Mathematical theorem.

$$\vec{V}(\vec{x}),\quad \nabla\cdot\vec{V} = S,\quad \nabla\times\vec{V} = \vec{C}$$

then, $\vec{V} = -\nabla\phi + \nabla\times\vec{A}$

$$\phi = \frac{1}{4\pi}\int d^3x' \frac{S(x')}{r},\quad r = |\vec{x} - \vec{x}'|$$

$$\vec{A} = \frac{1}{4\pi}\int d^3x' \frac{\vec{C}(x')}{r}$$

· **Uniqueness**: suppose there are two solutions : $\vec{V}_1$ & $\vec{V}_2$

$$\nabla\cdot\vec{V}_1 = \vec{S} \qquad \nabla\times\vec{V}_1 = \vec{C}$$
$$\nabla\cdot\vec{V}_2 = \vec{S} \qquad \nabla\times\vec{V}_2 = \vec{C}$$

Constant $\vec{W} = \vec{V}_1 - \vec{V}_2$

$$\nabla\cdot\vec{W} = \nabla\cdot\vec{V}_1 - \nabla\cdot\vec{V}_2 = 0$$
$$\nabla\cdot\vec{W} = \nabla\times\vec{V}_1 - \nabla\times\vec{V}_2 = 0$$

since $\nabla\times\vec{W} = 0$, $\vec{W} = -\nabla\psi$

$$\nabla\cdot\vec{W} = 0 = -\nabla^2\psi$$

Gauss theorem

$$\oint_S \psi(\cancel{\nabla\psi})\,d\vec{S} = \int_V d^3x\,\nabla\cdot(\psi\nabla\psi)$$
$$= \int d^3x\,(\nabla\psi)^2 + \int d^3x\,\psi\cancel{\nabla^2\psi}$$

ψ vanished at infinity

$$\Rightarrow \int d^3x\,(\nabla\psi)^2 = 0 \quad \text{for any volume.}$$

$$\nabla\psi = 0 = -\vec{W} \Rightarrow \vec{V}_1 = \vec{V}_2$$

$$\Rightarrow \vec{V} = -\nabla\phi + \nabla\times\vec{A} \quad \text{is unique}$$

physics

$$\vec{E} = q\frac{\vec{x}}{r^3} \quad \text{or} \quad \int d^3x\,\frac{\rho\vec{x}}{r^3}$$

$$\oint \vec{E}\cdot d\vec{S} = 4\pi Q_{inside}$$

$$\nabla\cdot\vec{E} = 4\pi\rho,\quad \nabla\times\vec{E} = 0$$

Potential & Potential Energy

irrotational

$$\nabla \times \vec{E} = 0 \rightarrow \vec{E} = -\nabla\phi$$

cf. meaning of curl

consider an object with a constant angular speed ω
velocity of a point at $\vec{x}$ in the object

$$\vec{v} = \vec{\omega} \times \vec{x}$$

$$\nabla\cdot\vec{v} = \nabla\cdot(\vec{\omega}\times\vec{x})$$

$$= (\nabla\times\vec{\omega})\cdot\vec{x} - \vec{\omega}\cdot(\nabla\times\vec{x}) = 0$$

$$\nabla\times\vec{v} = \nabla\times(\vec{\omega}\times\vec{x}) = \vec{\omega}\,\nabla\cdot\vec{x} - \vec{\omega}\cdot\nabla\vec{x}$$

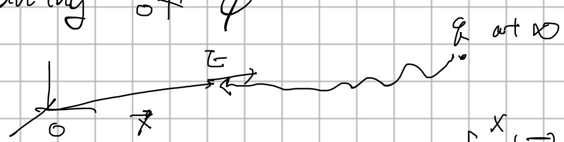
$$= 2\vec{\omega}$$

$$\Rightarrow \vec{\omega} = \frac{1}{2}\nabla\times\vec{v}$$

If $\nabla\times\vec{A} \neq 0$
A is in part the same as the velocity field of a rotation object.

$\Rightarrow$ non-vanishing curl $\equiv$ related to rotation or circulation motion.

① meaning of ϕ



$$W = \int_{\infty}^{x} d\vec{x}\cdot\left[-q\vec{E}(\vec{x})\right]$$

work done by me to bring the charge q at ∞ to the position x against E

$$\Rightarrow -q\int_{\infty}^{x} d\vec{x}\cdot\vec{E}(\vec{x}) = q\int d\vec{x}\cdot\nabla\phi$$

$$= q(\phi(\vec{x}) - \underbrace{\phi(\infty)}_{=0}) = q\phi(x)$$

$$\Rightarrow \phi = W/q$$

For a point charge

$$E = -\nabla\phi = \frac{q(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

$$\phi = \frac{q}{|\vec{x}-\vec{x}'|}$$

$$\nabla\cdot\vec{E} = 4\pi q\,\delta(\vec{x}-\vec{x}') \qquad \Rightarrow \nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\delta(\vec{x}-\vec{x}')$$

cf. cgs

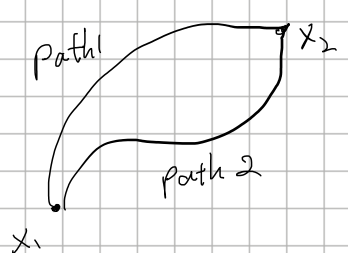
erg/esu = stat volt.

