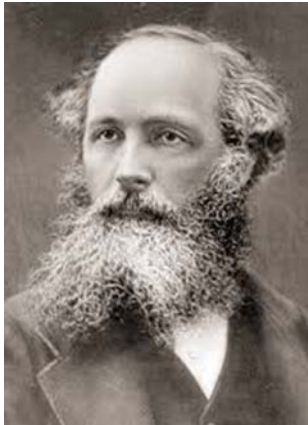


Chapter 2

VECTOR AND TENSOR ANALYSES

Lecture 6

E2.2 Microscopic Maxwell's Equations
E2.3 Macroscopic Maxwell's Equations



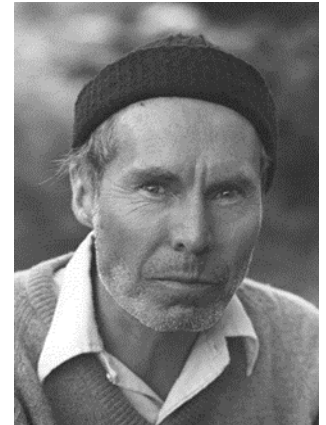
James Clerk Maxwell

(1831-1879)

Physics/Math

Maxwell's Equations

"EM Wave = Light"



John David Jackson

(1925-2016)

Physics (UIUC, UC Berkeley)

EM Text

"Classical Electrodynamics"

E2-4 Macroscopic Maxwell's Equation: Multipole-Expanded Bound Sources

In E2-2, we just took simple average of the microscopic sources. Here, we further take **multipole expansion of the microscopic bound sources**.

Macroscopic Charge Density and Displacement Field

Expanding the Dirac delta function in a molecular scale ($|\mathbf{r}_{\alpha(n)}| \ll |\mathbf{r} - \mathbf{r}_n|$),

$$\begin{aligned}\zeta_n(\mathbf{r}, t) &= \sum_{\alpha(n)} q_{\alpha(n)} \delta(\mathbf{r} - \mathbf{r}_n - \mathbf{r}_{\alpha(n)}) \\ &= \sum_{\alpha(n)} q_{\alpha(n)} \left[1 - \mathbf{r}_{\alpha(n)} \cdot \nabla + \frac{1}{2} (\mathbf{r}_{\alpha(n)} \cdot \nabla)^2 + \dots \right] \delta(\mathbf{r} - \mathbf{r}_n)\end{aligned}\quad \text{(E2.16)}$$

Taylor expansion

we define the molecular multipole moments:

$$q_n = \sum_{\alpha(n)} q_{\alpha(n)} : \text{Molecular Charge} \quad \text{(E2.17)}$$

$$\mathbf{p}_n(t) = \sum_{\alpha(n)} q_{\alpha(n)} \mathbf{r}_{\alpha(n)}(t) : \text{Molecular Dipole} \quad \text{(E2.18)}$$

$$\mathbf{Q}_n(t) = \frac{1}{2} \sum_{\alpha(n)} q_{\alpha(n)} \mathbf{r}_{\alpha(n)}(t) \mathbf{r}_{\alpha(n)}(t) : \text{Molecular Quadrupole} \quad \text{(E2.19)}$$

Thus the microscopic charge density of the n -th molecule is given by

$$\zeta_n(\mathbf{r}, t) = [q_n - \mathbf{p}_n(t) \cdot \nabla + \nabla \cdot \mathbf{Q}_n(t) \cdot \nabla + \dots] \delta[\mathbf{r} - \mathbf{r}_n(t)] \quad (\text{E2.20})$$

Now we decompose the total macroscopic charge density into free and bound ones:

$$\rho(\mathbf{r}, t) = \rho_f(\mathbf{r}, t) + \rho_b(\mathbf{r}, t) \quad (\text{E2.21})$$

