# Synchronous Filtering

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***Abstract*—A new filtering paradigm is introduced, called synchronous filtering, which notably includes complex or polyphase filters as a subset. The theory introduced demonstrates how a given time invariant filter can be implemented using a combination of modulators and a core filter, which itself could be time varying. That the equivalent synchronous filter processes the identical time-domain complete response as the original is also proven. Sub- and superheterodyne, as well as higher order, filters are discussed. Finally, a companding polyphase filter is introduced for the first time as another instance of a synchronous filter.**

***Index Terms*—Bandpass filters, complex filters, modulators, polyphase filters, synchronous filtering.**

## I. Introduction

THE DESIGN of linear filters is a topic that continues to occupy the energies of circuit designers as well as circuit and systems theorists. Most of the interest in recent history has been the result of the desire to create integrated circuitry that will implement filtering functions. Advances in basic analog gain blocks, such as operational amplifiers and transconductance amplifiers, have been one tangible result of this effort. At the other end of the spectrum, we have seen the introduction and perfection of new filtering paradigms, such as switched capacitor filters and delta sigma converters. In each case, the advances have pushed the state of the art further in at least some particular set of applications. One of the more interesting approaches to analog filtering that has been significantly developed in the past decade or so is that of complex, or polyphase, filtering, e.g., [1]–[3], [15]. A refinement of the more classic Weaver [4] and Hartley [5] filtering approaches, polyphase filters distinguish themselves by their combinaiton of modulators (multipliers) and multiple input-multiple output filters. With these filters, designers have been able to effectively create extremely selective filter systems in RF circuits with totally integrated circuitry. Although this paper concers itself with analog systems, it should be noted that multirate digital filters are conceptually the same, for example, [6]–[8].

The justification for these complex filters has been given using the idea of complex signals. In particular, a given real input signal is associated with a complex counterpart and is then modulated to produce a lower frequency complex signal that is applied to a complex filter. The resulting complex signal output is then finally down converted and/or demodulated. The explanation for the validity of this approach lies in a frequency-domain description of the modulation and complex filtering. [1] provides a good explanation along contemporary lines, while [9] provides a very nice description of the whole complex filtering concept. Although the explanation is a valid one, it is not easily translated into a time-domain equivalent. This leaves the somewhat unsatisfying disconnect between complex filtering and the rest of classic filtering approaches. In this paper, a new class of filter is proposed, called synchronous filters, which offers not only a firm time-domain derivation for complex filters, but also offers a way to expand the possibilities beyond those offered by complex filters.

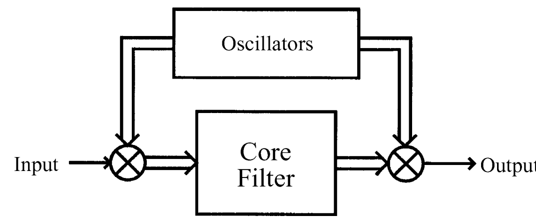


Fig. 1. Block diagram for a synchronous filter.

Fig. 1 shows the basic concept of a synchronous filter. Namely, a synchronous filter consists of a core filter preceeded and followed by modulators. In general, the modulators in the figure, represented by the multiplier symbols, will include weighted summers. The net result of this system is to implement an equivalent filter not having modulators. For example, we consider the case where a synchronous filter is used to implement a very high-frequency high-$Q$ bandpass filter. As will be shown later, the synchronous filter equivalent to the high-frequency high-$Q$ filter incorporates a core filter with a much lower frequency passband and a much lower $Q$. As a result, the synchronous filter allows one to implement the very difficult filter design (high-frequency high-$Q$) with a much easier filter design. In this paper, it is shown that the synchronous filter is time invariant, despite the modulators, and it produces the exact same complete response to arbitrary inputs and (equivalent) initial conditions. The key to the derivation lies in the state–space description for the original and the synchronous filters, as explained in Section II. Section III makes a precise connection between complex and synchronous filters. Section IV extends the results to higher order filters while Section V shows how the concept of synchronous filtering may be used to create companding complex filters.

## II. Development of Synchronous Filters

The concept of state–space-based filter design is an old one. It has been established rigourously and elegantly, yielding many modern filter topologies. The idea is simple namely, a given transfer function is used to set up a system of dynamical equations. These dynamical equations lead to a physical realization by associating electrical parameters, such as voltage or charge,

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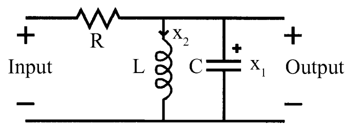


Fig. 2. Circuit schematic for a simple bandpass filter.

with each of the state variables. The result is a system of equations that may be realized by an electrical network. To make the point more clear and provide a basis for the later discussion, consider the transfer function for a bandpass filter

$$H(s) = \frac{\omega_0 s/Q}{s^2 + \omega_0 s/Q + \omega_0^2}. \tag{1}$$

Via standard techniques this transfer function leads to the set of dynamical equations below, which for a particular instance is a set taken from an equvalence class of sets

$$\frac{d}{dt}\bar{x} = A\bar{x} + \bar{b}u, \quad y = \bar{c}^T\bar{x} + du. \tag{2}$$

One particular instance might be the following form:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-\frac{\omega_0}{Q} & -\omega_0\\ \omega_0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\frac{\omega_0}{Q}\\ 0\end{bmatrix}u$$
$$y = [1 \quad 0]\begin{bmatrix}x_1\\ x_2\end{bmatrix} + (0)u. \tag{3}$$

Now suppose that we asociate a capacitor voltage and and inductor current with the state variables $x_1$ and $x_2$, respecitively. Then the state equations, with appropriate scaling, describe a Kirchhoff's current law (KCL) and a Kirchhoff's voltage law (KVL) equation corresponding to the standared *RLC* circuit realization for a bandpass filter cricuit, while the input–output equation asserts the fact that the output is taken to be the capacitor voltage. Fig. 2 shows this filter, with state variables labeled.

A second possible set of dynamical equations for the above transfer function is given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-\frac{\omega_0}{2Q} & -\omega_A\\ \omega_A & -\frac{\omega_0}{2Q}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \frac{\omega_0}{Q}\begin{bmatrix}1\\ -1\end{bmatrix}u$$
$$y = \frac{1}{2\lambda}\left[(-\gamma+\lambda) - (\gamma+\lambda)\right]\begin{bmatrix}x_1\\ x_2\end{bmatrix} + (0)u$$

where

$$\gamma = (1/2Q); \quad \lambda = \sqrt{1-\gamma^2}; \quad \omega_A = \lambda\omega_0 = \sqrt{1-\gamma^2}\omega_0. \tag{4}$$

A common circuit realization for these equations is had by implementing a pair of lossy integrators using the op amps and adding the necessary surrounding circuitry. Fig. 3 shows this.

One of the beauties of this state–space characterization is that it may be trasnformed with relative ease to yield a countless array of variations, each possessing its own possible circuit realization. Of course, as one might expect, only some realizations are considered useful in practice. Nevertheless, the search for new, potentially useful, realizations, has been an ongoing one. The class of exponential state–space filters introduced in

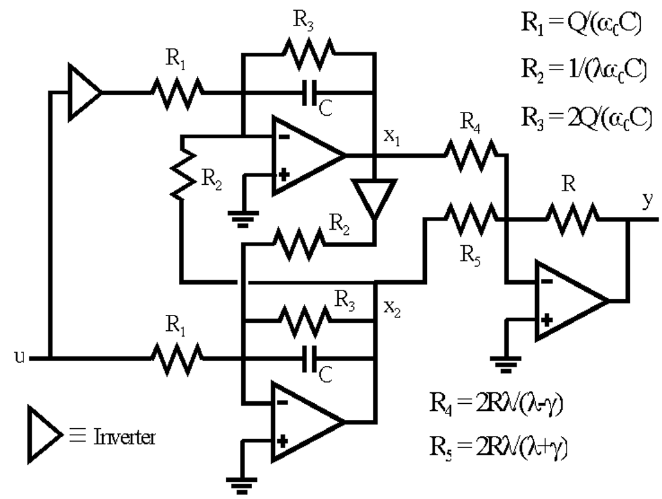


Fig. 3. Circuit schematic for the transformed equivalent to that in Fig. 2.

