DESIGN NOTE

The evaluation of standard uncertainty in the presence of limited resolution of indicating devices

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Abstract. Since the type A evaluation of standard uncertainty according to the rules given in the ISO *Guide to the Expression of Uncertainty in Measurement* does not take into account the uncertainty that arises from the limited resolution of indicating devices, we show in this design note that the treatment of this problem calls for a quantity about which no statistical information is available; therefore, its uncertainty has to be derived from a type B evaluation.

1. Introduction

In order to evaluate the uncertainty of a measured quantity according to the rules provided by the seminal ISO *Guide to the Expression of Uncertainty in Measurement* [1], an adequate model that includes all relevant input quantities needs to be established and all available information about these quantities has to be properly used to express their standard uncertainties and mutual covariances. The simplest case is that of a single quantity whose standard uncertainty is derived from observations sampled from a frequency distribution (a type A evaluation). However, one should be aware that the result of this evaluation does not heed the uncertainty, however small, that arises from the limited resolution of the indicating device used to obtain those observations. In that which follows we show that the treatment of this problem calls for a quantity about which no statistical information is available; therefore, its uncertainty has to be derived from a type B evaluation.

2. The theory

Let us begin by recalling that, in general, a measurand Y should be evaluated according to a model $Y = f(X_1, X_2, \ldots, X_N)$, where the X_i are input quantities. In the case of the direct measurement of a single quantity one should distinguish the measurand Y from the actually observed input quantity X, since to infer the former not only are the observed indicated values needed but also additional information about quantities that may cause systematic effects is necessary. Thus one should write, for example,

$$Y = X + C \tag{1}$$

where C is an input quantity that represents a correction for possible systematic deviations, information about which may come from the calibration certificate of the instrument. According to this model, the estimated value of Y is $y = x + c$, where x and c are the estimated values of the input quantities. Furthermore, if quantities X and C are independent, the standard uncertainty of the measurand is $u(y) = [u^2(x) + u^2(c)]^{1/2}$, where $u(x)$ and $u(c)$ are the standard uncertainties of x and c, respectively. However, let us assume that $c = 0$ and that $u(c)$ is negligible in comparison to $u(x)$, so that quantity C does not need to be taken into account. In this case one usually sets up the model $Y = X$, thereby expressing the fact that the only information required to infer the measurand concerns quantity X. However, as we will show below, this model may still be incomplete.

Suppose that the information about X consists of a series of independent observations x_k ($k = 1, 2, \ldots, n$) obtained under the same conditions of measurement. The value of X is then estimated by the arithmetic mean of these observations,

$$x = \overline{X} = \frac{1}{n}\sum_{k=1}^{n} x_k \tag{2}$$

whereas the standard uncertainty of this estimate is given by the experimental standard deviation of the mean:

$$u(x) = s(\overline{X}) = \left(\frac{1}{n(n-1)} \sum_{k=1}^{n} (x_k - \overline{X})^2 \right)^{1/2}. \qquad (3)$$

The *Guide* states that 'the individual observations (may) differ in value because of random variations in the influence quantities' (paragraph 4.2.2). This variability results in an incomplete knowledge about quantity X. It is this insufficient information that the uncertainty given by equation (3) reflects. From the model $Y = X$ one concludes that $y = x$ and $u(y) = u(x)$. However, consider the extreme case in which all measurements of X happen to give the same indication. Application of equation (3) then yields a vanishing uncertainty both for X and for Y. This last result implies that the measurand is perfectly known. However, suppose that the observations are carried out with an instrument whose indicating device does not have enough resolution for the scattering to be detected. (This situation is particularly likely to be encountered in industrial metrology). Since the corresponding lack of information is not taken into account by equation (3), it appears that in general at least one further input quantity is needed in order to infer Y reasonably.

The *Guide* addresses the matter of resolution in its annex F (paragraph F.2.2.1), where it is stated that this property can be described by a known parameter δ, such that any value recorded or displayed as x would be indicated by an ideal device as lying within the interval $(x - \delta/2, x + \delta/2)$. Usually, the parameter δ may be assumed to be equal to a step change in the least significant digit (for a digital display) or to the scale interval (for an analogue display), although in general more information about the device is needed to make a proper decision in this regard.

