2015 수학

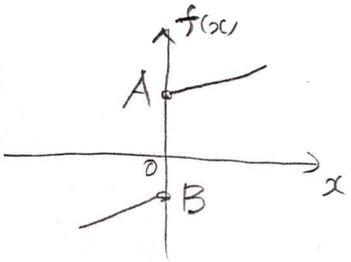
1. 로피탈 정리를 잘 쓰면,

(1) $$\lim_{x\to 0} \frac{\sqrt{1+\sin^2 x^2} - \cos^3 x^2}{x^3 \tan x} = \lim_{x\to 0} \frac{\frac{2x \sin x^2 \cos x^2}{\sqrt{1+\sin^2 x^2}} + 6x\cos^2 x^2 \cdot \sin x^2}{3x^2 \tan x + x^3 \sec^2 x}$$

$$= \lim_{x\to 0} \frac{2\cos x^2 \cdot \frac{\sin x^2}{x^2} + 6\cos^2 x^2 \cdot \frac{\sin x^2}{x^2}}{\frac{3\tan x}{x} + \sec^2 x}$$

$$= \frac{8}{4} = \underline{2}$$

(2) $$\lim_{x\to 0} \frac{\int_0^x \sin(xt^3)\,dt}{x^5} = \lim_{x\to 0} \frac{\sin x^4}{5x^4} = \underline{\frac{1}{5}}$$

2. 

각각의 경우에 x가 0 기준으로 오른쪽인지 왼쪽인지를 따지기.

$\lim_{x\to 0-} f(x^2-x) = A$, $\lim_{x\to 0-} \left(f(x^2)-f(x)\right) = A-B$, $\lim_{x\to 0+} f(x^3-x) = B$

$\lim_{x\to 0-} \left(f(x^3)-f(x)\right) = B-B=0$, $\lim_{x\to 1-} f(x^2-x)$ letting $x-1=t$,

$\lim_{t\to 0-} f(t^2+t) = B$

3. Let $\cos x = t$, $-\sin x\,dx = dt$.

$$\Rightarrow -\int (1-t^2)^2 \cdot t^4\,dt = -\int (t^4-2t^2+1)t^4\,dt = -\int t^8-2t^6+t^4\,dt$$

$$= -\frac{1}{9}t^9 + \frac{2}{7}t^7 - \frac{1}{5}t^5 + C$$

$$\therefore \int \sin^5 x \cos^4 x\,dx = \underline{-\frac{1}{9}\cos^9 x + \frac{2}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C}$$

4. $s^2Y - sy(0) - y'(0) + 3sY - 3y(0) + 2Y = \frac{1}{s+2}$

$s^2Y + 2s - 1 + 3sY + 6 + 2Y = \frac{1}{s+2}$, $\quad Y(s^2+3s+2) = \frac{1}{s+2} - (2s+5)$

$$Y(s) = \frac{1}{(s+1)(s+2)^2} - \frac{2s+5}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} - \left(\frac{D}{s+1} + \frac{E}{s+2}\right)$$

① $1 = As^2 + 4As + 4A + Bs^2 + 3Bs + 2B + Cs + C$

$(A+B)s^2 + (4A+3B+C)s + 4A + 2B + C = 0$

$4A+3B+C=0$ $\qquad B=-1,\ A=1,\ C=-1$

$4A+2B+C=1$

② $D+E=2$ $\qquad D=3 \quad E=-1$

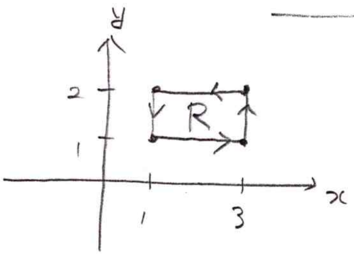
$2D+E=5$

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2} - \frac{3}{s+1} + \frac{1}{s+2}$$

$\therefore\ y(t) = e^{-t} - e^{-2t} - te^{-2t} - 3e^{-t} + e^{-2t}$

$= -2e^{-t} - te^{-2t}, \quad t>0.$

5.



$L(x,y) = xy$, $M(x,y) = xy$.

$$\int_1^2\int_1^3 (y-x)\,dx\,dy = \int_1^2 \left[xy - \frac{x^2}{2}\Big|_1^3 \right] dy$$

$$= \int_1^2 2y-4\,dy = y^2 - 4y\Big|_1^2 = 3 - 4 = -1$$

6.

