On the Propagation of Gaussian Beams of Light Through Lenslike Media Including those with a Loss or Gain Variation

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A theoretical analysis is made of Gaussian beam propagation in media where the refractive index and/or the gain constant varies quadratically with the distance from the optic axis. Use of a complex beam parameter simplifies the analysis. A differential equation for the complex beam parameter is deduced from the wave equation, and various of its solutions are discussed. This includes a discussion of light propagation in media with a gain profile, which are found capable of supporting stationary beams of constant diameter.

Introduction

This paper considers the propagation of optical modes or Gaussian beams of light of the kind produced by laser oscillators or used in microwave beam waveguides. It discusses the propagation of these beams through optical systems like lenses, gas lenses, and quite general lenslike media including media exhibiting a nonuniform loss or gain profile. An example of a medium with a nonuniform gain profile is the medium of a gas laser amplifier where the gain decreases with the distance from the axis of the laser tube.

The laws governing the propagation of the light beam will be formulated in terms of a complex valued beam parameter. This complex parameter gives information on the width of the beam (spot size), and on the curvature of the phase front in each beam cross section of interest. The advantage of using a complex beam parameter is apparent from the recent literature. Following the initial proposal by Collins various circle diagrams for Gaussian beams were introduced,[1,2] which are (or can be regarded as) plotted in the complex plane of a beam parameter of this kind. It was also found that the laws of beam propagation in homogeneous media can be written in a very simple form if a complex notation is used.[2–4] It will be seen that the use of a complex parameter also simplifies the analysis of beam propagation in lenslike media.

First, the paper recalls the laws of propagation for spherical waves in lossless lenslike media in the geometrical optics approximation. The variation of the phase fronts of these waves along the optic axis is described in terms of the elements of the ray matrix. These matrix elements can be obtained by tracing paraxial rays through the medium. Then Gaussian beams in lossless lenslike media are considered, and it is found that the law for the transformation of the curvature radius of spherical waves is formally the same as the transformation law for the complex beam parameter (*ABCD* law).

The paper continues by discussing wave propagation in lenslike media where loss or gain varies with the square of the distance from the optic axis. Assuming paraxial waves of small wavelengths, a parabolic equation is obtained from the scalar wave equation for these media. This parabolic equation is a generalization of the parabolic equation for a homogeneous medium which is used in diffraction theory.[5] Light beams with a Gaussian amplitude profile are obtained as a solution of this parabolic equation together with a differential equation of first order. It can be transformed into a linear differential equation of second order which has the form of the paraxial ray equation for the medium. The connection between the beam parameter equation and the differential equation used by Tien *et al.*[6] is studied and several solutions are discussed. Beam propagation in media with a gain profile is investigated in greater detail.

The paper is concluded by a brief discussion of analog electric circuits which correspond to the various lenslike media. The input and output admittances of these circuits (or impedances in the dual circuits) correspond to the complex parameters of the beam in the input and output planes of the optical system as in the lumped circuit analogs described by Deschamps and Mast.[2]

Propagation of Spherical Waves in the Geometrical Optics Approximation

Some well-known properties of spherical waves are recalled here for later comparison with the laws obeyed by Gaussian beams. Consider first the curvature of two phase fronts of a spherical wave propagating in the

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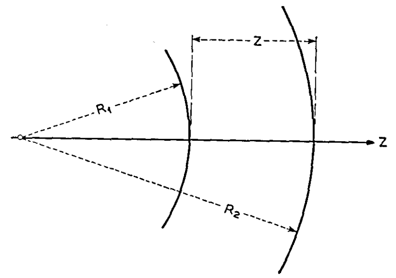

Fig. 1. Wavefronts of spherical wave.

z direction. Call the radii of curvature of the two phase fronts R_1 and R_2, respectively, and z the distance between the intersections of these phase fronts with the optic axis. It is quite obvious from Fig. 1 that the curvature radii are related by

$$R_2 = R_1 + z. \tag{1}$$

Here and throughout this paper a curvature radius is defined as positive if the phase front is concave as seen from the left ($z = -\infty$).

