ch.2 Spherical Harmonics

$$Y_{lm}(\theta,\phi)=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)\,e^{im\phi}$$

1. $Y_{l,-m}(\theta,\phi)=(-1)^m Y_{lm}^*(\theta,\phi)$

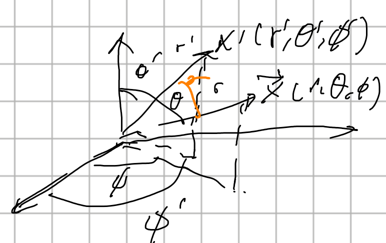
2. $\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\,d\theta\; Y_{lm}(\theta,\phi)\,Y_{l'm'}^*(\theta,\phi)=\delta_{ll'}\delta_{mm'}$

3. Closure selection

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)=\delta(\phi-\phi')\delta(\cos\theta-\cos\theta')$$

4. $g(\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}Y_{lm}(\theta,\phi)$, $A_{lm}=\int d\Omega\, Y_{lm}^*(\theta,\phi)\,g(\theta,\phi)$

5. Addition theorem for spherical harmonics



$$P_l(\cos\gamma)=\frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

6. generation function for $P_l(x)$, $-1\le x\le 1$

$$\frac{1}{\sqrt{1-2xt+t^2}}=\sum_{l=0}^{\infty}P_l(x)t^l \quad , |t|<1$$

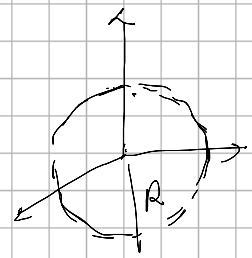
$$\frac{1}{|\vec{x}-\vec{x}'|}=\frac{1}{\sqrt{r^2-2rr'\,\hat{x}\cdot\hat{x}'+r'^2}}$$

$r_>$ ($r_<$) larger (smaller) of r & r'

$$=\frac{1}{r_>\sqrt{1-2\frac{r_<}{r_>}\underbrace{\hat{x}\cdot\hat{x}'}_{\cos\gamma}+\left(\frac{r_<}{r_>}\right)^2}}=\frac{1}{r_>}\sum_{l=0}^{\infty}P_l(\cos\gamma)\left(\frac{r_<}{r_>}\right)^l=\sum P_l(\cos\gamma)\frac{r_<^l}{r_>^{l+1}}$$

$$= 4\pi \sum_{l=0}^{m} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^{l}}{r_>^{l+1}} Y_{lm}^{*}(\theta', \phi') Y_{lm}(\theta, \phi)$$

Ex



Surface charge on a spherical shell with a results of R.

$\sigma = \sigma_0 \cos\theta, \quad r = R$

$\Phi = ?$

azimuthal symmetry : $\Phi(r, \theta, \phi) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$

$r > R \quad \Phi_{out} = A_0 + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \quad \because \Phi_{out}$ is finite at $r \to \infty$

$A_l = 0$ for $l \neq 0$

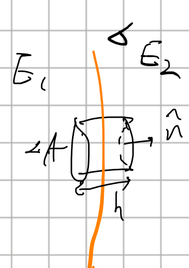
$r < R \quad \Phi_{in}(r, \theta) = \sum_l A_l' r^l P_l, \quad B_l = 0 \quad \because \Phi_{in}$ finite at $r = 0$.

BC 1. Potential must be continuous at $r = R$

$$\sum_l \frac{B_l}{R^{l+1}} P_l = \sum A_l' R^l P_l \Rightarrow \frac{B_l}{R^{l+1}} = A_l' R^l$$

BC2. Due to the surface charge, the normal component of E on the surface is discontinuous

cf.



$\nabla \cdot \vec{E} = 4\pi\rho$

$\int_V (\nabla \cdot \vec{E}) dV = 4\pi \int \rho dV$

$\int_S \vec{E} \cdot \hat{n} da = 4\pi \int \rho dV$

$h \to 0$

$[E_2 \hat{n} + E_1(-\hat{n})] \Delta A$

$= 4\pi \sigma \Delta A$

$\Rightarrow (\vec{E}_2 - \vec{E}_1) \cdot \hat{n} = 4\pi\sigma$

$$E_{n,out} - E_{n,in} = 4\pi\sigma$$

$$E_n = -\frac{\partial}{\partial r}\Phi$$

$$-\sum_l B_l\,[-(l+1)]\,r^{-(l+2)}P_l - \left(-\sum_l l A_l' r^{l-1} P_l\right)$$

$$= 4\pi\sigma_0 \cos\theta = 4\pi\sigma_0 P_1$$

For $l \neq 1$, $A_l' = 0$, $B_l = 0$

For $l = 1$, $A_1' + \frac{2B_1}{R^3} = 4\pi\sigma_0$

together with $\frac{B_1}{R^2} = A_1' R$

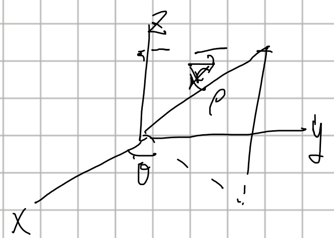
$$A_1' = \frac{4\pi}{3}\sigma_0, \quad B_1 = \frac{4\pi}{3}R^3\sigma_0$$

$$\Phi_{in} = A_1' r\cos\theta = \frac{4\pi}{3}\sigma r\cos\theta = \frac{4\pi}{3}\sigma z,$$ uniform field.

