Error propagation for the transformation of time domain into frequency domain.

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Abstract—**The aim for this paper is to propose an analytical procedure to calculate the random error in frequency domain (FD) for values achieved from time domain (TD) data by means of the Fourier transform. This error is calculated as a function of the errors and of the measurements parameters in TD. The formulation achieved is applied to calculate errors in harmonic signal analysis and in the reflection coefficient from TD measurements.**

I. INTRODUCTION.

In many electromagnetic problems such as: calculation of scattering parameters in TD [1], determination of magnetic field gradients [2], or in the modeling of electromagnetic interferences [3], the Fourier Transform (FT) is a part of their algorithm. Furthermore, in this operation is specially suitable to determinate the random error in the FD from the TD errors for the experimental data.

Up to now several efforts have been made in related subjects. Other authors [4 -5] review some methods to carry out the transformation from TD to FD but in these works only the systematic errors are estimated. However, it is possible to treat the problem of the random error estimation statistically, using Monte Carlo simulation techniques [6]. These provide only numerical solutions for the error and in addition are very expensive in processing time. Other advantage of an analytic formula as opposed to a numerical solution for the errors is that it gives an explicit relationship between the errors and the measurement parameters. This allows us to link the errors and the measurement parameters without having to repeat the whole calculation process.

The method here proposed has wide application on the study of performance limits and error characterization of problems that involve FT.

In this paper several examples are shown: One of which concerns with the harmonic analysis of power signals distorted by electrical machines. In this case the error level for the measurement system employed is established. A further example estimates the error in the Fourier transform of a voltage impulse. This result is an intermediate step in the error estimation of other parameters in the FD from TD measurements.

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A broadly employed parameter is the reflection coefficient, shown in the following example. This parameter is widely used in many fields of electrical engineering, for example in the characterization of material properties such as permittivity and permeability obtained from time domain measurement [7] or from antenna measurements [8].

II. THEORY

The method developed in this paper is based on the theory in the propagation of errors [9]. This concerns the way in which random errors are carried over from the data points to the final results.

The framework of this problem is as follows; we have a function, $y = f(u,v,...)$, of M random variables and we wish to determine the error in y from those, for the data variables u,v,... and from the functional dependence f(u,v,...).

We will restrict ourselves to functions whose most probable value is given by

$$\bar{y} = f(\bar{u}, \bar{v}, \ldots) \tag{1}$$

In the case that the errors are low enough to allow the expansion of y in first order Taylor´s series of the random variables u,v... for the scope of the errors, we can express the deviations of each sample y_i of the random variable y in terms of the deviations of the measured data u_i, v_i, ...

$$y_i - \bar{y} \cong (u_i - \bar{u})\left(\frac{\partial y}{\partial u}\right) + (v_i - \bar{v})\left(\frac{\partial y}{\partial v}\right) + \cdots \tag{2}$$

where each of the partial derivatives is evaluated with all other variables fixed at their mean values. Then in (2), the function is reduced to a linear combination of the data u,v,... Using basic statistic theory it is easy to obtain that the variance σ_y^2 for y is

$$\sigma_y^2 \cong \sigma_u^2\left(\frac{\partial y}{\partial u}\right)^2 + \sigma_v^2\left(\frac{\partial y}{\partial v}\right)^2 + 2\sigma_{uv}^2\left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial y}{\partial v}\right) + \cdots \tag{3}$$

where σ^2_{uv} is the covariance between the variables u, v.

Furthermore, if the random variables u,v,... are independent, the covariance between them is void and (3) is greatly simplified.

If we have another function z = g(u,v,...) of the random variables u,v,... and we assume these to be independent, then the covariance between them σ^2_{yz} is given by [10]

$$\sigma_{yz}^2 \cong \sigma_u^2\left(\frac{\partial y}{\partial u}\right)\left(\frac{\partial z}{\partial u}\right)+\sigma_v^2\left(\frac{\partial y}{\partial v}\right)\left(\frac{\partial z}{\partial v}\right)+\cdots \qquad (4)$$

III. The Method of Calculation.

In this paper a method has been developed to calculate how the errors in the Fourier transform, achieved from discrete time domain samples, are propagated.

In many practical problems we start with N time domain measurements $f(t_i)$, with noise characterized by standard errors $\sigma(t_i)$. Taking these errors into account as independent random variables and with gaussian density function, we intend to study how the errors propagate through the equation

$$f(\omega) = f_R(\omega) + jf_I(\omega) = \sum_{k=0}^{N-1} e^{-j(\omega\Delta t)\cdot k} f(\Delta t\cdot k)\Delta t \qquad (5)$$

where k is an integer and Δt is the sampling interval in the time domain. At this point it is convenient to remark that the results presented are also valid if the fast Fourier transform algorithm is employed to evaluate (5).

