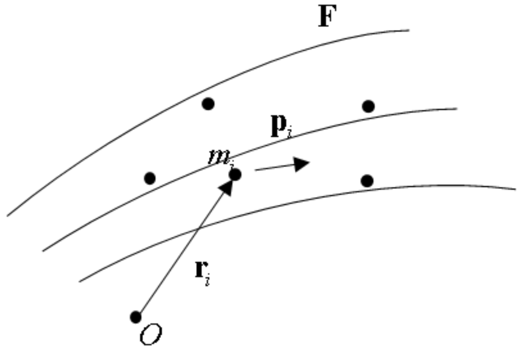
App.1. Classical Equations of Motion

Consider a system of point particles,

then the Newton's equation is given by

$$\sum_i \left(\mathbf{F}_i - \frac{d\mathbf{p}_i}{dt} \right) = 0 \quad \text{with} \, \mathbf{p}_i = m_i \frac{d\mathbf{r}_i}{dt}$$



$\Rightarrow$ Two major equations of motion

$$\left[\begin{array}{lcl} \text{Lagrangian Mechanics} & \rightarrow & \begin{pmatrix} \text{Feynman's} \\ \text{Path Integral} \end{pmatrix} \\ \text{Hamiltionian Mechanics} & \rightarrow & \left(\begin{array}{l} \text{Schrödinger Equation} \\ \text{Heisenberg Equation} \\ \text{Dirac Equation} \end{array} \right. \end{array} \right.$$

<u>Lagrangian Mechanics</u>

Using the Virtual Displacements $\delta \mathbf{r}_i$ of $\mathbf{r}_i$,

$$\sum_i \left(\mathbf{F}_i - \frac{d\mathbf{p}_i}{dt} \right) \bullet \delta \mathbf{r}_i = 0$$

Define the Generalized Coordinates $\mathbf{q}_i$'s,

$$\boxed{\delta \mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} \bullet \delta \mathbf{q}_j} \quad \text{where} \quad \boxed{\mathbf{r}_i(t) = \mathbf{r}_i(\mathbf{q}_i(t))}$$

$$\Rightarrow \boxed{\mathbf{v}_i = \frac{d\mathbf{r}_i}{dt} = \sum_j \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} \bullet \dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t}} \qquad (*)$$

i) $$\sum_i \mathbf{F}_i \bullet \delta \mathbf{r}_i = \sum_i \sum_j \mathbf{F}_i \bullet \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} \bullet \delta \mathbf{q}_j = \sum_j \mathbf{Q}_j \bullet \delta \mathbf{q}_j$$

$$\Rightarrow \boxed{\mathbf{Q}_j = \sum_i \mathbf{F}_i \bullet \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j}} \quad : \quad \text{Generalized Force}$$

ii) $$\sum_i \dot{\mathbf{p}}_i \bullet \delta \mathbf{r}_i = \sum_i m_i \frac{d^2 \mathbf{r}_i}{dt^2} \bullet \delta \mathbf{r}_i = \sum_i \sum_j \boxed{m_i \frac{d^2 \mathbf{r}_i}{dt^2} \bullet \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j}} \bullet \delta \mathbf{q}_j$$

with $$\sum_i m_i \ddot{\mathbf{r}}_i \bullet \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} = \sum_i \left[\frac{d}{dt}\left(m \dot{\mathbf{r}}_i \bullet \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} \right) - m_i \dot{\mathbf{r}}_i \bullet \underbrace{\frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} \right)} \right] \qquad (**)$$

Note that

$$\frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} \right) = \sum_k \frac{\partial^2 \mathbf{r}_i}{\partial \mathbf{q}_j \partial \mathbf{q}_k} \dot{\mathbf{q}}_k + \frac{\partial^2 \mathbf{r}_i}{\partial \mathbf{q}_j \partial t} = \frac{\partial \mathbf{v}_i}{\partial \mathbf{q}_j}$$

