Spring 2019

# EECE 588
# Lecture 6

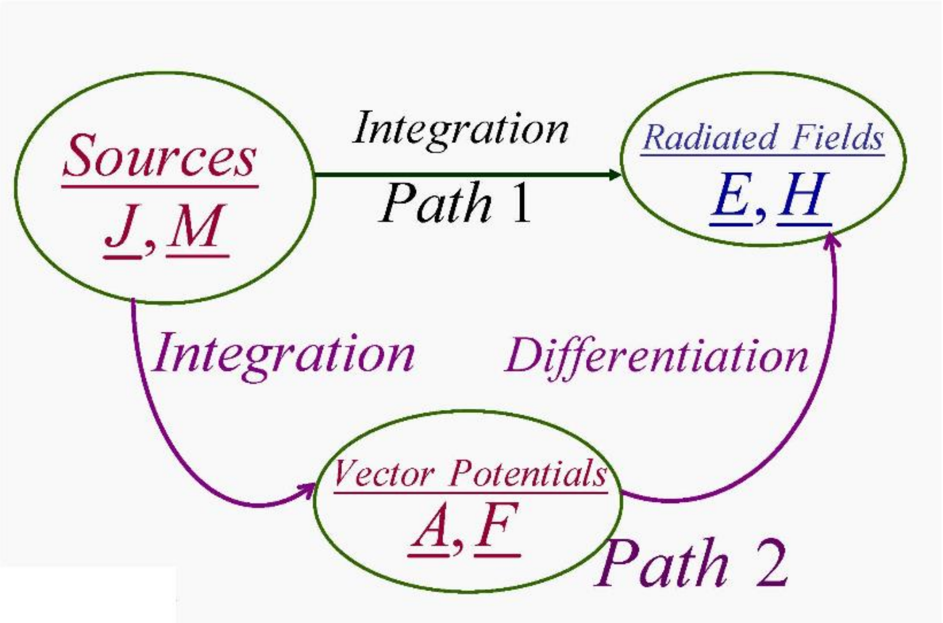
Prof. Wonbin Hong

# Maxwell's Equations

- In this chapter we solve the Maxwell's equations for given source terms.
- Essentially, we have a spatial distribution of an electric current **J**(**r**) and magnetic current **M**(**r**) and would like to find the electric and magnetic fields radiated by this field distribution (**E**, **H**).
- To do this, we define auxiliary potential functions:
  - These are the Vector Electric Potential **F** and the Vector magnetic Potential **M**.
  - From these auxiliary potentials, we can calculate **E** and **H**.

# Auxiliary Potentials

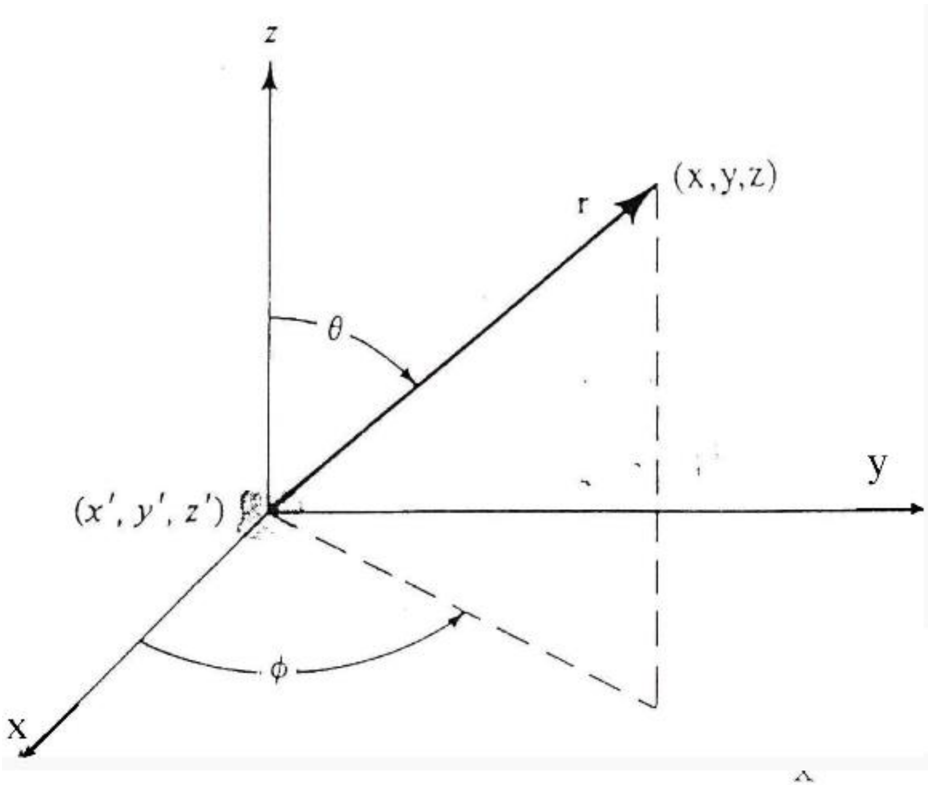
- Note that we can also calculate the **E** and **H** directly from **J** and **M**.
- So why bother with **F** and **A**?
  - Because the integrations involved are much simpler.
  - If you go through path 1, you can find **E** and **H** but it's going to be really tough.

