Fast Parameter Analysis of the Complex Exponential Signal by the Newton-Raphson Method

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***Abstract*—This paper proposes the algorithm to analyze parameters of the complex exponential signal by the Newton-Raphson Method. The parameters of complex exponential signal include angular velocity, damping, amplitude, and phase. This algorithm takes the time-domain data as the reference. According to differing positions on the complex plane, the relative algorithm will be established. The nonlinear equations could be quickly solved by the iterative calculation. This analysis result can demonstrate precise complex exponential parameters. This method has features of fast calculation and simple framework.**

***Index Terms*—complex exponential, parameter, Newton-Raphson Method**

I. INTRODUCTION

Parameter analysis is prevalent in a wide range of power-electronics, communications, energy-management, and stability. By the parameter analysis, the user can establish system models, and calculate the steady and transient performances of a system. In physical systems, dynamic behavior can be expressed as differential equations. The results of linear and time-invariant differential equations are mostly composed of exponential forms. Parameters of exponential forms include angular velocity, dampings, amplitudes, and phases. There are two types of exponential forms. If the damping is equal to zero, the mode is periodic; conversely, if the damping is not equal to zero, the mode will be aperiodic, and it will decay to zero with time.

Analysis technologies have been developed in time-domain and frequency-domain [1,2,3]. In time-domain, the technologies based on auto-regression have the advantages of simple calculation and easy accomplishment; consequently causing only little errors in analyzing the simple signal [4]. The technologies based on the evolutionary programming algorithm are quite robust for the complex signal; and are suitable for analyzing uncertain signals [5]. Besides, technologies based on the artificial neural network can deal with signal under noise. In frequency-domain, the polynomial method can find the pole and zero of the system [6]. The circle-fitting method uses the minimum error of a single mode to find parameters.

The complex exponential signals usually appear with transient state. In the fields of control and protection, the real-time result of the transient signal is needed in the system design. Because the numbers of analysis data must be sufficient, the above methods are limited to the application [7,8]. For improving the speed of analysis for complex exponential signals, this method references a small amount of data to calculate the unknown parameters. The parameters of exponential component contain angular velocity, damping, amplitude, and phase. According to different positions on the complex plane, components can be divided into three types: complex exponential, sinusoidal, and exponential forms. According to different forms of components, this paper establishes their relative algorithms. A complex exponential component includes four unknown parameters; the algorithm must establish four equations to find the solution. In the same way, it must establish three or two equations to find parameters of the sinusoidal or exponential components. This paper uses the Newton-Raphson (N-R) method to find solutions of the nonlinear equations. The advantages of this method are quick calculation, high accuracy, and components are decoupled.

This paper will evaluate the proposed method. The analyzed results of the single component, the multi-component signal, and the varied parameters will be discussed. The following sections completely illustrate theory and evaluation. Section 2 illustrates the way of the N-R method for complex exponential component analysis. Section 3 evaluates this method from different points of view.

II. THEORY

A. *Newton-Raphson Method*

Consider a function $y = f(x)$, assuming that the continuous derivative $f'(x)$ exists. Under this relationship, the independent variable $\overset{*}{x}$ will be found when the dependent variable $\overset{*}{y}$ is known. A value $\overset{0}{x}$ which approximates the solution is preset. The Taylor formula is given by

$$\overset{*}{y} = f(\overset{*}{x}) = f(\overset{0}{x}) + f'(\overset{0}{x})(\overset{*}{x} - \overset{0}{x}) + \frac{1}{2!} f''(\overset{0}{x})(\overset{*}{x} - \overset{0}{x})^2 + \dots \qquad (1)$$

Assuming $\overset{0}{x}$ is extremely close to $\overset{*}{x}$, the higher-order terms can be neglected, which results in

$$\overset{*}{y} \cong f(\overset{0}{x}) + f'(\overset{0}{x})(\overset{*}{x} - \overset{0}{x}) \tag{2}$$