Using the theory described in the previous section, it is straightforward to deduce that the standard error for the real part of $f(\omega)$ is:

$$\sigma_{f_R(\omega)} = \left(\sum_{k=0}^{N-1}(\Delta t\cos(\omega\Delta t\cdot k))^2\sigma^2_{f(\Delta t\cdot k)}\right)^{1/2} \qquad (6)$$

and for the imaginary part is:

$$\sigma_{f_I(\omega)} = \left(\sum_{k=0}^{N-1}(\Delta t\sin(\omega\Delta t\cdot k))^2\sigma^2_{f(\Delta t\cdot k)}\right)^{1/2} \qquad (7)$$

These formulas couple the FD error with the sampling interval (Δt) and the number of samples (N).

In many situations it is more appropriate to operate with the magnitude and the phase of the complex number $f(\omega)$ than with the real and imaginary parts. This is the case when $f(\omega)$ is only a intermediate step in the computation of the final result, as in the example shown later in which the reflection coefficient is evaluated from two time domain signals by the transformation into the FD.

TABLE. I.
Comparison Of The Errors.
Calculating on a PC Pentium-75.

						CPU time
Frequency (GHz)	1.40	4.19	6.98	9.77	12.56	
Error by Monte Carlo (%)	1.76	1.89	2.27	2.55	3.06	132 s
Error by this work (%)	1.75	1.90	2.23	2.55	3.06	0.038 s

To obtain the errors in such random variables from those found in (6)(7) there is an additional difficulty being that $f_R(\omega)$ and $f_I(\omega)$ are not independent random variables which force us to calculate the covariance $\sigma^2_{f_R f_I(\omega)}$ between them. If the TD random variables $f(t_i)$ are independent it is easy to deduce that:

$$\sigma^2_{f_R f_I(\omega)} = -\frac{1}{2}\sum_{k=0}^{N-1}(\Delta t)^2\sin(2\omega\Delta t\cdot k)\sigma^2_{f(\Delta t\cdot k)} \qquad (8)$$

From (6)(7)(8) and assuming that the approximation in (3) is still valid, it is possible to deduce for the standard errors in magnitude and phase of $f(\omega)$, the following expressions:

$$\sigma_{|f(\omega)|} = \frac{1}{|f(\omega)|}\left[f_R^2\sigma^2_{f_R(\omega)} + f_I^2\sigma^2_{f_I(\omega)} + 2f_R f_I\sigma^2_{f_R f_I(\omega)}\right]^{1/2} \qquad (9)$$

$$\sigma_{\phi(\omega)} = \frac{1}{|f(\omega)|^2}\left[f_I^2\sigma^2_{f_R(\omega)} + f_R^2\sigma^2_{f_I(\omega)} - 2f_R f_I\sigma^2_{f_R f_I(\omega)}\right]^{1/2} \qquad (10)$$

where $\phi(\omega)$ is the phase of $f(\omega)$.

IV. Results and Discussion

In order to verify the previously developed formulation, all the results have been contrasted with those obtained by the Monte Carlo simulation technique given in [6]. This simulation consists on the generation of synthetic results from a measured signal plus a random noise, which is constructed through an error model of the experimental system. In Table I the errors in the magnitude in the frequency domain for both calculation methods are shown for an experimental impulse signal.

The results achieved by the Monte Carlo method were obtained from $M=10^4$ simulations. The errors obtained by both methods are very close, with a difference < 0,2% in all the cases checked. However the CPU time for the technique proposed in this paper is much less than that for the Monte Carlo algorithm, given that in this case it is only necessary to calculate the proposed formulas, which requires O(5Nk) operations (N is the number of values in the time domain and k is the number of frequency values to be calculated), whilst in the Monte Carlo simulation the whole process must be repeated M times and the number of operations involved being $O((10N+N\log_2(N))M)$.

The method proposed has been applied to several practical examples: The first signal used as example is the power supply of a electric arc welder. This signal was measured in the time domain by a Tektronix TDS 320 digitizing oscilloscope with the following conditions:

- Time per division: 2 ms/div.
- Voltage per division: 10 V/div.
- Probe: 10X.
- Number of samples acquired: 1000.

