Chapter 2
VECTOR AND TENSOR ANALYSES

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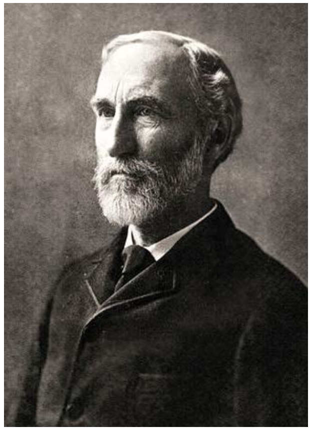
Lecture 4

2.8 Coordinate Systems

E2.1 Multipole Expansion Theory



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2.8 Coordinate Systems

Why Study Coordinate Systems?

Most of physics laws are expressed by partial differential equations such as classical and quantum wave equations. Depending on the **geometric symmetry** of the problem, the coordinate system should be chosen **to make it easier to solve the equations**.

The Cartesian coordinates is the simplest among the orthogonal coordinates, and has unique properties that all the unit vectors are constant, and perpendicular to each other.

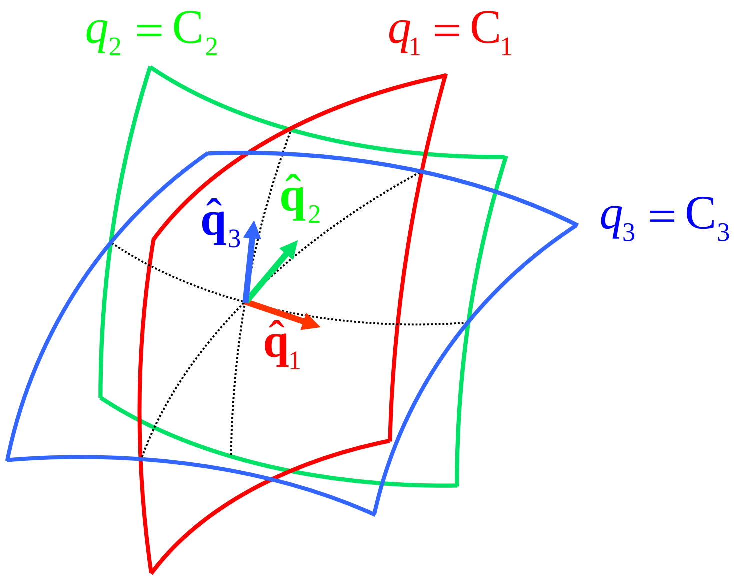
However, the Cartesian system is not the most convenient one for problems with spherical or cylindrical symmetries. In some cases, **not just orthogonal coordinates**, even **non-orthogonal (oblique or skew)** coordinates fit much better the problems, especially for **boundary value problems**.

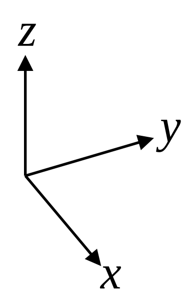
There are **eleven separable coordinate systems** in which the Laplace and wave equations are separable such that the partial differential equation can be split into ordinary differential equations.

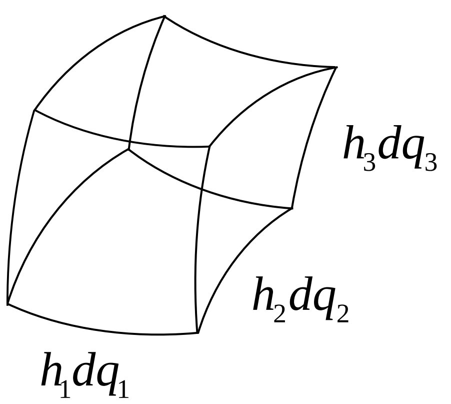
Orthogonal Curvilinear Coordinates

Orthogonal curvilinear coordinates (q_1, q_2, q_3) can define the Cartesian coordinates as

$$
\begin{aligned}
x(\mathbf{q}) &= x(q_1, q_2, q_3) \\
y(\mathbf{q}) &= y(q_1, q_2, q_3) \\
z(\mathbf{q}) &= z(q_1, q_2, q_3)
\end{aligned}
\tag{2.40}
$$







The differential length in (q_1, q_2, q_3) along $\hat{\mathbf{q}}_i$ is given by

$$dl_i = h_i dq_i \tag{2.41}$$

Now the differential length in an arbitrary axis can be defined as

$$dl = \sqrt{h_i^2 (dq_i)^2} = \sqrt{h_1^2 (dq_1)^2 + h_2^2 (dq_2)^2 + h_1^2 (dq_2)^2} \tag{2.42}$$

and then the differential volume is given by

$$dv = h_1 h_2 h_3 dq_1 dq_2 dq_3 \tag{2.43}$$

where the **scale factors** h_i are given below for some important coordinates

Coordinate System	Scale Factor	Differential Volume
Cartesian (x, y, z)	$(1,1,1)$	$dxdydz$
Cylindrical (ρ, ϕ, z)	$(1, \rho, 1)$	$\rho d\rho d\phi dz$
Spherical (ρ, θ, ϕ)	$(1, r, r\sin\theta)$	$r^2 \sin\theta dr d\theta d\phi$

Gradient
$$\nabla V = \mathbf{q}_i \frac{1}{h_i}\frac{\partial V}{\partial q_i} \tag{2.44}$$

