Week7 – Relaxation Time Approximation

ECE 695-O Semiconductor Transport Theory

Fall 2018

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Contents

- Relaxation Time Approximation

Boltzmann Transport Equation

- The Boltzmann transport equation that we found for the case that the distribution $f(\mathbf{k}, \mathbf{r}, t)$ is a small deviation from equilibrium distribution $f_0(\mathcal{E})$ looks like

$$\frac{\partial f}{\partial t} = -\frac{1}{\hbar}\nabla_{\mathbf{k}}\mathcal{E} \cdot \nabla_{\mathbf{r}} f - \frac{e}{\hbar}\left(\mathbf{E} + \frac{1}{\hbar}\nabla_{\mathbf{k}}\mathcal{E} \times \mathbf{B}\right) \cdot \nabla_{\mathbf{k}} f$$
$$+ \frac{1}{k_B T}\frac{1}{8\pi^3}\int d^3k' P_{\mathbf{kk'}} f_0(\mathcal{E}, \mathbf{r})[1 - f_0(\mathcal{E}', \mathbf{r})]\,[\phi(\mathbf{k}', \mathbf{r}, t) - \phi(\mathbf{k}, \mathbf{r}, t)]$$

where the small deviation $F(\mathbf{k}, \mathbf{r}, t) = -\phi(\mathbf{k}, \mathbf{r}, t)\,\frac{\partial f_0}{\partial \mathcal{E}}$.

- In steady state, $\frac{\partial f}{\partial t} = 0$, so the equation can be further simplified as

$$\frac{1}{\hbar}\nabla_{\mathbf{k}}\mathcal{E} \cdot \nabla_{\mathbf{r}} f + \frac{e}{\hbar}\left(\mathbf{E} + \frac{1}{\hbar}\nabla_{\mathbf{k}}\mathcal{E} \times \mathbf{B}\right) \cdot \nabla_{\mathbf{k}} f$$
$$= \frac{1}{k_B T}\frac{1}{8\pi^3}\int d^3k' P_{\mathbf{kk'}} f_0(\mathcal{E}, \mathbf{r})[1 - f_0(\mathcal{E}', \mathbf{r})]\,[\phi(\mathbf{k}', \mathbf{r}, t) - \phi(\mathbf{k}, \mathbf{r}, t)]$$

Relaxation Time Approximation

- This equation is hard to be handled analytically as is.
- One simple but effective approach is approximate scattering term:

$$\frac{1}{k_B T}\frac{1}{8\pi^3}\int d^3k' P_{\mathbf{kk}'} f_0(\mathcal{E},\mathbf{r})[1-f_0(\mathcal{E}',\mathbf{r})]\,[\phi(\mathbf{k}',\mathbf{r},t)-\phi(\mathbf{k},\mathbf{r},t)]$$

- And this is so-called relaxation time approximation:

$$\left.\frac{\partial f}{\partial t}\right)_{scatt.} = -\frac{f-f_0}{\tau} = -\frac{F}{\tau}$$

$$F(\mathbf{k},\mathbf{r},t) = -\phi(\mathbf{k},\mathbf{r},t)\frac{\partial f_0}{\partial \mathcal{E}}$$

$$\frac{\partial f_0}{\partial \mathcal{E}} = -\frac{1}{k_B T} f_0(1-f_0)$$

- This gives

$$\Rightarrow \frac{1}{\tau} = \left.\frac{\partial f}{\partial t}\right)_{scatt.}\left(-\frac{1}{F}\right) = \int d\mathbf{k}'\ P_{\mathbf{kk}'}\frac{[1-f_0(\mathcal{E}')]}{[1-f_0(\mathcal{E})]}\left[1-\frac{\phi(\mathbf{k}')}{\phi(\mathbf{k})}\right]$$

*$\mathbf{r}$ and t are dropped

Relaxation Time Approximation(2)

