Chapter 3 Maxwell's equation

3.1 Faraday Induction law
3.2 Dynamics of Maxwell's equation
3.3 Potentials

3.1 Faraday's Induction law

A. Basics

1831 Faraday's discovery: A transient current is induced in a circuit.

$$\underbrace{\mathcal{E}}_{\text{electromotive force}} = -\frac{1}{c}\frac{d\Phi}{dt}$$

Φ: flux of a magnetic field through a stationary circuit.

Electromotive force: the work done by the field
a charge is moved over a closed path.

$$\delta W = \oint \vec{F}\cdot d\vec{l} = q\oint \left(\vec{E} + \cancel{\frac{1}{c}\vec{v}\times\vec{B}}\right)\cdot d\vec{l}$$

0 $\quad \vec{v} \parallel d\vec{l}$

$$= q \underbrace{\oint \vec{E}\cdot d\vec{l}}_{\mathcal{E}}$$

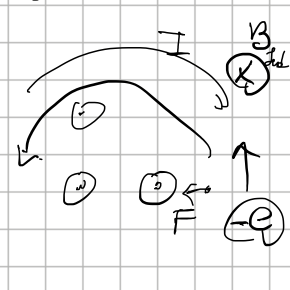
$$\oint \vec{E}\cdot d\vec{l} = -\frac{1}{c}\frac{d}{dt}\int \vec{B}\cdot d\vec{a} \xrightarrow{\text{stoke's theorem}} \int (\nabla\times\vec{E})\cdot d\vec{a} = \int d\vec{a}\cdot\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

$$\boxed{\nabla\times E = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}}$$

Faraday's induction law

⇒ ① $\vec{E} = -\nabla\phi$ is wrong in general

② the negative sign "−" : "diamagnetic property"



B) Maxwell's equation

microscopic description

$\nabla \cdot E = 4\pi\rho$

$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$

$\nabla \cdot B = 0$

$\nabla \times B = \frac{4\pi}{c}\vec{J} \longrightarrow \nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial E}{\partial t}$

$\nabla \cdot (\nabla \times \vec{B}) = \frac{4\pi}{c}\nabla \cdot J = 0$

this is against charge conservation

$\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0$

$\nabla \cdot (\nabla \times \vec{B}) = \frac{4\pi}{c}\nabla \cdot J + \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot E$

$= \frac{4\pi}{c}\nabla \cdot J + \frac{1}{c}\frac{\partial}{\partial t}4\pi\rho$

$= \frac{4\pi}{c}\left(\nabla \cdot J + \frac{\partial \rho}{\partial t}\right) = 0$

$\vec{D} = \vec{E} + 4\pi\vec{P}$

$\vec{H} = \vec{B} - 4\pi\vec{M}$

macroscopic description

$\nabla \cdot \vec{E} = 4\pi\rho_{free} - 4\pi\nabla \cdot \vec{P}$ or $\nabla \cdot \vec{D} = 4\pi\rho_{free}$

$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$

$\nabla \cdot \vec{B} = 0$

$\nabla \times \vec{B} = \frac{4\pi}{c}J_{free} + \frac{4\pi}{c}\underbrace{(c\nabla \times M)}_{\text{magnetization current}} + \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c}\frac{\partial P}{\partial t}$

$\Rightarrow \nabla \times \vec{H} = \frac{4\pi}{c}J_{free} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$

$0 = \frac{\partial \rho_p}{\partial t} + \nabla \cdot J_p = \frac{\partial}{\partial t}(-\nabla \cdot \vec{P}) + \nabla \cdot \vec{J}$

$= \nabla \cdot \left(-\frac{\partial \vec{P}}{\partial t} + \vec{J}\right)$

$\vec{J}_p = \frac{\partial \vec{P}}{\partial t}$

3.2 Dynamics of M.E

Mechanical properties of E&M field

a) linear momentum

$$-\vec{E}\times\left(\nabla\times\vec{E}+\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\right)-\vec{B}\times\underbrace{\left(\nabla\times\vec{B}-\frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right)}_{\frac{4\pi}{c}\vec{J}}+\vec{B}(\nabla\cdot\vec{B})+\vec{E}\underbrace{(\nabla\cdot\vec{E})}_{=4\pi\rho}$$

$$=\frac{4\pi}{c}\vec{J}\times\vec{B}+4\pi\rho\vec{E}_{\text{Lorentz}}$$

Left hand side of the above equation.