MKS

J/C = volt.

$$\text{volt} = J/C = \frac{10^7\,\text{erg}}{3\times10^9\,\text{esu}} = \frac{1}{300}\ \text{stat-volt.}$$

② $\nabla\times\vec{E} = 0 \Rightarrow \phi$ is not dependent on the path.



$$\phi_1 = \int_{x_1,\ \text{path}_1}^{x_2} \vec{F}\cdot d\vec{l}\ , \quad \vec{F} = -q\vec{E}$$

$$\phi_2 = \int_{x_1,\ \text{path}_2}^{x_2} \vec{F}\cdot d\vec{l}$$

$$\phi_1 - \phi_2 = \left(\int_{\text{path}_1} - \int_{\text{path}_2}\right)\vec{F}\cdot d\vec{l} = \oint_{\text{path}1-\text{path}2} \vec{F}\cdot d\vec{l} = -q\oint \vec{E}\cdot d\vec{l}$$

$$= -q\int_S (\nabla\times\vec{E})\cdot d\vec{a} \qquad \left(\text{cf. stoke's theorem } \oint d\vec{l}\cdot\vec{A} = \int_S d\vec{a}\cdot(\nabla\times\vec{A})\right)$$

$$= 0$$

$\phi_1 = \phi_2 \Rightarrow$ conservative

③ $\vec{E} = -\nabla\phi$, $\nabla\cdot\vec{E} = 4\pi\rho$

$\nabla^2\phi = -4\pi\rho$: Poisson equation.

$\rho = -\frac{1}{4\pi}\nabla^2\phi$

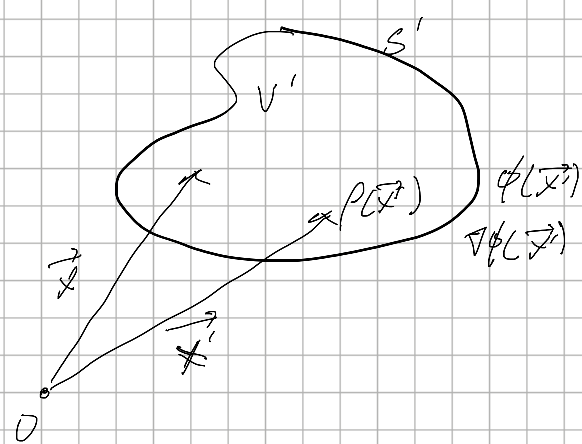
e.g.) $\phi(r) = q\,\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha}{2}r\right)$ Time-averaged potential of a neutral by Hydrogen

$r>0$ $\rho = \frac{q}{4\pi}\frac{\alpha^3}{2}e^{-\alpha r}$

$r\to 0$ $\rho \to \delta(r)$

④ $\nabla\cdot\vec{E} = 4\pi\rho$, $\nabla\times\vec{E} = 0$, $\phi = \int d^3x \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}$ with $\phi(\infty)=0$

for charge density known at all points in space



Green's theorem

$$\int_S d\vec{\sigma}\cdot(\phi\nabla'\psi - \psi\nabla'\phi)$$
$$= \int_{V'} d^3x'\,\nabla'\cdot(\phi\nabla'\psi - \psi\nabla'\phi)$$
$$= \int d^3x'(\phi\nabla'^2\psi - \psi\nabla'^2\phi)$$

ϕ, ψ continuous

Let ϕ electrostatic potential

& $\psi = \frac{1}{r}$, $\nabla^2\psi = \nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(r)$

$$-4\pi\int d^3x'\left(\phi\delta(r) - \frac{\rho(\vec{x}')}{r}\right) = \int d\sigma\cdot\left(\phi\frac{\vec{r}}{r^3} - \frac{\nabla\phi}{r}\right)$$, $r = |\vec{x}-\vec{x}'|$

inside V'

$$\phi(x) = \int_{V} d^3x'\frac{\rho(\vec{x}')}{r} + \frac{1}{4\pi}\int_{S} d\sigma'\cdot\left(-\phi\frac{\vec{r}}{r^3} + \frac{\nabla'\phi}{r}\right)$$