[10], being a superset of log-domain filters [11], is a prime example of how one migh contrive an equivalent filter in a subtle manner. While the mappings introduced in the development of exponential state–space filteres are more abstract, they are no more than variations on the old technique of transforming state variables. For example, if one were to replace the state variables in the system of (2) as shown below, the resulting set of dynamical equations will possess the same transfer function as before. Alternatively, one might say that the poles and zeros have remained unchanged

$$\text{Let } \bar{w} = M\bar{x}; \text{ then}$$
$$\frac{d}{dt}\bar{w} = MAM^{-1}\bar{w} + M\bar{b}u$$
$$y = \bar{c}^T M^{-1}\bar{w} + du. \tag{5}$$

In fact, such a transformation may be used to take the dynamical system of (3) into that shown in (4). Specifically, the transformation matrix $M$ may be given by

$$M = \begin{bmatrix}1 & (\gamma+\lambda)\\ -1 & (-\gamma+\lambda)\end{bmatrix}. \tag{6}$$

The interesting point here is that such a state-space transformation, defined by $M$, yields an alternative state-space representation that will almost certainly lead to an alternate circuit realization after associating circuit quantities with the state variables, a fact which is apparent by comparing Figs. 2 and 3. Unfortuantely, as is often the case in modern circuit design, if a certain transfer function is very difficult to realize due to limitations in available components, then a state-space transformation will rarely eliminate this problem. A good example would be that where the transfer function is a very high-frequency very high-$Q$ bandpass filter. Using available components, no monolithic realization is generally acceptable. Thus, state-space realizations have not been used to alleviate the problems. Instead, in modern radio receivers, a combination of modulators and lower frequency complex filters are used to obtain the needed selectivity and dynamic range. These realizations yield so-called high intermediate frequency (IF), low-IF, and zero-IF systems [12].

In each case, the operation of these filters is somewhat mysterious to the non-practitioner until he or she considers the actual frequency translation provided by the modulators, along with the phase spectra created by the so-called in-phase and quadrature modulation.

A point of interst here is that the bandpass filter described by (4) is almost identical to a second-order complex filter realized via the translation of a prototype low-pass filter, as discussed later in Section IV. The only difference lies in the $\bar{c}^T$ vector, which for the complex filter case is simply $(1/2)|1 - 1|$. The filter of (4) approaches this form for large values of $Q$. Thus, the intrinsic difference between a classic bandpass filter and a complex filter for the second-order case is that the ouput is created via slightly different combinations of the state variables. This difference between the filter types will be shown for the general case in Section IV. At this point is should be noted that both filter types have the same matrix, using (4) as the basis for the dynamical equations, and therefore both filter types may be used in the mathematical development below.

We are now ready to address a key question regarding the combination of modulation and filtering. Namely, can a fundamental equivalence be drawn between a high-$Q$ high-frequency filter and modulators and a lower $Q$ lower frequency filter? In order to investigate this question, suppose that a time-varying state-space transformation, similar to that above, were applied to the original dynamical equations. Then, we would have

$$\begin{aligned} &\text{Let } \bar{w} = M(t)\bar{x}; \text{ then} \\ M(t)\frac{d}{dx}\bar{x} &= M(t)AM^{-1}(t)\bar{w} + M(t)\bar{b}u \\ y &= \bar{c}^T M^{-1}(t)\bar{w} + du. \end{aligned} \tag{7}$$

This result shows the state equations with mixed variables. The next line shows how to fix this

$$\begin{aligned} &\text{Since } \frac{d}{dt}\bar{w} \\ &\quad = M(t)\frac{d}{dt}\bar{x} + \dot{M}(t)\bar{x}; \text{ then} \\ &M(t)\frac{d}{dx}\bar{x} + \dot{M}(t)\bar{x} \\ &\quad = M(t)AM^{-1}(t)\bar{w} + \dot{M}(t)\bar{x} + M(t)\bar{b}u \\ &\frac{d}{dt}\bar{w} \\ &\quad = M(t)AM^{-1}(t)\bar{w} + \dot{M}(t)M^{-1}\bar{w} + M(t)\bar{b}u \\ &\quad = \hat{A}\bar{w} + \bar{g}(t)u, \\ &\qquad \text{where } \hat{A} = M(t)AM^{-1}(t) + \dot{M}(t)M^{-1}(t); \\ &\qquad g(t) = M(t)\bar{b} \end{aligned} \tag{8}$$

In general, $\hat{\mathrm{A}}$ is a time-varying matrix; however, suppose that the new dynamical equations correspond intrinsically to a differnt (lower frequency, lower $Q$) filter. That is, suppose that $\hat{\mathrm{A}}$ is the constant state matrix for a lower frequency, lower $Q$, bandpass filter than the original filter. Considering the time-varying nature of $M(t)$, it is not obvious that such a scenario is possible. In fact, such a scenario is possible if the following matrix differential equation for $M(t)$ has a solution:

$$\dot{M}(t) = \hat{A}M(t) - M(t)A. \tag{9}$$

In general, this equation is difficult to solve. However, if $A$ and $\hat{\mathrm{A}}$ commute, then the solution is straightforward. In particular

$$M(t) = Ke^{(\hat{A}-A)t} = Ke^{\hat{A}t}e^{-At} \tag{10}$$

where $K$ is a constant matrix that must commute with $\hat{\mathrm{A}}$, which will be assumed to be the identity matrix unless otherwise stated.

Since not all choices of state matrices will commute with one another, it is wise to consider the appropriate choice of such matrices up front. The state matrix in (3), for example, will not commute with a similar version of itself, that is, a version pertaining to another filter with a different center frequency and $Q$. A state matrix that would commute with others in this case is given in (4), and is reproduced below

$$A = \begin{bmatrix} -\frac{\omega_0}{2Q} & -\omega_1 \\ \omega_1 & -\frac{\omega_0}{2Q} \end{bmatrix} = \omega_0 \begin{bmatrix} -\gamma & -\lambda \\ \lambda & -\gamma \end{bmatrix} = \begin{bmatrix} -\sigma & -\omega_A \\ \omega_A & -\sigma \end{bmatrix}$$

where

$$\sigma = \gamma\omega_0 = \frac{\omega_0}{2Q} \text{ and } \omega_A = \lambda\omega_0 = \sqrt{1-\gamma^2}\omega_0. \tag{11}$$

It is easy to show that this matrix commutes with a version of itself having different values of $\omega_0$ and $\gamma$ and $\lambda$. It is well known that the exponential matrix $\mathrm{e}^{\mathrm{A}t}$, typically referred to as the state transition matrix, corresponding to this state matrix is given by

$$\begin{aligned} e^{At} &= e^{-\sigma t}\Phi(-\omega_A t), \\ \text{where } \Phi(\Omega) &= \begin{bmatrix} \cos(\Omega) & \sin(\Omega) \\ -\sin(\Omega) & \cos(\Omega) \end{bmatrix}. \end{aligned} \tag{12}$$

Assuming that $\hat{\mathrm{A}}$ has a similar form to $A$, we may specify its state transition matrix

$$\hat{A} = \begin{bmatrix} -\hat{\sigma} & -\omega_{\hat{A}} \\ \omega_{\hat{A}} & -\hat{\sigma} \end{bmatrix} \Rightarrow e^{\hat{A}t} = e^{-\hat{\sigma}t}\Phi(-\omega_{\hat{A}}t). \tag{13}$$

