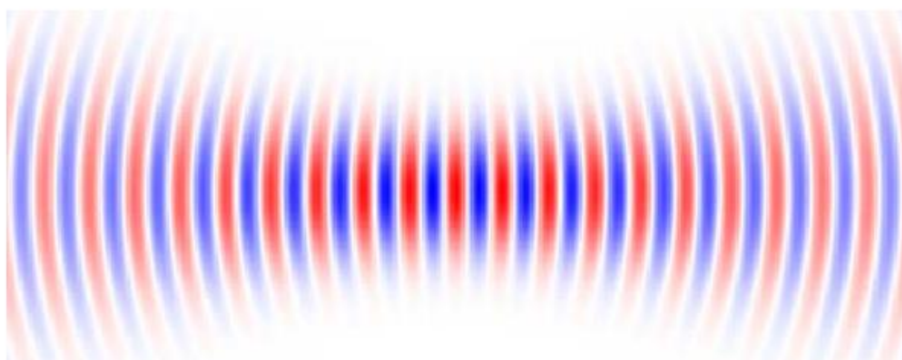
3.Beam Optics

Beam Optics

- Can light be spatially confined and transported in free space without angular spread? No!

- However, light can be confined in the form of beams that come as close as possible to spatially localized and nondiverging waves

- Gaussian Beam



From: www.rp-photonics.com/laser_beams.html

$$u(x, y, z) = \Psi(x, y, z)\mathrm{e}^{\mathrm{i}kz} .$$

$$\left(\nabla^2 + k^2\right) u(x, y, z) = 0$$

the following equation for u is obtained

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial \Psi}{\partial x}\mathrm{e}^{\mathrm{i}kz} , \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 \Psi}{\partial x^2}\mathrm{e}^{\mathrm{i}kz} , \\
\frac{\partial u}{\partial z} &= \frac{\partial \Psi}{\partial z}\mathrm{e}^{\mathrm{i}kz} + \mathrm{i}k\Psi\mathrm{e}^{\mathrm{i}kz} , \\
\frac{\partial^2 u}{\partial z^2} &= \frac{\partial^2 \Psi}{\partial z^2}\mathrm{e}^{\mathrm{i}kz} + 2\mathrm{i}k\frac{\partial \Psi}{\partial z}\mathrm{e}^{\mathrm{i}kz} - k^2\Psi\mathrm{e}^{\mathrm{i}kz} ; \\
\Rightarrow \left(\nabla^2 + k^2\right) u &= \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2}\right)\mathrm{e}^{\mathrm{i}kz} \\
&\quad + \frac{\partial^2 \Psi}{\partial z^2}\mathrm{e}^{\mathrm{i}kz} + 2\mathrm{i}k\frac{\partial \Psi}{\partial z}\mathrm{e}^{\mathrm{i}kz} = 0 .
\end{aligned}$$

$$\left|\frac{\partial^2 \Psi}{\partial z^2}\right| \ll \left|2k\frac{\partial \Psi}{\partial z}\right| = \frac{4\pi}{\lambda_n}\left|\frac{\partial \Psi}{\partial z}\right|$$

$$\Rightarrow \quad \left.\frac{|\Delta(\partial\Psi/\partial z)|}{|\partial\Psi/\partial z|}\right|_{\Delta z = \lambda_n} \ll 4\pi .$$

$$\boxed{\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + 2\mathrm{i}k\frac{\partial \Psi}{\partial z} = 0 .}$$

paraxial Helmholtz equation

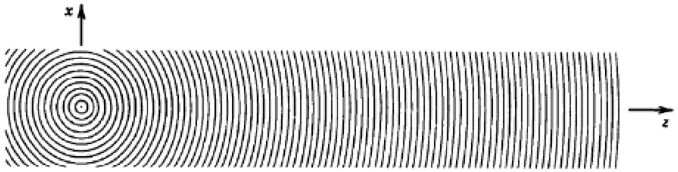
$$U(\mathbf{r}) = \frac{A}{z}\exp(-jkz)\exp[-jk_r(\frac{\rho}{2})] = A(\mathbf{r})\exp(-jkz)$$

Gaussian Beam

From the Paraxial Helmholtz equation: $$\boxed{\nabla_T^2 A - j2k\frac{\partial A}{\partial z} = 0}$$

$$\rightarrow \quad A(\mathbf{r}) = \frac{A_1}{z}\exp\left(-jk\frac{\rho^2}{2z}\right), \qquad \rho^2 = x^2 + y^2$$

Paraboloidal wave



$$\rightarrow \quad A(\mathbf{r}) = \frac{A_1}{q(z)}\exp\left[-jk\frac{\rho^2}{2q(z)}\right], \qquad q(z) = z - \xi$$

is also a solution to the Paraxial Helmholtz equation

$$\rightarrow \quad \boxed{A(\mathbf{r}) = \frac{A_1}{q(z)}\exp\left[-jk\frac{\rho^2}{2q(z)}\right], \qquad q(z) = z + jz_0}$$