Here the free and bound charge densities are given by

$$\rho_f(\mathbf{r}, t) = \left\langle \sum_{i(\text{free})} q_i \delta[\mathbf{r} - \mathbf{r}_i(t)] \right\rangle + \rho_{mol}(\mathbf{r}, t) \quad (\text{E2.22})$$

$$\rho_b(\mathbf{r}, t) = -\nabla \cdot \mathbf{P}(\mathbf{r}, t) + \nabla \cdot [\nabla \cdot \mathbf{Q}(\mathbf{r}, t)] + \dots \quad (\text{E2.23})$$

with the macroscopic multipole densities defined as

$$\rho_{mol}(\mathbf{r}, t) = \left\langle \sum_n \rho_n \delta[\mathbf{r} - \mathbf{r}_n(t)] \right\rangle: \text{Macroscopic Molecular Charge Density} \quad (\text{E2.24})$$

$$\mathbf{P}(\mathbf{r}, t) = \left\langle \sum_n \mathbf{p}_n(t) \delta[\mathbf{r} - \mathbf{r}_n(t)] \right\rangle: \text{Macroscopic Molecular Polarization} \quad (\text{E2.25})$$

$$\mathbf{Q}(\mathbf{r}, t) = \left\langle \sum_n \mathbf{Q}_n(t) \delta[\mathbf{r} - \mathbf{r}_n(t)] \right\rangle: \text{Macroscopic Quadrupole Density} \quad (\text{E2.25})$$

Substituting (E2.22) and (E2.23) into (E2.24), the macroscopic charge density is given by

$$\rho(\mathbf{r}, t) = \rho_f(\mathbf{r}, t) - \nabla \cdot \mathbf{P}(\mathbf{r}, t) + \nabla \cdot [\nabla \cdot \mathbf{Q}(\mathbf{r}, t)] + \dots \quad (\text{E2.26})$$

Next, substituting (E2.26) into the Poisson's equation in (E2.4), we find

$$\nabla \cdot [\varepsilon_0 \mathbf{E}(\mathbf{r}, t) + \mathbf{P}(\mathbf{r}, t) - \nabla \cdot \mathbf{Q}(\mathbf{r}, t) + \dots] = \rho_f(\mathbf{r}, t) \quad (\text{E2.27})$$

Introducing the general displacement field:

$$\mathbf{D}(\mathbf{r}, t) \equiv \varepsilon_0 \mathbf{E}(\mathbf{r}, t) + \mathbf{P}(\mathbf{r}, t) - \nabla \cdot \mathbf{Q}(\mathbf{r}, t) + \dots \quad \text{Displacement Field} \quad (\text{E2.28})$$

we finally obtain the usual form of the Poisson's equation:

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = \rho_f(\mathbf{r}, t) \quad (\text{E2.29})$$

Macroscopic Current Density

More involved, but in a quite similar manner the microscopic current density are expanded:

$$\begin{aligned}
 \mathbf{j}_n(\mathbf{r}, t) &= \sum_{\alpha(n)} q_{\alpha(n)} (\mathbf{v}_n + \mathbf{v}_{\alpha(n)}) \left[1 - \underbrace{\mathbf{r}_{\alpha(n)} \cdot \nabla}_{\text{}} + \frac{1}{2} \underbrace{(\mathbf{r}_{\alpha(n)} \cdot \nabla)^2}_{\text{}} + \dots \right] \delta(\mathbf{r} - \mathbf{r}_n) \\
 &= \mathbf{v}_n (\rho_n - \mathbf{p}_n \cdot \nabla + \nabla \cdot \mathbf{Q}_n \cdot \nabla + \dots) \delta(\mathbf{r} - \mathbf{r}_n) \\
 &\quad + \sum_{\alpha(n)} q_{\alpha(n)} \underbrace{\mathbf{v}_{\alpha(n)} \delta(\mathbf{r} - \mathbf{r}_n)}_{\text{}} - \sum_{\alpha(n)} q_{\alpha(n)} \underbrace{\mathbf{v}_{\alpha(n)} \mathbf{r}_{\alpha(n)} \cdot \nabla \delta(\mathbf{r} - \mathbf{r}_n)}_{\text{}} + \dots
 \end{aligned} \tag{E2.30}$$

