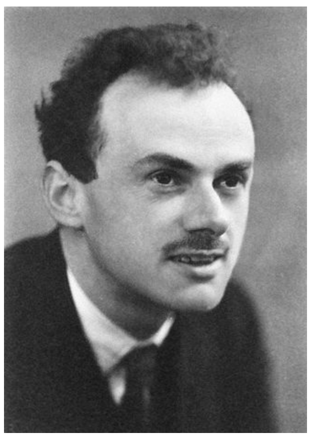
Lecture 1
DIRAC DELTA FUNCTION



Paul Dirac
(1902-1984)
Physics
BS in EE
Nobel Prize in Physics (1933)

1.1 Definition of Dirac Delta Function
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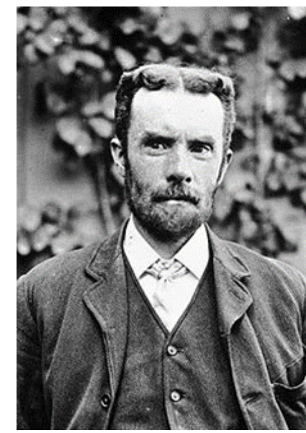

Oliver Heaviside
(1850-1925)
EE/Physics/Math
BS in EE
Vector Calculus
Transmission Line Eqs

1.1 Definition of Dirac Delta Function

- The Dirac delta function is the most important of singular functions or more generally called generalized functions, which are not defined by ordinary function theory.
- A singular function is always associated with a functional or "a function of a function".

For a "sufficiently well-behaved" test function $f(\mathbf{r})$, which is integrable such as

$$\int_{-\infty}^{\infty} d^n\mathbf{r}\ f(\mathbf{r}) = F \qquad (1.1)$$

$\int_{-\infty}^{\infty} d^n r\, f(r) = F$

the n-dimensional Dirac delta function is defined by using an integral property

$$\boxed{\int_{\Omega} d^n\mathbf{r}\ f(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}_0) = \begin{cases} f(\mathbf{r}_0), & \mathbf{r}_0 \in \Omega \\ 0, & \mathbf{r}_0 \notin \Omega \end{cases}} \qquad (1.2)$$

$\int_\Omega d^n r\, f(r)\delta(r-r_0) = \begin{cases} f(r_0), & r_0 \in \Omega \\ 0, & r_0 \notin \Omega \end{cases}$

leading to the normalization condition of the Dirac delta function, using $f(\mathbf{r}) = 1$ for all $\mathbf{r}$

$$\boxed{\int_{\Omega} d^n\mathbf{r}\,\delta(\mathbf{r}-\mathbf{r}_0) = \begin{cases} 1, & \mathbf{r}_0 \in \Omega \\ 0, & \mathbf{r}_0 \notin \Omega \end{cases}} \qquad (1.3)$$

$f(r) = 1$ (normalization condition)

$\int_\Omega d^n r\, \delta(r-r_0) = \begin{cases} 1 & r_0 \in \Omega \\ 0 & r_0 \notin \Omega \end{cases}$

In the spirit of generalized function, the n-dimensional Dirac delta function can be obtained using a sequence function $f_\varepsilon(\mathbf{r})$ of an ordinary function $f(\mathbf{r})$,

$$\boxed{f_\varepsilon(\mathbf{r}) = \frac{1}{\varepsilon^n} f\left(\frac{\mathbf{r}}{\varepsilon}\right)} \qquad (1.4)$$

$f_\varepsilon(r) = \frac{1}{\varepsilon^n} f\left(\frac{r}{\varepsilon}\right)$

Taking a limit of sequences, the delta function can be defined by a sequence function:

$\delta(r) \equiv \frac{1}{F} \lim_{\varepsilon\to 0} f_\varepsilon(r)$

$= \frac{1}{F} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^n} f\left(\frac{r}{\varepsilon}\right)$

$$\boxed{\delta(\mathbf{r}) \equiv \frac{1}{F} \lim_{\varepsilon\to 0^+} f_\varepsilon(\mathbf{r}) = \frac{1}{F} \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^n} f\left(\frac{\mathbf{r}}{\varepsilon}\right)} \qquad (1.5)$$

or equivalently $(\varepsilon \to 1/\eta)$

$\delta(r) \equiv \frac{1}{F} \lim_{\varepsilon\to 0} f_\varepsilon(r) = \frac{1}{F} \lim_{\eta\to\infty} \eta^n f(\eta r)$

$$\boxed{\delta(\mathbf{r}) \equiv \frac{1}{F} \lim_{\eta\to\infty} f_\eta(\mathbf{r}) = \frac{1}{F} \lim_{\eta\to\infty} \eta^n f(\eta\mathbf{r})} \qquad (1.6)$$

where $f(\mathbf{r})$ is a differentiable function having a finite, non-zero definite integral for normalization.

■ Unit Step Function versus Dirac Delta Function

$F_\varepsilon(x) = \int_{-\infty}^{x} ds \frac{1}{\varepsilon} f\left(\frac{s}{\varepsilon}\right) = \int_{-\infty}^{x/\varepsilon} ds\, f(s), \quad x>0$

Consider an integral function of $f(x)$ given by

$$F_\varepsilon(x) = \int_{-\infty}^{x} ds\, \frac{1}{\varepsilon} f\left(\frac{s}{\varepsilon}\right) = \int_{-\infty}^{x/\varepsilon} ds\ f(s), \quad x>0 \tag{1.7}$$

Taking its sequence as $\boxed{\varepsilon \to 0^+}$

$\lim_{\varepsilon\to0} F_\varepsilon(x) = \begin{cases} F & x>0 \\ 0 & x<0 \end{cases}$

$$\lim_{\varepsilon\to0} F_\varepsilon(x) = \begin{cases} F, & x>0 \\ 0, & x<0 \end{cases} = F\,u(x) \tag{1.8}$$

from which we can define the unit step function in two ways:

$u(x) = \frac{1}{F}\lim_{\varepsilon\to0} F_\varepsilon(x) = \begin{cases} 1 & x>0 \\ \frac{1}{2} & x=0 \\ 0 & x<0 \end{cases}$

$$\boxed{u(x) \equiv \frac{1}{F}\lim_{\varepsilon\to0} F_\varepsilon(x) = \begin{cases} 1, & x>0 \\ \frac{1}{2}, & x=0 \\ 0, & x<0 \end{cases}} \tag{1.9}$$

Unit Step (Heaviside) Function

Futhermore, we can also define the Dirac delta function from the unit step function:

$$\boxed{\delta(x) = \frac{d}{dx}u(x) = \begin{cases} \infty, & x=0 \\ 0, & x\neq0 \end{cases}} \tag{1.10}$$

$\delta(x) = \frac{d}{dx}u(x) = \begin{cases} \infty & x=0 \\ 0 & x\neq0 \end{cases}$

1.2 Sequence Functions

There are many sequence functions constructed from normalized ordinary functions:

$$f_\varepsilon(x)=\begin{cases}\dfrac{1}{\varepsilon}\left\{u\left[\dfrac{1}{\varepsilon}\left(x+\dfrac{1}{2}\right)\right]-u\left[\dfrac{1}{\varepsilon}\left(x-\dfrac{1}{2}\right)\right]\right\}=\dfrac{1}{\varepsilon}\left[u\left(x+\dfrac{\varepsilon}{2}\right)-u\left(x+\dfrac{\varepsilon}{2}\right)\right]\\[2ex] \dfrac{1}{\sqrt{\pi}}\dfrac{1}{\varepsilon}e^{-\left(\frac{x}{\varepsilon}\right)^2}\\[2ex] \dfrac{1}{\pi}\dfrac{1}{\varepsilon}\dfrac{\sin(x/\varepsilon)}{x/\varepsilon}=\dfrac{1}{\pi}\dfrac{\sin(x/\varepsilon)}{x}\\[2ex] \dfrac{1}{\pi}\dfrac{1}{\varepsilon}\dfrac{1}{(x/\varepsilon)^2+1}=\dfrac{1}{\pi}\dfrac{\varepsilon}{x^2+\varepsilon^2}\end{cases}\qquad(1.11)$$

