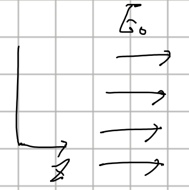
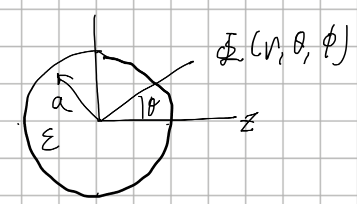
Ex. 2) Dielectric sphere in an uniform electric field.





azimuthal symmetry: $\Phi = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$

Inside the sphere, Φ is finite, $B_l = 0$, $\Phi_{in} = \sum A_l r^l P_l(\cos\theta)$

Outside the sphere, $\Phi_{out} = \sum_{l=0}^{\infty} \left[C_l r^l + \frac{D_l}{r^{l+1}} \right] P_l(\cos\theta)$

$r \to \infty \qquad \Phi_{out} = -E_0 z = -E_0 r\cos\theta = -E_0 r P_1(\cos\theta)$

$C_l = 0$, for $l \neq 1$, $C_1 = -E_0$

B.C. normal charge at $r=a$.

$(\vec{E}_{in} - \vec{E}_{out}) \times \hat{r} = 0$, $[D_{in} - D_{out}] \cdot \hat{r} = 0$

i) Continuous tangential E.

$$-\frac{1}{a} \frac{\partial}{\partial\theta} \Phi_{in} \Big|_{r=a} = -\frac{1}{a} \frac{\partial}{\partial\theta} \Phi_{out} \Big|_{r=a}$$

$$\sum A_l a^l \frac{dP_l}{d\theta} = \sum \left(C_l a^l + D_l / a^{l+1} \right) \frac{dP_l}{d\theta}$$

$l=1 \qquad A_1 a = -E_0 a + \frac{D_1}{a^2}$ ①

$l \neq 1 \qquad A_l a^l = D_l / a^{l+1}$ ②

ii) Continuous normal D

$$-\varepsilon \frac{\partial}{\partial r}\Phi_{in}\Big|_{r=a} = -\frac{\partial}{\partial r}\Phi_{out}\Big|_{r=a}$$

$$\varepsilon \sum_{\ell} A_\ell \, \ell \, a^{\ell-1} P_\ell = \sum_{\ell}\left[\ell C_\ell a^{\ell-1} - (\ell+1) D_\ell a^{-(\ell+2)}\right] P_\ell$$

$$\ell = 1 \qquad \varepsilon A_1 = -E_0 - \frac{2D_1}{a^3} \qquad ③$$

$$\ell \neq 1 \qquad \varepsilon A_\ell \, \ell \, a^{\ell} = -(\ell+1)\frac{D_\ell}{a^{\ell+1}} \qquad ④$$

①,②,③,④ $\Rightarrow A_\ell = 0,\ D_\ell = 0$ for $\ell \neq 1$

$$A_1 = -\frac{3}{\varepsilon+2} E_0$$

$$D_1 = \frac{\varepsilon-1}{\varepsilon+2} E_0 a^3$$

$$\Phi_{in} = -\frac{3}{\varepsilon+2} E_0 z$$

$$\Phi_{out} = -E_0 r\cos\theta + \underbrace{\frac{\varepsilon-1}{\varepsilon+2}\frac{E_0 a^3 \cos\theta}{r^2}}$$

initial field

field due to an induced dipole.

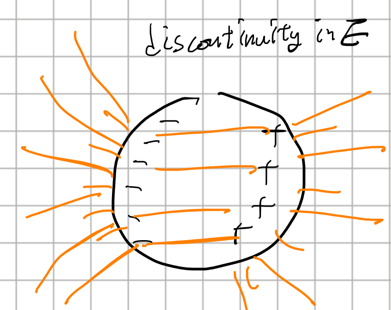
$$= \underbrace{\left(\frac{\varepsilon-1}{\varepsilon+2} E_0 a^3 \hat{z}\right)}_{=p\ :\ \text{displacement.}} \cdot \nabla\left(\frac{1}{r}\right)$$

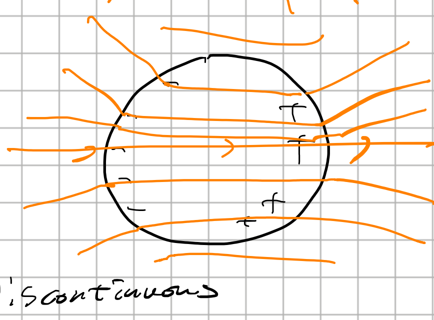
Polarization: dipole moment / unit volume

$$P = \frac{p}{\frac{4\pi a^3}{3}} = \frac{3}{4\pi}\frac{\varepsilon-1}{\varepsilon+2} E_0 \hat{z}$$

Surface charge density: $\vec{P}\cdot\hat{r} = \frac{3}{4\pi}\frac{\varepsilon-1}{\varepsilon+2} E_0 \cos\theta$

Charge density: $\rho_p = -\nabla\cdot\vec{P} = 0$





Polarizability

$$P = Np = N\alpha E \simeq \frac{\rho_m}{\frac{M}{N_0}} \alpha E$$

number density   dipole moment   polarizability

M : molecular weight

N_0 : Avogadro number

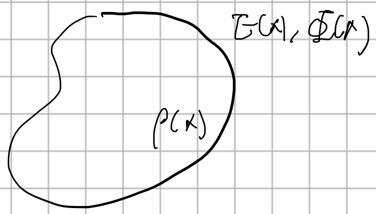
$$\frac{3}{4\pi} \frac{\varepsilon - 1}{\varepsilon + 2} E = \frac{N_0 \rho_m}{M}$$

$$\boxed{\frac{\varepsilon - 1}{\varepsilon + 2} = \frac{4\pi}{3} \frac{N_0 \rho_m}{M} \alpha.}$$

Clausius - Mossotti electron

If ε is measured α can be determined.