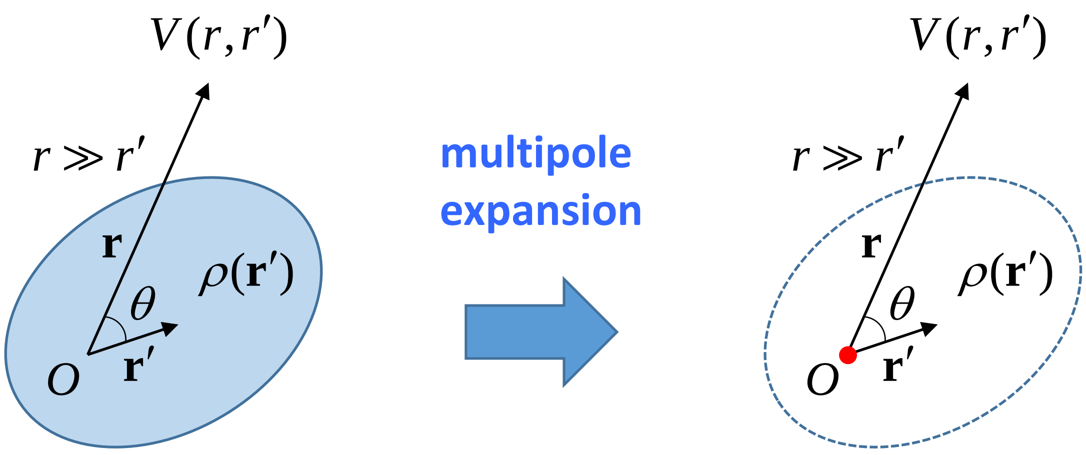
Divergence
$$\nabla \cdot V = \frac{1}{h_i}\frac{\partial V}{\partial q_i} \tag{2.45}$$

Curl
$$\nabla \times \mathbf{A} = \varepsilon_{ijk}\mathbf{u}_i \frac{1}{h_j}\frac{\partial A_k}{\partial q_j} \tag{2.46}$$

E2.1 Multipole Expansion Theory

In general, electromagnetic sources are moving charges and currents distributed over some volume. This means that the electromagnetic fields should be obtained by **integrating such distributed sources over some volume**, for example, using Coulomb's law.

However, the **multipole expansion theory** can be used to simply replace the continuous sources by **a few point sources in vacuum** at the origin.



Distributed Sources

Multipole Point Sources in Vacuum (concentrated at *O*)

E2.1.1 Electric Multipole Moments: Field Expansion (Conventional Method)

From the Coulomb's law in vacuum, the scalar potential is written as

$$V(\mathbf{r},t) = \int dv' \frac{\rho(\mathbf{r}',t)}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}'|} \tag{2.47}$$

where $\mathbf{r}$ and $\mathbf{r}'$ are the field and source points, respectively.
In the far-field region ($r \gg r'$), we can find the three largest terms:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\sqrt{r^2 - 2\mathbf{r}'\cdot\mathbf{r} + r'^2}} = \frac{1}{r}\frac{1}{\sqrt{1-2\alpha\cos\theta+\alpha^2}} \tag{2.48}$$

with a small parameter $\alpha = r'/r \ll 1$.

Considering the series expansion formula of the Legendre polynomial $P_n(\cos\theta)$:

$$\frac{1}{\sqrt{1-2\alpha\cos\theta+\alpha^2}} = \sum_{n=0}^{\infty} P_n(\cos\theta)\alpha^n \tag{2.49}$$

where the low-order Legendre polynomials are given by

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}\left(3x^2 - 1\right), \quad P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right), \quad \cdots \tag{2.50}$$

Substituting

Now we finally define the electric multipole moments as

$$
\begin{aligned}
V(\mathbf{r},t) &= \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\sum_{n=0}^{\infty}\int dv'\rho(\mathbf{r}',t)\left(\frac{r'}{r}\right)^n P_n(\cos\theta) \\
&\simeq \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\left[\int dv'\rho(\mathbf{r}',t)+\frac{1}{r}\int dv'\rho(\mathbf{r}',t)r'\cos\theta+\frac{1}{r^2}\int dv'\rho(\mathbf{r}',t)r'^2\,\frac{1}{2}\left(3\cos^2\theta-1\right)\right]
\end{aligned}
\tag{2.51}
$$

E2.1.2 Electric Multipole Moments: Source Expansion (This Lecture's Method)

From the **sifting** properties of Dirac delta function, the charge density is written as

$$\rho(\mathbf{r},t) = \int dv' \rho(\mathbf{r}',t)\delta(\mathbf{r}-\mathbf{r}') \tag{2.52}$$

where $\mathbf{r}$ and $\mathbf{r}'$ are the field and source points, respectively.

Using Taylor series expansion in the **far-field region** ($r \gg r'$), we have

$$\delta(\mathbf{r}-\mathbf{r}') = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(\mathbf{r}'\cdot\nabla)^n \delta(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\mathbf{r}'^n : \nabla^n \delta(\mathbf{r}) \tag{2.53}$$

from which we define the three lowest-order multipole moments as

$$\boxed{q(t) = \int dv' \rho(\mathbf{r}',t)} \quad \text{Electric Monopole (Total Charge)} \tag{2.54}$$

$$\boxed{\mathbf{d}(t) = \int dv' \rho(\mathbf{r}',t)\mathbf{r}'} \quad \text{Electric Dipole} \tag{2.55}$$

$$\boxed{\mathbf{Q}(t) = \int dv' \rho(\mathbf{r}',t)\mathbf{r}'\mathbf{r}'} \quad \text{Electric Quadrupole} \tag{2.56}$$

from which we define electric multipole moments as

$$\boxed{\rho(\mathbf{r},t)\delta(\mathbf{r}-\mathbf{r}') \simeq \left[q(t) - \mathbf{d}(t)\cdot\nabla + \frac{1}{2}\mathbf{Q}(t):\nabla\nabla \right]\delta(\mathbf{r})} \tag{2.57}$$

[Q] What are the MKS units of the three multipoles?