If a spherical wave passes through a lens of focal length f, it is transformed into another spherical wave. For an incoming wave with a radius of curvature R_1 at the lens the wave going out from the lens has a curvature radius R_2 given by

$$\frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f}. \tag{2}$$

Consider now paraxial light rays passing through a medium whose refractive index near the optic axis is described by

$$n(r,z) = n_0(z) - \frac{1}{2}\, n_2(z)\cdot r^2. \tag{3}$$

This kind of medium is called a lenslike medium. Examples are inhomogeneous crystals and gas lenses. The propagation of paraxial light rays in a lenslike medium is described by the paraxial ray equation[3,6,7]

$$\frac{d}{dz}\left(n_0 \frac{dx}{dz}\right) = -n_2 x, \tag{4}$$

where $x(z)$ indicates the x coordinate of the ray position in the plane z = const. A similar equation holds for the y coordinate. In Fig. 2 there is schematically indicated a ray path through a lenslike medium and an input and an output plane. The quantities of interest are also indicated. They are the ray position x_2 and the ray slope x_2' in the output plane and the corresponding ray parameters x_1 and x_1' in the input plane. The solution of the paraxial ray equation yields a linear relation between the input and output parameters which is conveniently written in matrix form[3,8]

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}. \tag{5}$$

The passage of rays through lenses and lens combinations can also be described in this matrix form, and the elements A, B, C, and D of the *ray matrix* are known for several optical structures.[3] The matrix elements are determined by the over-all focal length of the optical structure of interest and by the position of its principal planes.[3,8]

The rays associated with a spherical wave are perpendicular to the wavefront. The position x and slope x' of a paraxial ray are related to the radius of curvature R of the wavefront by

$$R = \frac{x}{x'}. \tag{6}$$

By tracing a paraxial ray through a given optical structure one can determine the transformation of a wavefront of a spherical wave passing through this structure. For a spherical wave with a radius of curvature R_1 at the input plane of the structure, one finds for the curvature radius R_2 at the output plane

$$R_2 = \frac{AR_1 + B}{CR_1 + D}. \tag{7}$$

Propagation of Gaussian Beams, *ABCD* Law

The properties of Gaussian beams of light in homogeneous media are well known. Near the optic axis the field distribution $E(r,z)$ of a fundamental Gaussian mode is described by[9]

$$E(r,z) = \frac{w_0}{w} \exp\left\{-j(kz + \varphi) - r^2\left(\frac{1}{w^2} + \frac{jk}{2R}\right)\right\}, \tag{8}$$

where a constant amplitude factor was dropped. The light is assumed to propagate in z direction, r measures the distance from the z axis (optic axis), $k = 2\pi/\lambda$ is the propagation constant in the medium, and φ (z) indicates an additional phase shift due to the geometry of the beam. The radius of the beam (spot size) is $w(z)$, and the radius of curvature of the phase front at z is $R(z)$.

As the light beam propagates in space it expands due to diffraction, but the transverse field distribution remains Gaussian. The law of expansion is[9]

$$w^2 = w_0^2\left[1 + \left(\frac{\lambda z}{\pi w_0^2}\right)^2\right], \tag{9}$$

where z is measured from the beam waist. At the waist the radius of the beam is w_0, and the phase front is plane. The radius of the phase-front curvature changes along the optic axis according to the law[9]

$$R = z\left[1 + \left(\frac{\pi w_0^2}{\lambda z}\right)^2\right], \tag{10}$$

and the additional phase shift is

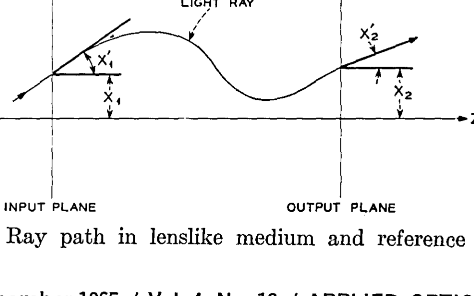

Fig. 2. Ray path in lenslike medium and reference planes.

$$\varphi = \tan^{-1}\left(\frac{\lambda z}{\pi w_0^2}\right). \tag{11}$$

It is the main object of this paper to determine the parameters w, R, and φ of a Gaussian beam which has passed through a lenslike medium of the most general type.