- The relaxation time $\tau(\mathbf{k})$ is a meaningful quantity provided that it is independent of the strength and type of the perturbation causing f to depart from f_0.
- What it means is that we can use this approximation if $\tau(\mathbf{k})$ is an intrinsic property of material itself and not influenced by the external perturbation applied to the system.
- In order for $\tau(\mathbf{k})$ to be independent of the external perturbation, or for the previous equation ($\frac{1}{\tau} = \int d\mathbf{k}' \; P_{\mathbf{k}\mathbf{k}'} \frac{[1-f_0(\mathcal{E}')]}{[1-f_0(\mathcal{E})]}\left[1 - \frac{\phi(\mathbf{k}')}{\phi(\mathbf{k})}\right]$) to be justified,

1) $f_0(\mathcal{E}') = f_0(\mathcal{E})$. This means $\mathcal{E}(\mathbf{k}') = \mathcal{E}(\mathbf{k})$, or the scattering process is elastic (no energy loss).
2) $\left[\frac{\phi(\mathbf{k}')}{\phi(\mathbf{k})}\right]$ must be independent of the type of perturbation.

Relaxation Time Approximation(3)

- In the cases where the relaxation time approximation is not justified (for instance inelastic scattering), another method must be used to solve BTE.
  - A variational calculation under the assumption that the return to the equilibrium from a steady state, when the perturbation is removed suddenly, would occur via a particular rapid relaxation mechanism.

<u>Relaxation time solution of BTE.</u>

- With the relaxation time approximation, the BTE can be written as

$$\frac{1}{\hbar}\nabla_{\mathbf{k}}\mathcal{E}\cdot\nabla_{\mathbf{r}}f+\frac{e}{\hbar}\left(\mathbf{E}+\frac{1}{\hbar}\nabla_{\mathbf{k}}\mathcal{E}\times\mathbf{B}\right)\cdot\nabla_{\mathbf{k}}f=-\frac{f-f_0}{\tau}$$

$$=\frac{\phi}{\tau}\frac{\partial f_0}{\partial\mathcal{E}}$$

Relaxation Time Approximation(4)

- $f = f_0 + F = f_0 - \phi \dfrac{\partial f_0}{\partial \mathcal{E}}$.
- Let $\phi = -\mathbf{v} \cdot \mathbf{G}$.
- Then, $f = f_0 + \mathbf{v} \cdot \mathbf{G} \dfrac{\partial f_0}{\partial \mathcal{E}}$ (*)

(*) $\mathbf{G}$ is small change in distribution function caused by external perturbation or excitation. (Like Taylor expansion)

$$\begin{aligned} f &= f_0 + \frac{\partial f_0}{\partial \mathbf{k}} \cdot \mathbf{G} \\ &= f_0 + \frac{\partial f_0}{\partial \mathcal{E}} \nabla_{\mathbf{k}} \mathcal{E} \cdot \mathbf{G} \\ &= f_0 + \frac{\partial f_0}{\partial \mathcal{E}} \mathbf{v} \cdot \mathbf{G} \end{aligned}$$

$$\mathbf{v} = \frac{1}{\hbar} \nabla_{\mathbf{k}} \mathcal{E}$$

$$\boxed{\frac{1}{\hbar} \nabla_{\mathbf{k}} \mathcal{E} \cdot \nabla_{\mathbf{r}} f} + \boxed{\frac{e}{\hbar} \left(\mathbf{E} + \frac{1}{\hbar} \nabla_{\mathbf{k}} \mathcal{E} \times \mathbf{B} \right) \cdot \nabla_{\mathbf{k}} f} = -\frac{f - f_0}{\tau}$$

$$= -\frac{\phi}{\tau} \frac{\partial f_0}{\partial \mathcal{E}}$$

Diffusion Term   Drift Term

Relaxation Time Approximation - Diffusion Term

- **Diffusion term**

$$\nabla_{\mathbf{r}} f = \nabla_{\mathbf{r}} f_0 + \cancel{\nabla_{\mathbf{r}}\left(\mathbf{v}\cdot\mathbf{G}\frac{\partial f_0}{\partial \mathcal{E}}\right)}$$

Let's ignore this term since it has small effect.