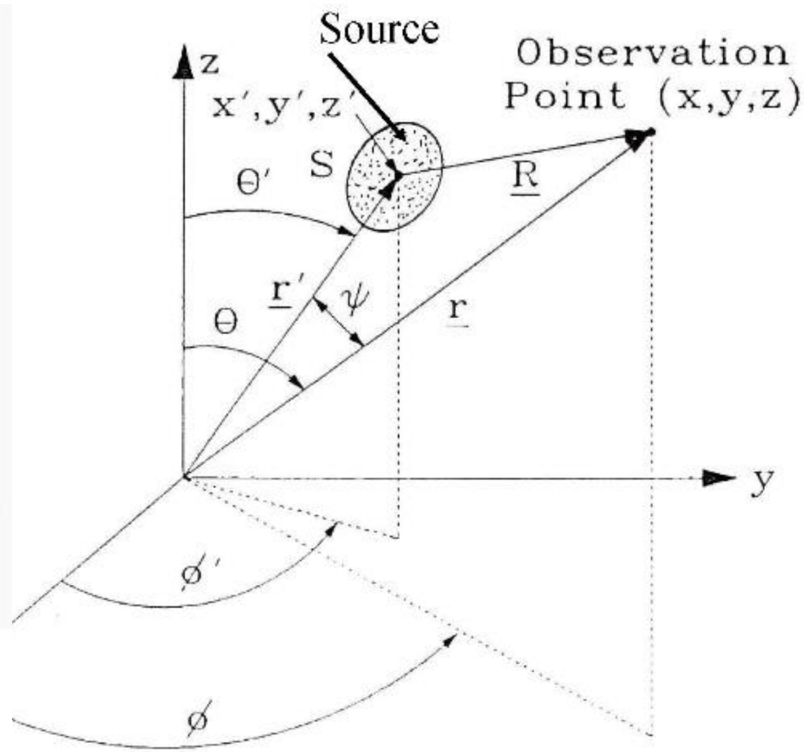



- If you go through the second path though, the integrations are much simpler and even though an extra differentiation step is involved, the overall process is much easier.

POSTECH

# Problem Setup



# Vector Magnetic Potential

- Remember that, magnetic field **B** is always solenoidal:

$$\nabla . \vec{B} = 0$$

- Therefore, **B** can be represented as the curl of another vector.
- Remember that the divergence of curl of any vector is zero:

$$\nabla . (\nabla \times \vec{A}) = 0 \Rightarrow \vec{B}_A = \mu \vec{H}_A = \nabla \times \vec{A}$$

$$\vec{H}_A = \frac{1}{\mu} \nabla \times \vec{A}$$

- Note that the subscript A denotes the field due to **A**.



POSTECH

# Field Calculations from Vector Magnetic Potential

- From the time harmonic Maxwell's equations we know:

$$\nabla \times \vec{E}_A = -j\omega\mu\vec{H}_A$$

$$\nabla \times \left(\vec{E}_A + j\omega\vec{A}\right) = 0$$

- Remember that:

$$\nabla \times \left(-\nabla \varphi_e\right) = 0$$

- Therefore:

$$\vec{E}_A + j\omega\vec{A} = -\nabla\varphi_e$$

$$\vec{E}_A = -j\omega\vec{A} - \nabla\varphi_e$$

POSTECH

# Field Calculations from Vector Magnetic Potential

- The function $\varphi_e$ represents an arbitrary electric scalar potential which is a function of position.
- Let us start from a well known identity:

$$\nabla\times\nabla\times\vec{A}=\nabla\left(\nabla.\vec{A}\right)-\nabla^2\vec{A}$$

$$\nabla\times\left(\mu\vec{H}_A\right)=\nabla\left(\nabla.\vec{A}\right)-\nabla^2\vec{A}\quad\Rightarrow\quad\mu\nabla\times\vec{H}_A=\nabla\left(\nabla.\vec{A}\right)-\nabla^2\vec{A}$$

- From Maxwell's equation, we know:

$$\nabla\times\vec{H}_A=\vec{J}+j\omega\varepsilon\vec{E}_A$$

- Therefore:

$$\mu\vec{J}+j\omega\mu\varepsilon\vec{E}_A=\nabla\left(\nabla.\vec{A}\right)-\nabla^2\vec{A}$$

$$\boxed{\vec{E}_A=-j\omega\vec{A}-\nabla\varphi_e}$$

POSTECH

# Field Calculations from Vector Magnetic Potential

$$\nabla^2\vec{A}+k^2\vec{A}=-\mu\vec{J}+\nabla\left(\nabla.\vec{A}\right)+\nabla(j\omega\mu\varepsilon\varphi_e)$$
$$=-\mu\vec{J}+\nabla(\nabla.\vec{A}+j\omega\mu\varepsilon\varphi_e)$$

- Note that we have defined the curl of vector **A** and now, we have the liberty to define its divergence (which is independent of its Curl). Remember the Helmholtz's Theorem in Vector Calculus.

$$\nabla.\vec{A}=-j\omega\mu\varepsilon\varphi_e \rightarrow \varphi_e=-\frac{1}{j\omega\mu\varepsilon}\nabla.\vec{A}$$

- This is known as the Lorentz condition or Lorentz Gauge:

$$\nabla^2\vec{A}+k^2\vec{A}=-\mu\vec{J}$$

# Side Note: Helmholtz's Theorem

- A vector field (vector point function) is determined to within an additive constant if both its divergence and its curl are specified everywhere.

