Chapter 2
VECTOR AND TENSOR ANALYSES

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Lecture 3

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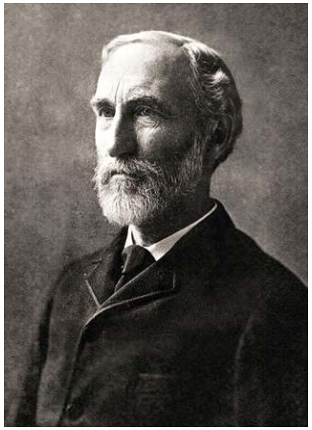

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2.1 Summation Convention and Special Symbols

Summation Convention

For the vector and tensor analyses, often it's convenient to use the summation (or Einstein) convention in which any repeated index implies a summation for that index, for example,

$$a_{ii} \to \sum_i a_{ii} \qquad a_i b_i \to \sum_i a_i b_i \tag{2.1}$$

Kronecker Delta

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{2.2}$$

Levi-Civita

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{even permutation} : (i, j, k) = (1,2,3),(2,3,1),(3,1,2) \\ -1, & \text{odd permutation} : (i, j, k) = (2,1,3),(3,2,1),(1,3,2) \\ 0, & \text{repeated index} : i = j,\ j = k,\ k = i \end{cases} \tag{2.3}$$

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}, \quad \varepsilon_{ijk}\varepsilon_{ijn} = 2\delta_{kn}, \quad \varepsilon_{ijk}\varepsilon_{ijk} = 6 \tag{2.4}$$

Product of Levi-Civitas

$$\varepsilon_{ijk}\varepsilon_{lmn} = \det\begin{bmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{bmatrix} \tag{2.5}$$

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}, \quad \varepsilon_{ijk}\varepsilon_{ijn} = 2\delta_{kn}, \quad \varepsilon_{ijk}\varepsilon_{ijk} = 6 \tag{2.6}$$

2.2 Vectors and Tensors

Scalars and vectors are 0th and 1st-order **tensors (polyads)**. The 2nd- and 3rd-order tensors are also called **dyad** and triad. The dyad, one of the most used tensors in both physics and engineering, defines **the relation between two vectors**.

Coordination-Free Equations

Physics laws should be independent of coordinates. With vectors and dyads, we can write equations in **coordinate-free forms**. Obviously, the Cartesian tensor is the simplest, and we can use it to derive the coordinate-free equations.

$$\text{Notations} \quad \left[A:\text{scalar}\right] \quad \begin{bmatrix} \mathbf{A}:\text{vector} \\ \mathbf{u}:\text{unit vector} \end{bmatrix} \quad \begin{bmatrix} \overline{\mathbf{T}}:\text{dyad} \\ \overline{\mathbf{I}}:\text{unit dyad} \end{bmatrix}$$

Vector

An n-dimensional vector has n scalar components :

$$\mathbf{A} = A_i\mathbf{u}_i = (A_1, A_2, \cdots A_n) = A_1\mathbf{u}_1 + A_2\mathbf{u}_2 + \cdots + A_n\mathbf{u}_n \tag{2.7}$$

The position vector is given by

$$\mathbf{r} = \mathbf{u}_i x_i \tag{2.8}$$

with unit vectors defined as

$$\mathbf{u}_i = \frac{\partial \mathbf{r} / \partial x_i}{\left|\partial \mathbf{r} / \partial x_i\right|} \tag{2.9}$$

Dyad

An n-dimensional dyad has n vector components (n x n scalar components) :

$$\overline{\mathbf{T}} = \mathbf{A}_i\mathbf{u}_i = T_{ij}\mathbf{u}_i\mathbf{u}_j \tag{2.10}$$

Note that the i-th component of a dyad is a vector while that of a vector is a scalar

For an example of a 2D dyad,

$$\begin{aligned}\overline{\mathbf{T}} &= \mathbf{A}_1\mathbf{u}_1 + \mathbf{A}_2\mathbf{u}_2 = (A_{11}\mathbf{u}_1 + A_{12}\mathbf{u}_2)\mathbf{u}_1 + (A_{21}\mathbf{u}_1 + A_{22}\mathbf{u}_2)\mathbf{u}_2 \\ &= A_{11}\mathbf{u}_1\mathbf{u}_1 + A_{12}\mathbf{u}_2\mathbf{u}_1 + A_{21}\mathbf{u}_1\mathbf{u}_2 + A_{22}\mathbf{u}_2\mathbf{u}_2\end{aligned}$$

Here, it should be noted that $\mathbf{u}_i\mathbf{u}_j \neq \mathbf{u}_j\mathbf{u}_i$ for $i \neq j$.