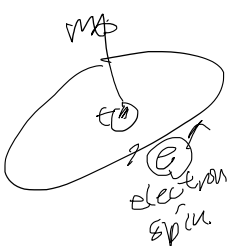
Here the second term can be written as

$$\begin{aligned}
 &\frac{\partial}{\partial t} \left[\sum_{\alpha(n)} q_{\alpha(n)} \mathbf{r}_{\alpha(n)} \delta(\mathbf{r} - \mathbf{r}_n) \right] + \sum_{\alpha(n)} q_{\alpha(n)} \mathbf{r}_{\alpha(n)} \mathbf{v}_n \cdot \nabla \delta(\mathbf{r} - \mathbf{r}_n) \\
 &= \frac{\partial}{\partial t} [\mathbf{p}_n \delta(\mathbf{r} - \mathbf{r}_n)] + \mathbf{p}_n \mathbf{v}_n \cdot \nabla \delta(\mathbf{r} - \mathbf{r}_n)
 \end{aligned} \tag{E2.31}$$

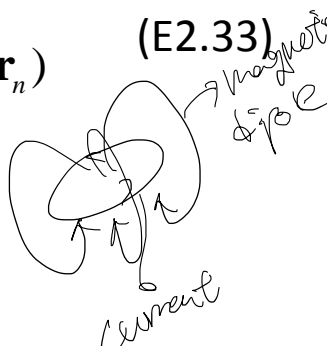
Here the third term can also be written as

$$\begin{aligned}
 & - \sum_{\alpha(n)} q_{\alpha(n)} \mathbf{v}_{\alpha(n)} \mathbf{r}_{\alpha(n)} \cdot \nabla \delta(\mathbf{r} - \mathbf{r}_n) \quad \text{--- } \sum_{\alpha(n)} q_{\alpha(n)} \mathbf{v}_{\alpha(n)} \mathbf{r}_{\alpha(n)} \cdot \nabla \delta(\mathbf{r} - \mathbf{r}_n) \\
 &= - \frac{\partial}{\partial t} [\mathbf{Q}_n \cdot \nabla \delta(\mathbf{r} - \mathbf{r}_n)] - \frac{1}{2} \sum_{\alpha(n)} q_{\alpha(n)} \underbrace{[\mathbf{v}_{\alpha(n)} \mathbf{r}_{\alpha(n)} - \mathbf{r}_{\alpha(n)} \mathbf{v}_{\alpha(n)}]}_{\text{}} \cdot \nabla \delta(\mathbf{r} - \mathbf{r}_n) \\
 &\quad - \mathbf{Q}_n \cdot \nabla [\mathbf{v}_n \cdot \nabla \delta(\mathbf{r} - \mathbf{r}_n)] \\
 &\quad \text{--- } \frac{\partial}{\partial t} [\mathbf{Q}_n \cdot \nabla \delta(\mathbf{r} - \mathbf{r}_n)] - \frac{1}{2} \sum_{\alpha(n)} q_{\alpha(n)} [\mathbf{v}_{\alpha(n)} \mathbf{r}_{\alpha(n)} - \mathbf{r}_{\alpha(n)} \mathbf{v}_{\alpha(n)}] \cdot \nabla \delta(\mathbf{r} - \mathbf{r}_n) \\
 &\quad \text{--- } \mathbf{Q}_n \cdot \nabla [\mathbf{v}_n \cdot \nabla \delta(\mathbf{r} - \mathbf{r}_n)]
 \end{aligned} \tag{E2.32}$$

By substituting (E2.31) and (E2.32) into (E2.30),



Handwritten notes: "Paramagnetic Spin Resonance" and "electron spin".

