# Chapter 3
# COMPLEX VARIABLES

**Augustin-Louis Cauchy**
(1789-1857)
Math/Physics
Complex Analysis
Stress Tensor

## Lecture 10

2.6 Residue Theory
2.7 Evalution of Definite Integrals

**David Hilbert**
(1862-1943)
Math
Hilbert Space
(Vector Space for QM)

## 2.6 Residue Theory

For many physics and engineering problems, we need to evaluate definite integrals in closed forms. The residue theory is a powerful tool for finding these integrals without brute-force numerical simulations that are often time-consuming and computationally expensive.

$f(z) = \sum_{n=-\infty}^{\infty} A_n(z_0)(z-z_0)^n$

**[Definition] Residue** $A_{-1}(z_0)$

In the Laurent expansion of $f(z)$ at $z = z_0$, $f(z) = \sum_{n=-\infty}^{\infty} A_n(z_0)(z-z_0)^n$, we define $A_{-1}(z_0)$ as the residue of $f(z)$ at $z = z_0$:

$A_{-1}(z_0) = \frac{1}{2\pi i}\oint_C dz\, f(z)$

$$\boxed{A_{-1}(z_0) = \frac{1}{2\pi i}\oint_C dz\, f(z)} \quad \textbf{Residue} \tag{3.34}$$

where $C$ is a circular contour centered at $z_0$ (Fig. 3-5).

[Proof] Since the integrals of the analytic part of Laurent expansion becomes zero,

$$2\pi i A_{-1}(z_0) = \sum_{n=1}^{\infty} A_{-n}(z_0)\oint_C \frac{dz}{(z-z_0)^n} \tag{3.35}$$

Let $z - z_0 = re^{i\theta}$, then we have

$$\oint_{C_1} \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i, & n = 1 \\ ie^{1-n}\int_0^{2\pi} d\theta_1\, e^{i(1-n)\theta_1} = \frac{e^{1-n}}{1-n}\left[e^{i2\pi(1-n)} - 1\right] = 0, & n \neq 1 \end{cases}$$

## Summation of Residues for Multiple Poles

The contour integral of $f(z)$ on $C$ enclosing $M$ poles at $z = z_m$ $(m = 1, 2, \cdots M)$ is given by the sum of integrals on the small contours centered at $z_m$'s (Fig. 3-6):

$$\boxed{\oint_C dz\, f(z) = 2\pi i \sum_{m=1}^{N} A_{-1}(z_m)} \tag{3.34}$$

[Proof] Using the Cauchy-Goursat theorem with a simply-connected contour,

$$\oint_C dz\, f(z) + \sum_{m=1}^{\infty} \oint_{-C_m} dz\, f(z) = 0$$

$$\rightarrow \oint_C dz\, f(z) = \sum_{m=1}^{\infty} \oint_{C_m} dz\, f(z) = 2\pi i \sum_{m=1}^{\infty} A_{-1}(z_m)$$

$\oint_C dz f(z) = 2\pi i \sum_{m=1} A_{-1}(z)$

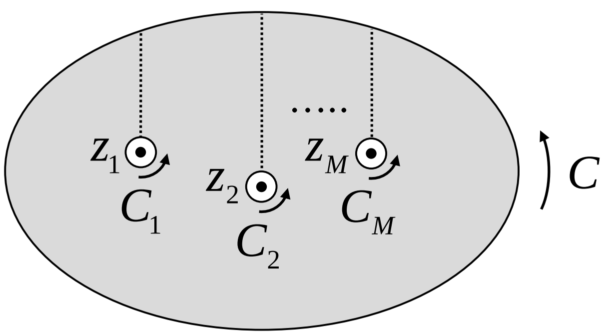


Fig. 3-6 Contour for Residue theory

## Generalized Jordan's Lemma (Han's Lemma, this Lecture)

If $f(z)$ is continuous on a circular section $C_R$ ($z = Re^{i\theta}, 0 \le \theta_1 \le \theta \le \theta_2 \le \pi$) in the upper half plane, bounded by

$$\lim_{R\to\infty} R^n f(Re^{i\theta}) = 0, \quad n \ge 1 \tag{3.35}$$

then the contour integral with a complex number $k = k' + ik''$ vanishes:

$$\boxed{\lim_{R\to\infty} \int_{C_R} dz\, f(z) e^{ikz} = 0, \quad \mathrm{Re}[kRe^{i\theta}] < 0 \to k' \sin\theta + k'' \cos\theta > 0} \tag{3.36}$$