Because $f(x)$ is nonlinear, the right-hand side of (2) will incline to but not equal $\overset{*}{y}$. Thus, the values of both sides of (2) will be equal, via a series of renewing the independent variable. This method is just the iterative procedure. Let the independent variable in v^{th} iteration be $\overset{v}{x}$, then in the v+1[st] iteration, the equation can be satisfied as

$$\overset{*}{y} = f(\overset{v}{x}) + f'(\overset{v}{x})(\overset{v+1}{x} - \overset{v}{x}) \tag{3}$$

The next iterative value can be found accordingly:

$$\overset{v+1}{x} = \overset{v}{x} + \frac{\overset{*}{y} - f(\overset{v}{x})}{f'(\overset{v}{x})} \tag{4}$$

After continuous iteration by (4), the independent variable will incline to $\overset{*}{x}$. In practice, when the independent variable converges in a tolerable range, the equation is regarded as being solved. Consider K nonlinear equations which are formed as

$$y_k = f_k(x_1, x_2, ..., x_K)\ ,\ k = 1,2,..,K \tag{5}$$

Using the iteration method to find the independent variables, then (4) can be expressed as a matrix form [9]

$$\overset{v+1}{\mathbf{X}} = \overset{v}{\mathbf{X}} + \left(\mathbf{J}(\overset{v}{\mathbf{X}})\right)^{-1}\left(\overset{*}{\mathbf{Y}} - \mathbf{F}(\overset{v}{\mathbf{X}})\right) \tag{6}$$

Where

$$\overset{v}{\mathbf{X}} = [x_1, x_2, .., x_K]^T$$

$$\overset{*}{\mathbf{Y}} = \left[\overset{*}{y}_{1,}\ \overset{*}{y}_2, .., \overset{*}{y}_K\right]^T$$

$$\mathbf{F}(\overset{v}{\mathbf{X}}) = \left[f_1(\overset{v}{\mathbf{X}}), f_2(\overset{v}{\mathbf{X}}), .., f_K(\overset{v}{\mathbf{X}})\right]^T$$

$$\mathbf{J}(\mathbf{X}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_K} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_K} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_K}{\partial x_1} & \frac{\partial f_K}{\partial x_2} & \cdots & \frac{\partial f_K}{\partial x_K} \end{bmatrix}$$

B. N-R Method for Complex Exponential Component

When the signal contains K independent complex exponential components, unknown parameters include angular velocity ω, damping α, amplitude A, and phase ϕ. A complex exponential signal is described as [10]

$$f(n) = \sum_{k=1}^{K} A_k e^{-\alpha_k nT} \sin(\omega_k nT + \phi_k)\ ,\ n = 0,1,2,... \tag{7}$$

Different forms will influence the performance of a signal in the time-domain. According to different positions of characteristic roots, the complex exponential forms can be divided into three types, as shown in Fig. 1. If both the angular velocity and damping are not equal to zero, the form is complex exponential; when only the damping is zero, the form is sinusoidal; and the component is an exponential form when only its angular velocity is zero.

1) Complex exponential component

For a complex exponential component, its angular velocity and damping are nonzero. The positions of it characteristic roots are shown as f_{ES} in Fig. 1. The component can be described as

$$f_{ES}(n) = Ae^{-\alpha nT} \sin(\omega nT + \phi)\ ,\ n = 0,1,2,... \tag{8}$$

The parameters of this component can be found by referring to the four adjacent data, showing in Fig. 2. The relationship between sequence data and their parameters can be written as

$$\begin{aligned} f_{ES}(n) &= Ae^{-\alpha nT} \sin(\omega nT + \phi) \\ f_{ES}(n-1) &= Ae^{-\alpha(n-1)T} \sin\left(\omega(n-1)T + \phi\right) \\ f_{ES}(n-2) &= Ae^{-\alpha(n-2)T} \sin\left(\omega(n-2)T + \phi\right) \\ f_{ES}(n-3) &= Ae^{-\alpha(n-3)T} \sin\left(\omega(n-3)T + \phi\right) \end{aligned} \tag{9}$$