We can show that an n-dimensional dyad is given by ***n* terms of dyadic (or direct) products** of two vectors,

$$\overline{\mathbf{T}} = \mathbf{A}_i\mathbf{B}_i = \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2 + \cdots \mathbf{A}_n\mathbf{B}_n \tag{2.11}$$

Matrix Representation of Dyads

We can also use a matrix notation for a dyad :

$$\overline{\mathbf{T}} = \mathbf{A}_i\mathbf{B}_i^t = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}\begin{bmatrix} B_1 & B_2 & B_3 \end{bmatrix} = \begin{bmatrix} A_1B_1 & A_1B_2 & A_1B_3 \\ A_2B_1 & A_2B_2 & A_2B_3 \\ A_3B_1 & A_3B_2 & A_3B_3 \end{bmatrix} \tag{2.12}$$

Dyadic Transpose

Unlike the matrix formalism of vectors (row and column vectors), for the dyadic notation, transposed vectors make no difference in its representation.

$$\begin{aligned} &\mathbf{A}^t = \mathbf{A} \\ &\mathbf{T}^t = \mathbf{u}_j \mathbf{u}_i T_{ji} = \mathbf{u}_i \mathbf{u}_j T_{ij} \\ &(\mathbf{AB})^t = \mathbf{B}^t \mathbf{A}^t = \mathbf{BA} \end{aligned} \tag{2.13}$$

However, a dyad becomes from its transpose in general,

$$\begin{aligned} &\mathbf{AB} = \mathbf{BA}: \ \text{symmetric dyad (commuting)} \\ &\mathbf{AB} \neq \mathbf{BA}: \ \text{asymmetric dyas (non-commuting)} \end{aligned} \tag{2.14}$$

Dot, Cross, and Direct Products

$$\text{vector-vector} \begin{cases} \mathbf{A}\cdot\mathbf{B} = A_i B_i & : \text{dot product (projection)} \\ \mathbf{A}\times\mathbf{B} = \varepsilon_{ijk}\mathbf{u}_i A_j B_k & : \text{cross product (directional area)} \\ \mathbf{AB} = A_i B_j \mathbf{u}_i \mathbf{u}_j & : \text{direct product} \end{cases} \tag{2.15}$$

$$\text{vector-dyad} \begin{cases} \mathbf{A}\cdot\mathbf{BC} = (\mathbf{A}\cdot\mathbf{B})\mathbf{C}, \quad \mathbf{AB}\cdot\mathbf{C} = \mathbf{A}(\mathbf{B}\cdot\mathbf{C}) \\ \mathbf{A}\times\mathbf{BC} = (\mathbf{A}\times\mathbf{B})\mathbf{C}, \quad \mathbf{AB}\times\mathbf{C} = \mathbf{A}(\mathbf{B}\times\mathbf{C}) \end{cases} \tag{2.16}$$

Combined Products

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \varepsilon_{ijk}A_i B_j C_k \quad : \text{volume} \tag{2.17}$$

$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\times\mathbf{C} &= (\mathbf{BA}-\mathbf{AB})\cdot\mathbf{C} = \mathbf{C}\cdot(\mathbf{AB}-\mathbf{BA})\\ \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) &= \mathbf{C}\cdot(\mathbf{BA}-\mathbf{AB}) = (\mathbf{AB}-\mathbf{BA})\cdot\mathbf{C}\end{aligned} \tag{2.18}$$

$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = \mathbf{A}\cdot(\mathbf{CD}-\mathbf{DC})\cdot\mathbf{B} = \mathbf{C}\cdot(\mathbf{AB}-\mathbf{BA})\cdot\mathbf{D} \tag{2.19}$$

Unit Dyad

$$\overline{\mathbf{I}} = \hat{\mathbf{u}}_i\hat{\mathbf{u}}_i \qquad \overline{\mathbf{I}}\cdot\overline{\mathbf{I}} = \overline{\mathbf{I}} \qquad \overline{\mathbf{I}}\times\overline{\mathbf{I}} = 0 \tag{2.20}$$

$$\mathbf{A}\cdot\overline{\mathbf{I}} = \overline{\mathbf{I}}\cdot\mathbf{A} = \mathbf{A} \tag{2.21}$$

$$\mathbf{A}\times\overline{\mathbf{I}} = \overline{\mathbf{I}}\times\mathbf{A} = \mathbf{u}_i\mathbf{u}_j\varepsilon_{ijk}A_k : \text{ anti-symmetric dyadic} \tag{2.22}$$

$$(\mathbf{A}\times\mathbf{B})\times\overline{\mathbf{I}} = \overline{\mathbf{I}}\times(\mathbf{A}\times\mathbf{B}) = \mathbf{BA}-\mathbf{AB} \tag{2.23}$$

where (2.23) can be derived from (2.18) by $\mathbf{C}\rightarrow\mathbf{I}$.

