1-D Green's Function

$$\left(\frac{d^2}{dx^2} + k_0^2\right) g(x,x') = -\delta(x-x')$$

$$\therefore \; g(x,x') = \begin{cases} A\, e^{ik_0(x-x')}, & x > x' \\ B\, e^{-ik_0(x-x')}, & x < x' \end{cases}$$

with $\left(\begin{array}{l} \text{i) } g(x,x') : \text{continuous} \\ \text{ii) } \frac{d}{dx} g(x,x') : \text{discontinuous} \end{array} \right)$ at $x = x'$

From i), $\quad g(x'^+, x') = g(x'^-, x')$, $\quad \therefore \; A = B$.

From ii), $\quad \int_{x'^-}^{x'^+} dx \left(\frac{d^2}{dx^2} + k_0^2\right) g(x,x') = -1$

ie $\quad \left.\frac{d g(x,x')}{dx}\right|_{x'^-}^{x'^+} = -1$, $\quad \therefore \; A = \frac{i}{2k_0}$

$$\therefore \; g(x,x') = \begin{cases} \frac{i}{2k_0} e^{ik_0(x-x')}, & x > x' \\ \frac{i}{2k_0} e^{-ik_0(x-x')}, & x < x' \end{cases}$$

$$\Rightarrow \boxed{g(x,x') = \frac{i}{2k_0} e^{ik_0|x-x'|}} \quad ; \text{ 1-D Green's Function}$$

where $\underline{\underline{\mathrm{Im}(k_0) > 0}}$ for the Radiation Condition, $g(\pm\infty, x') = 0$

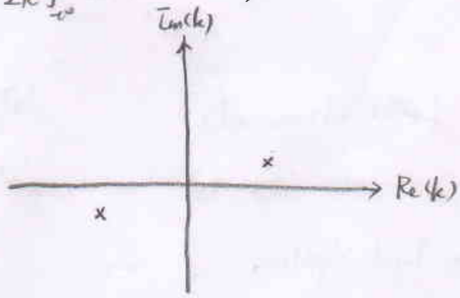
Alternatively, can use F.T.,

Let $\quad g(x,x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, g(k,x')\, e^{ikx}$, then

$$(-k^2 + k_0^2)\, g(k,x') = -e^{ik(x-x')} \quad \left(\text{use } \delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(x-x')}\right)$$

$$\therefore \; g(k,x') = \frac{e^{ik(x-x')}}{k_0^2 - k^2}$$

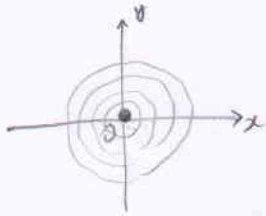
$$\Rightarrow \boxed{g(x,x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \, \frac{e^{ik(x-x')}}{k_0^2 - k^2}}$$



2-D Green's Function

$$(\nabla_t^2 + k_0^2)\, g(\mathbf{r}_t, \mathbf{r}_t') = -\delta(\mathbf{r}_t - \mathbf{r}_t')$$

A) "Source at $\mathbf{r}_t' = 0$



$$(\nabla_t^2 + k_0^2)\, g(\mathbf{r}_t) = -\delta(\mathbf{r}_t)$$

At $\mathbf{r}_t \neq 0$, $g(\mathbf{r}_t) \propto (J_n, N_n, H_n^{(1)}, H_n^{(2)})\, e^{in\phi}$

Consider (i) Outgoing Wave
(ii) Cylindrical Symmetry : $g(r_t, \phi) = g(r_t, \phi + \phi')$ for arbitrary ϕ'

$\Rightarrow$ $g(\mathbf{r}_t) = C\, H_0^{(1)}(k_0 r_t)$

To find C, take $\int d\mathbf{r}_t$ around $r_t = 0$,

$$\lim_{r_t \to 0} \left[\int_0^{2\pi} d\phi\, 2\pi r_t \frac{d}{dr_t}\left(C H_0^{(1)}(k_0 r_t)\right) + \int_0^{2\pi} d\phi\, \pi r_t^2 k_0^2\, C H_0^{(1)}(k_0 r_t) \right] = -1$$

$\Rightarrow$ $C = \dfrac{i}{4}$

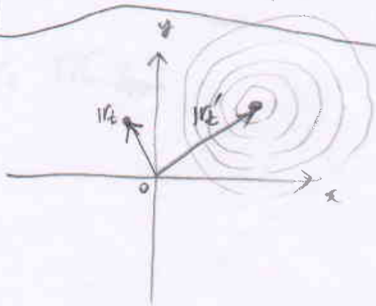
$\Rightarrow$ $\boxed{g(\mathbf{r}_t) = \dfrac{i}{4} H_0^{(1)}(k_0 r_t)}$