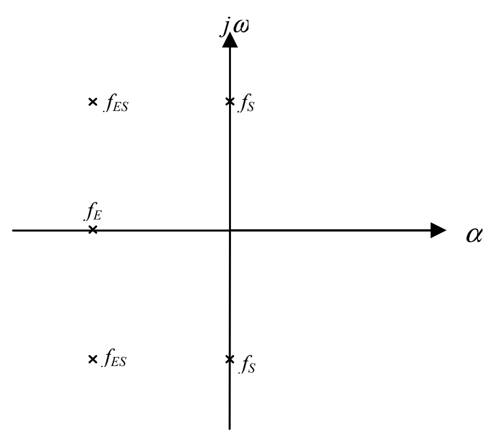


Fig. 1. Components on the complex plane

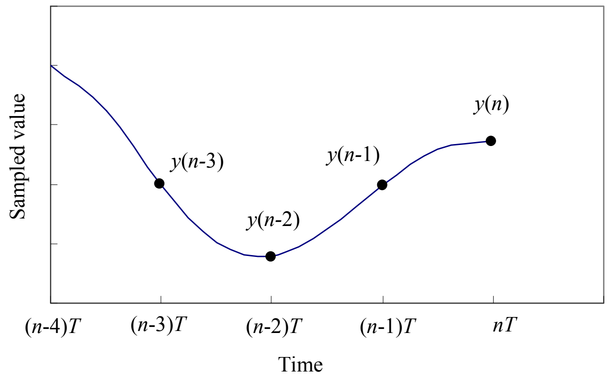


Fig. 2. Reference data of N-R method

Equation (9) is a set of nonlinear equations, and it can be solved by the N-R method. The matrix of this form can be written as [11]

$$\mathbf{X}_{ES} = [\omega \;\; \alpha \;\; A \;\; \phi]^T$$

$$\mathbf{Y}_{ES} = [y(n) \;\; y(n\text{-}1) \;\; y(n\text{-}2) \;\; y(n\text{-}3)]^T$$

$$\mathbf{F}_{ES} = [f_{ES}(n) \;\; f_{ES}(n\text{-}1) \;\; f_{ES}(n\text{-}2) \;\; f_{ES}(n\text{-}3)]^T$$

$$\mathbf{J}_{ES} = \begin{bmatrix} \frac{\partial f_{ES}(n)}{\partial \omega} & \frac{\partial f_{ES}(n)}{\partial \alpha} & \frac{\partial f_{ES}(n)}{\partial A} & \frac{\partial f_{ES}(n)}{\partial \phi} \\ \frac{\partial f_{ES}(n-1)}{\partial \omega} & \frac{\partial f_{ES}(n-1)}{\partial \alpha} & \frac{\partial f_{ES}(n-1)}{\partial A} & \frac{\partial f_{ES}(n-1)}{\partial \phi} \\ \frac{\partial f_{ES}(n-2)}{\partial \omega} & \frac{\partial f_{ES}(n-2)}{\partial \alpha} & \frac{\partial f_{ES}(n-2)}{\partial A} & \frac{\partial f_{ES}(n-2)}{\partial \phi} \\ \frac{\partial f_{ES}(n-3)}{\partial \omega} & \frac{\partial f_{ES}(n-3)}{\partial \alpha} & \frac{\partial f_{ES}(n-3)}{\partial A} & \frac{\partial f_{ES}(n-3)}{\partial \phi} \end{bmatrix} \tag{10}$$

2) Sinusoidal component

For a sinusoidal component, only its damping is nonzero. The positions of its characteristic roots are shown as f_S in Fig. 1. The component can be described as

$$f_S(n) = A\sin(\omega nT + \phi) \;\;,\;\; n = 0,1,2,... \tag{11}$$

The parameters of this component can be found by referring to the three adjacent data. The relationship between sequence data and their parameters can be written as

$$\begin{aligned} f_S(n) &= A\sin(\omega nT + \phi) \\ f_S(n-1) &= A\sin(\omega(n-1)T + \phi) \\ f_S(n-2) &= A\sin(\omega(n-2)T + \phi) \end{aligned} \tag{12}$$