POSTECH

# Calculating A from J and E and H from A

$$\boxed{\nabla^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J}}$$

$$\boxed{\vec{E}_A = -\nabla \varphi_e - j\omega \vec{A} = -j\omega \vec{A} - j\frac{1}{\omega\mu\varepsilon}\nabla\left(\nabla . \vec{A}\right)}$$

$$\boxed{\vec{H}_A = \frac{1}{\mu}\nabla \times \vec{A}}$$

POSTECH

# Vector Electric Potential

- Although magnetic currents do not exist equivalent magnetic currents can be used in certain problems.
- The fields generated by a harmonic current in a homogeneous region with **J**=0 and **M**≠0.

$$\nabla.\vec{D} = 0 \Rightarrow \vec{E}_F = -\frac{1}{\varepsilon}\nabla\times\vec{F}$$

$$\nabla\times\vec{H}_F = j\omega\varepsilon\vec{E}_F \Rightarrow \nabla\times\left(\vec{H}_F + j\omega\vec{F}\right) = 0$$

$$\boxed{\vec{H}_F = -\nabla\varphi_m - j\omega\vec{F}}$$

$\Phi_m$: arbitrary magnetic scalar potential.

# Vector Electric Potential

- Let's start by taking the curl of the equation.

$$\nabla \times \vec{E}_F = -\frac{1}{\varepsilon}\nabla \times \nabla \times \vec{F} = -\frac{1}{\varepsilon}\left[\nabla\nabla.\vec{F} - \nabla^2\vec{F}\right]$$

- From Maxwell's equations, we have:

$$\nabla \times \vec{E}_F = -\vec{M} - j\omega\mu\vec{H}_F$$

- Therefore:

$$\nabla^2\vec{F} + j\omega\mu\varepsilon\vec{H}_F = \nabla\nabla.\vec{F} - \varepsilon\vec{M}$$

$$\nabla^2\vec{F} + k^2\vec{F} = -\varepsilon\vec{M} + \nabla\left(\nabla.\vec{F}\right) + \nabla\left(j\omega\mu\varepsilon\varphi_m\right)$$

**We Choose** $$\nabla.\vec{F} = -j\omega\mu\varepsilon\varphi_m \rightarrow \varphi_m = -\frac{1}{j\omega\mu\varepsilon}\nabla.\vec{F}$$

# Calculating EM fields due to a magnetic current

$$\nabla^2 \vec{F} + k^2 \vec{F} = -\varepsilon \vec{M}$$

$$\vec{H}_F = -j\omega\vec{F} - \frac{j}{\omega\mu\varepsilon}\nabla(\nabla.\vec{F})$$