It is convenient to combine the beam parameters w and R in a complex-valued parameter q defined by

$$\frac{1}{q} = \frac{1}{R} - j\frac{\lambda}{\pi w^2}. \tag{12}$$

For homogeneous media the transformation laws (9) and (10) can then be written as one equation in the simple form[2–4]

$$q_2 = q_0 + z, \tag{13}$$

where

$$q_0 = j\frac{\pi w_0^2}{\lambda}. \tag{14}$$

Instead of defining z as the distance from the beam waist one can now redefine z as the distance between any two reference planes in the medium. Because of the linearity of (13) the relation between the input parameter q_1 and the output parameter q_2 can then be written as

$$q_2 = q_1 + z. \tag{15}$$

The parameters of a Gaussian beam are transformed when it passes through an ideal thin lens. The beam radii remain unchanged, i.e., the beam radius w_1 immediately to the left of the lens will be equal to the beam radius w_2 immediately to the right of it. The curvature radii R_1 and R_2 of the corresponding phase fronts are transformed in the same way as the curvature radii of spherical waves, i.e., they are related by (2). Therefore, the lens transformation law for the complex beam parameter is [2,3]

$$\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{f}. \tag{16}$$

The laws for the transformation of Gaussian beams are seen to be formally the same as the transformation laws obeyed by the spherical waves of geometrical optics. Equation (15) is formally the same as (1), and (16) is formally the same as (2). The complex beam parameter q plays a role corresponding to the one played by the radius R of the spherical waves and might be called a complex curvature radius.

Like other optical structures a lenslike medium can be regarded as composed of a set of thin lenses. The number of lenses required for a fair representation of the medium may, of course, be very large. But the fact remains that (15) and (16) can be used to trace a Gaussian beam through each lens element, while (1) and (2) are used to trace a spherical wave. As a result of ray tracing one has the law (7) for the transformation of a spherical wave by the optical structure. Because of the formal equivalence discussed above, the transformation law for Gaussian beams must be

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}. \tag{17}$$

This law has been called the $ABCD$ law.[3] It allows one to trace Gaussian beams through any optical structure for which the elements A, B, C, and D of the ray matrix (or the focal length and the principal planes) are known. Its application to some optical structures of interest has been given in ref. 3.

Almost-Plane Waves in Lenslike Media

Pierce[10] starts from the wave equation to derive the propagation laws for Gaussian beams in free space and in sequences of lenses. A modification of his results was used by Tien *et al.*[6] for their study of beam propagation in various lenslike media. The present derivation of the propagation laws for Gaussian beams in quite general lenslike media starts also with the scalar wave equation*

$$\Delta E + k^2 E = 0, \tag{18}$$

where E is a scalar field or potential, and k is defined by

$$k^2 = \omega^2\epsilon\mu\left(1 - j\frac{\sigma}{\epsilon\omega}\right). \tag{19}$$

Here ω is the angular frequency of the field, ϵ is the dielectric constant of the medium, σ the conductivity, and μ the permeability. For fields oscillating like exp $(j\omega t)$ the values of σ are positive for lossy media, and negative for media with gain. A lenslike medium of the most general type is characterized by a quadratic radial dependence of k^2 near the optic axis, i.e., one can write

$$k^2(r,z) = k_0^2 - k_0 k_2 r^2. \tag{20}$$

The medium constants $k_0(z)$ and $k_2(z)$ are generally complex valued to allow for loss or gain. They are assumed to be slow functions of z.

