Chapter 3
Complex Variables

Leonard Euler
(1707-1783)
Math/Physics
Calculus, Euler-Lagrange Eq.
" $i, \pi, e, f(x), \Sigma$ "
$\cos z, \sin z, \tan z$
$e^{i\pi} + 1 = 0$

Lecture 7

3.1 Complex Numbers and Functions
3.2 Analytic Function
3.3 Cauchy's Theorem

Augustin-Louis Cauchy
(1789-1857)
Math/Physics
Complex Analysis
Stress Tensor

3.1 Complex Numbers and Functions

A complex number z is an ordered pair of two real numbers (x, y), which is subject to a multiplication rule:

$$(a,b)\cdot(c,d) = (ac - bd, bc + ad) \tag{3.1}$$

By introducing a symbol i, where $i^2 = -1$, we can write $z = (x, y) = x + iy$. Retaining all the arithmetic rules of real numbers, the multiplication rule of complex numbers is given by that of real numbers:

$$(a+ib)\cdot(c+id) = ac - bd + i(bc + ad) \tag{3.2}$$

A complex number can also be defined in a complex polar coordinates (r, θ):

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta} \tag{3.3}$$

where $r = (x^2 + y^2)^{1/2}$ and $\theta = \tan^{-1}(y / x)$ are called the magnitude (or modulus) and the phase (or argument) of z, respectively. Here we use the sign convention: θ increases along a counterclockwise direction.

We define a function of complex variables as a mapping from z to $w = f(z)$ plane:

$$f(z) = u(z) + iv(z) \tag{3.4}$$

where $u(z)$ and $v(z)$ are real-valued functions. Then all the elementary functions of real variables can be extended into complex variables.

For instance, the exponential function can be defined by a power series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots \tag{3.5}$$

and the Euler formula relates the exponentials with sinusoidal and hyperbolic functions:

$$\begin{aligned} e^{\pm iz} &= \cos z \pm i \sin z \\ e^{\pm z} &= \cosh z \pm \sinh z \end{aligned} \tag{3.6}$$

Simply-Connected and Multiply-Connected Regions

A region R in the z plane is said to be a **simply-connected** region if any closed curve in R encloses only points belonging to R. However, another region R' might be a **multiply-connected** region where some closed curves in R' can also enclose points not belonging to R' (Fig 2-1).

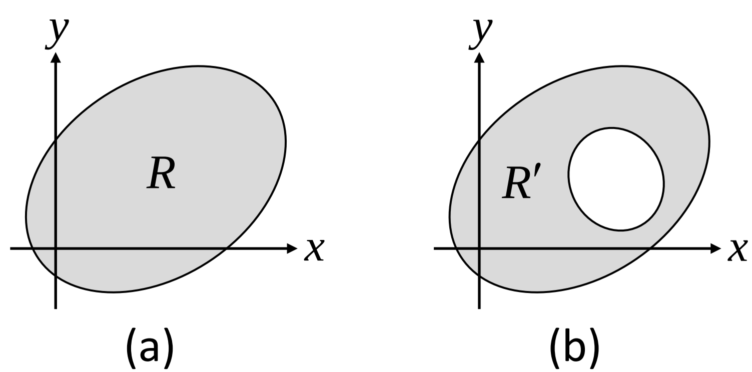


Fig. 3-1 (a) Simply-connected region and (b) multiply-connected region.

3.2 Analytic Function

For physics and engineering problems, we are interested mainly in **continuous** and **differentiable** functions. In complex analysis, the **"well-behaved" analytic function** plays the most important role for physical problems.

Continuity and Differentiability

[Def: Continuity] $f(z)$ is continuous at z if there exists the limit:

$$\lim_{\Delta z \to 0} f(z+\Delta z) = f(z) \qquad (3.7)$$

$\lim_{\Delta z \to 0} f(z+\Delta z) = f(z)$

[Def: Differentiability] $f(z)$ is differentiable at z if there exists the limit:

$$f'(z) = \frac{df(z)}{dz} \equiv \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} \qquad (3.8)$$

$f'(z) = \frac{df(z)}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$

$= \lim_{\Delta z \to 0} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$

where the increments are defined as $\Delta f = f(z+\Delta z) - f(z)$ and $\Delta z = \Delta x + i\Delta y$, and the existence of the limit means that $f'(z)$ has **the same single value no matter which direction** we take to approach to z. In other words, $f'(z)$ should not depends on the direction for differentiation.