$$\vec{E}_F = -\frac{1}{\varepsilon}\nabla\times\vec{F}$$

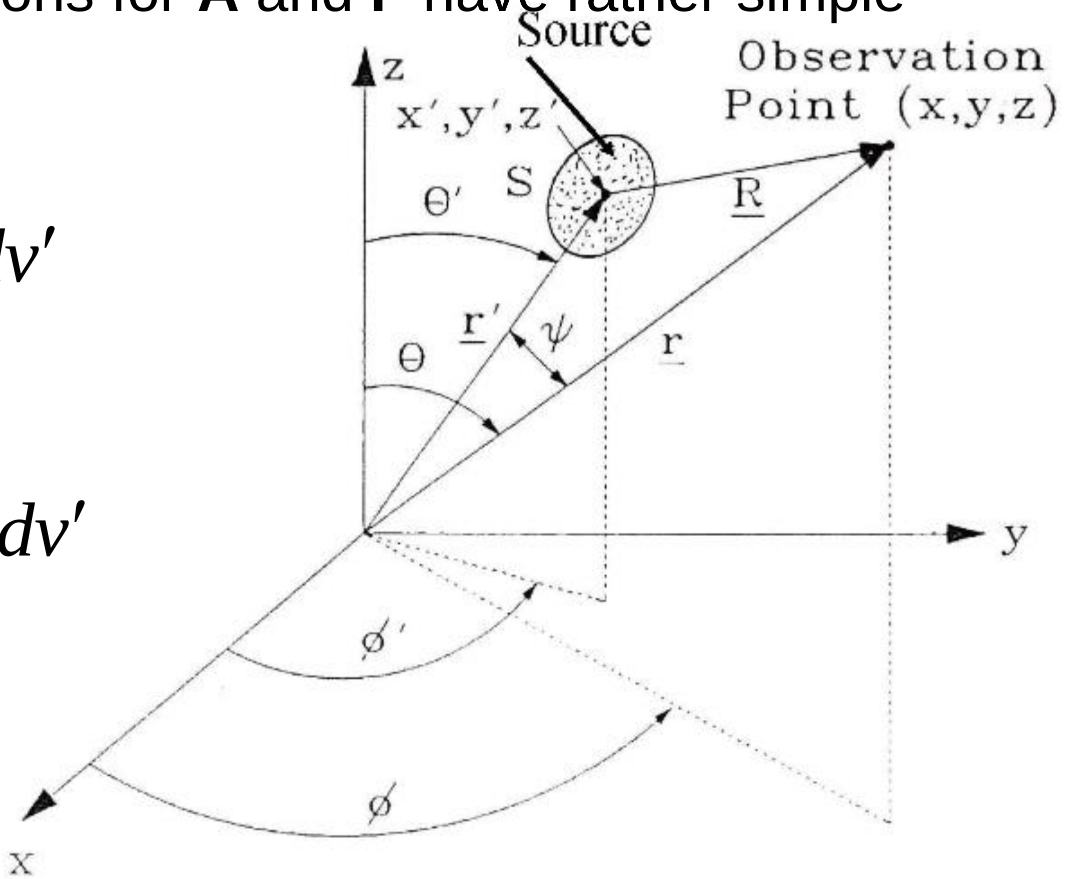
POSTECH

# Solution of Wave Equations Using Auxiliary Potential

- Now, it is time to solve these differential equations.
- The two differential equations for **A** and **F** have rather simple solutions.

$$\vec{A} = \frac{\mu}{4\pi} \iiint_V \vec{J} \frac{e^{-jk|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} dv'$$

$$\vec{F} = \frac{\varepsilon}{4\pi} \iiint_V \vec{M} \frac{e^{-jk|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} dv'$$



# Solution of E & H based on A and F

$$\vec{E} = \vec{E}_A + \vec{E}_F = -j\omega\vec{A} - \frac{j}{\omega\mu\varepsilon}\nabla\left(\nabla.\vec{A}\right) - \frac{1}{\varepsilon}\nabla\times\vec{F}$$

$$\vec{E} = \vec{E}_A + \vec{E}_F = \frac{1}{j\omega\varepsilon}\nabla\times\vec{H}_A - \frac{1}{\varepsilon}\nabla\times\vec{F}$$

$$\vec{H} = \vec{H}_A + \vec{H}_F = \frac{1}{\mu}\nabla\times\vec{A} - j\omega\vec{F} - \frac{j}{\omega\mu\varepsilon}\nabla(\nabla.\vec{F})$$

$$\vec{H} = \vec{H}_A + \vec{H}_F = \frac{1}{\mu}\nabla\times\vec{A} - \frac{1}{j\omega\mu}\nabla\times\vec{E}_F$$

# Far Field Radiation

- The fields radiated by antennas of finite dimensions have spherical wave fronts.
- In such cases, we can represent **A** as follows:

$$\vec{A} = \hat{r}A_r(r,\theta,\phi) + \hat{\theta}A_\theta(r,\theta,\phi) + \hat{\phi}A_\varphi(r,\theta,\phi)$$

- The amplitude variations of $r$ in each component of the above equation takes the form of $1/r^n$.
- We ignore the higher order powers $n > 1$.

$$\vec{A} \approx \left\{\hat{r}A_r'(\theta,\phi) + \hat{\theta}A_\theta'(\theta,\phi) + \hat{\phi}A_\varphi'(\theta,\phi)\right\}\frac{e^{-jkr}}{r}, r \to \infty$$

POSTECH

# Far Field Approximation

- The $r$ variations are separable from those of $\theta$ and $\phi$.

$$\vec{E}_A = -j\omega\vec{A} - \frac{j}{\omega\mu\varepsilon}\nabla\left(\nabla.\vec{A}\right)$$

- Substituting the expression for $\vec{A}$ into the equation above,

$$\vec{E} = \frac{1}{r}\left\{-j\omega e^{-jkr}[\hat{r}(0) + \hat{\vartheta}A'_\theta(\theta,\varphi) + \hat{\varphi}A'_\varphi(\theta,\varphi)]\right\} + \frac{1}{r^2}\{\ldots\} + \ldots$$

- Similarly,

$$\vec{H} = \frac{1}{r}\left\{j\frac{\omega}{\eta}e^{-jkr}[\hat{r}(0) + \hat{\vartheta}A'_\varphi(\theta,\varphi) - \hat{\varphi}A'_\theta(\theta,\varphi)]\right\} + \frac{1}{r^2}\{\ldots\} + \ldots$$

$$\eta = \sqrt{\mu/\varepsilon}$$

# Far Field Radiation due to J

- Neglecting higher order terms of $1/r^n$ the radiated **E** and **H** fields have only $\theta$ and $\phi$ components.

$$\left.\begin{aligned} E_r &\approx 0 \\ E_\theta &\approx -j\omega A_\theta \\ E_\phi &\approx -j\omega A_\phi \end{aligned}\right] \to \vec{E}_A \approx -j\omega \vec{A}$$

$$\left.\begin{aligned} H_r &\approx 0 \\ H_\theta &\approx j\omega/\eta A_\phi = -E_\phi/\eta \\ H_\phi &\approx -j\omega/\eta\ A_\theta = +E_\theta/\eta \end{aligned}\right\} \to \vec{H}_A \approx -\frac{\hat{r}}{\eta} \times \vec{E}_A = -j\frac{\omega}{\eta}\hat{r} \times \vec{A}$$