The complex envelope of the Gaussian Beam

Gaussian Beam

$$A(\mathbf{r}) = \frac{A_1}{q(z)} \exp\left[-jk\frac{\rho^2}{2q(z)}\right], \qquad q(z) = z + jz_0$$

q-parameter

Rayleigh length

$$\frac{1}{q(z)} = \frac{1}{z + jz_0}$$

$$= \frac{z}{z^2 + z_0^2} - j\frac{z_0}{z^2 + z_0^2}$$

The complex envelope of the Gaussian Beam

$$\frac{1}{q(z)} = \frac{1}{R(z)} - j\frac{\lambda}{\pi W^2(z)}$$

$W(z)$: Beam width

$R(z)$: Wavefront radius of curvature

$$\frac{1}{R} = \frac{z}{z^2 + z_0^2}$$

$$\frac{\lambda}{\pi w^2} = \frac{z_0}{z^2 + z_0^2}$$

$$U(\mathbf{r}) = A_0 \frac{W_0}{W(z)} \exp\left[-\frac{\rho^2}{W^2(z)}\right] \exp\left[-jkz - jk\frac{\rho^2}{2R(z)} + j\zeta(z)\right]$$

where $A_0 = A_1/jz_0$

Gaussian Beam

$$U(\mathbf{r}) = A_0 \frac{W_0}{W(z)} \exp\left[-\frac{\rho^2}{W^2(z)}\right] \exp\left[-jkz - jk\frac{\rho^2}{2R(z)} + j\zeta(z)\right]$$

$$W(z) = W_0\left[1 + \left(\frac{z}{z_0}\right)^2\right]^{1/2}$$

$$R(z) = z\left[1 + \left(\frac{z_0}{z}\right)^2\right]$$

$$\zeta(z) = \tan^{-1}\frac{z}{z_0}$$

$$W_0 = \left(\frac{\lambda z_0}{\pi}\right)^{1/2}$$

Beam parameters

$$\frac{W_0}{W(z)}\exp\left(-\frac{r^2}{W^2(z)}\right)$$ Amplitude factor

$$\times \exp\left[-j\left(kz - \tan^{-1}\left(\frac{z}{z_0}\right)\right)\right]$$ Longitudinal phase

$$\times \exp\left(-j\frac{kr^2}{2R(z)}\right)$$ Radial phase

Intensity of a Gaussian Beam

From the definition of an optical intensity: $I(\mathbf{r}) = |U(\mathbf{r})|^2$

$$\boxed{I(\rho, z) = I_0 \left[\frac{W_0}{W(z)}\right]^2 \exp\left[-\frac{2\rho^2}{W^2(z)}\right],}$$

where $I_0 = |A_0|^2$

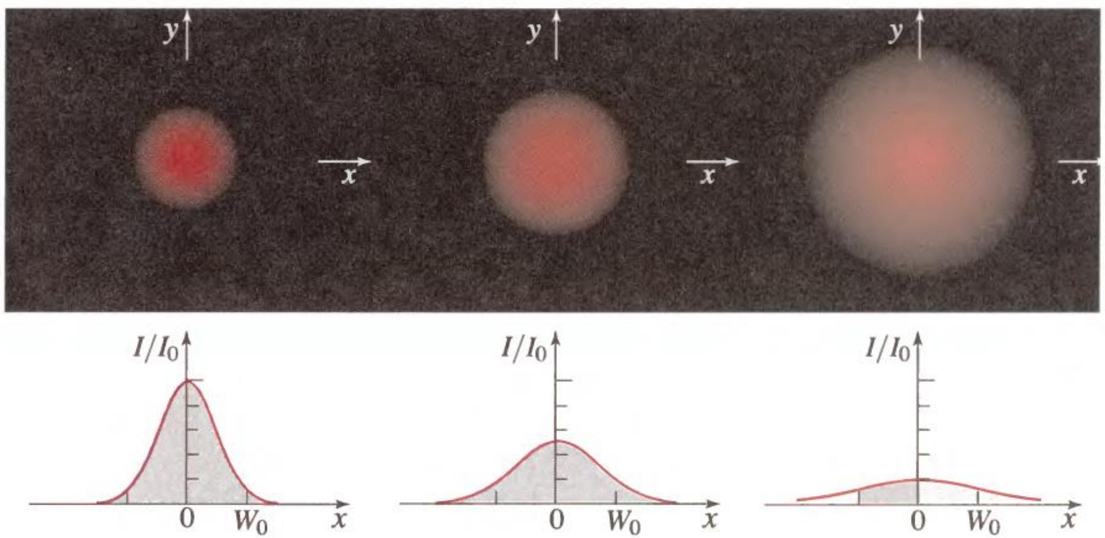


Figure 3.1-1 The normalized beam intensity I/I_0 as a function of the radial distance ρ at different axial distances: (*a*) $z = 0$; (*b*) $z = z_0$; (*c*) $z = 2z_0$.

- At each value of z, the intensity is a Gaussian function of the radial distance ρ.
- The width of the Gaussian distribution increases as the axial distance *z*

Intensity of a Gaussian Beam

On the beam axis ($\rho = 0$):

$$\boxed{I(0, z) = I_0\left[\frac{W_0}{W(z)}\right]^2 = \frac{I_0}{1 + (z/z_0)^2}}$$

