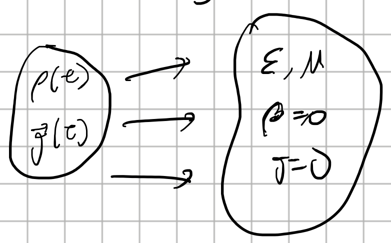
Ch4 Propagation of Electromagnetic wave

4.1 Basics

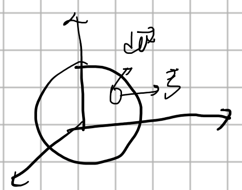


$$E_\omega = -k^2 \frac{e^{ikR}}{R} \hat{R} \times (\hat{R} \times \vec{p})$$

$$B_\omega = k^2 \frac{e^{ikR}}{R} (\hat{R} \times \vec{p})$$

We are interested in EM wave propagation generated by an external source through media

radiation: energy delivered to a far region by E, B field



$$\int d\vec{\sigma} \cdot \vec{S} = \int R^2 d\theta d\phi \left[\frac{c}{4\pi} \left(\frac{k^2 p^2}{R} \right)^2 \sin^2\theta + O\left(\frac{1}{R^x}\right) \right] \to \text{finite}$$

$$\vec{\nabla} \times \vec{H} - \frac{1}{c} \frac{\partial \vec{D}}{\partial t} = 0, \quad \vec{\nabla} \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0, \quad \vec{D} = \varepsilon \vec{E}, \quad \vec{B} = \mu \vec{H}$$

$$\nabla \cdot \vec{D} = 0, \quad \nabla \cdot \vec{B} = 0$$

$$\underbrace{\vec{\nabla} \times \left(\frac{\vec{\nabla} \times \vec{E}}{\mu} \right)} + \frac{1}{c} \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{H}) = 0$$

$$\hookrightarrow \frac{1}{\mu} \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) + \vec{\nabla}\left(\frac{1}{\mu}\right) \times (\vec{\nabla} \times \vec{E})$$

$$\Rightarrow \nabla^2 \vec{E} - \frac{\varepsilon\mu}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} + [\vec{\nabla}(\log \mu)] \times (\vec{\nabla} \times \vec{E}) + \vec{\nabla}(\vec{E} \cdot \vec{\nabla} \log \varepsilon) = 0$$

$$\nabla^2 \vec{H} - \frac{\varepsilon\mu}{c^2} \frac{\partial^2 \vec{H}}{\partial t^2} + [\vec{\nabla}(\log \varepsilon)] \times (\vec{\nabla} \times \vec{H}) + \vec{\nabla}(\vec{H} \cdot \vec{\nabla} \log \mu) = 0$$

If the medium is uniform, like air $\Rightarrow \vec{\nabla}(\log \varepsilon) = \vec{\nabla}(\log \mu) = 0$

$$\nabla^2 \vec{E} - \frac{\varepsilon\mu}{c^2} \frac{\partial^2}{\partial t^2} \vec{E} = 0$$

$$\nabla^2 \vec{H} - \frac{\varepsilon\mu}{c^2} \frac{\partial^2}{\partial t^2} \vec{H} = 0$$

wave equation with speed of

$$v = \frac{c}{\sqrt{\varepsilon\mu}}, \quad c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

$n = \sqrt{\varepsilon\mu}$: index of refraction $v = \frac{c}{n}$

$\to$ existance of E & B wave

① 1-D case

$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\psi = 0$ general solution $\psi = \psi(x \pm vt)$

ψ: any type of function

harmonic waves when ψ is sin

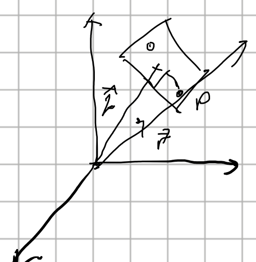
$\psi = \sin(k(x \pm vt)) = \sin(kx \pm kvt) = \sin(kx \pm \omega t)$

k: scaling factor

$k = \frac{2\pi}{\lambda}$, $v = \frac{\lambda}{T} = \nu\lambda$

② 3D case

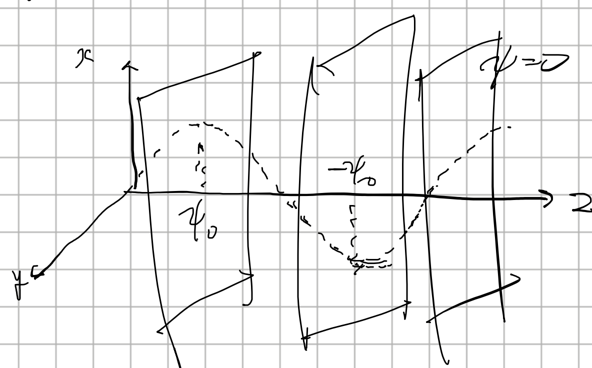
$\psi = f(\vec{k}\cdot\vec{r} - \omega t)$