[Proof] Consider the absolute value of the integral:

$$\left| \lim_{R\to\infty} \int_{C_R} dz\, f(z) e^{iaz} \right| = \left| \lim_{R\to\infty} \int_{\theta_1}^{\theta_2} d\theta\, iRe^{i\theta} f(Re^{i\theta}) e^{i(k'+ik'')R(\cos\theta + i\sin\theta)} \right|$$

$$\le \lim_{R\to\infty} \int_{\theta_1}^{\theta_2} d\theta\, R \left| f(Re^{i\theta}) \right| e^{-R(k'\sin\theta + k''\cos\theta)} = 0$$

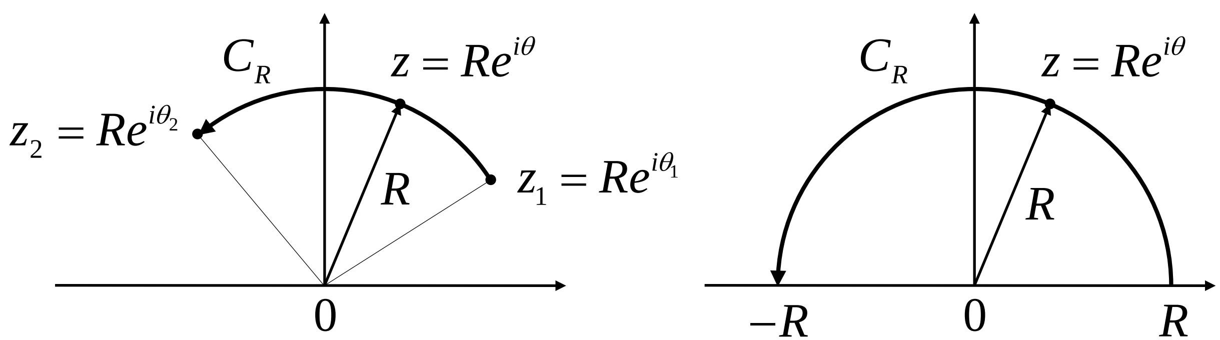


Fig. 3-6 Contours for Han's and Jordan's Lemmas

## Jordan's Lemma

If $f(z)$ is continuous on a semi-circular section $C_R$ $(z = Re^{i\theta}, 0 \le \theta \le \pi)$ in the upper half plane (Fig. 3-~~6),~~ bounded by

$$\lim_{R\to\infty} R^n f(Re^{i\theta}) = 0, \quad n \ge 1 \tag{3.37}$$

then we have

$$\boxed{\lim_{R\to\infty} \int_{C_R} dz\, f(z) e^{iaz} = 0, \quad a > 0} \qquad \lim_{R\to\infty} \int_{C_R} dz\, f(z) e^{iaz} = 0 \tag{3.38}$$

[Proof] This is just a special case of the generalized Jordan's Lemma with a real $k = a > 0$.

## Estimation Lemma

If $f(x)$ is continuous on a semi-circular section $C_R$ $(z = Re^{i\theta}, 0 \le \theta \le \pi)$ in the upper half plane, bounded by

$$\lim_{R\to\infty} R^n f(Re^{i\theta}) = 0, \quad n \ge 1 \tag{3.39}$$

then we have

$$\boxed{\lim_{R\to\infty} \int_{C_R} dz\, f(z) = 0, \quad a > 0} \qquad \lim_{R\to\infty} \int_{C_R} dz\, f(z) = 0 \tag{3.40}$$

[Proof] This is a special case of the Jordan's Lemma with a real $k = a = 0$.