None zero. Thus, in first order approx. , $\nabla_{\mathbf{r}} f \approx \nabla_{\mathbf{r}} f_0$.

$$f_0 = \frac{1}{1+e^{(\mathcal{E}-\mathcal{E}_F)/k_B T}} = \frac{1}{1+e^{\eta}}$$

where $\eta = {(\mathcal{E}-\mathcal{E}_F)}/{k_B T}$.

- Then,

$$\nabla_{\mathbf{r}} f_0 = \frac{\partial f_0}{\partial \eta}\nabla_{\mathbf{r}}\eta = \frac{\partial f_0}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial \eta}\nabla_{\mathbf{r}}\left(\frac{\mathcal{E}-\mathcal{E}_F}{k_B T}\right)$$

$$= k_B T\frac{\partial f_0}{\partial \mathcal{E}}\nabla_{\mathbf{r}}\left(\frac{\mathcal{E}-\mathcal{E}_F}{k_B T}\right)$$

Relaxation Time Approximation – Diffusion Term(2)

$$\nabla_{\mathbf{r}}\left(\frac{\mathcal{E}-\mathcal{E}_F}{k_B T}\right)=\frac{1}{k_B T}\nabla_{\mathbf{r}}(\mathcal{E}-\mathcal{E}_F)-\frac{\mathcal{E}-\mathcal{E}_F}{k_B T^2}\nabla_{\mathbf{r}}T$$

- Thus, the diffusion term becomes

$$\mathbf{v}\cdot\nabla_{\mathbf{r}}f_0\cong\frac{\partial f_0}{\partial\mathcal{E}}\mathbf{v}\cdot\left\{\nabla_{\mathbf{r}}(\mathcal{E}-\mathcal{E}_F)-\frac{\mathcal{E}-\mathcal{E}_F}{T}\nabla_{\mathbf{r}}T\right\}$$

Relaxation Time Approximation – Drift Term

- **Drift term**

$$\left(\mathbf{E}+\overbrace{\boxed{\frac{1}{\hbar}\nabla_{\mathbf{k}}\mathcal{E}}}^{=\mathbf{v}}\times\mathbf{B}\right)\cdot\nabla_{\mathbf{k}}f=(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{k}}\left(f_0+\mathbf{v}\cdot\mathbf{G}\frac{\partial f_0}{\partial\mathcal{E}}\right)$$

$$=\mathbf{E}\cdot\underbrace{\nabla_{\mathbf{k}}f_0}_{=\frac{\partial f_0}{\partial\mathcal{E}}\nabla_{\mathbf{k}}\mathcal{E}}+\cancel{\mathbf{E}\cdot\nabla_{\mathbf{k}}\left(\frac{\partial f_0}{\partial\mathcal{E}}\mathbf{v}\cdot\mathbf{G}\right)}$$

ignore higher order

$$+\cancel{(\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{k}}f_0}+\boxed{(\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{k}}\left(\frac{\partial f_0}{\partial\mathcal{E}}\mathbf{v}\cdot\mathbf{G}\right)}$$

This is zero from the vector identity:
$(A\times B)\cdot A=0$ since
$\nabla_{\mathbf{k}}f_0=\frac{\partial f_0}{\partial\mathcal{E}}\nabla_{\mathbf{k}}\mathcal{E}\ //\ \mathbf{v}$

$$(\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{k}}\left(\frac{\partial f_0}{\partial\mathcal{E}}\mathbf{v}\cdot\mathbf{G}\right)=\underbrace{\frac{\partial f_0}{\partial\mathcal{E}}(\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{k}}(\mathbf{v}\cdot\mathbf{G})}+\cancel{(\mathbf{v}\cdot\mathbf{G})(\mathbf{v}\times\mathbf{B})\cdot\underline{\nabla_{\mathbf{k}}\left(\frac{\partial f_0}{\partial\mathcal{E}}\right)}}$$

Only this term survives.