Equation (12) can be solved by the N-R method. The matrix of this form can be written as

$$\mathbf{X}_S = [\omega \;\; A \;\; \phi]^T$$

$$\mathbf{Y}_S = [y(n) \;\; y(n\text{-}1) \;\; y(n\text{-}2)\;]^T$$

$$\mathbf{F}_S = [f_S(n) \;\; f_S(n\text{-}1) \;\; f_S(n\text{-}2)]^T$$

$$\mathbf{J}_S = \begin{bmatrix} \frac{\partial f_S(n)}{\partial \omega} & \frac{\partial f_S(n)}{\partial A} & \frac{\partial f_S(n)}{\partial \phi} \\ \frac{\partial f_S(n-1)}{\partial \omega} & \frac{\partial f_S(n-1)}{\partial A} & \frac{\partial f_S(n-1)}{\partial \phi} \\ \frac{\partial f_S(n-2)}{\partial \omega} & \frac{\partial f_S(n-2)}{\partial A} & \frac{\partial f_S(n-2)}{\partial \phi} \end{bmatrix} \tag{13}$$

3) Exponential component

For an exponential component, only its angular velocity is nonzero. The positions of its characteristic roots are shown as f_E in Fig. 1. The component can be described as

$$f_E(n) = Ae^{-\alpha nT} \;\;,\;\; n = 0,1,2,... \tag{14}$$

The parameters of this component can be found by referring to the two adjacent data. The relationship between sequence data and their parameters can be written as

$$\begin{aligned} f_E(n) &= Ae^{-\alpha nT} \\ f_E(n-1) &= Ae^{-\alpha(n-1)T} \end{aligned} \tag{15}$$

Equation (15) can be solved by the N-R method. The matrix of this form can be written as

$$\mathbf{X}_E = [\alpha \;\; A]^T$$

$$\mathbf{Y}_E = [y(n) \;\; y(n\text{-}1)]^T$$

$$\mathbf{F}_E = [f_E(n) \;\; f_E(n\text{-}1)]^T$$

$$\mathbf{J}_E = \begin{bmatrix} \frac{\partial f_E(n)}{\partial \alpha} & \frac{\partial f_E(n)}{\partial A} \\ \frac{\partial f_E(n-1)}{\partial \alpha} & \frac{\partial f_E(n-1)}{\partial A} \end{bmatrix} \tag{16}$$

4) Multi-component

Consider a signal with multiple components. Because complex exponential component are independent, the following relationship exists

$$\left.\frac{\partial f_k}{\partial x_{k'}}\right|_{k \neq k'} = 0 \;\;,\;\; x = \omega, \alpha, A, \phi \;,\;\; k = 1,2,..,K \tag{17}$$

From (17), each component is decoupled mutually, and Jacobian matrix of a multi-component signal can be written as

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{11} & \mathbf{J}_{12} & \cdots & \mathbf{J}_{1K} \\ \mathbf{J}_{21} & \mathbf{J}_{22} & \cdots & \mathbf{J}_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{J}_{K1} & \mathbf{J}_{K2} & \cdots & \mathbf{J}_{KK} \end{bmatrix} = \begin{bmatrix} \mathbf{J}_{11} & 0 & \cdots & 0 \\ 0 & \mathbf{J}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{J}_{KK} \end{bmatrix} \quad (18)$$

Because of the relationship in a decoupled matrix, the following advantages can be reached: Firstly, the parameter modification of each component just considers its Jacobian matrix, which can improve the processing speed. Secondly, components in a signal become modules, which can simplify the framework.

III. Ability Evaluation

This section evaluates the ability of the method in three parts. The first part discusses analyzed results for the single component; the second part illustrates the result of the multi-component signal analysis; and the last part shows the analyzed result under varied parameters.