In any case, let X' represent the quantity that corresponds to the ideally indicated values. If systematic effects are ignored, the model $Y = X'$ would be free from constraints on available information arising from resolution limitations and equation (3) would thus be appropriate to evaluate the uncertainty of Y. However, since an ideal indicating device does not exist, we need another way of expressing this model. This can be done by introducing a correction quantity Z to account for the unobservable differences between the ideal and actual indications, namely, $Z = X' - X$. One then has

$$Y = X + Z. \qquad (4)$$

As before, the information about input quantity X can be treated by a type A evaluation (equations (2) and (3)), whereas for Z a type B evaluation must be used. The only available information about this quantity, $|Z| \leq \delta/2$, is consistent with a rectangular probability density function. Accordingly, we take $z = 0$ and $u(z) = \delta/\sqrt{12}$. From equation (4), the value of measurand Y is estimated by

$$y = x + z = x = \overline{X} \qquad (5)$$

and (since X and Z are independent) its standard uncertainty is

$$u(y) = [u^2(x) + u^2(z)]^{1/2} = \left(s^2(\overline{X}) + \frac{\delta^2}{12} \right)^{1/2}. \qquad (6)$$

Table 1. Results of ten repeated measurements of a distance Y.

Measurement k	Micrometer values $x_{k,m}$ (mm)	Caliper values $x_{k,c}$ (mm)
1	7.489	7.5
2	7.503	7.5
3	7.433	7.4
4	7.549	7.5
5	7.526	7.5
6	7.396	7.4
7	7.543	7.5
8	7.509	7.5
9	7.504	7.5
10	7.383	7.4

Table 2. Measurement estimates and standard uncertainties.

	Micrometer	Caliper
x (or y) (mm)	7.484	7.47
δ (mm)	0.001	0.1
$u(x)$ (mm) (equation (3))	0.019	0.02
$u(y)$ (mm) (equation (6))	0.019	0.03
$s^2(\overline{X})$ (μm^2)	349	233
$\delta^2/12$ (μm^2)	0	833

Note that the models represented by equations (1) and (4) are similar. If it happens that $u(c)$ or $u(z)$ are respectively negligible compared to $u(x)$, the resulting uncertainty of the measurand would be the same as if it had been calculated from a model without the added input quantity. These models differ in that the information about the random process that gives rise to the variability in the correction C usually consists of statistical data obtained during the calibration process, so that $u(c)$ can be type-A evaluated. Instead, in the resolution problem only the limits of the interval that contains the unobservable statistical material are known; therefore, $u(z)$ must be type-B evaluated.

3. An example

As an illustration, consider ten repeated measurements of some distance Y. Table 1 shows the indications as observed with a micrometer (scale interval $\delta_m = 1\ \mu$m) and with a caliper (scale interval $\delta_c = 0.1$ mm). Both instruments are assumed to be free from systematic deviations. Table 2 shows the respective estimates x (or y), the standard uncertainties $u(x)$ (equation (3)) and $u(y)$ (equation (6)) and the terms $s^2(\overline{X})$ and $\delta^2/12$. It may be seen that, for the micrometer, the contribution of the latter term is negligible, whereas for the caliper the term $\delta_c^2/12$ is more than three times larger than the corresponding statistical expression $s^2(\overline{X})$. These results are intuitively correct, for it is to be expected that the uncertainty of a measurement result should increase as a result of the imperfections of the instrument.

On the other hand, when the outcome of repeated measurements is a set consisting of only a few different

Table 3. Other possible measurements of distance Y.

Measurement k	Micrometer values $x_{k.m}$ (mm)	Caliper values $x_{k.c}$ (mm)
1	7.500	7.5
2	7.500	7.5
3	7.400	7.4
4	7.500	7.5
5	7.500	7.5
6	7.400	7.4
7	7.500	7.5
8	7.500	7.5
9	7.500	7.5
10	7.400	7.4

values in a narrow range, the possibility that $u(x)$ does actually provide an adequate picture of the ideal scattering region cannot be excluded. However, one cannot decide on this without further information. For example, suppose that the measurements are as given in table 3. Obviously, the term $u(x)$ has now the same value for both instruments. However, only for the micrometer is this result also equal to $u(y)$, since for the caliper the term $\delta_c^2/12$ should still need to be added. It is only the *additional* information provided by the *simultaneous* micrometer readings that would allow us to drop this term. However, of course, one would seldom have these simultaneous measurements in practice.

4. Conclusion

It is always advisable to include the second term in equation (6). It may happen that the uncertainty in Z is negligible compared to that of X (as in the case of the micrometer) but this can only be decided *after* the evaluation has been performed. The hypothetical example we have considered in this design note illustrates that the uncertainty in measurement should not be interpreted as some sought-after hidden quantity, but rather as a quantitative description of the incomplete knowledge that we have about the measurand. In fact, this view underlies the definition of uncertainty in the *Guide*.

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Reference

[1] ISO 1993 *Guide to the Expression of Uncertainty in Measurement* (Published by the ISO in the name of the BIPM, IEC, IFCC, IUPAC, IUPAP and OIML (Geneva))