$\nabla_{\mathbf{k}}\left(\frac{\partial f_0}{\partial\mathcal{E}}\right)=\frac{\partial^2 f_0}{\partial\mathcal{E}^2}\nabla_{\mathbf{k}}\mathcal{E}\ //\ \mathbf{v}$ and from the vector identity, it is zero.

Relaxation Time Approximation – Drift Term(2)

- **Drift term**

$$\left(\mathbf{E}+\frac{1}{\hbar}\nabla_{\mathbf{k}}\mathcal{E}\times\mathbf{B}\right)\cdot\nabla_{\mathbf{k}}f \cong \mathbf{E}\cdot\frac{\partial f_0}{\partial\mathcal{E}}\nabla_{\mathbf{k}}\mathcal{E}+\frac{\partial f_0}{\partial\mathcal{E}}(\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{k}}(\mathbf{v}\cdot\mathbf{G})$$

$$=\hbar\frac{\partial f_0}{\partial\mathcal{E}}\mathbf{E}\cdot\mathbf{v}+\frac{1}{\hbar}\frac{\partial f_0}{\partial\mathcal{E}}(\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{k}}(\nabla_{\mathbf{k}}\mathcal{E}\cdot\mathbf{G})$$

Apply here

Vector Identity: $A\cdot(B\times C)=B\cdot(C\times A)=C\cdot(A\times B)$
Since they are all the same volume.

$$=\hbar\frac{\partial f_0}{\partial\mathcal{E}}\mathbf{E}\cdot\mathbf{v}+\frac{1}{\hbar}\frac{\partial f_0}{\partial\mathcal{E}}\mathbf{v}\cdot[\mathbf{B}\times\nabla_{\mathbf{k}}(\nabla_{\mathbf{k}}\mathcal{E}\cdot\mathbf{G})]$$

Relaxation Time Approximation(5)

- **Diffusion term**

$$\mathbf{v}\cdot\nabla_{\mathbf{r}} f_0 = \mathbf{v}\cdot T\frac{\partial f_0}{\partial \mathcal{E}}\nabla_{\mathbf{r}}\left(\frac{\mathcal{E}-\mathcal{E}_F}{T}\right) \cong \frac{\partial f_0}{\partial \mathcal{E}}\mathbf{v}\cdot\left\{\nabla_{\mathbf{r}}(\mathcal{E}-\mathcal{E}_F)-\frac{\mathcal{E}-\mathcal{E}_F}{T}\nabla_{\mathbf{r}}T\right\}$$

- **Drift term**

$$\left(\mathbf{E}+\frac{1}{\hbar}\nabla_{\mathbf{k}}\mathcal{E}\times\mathbf{B}\right)\cdot\nabla_{\mathbf{k}}f \cong \hbar\frac{\partial f_0}{\partial \mathcal{E}}\mathbf{E}\cdot\mathbf{v}+\frac{\partial f_0}{\partial \mathcal{E}}(\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{k}}(\mathbf{v}\cdot\mathbf{G})$$

$$= \hbar\frac{\partial f_0}{\partial \mathcal{E}}\mathbf{E}\cdot\mathbf{v}+\frac{1}{\hbar}\frac{\partial f_0}{\partial \mathcal{E}}\mathbf{v}\cdot[\mathbf{B}\times\nabla_{\mathbf{k}}(\nabla_{\mathbf{k}}\mathcal{E}\cdot\mathbf{G})]$$