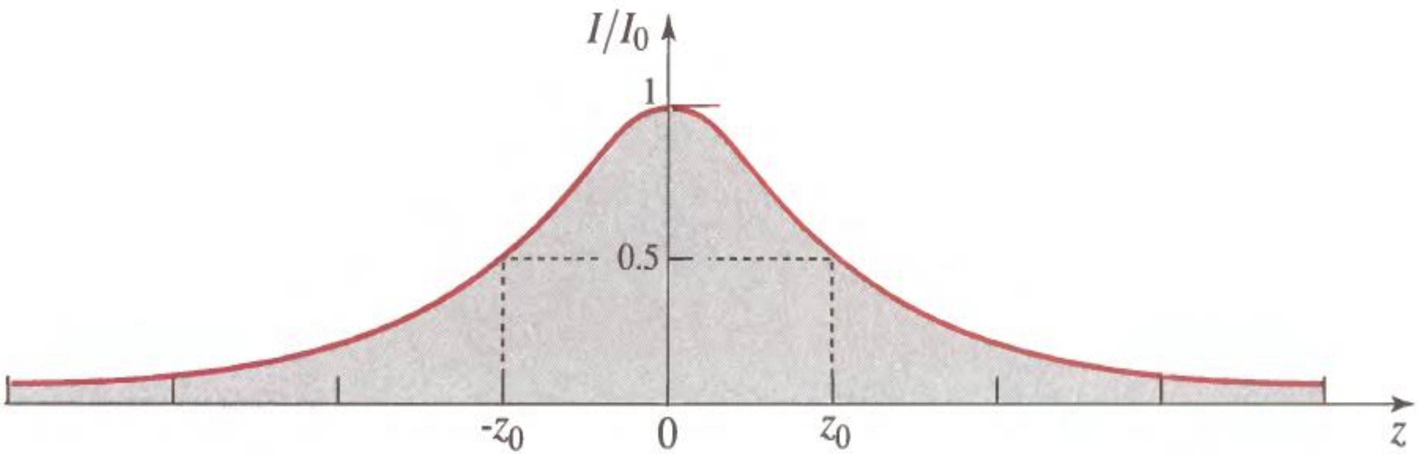


Figure 3.1-2 The normalized beam intensity I/I_0 at points on the beam axis ($\rho = 0$) as a function of distance along the beam axis, z.

When $|z| \gg z_0$, $I(0, z) \approx I_0 z_0^2/z^2$: an inverse-square law

Power of a Gaussian Beam

From the definition of a optical power:

$$P(t) = \int_A I(\mathbf{r}, t)\, dA.$$

The total optical power carried by the beam:

Beam area

$$P = \int_0^\infty I(\rho, z) 2\pi\rho\, d\rho = \tfrac{1}{2} I_0 (\pi W_0^2)$$

Independent of z

$$\boxed{I(\rho, z) = \frac{2P}{\pi W^2(z)} \exp\left[-\frac{2\rho^2}{W^2(z)}\right]}$$

Gaussian beam intensity

$$\frac{1}{P}\int_0^{\rho_0} I(\rho, z) 2\pi\rho\, d\rho = 1 - \exp\left[-\frac{2\rho_0^2}{W^2(z)}\right]$$

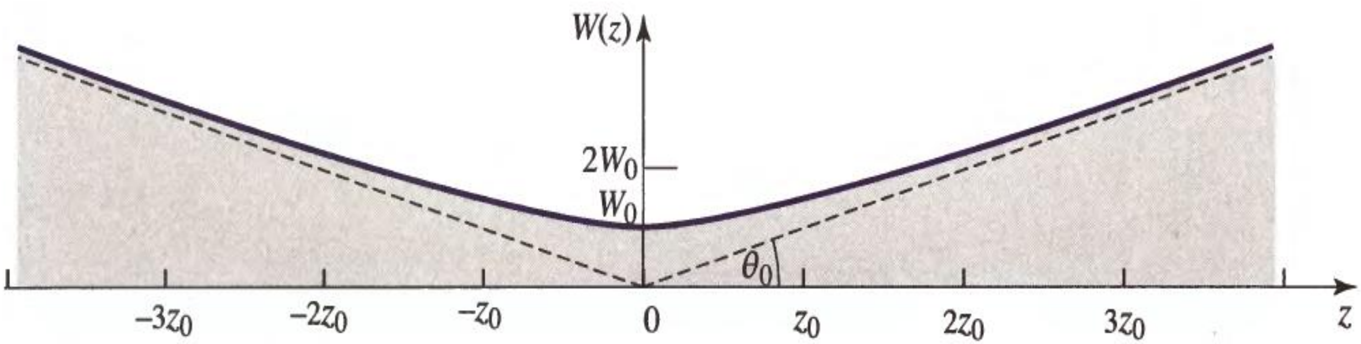
Ratio of the power carried within a circle of radius ρ

Gaussian Beam Width

The beam intensity drops by the factor $1/e^2 \simeq 0.135$ at the radial distance $\rho = W(z)$

$$\boxed{W(z) = W_0 \left[1 + \left(\frac{z}{z_0} \right)^2 \right]^{1/2}}$$

Beam width (or radius)



Waist radius = W_0

Spot size = $2W_0$

es

Figure 3.1-3 The beam width $W(z)$ assumes its minimum value W_0 at the beam waist ($z = 0$), reaches $\sqrt{2}W_0$ at $z = \pm z_0$, and increases linearly with z for large z.

Gaussian Beam Divergence

$z \to \infty$

$$W(z) = W_0\left[1+\left(\frac{z}{z_0}\right)^2\right]^{1/2} \approx \frac{W_0}{z_0} z = \theta_0 z \qquad (\text{For } z \gg z_0)$$