Cauchy–Riemann Conditions

For the existence of $f(z)$, the limit should have the same single value for all the possible approach $(\Delta z = \Delta x + i\Delta y \to 0)$ to z, which is the condition of **isotropic derivative**.

independent to direction.

Considering two possible paths of $\Delta z = \Delta x$ and $\Delta z = i\Delta y$,

$$\lim_{\substack{\Delta x \to 0 \\ (y=0)}} \frac{\Delta u + i\Delta v}{\Delta x} = \lim_{\substack{(x=0) \\ \Delta y \to 0}} \frac{\Delta u + i\Delta v}{i\Delta y} \quad \to \quad \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \tag{3.9}$$

we have

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \qquad \textbf{Cauchy-Riemann Condition} \tag{3.10}$$

which is actually the necessary condition for differentiability at a point z. The sufficient conditions for differentiability are given by 1) the Cauchy-Riemann condition and 2) the continuity at z.

Analytic Function

differentiable at z and in its neighborhood

A function $f(z)$ is said to be analytic (regular or holomorphic) at z if $f(z)$ is **differentiable at z and also in its neighborhood**. So there is a slight difference between differentiability and analyticity: **the analyticity is a stronger condition than differentiability**.

If $f(z)$ is analytic at a point, the point is said to be **regular**. If $f(z)$ is not analytic at a point, the point is said to be **singular**.

3.3 Cauchy's Theorem

$$\int_{Z_A}^{Z_B} dz f(z) = \lim_{N\to\infty} \sum_{n=0}^{N} f(z_n)(z_n - z_{n-1})$$

Contour Integral

A contour integral of a complex function $f(z)$ over a contour from z_A to z_B in the z-plane is defined as a limit of consecutive infinitesimal summation:

$$\int_{z_A}^{z_B} dz\, f(z) = \lim_{N\to\infty} \sum_{n=0}^{N} f(z_n)(z_n - z_{n-1}) \tag{3.11}$$

with a set of intermediate points between z_A and z_B.

Cauchy-Goursat Theorem (Cauchy Theorem)

If $f(z)$ is analytic within and on a closed contour C, then

$$\boxed{\oint_C dz\, f(z) = 0} \tag{3.12}$$

Although an approximate proof is available from many textbooks, the formal and rigorous proof of Cauchy-Goursat theorem is not easy. You can find it in some advanced textbooks.

The original **Cauchy's theorem, a weak version of the Cauchy-Gousart theorem**, requires an additional condition that $f'(z)$ is continuous in the region, which had been shown to be unnecessary by Goursat. The standard reference direction of a closed contour is usually ***counterclockwise***.

Morera Theorem : The Converse of the Cauchy's Theorem

If $f(z)$ is continuous in a region R, and if $\oint_C dz\, f(z) = 0$ for any closed contour C in R, then $f(z)$ is analytic in R.

Path Independence

If $f(z)$ is analytic within and on a closed contour C in a region R, then

$$\boxed{\int_{C_1} dz\, f(z) = \int_{C_2} dz\, f(z)} \tag{3.13}$$

where the closed contour consists of $-C_1$ and C_2 as shown in Fig. 3-2. This is a direct consequence of the Cauchy-Goursat theorem.

Antiderivative Theorem

If $f(z)$ is analytic in R, and if $F(z)$ is defined as the antiderivative of $f(z)$:

$$F(z) = \int_{z_0}^{z} dz' f(z') \tag{3.14}$$

then $F(z)$ is also analytic and $F'(z) = f(z)$.

Note that the integration path can be taken to be a straight line between z_0 and z because of the path independence of the analytic function. This means that the proof is easy.