$$\left\{\vec{E}(\nabla\cdot\vec{E})-\vec{E}\times(\nabla\times\vec{E})\right\}+\left\{\vec{B}\nabla\cdot\vec{B}-\vec{B}\times(\nabla\times\vec{B})\right\}+\underbrace{\frac{1}{c}\left[-\vec{E}\times\frac{\partial\vec{B}}{\partial t}+\vec{B}\frac{\partial\vec{E}}{\partial t}\right]}_{=-\frac{1}{c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B})}$$

$$\left[\vec{E}(\nabla\cdot\vec{E})-\vec{E}\times(\nabla\times\vec{E})\right]_i=E_i\frac{\partial}{\partial x_j}E_j-E_j\frac{\partial}{\partial x_i}E_j+E_j\frac{\partial}{\partial x_j}E_i$$

(cf. $[\vec{E}\times(\nabla\times\vec{E})]_i=\varepsilon_{ijk}E_j(\nabla\times\vec{E})_k$

$=\varepsilon_{ijk}E_j\varepsilon_{klm}\frac{\partial}{\partial x_l}E_m$ $\qquad \varepsilon_{ijk}\varepsilon_{klm}=\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}$

$=E_j\frac{\partial}{\partial x_i}E_j-E_j\frac{\partial}{\partial x_j}E_i$)

$$=\frac{\partial}{\partial x_j}E_iE_j-\frac{1}{2}\frac{\partial}{\partial x_i}E_jE_j=\frac{\partial}{\partial x_j}\underbrace{\left(E_{ij}-\frac{1}{2}\delta_{ij}E_jE_j\right)}_{}$$

$$\overleftrightarrow{T}_E=\vec{E}\vec{E}-\frac{1}{2}\vec{E}^2\overleftrightarrow{I}\qquad\qquad \overleftrightarrow{I}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\vec{E}(\nabla\cdot\vec{E})-\vec{E}\times(\nabla\times\vec{E})=\nabla\cdot\overleftrightarrow{T}_E$$

Likewise

$$\vec{B}(\nabla\cdot\vec{B})-\vec{B}\times(\nabla\times\vec{B})=\nabla\cdot\overleftrightarrow{T}_B$$

$$\nabla\cdot\overleftrightarrow{T} = \frac{1}{c}\vec{J}\times\vec{B} + \rho\vec{E} + \frac{1}{4\pi c}\frac{\partial}{\partial t}\vec{E}\times\vec{B}$$

$$\overleftrightarrow{T} = \frac{1}{4\pi}\left(\overleftrightarrow{T}_E + \overleftrightarrow{T}_B\right) = \frac{1}{4\pi}\left\{\vec{E}\vec{E} + \vec{B}\vec{B} - \frac{1}{2}\overleftrightarrow{I}\left(E^2+B^2\right)\right\}$$ electromagnetic tensor.

$$\int dv\,\nabla\cdot\overleftrightarrow{T} = \int dv\left(\frac{1}{c}\vec{J}\times\vec{B} + \rho\vec{E}\right) + \frac{d}{dt}\int dv\,\frac{\vec{E}\times\vec{B}}{4\pi c}$$

$$\oint d\vec{a}\cdot\overleftrightarrow{T} = \frac{d}{dt}\underbrace{P_{Mech}}_{\text{Mechanical moment}} + \frac{d}{dt}P_{EM\ field}$$

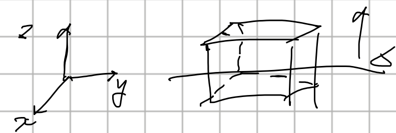
$$P_{EM\ field} = \frac{\vec{E}\times\vec{B}}{4\pi c}$$

Ex) Find the force acting on the charged conducting surface

outside $\quad E_z = 4\pi\sigma, \quad B=0$

$E_x = E_y = 0$.

inside. $\quad E = B = 0$



$$\overleftrightarrow{T}_{out} = \frac{1}{4\pi}\begin{pmatrix} -\frac{E_z^2}{2} & 0 & 0 \\ 0 & -\frac{E_z^2}{2} & 0 \\ 0 & 0 & -\frac{E_z^2}{2}\end{pmatrix} \qquad \overleftrightarrow{T}_{in} = 0$$

$$F_x = \int d\vec{a}\cdot\overleftrightarrow{T} = \int da\,\hat{x}\cdot\overleftrightarrow{T} = \left(\int_1 d\vec{a}_1 + \int_2 d\vec{a}_2\right)\cdot\overleftrightarrow{T} = 0$$

Likewise $F_y = 0$

$$F_z = \int_{out} da\,\hat{z}\cdot\overleftrightarrow{T} = \int da\,\hat{z}\cdot\overleftrightarrow{T}_{out} = 2\pi\sigma^2 A \Rightarrow F_z/A = 2\pi\sigma^2$$

$$\hat{x}\cdot\overleftrightarrow{T} = (1,0,0)\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1\end{pmatrix}(2\pi\sigma^2) = -2\pi\sigma^2\hat{x}$$