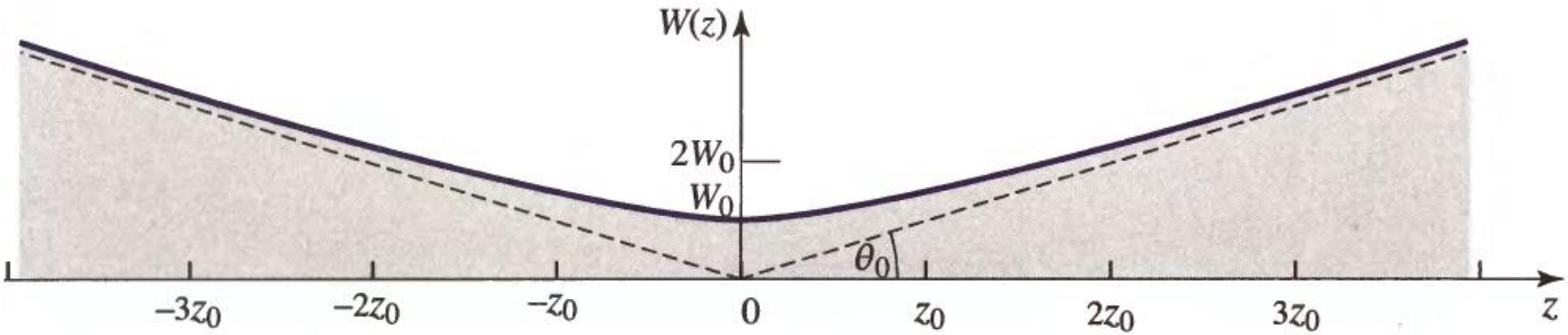
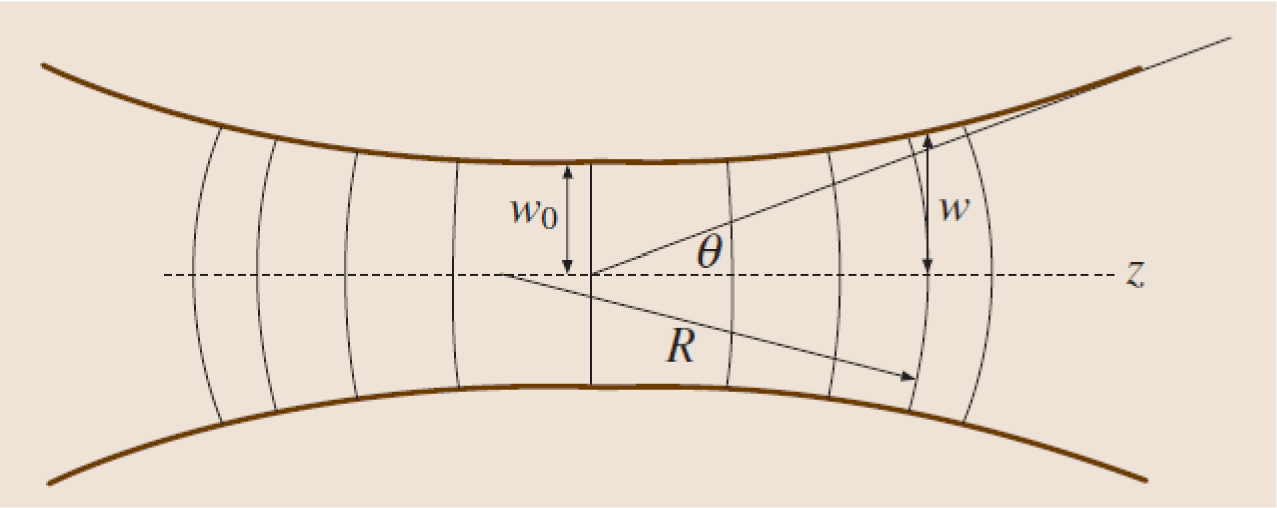


Figure 3.1-3 The beam width $W(z)$ assumes its minimum value W_0 at the beam waist ($z = 0$), reaches $\sqrt{2}W_0$ at $z = \pm z_0$, and increases linearly with z for large z.

$$2\theta_0 = \frac{4}{\pi}\frac{\lambda}{2W_0}$$: Angular divergence of the beam

Note: Squeezing the spot size leads to increased beam divergence



$$w(z \gg z_0) = \frac{w_0 z}{z_0} = \frac{\lambda_0 z}{\pi n w_0}$$

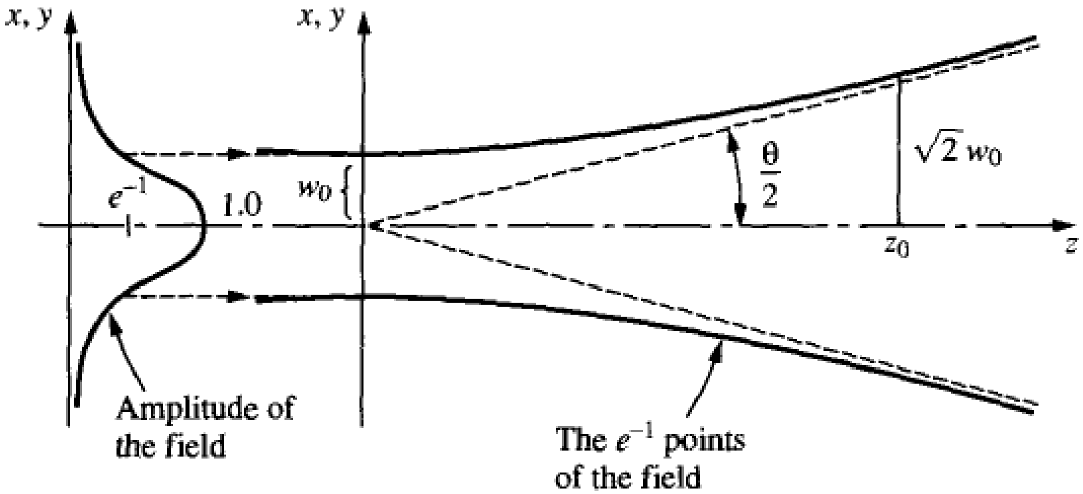


FIGURE 3.2. Spreading of a $TEM_{0,0}$ mode.

Depth of Focus

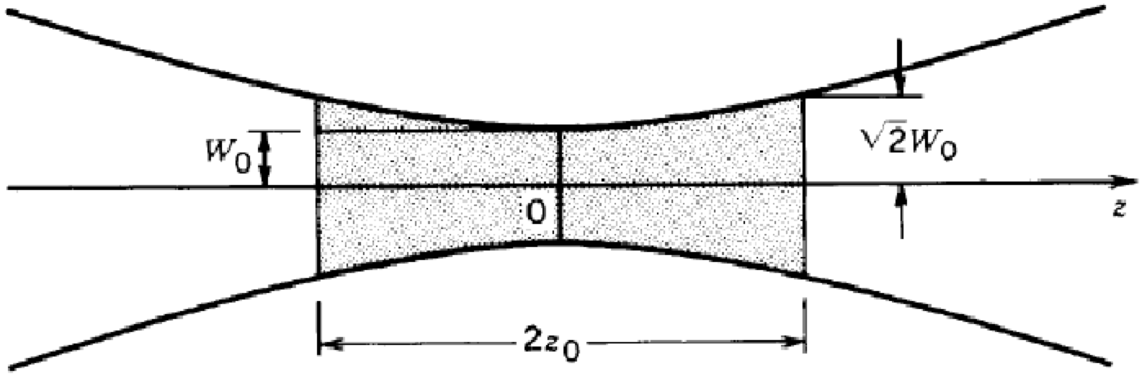


Figure 3.1-4 The depth of focus of a Gaussian beam.