- **BTE**

$$\frac{\phi}{\tau}\frac{\partial f_0}{\partial \mathcal{E}} = \frac{1}{\hbar}\nabla_{\mathbf{k}}\mathcal{E}\cdot\nabla_{\mathbf{r}}f+\frac{e}{\hbar}\left(\mathbf{E}+\frac{1}{\hbar}\nabla_{\mathbf{k}}\mathcal{E}\times\mathbf{B}\right)\cdot\nabla_{\mathbf{k}}f$$

$$\frac{1}{\tau}\frac{\partial f_0}{\partial \mathcal{E}}\mathbf{v}\cdot\mathbf{G} = -T\frac{\partial f_0}{\partial \mathcal{E}}\mathbf{v}\cdot\nabla_{\mathbf{r}}\left(\frac{\mathcal{E}-\mathcal{E}_F}{T}\right)-e\frac{\partial f_0}{\partial \mathcal{E}}\mathbf{v}\cdot\mathbf{E}-\frac{e}{\hbar^2}\frac{\partial f_0}{\partial \mathcal{E}}\mathbf{v}\cdot[\mathbf{B}\times\nabla_{\mathbf{k}}(\nabla_{\mathbf{k}}\mathcal{E}\cdot\mathbf{G})]$$

$$\mathbf{G}(\mathbf{k},\mathbf{r}) = -e\tau\mathbf{E}-\tau T\,\nabla_{\mathbf{r}}\left(\frac{\mathcal{E}-\mathcal{E}_F}{T}\right)-\frac{e\tau}{\hbar^2}\underbrace{[\mathbf{B}\times\nabla_{\mathbf{k}}(\nabla_{\mathbf{k}}\mathcal{E}\cdot\mathbf{G})]}$$

Magnetic term also has $\mathbf{G}$ component

Relaxation Time Approximation(6)

$$\mathbf{G}(\mathbf{k},\mathbf{r}) = -e\tau\mathbf{E} - \tau T\,\nabla_{\mathbf{r}}\left(\frac{\mathcal{E}-\mathcal{E}_F}{T}\right) - \frac{e\tau}{\hbar^2}\underbrace{[\mathbf{B}\times\nabla_{\mathbf{k}}(\nabla_{\mathbf{k}}\mathcal{E}\cdot\mathbf{G})]}$$

Interestingly, magnetic term also has $\mathbf{G}$ component. This is more like the dependence on the magnitude of $\mathbf{G}$, rather than the direction of $\mathbf{k}$. (no angular dependence)

- Let's assume that $\mathbf{G}(\mathbf{k},\mathbf{r}) = \mathbf{G}(\mathcal{E},\mathbf{r})$.

$$\begin{aligned}\nabla_{\mathbf{k}}(\nabla_{\mathbf{k}}\mathcal{E}\cdot\mathbf{G}) &= (\nabla_{\mathbf{k}}\mathcal{E}\cdot\nabla_{\mathbf{k}})\mathbf{G} + \boxed{\nabla_{\mathbf{k}}\mathcal{E}\times(\nabla_{\mathbf{k}}\times\mathbf{G})}\,(*) \\ &\quad +(\mathbf{G}\cdot\nabla_{\mathbf{k}})\nabla_{\mathbf{k}}\mathcal{E} + \cancel{\mathbf{G}\times(\nabla_{\mathbf{k}}\times\nabla_{\mathbf{k}}\times\mathcal{E})}\end{aligned}$$

=0 : curl of curl

Vector Identity: $\nabla(A\cdot B) = (A\cdot\nabla)B + (B\cdot\nabla)A + A\times(\nabla\times B) + B\times(\nabla\times A)$

(*) $$\nabla_{\mathbf{k}}\mathcal{E}\times(\nabla_{\mathbf{k}}\times\mathbf{G}) = \nabla_{\mathbf{k}}\mathcal{E}\left(\nabla_{\mathbf{k}}\mathcal{E}\cdot\frac{\partial\mathbf{G}}{\partial\mathcal{E}}\right) - \frac{\partial\mathbf{G}}{\partial\mathcal{E}}(\nabla_{\mathbf{k}}\mathcal{E}\cdot\nabla_{\mathbf{k}}\mathcal{E})$$