Symmetric and Anti-Symmetric Tensors

$$\mathbf{T} = \mathbf{T}^t : \text{ symmetric tensor}$$
$$\mathbf{T} = -\mathbf{T}^t : \text{ anti-symmetric tensor} \rightarrow T_{ii} = 0,\ T_{ij} = -T_{ji} \tag{2.24}$$

An arbitrary tensor can be decomposed into symmetric and anti-symmetric components,

$$\overline{\mathbf{T}} = \frac{1}{2}(\overline{\mathbf{T}} + \overline{\mathbf{T}}^t) + \frac{1}{2}(\overline{\mathbf{T}} - \overline{\mathbf{T}}^t) \tag{2.25}$$

and we obtain dyadic symmetry decomposition:

$$\mathbf{AB} = \frac{1}{2}(\mathbf{AB} + \mathbf{BA}) + \frac{1}{2}(\mathbf{AB} - \mathbf{BA}) = \frac{1}{2}[\mathbf{A},\mathbf{B}]_+ + \frac{1}{2}[\mathbf{A},\mathbf{B}]_- \tag{2.26}$$

where the two dyadic operators are defined as:

$$[\mathbf{A},\mathbf{B}]_+ = \mathbf{AB} + \mathbf{BA} : \text{ anti-commutator}$$
$$[\mathbf{A},\mathbf{B}]_- = \mathbf{AB} - \mathbf{BA} : \text{ commutator} \tag{2.27}$$

2.3 Differential Vector Operators

We consider three orthogonal unit vectors $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ in Cartesian coordinates.

Del Operator

$$\nabla = \mathbf{u}_i \partial_i \tag{2.28}$$

Gradient

$$\boxed{\nabla \varphi(\mathbf{r}) = \mathbf{u}_i \partial_i \varphi(\mathbf{r})} \tag{2.29}$$

$$\varphi(\mathbf{r} + d\mathbf{r}) - \varphi(\mathbf{r}) = d\mathbf{r} \cdot \nabla \varphi(\mathbf{r}) \tag{2.30}$$

Divergence

$$\boxed{\nabla \cdot \mathbf{A}(\mathbf{r}) = \partial_i A_i(\mathbf{r}) = \lim_{V \to 0} \frac{1}{V} \oint_S d\mathbf{s} \cdot \mathbf{A}(\mathbf{r})} \tag{2.31}$$

Curl

$$\boxed{\nabla \times \mathbf{A}(\mathbf{r}) = \varepsilon_{ijk} \mathbf{u}_i \partial_j A_k = \lim_{V \to 0} \frac{1}{V} \oint_S d\mathbf{s} \times \mathbf{A}(\mathbf{r})} \tag{2.32}$$

Laplacian

From a vector identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla \nabla \cdot \mathbf{A} - \nabla^2 \mathbf{A} \tag{2.33}$$

the Laplacian operator is defined as

$$\boxed{\nabla^2 = \nabla \cdot \nabla = \nabla \nabla \cdot - \nabla \times \nabla \times} \tag{2.34}$$

Differential Vector Identities

Gradient

$$\nabla(\psi\phi) = \phi\nabla\psi + \psi\nabla\phi$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\cdot\nabla\mathbf{B} + \mathbf{B}\cdot\nabla\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A})$$

Divergence

$$\nabla\cdot(\psi\mathbf{A}) = (\nabla\psi)\cdot\mathbf{A} + \psi\nabla\cdot\mathbf{A}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

Curl

$$\nabla\times(\psi\mathbf{A}) = (\nabla\psi)\times\mathbf{A} + \psi\nabla\times\mathbf{A}$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$$

Second Derivative

$$\nabla\cdot\nabla F = \nabla^2 F$$

$$\nabla\times\nabla\times\mathbf{A} = \nabla^2\mathbf{A} - \nabla(\nabla\cdot\mathbf{A})$$

$$\nabla\times\nabla F = 0$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0$$

$$\nabla\cdot(F\nabla G) = \nabla F\cdot\nabla G + F\nabla^2 G$$

$$\nabla\cdot(F\nabla G - G\nabla F) = F\nabla^2 G - G\nabla^2 F$$

$$\nabla^2(FG) = F\nabla^2 G + 2\nabla F\cdot\nabla G + G\nabla^2 F$$

$$\nabla^2(F\mathbf{A}) = (\nabla^2 F)\mathbf{A} + 2\nabla F\cdot\nabla\mathbf{A} + F\nabla^2\mathbf{A}$$