The main interest here is in waves that propagate primarily in the z direction, or almost-plane waves. Accordingly, one writes

$$E = \psi(x,y,z)\cdot\exp(-j\Phi), \tag{21}$$

with

$$\Phi'(z) = k_0(z), \tag{22}$$

and the prime indicating differentiation with respect to z. For k_0 = const, the exponential in (21) becomes the usual $\exp(-jk_0 z)$. The function ψ is assumed to vary so slowly with z that its second derivative can be neglected. If one inserts (20), (21), and (22) into the wave equation (18) and neglects ψ'' one obtains

$$\Delta_t\psi - 2jk_0\psi' - jk_0'\psi - k_0 k_2 r^2\psi = 0, \tag{23}$$

where

* A treatment of the exact vector problem has to start from Maxwell's equations, where one obtains this form of the wave equation for the electric field E in a medium of constant permeability u and space-dependent dielectric constant ϵ by neglecting the term grad div E = -grad $\left[\frac{1}{\epsilon}\,(\text{grad}\,\epsilon . E)\right]$.

$$\Delta_t = \Delta - \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{24}$$

The parabolic equation (23) is a generalized form of the parabolic equation for lossless homogeneous media which was used by Leontovich *et al.*[5] in their analyses of various diffraction phenomena. This parabolic equation takes into account the *transverse diffusion* of the field, but it neglects *longitudinal diffusion*. It yields good results for paraxial waves whose wavelengths are much smaller than their transverse dimensions.

A wave with a Gaussian transverse field distribution is an exact solution of Eq. (23). To determine the parameters of the Gaussian beam one can write

$$\psi = \exp - j\left(P + \frac{1}{2}Qr^2\right). \tag{25}$$

using a complex phase parameter P, and a complex beam parameter Q which corresponds to the parameter $1/q$ used above. Both parameters are functions of z. By inserting (25) into (23) one gets

$$Q^2r^2 + 2jQ + k_0(2P' + Q'r^2) + jk_0' + k_0k_2r^2 = 0. \tag{26}$$

Now one compares terms with equal powers of r, and obtains an expression for the complex phase P

$$P' = -j\left(\frac{Q}{k_0} + \frac{k_0'}{2k_0}\right), \tag{27}$$

as well as a differential equation for the complex beam parameter Q

$$Q^2 + k_0Q' + k_0k_2 = 0. \tag{28}$$

Equation (28) is a nonlinear differential equation of the Ricatti type. Its connection to the paraxial ray equation, and several of its solutions will be discussed below.

Beam Parameter Equation and Paraxial Ray Equation

Solutions of (28) describe the propagation properties of Gaussian beams. These solutions can also be constructed from solutions of the paraxial ray equation. The connection between these two equations is made by

$$Q = k_0 \frac{x'}{x}. \tag{29}$$

If this is inserted into (28) one arrives at

$$(k_0x')' + k_2x = 0. \tag{30}$$

This equation is formally the same as the paraxial ray equation (4) except that one has, in general, complex-valued k_0 and k_2 instead of the real n_0 and n_2 which describe lossless media. In media with loss or gain the variable x is also complex-valued, and its physical meaning is not so clear as its meaning as a ray position in lossless media. However, the general solutions of (30) can be combined according to (29) to yield solutions of (28) for the complex beam parameter which is a measure for the width and the phase-front curvature of Gaussian beams. Equation (29) is essentially the *ABCD* law (17) extended in validity to media with loss or gain.

The example of a medium with k_0 = const and k_2 = const will help to illustrate the preceding remarks. For $k_2 \neq 0$ such a medium can support Gaussian beams whose diameter does not change along the optic axis. The Q parameter of this kind of a beam is obtained from (28) by postulating that $Q' = 0$. One finds

$$Q_m = -j\sqrt{k_0k_2}. \tag{31}$$

The sign of the square root has been so chosen that a real and positive beam radius w is obtained. The general solution for a beam of varying diameter is easily derived from (30). The general solution of this equation is

$$x = ae^{j\gamma z} + be^{-j\gamma z}, \tag{32}$$

where a and b are constants which depend on the initial conditions at $z = 0$, and

$$\gamma = \sqrt{\frac{k_2}{k_0}}. \tag{33}$$

To get a general solution for the complex beam parameter one inserts (32) and its derivative with respect to z into (29). After some rearranging one arrives at

$$Q_2 = Q_m \frac{Q_1 + Q_m + (Q_1 - Q_m)e^{2j\gamma z}}{Q_1 + Q_m - (Q_1 - Q_m)e^{2j\gamma z}}, \tag{34}$$

where Q_1 is the beam parameter at the input plane at $z = 0$ and Q_2 is the output parameter at z. Use of this general formula will be made later when the effect of a medium with a gain profile on Gaussian beams is studied in greater detail.