Vector Identity: $A\times(B\times C) = B(A\cdot C) - C(A\cdot B)$

Relaxation Time Approximation(7)

cancel to each other

$$\nabla_{\mathbf{k}}(\nabla_{\mathbf{k}}\mathcal{E}\cdot\mathbf{G}) = (\nabla_{\mathbf{k}}\mathcal{E}\cdot\nabla_{\mathbf{k}}\mathcal{E})\frac{\partial\mathbf{G}}{\partial\mathcal{E}} + \nabla_{\mathbf{k}}\mathcal{E}\left(\nabla_{\mathbf{k}}\mathcal{E}\cdot\frac{\partial\mathbf{G}}{\partial\mathcal{E}}\right) - \frac{\partial\mathbf{G}}{\partial\mathcal{E}}(\nabla_{\mathbf{k}}\mathcal{E}\cdot\nabla_{\mathbf{k}}\mathcal{E})$$
$$+(\mathbf{G}\cdot\nabla_{\mathbf{k}})\nabla_{\mathbf{k}}\mathcal{E}$$

$$= \nabla_{\mathbf{k}}\mathcal{E}\left(\nabla_{\mathbf{k}}\mathcal{E}\cdot\frac{\partial\mathbf{G}}{\partial\mathcal{E}}\right) + (\mathbf{G}\cdot\nabla_{\mathbf{k}})\nabla_{\mathbf{k}}\mathcal{E}$$

- Then the magnetic term $\mathbf{v}\cdot\mathbf{B}\times\nabla_{\mathbf{k}}(\nabla_{\mathbf{k}}\mathcal{E}\cdot\mathbf{G})$ becomes

$$\mathbf{v}\cdot\mathbf{B}\times\left[\underbrace{\nabla_{\mathbf{k}}\mathcal{E}}_{\sim\mathbf{v}}\left(\nabla_{\mathbf{k}}\mathcal{E}\cdot\frac{\partial\mathbf{G}}{\partial\mathcal{E}}\right) + (\mathbf{G}\cdot\nabla_{\mathbf{k}})\nabla_{\mathbf{k}}\mathcal{E}\right] = \mathbf{v}\cdot\mathbf{B}\times(\mathbf{G}\cdot\nabla_{\mathbf{k}})\nabla_{\mathbf{k}}\mathcal{E}$$

They cancel to each other

- If we collect all together,

$$\mathbf{G}(\mathcal{E},\mathbf{r}) = -e\tau\mathbf{E} - \tau T\,\nabla_{\mathbf{r}}\left(\frac{\mathcal{E}-\mathcal{E}_F}{T}\right) - \frac{e\tau}{\hbar^2}\mathbf{B}\times(\mathbf{G}\cdot\nabla_{\mathbf{k}})\nabla_{\mathbf{k}}\mathcal{E}$$

Relaxation Time Approximation(8)

$$\mathbf{G}(\mathcal{E},\mathbf{r}) = \underbrace{-e\tau\mathbf{E}}_{\text{Drift term}} \underbrace{- \tau T\, \nabla_{\mathbf{r}}\left(\frac{\mathcal{E}-\mathcal{E}_F}{T}\right)}_{\text{Temperature term}} - \frac{e\tau}{\hbar^2}\mathbf{B}\times(\mathbf{G}\cdot\nabla_{\mathbf{k}})\nabla_{\mathbf{k}}\mathcal{E}$$

- Now, we define "electro thermal field" as

$$\boldsymbol{\mathcal{F}} = \mathbf{E} + \frac{T}{e}\,\nabla_{\mathbf{r}}\left(\frac{\mathcal{E}-\mathcal{E}_F}{T}\right)$$

- Then,

$$\mathbf{G}(\mathcal{E},\mathbf{r}) = -e\tau\boldsymbol{\mathcal{F}} - \frac{e\tau}{\hbar^2}\mathbf{B}\times(\mathbf{G}\cdot\nabla_{\mathbf{k}})\nabla_{\mathbf{k}}\mathcal{E}$$