$$\Phi_{out} = \frac{B_1}{r^2}P_1 = \frac{4\pi}{3}R^3\sigma_0\frac{\cos\theta}{r^2} \propto \frac{1}{r^2}$$

dipole field.

d) Laplace equation in cylindrical coordinates.



$$\nabla^2 \Phi(\rho, \theta, z) = 0$$

$$\frac{\partial^2}{\partial \rho^2}\Phi + \frac{1}{\rho}\frac{\partial}{\partial \rho}\Phi + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\Phi(\rho, \theta, z) = R(\rho)\,\Theta(\theta)\,Z(z)$$

$$d^2 Z / dz^2 = k^2 Z(z) \rightarrow Z = e^{\pm kz}$$

$$d^2 \Theta / d\theta^2 = -\nu^2 \Theta \rightarrow \Theta = e^{\pm i\nu\theta}$$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right)R = 0$$

For $k\rho = x$, $\frac{d^2 R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)R = 0$

Bessel's equation

Solution; Bessel's function

1^{st} kind : $J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(j+\nu+1)} \left(\frac{x}{2}\right)^{2j}$

$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{j} \frac{(-1)^j}{j!\,(j-\nu+1)} \left(\frac{x}{2}\right)^{2j}$$

In our case, ν is integer $\because$ $\Theta(\theta) = \Theta(\theta + 2\pi)$

$$\nu = m$$

$$I_m = (-1)^m J_m(x)$$

2^{nd} kind Neumann function

$$N_\nu = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

3^{rd} kind Hankel function

$$H_\nu^{(1)} = J_\nu + i N_\nu$$

$$H_\nu^{(2)} = J_\nu - i N_\nu$$

$\{H_\nu^{(1)}, H_\nu^{(2)}\}$, $\{J_\nu, N_\nu\}$ are sets of orthogonal functions

usually used.

$$\Phi(\rho, \theta, z) = \sum_{m,k} \left[A_m J_m(k\rho) + B_m N_m\right]\left[C_m \sin(m\theta) + D_m \cos(m\theta)\right]$$

$$\left[E \sinh(kz) + F\cosh(kz)\right], \quad m = \text{integer}.$$

cf. useful properties of Bessel's function

I. $x \to 0$ $\qquad J_\nu(x) \to \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu$

$$N_\nu(x) \to \begin{cases} 2/\pi \left[\ln\left(\frac{x}{2}\right) + 0.57722 \cdots\right] & \nu = 0 \\ -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu & \nu \neq 0 \end{cases}$$

$x \to \infty$

$$J_\nu(x) \to \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

$$N_\nu(x) \to \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

2. Each Bessel function has an infinite number of roots.

$J_m(x_{mn}) = 0$, x_{mn}: the n^{th} root of $J_m(x) = 0$

$x_{01} = 2.405\ldots$, $x_{02} = 5.52\ldots$ $x_{11} = 3.832\ldots$ $x_{12} = 7.016\ldots$

3. $0 \le \rho \le a$

$$\int_0^a d\rho\ \rho J_m\left(x_{mn'} \frac{\rho}{a}\right) J_m\left(x_{mn} \frac{\rho}{a}\right) = \frac{a^2}{2}\left[J_{m+1}(x_{mn})\right]^2 \delta_{nn'}$$

$$f(\rho) = \sum_{n=1}^{\infty} A_{mn} J_m\left(x_{mn} \frac{\rho}{a}\right)$$

$$A_{mn} = \frac{2}{a^2 J_{m+1}^2(x_{mn})} \int_0^a d\rho\ \rho f(\rho) J_m\left(x_{mn} \frac{\rho}{a}\right)$$

4. $\int_0^\infty dx\ x J_m(kx) J_m(k'x) = \frac{1}{k}\delta(k-k')$

5. If we put $d^2 Z/dz^2 = -k^2 Z$

$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} - \left(k^2 + \frac{\nu^2}{\rho^2}\right)R = 0 \xrightarrow{k\rho = x} \frac{d^2 R}{dx^2} + \frac{1}{x}\frac{dR}{dx} - \left(1 + \frac{\nu^2}{x^2}\right)R = 0$$

→ modified Bessel equation

Solution:

$$I_\nu(x) = \frac{1}{i^\nu} J_\nu(ix)$$

$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix)$$