Addition Theorem

$*$ $H_0^{(1,2)}(k_0 |\mathbf{r}_t - \mathbf{r}_t'|) = \sum_{n=-\infty}^{\infty} J_n(k_0 r_<)\, H_n^{(1,2)}(k_0 r_>) \times e^{in(\phi - \phi')}$

$J_0(k_0 |\mathbf{r}_t - \mathbf{r}_t'|) = \sum_{n=-\infty}^{\infty} J_n(r_t)\, J_n(r_t') \times e^{in(\phi-\phi')}$

B) Source at $\mathbf{r}_t' \neq 0$



$$\boxed{g(\mathbf{r}_t, \mathbf{r}_t') = \frac{i}{4} H_0^{(1)}(k_0 |\mathbf{r}_t - \mathbf{r}_t'|)}$$

Since $g(\mathbf{r}_t, \mathbf{r}_t')$ is periodic,

$$g(\mathbf{r}_t, \mathbf{r}_t') = \begin{cases} \sum_n f_n(k_0 r_t')\, J_n(k_0 r_t)\, e^{in(\phi-\phi')}, & r_t < r_t' \\ \sum_n g_n(k_0 r_t')\, H_n^{(1)}(k_0 r_t)\, e^{in(\phi-\phi')}, & r_t > r_t' \end{cases}$$

why consider this!
$\Rightarrow$ can use multiple local coordinate

Since $g(\mathbf{r}_t, \mathbf{r}_t')$ is continuous at $r_t = r_t'$,

$$f_n(k_0 r_t')\, J_n(k_0 r_t') = g_n(k_0 r_t')\, H_n^{(1)}(k_0 r_t')$$

$$\Rightarrow \begin{cases} f_n(k_0 r_t') \propto H_n^{(1)}(k_0 r_t') \\ g_n(k_0 r_t') \propto J_n(k_0 r_t') \end{cases}$$

$$\boxed{g(\mathbf{r}_t, \mathbf{r}_t') = \sum_n J_n(k_0 r_<)\, H_n^{(1)}(k_0 r_>) \times e^{in(\phi-\phi')}}$$

$$\Rightarrow g(\mathbf{r}_t, \mathbf{r}_t') = \begin{cases} \sum_n A_n H_n^{(1)}(k_0 r_t')\, J_n(k_0 r_t)\, e^{in(\phi-\phi')}, & r_t < r_t' \\ \sum_n A_n J_n(k_0 r_t')\, H_n^{(1)}(k_0 r_t)\, e^{in(\phi-\phi')}, & r_t > r_t' \end{cases}$$

$A_n = \dfrac{i}{4}$

3-D Green's Function

$$(\nabla^2 + k_0^2)\, g(\mathbf{r}) = -\delta(\mathbf{r})$$

At $\mathbf{r} \neq 0$,

$$(\nabla^2 + k_0^2)\, g(\mathbf{r}) = 0$$

Due to the spherical symmetry,

$$g(\mathbf{r}) = C \frac{e^{ikr}}{r}$$

To find C, take $\int d\mathbf{r}^3$, around $\mathbf{r} = 0$.

$$\lim_{r \to 0} \left[\int \underbrace{d\mathbf{a}}_{4\pi r^2} \cdot \nabla \left(C \frac{e^{ikr}}{r} \right) + \cancel{\int \underbrace{d\mathbf{r}^3}_{dr 4\pi r^2} k_0^2 \, C \frac{e^{ikr}}{r}}^{\,0} \right] = -1$$

$$\therefore \lim_{r \to 0} 4\pi r^2 \cdot C \frac{(\cancel{ikr}^{\,0} - 1)\, \cancel{e^{ikr}}^{\,1}}{r^2} = -1$$

$$\therefore C = \frac{1}{4\pi}$$

$$\Rightarrow \quad g(\mathbf{r}) = \frac{1}{4\pi} \frac{e^{ikr}}{r}$$

$$\Rightarrow \quad \boxed{g(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi} \frac{e^{ik|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|}} \quad : \text{3-D Green's Function}$$

Ch. 2 Green's Function

< " Spatial and Temporal Impulse Response Function"

In a Homogeneous medium, $\Rightarrow$ Superposition Principle !.