Sokhotski–Plemelj's theorem

■ **Dirac Identity*** **(Sokhotski–Plemelj theorem)**

A singular function can be regularized using the Dirac delta function:

$$\frac{1}{x}=\lim_{\varepsilon\to 0^+}\frac{1}{x\pm i\varepsilon}=P\left(\frac{1}{x}\right)\mp i\pi\delta(x)\qquad(1.12)$$

* We will come back to the Dirac identity later.

$$\frac{1}{x}=\lim_{\varepsilon\to 0^+}\frac{1}{x\pm i\varepsilon}=P\left(\frac{1}{x}\right)\mp i\pi\delta(x)$$

1.3 Properties of Dirac Delta Function

Dimension

$\delta(\mathbf{r}) \quad [1/\mathrm{m}^n]$ $\delta(r)$

Scaling

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$

$\delta(ax) = \frac{1}{|a|}\delta(x)$

Inversion

$$\delta(x) = \delta(-x)$$

$\delta(x) = \delta(-x)$

Function Argument : $\boxed{f(x_m) = 0}$

$$\delta[f(x)] = \sum_m \frac{1}{|f'(x_m)|}\delta(x - x_m)$$

$\delta[f(x)] = \sum_m \frac{1}{|f'(x_m)|}\delta(x - x_m)$

$$\delta(x^2 - a^2) = \frac{1}{2|a|}\left[\delta(x+a) + \delta(x-a)\right]$$

Taylor Series

$$\delta(\mathbf{r} + \mathbf{a}) = \sum_{m=0}^{\infty} \frac{1}{m!}(\mathbf{a}\cdot\nabla)^m \delta(\mathbf{r})$$

$\delta(r+a) = \sum_{m=0}^{\infty} \frac{1}{m!}(a\cdot\nabla)^m \delta(r)$

Coordinates

$$\delta(\mathbf{r}_t) = \delta(x, y) = \delta(x)\delta(y)$$

$$\delta(\mathbf{r}) = \delta(x, y, z) = \delta(x)\delta(y)\delta(z)$$

$$\delta(\rho, \phi) = \frac{1}{\rho}\delta(\rho)\delta(\phi)$$

$$\begin{aligned}
\delta(\mathbf{r} - \mathbf{r}') &= \delta(x - x')\delta(y - y')\delta(z - z') \\
&= \frac{1}{\rho}\delta(\rho - \rho')\delta(\phi - \phi')\delta(z - z') \\
&= \frac{1}{r^2}\delta(r)\delta(\cos\theta - \cos\theta')\delta(\phi - \phi') \\
&= \frac{1}{r^2\sin\theta}\delta(r)\delta(\theta - \theta')\delta(\phi - \phi')
\end{aligned}$$

$$\boxed{\delta'(r) = \frac{1}{r}\delta(r)}$$

1.4 Dirac Comb Function*

infinite periodic array. A Dirac delta function

The Dirac Comb is an **infinite periodic array of Dirac delta functions** with a period R,

$$\boxed{\delta_R(x) = \sum_{n=-\infty}^{\infty} \delta(x - nR)} \tag{1.13}$$

and its Fourier series is given by

$$\boxed{\delta_R(x) = \frac{1}{R} \sum_{n=-\infty}^{\infty} e^{i\frac{2\pi n}{R}x}} \tag{1.14}$$

By taking Fourier transform of (1.12) or (1.13), we can easily show that it becomes another Dirac comb in the k space,

$$\boxed{\tilde{\delta}_R(k) = K \sum_{n=-\infty}^{\infty} \delta(k - nK)} \tag{1.15}$$

* We will come back to the Dirac comb for its applications to electrodynamics and solid state physics.