## Calculation of Residues

Even if $f(z)$ is not explicitly given by a Laurent series, we can still calculate the residue for an $m^{th}$-order pole:

$$\boxed{A_{-1}(z_0) = \lim_{z\to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]} \tag{3.41}$$

As a simple case, for a simple pole, we have a simple formula

$$\boxed{A_{-1}(z_0) = \lim_{z\to z_0}(z-z_0)\, f(z)} \tag{3.42}$$

$$A_{-1}(z_0) = \lim_{z\to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$$

[Ex-1] Find the Laurent series and the residues of

$$f(z) = \frac{3z^2 - 12z + 11}{(z-1)(z-2)^2}$$

Sol) For a complete expression of the Laurent series,

$$f(z) = \frac{A_{-1}(1)}{z-1} + \frac{A_{-1}(2)}{z-2} + \frac{A_{-2}(2)}{(z-2)^2} \quad \rightarrow \quad A_{-1}(1) = 2, \quad A_{-1}(2) = 1, \quad A_{-2}(2) = -1$$

For the residues only, we can use the residue formulas,

$$A_{-1}(1) = \lim_{z\to 1}(z-1)\, f(z) = 2$$

$$A_{-1}(2) = \lim_{z\to 2}\frac{d}{dz}\left[(z-2)^2 f(z)\right] = 1$$

## 2.7 Evaluation of Definite Integrals

The residue theorem is very useful to evaluate definite integrals of a function $f(x)$, and there are only **two standard forms of the definite integrals**. There are still many other types of definite integrals which cannot be solved by standard, general prescription. Unfortunately, in these cases, a particular contour and its related techniques should be devised for each problem.

**Type-1: Fourier Transform**

Consider $f(z)$ under two conditions:

1) meromorphic in the upper half plane with a finite number $M$ of poles $z_m$ $(m=1,2,\cdots M)$
2) bounded by $\lim_{R\to\infty} R^n f(Re^{i\theta})=0, \quad n\geq 1$

Then we define a Fourier transform integral

$$I=\int_{-\infty}^{\infty} dx\, f(x)e^{ikx}, \quad k>0 \tag{3.43}$$

Using the residue theorem, an integral on a closed contour $C$ (Fig. 3-7) is given by

$$\lim_{R\to\infty}\oint_C dz\, f(z)e^{ikx}=\lim_{R\to\infty}\int_{C_R} dz\, f(z)e^{ikx}+I=2\pi i\sum_{m=1}^{M} A_{-1}(z_m) \tag{3.44}$$

From the Jordan's lemma, the first integral vanishes, and thus we have

$$\int_{-\infty}^{\infty} dx\, f(x)\mathrm{e}^{ikx} = 2\pi i \sum_{m=1}^{M} A_{-1}(z_m), \quad k>0 \tag{3.45}$$

As a special case, from the estimation lemma, we can still evaluate the integral for $k=0$,

$$\int_{-\infty}^{\infty} dx\, f(x) = 2\pi i \sum_{m=1}^{M} A_{-1}(z_m) \tag{3.45}$$

If the integrand has a pole in the real axis, we can add a small half circle in the closed contour, either clock-wise or counterclock-wise, to evaluate, in this case, the principal value of the definite integral (Fig. 3-x).

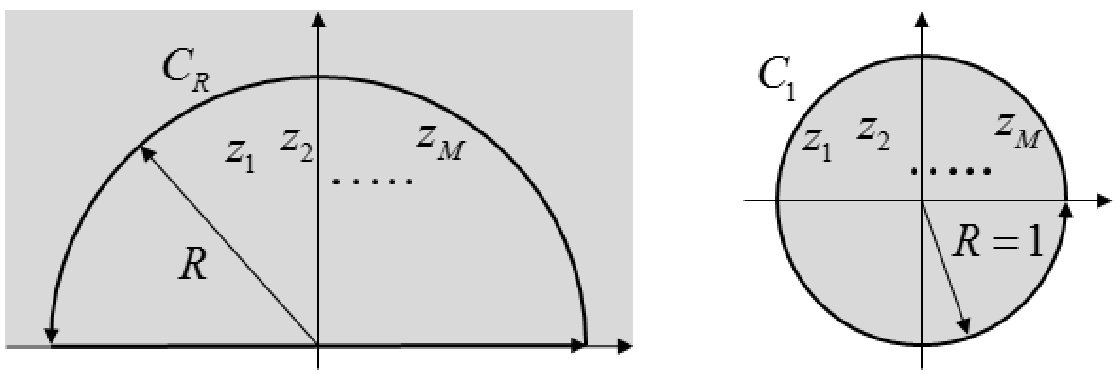


Fig. 3-7 Contours for type-1 and type-2 definite integrals