$$\begin{cases} \nabla \times \mathbb{E} = i\omega\mu \mathbb{H} \\ \nabla \times \mathbb{H} = -i\omega\epsilon \mathbb{E} + \mathbb{J} \end{cases}$$

$$\Rightarrow \boxed{\nabla \times \nabla \times \mathbb{E} - k^2 \mathbb{E} = i\omega\mu \mathbb{J}}$$

$$\nabla \times \nabla \times \mathbb{E} = \nabla \nabla \cdot \mathbb{E} - \nabla^2 \mathbb{E}$$

$$= \frac{1}{\epsilon} \nabla \rho - \nabla^2 \mathbb{E}$$

From $\nabla \cdot \mathbb{J} + \frac{\partial \rho}{\partial t} = 0$, ie $\nabla \cdot \mathbb{J} = i\omega\rho$

$$\nabla \times \nabla \times \mathbb{E} = \frac{1}{i\omega\epsilon} \nabla \nabla \cdot \mathbb{J} - \nabla^2 \mathbb{E}$$

$$\therefore \nabla^2 \mathbb{E} + k^2 \mathbb{E} = \frac{1}{i\omega\epsilon} \nabla \nabla \cdot \mathbb{J} - i\omega\mu \mathbb{J}$$

$$= -i\omega\mu \left[\frac{1}{\omega^2 \epsilon\mu} \nabla \nabla \cdot \mathbb{J} + \mathbb{J} \right]$$

$$\Rightarrow \boxed{\nabla^2 \mathbb{E} + k^2 \mathbb{E} = -i\omega\mu \left[\frac{1}{k^2} \nabla \nabla + \mathbb{I} \right] \cdot \mathbb{J}}$$

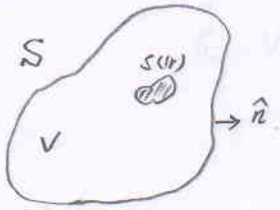
where $\mathbb{I}$: unit dyad (3 x 3 unit diagonal matrix)

$\Rightarrow$ Can define Vector Green's Function ?

2.1 Scalar Green's Function

In a Bound Homogeneous medium,

$$(\nabla^2 + k^2)\,\psi(\mathbf{r}) = -s(\mathbf{r}) \quad \text{in } V.$$



Define the scalar Green's Function,

$$(\nabla^2 + k^2)\,g(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}') \quad \text{in } V.$$

$$g(\mathbf{r},\mathbf{r}')\,\nabla^2\psi(\mathbf{r}) - \psi(\mathbf{r})\,\nabla^2 g(\mathbf{r},\mathbf{r}') = -g(\mathbf{r},\mathbf{r}')\,s(\mathbf{r}) + \psi(\mathbf{r})\,\delta(\mathbf{r}-\mathbf{r}')$$

Taking $\int_V dv$, and using Green's theorem,

$$\int_S d^2r\;\hat{n}\cdot[\,g(\mathbf{r},\mathbf{r}')\,\nabla\psi(\mathbf{r}) - \psi(\mathbf{r})\,\nabla g(\mathbf{r},\mathbf{r}')\,] = \int_V d^3r\;g(\mathbf{r},\mathbf{r}')\,s(\mathbf{r}) + \psi(\mathbf{r}')$$

From the Reciprocity therom, $g(\mathbf{r},\mathbf{r}') = g(\mathbf{r}',\mathbf{r})$

$$\Rightarrow \boxed{\psi(\mathbf{r}) = \underbrace{\int_V d^3r'\;g(\mathbf{r},\mathbf{r}')\,s(\mathbf{r}')}_{\text{particular solution}} + \underbrace{\int_S d^2r'\left[g(\mathbf{r},\mathbf{r}')\frac{\partial}{\partial n}\psi(\mathbf{r}') - \psi(\mathbf{r}')\frac{\partial}{\partial n}g(\mathbf{r},\mathbf{r}')\right]}_{\text{Homogeneous Solution}}}$$

with B.C's for $\psi(\mathbf{r})$

and for $g(\mathbf{r},\mathbf{r}')$
$$\left[\begin{array}{ll} g(\mathbf{r},\mathbf{r}') = 0 & ;\ \text{Dirichlet's B.C} \\ \frac{\partial}{\partial n}g(\mathbf{r},\mathbf{r}') = 0 & ;\ \text{Neumann's B.C} \\ g(\mathbf{r},\mathbf{r}') + f(\mathbf{r})\frac{\partial}{\partial n}g(\mathbf{r},\mathbf{r}') = 0 & ;\ \text{Mixed B.C} \end{array}\right.$$