7

8. (a) $\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \int_0^T f(t)e^{-st}dt + \int_T^\infty f(t)e^{-st}dt.$

두 번째 term 에서, $f(t) = f(t-T)$ 이므로 ($\because f(t)$ is a periodic function)

$\int_T^\infty f(t)e^{-st}dt = \int_T^\infty f(t-T)e^{-st}dt$, , $t-T=\tau$ 라고 하면

$$\int_T^\infty f(t-T)e^{-st}dt = \int_0^\infty f(\tau)e^{-s(\tau+T)}d\tau = e^{-sT}\int_0^\infty f(\tau)e^{-s\tau}d\tau.$$

$$\therefore \mathcal{L}\{f(t)\} = \int_0^T f(t)e^{-st}dt + e^{-sT}\mathcal{L}\{f(t)\},$$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt$$

(b) $$\sum_{n=-\infty}^{\infty}|x[n]|^2 = \sum_{n=-\infty}^{\infty} x[n]x^*[n] = \frac{1}{4\pi^2}\sum_{n=-\infty}^{\infty}\int_0^{2\pi} X(e^{j\omega_0})e^{j\omega_0 n}d\omega_0 \int_0^{2\pi} X^*(e^{j\omega_1})e^{-j\omega_1 n}d\omega_1$$

$$= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} X(e^{j\omega_0})X^*(e^{j\omega_1})\sum_{n=-\infty}^{\infty} e^{-j(\omega_1-\omega_0)n}d\omega_0\, d\omega_1$$

$$= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} X(e^{j\omega_0})X^*(e^{j\omega_1})\, 2\pi\,\delta(\omega_1-\omega_0)\, d\omega_0\, d\omega_1$$

$$= \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega_1})X^*(e^{j\omega_1})\, d\omega_1$$

$$= \frac{1}{2\pi}\int_0^{2\pi} |X(e^{j\omega})|^2 d\omega.$$

$$\therefore \sum_{n=-\infty}^{\infty}|x[n]|^2 = \frac{1}{2\pi}\int_0^{2\pi}|X(e^{j\omega})|^2 d\omega$$

∴ 2015 통신

1. $y[n] + a_1 y[n-1] + a_2 y[n-2] = 0$

$\Rightarrow Y(z) + a_1 (z^{-1} Y(z) + y[-1]) + a_2 (z^{-2} Y(z) + z^{-1} y[-1] + y[-2]) = 0$

$Y(z) (1 + a_1 z^{-1} + a_2 z^{-2}) + a_1 y[-1] + a_2 z^{-1} y[-1] + a_2 y[-2] = 0.$

$$Y(z) = \frac{-(a_1 y[-1] + a_2 y[-2] + a_2 y[-1] z^{-1})}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

(a) $1 + a_1 z^{-1} + a_2 z^{-2} = (1 - e^{j\Omega_0} z^{-1})(1 - e^{-j\Omega_0} z^{-1})$

$= 1 - (e^{j\Omega_0} + e^{-j\Omega_0}) z^{-1} + z^{-2}.$

∴ $\underline{a_1 = -2\cos\Omega_0, \; a_2 = 1}$

(b) $y[n] = A\cos\theta \cos(\Omega_0 n) - A\sin\theta \sin(\Omega_0 n)$

$$\Rightarrow Y(z) = A\cos\theta \left(\frac{1 - z^{-1}\cos\Omega_0}{1 - 2(\cos\Omega_0) z^{-1} + z^{-2}} \right) - A\sin\theta \left(\frac{z^{-1}\sin\Omega_0}{1 - 2(\cos\Omega_0) z^{-1} + z^{-2}} \right).$$

So, $A\cos\theta - z^{-1}(A\cos\theta\cos\Omega_0 + A\sin\theta\sin\Omega_0)$

$= A\cos\theta - A z^{-1} \cos(\theta - \Omega_0)$,

$$∴ \begin{cases} -a_1 y[-1] - a_2 y[-2] = A\cos\theta. \\ a_2 y[-1] = A\cos(\theta - \Omega_0) \end{cases}$$

(c) 몰라

2.

(a) $e^{j\theta} = \cos\theta + j\sin\theta$

(b) $Re\{x\} = \dfrac{x+x^*}{2}$

(c) $\dfrac{1}{2T}\int_0^T e^{j(2\pi f_k t+\theta_k)} e^{-j(2\pi f_{k'} t+\theta_{k'})} dt = \dfrac{e^{j(\theta_k-\theta_{k'})}}{2T} \cdot \dfrac{1}{j2\pi(f_k-f_{k'})}\left[e^{j2\pi T(f_k-f_{k'})}-1\right] = 0$

$\Rightarrow e^{j2\pi T(f_k-f_{k'})} = 1 = e^{j2\pi n}$, for $n \in \mathbb{Z}$.