A. *Analysis Result for the Single Component*

Types of components include complex exponential, sinusoidal, and exponential forms. This section discusses analyzed results for them in turn.

1) Complex exponential component: The complex exponential component can be expressed as

$$f_{ES}(t) = 110e^{-10t}\sin(377t + 45^{\circ}) \quad (19)$$

Its waveform is shown in Fig. 3. The sample period is sat T= 9.77*10^{-4} sec. The analyzed results are recorded in Fig. 3. Satisfactory results will be calculated by this method. In this example, the convergence is reached after 5 iterations. Real values can be obtained only after a few iterations. Then satisfactory results will be calculated.

2) Sinusoidal component: The Sinusoidal component can be expressed as

$$f_S(t) = 110\sin(377t + 45^{\circ}) \quad (20)$$

Its waveform is shown in Fig. 4. The sample period is sat T= 9.77*10^{-4} sec. The analyzed results are recorded in Fig. 4. Satisfactory results will be calculated by this method, too.

3) Exponential component: The exponential component can be expressed as

$$f_E(t) = 110e^{-10t} \quad (21)$$

Its waveform is shown in Fig. 5. The sample period is sat T= 9.77*10^{-4} sec. The analyzed results are recorded in Fig. 5. Satisfactory results will be calculated by this method, too.