Before proceeding, note the following properties of $\Phi()$:

$$\begin{aligned} \Phi(\Theta_1)\,\Phi(\Theta_2) = \Phi(\Theta_2)\,\Phi(\Theta_1) &= \Phi(\Theta_1 + \Theta_2) \\ \Phi(\Theta)^{-1} = \Phi^T(\Theta) &= \Phi(-\Theta) \end{aligned} \tag{14}$$

Thus, we may write

$$\begin{aligned} M(t) &= Ke^{\hat{A}t}e^{-At} \\ &= Ke^{\hat{\sigma}t}\Phi(-\omega_{\hat{A}}t)[e^{-\sigma t}\Phi(-\omega_A t)]^{-1} \\ &= Ke^{(\sigma-\hat{\sigma})t}\Phi(-\omega_{\hat{A}}t)\Phi(\omega_A t) \\ &= Ke^{(\sigma-\hat{\sigma})t}\Phi(\omega_M t) \\ &= Ke^{(\sigma-\hat{\sigma})t}\begin{bmatrix} \cos(\omega_M t) & \sin(\omega_M t) \\ -\sin(\omega_M t) & \cos(\omega_M t) \end{bmatrix}, \\ &\qquad \text{where } \omega_M = \omega_A - \omega_{\hat{A}} \end{aligned} \tag{15}$$

Now, by letting $\sigma - \hat{\sigma} = 0$, we get $M(t)$ equal to a unitary matrix with sinusoidal entries premultiplied by $K$. We can achieve an intuitively pleasing generalization by allowing $K$ to be given by the unitary matrix below, which possesses the necessary commutativity property

$$K = \Phi(\Theta) = \begin{bmatrix} \cos(\Theta) & \sin(\Theta) \\ -\sin(\Theta) & \cos(\Theta) \end{bmatrix}$$
$$\therefore M(t) = \Phi(\Theta)\Phi(\omega_M t) = \begin{bmatrix} \cos(\omega_M t + \Theta) & \sin(\omega_M t + \Theta) \\ -\sin(\omega_M t + \Theta) & \cos(\omega_M t + \Theta) \end{bmatrix}. \quad (16)$$

This result proves ultimately that the synchronous filters derived here are time invariant, which is a fact proven later. In the meantime, we shall assume $\Theta$ equal to 0 to minimize the clutter in the mathematics.

Using the results of these calculations, we may specify the form of the transformed dynamical equations in (8) more completely. This is done below as

$$\frac{d}{dx}\bar{w} = \hat{A}\bar{w} + \bar{g}(t)u$$
$$y = \bar{h}^T(t)\bar{w}$$
$$\bar{g}(t) = M(t)\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = b_1 v_1(t) + b_2 v_2(t)$$

where

$$\bar{h}^T(t) = x\,[\,c_1 \quad c_2\,]\,M^{-1}(t) = c_1 v_1^T(t) + c_2 v_2^T(t)$$
$$v_1(t) = \begin{bmatrix} \cos(\omega_M t) \\ -\sin(\omega_M t) \end{bmatrix}$$
$$v_2(t) = \begin{bmatrix} \sin(\omega_M t) \\ \cos(\omega_M t) \end{bmatrix}. \quad (17)$$

This result is quite interesting, since it provides a mathematical equivalence between a given bandpass filter and another filter with modulators $\bar{g}(t)$ and $\bar{h}(t)$, at the input and the output. Taking a closer look at the modulators, we see that they are quadrature modulators of the type well known to RF designers, for example. Specifically, we have

$$\bar{g}(t) = b_1 v_1(t) + b_2 v_2(t) = \begin{bmatrix} b_1\cos(\omega_M t) + b_2 \sin(\omega_M t) \\ -b_1 \sin(\omega_M t) + b_2\cos(\omega_M t) \end{bmatrix} = b_0 \begin{bmatrix} \sin(\omega_M t + \beta) \\ \cos(\omega_M t + \beta) \end{bmatrix}$$

where

$$b_0 = \sqrt{b_1^2 + b_2^2}$$
$$\beta = \tan^{-1}\left(\frac{b_1}{b_2}\right)$$
$$\bar{h}^T(t) = c_1 v_1^T(t) + c_2 v_2^T(t) = c_0\,[\,\sin(\omega_M t + \alpha) \quad \cos(\omega_M t + \alpha)\,]$$
$$\text{where } c_0 = \sqrt{c_1^2 + c_2^2}; \quad \alpha = \tan^{-1}\left(\tfrac{c_1}{c_2}\right) \quad (18)$$

For the system of (4), we have

$$\beta = \tan^{-1}\left(\frac{1}{-1}\right) = \frac{\pi}{4}$$
$$\alpha = \tan^{-1}\left(\frac{-\gamma+\lambda}{-(\gamma+\lambda)}\right) = -\tan^{-1}\left(\frac{\lambda-\gamma}{\lambda+\gamma}\right). \quad (19)$$

Notice that for large $Q, \alpha \approx \beta = -\pi/4$. Considering the fact that $\omega_{\hat{\mathrm{A}}} = \omega_{\mathrm{A}} - \omega_M$, the core filter $Q$ is $(\omega_{\hat{\mathrm{A}}})/(\omega_{\mathrm{A}})$ times the $Q$ of the original filter. For low IF systems $\omega_{\hat{\mathrm{A}}} \ll \omega_{\mathrm{A}}$, which guarantees that the $Q$ of the core filter is much lower than that of the original filter.

Using the results derived thus far, the filter described in (17) may be put into a form that completely captures the synchronous filter concept. Specifically, we may rewrite (17) as follows:

$$\frac{d}{dt}\bar{w} = \hat{A}\bar{w} + \hat{B}\bar{u}_M$$
$$y = \bar{h}^T(t)\bar{w}$$

where

$$\hat{B} = \begin{bmatrix} b_0 & 0 \\ 0 & b_0 \end{bmatrix}$$
$$\bar{u}_M = \begin{bmatrix} \sin(\omega_M t + \beta) \\ \cos(\omega_M t + \beta) \end{bmatrix} u(t)$$
$$\bar{h}^T(t) = c_0\,[\,\sin(\omega_M t + \alpha) \quad \cos(\omega_M t + \alpha)\,]. \quad (20)$$

Thus, we see that the original filter, of Figs. 2 and 3, has been replaced by a multiple input filter, having center frequency, $\hat{\omega}_0 \approx \omega_{\hat{\mathrm{A}}}$, and quality factor $Q_C = (\omega_{\hat{\mathrm{A}}}/\omega_{\mathrm{A}})Q$, preceded and followed by quadrature modulators. This is the synchronous filter form shown in Fig. 1.