With the previous specifications the standard errors in time and voltage for the measured signal are given by σ_v=4v and σ_t=2.5ns. The signal was sampled at N equally spread time intervals $V(t_i)$ and the total error $\sigma^2_{V(t_i)}$ in voltage at any time, as shown in [6], is given by

$$\sigma^2_{V(t_i)} = \sigma^2_v + \left(\frac{\partial V(t_i)}{\partial t}\right)^2 \sigma^2_t \qquad (11)$$

As the signal measured is of low frequency and is not close to the bandwidth of the measurement system, the second term in (11) can be neglected and so the amount of error is thus independent of time, consequently we have ergodic white noise. Furthermore, as in this case we are working with a periodic signal, we must include a normalization constant (1/T) in (5), where T is the signal period (T=N×Δt). Equations (6)(7)(8)(9)(10) are now greatly simplified, and we achieve

$$\sigma_{|V(\omega)|} = \frac{\sigma_v}{\sqrt{2N}} \qquad (12)$$

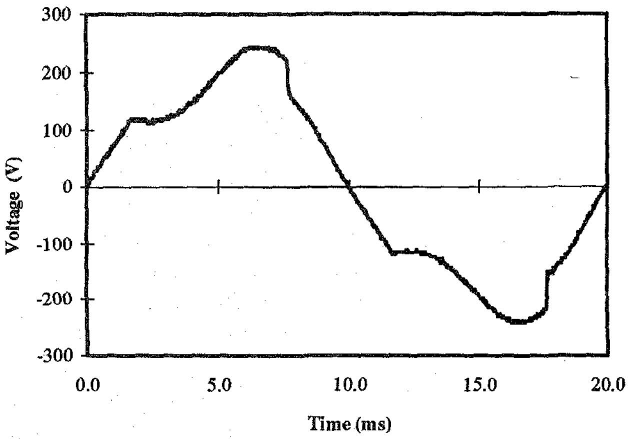

Fig. 1. Signal measured from the supply power of a welder in time domain.

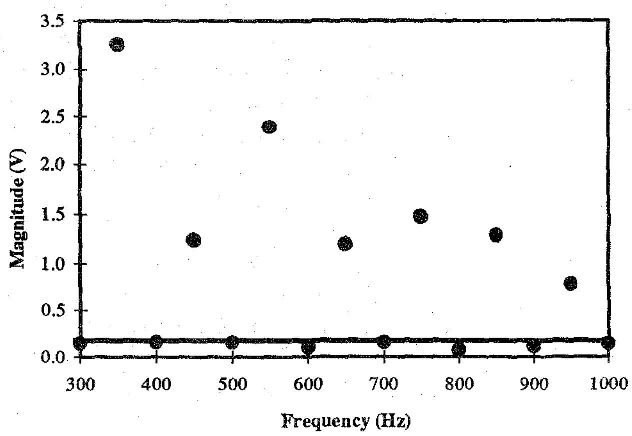

Fig. 2. The ● points are the values of the Fourier transform magnitude of the signal shown in Fig. 1. The solid line is the error for each frequency.

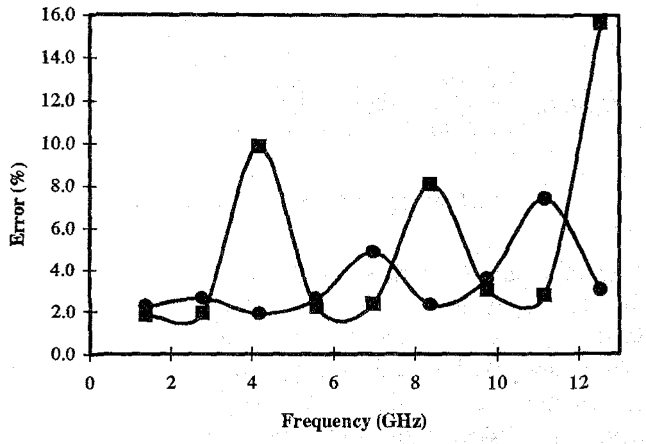

Fig. 3. The points indicated by ● are the errors in the real part of the spectrum of the impulse signal calculated using (6) an the ■ point are the errors in the imaginary part.

$$\sigma_{\Phi(\omega)} = \frac{\sigma_{|V(\omega)|}}{|V(\omega)|^2} \qquad (13)$$

Using (12) the error in the magnitude is calculated as $\sigma_{|V|} = 0{,}18\ \mathrm{V}$, which is frequency independent as is expected. The result is in agreement with those achieved by the more general equation (9) and by the Monte Carlo simulation.

In Fig. 1 the time domain signal measured from the power supply of a welder is shown. This was acquired with N=1000 samples per period. Fig. 2 represents the magnitude of the Fourier transform of the signal in Fig. 1, and its corresponding error achieved by (12), versus frequency.

An interesting point can be noted from (13), in that the error in the phase is greatly increased when the magnitude of the measured signal decreases.