<u>**Depth of focus**</u> (confocal parameter): The axial distance within which the beam radius lies within a factor $\sqrt{2}$ of its minimum value (i.e., its area lies within a factor of 2 of its minimum)

$$\boxed{2z_0 = \frac{2\pi W_0^2}{\lambda}}$$

Waist beam area

Note that a small spot size and a long depth of focus cannot be obtained simultaneously!

Phase

$$\varphi(\rho, z) = kz - \zeta(z) + \frac{k\rho^2}{2R(z)} \xrightarrow{\rho = 0} \boxed{\varphi(0, z) = kz - \zeta(z)}$$

Phase of a plane wave

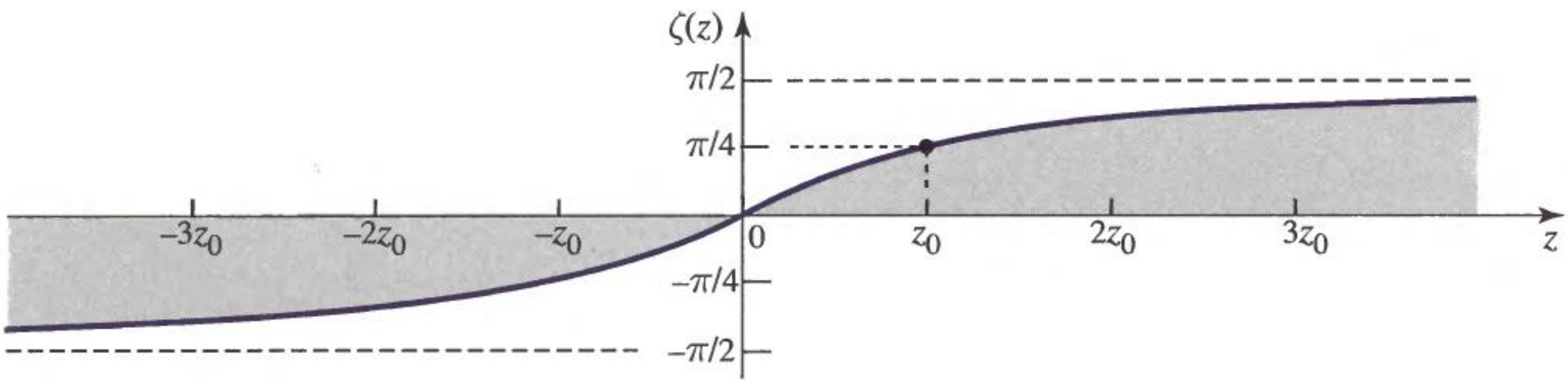


Figure 3.1-5 The function $\zeta(z)$ represents the phase retardation of the Gaussian beam relative to a uniform plane wave at points on the beam axis.

The total accumulated excess retardation (compared to the corresponding plane wave) as the wave travels from $z = -\infty$ to $z = \infty$ is π → Guoy effect

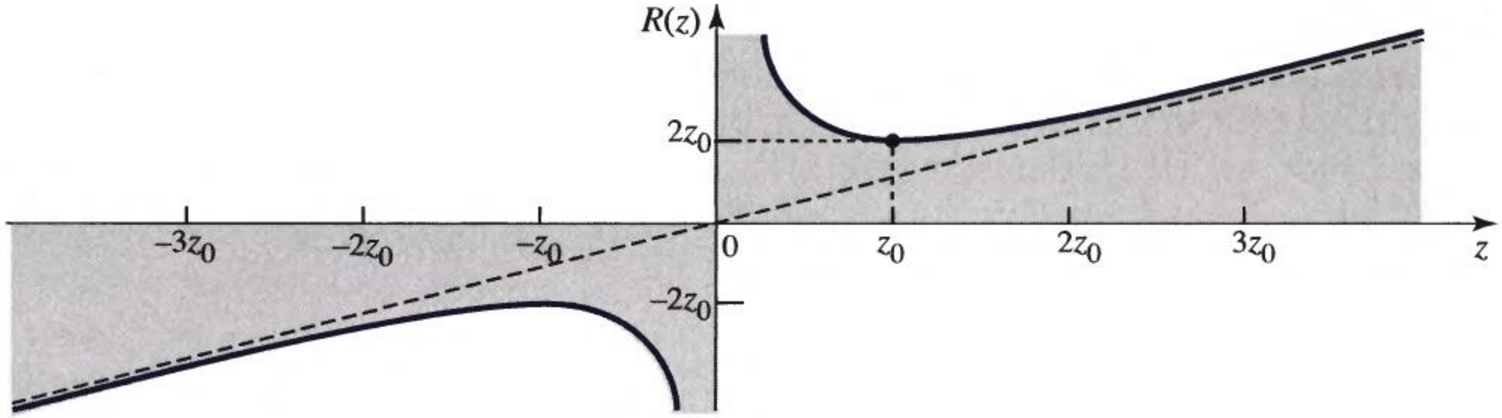


Figure 3.1-6 The radius of curvature $R(z)$ of the wavefronts of a Gaussian beam as a function of position along the beam axis. The dashed line is the radius of curvature of a spherical wave.