As one would expect, the analysis shows that the modulators operate with a frequency near the difference between the resonant frequencies of the two filters. However, one might also wonder why the modulators only operate at a frequency near the difference and not also the sum. In fact, one may find a sub- and a super-heterodyne solution. To see how to find the other form and prepare for the discussion later, consider the following alternate approach to finding $\hat{\mathrm{A}}$ in the analysis above

$$\hat{A} = M(t)AM^{-1}(t) + \dot{M}(t)M^{-1}(t)$$
$$= A + \frac{d}{dt}[K\Phi(\omega_M t)][K\Phi(\omega_M t)]^{-1}$$
$$= \begin{bmatrix} -\sigma & -\omega_A \\ \omega_A & -\sigma \end{bmatrix} + \begin{bmatrix} 0 & \omega_M \\ -\omega_M & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -\sigma & -\omega_{\hat{A}} \\ \omega_{\hat{A}} & -\sigma \end{bmatrix}, \qquad \text{where } \omega_{\hat{A}} = \omega_A - \omega_M. \quad (21)$$

Suppose that one were to transform dynamical equations whose state matrix were in the form shown in (11) using the unitary matrix $T$ below

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow TAT^{-1} = A^T$$
$$= \begin{bmatrix} -\sigma & \omega_A \\ -\omega_A & -\sigma \end{bmatrix} \Rightarrow e^{A^T t} = e^{-\sigma T}\Phi(\omega_A t). \quad (22)$$

This would suggest a set of dynamical equations of the form

$$\frac{d}{dx}\bar{x} = A^T\bar{x} + T\bar{b}u; \quad y = \bar{c}^T T\bar{x} + du. \quad (23)$$

Note that we have exploited the fact that $T$ is its own inverse in writing (23). Also, we have reused the name of the state variables. Suppose that we now apply a time-varying transformation that is the transpose of the matrix used for $M(t)$ earlier. This yields the following transformed state matrix:

$$\begin{aligned}
A &\rightarrow A^T = TAT\\
M(t) &\rightarrow M^T(t) = TM(t)T\\
\hat{A} &= TM(t)T(TAT)TM^{-1}(t)T + T\dot{M}(t)T(TM^{-1}(t)T)\\
&= TM(t)AM^{-1}(t)T + T\dot{M}(t)M^{-1}(t)T\\
&= TAT + T\dot{M}(t)M^{-1}(t)T\\
&= T\begin{bmatrix} -\sigma & -\omega_A \\ \omega_A & -\sigma \end{bmatrix} T + T\begin{bmatrix} 0 & \omega_M \\ -\omega_M & 0 \end{bmatrix} T\\
&= \begin{bmatrix} -\sigma & \omega_A \\ -\omega_A & -\sigma \end{bmatrix} + \begin{bmatrix} 0 & -\omega_M \\ \omega_M & 0 \end{bmatrix}\\
&= \begin{bmatrix} -\sigma & -\omega_{\hat{A}} \\ \omega_{\hat{A}} & -\sigma \end{bmatrix}, \qquad \text{where, } \omega_{\hat{A}} = -\omega_A + \omega_M
\end{aligned} \tag{24}$$

Now, the modulation frequency is the sum of the two resonant frequencies as opposed to the difference. As before, in the specification of $M(t), K$ may be chosen as in (16), yielding an arbitrary phase shift; however, we will assume $K$ to be the identity matrix to simplify the form of the results. The final form of the transformed state equations, analogous to those in (20), is shown below as

$$\begin{aligned}
\frac{d}{dt}\bar{w} &= \hat{A}\bar{w} + \hat{B}\bar{u}_M\\
y &= \bar{h}^T(t)\bar{w}
\end{aligned}$$

where

$$\begin{aligned}
\hat{B} &= \begin{bmatrix} b_0 & 0 \\ 0 & b_0 \end{bmatrix}\\
\bar{u}_M &= \begin{bmatrix} \cos(\omega_M t + \beta) \\ \sin(\omega_M t + \beta) \end{bmatrix} u(t)\\
\bar{h}^T(t) &= c_0 \left[ \cos(\omega_M t + \alpha) \quad \sin(\omega_M t + \alpha) \right].
\end{aligned} \tag{25}$$

The subtle difference between the modulating vectors in (20) and (25) is all that separates the sub-heterodyne version from the super-heterodyne version. The similarity of the mathematical results in (20) and (25) permits a fairly straightforward discussion of the consequences of phase and magnitude errors in the modulating vectors. Practitioners have come to appreciate that such errors lead to imperfect image rejection. While the calculation of such effects has been shown before, the Appendix demonstrates how one may use the synchronous filter setting to rederive them.

Thus far, we have obtained exact equivalence between the synchronous filter and its original time invariant bandpass equivalent. The synchronous filter maintains the number of state variables and we can show that the complete response of the synchronous filter must be the same as that of the original system. Clearly, the transfer functions are the same, since the dynamical equations have been transformed in such a way as to guarantee this. Hence, the zero state responses must be the same. One may ask how the zero input response is maintained. To see this, consider the general form of the zero input response in the original system, $y_{ZI}(t)$, as specified using the state transition matrix $\Phi(t)$

$$y_{ZI}(t) = \bar{c}^T\Phi(t)\bar{x}(0), \qquad \text{where } \Phi(t) = e^{At}. \tag{26}$$

The initial conditions specified by the initial value of the state vector completely determine the zero input response. These initial conditions correspond to the initial stored energy on capacitors and inductors in the original, un-transformed system. Using the transformation matrix $M(t)$, we can rewrite (26) as follows:

$$\begin{aligned}
y_{ZI}(t) &= \bar{c}^T\Phi(t)(\bar{x})(0)\\
&= \bar{c}^T M^{-1}(t)M(t)\Phi(t)M^{-1}(0)M(0)\bar{x}(0)\\
&= \bar{h}^T(t)(M(t)\Phi(t)M^{-1}(0))\bar{w}(0).
\end{aligned} \tag{27}$$

Using the relations derived in (14) and (21), and the Cayley–Hamilton Theorem, we have

$$\begin{aligned}
\hat{A} &= M(t)AM^{-1}(t) + \dot{M}(t)M^{-1}(t)\\
&= M(t)AM^{-1}(t) + \begin{bmatrix} 0 & \omega_M \\ -\omega_M & 0 \end{bmatrix}\\
\therefore\ e^{\hat{A}t} &= M(t)e^{At}M^{-1}(t)\Phi(\omega_M t)\\
&= M(t)e^{At}M^{-1}(0)\Phi(-\omega_M t)\Phi(\omega_M t)\\
&= M(t)e^{At}M^{-1}(0).
\end{aligned} \tag{28}$$

Combining these results, we have

$$\begin{aligned}
y_{ZI}(t) &= \bar{c}^T\Phi(t)\bar{x}(0)\\
&= \bar{h}^T(t)(M(t)\Phi(t)M^{-1}(0))\bar{w}(0)\\
&= \bar{h}^T(t)e^{\hat{A}t}\bar{w}(0)\\
&= \bar{h}^T(t)\hat{\Phi}(t)\bar{w}(0), \qquad \text{where } \hat{\Phi}(t) = e^{\hat{A}t}.
\end{aligned} \tag{29}$$

This result demonstrates that the initial conditions in the core filter will completely specify the zero input response of the overall synchronous filter. Furthermore, the response is invariant to the phase of the modulators. However, note that the initial condition vector in the core filter is dependent upon the value of $M(t)$ at $t = 0$. Therefore, although the overall synchronous filter is invariant to the choice of $K = M(0)$, the initial condition vector for the embedded (core) filter is dependent upon not only the initial condition vector for the original filter but also the value of $K$. Some thought will convince the reader that this is quite reasonable and should be expected.

## III. Complex Filters

There is a close correspondence between complex filters and a subclass of synchronous filters that can be seen by a closer examination of the mathematics introduced above. In order to see this, consider the case of a typical modulated signal $f(t)$. Regardless of the modulation method, this real signal may be expressed as a combination of two independent signals modulated by a cosine and a sine carrier, respectively. Specifically, we can say

$$f(t) = f_1(t)\cos(\omega_0 t) + f_2(t)\sin(\omega_0 t). \tag{30}$$

In the standard complex filtering context, this signal coexists with a number of other interfering signals that are removed through the complex filtering process. Ultimately, then, this signal is translated by $\omega_0$ in the frequency domain, yielding a complex baseband signal where images are removed by the complex filtering operation. The so-called low-IF approach to this is to multiply $f(t)$ by the complex exponential $\exp(-j\omega_M t)$, and apply a bandpass filter with center frequency $\omega_0 - \omega_M$ to each of the real (in-phase) and imaginary (quadrature) signals. In particular, the net input to the complex filter is

$$\begin{aligned} f_{\mathrm{BP-in}}(t) &= (\cos(\omega_M t) - j\sin(\omega_M t))f(t) \\ &= f_1(t)\cos(\omega_M t)\cos(\omega_0 t) \\ &\quad + f_2(t)\cos(\omega_M t)\sin(\omega_0 t) \\ &\quad - j[f_1(t)\sin(\omega_M t)\cos(\omega_0 t) \\ &\quad + f_2\sin(\omega_M t)\sin(\omega_0 t)]. \end{aligned} \tag{31}$$