For Unbound Homogeneous medium, $V \to \infty$

$\Rightarrow$ Radiation Condition $\lim_{r\to\infty}\left(r\frac{\partial\psi}{\partial r} + jk\psi\right) = 0$, ie $\left[\begin{array}{l} \lim_{r\to\infty}\psi = 0 \\ \lim_{r\to\infty} r\frac{\partial\psi}{\partial r} = 0 \end{array}\right.$

$$\Rightarrow \boxed{\psi(\mathbf{r}) = \int_V d^3r'\;g(\mathbf{r},\mathbf{r}')\,s(\mathbf{r}')} \quad \text{for } V\to\infty.$$

<cf> $(\nabla^2 + k^2)\,\psi(\mathbf{r}) = -S(\mathbf{r})$

$$= -\int d^3\mathbf{r}'\; S(\mathbf{r}')\,\delta(\mathbf{r}-\mathbf{r}')$$

$S_i\,,\ V_i$

$\Rightarrow$ Superposition !

$\Rightarrow$ To find $g(\mathbf{r},\mathbf{r}')$, start from Homogeneous scalar wave eq.

i.e. $(\nabla^2 + k^2)\,g(\mathbf{r},\mathbf{r}') = 0$ at $\mathbf{r} \neq \mathbf{r}'$

Homogeneous Scalar Wave Equations

i) Cartesian Coordinates

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\psi(\mathbf{r}) = 0.$$

$$\Rightarrow \psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}$$

where $\begin{cases} \mathbf{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z} \\ k^2 = k_x^2 + k_y^2 + k_z^2 \end{cases}$

ii) Cylindrical Coordinates

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\psi(\mathbf{r}) = 0$$

Use the sepation of variables, $\psi(\mathbf{r}) = F_n(\rho)\,e^{in\phi}\,e^{ik_z z}$ with intergen n.

$$\Rightarrow \left(\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} - \frac{n^2}{\rho^2} + k_\rho^2\right)F_n(\rho) = 0$$

where $k_\rho^2 = k^2 - k_z^2$

$$\Rightarrow F_n(\rho) = \begin{cases} A\,J_n(k_\rho\rho) + B\,N_n(k_\rho\rho) & ; \text{ standing Wave} \\ A\,H_n^{(1)}(k_\rho\rho) + B\,H_n^{(2)}(k_\rho\rho) & ; \text{ Traveling Wave} \\ A\,I_n(\alpha_n\rho) + B\,K_n(\alpha_n\rho) & ; \text{ Evanescent Wave} \end{cases}$$

iii) Spherical Coordinate

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}+k^2\right]\psi(r)=0$$

Let $\psi(r) = b_n(kr)\, P_n^m(\cos\theta)\, e^{im\phi}$

$$\begin{cases}\left\{\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}+\left[n(n+1)-\frac{m^2}{\sin^2\theta}\right]\right\}P_n^m(\cos\theta)=0\\ \left[\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}+k^2-\frac{n(n+1)}{r^2}\right]b_n(kr)=0\end{cases}$$

$\Rightarrow$ $b_n(kr) = \sqrt{\frac{\pi}{2kr}}\, B_{n+\frac{1}{2}}(kr)$; Spherical Bessel Function

$P_n^m(\cos\theta)$; Associate Legendre Polynomial

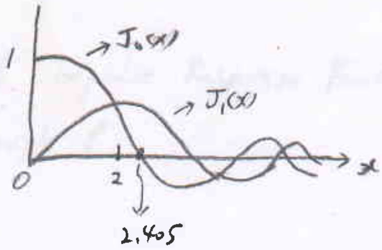
$$\begin{cases} j_0(kr)=\frac{\sin kr}{kr}\\ n_0(kr)=-\frac{\cos kr}{kr}\end{cases}\qquad \begin{cases} j_1(kr)=-\frac{\cos kr}{kr}+\frac{\sin kr}{(kr)^2}\\ n_1(kr)=-\frac{\sin kr}{kr}-\frac{\cos kr}{kr^2}\end{cases}$$

; Spherical Bessel, Neumann Functions

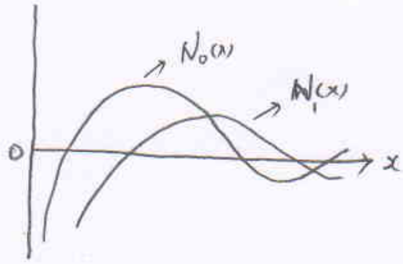
$$\begin{cases} h_0^{(1)}(kr)=\frac{e^{ikr}}{ikr}\\ h_0^{(2)}(kr)=-\frac{e^{-ikr}}{ikr}\end{cases}\qquad \begin{cases} h_1^{(1)}(kr)=-\left(1+\frac{i}{kr}\right)\frac{e^{ikr}}{kr}\\ h_1^{(2)}(kr)=-\left(1-\frac{i}{kr}\right)\frac{e^{-ikr}}{kr}\end{cases}$$