$$\mathbf{j}_n(\mathbf{r}, t) = \rho_n \mathbf{v}_n + \frac{\partial}{\partial t} [(\mathbf{p}_n - \mathbf{Q}_n \cdot \nabla) \delta(\mathbf{r} - \mathbf{r}_n) + \dots] + \left\{ \frac{1}{2} \sum_{\alpha(n)} q_{\alpha(n)} [\mathbf{r}_{\alpha(n)} \mathbf{v}_{\alpha(n)} - \mathbf{v}_{\alpha(n)} \mathbf{r}_{\alpha(n)}] + [\mathbf{p}_n \mathbf{v}_n - \mathbf{v}_n \mathbf{p}_n] \right\} \cdot \nabla \delta(\mathbf{r} - \mathbf{r}_n) + [\mathbf{Q}_n \cdot \nabla \mathbf{v}_n - \mathbf{v}_n \nabla \cdot \mathbf{Q}_n] \cdot \nabla \delta(\mathbf{r} - \mathbf{r}_n) + \dots \quad (\text{E2.33})$$


Handwritten notes: "magnetic dipole" and "current".

Using a vector identity:

$$\begin{aligned} \nabla \times [\mathbf{r}_{\alpha(n)} \times \mathbf{v}_{\alpha(n)} \delta(\mathbf{r} - \mathbf{r}_n)] &= \nabla \cdot [(\mathbf{v}_{\alpha(n)} \mathbf{r}_{\alpha(n)} - \mathbf{r}_{\alpha(n)} \mathbf{v}_{\alpha(n)}) \delta(\mathbf{r} - \mathbf{r}_n)] \\ &= [\nabla \delta(\mathbf{r} - \mathbf{r}_n)] \cdot (\mathbf{v}_{\alpha(n)} \mathbf{r}_{\alpha(n)} - \mathbf{r}_{\alpha(n)} \mathbf{v}_{\alpha(n)}) \\ &= (\mathbf{r}_{\alpha(n)} \mathbf{v}_{\alpha(n)} - \mathbf{v}_{\alpha(n)} \mathbf{r}_{\alpha(n)}) \cdot [\nabla \delta(\mathbf{r} - \mathbf{r}_n)] \end{aligned} \quad (\text{E2.34})$$

the microscopic current density of the nth molecule becomes

$$\begin{aligned} \mathbf{j}_n(\mathbf{r}, t) &= \rho_n \mathbf{v}_n + \frac{\partial}{\partial t} [(\mathbf{p}_n - \mathbf{Q}_n \cdot \nabla + \dots) \delta(\mathbf{r} - \mathbf{r}_n)] + \nabla \times [\mathbf{m}_n \delta(\mathbf{r} - \mathbf{r}_n)] \\ &\quad + \nabla \times \langle \mathbf{p}_n \times \mathbf{v}_n \delta(\mathbf{r} - \mathbf{r}_n) \rangle + [\mathbf{Q}_n \cdot \nabla \mathbf{v}_n - \mathbf{v}_n \nabla \cdot \mathbf{Q}_n] \cdot \nabla \delta(\mathbf{r}, t) + \dots \end{aligned} \quad (\text{E2.35})$$

Here, we have introduced the molecular magnetic moment

$$\mathbf{m}_n = \frac{1}{2} \sum_{\alpha(n)} q_{\alpha(n)} \mathbf{r}_{\alpha(n)} \times \mathbf{v}_{\alpha(n)} \quad \text{Molecular Magnetic Moment} \quad (\text{E2.36})$$

Now we have the macroscopic current density

$$\mathbf{J}(\mathbf{r}, t) = \mathbf{J}_f(\mathbf{r}, t) + \mathbf{J}_b(\mathbf{r}, t) \quad (\text{E2.37})$$

where we have defined

$$\mathbf{J}_f(\mathbf{r}, t) = \left\langle \sum_{i(\text{free})} q_i \mathbf{v}_i \delta[\mathbf{r} - \mathbf{r}_i(t)] \right\rangle + \left\langle \sum_{n(\text{mol})} q_n \mathbf{v}_n \delta[\mathbf{r} - \mathbf{r}_n(t)] \right\rangle \quad (\text{E2.38})$$

current due to mobile ions/molecules
Macroscopic Free Current Density

$$\mathbf{J}_b(\mathbf{r}, t) = \frac{\partial}{\partial t} [\mathbf{P}(\mathbf{r}, t) - \nabla \cdot \mathbf{Q}(\mathbf{r}, t) + \dots] + \nabla \times \mathbf{M}(\mathbf{r}, t) + \nabla \times \langle \mathbf{p}_n \times \mathbf{v}_n \delta(\mathbf{r} - \mathbf{r}_n) \rangle \quad (\text{E2.39})$$