Making the practical assumption that the bandwidths of $f_1(t)$ and $f_2(t)$ are small compared to $\omega_M$, then upon bandpass filtering (low-IF) the real and imaginary signals only have significant energy at the difference frequency $\omega_0 - \omega_M$, leaving

$$\begin{aligned} f_{\mathrm{BP-out}}(t) &= f_1(t)\cos((\omega_0 - \omega_M)t) + f_2(t)\sin((\omega_0 - \omega_M)t) \\ &\quad + j[f_1(t)\sin((\omega_0 - \omega_M)t) - f_2(t)\cos((\omega_0 - \omega_M)t)]. \end{aligned} \tag{32}$$

A factor of $(1/2)$ has been neglected in writing the above result. Finally, by multiplying by the complex exponential $\exp(-j\omega_L t)$, where $\omega_L = \omega_0 - \omega_M$, we have

$$\begin{aligned} e^{j\omega_L t} f_{\mathrm{BP-out}}(t) &= [\cos(\omega_L t) - j\sin(\omega_L t)][f_1(t)\cos(\omega_L t) \\ &\quad + f_2(t)\sin(\omega_L t) + j(f_1(t)\sin(\omega_L t) \\ &\quad - f_2(t)\cos(\omega_L t))] \\ &= f_1(t)[cos^2(\omega_L t) + \sin^2(\omega_L t)] \\ &\quad - jf_2(t)[sin^2(\omega_L t) + \cos^2(\omega_L t)] \\ &= f_1(t) - jf_2(t). \end{aligned} \tag{33}$$

This is the complex baseband signal recovered in a typical complex filtering scenario.

Now let us consider the synchronous filter situation. Recall that the output is obtained by scaling the state variables by cosine and sine terms, respectively. In a given filter scenario, suppose that filter inputs are translated down in frequency to a (low-IF) frequency of $\omega_L$. In fact, the inputs to the core filter see signals translated both down to $\omega_L$ and up to $2\omega_0 - \omega_1$. However, as in the case of complex filters, the higher frequency components are rejected by the core (complex) filter, so at the core filter outputs we only have signals whose energy is centered near $\omega_L$. The final output is derived by, for example, the following computation:

$$y(t) = x_1(t)\cos(\omega_M t) + x_2(t)\sin(\omega_M t). \tag{34}$$

Because we know that the synchronous filter exactly produces the correct bandpass filtered result, then, letting $f(t)$ denote that signal, we have

$$\begin{aligned} y(t) &= x_1(t)\cos(\omega_M t) + x_2(t)\sin(\omega_M t) \\ &= f_1(t)\cos(\omega_0 t) + f_2(t)\sin(\omega_0 t). \end{aligned} \tag{35}$$

This suggests that the state variable outputs must obey the following relations:

$$\begin{aligned} x_1(t) &= f_1(t)\cos(\omega_L t) + f_2(t)\sin(\omega_L t) \\ x_2(t) &= -f_1(t)\sin(\omega_L t) + f_2(t)\cos(\omega_L t), \\ &\qquad \text{where } \omega_L + \omega_M = \omega_0. \end{aligned} \tag{36}$$

Given this fact, we may recover the individual components of $f(t)$ with the following computations:

$$\begin{aligned} f_1(t) &= x_1(t)\cos(\omega_L t) - x_2(t)\sin(\omega_L t) \\ f_2(t) &= x_1(t)\sin(\omega_L t) + x_2(t)\cos(\omega_L t). \end{aligned} \tag{37}$$

Hence, we have recovered the real and imaginary parts of the baseband complex signal just as in the case of the complex filtering operation. Thus, synchronous and complex filters are identical when viewed from this perspective. On the other hand, synchronous filters provide a larger class of filters that accomplish the task of down conversion and filtering.

## IV. Higher Order Filters

Thus far, we have restricted our attention to the most basic bandpass filters. Nevertheless, the above discussion may be generalized to higher order filters in a fairly straightforward way. However, when one considers the general problem of realizing a bandpass filter, it is imperative to consider the method by which one obtains a characterization of the filter in the first place. For higher order filters there are really only two classical approaches. The first is based on what might be called a geometric mapping of a low-pass prototype, and the second is based on a translation of a low-pass prototype. In each case the full body of design knowledge regarding low-pass filters is exploited in the creation of the bandpass filter. The second, or translational, approach has become popular due to the study of complex filtering in recent years. Here a low-pass filter is simply translated to another center point in the frequency domain. Translation in one direction only results in a filter that is necessarily complex. Although RF designers routinely design such filters, they always accomplish the physical circuitry with real filters. It should be noted that one may directly specify a complex filter as shown in [9]. The approach in [15] is more general than a simple translation of low-pass filter prototypes. While such filters add an important dimension to the design of complex filters, a proper discussion of that broader class of filters is beyond the scope of this work. In light of this, consider the translated filter created by taking two identical low-pass prototypes and translating each by the same amount in opposite directions in the frequency domain. The symmetry of this operation yields a real filter of the type that is actually implemented in complex filtering. Let us consider the mathematical operation involved in this process.

Consider an arbitrary, $N$th-order, prototype low-pass filter characterized by the dynamical equations

$$\frac{d}{dt}\bar{x} = A_{LP}\bar{x} + \bar{b}_{LP}u \quad y = \bar{c}_{LP}^T\bar{x}. \tag{38}$$

When we create a pair of these filters, symmetrically translate, and sum the outputs, we have the translated version of the filters suggested above. This composite filter is given by the following dynamical equations:

$$\begin{aligned}\frac{d}{dx}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\end{bmatrix} &= \begin{bmatrix}A_{LP}-j\Omega_0 & 0\\ 0 & A_{LP}+j\Omega_0\end{bmatrix}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\end{bmatrix}\\ &\quad + \begin{bmatrix}\bar{b}_{LP}\\ \bar{b}_{LP}\end{bmatrix}u\\ y &= [\bar{c}_{LP}^T \quad \bar{c}_{LP}^T]\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\end{bmatrix}.\end{aligned} \tag{39}$$

The symbol $\Omega_0$ denotes the identity matrix scaled by the frequency translation $\omega_0$. It is trivial to transform this system into one with real parameters as shown below

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\end{bmatrix} &= \begin{bmatrix}A_{LP} & -\Omega_0\\ \Omega_0 & A_{LP}\end{bmatrix}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\end{bmatrix} + \begin{bmatrix}\bar{b}_{LP}\\ \bar{b}_{LP}\end{bmatrix}u\\ y &= [\bar{c}_{LP}^T \quad \bar{c}_{LP}^T]\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\end{bmatrix}.\end{aligned} \tag{40}$$

Note that this system of dynamical equations possesses a state matrix with the type of symmetry obtained earlier, although on a block basis.