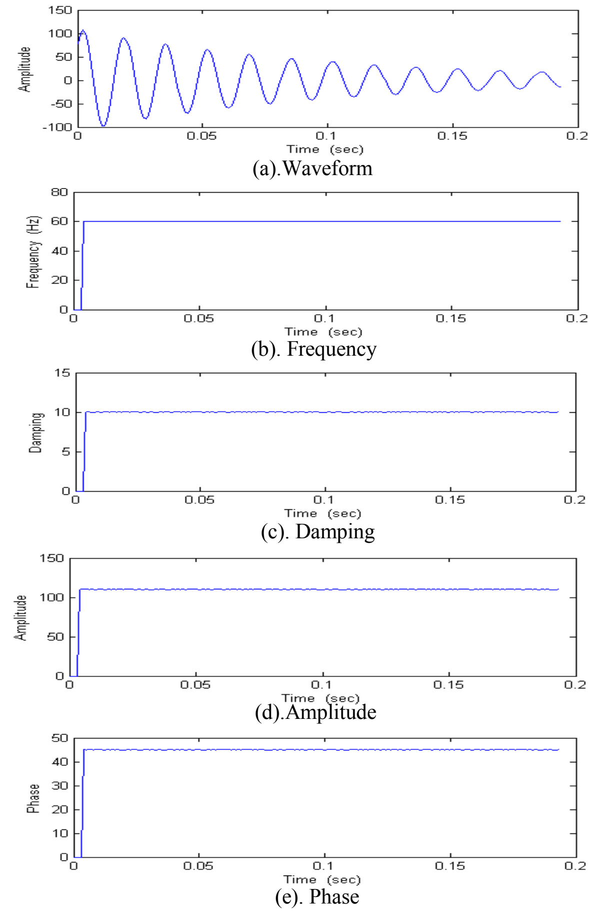


Fig 3. Analysis for complex exponential component

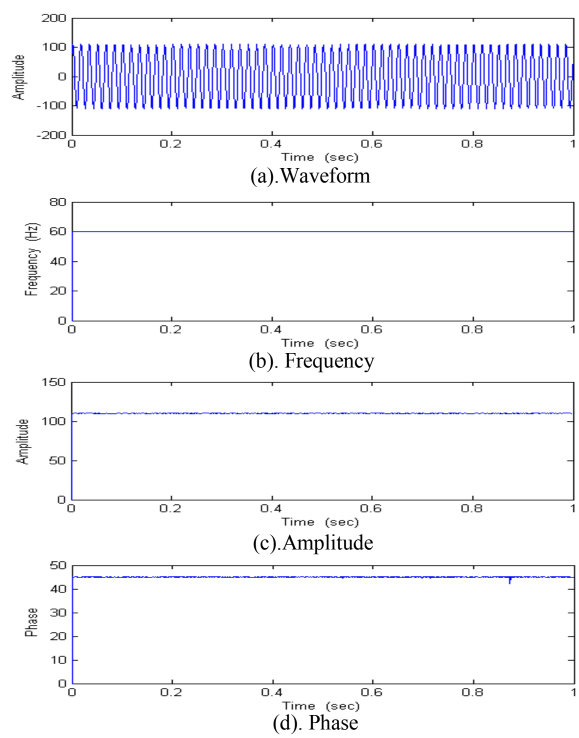


Fig 4. Analysis for sinusoidal component

B. Result of the Multi-Component Signal

The responses of a circuit mostly include transient and steady state responses. The steady-state response is a sinusoidal form, and the transient response is an exponential form. This section takes a signal with a sinusoidal component and an exponential component as an example. The signal is described as

$$f(t) = 110\, e^{-30t} - 56.56 \sin(377\, t + 45^\circ) \qquad (22)$$

Its waveform is shown in Fig. 6. From (18), the Jacobian matrix of the equations is decoupled. It decomposes the Jacobian matrix from a 5*5 matrix to a 3*3 matrix and a 2*2 matrix. The calculation of inverse matrix will be decreased significantly. The sample period is sat T= $9.77*10^{-4}$ sec. The analyzed results are recorded in Fig. 6. The exponential component decays to a quite little value after 0.3 sec. Jacobian matrix is in the bad condition, which causes errors in analysis. This default can be prevented by checking the determinant. If the determinant descends to close to zero, it can be regarded as the component is ignorable. The parameter analysis could not consider this component, and the result will be similar to it from one component.

C. Result Under Varied Parameters

When an electrical apparatus is in operation, the impedance usually varies to follow the different conditions. That is, the impedance of an electrical apparatus is time-varied. Detection of the time-varied impedance can be contributive to the control and protection of the apparatus [12]. This section calculates the time-varied impedance by analyzing signals of voltage and current. Consider an impedance; its varied value is expressed as

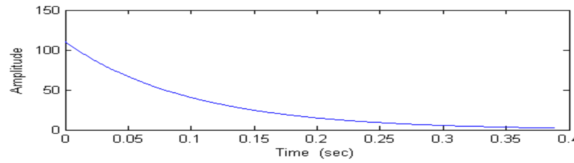
$$Z(t) = (10 + 100t)\angle 60^\circ \qquad (23)$$

The supply voltage is given by $v(t) = 110\sin(377t)$ and sample period is sat T= $9.77*10^{-4}$ sec. Parameters of input and output signals can be found by this method. The analyzed results are shown in Fig. 7. It shows that the method can track variation of the impedance, and its result is extremely close to reality.

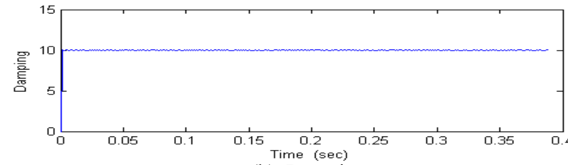
IV. Conclusion

This paper proposes a method to analyze parameters of the complex exponential signal by the Newton-Raphson method. Parameters of the complex exponential signal contain angular velocity, damping, amplitude, and phase. According to the different position of characteristic roots on the complex plane, the component forms are divided into three types. According to different types, the unique algorithms are established. These algorithms can obtain accurate results of their specific forms. This method has the features of:

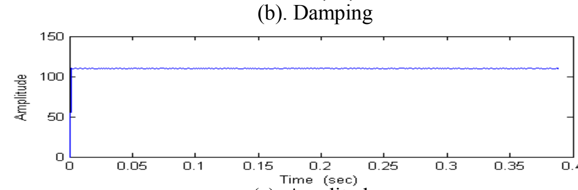
1) Fast calculation: The N-R method has the feature of fast convergence, which enables solutions to be found quickly.