Refractive Indices and Gain Constants

For a more detailed study of beam propagation the complex propagation constant is separated into its real and imaginary parts

$$k = \beta + j\alpha, \tag{35}$$

where β is the real propagation constant, and α is the gain constant (which is negative for loss). At optical and infrared frequencies the gain per wavelength is usually very small ($\alpha \ll \beta$), and one can write to a good approximation

$$k^2 = \beta^2 + 2j\alpha\beta. \tag{36}$$

To find a relation between α and the conductivity one compares (36) with (19) and gets

$$\alpha = -\frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}. \tag{37}$$

In a medium described by (35) the power of a plane wave grows like

$$P_2 = P_1 \colon e2\alpha z. \tag{38}$$

This power gain is often measured in decibels (dB). The gain constant α is related to the dB gain per distance z by

$$\alpha = \frac{\text{dB gain}}{8.686(z)} \tag{39}$$

as one easily computes.

Near the optic axis the propagation constant of a lenslike medium changes generally as in

$$\beta = \beta_0 - \frac{1}{2}\beta_2 r^2. \tag{40}$$

Closely related to β is the refractive index of the medium

$$n = \frac{\lambda}{2\pi}\beta = n_0 - \frac{1}{2}n_2 r^2, \tag{41}$$

where λ is the wavelength in free space. In the most general type of a lenslike medium one has also a gain constant that varies quadratically with the distance from the optic axis

$$\alpha = \alpha_0 - \frac{1}{2}\alpha_2 r^2. \tag{42}$$

It is not difficult to relate the above gain and propagation constants to the complex constants used in the beam parameter equations. It follows from (20) that, near the axis, the complex propagation constant is

$$k = k_0 - \frac{1}{2}k_2 r^2. \tag{43}$$

Consequently,

$$k_0 = \beta_0 + j\alpha_0, \tag{44}$$

and

$$k_2 = \beta_2 + j\alpha_2. \tag{45}$$

Keeping in mind that the gain per wavelength is small, one can write for the product k_0k_2

$$k_0k_2 = \beta_0\beta_2 + j(\alpha_0\beta_2 + \alpha_2\beta_0). \tag{46}$$

Solution for Beams in Free Space

In this section the solution of the beam parameter equation for Gaussian beams in free space is discussed briefly. This will serve to connect the present method to the published literature on the subject, and to demonstrate that the parabolic equation (23) leads to the same results as the theory of Fresnel diffraction.[9]

In free space one has $\alpha_0 = \alpha_2 = \beta_2 = 0$ and $\beta_0 = 2\pi/\lambda$. The beam parameter q defined in (12) is related to the parameter Q defined in (25) by

$$\frac{1}{q} = \frac{\lambda}{2\pi}Q. \tag{47}$$

If this is introduced in (28), the beam parameter equation becomes

$$\frac{dq}{dz} = 1. \tag{48}$$

The solution of (48) is, clearly,

$$q_2 = q_1 + z, \tag{49}$$

which is seen to be identical to (15).

The complex phase shift on the axis is computed by means of (27). Together with (47) and (49) this gives

$$P_2 - P_1 = -j\int_0^z \frac{dz}{q_1 + z} = j\ln\frac{q_1}{q_1 + z}. \tag{50}$$

If one measures z from the beam waist, one has a $q_1 = q_0$ that is purely imaginary and given by (14). As is well known, the real and imaginary parts of the phase shift can then be separated by using the identity

$$\ln(u + jv) = \ln\sqrt{u^2 + v^2} + j\tan^{-1}\frac{v}{u}. \tag{51}$$

The imaginary part of the phase shift produces the factor w_0/w appearing in (8), and the real part predicts the same additional phase shift as (11).