It has been noted by others—e.g., [15]—and is clear by considering the translation operation, that the translated low-pass filter is a bandpass filter whose dc response is not necessarily 0, since a low-pass filter typically only has a gain of zero identically at infinite frequencies. A more classic bandpass filter, having a dc gain of 0, is obtained by the geometric approach which has been used for many years to obtain bandpass filters from low-pass prototypes. In this approach, the frequency-domain variable, s, in the transfer function of the low-pass prototype is replaced, or mapped, by the function $(s^2+\omega_0{}^2)/(s)$. The resulting transfer function is then realized in circuit form by using any of a variety of methods. In our case, we wish to find the state-space realization that results from this mapping on s. It can be shown [13] that this realization is given by the following system of dynamical equations:

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\end{bmatrix} &= \begin{bmatrix}A_{LP} & -\Omega_0\\ \Omega_0 & 0\end{bmatrix}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\end{bmatrix}\\ &\quad + \begin{bmatrix}\bar{b}_{LP}\\ 0\end{bmatrix}u\\ y &= [\bar{c}_{LP}^T \quad 0]\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\end{bmatrix}.\end{aligned} \tag{41}$$

Despite its obvious similarity to the translated filter, this system, as shown, fails to possess a state matrix with the symmetry properties described earlier. In fact, it resembles (3). Fortunately, because of the scaled identity matrices comprising the off-diagonal entries, this block matrix may be transformed to the desired form via a simple extension of the similarity transformation used in the second-order case studied in Section II. Specifically, we may use the following transforming matrix:

$$M_0 = \begin{bmatrix}I & \Lambda-\Gamma\\ -I & \Lambda+\Gamma\end{bmatrix},$$
$$\text{where } \Gamma = \tfrac{1}{2\omega_0}A_{LP}; \quad \Lambda = \sqrt{I-\Gamma^2}. \tag{42}$$

Applying this transformation to the matrices in the dynamical equations above yields

$$\begin{aligned}A &\to \begin{bmatrix}\frac{1}{2}A_{LP} & -\Omega_A\\ \Omega_A & \frac{1}{2}A_{LP}\end{bmatrix}\\ \bar{b} &\to \begin{bmatrix}I\\ -I\end{bmatrix}\bar{b}_{LP}\\ \bar{c}^T &\to \bar{c}_{LP}^T[-\Gamma+\Lambda \ \ -\Gamma-\Lambda]\\ \Omega_A &= \omega_0\Lambda = \omega_0\sqrt{I-\frac{A_{LP}^2}{(4\omega_0^2)}}.\end{aligned} \tag{43}$$

The two important differences between the dynamical system based on these matrices and the system obtained by the translation approach is that the off-diagonal matrices in the block state matrix are not diagonal and that the output operator, $\bar{c}^T$, has slightly skewed entries (as in the second-order example earlier).

Now that we have dynamical equations representing the two key classes of bandpass filters that are considered in modern electronic systems, we are in a position to consider the effect of a time-varying transformation that will lead to synchronous filter realizations. Suppose that one were to define the mapping $M(t)$ analogous to before, as follows:

$$\begin{aligned}M(t) &= \Phi(\Theta)\Phi(\omega_M t)\\ &= \begin{bmatrix}\cos(\omega_M t+\Theta)\cdot I & \sin(\omega_M t+\Theta)\cdot I\\ -\sin(\omega_M t+\Theta)\cdot I & \cos(\omega_M t+\Theta)\cdot I\end{bmatrix},\\ \text{where}\Phi(\Theta) &= \begin{bmatrix}\cos(\Theta)\cdot I & \sin(\Theta)\cdot I\\ -\sin(\Theta)\cdot I & \cos(\Theta)\cdot I\end{bmatrix}.\end{aligned} \tag{44}$$

$I$ denotes an identity matrix of dimension equal to $\mathrm{A_{LP}}$. Applying this time-varying similarity transformation with $\Theta = \pi/4$ to the translated filter described in (40) above yields a set of dynamical equations with the matrix operators given by

$$\begin{aligned}\hat{A} &= \begin{bmatrix}A_{LP} & -(\Omega_0-\Omega_M)\\ \Omega_0-\Omega_M & A_{LP}\end{bmatrix}\\ \bar{b} &= \begin{bmatrix}\cos(\omega_M t)\cdot I\\ \sin(\omega_M t)\cdot I\end{bmatrix}\bar{b}_{LP}\\ \bar{c}^T &= \bar{c}_{LP}^T[\cos(\omega_M t)\cdot I \ \ \sin(\omega_M t)\cdot I].\end{aligned} \tag{45}$$

Applying the transformation with $\Theta = -\pi/4$ to the dynamical equations in (43) derived from the geometric transformation yields the following matrix operators:

$$\begin{aligned}\hat{A} &= \begin{bmatrix}\frac{1}{2}A_{LP} & -(\Omega_A-\Omega_M)\\ \Omega_A-\Omega_M & \frac{1}{2}A_{LP}\end{bmatrix}\\ \bar{b} &= \begin{bmatrix}\cos(\omega_M t)\cdot I\\ -\sin(\omega_M t)\cdot I\end{bmatrix}\bar{b}_{LP}\\ \bar{c}^T &= \bar{c}_{LP}^T[\cos(\omega_M t)\cdot(-\Gamma-\Lambda) \ \ \sin(\omega_M t)\\ &\quad \cdot(-\Gamma+\Lambda)].\end{aligned} \tag{46}$$

Finally, we may rewrite the dynamical equations in multiple input form, capturing all of the aspects of the general high-order synchronous filter. In the process it is possible to identify a two dimensional input vector as input to the higher order system. Specifically

$$\frac{d}{dt}\bar{w} = \hat{A}\bar{w} + \hat{B}\bar{u}_M$$
$$y = \bar{h}^T(t)\bar{w},$$

where

$$\hat{B} = \begin{bmatrix} \bar{b}_{LP} & \bar{0} \\ \bar{0} & \bar{b}_{LP} \end{bmatrix}$$
$$\bar{u}_M = \begin{bmatrix} \sin(\omega_M t + \Theta) \\ \cos(\omega_M t + \Theta) \end{bmatrix} u(t)$$
$$\bar{h}^T(t) = [\sin(\omega_M t + \Theta) \ \ \cos(\omega_M t + \Theta)] \begin{bmatrix} \bar{c}_{LP} & \bar{0}^T \\ \bar{0}^T & \bar{c}_{LP} \end{bmatrix} \hat{C}. \tag{47}$$

In (47), $\bar{O}$ denotes an $N \times 1$ vector of zeros, making $\hat{B}$ an $N \times 2$ matrix. $\Theta$ may be chosen arbitrarily and may be assumed to be zero for convenience, although a value of $\pi/2$ makes the modulating vectors look like their counterparts in (45) and (46). The distinction between the translational and the geometric mapping cases lies in the difference in their state matrices and in $\hat{C}$, which is a $2N \times 2N$ identity for the translational case and block diagonal with $\hat{C} = \mathrm{diag}(-\Gamma - \Lambda, -\Gamma + \Lambda)$ in the geometric mapping case.

Observe that in each case the core filter comprises a pair of low-pass filter prototypes, given by the blocks on the diagonal of the state matrix. The equal and opposite off-diagonal terms in each state matrix will result in cross coupling gains between the filter pair. This is precisely how $Gm$–$C$ realizations for complex filters are achieved. Although the state matrices are quite similar, there are some small differences between the translational and the geometric filters. The off-diagonal blocks in the translational case are simply scaled identity matrices, where in the geometric case the off-diagonal terms only approximate diagonal matrices. Thus, the cross coupling between the low-pass filters will be more involved. In addition the matrix, $\Omega_{\mathrm{A}}$, may be complex in general, requiring some additional mathematical preprocessing before a physical realization is achieved. However, in virtually all realistic cases, $\omega_{\hat{\mathrm{A}}} \ll \omega_0$, and therefore $\Omega_{\mathrm{A}}$ will be a real matrix. Another difference between the cases lies in the fact that the output is created by slightly asymmetric gains on the state variables from the respective low-pass filters. One would assume that the geometric filter realization would therefore be more elaborate; however, there may be other benefits that outweigh this. Finally, the result in (47) applies to the sub-heterodyne case, which is analogous to (20) for the second-order case. It is quite straightforward to show that the super-heterodyne case is given by (47) with sine and cosine terms reversed. This is completely analogous to (25) for the second-order case.