$0 = \overrightarrow{OQ}\cdot\overrightarrow{QP}$

$\hat{k}\cdot\vec{r} = \overline{OQ}$ = constant

⇒ $\hat{k}\cdot\vec{r}$ = const plane is perpendicular to the $\hat{k}$ (propagation vector of EM wave)



$\hat{k} = \hat{z}$

plane harmonic wave

$\psi = \psi_0 \underbrace{\cos}_{\text{harmonic}}(\vec{k}\cdot\vec{r} - \omega t + \phi(t))$

A general form

$\psi(\vec{r}, t) = a(\vec{r})\cos(\vec{k}\cdot\vec{r} - \omega t)$

"$\vec{k}\cdot\vec{r}$ = const" surface = co-phase surface

$= \mathrm{Re}\{u(r)e^{-i\omega t}\}$, $u(r) = a(r)e^{i\vec{k}\cdot\vec{r}}$

$$\nabla^2 \psi - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi = 0 \Rightarrow \nabla^2 u + \frac{\omega^2}{v^2} u = 0$$

with an understanding that we take a real part of the final expression for physical quantity in question.

③ spherical wave (a wave from a point source)

$$\psi = \psi(r, t)$$

$$\nabla^2 \psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)$$

$$\nabla^2 \psi - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\psi = \frac{\partial^2}{\partial r^2}(r\psi) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}(r\psi) = 0$$

$$r\psi = \psi_1(r - vt) + \psi_2(r + vt)$$

$$\psi = \frac{1}{r}\psi_1(r - vt) + \frac{1}{r}\psi_2(r + vt)$$

$\frac{1}{r}\psi_1(r-vt)$: outgoing from a source

$\frac{1}{r}\psi_2(r+vt)$: incoming to a source

④ phase velocity

The phase $\vec{k}\cdot\vec{r} - \omega t$ is the same for $(\vec{r}, t)$ and $(\vec{r}+d\vec{r}, t+dt)$

if $\omega dt - \vec{k}\cdot d\vec{r} = 0$

$$\vec{k}\cdot\vec{r} - \omega t = \vec{k}\cdot(\vec{r} + d\vec{r}) - \omega(t + dt)$$

$$d\vec{r} = \hat{s}\, ds$$

$$\frac{ds}{dt} = \frac{\omega}{\vec{k}\cdot\hat{s}}$$

If $\hat{s}$ is taken in the normal to the co-phase surface

$$\hat{s} = \hat{k}$$

$$\frac{ds}{dt} = \frac{\omega}{k} = v_{phase} = \frac{c}{n}$$ (the speed of co-phase surface)

⑤ group velocity

If the wave has many frequencies in a certain frequency region

$$\bar{\omega} - \frac{\Delta\omega}{2} \le \omega \le \bar{\omega} + \frac{\Delta\omega}{2} \qquad \Delta\omega: \text{band width}$$

$$\psi(\vec{r}, t) = \int d\omega\, a_\omega(\vec{r}) \cos(\vec{k}\cdot\vec{r} - \omega t)$$

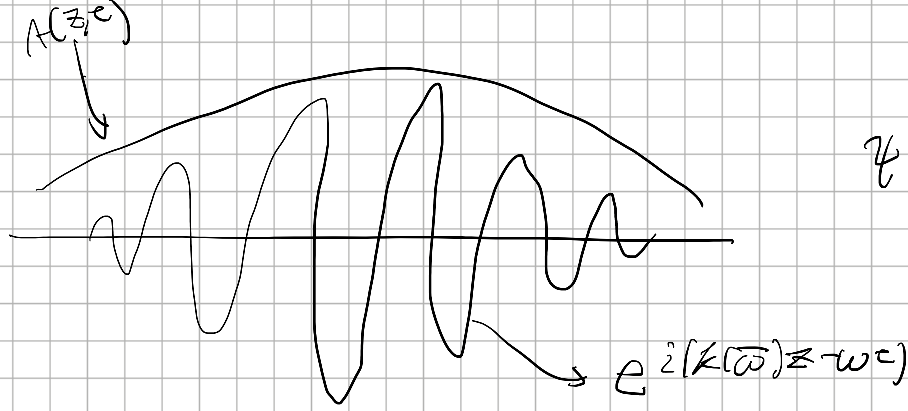
$$= Re \int_{\bar{\omega}-\frac{\Delta\omega}{2}}^{\bar{\omega}+\frac{\Delta\omega}{2}} d\omega\, a_\omega(\vec{r})\, \exp[-i(\omega t - \vec{k}\cdot\vec{r})]$$

