Chapter 3
COMPLEX VARIABLES

Augustin-Louis Cauchy
(1789-1857)
Math/Physics
Complex Analysis
Stress Tensor

Lecture 11

3.7 Evaluation of Definite Integrals
E3.1 Causality and Nonlocality in Classical Electrodynamics

David Hilbert
(1862-1943)
Math
Hilbert Space
(Vector Space for QM)

3.7 Evaluation of Definite Integrals

The residue theorem is very useful to evaluate definite integrals of a function $f(x)$, and there are only **two standard forms of the definite integrals**.

There are still many other types of definite integrals which cannot be solved by standard, general prescription. Unfortunately, in these cases, a particular contour and its related techniques should be devised for each problem.

Type-1: Fourier Transform

Consider $f(z)$ under two conditions:

1) meromorphic in the upper half plane with a finite number M of poles z_m $(m = 1, 2, \cdots M)$
2) bounded by $\lim_{R\to\infty} R^n f(Re^{i\theta}) = 0, \quad n \geq 1$

Then we define a Fourier transform integral

$I = \int_{-\infty}^{\infty} dx\, f(x) e^{ikx},$

$$\boxed{I = \int_{-\infty}^{\infty} dx\, f(x) e^{ikx}, \quad k > 0} \tag{3.43}$$

Using the residue theorem, an integral on a closed contour C (Fig. 3-7) is given by

$$\lim_{R\to\infty} \oint_C dz\, f(z) e^{ikx} = \cancel{\lim_{R\to\infty} \int_{C_R} dz\, f(z) e^{ikx}} + I = 2\pi i \sum_{m=1}^{M} A_{-1}(z_m) \tag{3.44}$$

From the Jordan's lemma, the first integral vanishes, and thus we have

$$\int_{-\infty}^{\infty} dx\, f(x) e^{ikx} = 2\pi i \sum_{m=1}^{M} A_{-1}(z_m), \quad k > 0 \tag{3.45}$$

As a special case, from the estimation lemma, we can still evaluate the integral for $k = 0$,

$$\int_{-\infty}^{\infty} dx\, f(x) = 2\pi i \sum_{m=1}^{M} A_{-1}(z_m) \tag{3.46}$$

If the integrand has a pole in the real axis, we can add a small half circle in the closed contour, either clock-wise or counterclock-wise, to evaluate, in this case, the principal value of the definite integral (Fig. 3-5).

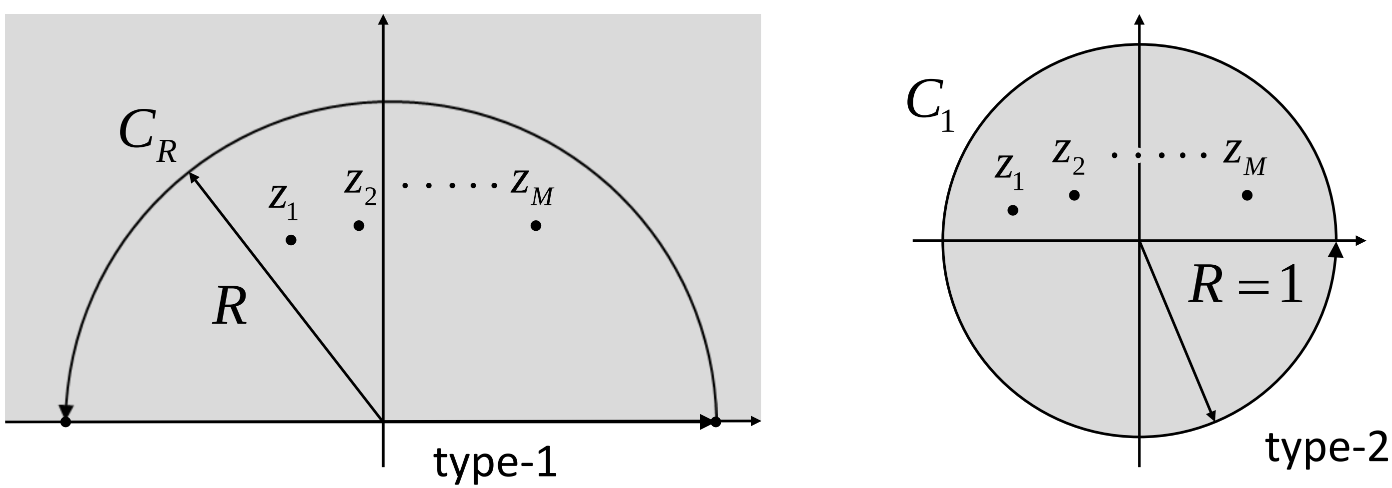


Fig. 3-7 Contours for type-1 and type-2 definite integrals

Type-2: Integrands with Sinusoidal Functions

Consider a single-valued, finite function $f(\sin\theta, \cos\theta)$ for all θ, and define a definite integral

$$I = \int_0^{2\pi} d\theta\, f(\sin\theta, \cos\theta) \tag{3.47}$$

Since the integral path is a unit circle ($z = e^{i\theta}$) in the z-plane (Fig. 3-7) in which there are a finite number M of poles z_m ($m = 1, 2, \cdots M$), we can use the Euler formula:

$$\sin\theta = \frac{z - z^{-1}}{2i}, \quad \cos\theta = \frac{z + z^{-1}}{2}, \quad dz = iz\,d\theta$$