$|f_k - f_{k'}| = 0$ 인 경우, 원래의 식에서 $\dfrac{e^{j(\theta_k-\theta_{k'})}}{2T}\int_0^T 1\,dt \neq 0$ 이므로

$\underline{\min\{|f_k-f_{k'}|\} = \dfrac{1}{T}}$.

(d) $\dfrac{1}{2T}\int_0^T e^{-j(2\pi f_k t+\theta_k)} e^{-j(2\pi f_{k'} t+\theta_{k'})} dt = \dfrac{e^{-j(\theta_k+\theta_{k'})}}{2T} \cdot \dfrac{1}{-j2\pi(f_k+f_{k'})}\left[e^{-j2\pi T(f_k+f_{k'})}-1\right]$

$= \dfrac{e^{-j(\theta_k+\theta_{k'})}}{2T(f_k+f_{k'})\pi} \cdot e^{-j\pi T(f_k+f_{k'})} \cdot \sin(2\pi T(f_k+f_{k'}))$, 여기서 magnitude의 최대치가

$\dfrac{1}{2\pi} \cdot \dfrac{1}{T(f_k+f_{k'})}$ 인데 $Tf_k \gg 1$, $\forall k$ 이므로 approximately equal to zero.

(e) $A = \dfrac{1}{T}\int_0^T \left(Re\left\{\sum_{k=1}^{K}\sqrt{2P_k}\,d_k e^{j(2\pi f_k t+\theta_k)}\right\} + N(t)\right) \cdot e^{-j(2\pi f_{k'} t+\theta_{k'})} dt$, $N_0 = 0$ 이므로 noise의 power가 0, 즉 $N(t)=0$. , $Re\left\{\sum_{k=1}^{K}\sqrt{2P_k}\,d_k e^{j(2\pi f_k t+\theta_k)}\right\} = \sum_{k=1}^{K}\sqrt{2P_k}\,d_k Re\left\{e^{j(2\pi f_k t+\theta_k)}\right\}$.

$\therefore A = \dfrac{1}{T}\int_0^T \sum_{k=1}^{K}\sqrt{2P_k}\,d_k \cos(2\pi f_k t+\theta_k)\left(\cos(2\pi f_{k'} t+\theta_{k'}) - j\sin(2\pi f_{k'} t+\theta_{k'})\right) dt$

$k \neq k'$ 의 경우, 모든 term이 approximately zero가 되고 (by (d)), $k=k'$의 경우에는

$A = \dfrac{1}{T}\int_0^T \sqrt{2P_{k'}}\,d_{k'} \cos^2(2\pi f_{k'} t+\theta_{k'}) dt = \dfrac{\sqrt{2P_{k'}}\,d_{k'}}{T}\int_0^T \dfrac{1+\cos(4\pi f_{k'} t+2\theta_{k'})}{2} dt$

$= \sqrt{P_{k'}/2}\cdot d_{k'} \cdot \int_0^T \dfrac{1}{T} dt = \underline{\sqrt{\dfrac{P_{k'}}{2}}\,d_{k'}}$

$\therefore$ (f) $Re\{N_{k'}\} = \frac{1}{T}\int_0^T N(t)\, Re\{e^{-j(2\pi f_{k'}t + \theta_{k'})}\}dt = \frac{1}{T}\int_0^T N(t)\cos(2\pi f_{k'}t + \theta_{k'})\, dt.$

$E[Re\{N_{k'}\}] = \frac{1}{T}\int_0^T E[N(t)]\cos(2\pi f_{k'}t + \theta_{k'})\, dt = \underline{0.}$

$Var(Re\{N_{k'}\}) = E[Re\{N_{k'}\}^2] = \frac{1}{T^2}\int_0^T\int_0^T E[N(t_1)N(t_2)]\cos(2\pi f_{k'}t_1 + \theta_{k'})\cos(2\pi f_{k'}t_2 + \theta_{k'})\, dt_1 dt_2$

$= \frac{N_0}{2T^2}\int_0^T \cos^2(2\pi f_{k'}t + \theta_{k'})\, dt = \frac{N_0}{2T^2}\cdot\int_0^T \frac{1+\cos(4\pi f_{k'}t + \theta_{k'})}{2}dt = \underline{\frac{N_0}{4T}}$

(g) $Im\{N_{k'}\} = \frac{1}{T}\int_0^T N(t)\, Im\{e^{-j(2\pi f_{k'}t + \theta_{k'})}\}dt = -\frac{1}{T}\int_0^T N(t)\sin(2\pi f_{k'}t + \theta_{k'})\, dt.$

$E[Im\{N_{k'}\}] = \frac{-1}{T}\int_0^T E[N(t)]\sin(2\pi f_{k'}t + \theta_{k'})dt = \underline{0}$