Macroscopic Bound Current Density

$$+ \left\langle [\mathbf{Q}_n \cdot \nabla \mathbf{v}_n - \mathbf{v}_n \nabla \cdot \mathbf{Q}_n] \cdot \nabla \delta(\mathbf{r}, t) \right\rangle + \dots \rightarrow \text{can be ignored} \\ \text{~ vibrational velocities} \\ \text{~ zero average}$$

Macroscopic Magnetization (E2.39)

Gomez: Power

$$\mathbf{M}(\mathbf{r}, t) = \left\langle \sum_{n(\text{mol})} \mathbf{m}_n(t) \delta[\mathbf{r} - \mathbf{r}_n(t)] \right\rangle$$

In (E2.38), the second term is the **drift current due to mobile ionic molecules**. If there is no ionic molecule, then only the free charges such as conduction electrons and holes (in semiconductors) contribute to the free current density.

In (E2.39), the third and fourth terms can be ignored in most cases because the molecular velocities are basically thermal or vibrational velocities ($\ll c$), and also because the molecular velocities are random in nature, resulting in zero average in the macroscopic scale.

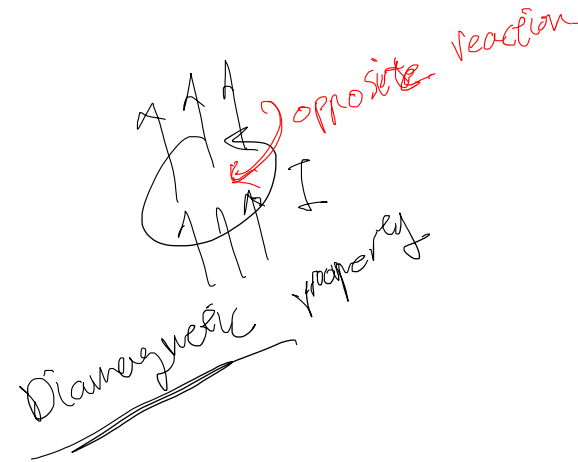
Now we define the magnetic field as

$$\mathbf{H}(\mathbf{r}, t) = \frac{1}{\mu_0} \mathbf{B}(\mathbf{r}, t) - \mathbf{M}(\mathbf{r}, t) + \cdots \quad \text{Magnetic Field} \quad (\text{E2.40})$$

Long derivation! Finally we now obtain the conventional macroscopic Maxwell's equations:

$$\begin{aligned} \nabla \times \mathbf{E}(\mathbf{r}, t) &= -\frac{\partial}{\partial t} \mathbf{B}(\mathbf{r}, t) \\ \nabla \times \mathbf{H}(\mathbf{r}, t) &= \frac{\partial}{\partial t} \mathbf{D}(\mathbf{r}, t) + \mathbf{J}_f(\mathbf{r}, t) \\ \nabla \cdot \mathbf{D}(\mathbf{r}, t) &= \rho_f(\mathbf{r}, t) \\ \nabla \cdot \mathbf{B}(\mathbf{r}, t) &= 0 \end{aligned} \quad \text{Macroscopic Maxwell's Equations} \quad (\text{E2.41})$$