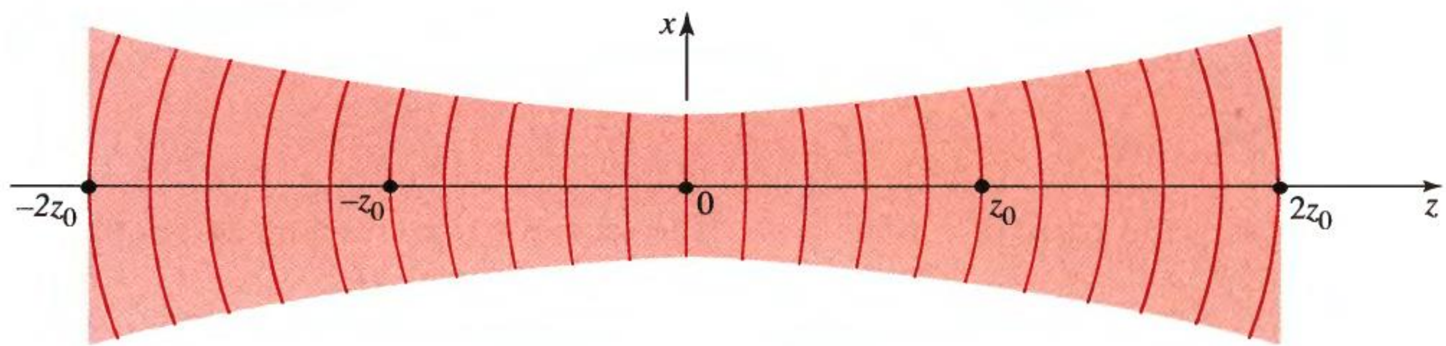


Figure 3.1-7 Wavefronts of a Gaussian beam.

Wavefront Bending

$$\varphi(\rho, z) = kz - \zeta(z) + \boxed{\frac{k\rho^2}{2R(z)}}$$

Responsible for wavefront bending

Deviation from the phase at off-axis points in a given transverse plane from that at the axial point

Wavefronts (= surfaces of constant phase) are given by

$$k[z + \rho^2/2R(z)] - \zeta(z) = 2\pi q$$

Since $\zeta(z)$ and $R(z)$ are relatively slowly varying,

$$\boxed{z + \rho^2/2R = q\lambda + \zeta\lambda/2\pi}$$

Equation of a paraboloidal surface of radius of curvature *R*

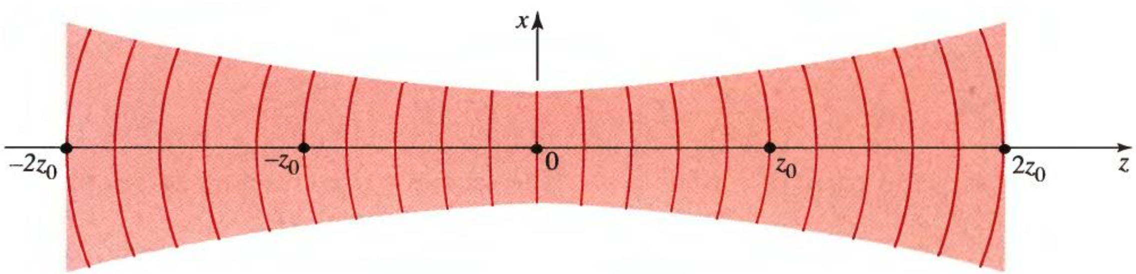


Figure 3.1-7 Wavefronts of a Gaussian beam.

Gaussian Beam

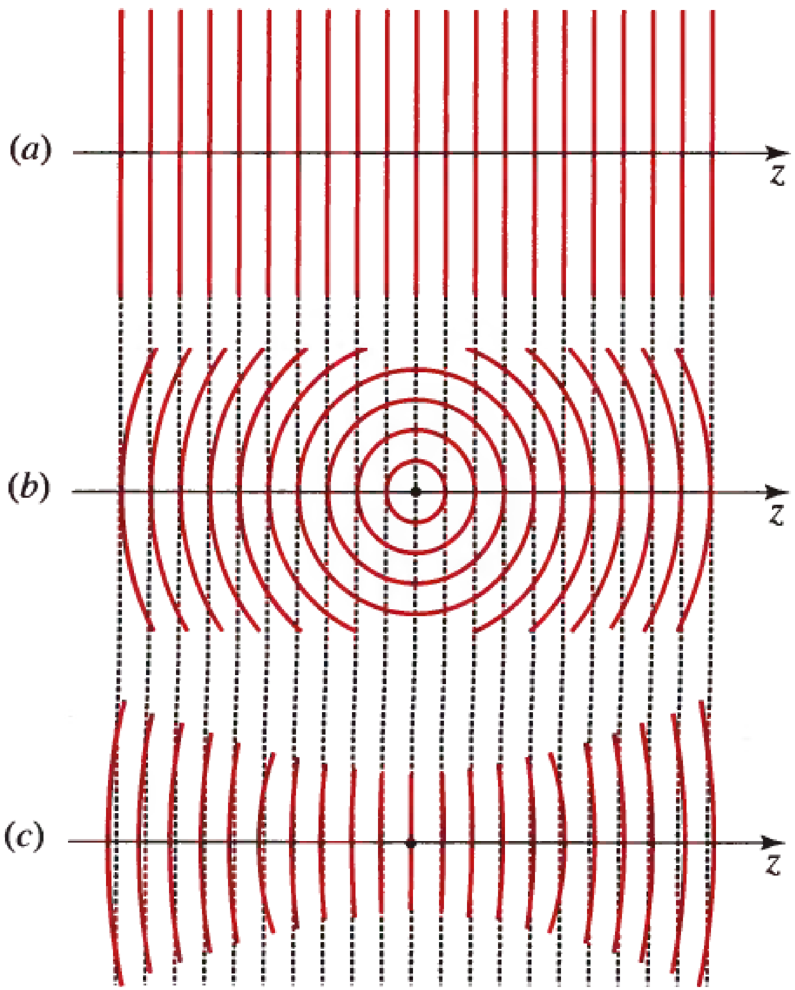


Figure 3.1-8 Wavefronts of (a) a uniform plane wave; (b) a spherical wave; (c) a Gaussian beam. At points near the beam center, the Gaussian beam resembles a plane wave. At large z the beam behaves like a spherical wave except that its phase is retarded by $\pi/2$ (a quarter of the distance between two adjacent wavefronts).