# Far Field Radiation due to M

- Neglecting higher order terms of $1/r^n$ the radiated $\vec{E}$ and $\vec{H}$ fields, due to F, have only $\theta$ and $\phi$ components:

$$\left.\begin{aligned} H_r &\approx 0 \\ H_\theta &\approx -j\omega F_\theta \\ H_\phi &\approx -j\omega F_\phi \end{aligned}\right\} \rightarrow \vec{H}_F \approx -j\omega \vec{F}$$

$$\left.\begin{aligned} E_r &\approx 0 \\ E_\theta &\approx -j\omega\eta F_\phi = \eta H_\phi \\ E_\phi &\approx +j\omega\eta F_\theta = -\eta H_\theta \end{aligned}\right\} \rightarrow \vec{E}_F \approx -\eta \hat{r} \times \vec{H}_F = j\omega\eta \hat{r} \times \vec{F}$$

# Duality Theorem

- If two equations that describe the behavior of two different variables are of the same mathematical form their solutions will also be identical.
- The variables in the two equations that occupy identical positions are known as dual quantities and a solution of one can be formed by a systematic interchange of symbols to the other.
- This concept is known as Duality Theorem.

# Duality Theorem

Table 3.1 DUAL EQUATIONS FOR ELECTRIC (J) AND MAGNETIC (M) CURRENT SOURCES

| Electric Sources ($\mathbf{J} \neq 0, \mathbf{M} = 0$) | Magnetic Sources ($\mathbf{J} = 0, \mathbf{M} \neq 0$) |
|---|---|
| $\nabla \times \mathbf{E}_A = -j\omega\mu\mathbf{H}_A$ | $\nabla \times \mathbf{H}_F = j\omega\epsilon\mathbf{E}_F$ |
| $\nabla \times \mathbf{H}_A = \mathbf{J} + j\omega\epsilon\mathbf{E}_A$ | $-\nabla \times \mathbf{E}_F = \mathbf{M} + j\omega\mu\mathbf{H}_F$ |
| $\nabla^2\mathbf{A} + k^2\mathbf{A} = -\mu\mathbf{J}$ | $\nabla^2\mathbf{F} + k^2\mathbf{F} = -\epsilon\mathbf{M}$ |
| $\mathbf{A} = \frac{\mu}{4\pi}\iiint_V \mathbf{J}\frac{e^{-jkR}}{R}\,dv'$ | $\mathbf{F} = \frac{\epsilon}{4\pi}\iiint_V \mathbf{M}\frac{e^{-jkR}}{R}\,dv'$ |
| $\mathbf{H}_A = \frac{1}{\mu}\nabla \times \mathbf{A}$ | $\mathbf{E}_F = -\frac{1}{\epsilon}\nabla \times \mathbf{F}$ |
| $\mathbf{E}_A = -j\omega\mathbf{A} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla \cdot \mathbf{A})$ | $\mathbf{H}_F = -j\omega\mathbf{F} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla \cdot \mathbf{F})$ |

# Duality Theorem

Table 3.2 DUAL QUANTITIES FOR ELECTRIC (J) AND MAGNETIC (M) CURRENT SOURCES

| Electric Sources ($\mathbf{J} \neq 0, \mathbf{M} = 0$) | Magnetic Sources ($\mathbf{J} = 0, \mathbf{M} \neq 0$) |
|---|---|
| $\mathbf{E}_A$ | $\mathbf{H}_F$ |
| $\mathbf{H}_A$ | $-\mathbf{E}_F$ |
| $\mathbf{J}$ | $\mathbf{M}$ |
| $\mathbf{A}$ | $\mathbf{F}$ |
| $\epsilon$ | $\mu$ |
| $\mu$ | $\epsilon$ |
| $k$ | $k$ |
| $\eta$ | $1/\eta$ |
| $1/\eta$ | $\eta$ |

# RECIPROCITY AND REACTION THEOREMS

- Reciprocity theorem from our circuit courses:
  - In any network composed of linear bilateral lumped elements, if one places a constant current (voltage) generator between two nodes (in any branch) and places a voltage (current) meter between any other two nodes (in any other branch), makes observation of the meter reading, then interchanges the locations of the source and the meter, the meter reading will be unchanged.
- Reciprocity theorem does also apply to EM theory.

# Reciprocity

- Let's assume that within a linear and isotropic medium, but not necessarily homogeneous, we have:
  - $\mathbf{J_1}$, $\mathbf{M_1}$ and $\mathbf{J_2}$ and $\mathbf{M_2}$.
  - These currents radiate simultaneously or individually and at the same frequency.
  - These sources and fields will satisfy

$$-\nabla.\left(\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1\right) = \vec{E}_1.\vec{J}_2 + \vec{H}_2.\vec{M}_1 - \vec{E}_2.\vec{J}_1 - \vec{H}_1.\vec{M}_2$$

- Taking the volume integral of both sides of this equation:

$$-\oiint_S (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1).ds' = \oiiint_V \left(\vec{E}_1.\vec{J}_2 + \vec{H}_2.\vec{M}_1 - \vec{E}_2.\vec{J}_1 - \vec{H}_1.\vec{M}_2\right)dv'$$