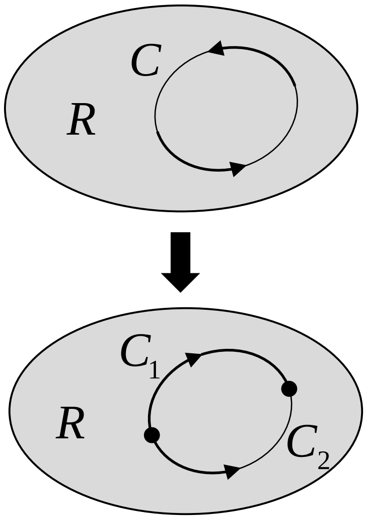


Fig. 3-2 Path independence of contour integral.

Cauchy's Integral Formula: Conversion from "Multiply-Connected" to "Simply-Connected"

If $f(z)$ is analytic within and on a closed contour C, then for any **interior point** z_0

$$\boxed{f(z_0) = \frac{1}{2\pi i}\oint_C dz \frac{f(z)}{z - z_0}} \qquad (3.15)$$

$f(z_0) = \frac{1}{2\pi i}\oint_C dz \frac{f(z)}{z-z_0}$

Proof) Having a singular point z_0 within the contour C, the interior region is multiply-connected. So we convert it into a simply-connected region by using a modified contour $\Gamma = C_\varepsilon + L_{\varepsilon 1} + C'_\varepsilon + L_{\varepsilon 2}$ for an **effectively simply-connected** interior region with no singular point, as shown in Fig. 3-3.

From the Cauchy's theorem (3.12) in the limit of $\varepsilon \to 0$

$$\oint_\Gamma dz \frac{f(z)}{z - z_0}\bigg|_{\varepsilon \to 0} = \left[\int_C dz + \cancel{\int_{L_1} dz} + \int_{C'} dz + \cancel{\int_{L_2} dz}\right]\frac{f(z)}{z - z_0} = 0$$

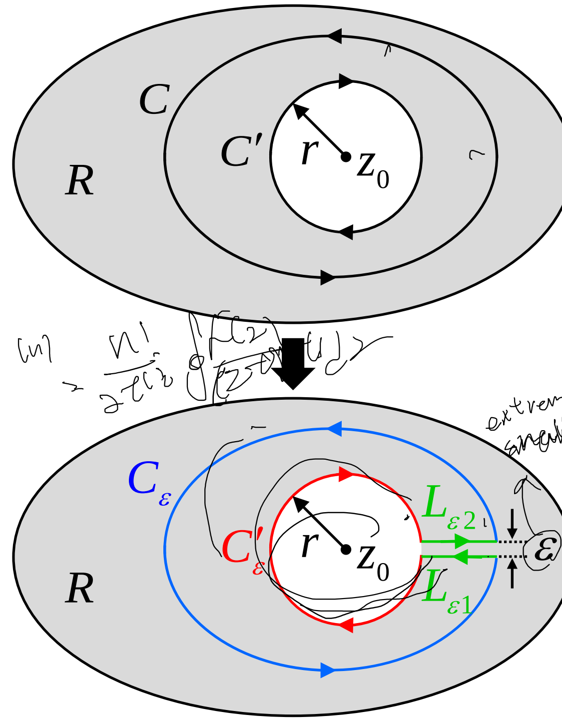
With $z = z_0 + re^{i\theta}$ and $dz = ire^{i\theta}d\theta$ on C'

$$\oint_C dz \frac{f(z)}{z - z_0} = -\oint_{C'} dz \frac{f(z)}{z - z_0} = -\int_{2\pi}^{0} ire^{i\theta} d\theta \frac{f(z_0 + re^{i\theta})}{re^{i\theta}}$$

$$\xrightarrow{r \to 0} 2\pi i f(z_0)$$

QED

$(n) \quad \frac{n!}{2\pi i}\oint \frac{f(z)}{(z-z_0)^{n+1}}dz$



$\Gamma = C_\varepsilon + L_1 + C'_\varepsilon + L_2$

Fig. 3-3 Contour coversion for Cauchy's Integral Formula.