The second application is the calculation of errors in the spectrum of a voltage impulse signal in the time domain V(t). This signal has 8 mV. amplitude and 28 ps width. This impulse was measured with a 20 GHz digitizing oscilloscope HP 54120B with 128 averages per sample and 12 GHz bandwidth mode. In these measurement conditions, the errors in voltage and time are characterized by normal probability distributions with standard errors given by σ_v = 44 μV and σ_t = 0.25 ps and therefore the total error is given by (11).

A detailed study of the measurement system [6] shows that errors for each $V(t_i)$ are independent and gaussian distributed and thus satisfy the conditions for the application of the equations presented in the previous section.

An interesting result is shown in Fig. 3. Here the errors in the real and imaginary part of the spectrum of the impulse signal previously described are represented. We regard a well-known experimental result, which is that when the real part is best measured the imaginary part is worse and vice-verse.

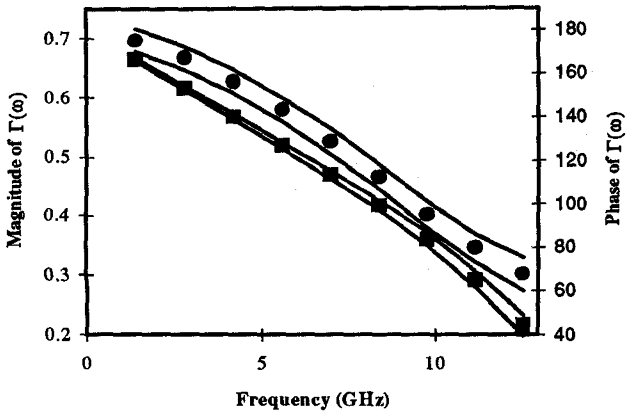

Fig. 4. The ● points are the values of the reflection coefficient magnitude and the solid lines are the error bands over it. The ■ points are the values of the reflection coefficient phase (degrees) and the solid lines are the error bands.

This was achieved using (6)(7) and is closely coincident with the Monte Carlo simulation, although is not shown in the figure because both are entirely superposed.

This result suggests an attractive research path in the optimization of measurements in the TD to achieve the minimum error in the real or in the imaginary part for a frequency range under study.

This technique can also be applied to the estimation of errors in others parameter achieved in the FD from TD measurements. A particularly useful result is the reflection coefficient $\Gamma(\omega)$, in the FD. This is obtained as the quotient between the reflected signal and the incident signal in a device under test. With the results presented in (9)(10), the error in $\Gamma(\omega)$ can be estimated from measurements of the incident signal and the reflected signal in the TD.

Fig. 4 shows the magnitude and phase of $\Gamma(\omega)$ and their error bands in function of the frequency. We can see that while in the magnitude of $\Gamma(\omega)$ the absolute error is almost constant with the frequency, in the phase the error is greatly increased with it.

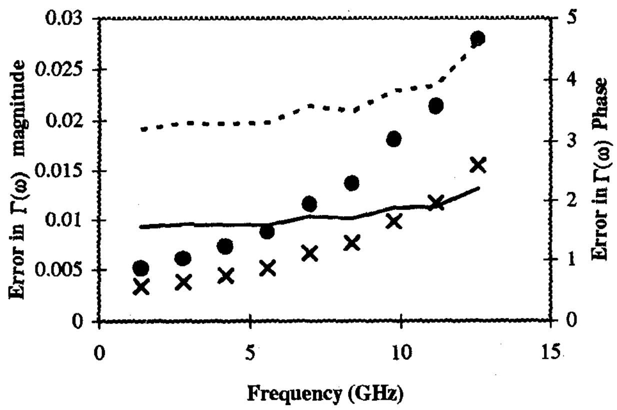

Fig. 5. The dashed line (512 samples) and the solid line (256 samples) are the errors in the magnitude of the reflection coefficient, and the ● and × points are the errors in the phase (Degrees).

Finally, an easy example is indicated of error minimization for results obtained in the FD from the optimization of parameters in the TD. This is based on the data of the previous example, in which the incident and reflected signal were sampled with 512 samples, the non truncation criterion being broadly satisfied. If the number of samples is decreased without truncating the signal, the error according to (6)(7) must decrease also. This is shown in Fig. 5, that represents the errors in magnitude and phase for $\Gamma(\omega)$ for same incident and reflected signals sampled with 256 and 512 samples.

V. Conclusion

The technique developed provides a connection between time domain and frequency domain errors when the TD errors are independent. This covers the error transmission in a frequently used operation such as the Fourier transform, allowing the analytical treatment of the error propagation.

This method allows us to improve the process time with respect to that of the Monte Carlo simulation technique.

The formulation implemented establishes the performance limits in several measurement techniques by means of error characterization.

An interesting branch of the work proposed, now in development within this research team, is to relate the errors in FD with TD measurement parameters, which permits their optimization in order to minimize FD errors.

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