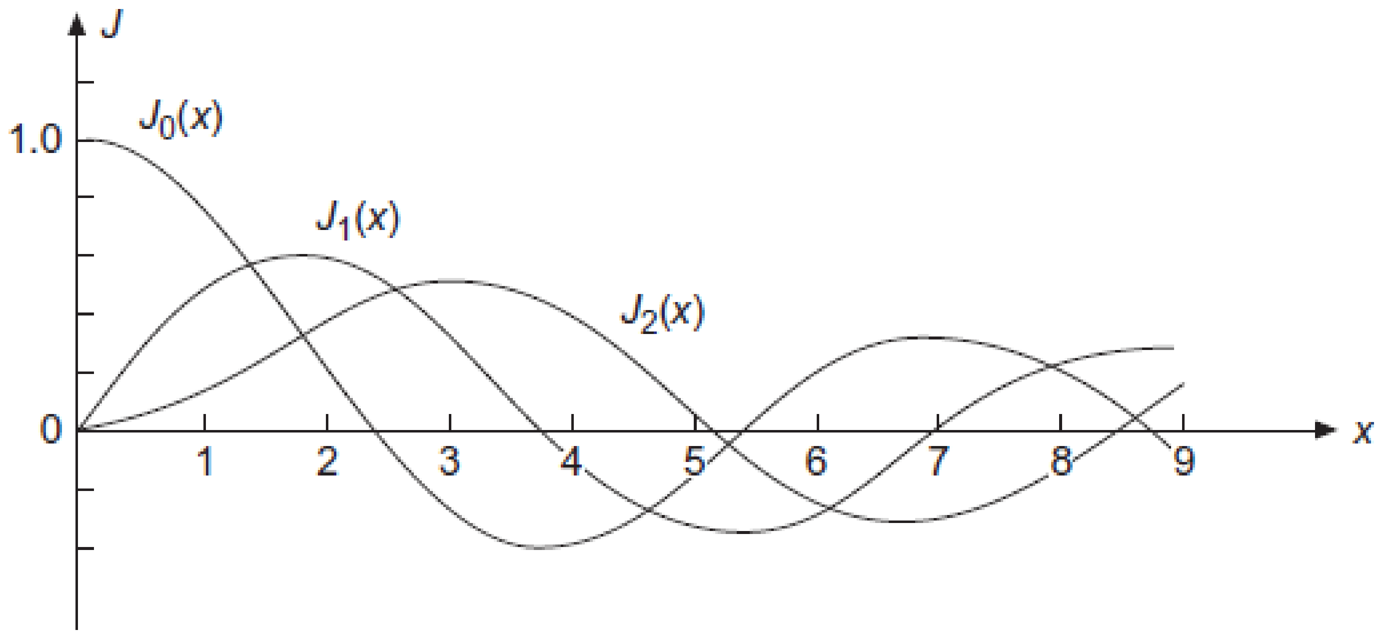
Bessel Function of the Second Kind, Order v

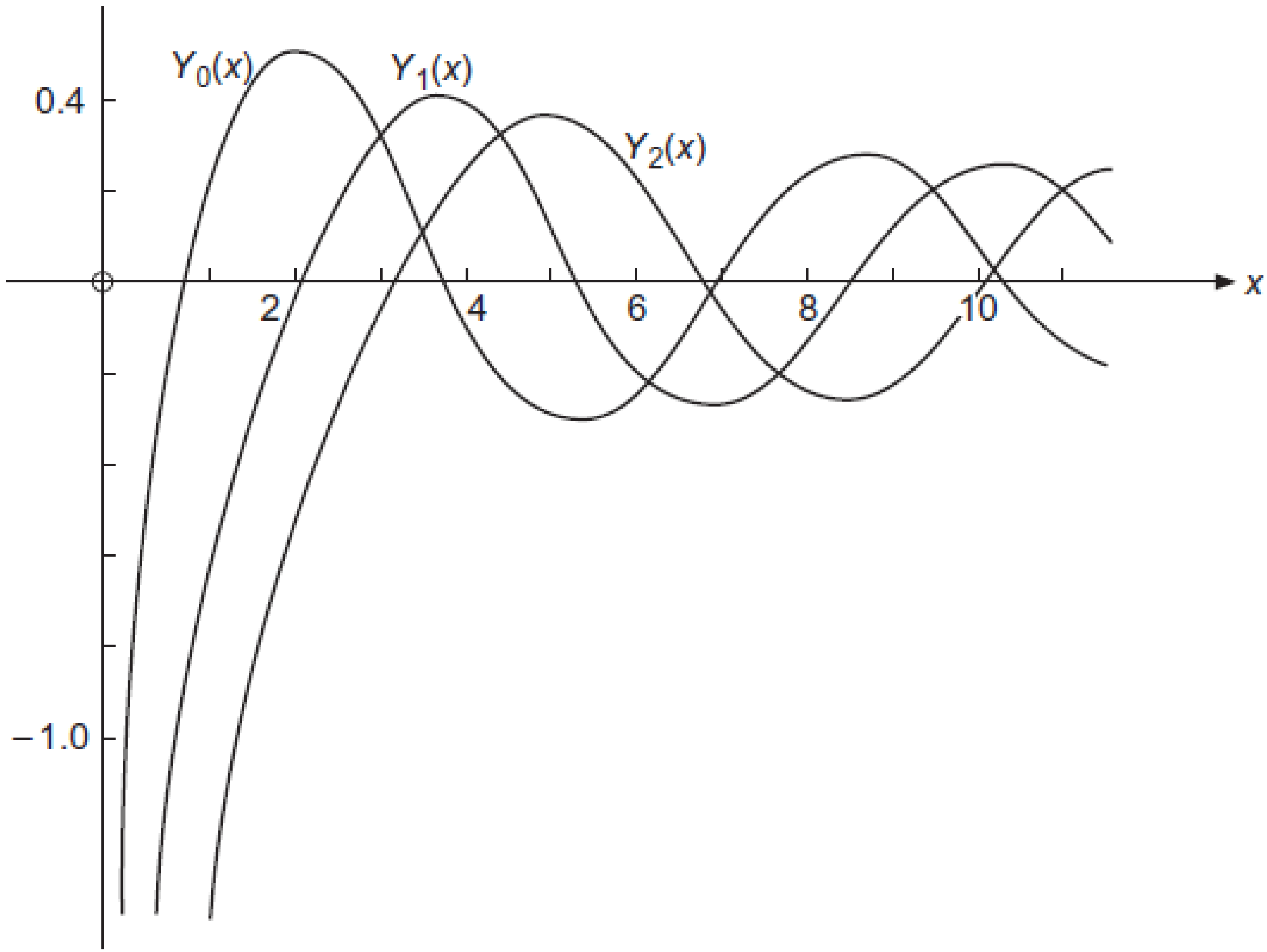
Now we have the Bessel function of second kind:

$$Y_{\nu}(x) = \frac{\cos\nu\pi J_{\nu}(x) - J_{-\nu}(x)}{\sin n\pi} \tag{6.15}$$

and finally the general solution is given by

$$y(x) = A_{\nu} J_{\nu}(x) + B_{\nu} Y_{\nu}(x) \tag{6.16}$$


J
1.0
0
$J_0(x)$
$J_1(x)$
$J_2(x)$
1
2
3
4
5
6
7
8
9
x



$Y_0(x)$
$Y_1(x)$
$Y_2(x)$
0.4
−1.0
2
4
6
8
10
x