Relaxation Time Approximation(9)

- For the case of an ellipsoidal energy band,

$$\mathcal{E} = \mathcal{E}_c + \frac{\hbar^2 k_x^2}{2m_x^*} + \frac{\hbar^2 k_y^2}{2m_y^*} + \frac{\hbar^2 k_z^2}{2m_z^*}$$

- Then,

$$\mathbf{M}^{-1} = \begin{pmatrix} \frac{1}{m_x^*} & 0 & 0 \\ 0 & \frac{1}{m_y^*} & 0 \\ 0 & 0 & \frac{1}{m_z^*} \end{pmatrix}$$

and we can express $\mathcal{E}$ as

$$\mathcal{E} = \mathcal{E}_c + \frac{\hbar^2}{2} \mathbf{k}\, \mathbf{M}^{-1}\, \mathbf{k}$$

Relaxation Time Approximation(10)

- Then,

$$(\mathbf{G}\cdot\nabla_{\mathbf{k}})\nabla_{\mathbf{k}}\mathcal{E} = \mathbf{M}^{-1}\cdot\mathbf{G}$$

- Thus,

$$\begin{aligned}\mathbf{G}(\mathcal{E},\mathbf{r}) &= -e\tau\boldsymbol{\mathcal{F}} - \frac{e\tau}{\hbar^2}\mathbf{B}\times(\mathbf{G}\cdot\nabla_{\mathbf{k}})\nabla_{\mathbf{k}}\mathcal{E} \\ &= -e\tau\boldsymbol{\mathcal{F}} - \frac{e\tau}{\hbar^2}\mathbf{B}\times(\mathbf{M}^{-1}\cdot\mathbf{G})\end{aligned}$$

- After a few algebraic manipulation,

$$\Rightarrow \mathbf{G} = -e\tau\left\{\frac{\boldsymbol{\mathcal{F}} + e\tau(\mathbf{M}^{-1}\cdot\boldsymbol{\mathcal{F}})\times\mathbf{B} + (e\tau)^2(\det\mathbf{M}^{-1})(\boldsymbol{\mathcal{F}}\cdot\mathbf{B})(\mathbf{M}\cdot\mathbf{B})}{1+(e\tau)^2\,(\det\mathbf{M}^{-1})(\mathbf{M}\cdot\mathbf{B})\cdot\mathbf{B}}\right\}$$

Relaxation Time Approximation(11)

- For a spherical energy band where $m_x^* = m_y^* = m_z^* = m^*$,

$$\Rightarrow \mathbf{G} = -e\tau \left\{ \frac{\boldsymbol{\mathcal{F}} + e\tau(\mathbf{M}^{-1} \cdot \boldsymbol{\mathcal{F}}) \times \mathbf{B} + (e\tau)^2(\det \mathbf{M}^{-1})(\boldsymbol{\mathcal{F}} \cdot \mathbf{B})(\mathbf{M} \cdot \mathbf{B})}{1 + (e\tau)^2 (\det \mathbf{M}^{-1})(\mathbf{M} \cdot \mathbf{B}) \cdot \mathbf{B}} \right\}$$

$$\frac{\boldsymbol{\mathcal{F}}}{m^*} \qquad \left(\frac{1}{m^*}\right)^3 \qquad m^*\mathbf{B}$$

becomes

$$\mathbf{G} = -e\tau \left\{ \frac{\boldsymbol{\mathcal{F}} + \frac{e\tau}{m^*}\boldsymbol{\mathcal{F}} \times \mathbf{B} + \left(\frac{e\tau}{m^*}\right)^2 (\boldsymbol{\mathcal{F}} \cdot \mathbf{B})\mathbf{B}}{1 + \left(\frac{e\tau}{m^*}\right)^2 \mathbf{B} \cdot \mathbf{B}} \right\}$$

Ohmic contribution of transport (including thermoelectric effect)

Hall contribution

Magnetoresistanc