# Reciprocity

- For source free regions:

$$\nabla.\left(\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1\right) = 0$$

$$\oiint_S (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1).ds' = 0$$

- Let's consider fields (**$E_1$**, **$H_1$**, **$E_2$**, **$H_2$**) and the sources (**$J_1$**, **$M_1$**, **$J_2$**, **$M_2$**) are within a medium that is enclosed by a sphere of infinite radius.
- Let's assume sources are positioned within a finite region and that the fields are observed in the far field (ideally at infinity):

$$\oiint_S (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1).ds' = 0$$

# Why?

- Why do we have?

$$\oiint_S (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1).ds' = 0$$

- Note that $\vec{E}$ and $\vec{H}$ decay as $1/r$ and we are multiplying the equation by $ds$ which has a term $r^2$ in it.
- The reason is, if sources are limited to a finite region, we will have a spherical wave radiated in the far field, therefore $\vec{E}$ will have only $\phi$ and $\theta$ components:

$$E_\theta = \eta H_\varphi \qquad\qquad E_\varphi = -\eta H_\theta$$

$$-\eta \oiint_{r\to\infty} \left( H_\theta^1 H_\theta^2 + H_\varphi^1 H_\varphi^2 - H_\theta^2 H_\theta^1 - H_\varphi^2 H_\varphi^1 \right) ds = 0$$

# Reciprocity

- Therefore, we have:

$$\iiint_V (\vec{E}_1.\vec{J}_2 + \vec{H}_2.\vec{M}_1 - \vec{E}_2.\vec{J}_1 - \vec{H}_1.\vec{M}_2)dv' = 0$$

$$\iiint_V (\vec{E}_1.\vec{J}_2 - \vec{H}_1.\vec{M}_2)dv' = \iiint_V (\vec{E}_2.\vec{J}_1 - \vec{H}_2.\vec{M}_1)dv'$$

- We can define reaction as:

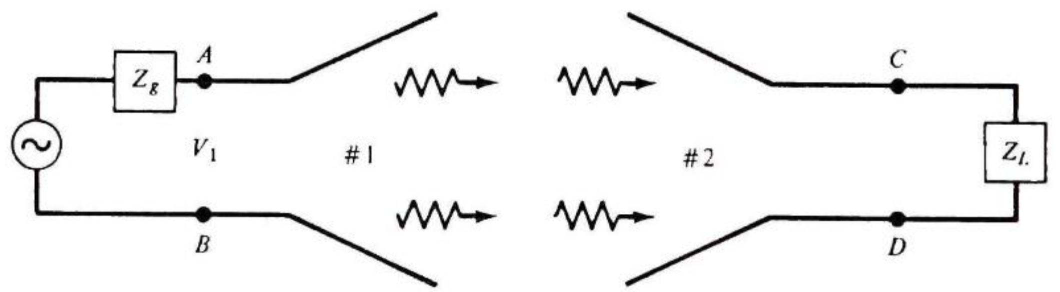
$$<1,2> = \iiint_V (\vec{E}_1.\vec{J}_2 - \vec{H}_1.\vec{M}_2)dv'$$

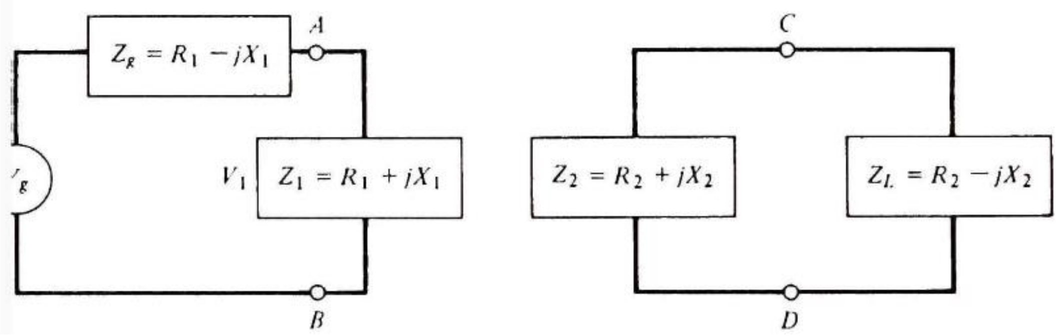
$$<2,1> = \iiint_V (\vec{E}_2.\vec{J}_1 - \vec{H}_2.\vec{M}_1)dv'$$

# Reciprocity For Two Antennas

- Let's assume that two antennas are separated by a linear and isotropic, but not necessarily homogeneous, medium.
- Ant 1 is used as the transmitter and 2 as the receiver.