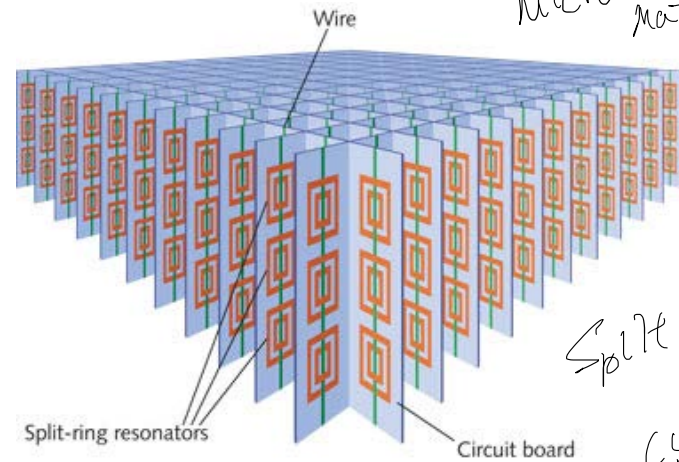
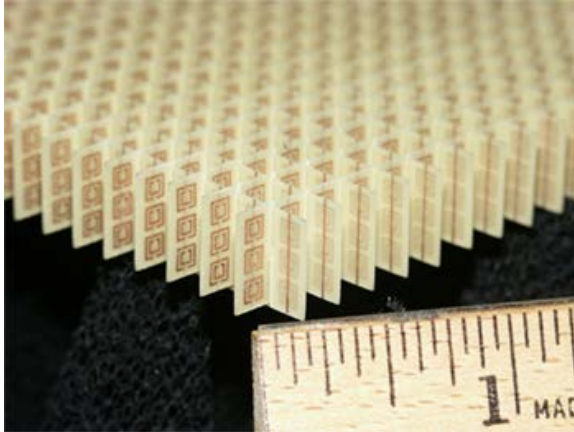
with the constitutive relations and charge conservation or current continuity equation



$$\begin{aligned} \mathbf{D}(\mathbf{r}, t) &= \varepsilon_0 \mathbf{E}(\mathbf{r}, t) + \mathbf{P}(\mathbf{r}, t) \\ \mathbf{H}(\mathbf{r}, t) &= \frac{1}{\mu_0} \mathbf{B}(\mathbf{r}, t) - \mathbf{M}(\mathbf{r}, t) \end{aligned} \quad \text{Constitutive Relations} \quad (\text{E2.42})$$

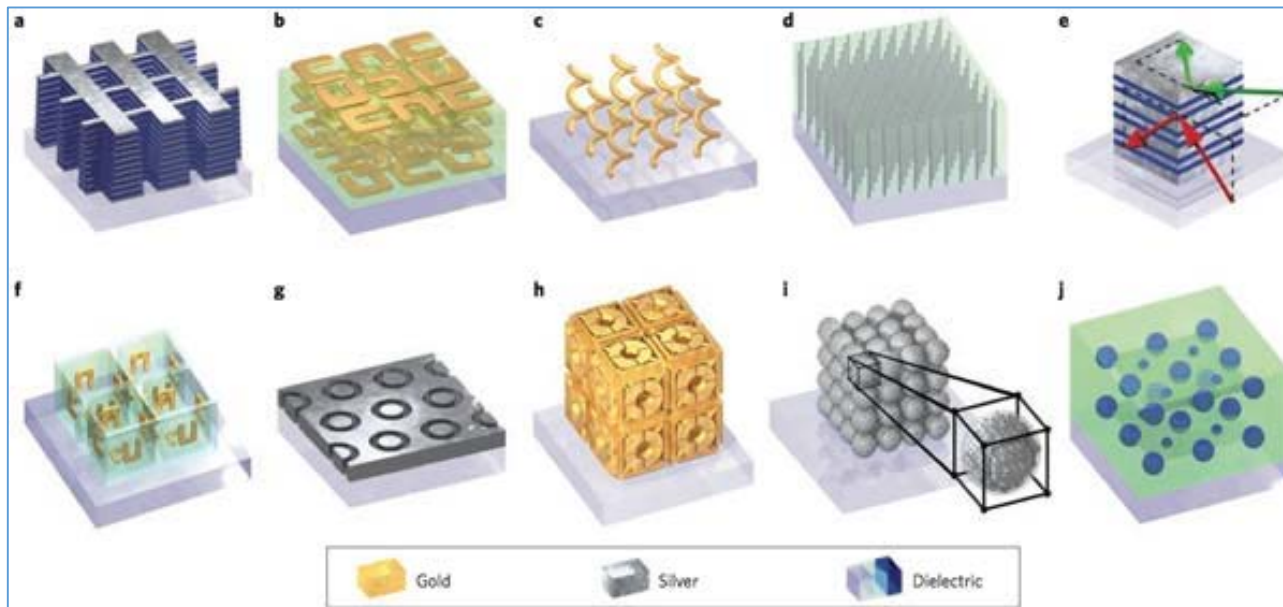
$$\nabla \cdot \mathbf{J}_f(\mathbf{r}, t) + \frac{\partial}{\partial t} \rho_f(\mathbf{r}, t) = 0 \quad \text{Current Continuity} \quad (\text{E2.43})$$