; Spherical Hankel Function

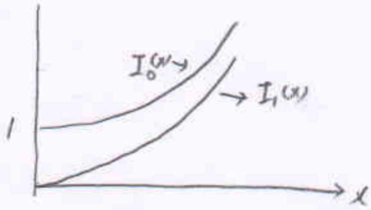
where
$$\begin{cases} H_n^{(1)}(k_\rho \rho) = J_n(k_\rho \rho) + i N_n(k_\rho \rho) \\ H_n^{(2)}(k_\rho \rho) = J_n(k_\rho \rho) - i N_n(k_\rho \rho) \end{cases}$$



$$\begin{cases} J_n(k_\rho \rho) = \frac{1}{2}\left[H_n^{(1)}(k_\rho \rho) + H_n^{(2)}(k_\rho \rho) \right] \\ N_n(k_\rho \rho) = \frac{1}{2i}\left[H_n^{(1)}(k_\rho \rho) - H_n^{(2)}(k_\rho \rho) \right] \end{cases}$$

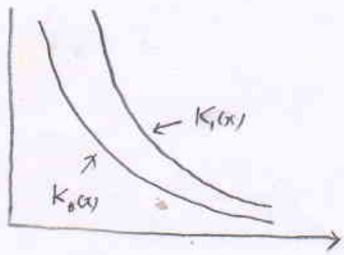


$$\begin{cases} I_n(\alpha_\rho \rho) = i^{-n} J_n(i \alpha_\rho \rho) \\ K_n(\alpha_\rho \rho) = \frac{\pi}{2} i^{n+1} H_n^{(1)}(i \alpha_\rho \rho) \end{cases}$$



$$\begin{aligned} B_n'(k_\rho \rho) &= B_{n-1}(k_\rho \rho) - \frac{n}{k_\rho \rho} B_n(k_\rho \rho) \\ &= -B_{n+1}(k_\rho \rho) + \frac{n}{k_\rho \rho} B_n(k_\rho \rho) \end{aligned}$$

→ Recurrence Relation



For $k_\rho \rho \to 0$,

$$\begin{cases} J_0(k_\rho \rho) \sim 1 \\ N_0(k_\rho \rho) \sim \frac{2}{\pi} \ln(k_\rho \rho) \end{cases} \qquad \begin{cases} H_0^{(1)}(k_\rho \rho) \sim \frac{2i}{\pi} \ln(k_\rho \rho) \\ H_0^{(2)}(k_\rho \rho) \sim -\frac{2i}{\pi} \ln(k_\rho \rho) \end{cases}$$

$$\begin{cases} J_n(k_\rho \rho) \sim \frac{1}{n!}\left(\frac{k_\rho \rho}{2}\right)^n \\ N_n(k_\rho \rho) \sim -\frac{(n-1)!}{\pi}\left(\frac{2}{k_\rho \rho}\right)^n \end{cases} \qquad \begin{aligned} H_n^{(1)}(k_\rho \rho) &\sim -\frac{i(n-1)!}{\pi}\left(\frac{2}{k_\rho \rho}\right)^n \\ H_n^{(2)}(k_\rho \rho) &\sim \frac{i(n-1)!}{\pi}\left(\frac{2}{k_\rho \rho}\right)^n \end{aligned}$$

For $k_\rho \rho \to \infty$

$$J_n(k_\rho \rho) \sim \sqrt{\frac{2}{\pi k_\rho \rho}} \cos\left(k_\rho \rho - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

$$N_n(k_\rho \rho) \sim \sqrt{\frac{2}{\pi k_\rho \rho}} \sin\left(k_\rho \rho - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

$$H_n^{(1)}(k_\rho \rho) \sim \sqrt{\frac{2}{\pi k_\rho \rho}}\, e^{i\left(k_\rho \rho - \frac{n\pi}{2} - \frac{\pi}{4}\right)}$$

$$H_n^{(2)}(k_\rho \rho) \sim \sqrt{\frac{2}{\pi k_\rho \rho}}\, e^{-i\left(k_\rho \rho - \frac{n\pi}{2} - \frac{\pi}{4}\right)}$$