Thus we obtain by the Cauchy theorem

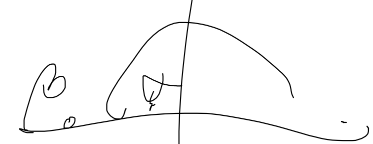
$$\boxed{I = -i\int_{C_1} \frac{dz}{z} f\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right) = 2\pi \sum_{m=1}^{M} A_{-1}(z_m)}$$

Ex) Type-1 definite integral: $I = \int_{-\infty}^{\infty} dx \frac{1}{1+x^2}$

$$I = \left[\oint_C dz - \int_{C_R} dz \right] \frac{1}{1+z^2} = \oint_C dz \frac{1}{1+z^2} = \oint_C dz \frac{1}{(z+i)(z-i)} = 2\pi i A_{-1}(i) = 2\pi i \frac{1}{2i} = \pi$$

Thus we obtain the definite integral of a **Lorentz function**

$$\boxed{\int_{-\infty}^{\infty} dx \frac{1}{1+x^2} = \pi}$$



Ex) Type-1 definite integral: $I = \int_0^{2\pi} \frac{d\theta}{a+\cos\theta}, \quad a > 1$

$$I = -i \oint_{|z|=1} \frac{dz}{z} \frac{1}{a + \frac{z}{2} + \frac{1}{2z}} = -2i \oint_{|z|=1} dz \frac{1}{(z-\alpha)(z-\beta)} \quad \text{with} \quad \left[\begin{array}{ll} \alpha = -a + \sqrt{a^2-1}, & |\alpha| < 1 \\ \beta = -a - \sqrt{a^2-1}, & |\beta| > 1 \end{array} \right.$$

So we have only one pole at $z = \alpha$ within the unit circle, and

$$\boxed{I = \int_0^{2\pi} \frac{d\theta}{a+\cos\theta} = -2i(2\pi i) \frac{1}{\alpha - \beta} = \frac{2\pi}{\sqrt{a^2-1}}}$$

$PV\int_{-\infty}^{\infty} dx \frac{f(x)}{x-x_0} = i\pi f(x_0)$

Ex) $I = \int_{-\infty}^{\infty} dx \frac{e^{ix}}{x}$

Here, we can directly use the Hilbert transform: $PV\int_{-\infty}^{\infty} dx \frac{f(x)}{x-x_0} = i\pi f(x_0)$

$PV\int_{-\infty}^{\infty} dx \frac{e^{ix}}{x} = i\pi f(0) = i\pi$

$$PV\int_{-\infty}^{\infty} dx \frac{e^{ix}}{x} = i\pi f(0) = i\pi$$

$$= PV\left[\cancel{\int_{-\infty}^{\infty} dx \frac{\cos x}{x}} + i\int_{-\infty}^{\infty} dx \frac{\sin x}{x} \right] = PV\left[i\int_{-\infty}^{\infty} dx \frac{\sin x}{x} \right] = i\int_{-\infty}^{\infty} dx \frac{\sin x}{x}$$

Now we have the definite integral of the sinc function:

$$\boxed{\int_{-\infty}^{\infty} dx \frac{\sin x}{x} = \pi}$$

In this example, note that the **sinc function** does not have a pole. It is a **regular function**. This is the reason why we dropped the PV symbol in the above result.

[Q] How do we know that the sinc function is a regular function?

E3-1 Causality and Nonlocality in Classical Electrodynamics*

The **causality** means **"No Effect before Cause"** or **"No Response before Excitation"**.
In other words, the **output function** $f(t)$ is a function of **the input function** $h(t')$ at $t > t'$.

The causality has **no proof**, and it is just **Nature's Law or Philosophy**, but once we decide to accept the causality, we can *derive* the **relation between** $f(t)$ **and** $h(t')$, which can be written as

$$\boxed{f(t) = \lim_{\Delta t \to 0} f[t; h(t-\Delta t), h(t-2\Delta t), h(t-3\Delta t), \cdots]} \tag{3.48}$$

In a linear system, the dependence of the effect on **the past history of the cause** can be represented by linear combinations of $h(t - n\Delta t)$ with expansion coefficient a_n :

$$f(t) = \lim_{\Delta t \to 0} \sum_{n=1}^{\infty} a_n h(t - n\Delta t) = \int_0^{\infty} dt' \frac{a(t')}{\Delta t} h(t - t') \tag{3.49}$$

Defining the response function as $r(t') = a(t') / \Delta t$, we have

$$\boxed{\begin{aligned} f(t) &= \int_0^{\infty} dt' r(t') h(t - t') \\ f(t) &= \int_{-\infty}^{t} dt' r(t - t') h(t') \end{aligned}} \quad (t - t' \to t') \tag{3.50}$$

*Here, we consider only linear systems.

This results exactly tells us the causality that **there is no response before excitation**, and can equivalently be written as

$$\boxed{f(t)=\int_{-\infty}^{\infty}dt'g(t-t')h(t')} \tag{3.51}$$

with an explicit causality condition

$$g(t-t')=0 \text{ for } t<t' \tag{3.52}$$

Using the convolution theorem, it can be written in the frequency domain as

$$\boxed{f(\omega)=r(\omega)h(\omega)} \tag{3.53}$$

where the causality condition should be always kept in mind. The causality is a kind of temporal nonlocality, and we can define spatial nonlocality in a similar manner.