Metamaterials: Artificial EM Atoms and Molecules

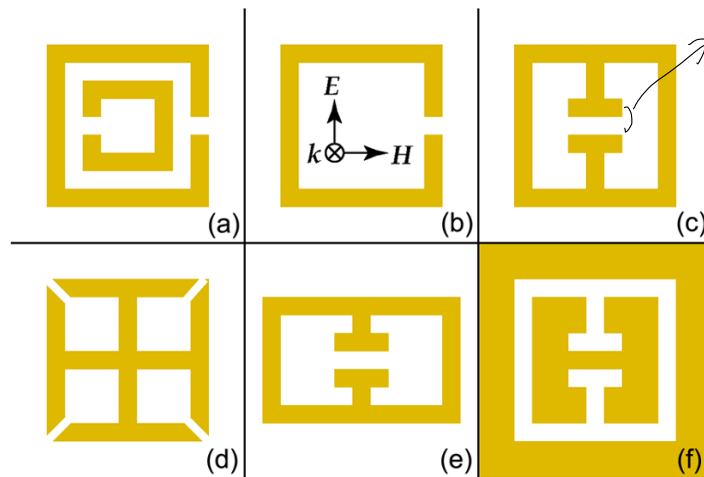
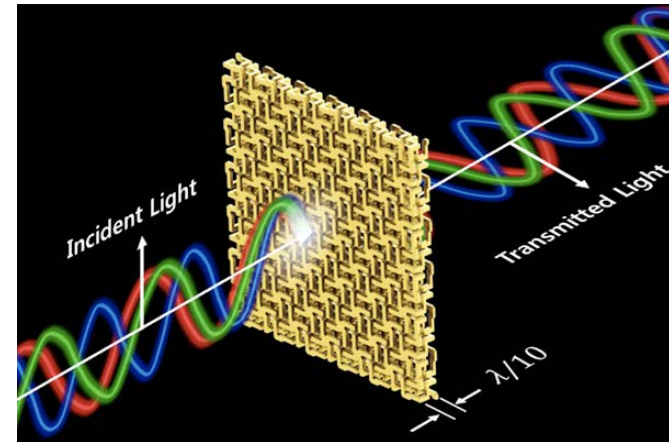
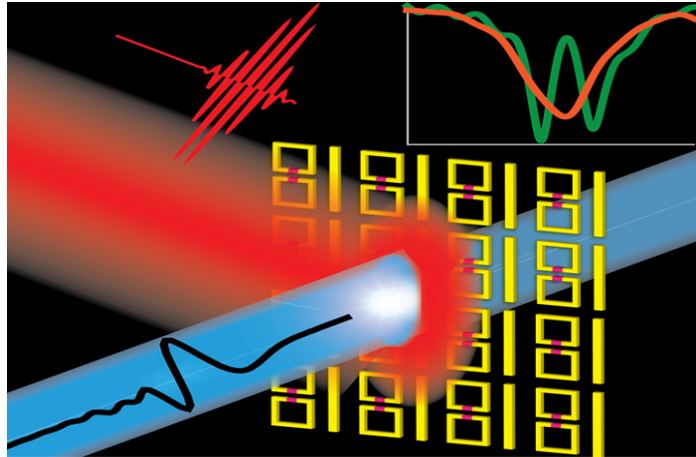


Spatial average
 Micro → Macro
 Material Response
 ($\epsilon, \mu \rightarrow \epsilon_{eff}, \mu_{eff}$)

Split Ring Resonator
 (SRR)



THz Metamaterials



capacitor without other structure change.