For simplicity, 1-D case

$$\psi(z, t) = \int_{\Delta\omega} a_\omega e^{-i(\omega t - kz)} d\omega$$

$$k = n(\omega)\cdot\frac{\omega}{c} = k(\bar{\omega}) + \frac{\partial k}{\partial \omega}\Big|_{\bar{\omega}} (\omega - \bar{\omega}) + \cdots \quad \text{for } \frac{\Delta\omega}{\bar{\omega}} \ll 1$$

$$\psi = e^{-i(\bar{\omega} t - k(\bar{\omega}) z)} \underbrace{\int_{\Delta\omega} a_\omega e^{-i(\omega - \bar{\omega})\left(t - \left(\frac{dk}{d\omega}\right)_{\bar{\omega}} z\right)}}_{A(z, t)}$$



$\psi = A(z,t)\, e^{i[k(\bar{\omega})z-\omega t]}$

for fixable

$(t, z) \rightarrow (t+dt, z+dz)$

If $t+dt \rightarrow (z+dz)\frac{dt}{d\omega} = t + \left(\frac{\quad}{d\omega}\right) z$

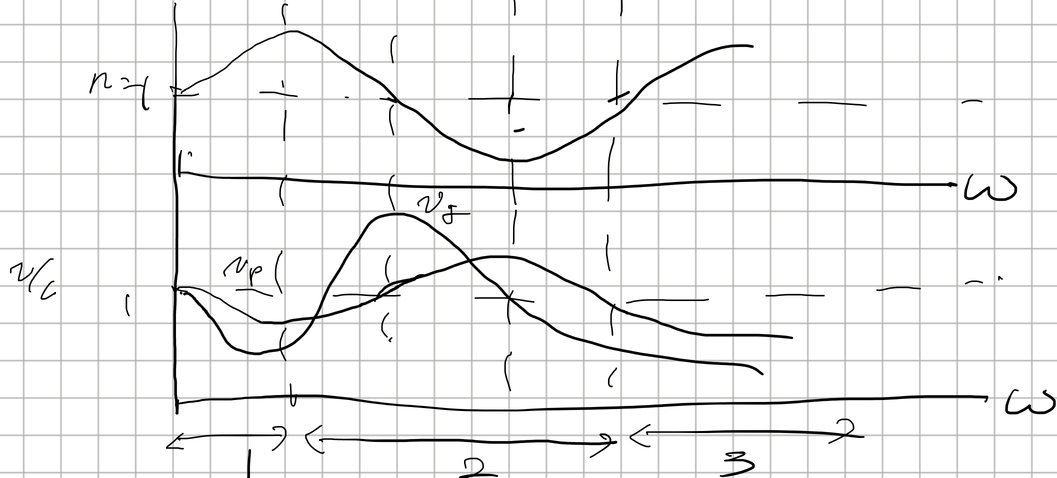
$A(z,t)$ remain constant.

$\frac{dz}{dt} = \left(\frac{d\omega}{dk}\right)_{\bar{\omega}}$ group velocity.

$k = n(\omega)\frac{\omega}{c}$, $\omega = \frac{c(k)}{n(k)}$

$$v_g = \frac{d\omega}{dk} = \frac{c}{n} + \frac{ck}{-n^2}\frac{dy}{dk} = \frac{c}{n} - \frac{ck}{n^2}\frac{dn}{d\omega}\cdot\left(\frac{dy}{dk}\right)_{v_g}$$

$$v_g = \frac{d\omega}{dk} = \frac{c}{n + \omega\frac{dn}{d\omega}}$$



1: normal dispersion region

2: anomalous region

3: normal.