(a).Waveform

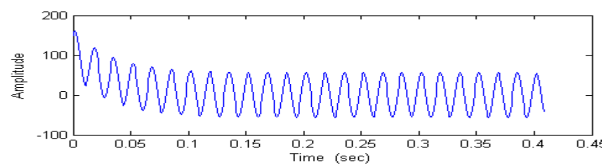


(b). Damping

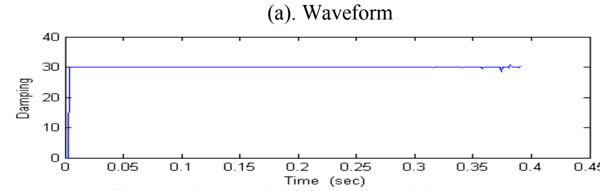


(c). Amplitude

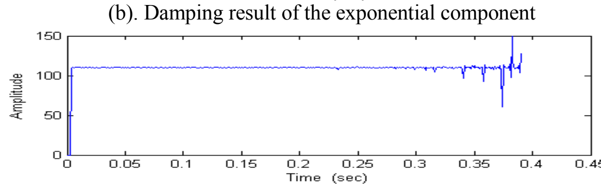
Fig 5. Analysis for exponential component



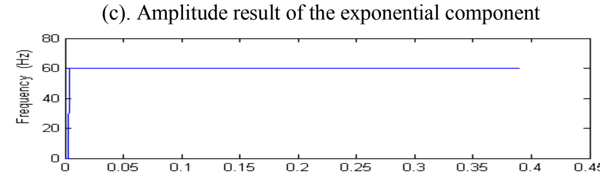
(a). Waveform



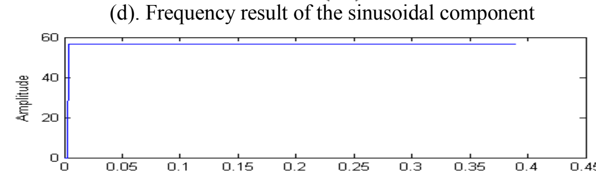
(b). Damping result of the exponential component



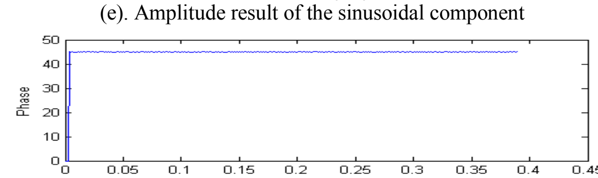
(c). Amplitude result of the exponential component



(d). Frequency result of the sinusoidal component

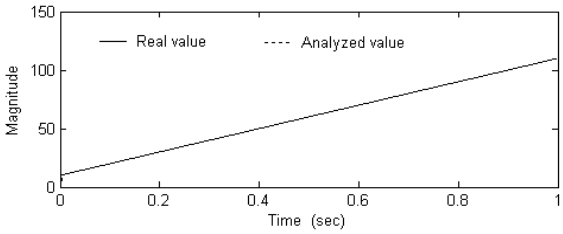


(e). Amplitude result of the sinusoidal component

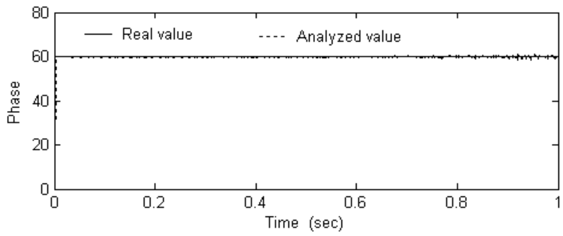


(f). Phase result of the sinusoidal component

Fig 6. Analysis for multi-component signal



(a). Magnitude



(b). Phase

Fig 7. Real-time analysis for impedance

2) Simple framework: Since Jacobian matrix of this method is decoupled, the procedure will be simplified and the speed will be improved.

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