B) Electrostatic Energy in dielectric media



Consider a small charge in every δw when a small increment of real charge $\delta \rho$ is added to the system

$$\delta w = \int \delta\rho\, \Phi(x)\, dx$$

potential due to the charge density $\rho(x)$ already present.

$$\nabla \cdot \vec{D} = 4\pi \rho_{real}$$
$$\nabla \cdot (\delta D) = 4\pi \delta\rho$$

$$\delta w = \int \frac{1}{4\pi} \nabla\cdot(\delta D)\, \Phi(x)\, d^3x$$

$$\left(\int d^3x\, \nabla\cdot(\Phi \cdot \delta D) = \int d^3x \left(\nabla\Phi\cdot\delta D + \Phi\,\nabla\cdot(\delta D)\right) \right)$$

Gaussian theorem & $\Phi \to 0 \cdot \vec{x} \to \infty$

$$= \frac{1}{4\pi} \int \vec{E}\cdot\delta D\, d^3x$$

Total electrostatic energy needed to build up a distribution from $D=0$ to $D=D$

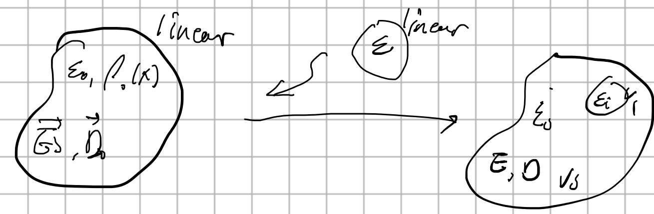
$$W = \int \delta w = \frac{1}{4\pi}\int d^3x \int_0^D \vec{E}\cdot\delta D$$
$$= \frac{1}{8\pi}\int d^3x \int \delta(\vec{E}\cdot\vec{D}) = \boxed{\frac{1}{8\pi}\int d^3x\, \vec{E}\cdot\vec{D}}$$

medium is uniform & linear

$$\vec{E}\cdot\delta D = \frac{1}{2}\delta(\vec{E}\cdot\vec{D})$$

ε is constant

* change in energy due to the introduction of a dielectric object



$$\vec{D}_0 = \varepsilon \vec{E}_0$$

$$W_0 = \frac{1}{8\pi}\int \vec{E}_0\cdot\vec{D}_0\, d^3x \qquad W_1 = \frac{1}{8\pi}\int \vec{E}\cdot\vec{D}\, d^3x$$

$$\Delta W = W_1 - W_0 = \frac{1}{8\pi}\int d^3x\,(\vec{E}\cdot\vec{D} - \vec{E}_0\cdot\vec{D}_0)$$

$$= \frac{1}{8\pi}\int d^3x\,(\vec{E}\cdot\vec{D}_0 - \vec{E}_0\cdot\vec{D}) + \underline{\frac{1}{8\pi}\int d^3x\,(\vec{E}+\vec{E}_0)\cdot(\vec{D}-\vec{D}_0)}$$

$$\nabla\times E = 0, \quad \nabla\times E_0 = 0$$

$$\nabla\times(E+E_0) = 0$$

$$E+E_0 = -\nabla\phi$$

$$= -\int d^3x\, \nabla\phi\cdot(D-D_0)$$

$$= -\left[\int d^3x\, \nabla\cdot(\phi(\vec{D}-\vec{D}_0)) - \int d^3x\, \phi\nabla\cdot(\vec{D}-\vec{D}_0)\right]$$

$$= 0 + \int d^3x\, \phi\, \nabla\cdot(\vec{D}-\vec{D}_0) \qquad \nabla\cdot D = 4\pi\rho_{real}, \quad \nabla\cdot D_0 = 4\pi\rho_{real}$$

$$= 0 + 0$$

$$\Rightarrow \frac{1}{8\pi}\int_{V_0} d^3x\,\cancelto{0}{(\vec{E}\cdot\vec{D}_0 - \vec{E}_0\cdot\vec{D})} \qquad D = \varepsilon_0 E, \quad D_0 = \varepsilon_0 E_0$$

$$+ \frac{1}{8\pi}\int_{V_1} d^3x\,(\vec{E}\cdot\vec{D}_0 - \vec{E}_0\cdot\vec{D}) \qquad \vec{D} = \varepsilon_1\vec{E}, \quad \vec{D}_0 = \varepsilon_0\vec{E}_0$$

$$= -\frac{1}{8\pi}\int_{V_1} d^3x\,(\varepsilon_1-\varepsilon_0)\,\vec{E}_0\cdot\vec{E}_1$$

$$= -\frac{1}{2}\int d^3x\,\underbrace{\left(\frac{\varepsilon_1-1}{4\pi}\vec{E}\right)}_{\vec{P}}\cdot\vec{E}_0 \qquad \varepsilon_0 = 1 \qquad = -\frac{1}{2}\int_{V_1} d^3x\, \vec{P}\cdot\vec{E}_0$$

A permanent dipole in an external field

$\Delta W = -\vec{p} \cdot \vec{E}_0$

difference of 1/2 2.

$$-\frac{1}{2} \int_V d^3x \, \vec{P} \cdot \vec{E} = \Delta W_A + \left(-\int_{V_1} \vec{P} \cdot \vec{E} \, d^3x\right)$$