The Lossless Lenslike Medium

For a lossless medium one has $\alpha_0 = \alpha_2 = 0$. If the definitions of the refractive index (41) and of the beam parameter q (47) are introduced, the beam parameter equation (28) can be written in the form

$$\frac{1}{q^2} + n_0\left(\frac{1}{q}\right)' + n_0n_2 = 0. \tag{52}$$

Because of the recent interest in gas lenses, several studies have been made[3,6,7] of Gaussian beam propagation in lenslike media. A familiar result for media with constant n_0 and n_2 is the guided beam of constant diameter $2w_m$ whose parameter q_m is obtained from (52) by putting $q' = 0$.

$$\frac{1}{q_m} = -j\sqrt{n_0n_2} = -j\frac{\lambda}{\pi w_m^2}. \tag{53}$$

Tien *et al.*[6] have studied the propagation of Gaussian beams through various media, each characterized by refractive indices $n_0(z)$ and $n_2(z)$ with a certain z dependence of practical interest. They have based their work on a differential equation for the beam radius w. In order to see the connection between their differential equation and the beam parameter equation (52) one introduces in the latter the definition of the q parameter given in (12). By comparing the imaginary parts of (52) one then obtains

$$\frac{1}{R} = n_0\frac{w'}{w}, \tag{54}$$

which is a relation that has already been given by Pierce.[10] One then compares the real parts of (52), eliminates R by means of (54), and arrives at

$$\frac{n_0}{w}(n_0w')' = -n_0n_2 + \frac{\lambda^2}{\pi^2w^4}. \tag{55}$$

This is exactly the differential equation (12) of Tien *et al.*[6] It is interesting to note that the differential equation (52) for the complex beam parameter q has a simpler form than the differential equation (55) for the real beam radius w.

Medium with a Gain Profile

In this section the propagation of Gaussian beams in a medium with a quadratic gain profile is studied in greater detail. This is a medium where the gain decreases with the distance from the axis according to (42). The plasma medium of various gas laser tubes is of that nature.[11]

For simplicity it is assumed here that the gain constants are independent of z, i.e., $\alpha_0 =$ const and $\alpha_2 =$ const. The refractive index of the medium is taken to be that of free space, $\beta_0 = 2\pi/\lambda$ and $\beta_2 = 0$. With these assumptions (46) becomes

$$k_0 k_2 = j\frac{2\pi}{\lambda}\alpha_2. \tag{56}$$

From this and (31) it follows that a medium with a gain profile is capable of supporting a beam of constant diameter even though there is no refractive index variation. The q parameter of this stationary beam is computed from (31) and (56) as

$$\frac{1}{q_m} = \frac{\lambda}{2\pi} Q_m = \frac{1}{2}(1 - j)\sqrt{\frac{\alpha_2\lambda}{\pi}}. \tag{57}$$

Inspecting this expression one concludes that the stationary beam of constant width has phase fronts whose curvature is finite, positive, and independent of z. It is indicated by (57) that the radius of curvature R_m of these wavefronts is

$$R_m = \frac{\pi w_m^2}{\lambda} = 2\sqrt{\frac{\pi}{\alpha_2\lambda}}, \tag{58}$$

and that the radius of the beam w_m is given by

$$w_m^2 = 2\sqrt{\frac{\lambda}{\pi\alpha_2}}. \tag{59}$$

One notes that the real and imaginary parts of $1/q_m$ are of equal magnitude. This reminds one of a Gaussian beam in free space where the parameter $1/q$ has the same property at a distance $z = \pi w_0^2/\lambda$ (the *confocal distance*) from the beam waist. In this particular beam cross section the phase-front curvature is stationary and the corresponding radius of curvature goes through its minimum value. In free space the beam diameter expands at the confocal distance, but the gain profile of the present medium counteracts this tendency. If the diffractional and geometrical expansion of the beam balances the size-reducing effect of the gain profile, the beam diameter remains stationary. This leads to a stationary beam with the properties discussed above.