Transmission through a Thin Lens

Transmission through optical components:
The Gaussian beam remains a Gaussian beam (within the paraxial approximation) → Only the beam width and curvature changed

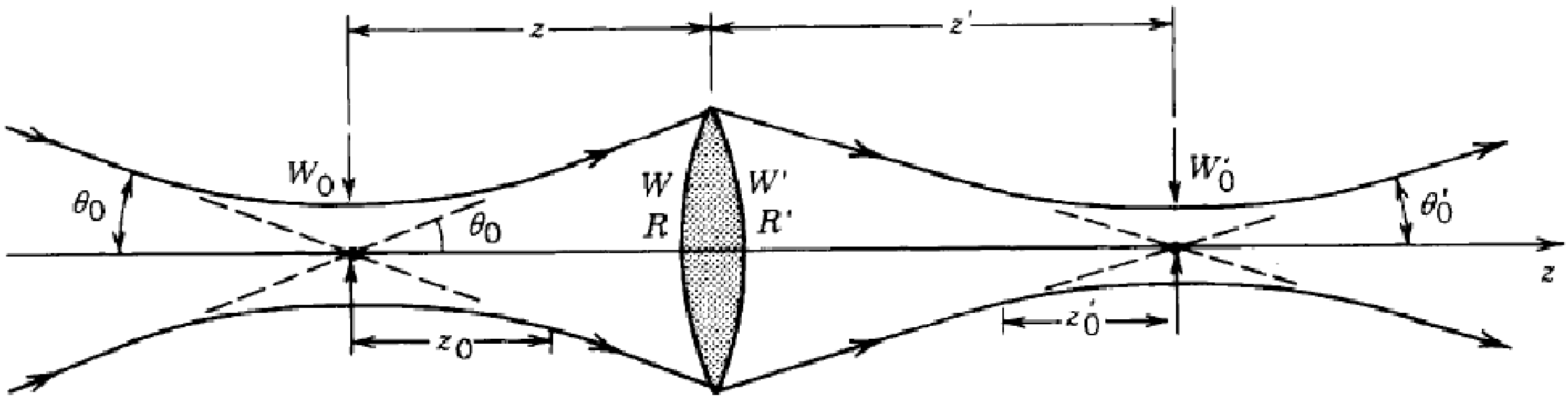


Figure 3.2-1 Transmission of a Gaussian beam through a thin lens.

Remember!

$$\boxed{t(x, y) \approx h_0 \exp\left[jk_o \frac{x^2 + y^2}{2f}\right]}$$

Complex amplitude transmittance of a thin lens

Transmission through a Thin Lens

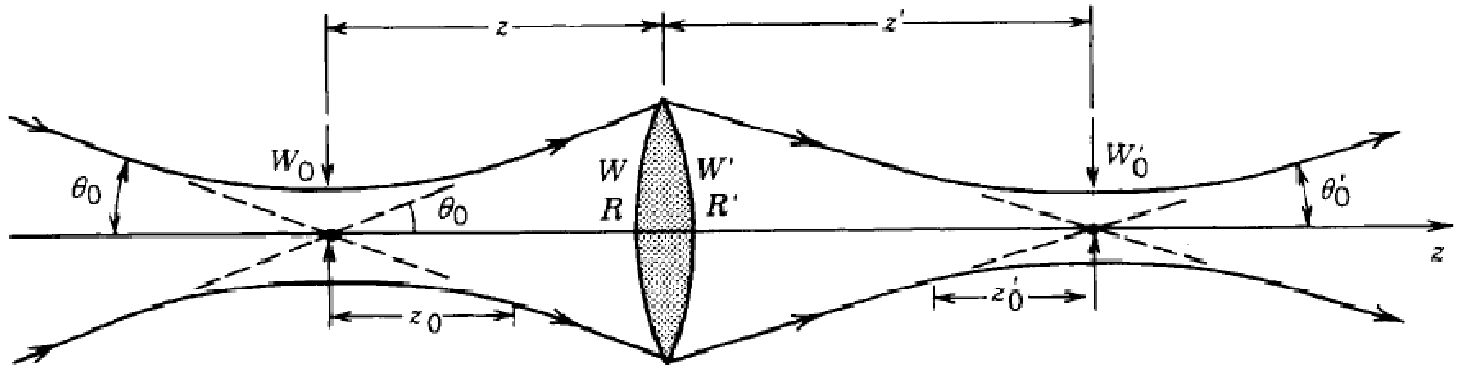


Figure 3.2-1 Transmission of a Gaussian beam through a thin lens.