- $Z_g = Z_1^* = R_1 - jX_1$
- $Z_L = Z_2^* = R_2 - jX_2$




POSTECH

# Reciprocity For Two Antennas

- The power delivered by the generator to antenna1 is given by:

$$P_1 = 1/2\,\mathrm{Re}[V_1 I_1^{\,*}] = 1/2\,\mathrm{Re}\left\{\left(\frac{V_g Z_1}{Z_1 + Z_g}\right)\frac{V_g^{\,*}}{(Z_1 + Z_g)^*}\right\} = \frac{|V_g|^2}{8R_1}$$

- If the transfer admittance of the combined network consisting of the generator impedance, antennas, and load impedance is $Y_{21}$, the current through the load is $V_g Y_{21}$ and the power delivered to the load is:

$$P_2 = 1/2\,\mathrm{Re}[Z_2 (V_g Y_{21})(V_g Y_{21})^*] = 1/2\,R_2 \left|V_g\right|^2 \left|Y_{21}\right|^2$$

# Reciprocity For Two Antennas

- Therefore, the ratio of P2/P1 is:

$$\frac{P_2}{P_1} = 4R_1R_2\,|Y_{21}|^2$$

- In similar fashion we can show that when antenna 2 is transmitting and antenna 1 is receiving P1/P2:

$$\frac{P_1}{P_2} = 4R_2R_1\,|Y_{12}|^2$$

- Under reciprocity conditions ($Y_{12} = Y_{21}$) the power delivered in either direction is the same.

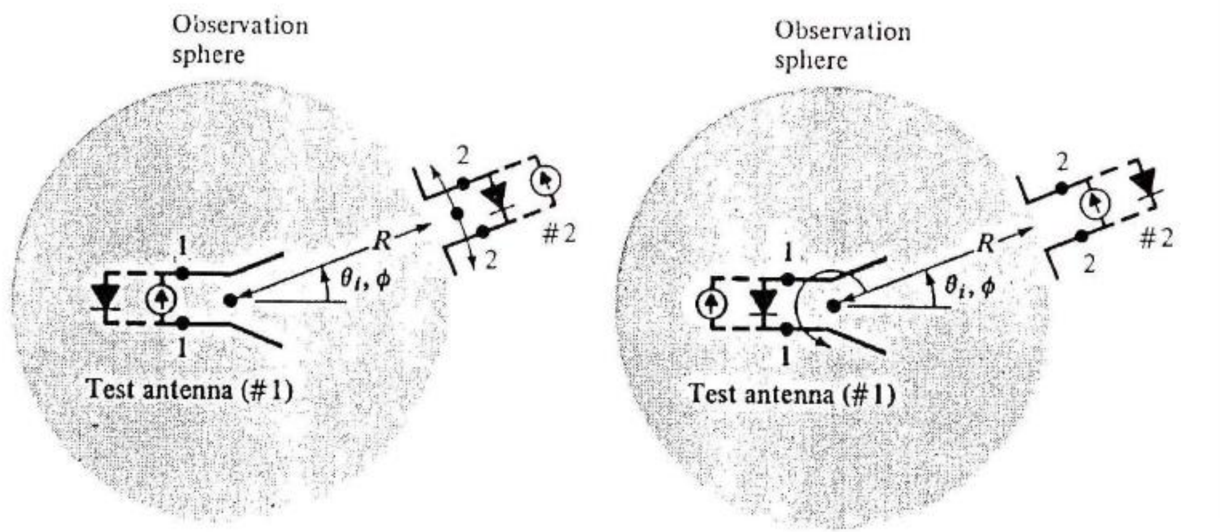
# Reciprocity for Antenna Radiation Patterns

- The radiation pattern of the antenna is a very important antenna characteristics.
- Although it may be convenient to measure the radiation pattern of the antenna in receiving (or transmitting) mode, the radiation pattern is the same in transmitting (or receiving) mode.
- Reciprocity for antenna radiation patterns is general provided that materials used for the antennas and feeds and the media of wave propagation are linear.
- The only other restriction for reciprocity to hold is for the antennas in the transmit and receive modes to be polarization matched, including the sense of rotation.

# Reciprocity for Antenna Radiation Patterns

- Referring to the figure, antenna 1 is the antenna under test.
- The probe antenna 2 is oriented to transmit or receive maximum radiation.
- We have:

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$
$$V_2 = Z_{21}I_1 + Z_{22}I_2$$



(a) (b)

POSTECH

# Reciprocity for Antenna Radiation Patterns

- If current $I_1$ is applied at the terminals 1-1 and voltage $V_2$ ($V_{2oc}$) is measured at the open ($I_2 = 0$) terminals of antenna 2, then an equal voltage $V_{1oc}$ will be measured at the open ($I_1 = 0$) terminals of antenna1 provided the current $I_2$ of antenna 2 is equal to $I_1$.

$$Z_{21} = \left.\frac{V_{2oc}}{I_1}\right|_{I_2=0} \qquad Z_{12} = \left.\frac{V_{1oc}}{I_2}\right|_{I_1=0}$$

- If the medium between the two antennas is linear, passive, isotropic, and the waves monochromatic, then because of reciprocity, we will have:

$$Z_{21} = \left.\frac{V_{2oc}}{I_1}\right|_{I_2=0} = Z_{12} = \left.\frac{V_{1oc}}{I_2}\right|_{I_1=0} = Z_{12}$$