For several practical laser tubes it is reasonable to assume that the gain constant has its largest value at the tube axis, decreases quadratically, and is zero at the tube walls.[11] If one calls the tube radius r_0 one can, then, write for the gain constant

$$\alpha = \alpha_0\left(1 - \frac{r^2}{r_0^2}\right). \tag{60}$$

Comparing with (42) one has

$$\alpha_2 = 2\alpha_0/r_0^2, \tag{61}$$

while (58) and (59) become

$$R_m = r_0\sqrt{\frac{2\pi}{\lambda\alpha_0}}, \qquad w_m^2 = r_0\sqrt{\frac{2\lambda}{\pi\alpha_0}}. \tag{62}$$

The reported[11] gain of a He–Xe amplifier tube of 4-mm diam ($r_0 = 2$ mm) at a wavelength of 3.5 μ is about 100 dB/m. From this value one computes with (39) a gain constant on the axis of approximately

$$\alpha_0 = 11.5\ m^{-1}. \tag{63}$$

By inserting these values in (62) one calculates for R_m and w_m the values

$$R_m = 79 \text{ cm}, \qquad w_m = 0.94 \text{ mm}. \tag{64}$$

One notes that, at these high gain values, the stationary beam radius is smaller than the radius of the tube, and the tube can guide its own beam, provided there is no saturation. For small gain values the stationary radius w_m is larger than r_0, and the tube cannot support a stationary beam. But even then w_m can be used as a parameter to characterize the effect of the gain profile on the expansion or contraction of Gaussian beams with radii w that are smaller than r_0.

For the stationary beam the formula (27) for the complex phase shift becomes

$$P' = -\frac{j}{q_m} = -(1 + j)\frac{z}{R_m}. \tag{65}$$

If this is integrated, one gets

$$P_2 = P_1 - (1 + j)\frac{1}{R_m}. \tag{66}$$

The imaginary part of this phase shift indicates the reduction in gain due to the gain profile. One concludes that the amplitude of the stationary beam grows along the axis like

$$\exp\left(\alpha_0 - \frac{1}{R_m}\right)z. \tag{67}$$

A stationary beam is obtained only when the light is so injected into the medium that the incoming beam matches the stationary beam in diameter and phase-front curvature. If these matching conditions are not satisfied, the beam will generally fluctuate along z in diameter and curvature. To study this situation in more detail one has to evaluate the constant γ given in (33). For the medium with a gain profile one gets

$$\gamma = \sqrt{\frac{k_2}{k_0}} = \frac{j}{q_m} = \frac{1 + j}{R_m}. \tag{68}$$

This is substituted in (34) together with (47) to yield the result

$$\frac{1}{q_2} = \frac{1}{q_m}\,\frac{(q_1 + q_m) - (q_1 - q_m)\exp 2(j - 1)\dfrac{z}{R_m}}{(q_1 + q_m) + (q_1 - q_m)\exp 2(j - 1)\dfrac{z}{R_m}}. \tag{69}$$

The above formula describes the passage of a Gaussian beam through a medium with a gain profile for any input condition. For $q_1 = q_m$ one gets $q_2 = q_m$ which is, of course, exactly what has just been said about matching of the input beam. For very large $z(z \to \infty)$ one also gets $q_2 = q_m$ no matter what the input conditions. This means that the parameters of a beam will first fluctuate more or less around the stationary values w_m and R_m but will finally settle down to equilibrium. All these remarks assume, of course, that the light beam remains a paraxial wave while fluctuating, and that there is no interference from the tube walls or from other obstacles.