## V. Novel Synchronous Filters

Thus far, we have presented a complete theory for the development of complex filters that differs from the standard approach. The framework for synchronous filtering allows us to go beyond that which has already been developed. While there may be many new directions that will prove interesting, one particular case is now presented which shows the possibilities. Suppose that we start with the system of (40) or (43) derived from some desirable low-pass prototype. Now further suppose that we employ a time-varying transformation as in (44). However, this time let $M(t)$ be defined as before with the introduction of a separate time-varying scalar factor, $f(t)$. Specifically, let us define $M_{\mathrm{new}}(t)$ as below.

$$\begin{aligned} M_{\mathrm{new}}(t) &= f(t)M(t) = f(t)K\Phi(\omega_M t) \\ &= f(t)\Phi(\Theta)\Phi(\omega_M t). \end{aligned} \tag{48}$$

As before, we shall choose the value of $\Theta$ for convenience. The only restriction placed upon $f(t)$ is that it be a strictly positive time function so that $M_{\mathrm{new}}(t)$ possesses an inverse. Before explaining the rationale for this change, let us derive the resulting synchronous filter equations. Because of the new definition for $M(t)$, we must now compute the resulting state matrix for the transformed system. Using the derivation in (8), and exploiting the fact that $f(t)$ commutes with $M(t)$ and $A$ we have

$$\begin{aligned} \hat{A}_{\mathrm{new}}(t) &= f(t)M(t)AM^{-1}(t)\frac{1}{f(t)} \\ &\quad + \frac{d}{dt}[f(t)M(t)]M^{-1}(t)\frac{1}{f(t)} \\ &= M(t)AM^{-1}(t) + [f(t)\dot{M}(t) \\ &\quad + \dot{f}(t)M(t)]M^{-1}(t)\frac{1}{f(t)} \\ &= M(t)AM^{-1}(t) + \dot{M}(t)M^{-1}(t) + \frac{\dot{f}(t)}{f(t)}I \\ &= \hat{A} + \frac{\dot{f}(t)}{f(t)}I. \end{aligned} \tag{49}$$

Note that $I$ represents a $2N \times 2N$ identity matrix in the final result, where $N$ is the order of the prototype low-pass filter as discussed in Section IV. The state matrix for the new system, $\hat{\mathrm{A}}_{\mathrm{new}}(t)$, is now given by the time invariant matrix, $\hat{\mathrm{A}}$, derived earlier plus a time dependent diagonal matrix. With these preliminaries we may now specify the complete synchronous filter as shown below, where all parameters are defined within the discussion of (47)

$$\frac{d}{dt}\bar{w} = (\hat{A} + \frac{\dot{f}(t)}{f(t)}I)\bar{w} + \hat{B}\bar{u}_M$$
$$y = \bar{h}^T(t)\bar{w}$$

where

$$\hat{B} = \begin{bmatrix} \bar{b}_{LP} & \bar{0} \\ \bar{0} & \bar{b}_{LP} \end{bmatrix}$$
$$\bar{u}_M = \begin{bmatrix} \sin(\omega_M t + \Theta) \\ \cos(\omega_M t + \Theta) \end{bmatrix} f(t)u(t)$$
$$\begin{aligned} \bar{h}^T(t) &= \frac{1}{f(t)}[\sin(\omega_M t + \Theta) \ \ \cos(\omega_M t + \Theta)] \\ &\quad \times \begin{bmatrix} \bar{c}_{LP} & \bar{0}^T \\ \bar{0}^T & \bar{c}_{LP} \end{bmatrix} \hat{C}. \end{aligned} \tag{50}$$

This is, of course, quite similar to that derived earlier, but with one key change. The front-end modulator, $\bar{g}(t)$, incorporates a time-varying gain. The output modulator, $\bar{h}(t)$, incorporates the

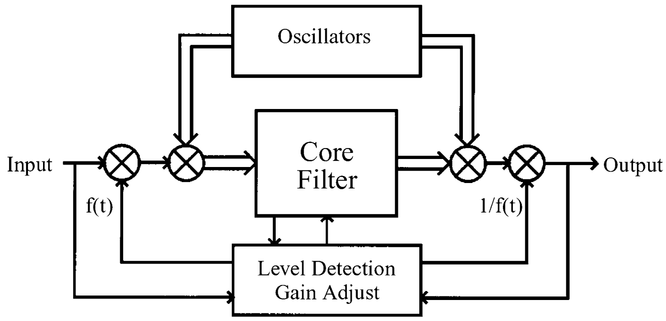


Fig. 4. Block diagram for a companding complex filter system.

reciprocal of this gain. It can now be seen that we have augmented the original synchronous filter with a companding feature. In practice, $f(t)$ may be adjusted continuously. It is often advantageous to maximize dynamic range via some sort of automatic gain control. However, the time-varying gain changes can change the signal in such a way as to produce distortion when filtering is part of the companding channel. These issues have been described and addressed in [16][17] and [18], for example. Here, we see that the issue of companding is addressed with a fairly simple time-varying addition to the state equations. Further inspection shows that this time-varying change in the state matrix amounts to a time-varying $Q$ factor in the underlying core filter. Electronically changing the $Q$ of a filter is relatively simple in most integrated filters and, therefore, this change does not pose a serious implementation problem. The benefits resulting from the companding nature of the system will almost certainly outweigh the cost of the increased complexity.

Fig. 4 shows a block diagram of the companding system just described. Because of the demonstrated equivalence between complex and synchronous filters, this demonstrates an augmented complex filtering system not described elsewhere. While the derivation of this system has been shown for the sub-heterodyne filter case, the idea can be easily extended to a super-heterodyne version.

## VI. Conclusion

Synchronous filtering has been introduced and articulated. The idea stems from the concept of a time-varying transformation of the dynamical equations for a linear filter. Since high-$Q$ high-frequency bandpass filters present the most interesting and challenging design problem currently, the synchronous filters pertaining to this case have been fully explored. Nevertheless, the mathematics derived here would be applicable to a wider variety of applications. It has been shown that synchronous filters represent an exact equivalent to other filters, both in terms of their transfer functions and also in terms of their zero input responses. This theory shows an equivalence between synchronous filters and commonly used complex, or polyphase, filters. Furthermore, new possibilities for the realization of polyphase filters are suggested by the mathematics shown here. Finally, new possibilities not shown before are also suggested by the use of this paradigm. An exact companding polyphase filter was proposed for the first time as an extension of the basic concept.

## Appendix

Before considering the case with system imperfections, let us examine the behavior of the ideal system from a different viewpoint. In particular, we may write the zero-state solution for the synchronous filter using the following classical solution to the dynamical equations, where we have assumed that the input to the system comprises the sum of relatively narrowband signals whose spectra are centered at the modulating frequency $\omega_M$, plus and minus the core filter center frequency $\omega_{\hat{A}}$. In this way, we consider an input comprising what is commonly thought to be the desired signal and its image. Accordingly, we may write the input $\mathrm{u}(t)$, as

$$\begin{aligned} u(t) &= [f_1(t)\cos((\omega_M-\omega_{\hat{A}})t) + f_2(t)\sin((\omega_M-\omega_{\hat{A}})t)] \\ &\quad + [g_1(t)\cos((\omega_M+\omega_{\hat{A}})t) \\ &\quad + g_2(t)\sin((\omega_M+\omega_{\hat{A}})t)] \\ &= [\cos((\omega_M-\omega_{\hat{A}})t)\ \ \sin((\omega_M-\omega_{\hat{A}})t)]\bar{f}(t) \\ &\quad + [\cos((\omega_M+\omega_{\hat{A}})t)\ \ \sin((\omega_M+\omega_{\hat{A}})t)]\bar{g}(t) \end{aligned}$$

where

$$\bar{f}(t) \equiv [f_1(t)\ \ f_2(t)]^T; \quad \bar{g}(t) \equiv [g_1(t)\ \ g_2(t)]^T. \tag{51}$$