$Var(Im\{N_{k'}\}) = E[Im\{N_{k'}\}^2] = \underline{\frac{N_0}{4T}}$

$E[Im\{N_{k'}\}Re\{N_{k'}\}] = -\frac{1}{T^2}\int_0^T\int_0^T E[N(t_1)N(t_2)]\cos(2\pi f_{k'}t_1 + \theta_{k'})\sin(2\pi f_{k'}t_2 + \theta_{k'})\, dt$

$= -\frac{N_0}{2T^2}\int_0^T \sin(2\pi f_{k'}t + \theta_{k'})\cos(2\pi f_{k'}t + \theta_{k'})\, dt = 0.$

$\therefore$ uncorrelated.

$\Rightarrow$ Gaussian distribution의 Fourier transform은 Gaussian distribution 이므로 approximately independent, identically distributed.

2015 제어 필수

1.

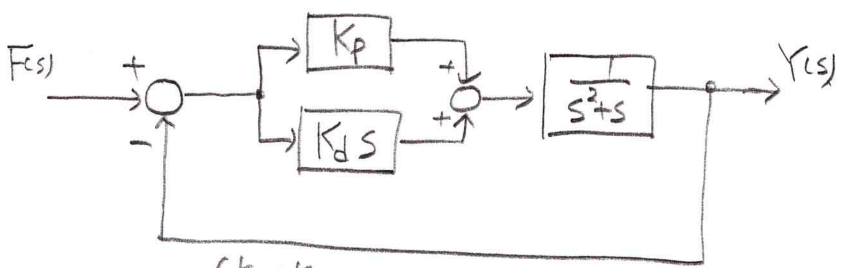
a) $f(t) - B\frac{dy(t)}{dt} = M\frac{d^2y(t)}{dt^2}$ $\quad \therefore \underline{\frac{d^2y(t)}{dt} = -\frac{B}{M}\frac{dy(t)}{dt} + \frac{1}{M}f(t)}$

b) With zero initial condition,

$s^2Y(s) = -\frac{B}{M}sY(s) + \frac{1}{M}F(s)$, $\quad (Ms^2+Bs)Y(s) = F(s)$

$\therefore \underline{\frac{Y(s)}{F(s)} = \frac{1}{Ms^2+Bs}}$

c)



$$\frac{Y(s)}{F(s)} = \frac{\frac{(k_p+k_d s)}{s^2+s}}{1+\frac{(k_p+k_d s)}{s^2+s}} = \frac{k_d s + k_p}{s^2+(1+k_d)s+k_p},$$

$(s+2)(s+5) = s^2+7s+10 = s^2+(1+k_d)s+k_p$

$\therefore \underline{k_d=6,\ k_p=10}$

d). In block diagram in problem c), we have

$$Y(s) = F(s)\left(\frac{6s+10}{s^2+7s+10}\right) = \frac{6s+10}{s(s+2)(s+5)} = \frac{A}{s}+\frac{B}{s+2}+\frac{C}{s+5}.$$

$As^2+7As+10A+Bs^2+5Bs+Cs^2+2Cs$

$= (A+B+C)s^2+(7A+5B+2C)s+10A = 6s+10. \rightarrow \underline{A=1}$

$B+C=-1$
$5B+2C=-1$
$2B+2C=-2.$

$\underline{B=\frac{1}{3}},\ \underline{C=-\frac{4}{3}}$ $\quad Y(s) = \frac{1}{s}+\frac{1}{3}\cdot\frac{1}{s+2}-\frac{4}{3}\cdot\frac{1}{s+5}$

$$\therefore \underline{y(t) = \left(1+\frac{1}{3}e^{-2t}-\frac{4}{3}e^{-5t}\right)u_s(t)}$$

· 제어 선택

1. observable at t_0 : $x(t_0)$를 결정적는 output $y[t_0, t_1]$, input $x[t_0, t_1]$를 알 수 있는 finite $t_1 > t_0$ 가 존재.

Stable at t_0 in the sense of Lyapunov : zero-input response ($\dot{x} = Ax$) is stable in the sense of Lyapunov if every finite initial state x_0 excites a bounded response. ⇐ 그냥 다 외워.

2. 1) $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -k_1 & -k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ -2-k_1 & -3-k_2 \end{bmatrix} x(t)$

$\begin{vmatrix} \lambda & -1 \\ 2+k_1 & \lambda+3+k_2 \end{vmatrix} = \lambda^2 + (3+k_2)\lambda + (2+k_1) = \lambda^2 + 4\lambda + 8$

$\therefore \underline{k_1 = 6,\ k_2 = 1}$

2)