Equation (69) can be so cast as to show directly the fluctuations of both the wavefront curvature and the beam diameter. In order to do that one defines two angles δ and φ by

$$\frac{q_1 - q_m}{q_1 + q_m} = \exp 2(\delta - j\varphi). \tag{70}$$

The angles δ and φ are determined by the initial conditions of the beam. If one uses, in addition, the abbreviations

$$\sigma = 2\left(\frac{z}{R_m} - \delta\right),\quad \rho = 2\left(\frac{z}{R_m} - \varphi\right), \tag{71}$$

one can write (69) in the form

$$\frac{1}{q_2} = \frac{1}{q_m}\frac{e^{\sigma} - e^{j\rho}}{e^{\sigma} + e^{j\rho}} = \frac{1}{q_m}\frac{sh\ \sigma - j\sin\rho}{ch\sigma + \cos\rho}. \tag{72}$$

Comparison of the real parts of (72) leads to an expression for the fluctuation of the curvature radius R_2 of the beam phase front

$$\frac{R_2}{R_m} = \frac{ch\ \sigma + \cos\rho}{sh\ \sigma - \sin\rho}. \tag{73}$$

The imaginary parts of (73) indicate the fluctuations of the beam radius w_2

$$\frac{w_2^2}{w_m^2} = \frac{ch\ \sigma + \cos\rho}{sh\ \sigma + \sin\rho}. \tag{74}$$

Electric Circuit Analogs

The equations (15) and (16) for the complex beam parameter have the same form as the well-known formulas of the input impedance of a series circuit of two impedances and of a parallel circuit of two impedances. On this basis Deschamps and Mast[2] have discussed lumped electric circuits which are analogs of systems of lenses. Two dual circuit analogs of a system of lenses are shown in Fig. 3.

The first is an admittance analog. Here the admittance Y of each section is related to the complex beam parameter in the corresponding beam cross section by

$$Y = \frac{j}{q} = \frac{\lambda}{\pi w^2} + \frac{j}{R}. \tag{75}$$

The admittance Y_1 corresponds to the beam parameter q_1 in the input plane. The lens spacing d_1 is represented by a capacitance $C_1 = 1/\omega d_1$ connected in series. The first lens of focal length f_1 is represented by an inductance $L_1 = f_1/\omega$ connected in parallel, and so on. Finally, the admittance Y_2 of the circuit analog is a measure for the beam parameter q_2 in the output plane.

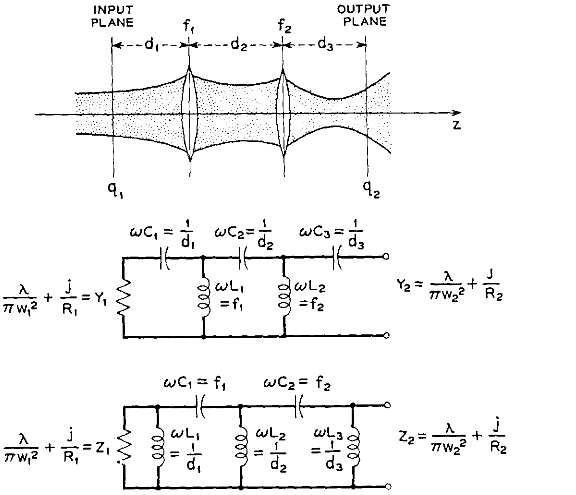

Fig. 3. Gaussian beam propagating through lens system and corresponding circuit analogs.

Below the admittance analog there is shown the dual impedance analog. In the latter all admittances from above are replaced by impedances

$$Z = \frac{j}{q}, \tag{76}$$

series circuits are replaced by parallel circuits, and vice versa.

The analog of a lenslike medium is a distributed electric circuit or transmission line. The voltage V on such a line obeys the differential equation

$$\frac{d}{dz}\left(\frac{1}{Z_L}\frac{dV}{dz}\right) = Y_L V, \tag{77}$$

where Z_L is the line impedance per unit length, and Y_L is the line admittance per unit length. Equation (77) corresponds to the paraxial ray equation (30) if one puts

$$\frac{j}{Z_L} = k_0(z) \tag{78}$$

and

$$Y_L = jk_2(z). \tag{79}$$

The transmission line is analogous to the lenslike medium if one interprets the voltage V as ray position x and relates the line current I to the ray slope by

$$I(z) = \frac{x'}{Z_L}. \tag{80}$$

For media with loss or gain, of course, the analogous transmission lines are lossy or show gain, and the line impedances and admittance are complex-valued.

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