# Reciprocity for Antenna Radiation Patterns

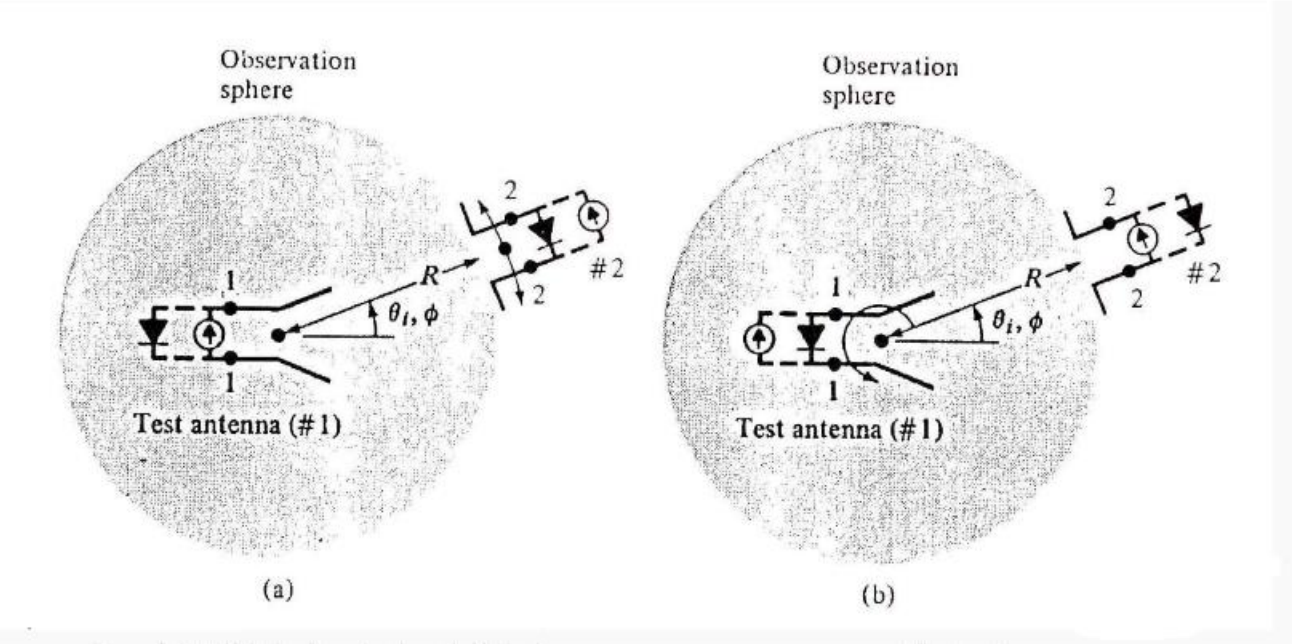
- If in addition $I_1 = I_2$ then:

$$V_{2oc} = V_{1oc}$$

- The above are valid for any position and any sort of configuration of operation between the two antennas.
- Let's assume that antenna 1 is held stationary while antenna 2 is allowed to move on surface of a sphere with constant radius.
- Let's also assume another mode when antenna 2 is stationary while antenna 1 can rotate about a point.

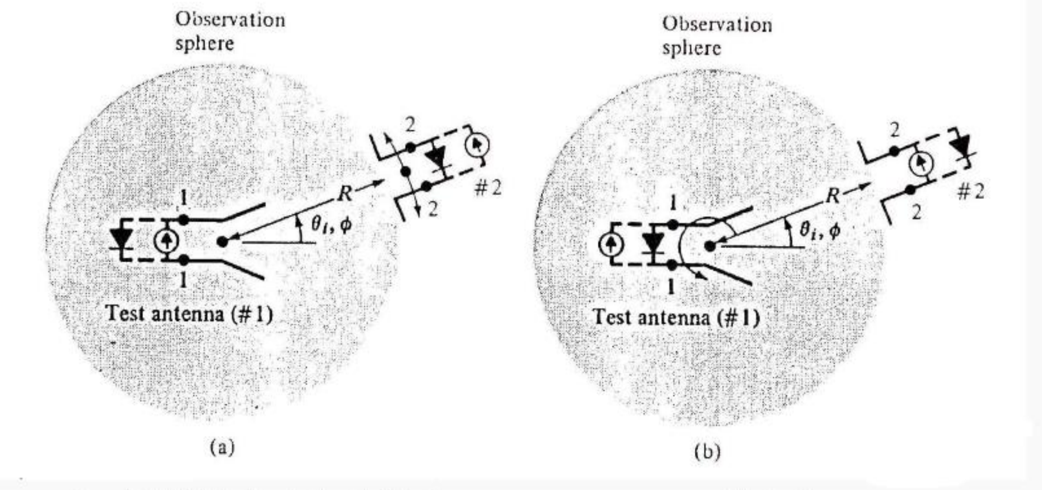
# Reciprocity for Antenna Radiation Patterns

- Antenna 1 can be used either as transmitter or as receiver.
- If under transmitting mode, antenna 2 is moved on the surface of the sphere of constant radius, we have essentially measured the radiation pattern of the antenna in TX mode.



# Reciprocity for Antenna Radiation Patterns

- On the other hand, if antenna 2 is used as the transmitter and ant. 1 as the receiver and antenna 2 is moved on the surface of the sphere, the radiation pattern of the antenna under the receiving mode is measured.
- Because of reciprocity, the two measurements will be identical.
- Therefore, the radiation patterns of the antenna under TX and RX modes are identical.




POSTECH