Supposing that we look at the superheterodyne synchronous filter, the modulated input to its core filter will be given by, using (25) with $\beta = 0$

$$\begin{aligned} \bar{u}_M(t) &= \begin{bmatrix} \cos(\omega_M t) \\ \sin(\omega_M t) \end{bmatrix} u(t) \\ &= \begin{bmatrix} \cos(\omega_M t) \\ \sin(\omega_M t) \end{bmatrix} \\ &\quad \times [\cos((\omega_M-\omega_{\hat{A}})t) \quad \sin((\omega_M-\omega_{\hat{A}})t)]\bar{f}(t) \\ &\quad + \begin{bmatrix} \cos(\omega_M t) \\ \sin(\omega_M t) \end{bmatrix} \\ &\quad [\cos((\omega_M+\omega_{\hat{A}})t) \quad \sin((\omega_M+\omega_{\hat{A}})t)]\bar{g}(t) \\ &= [\Phi(-\omega_{\hat{A}}t) + \Phi(-2\omega_M \\ &\quad + \omega_{\hat{A}})P]\bar{f}(t) + [\Phi(\omega_{\hat{A}}t) + \Phi(-2\omega_M - \omega_{\hat{A}})P]\bar{g}(t) \\ &\qquad \text{where } P \equiv \mathrm{diag}(1,-1). \end{aligned} \tag{52}$$

The state vector zero state response of the core filter will be given by

$$\begin{aligned} \bar{x}_{ZS} &= \int_0^t e^{\hat{A}(t-\tau)} u_M(\tau)d\tau \\ &= \int_0^t e^{-\hat{\sigma}(t-\tau)}\Phi(-\omega_{\hat{A}}(t-\tau))u_M(\tau)d\tau \\ &= \int_0^t e^{-\hat{\sigma}(t-\tau)}\Phi(-\omega_{\hat{A}}t)\Phi(\omega_{\hat{A}}\tau)u_M(\tau)d\tau. \end{aligned} \tag{53}$$

Combining the two previous results yields

$$\begin{aligned}\bar{x}_{ZS} &= \Phi(-\omega_{\hat{A}}t)\int_0^t e^{-\hat{\sigma}(t-\tau)}\Phi(\omega_{\hat{A}}\tau)u_M(\tau)d\tau \\ &= \Phi(-\omega_{\hat{A}}t)\left[\int_0^t e^{-\hat{\sigma}(t-\tau)}\Phi(\omega_{\hat{A}}\tau)\Phi(-\omega_{\hat{A}}\tau)\bar{f}(\tau)d\tau\right. \\ &\quad + \int_0^t e^{-\hat{\sigma}(t-\tau)}\Phi(\omega_{\hat{A}}\tau) \\ &\quad \left.\times\ \Phi((-2\omega_M+\omega_{\hat{A}})\tau)P\bar{f}(\tau)d\tau+\cdots\right] \\ &= \Phi(-\omega_{\hat{A}}t)\left[\int_0^t e^{-\hat{\sigma}(t-\tau)}\bar{f}(\tau)d\tau\right. \\ &\quad + \int_0^t e^{-\hat{\sigma}(t-\tau)}\Phi((-2\omega_M+2\omega_{\hat{A}})\tau)P\bar{f}(\tau)d\tau \\ &\quad + \int_0^1 e^{-\hat{\sigma}(t-\tau)}\Phi(2\omega_{\hat{A}}\tau)\bar{g}(\tau)d\tau \\ &\quad \left.+ \int_0^t e^{-\hat{\sigma}(t-\tau)}\Phi(-2\omega_M\tau)P\bar{g}(\tau)d\tau\right]. \end{aligned} \tag{54}$$

Each of the convolution integrals in the final result describes a filtering operation. Only the first of these integrals, however, represents a low-pass filtering operation on the signal vectors, whose components are assumed to be low frequency, bandlimited signals. The typical scenario is that the other integrals will be approximately zero due to low-pass filtering operations on significantly up-converted signals. Since we are simply trying to offer an approximate analysis in this Appendix that will be appropriate to investigate non ideal effects, it is reasonable to make the following approximation:

$$\bar{x}_{ZS} \approx \Phi(-\omega_{\hat{A}}t)\int_0^t e^{-\hat{\sigma}(t-\tau)}\bar{f}(\tau)d\tau. \tag{55}$$

Using this result in the input-output equation of (25) yields the expected result that the zero state response of the filter is an up-converted version of the state vector solution above. Notice that the "image" signal components of $\bar{g}(t)$ have been rejected in the process.

Now suppose that the modulators at the front-end of a practical system were not exactly in quadrature. This would lead to a modulated input that might be written in the following form:

$$\begin{aligned}\bar{u}_M(t) &= \begin{bmatrix}\cos(\omega_M t-\Theta)\\ \sin(\omega_M t+\Theta)\end{bmatrix}u(t) \\ &= \cos(\Theta)\begin{bmatrix}\cos(\omega_M t)\\ \sin(\omega_M t)\end{bmatrix}u(t) \\ &\quad + \sin(\Theta)\begin{bmatrix}\sin(\omega_M t)\\ \cos(\omega_M t)\end{bmatrix}u(t) \\ &= \cos(\Theta)[I+\tan(\Theta)T]\begin{bmatrix}\cos(\omega_M t)\\ \sin(\omega_M t)\end{bmatrix}u(t). \end{aligned} \tag{56}$$

$I$ represents an identity matrix and $T$ is defined in (22). The multiplier $\cos(\Theta)$ is simply a gain factor and, since one typically assumes that the signals are close to being in quadrature, can be assumed to be approximately 1. Clearly, when this result is substituted for the earlier ideal quadrature vector, we get a main term that matches that derived in the ideal case above plus a new term scaled by $\tan(\Theta)$. To see what the new term looks like observe that

$$\begin{aligned}&\Phi(\omega_{\hat{A}})T\begin{bmatrix}\cos(\omega_M t)\\ \sin(\omega_M t)\end{bmatrix}u(t) \\ &\quad = \Phi(\omega_{\hat{A}})T[\Phi(-\omega_{\hat{A}}t)+\Phi(-2\omega_M+\omega_{\hat{A}})P]\bar{f}(t) \\ &\qquad + \Phi(\omega_{\hat{A}})T[\Phi(\omega_{\hat{A}}t)+\Phi(-2\omega_M-\omega_{\hat{A}})P]\bar{g}(t) \\ &\quad = [T\Phi(-2\omega_{\hat{A}}t)+T\Phi(2\omega_M)P]\bar{f}(t) \\ &\qquad + [T+T\Phi(2\omega_M-2\omega_{\hat{A}})P]\bar{g}(t). \end{aligned} \tag{57}$$

Note that the special properties of T and $\Phi()$ have been exploited in writing the last line of (57). Thus, following the same steps as before we have

$$\begin{aligned}\bar{x}_{ZS} &\approx \Phi(-\omega_{\hat{A}}t)\int_0^t e^{-\hat{\sigma}(t-\tau)}\bar{f}(\tau)d\tau \\ &\quad +\Phi(-\omega_{\hat{A}}t)\tan(\Theta)\int_0^t e^{-\hat{\sigma}(t-\tau)}T\bar{g}(t)d\tau. \end{aligned} \tag{58}$$

This result shows that the final synchronous filter output will be the sum of the desired filtered signal plus a term related to the image signal, where $g_1(t)$ and $g_2(t)$ are interchanged due to the operation of the T matrix in the second integral. Hence, the image rejection is essentially given by $\tan(\Theta)$, where $2\Theta$ is the error angle from ideal quadrature. A similar analysis shows that a small gain error between the two components of the modulating vector yields a similar result. Extension to the higher order case is simple starting from (47).

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