After the transmission, the phase changes to:

$$kz + k\frac{\rho^2}{2R} - \zeta - k\frac{\rho^2}{2f} = \boxed{kz + k\frac{\rho^2}{2R'} - \zeta} \qquad \text{where} \quad \frac{1}{R'} = \frac{1}{R} - \frac{1}{f}$$

$$\boxed{\begin{gathered} W' = W \\ 1/R - 1/R' = 1/f \end{gathered}}$$

Transmission through a Thin Lens

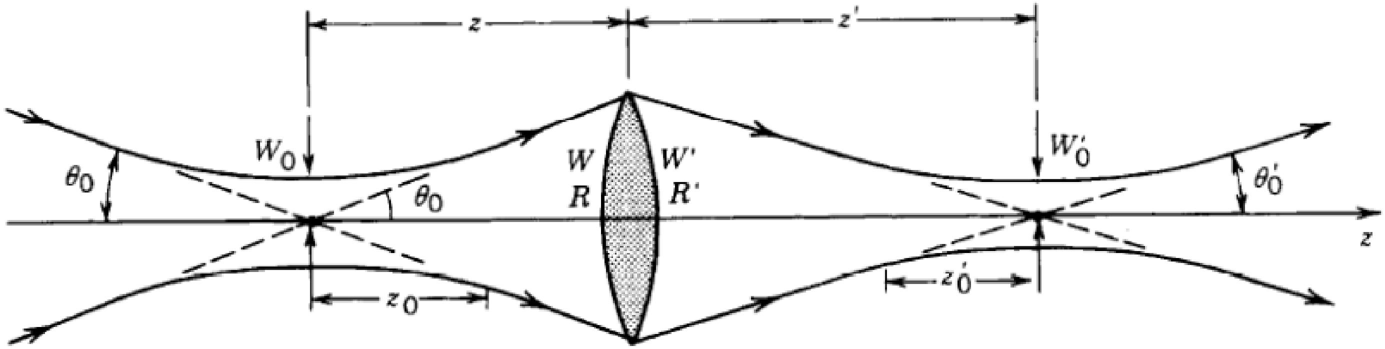
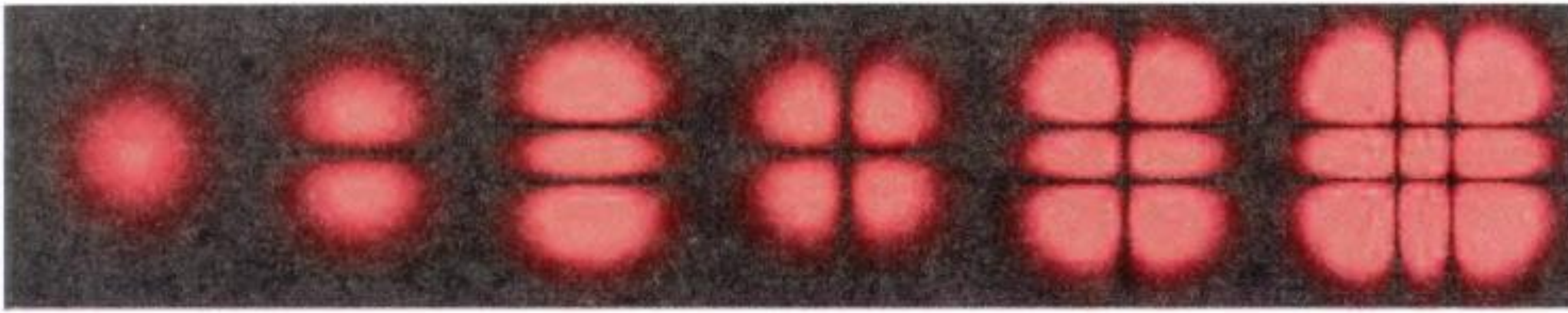


Figure 3.2-1 Transmission of a Gaussian beam through a thin lens.

Waist radius	$W_0' = MW_0$
Waist location	$(z' - f) = M^2(z - f)$
Depth of focus	$2z_0' = M^2(2z_0)$
Divergence	$2\theta_0' = \dfrac{2\theta_0}{M}$
Magnification	$M = \dfrac{M_r}{(1 + r^2)^{1/2}}$
$r = \dfrac{z_0}{z - f}$,	$M_r = \left\lvert \dfrac{f}{z - f} \right\rvert$.

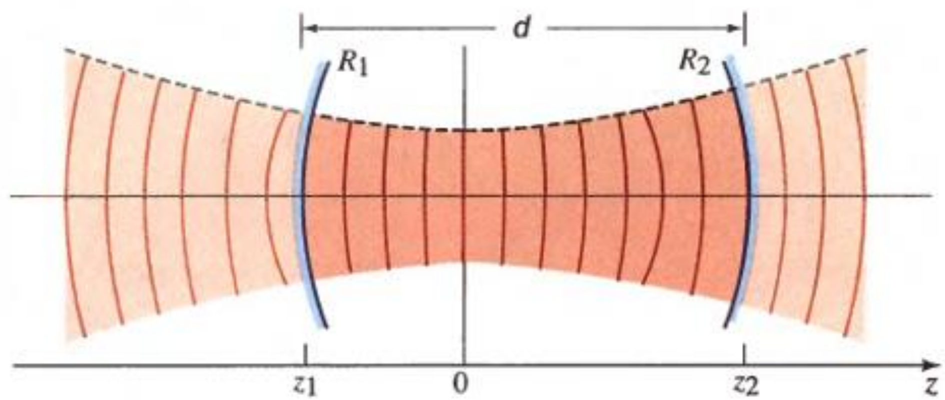
Parameter transformation by a lens



(0, 0) (0, 1) (0, 2) (1, 1) (1, 2) (2, 2)

Figure 3.3-2 Intensity distributions of several low-order Hermite–Gaussian beams in the transverse plane. The order (l, m) is indicated in each case.

Problem



Distance between two mirror: d

Curvature R_1 ,R_2

Center of Gaussian beam loacate z=0

$$\mathrm{R}_1 = \mathrm{z}_1 + \frac{z_0^2}{z_1} \qquad \text{---(A)}$$

$$\mathrm{R}_2 = \mathrm{z}_2 + \frac{z_0^2}{z_2} \qquad \text{---(B)} \qquad (\text{here, } \mathrm{z}_1+\mathrm{z}_2 = \mathrm{d})$$

Show that $\mathrm{z}_1 = \dfrac{d(R_2-d)}{R_2+R_1-2d}$ (Hint, (A)+(B))