From (*)

$$\frac{\partial \mathbf{v}_i}{\partial \dot{\mathbf{q}}_j} = \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j}$$

$$\Rightarrow \sum_i m_i \ddot{\mathbf{r}}_j \bullet \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} = \sum_i \left[\frac{d}{dt}\left(m_i \mathbf{v}_i \bullet \frac{\partial \mathbf{v}_i}{\partial \dot{\mathbf{q}}_i} \right) - m_i \mathbf{v}_i \frac{\partial \mathbf{v}_i}{\partial \mathbf{q}_j} \right] \quad (\text{from} (**))$$

$$\Rightarrow \sum_i \dot{\mathbf{p}}_i \bullet \delta \mathbf{r}_i = \sum_j \left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\mathbf{q}}_j} \left(\sum_i \frac{1}{2} m \mathbf{v}_i^{\,2} \right) \right) - \frac{\partial}{\partial \mathbf{q}_j} \left(\sum_i \frac{1}{2} m \mathbf{v}_i^{\,2} \right) \right] \bullet \delta \mathbf{q}_j$$

Define $\boxed{K = \sum_i \frac{1}{2} m_i \mathbf{v}_i^{\,2}}$: Kinetic Energy

$$\Rightarrow \sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = \sum_j \left[\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\mathbf{q}}_j} \right) - \frac{\partial K}{\partial \mathbf{q}_j} - \mathrm{Q}_j \right] \bullet \delta \mathbf{q}_j = 0$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\mathbf{q}}_j} \right) - \frac{\partial K}{\partial \mathbf{q}_j} = \mathbf{Q}_j}$$

For $\mathbf{F}_i = -\nabla_i V = -\dfrac{\partial V}{\partial \mathbf{r}_i}$

$$\mathbf{Q}_j = \sum_i \mathbf{F}_i \bullet \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} = -\sum_i \frac{\partial V}{\partial \mathbf{r}_i} \bullet \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j}$$

$\Rightarrow \boxed{\mathbf{Q}j = -\dfrac{\partial V}{\partial \mathbf{q}_j}}$: Generalized Force

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial K}{\partial \mathbf{q}_j}\right) - \frac{\partial (K - V)}{\partial \mathbf{q}_j}} = 0$$

For $V = V(\mathbf{q}_1, \mathbf{q}_2, \ldots)$: velocity independent

$\Rightarrow \boxed{\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\mathbf{q}}_j}\right) - \dfrac{\partial L}{\partial \mathbf{q}_j} = 0}$: Euler-Lagrangian Equation

with $\boxed{L(\mathbf{q}, \dot{\mathbf{q}}) = K - V}$: Lagrangian

Compare with $\dfrac{d\mathbf{p}_j}{dt} = \mathbf{F}_j$

$$\Rightarrow \boxed{\begin{aligned} \mathbf{p}_j &= \frac{\partial L}{\partial \dot{\mathbf{q}}_j} \\ \mathbf{F}_j &= \frac{\partial L}{\partial \mathbf{q}_j} \end{aligned}}$$

Hamiltonian Mechanics

From the Lagrangian

$$\boxed{\mathbf{p}_i = \frac{\partial L}{\partial \dot{\mathbf{q}}_i}}$$

Define $H = \sum_i \dot{\mathbf{q}}_i \mathbf{p}_i - L$

In ordinary coordinates,

$$\dot{\mathbf{q}}_i \mathbf{p}_i = \mathbf{v}_i \bullet m_i \mathbf{v}_i = m_i \mathbf{v}_i{}^2 = 2K$$

$$\boxed{H(\mathbf{p},\mathbf{q},t) = \sum_i \dot{\mathbf{q}}_i \mathbf{p}_i - L(\mathbf{q},\dot{\mathbf{q}},t) = K + V} \quad : \quad \text{Hamiltonian}\left(\text{Total Energy}\right)$$

Consider an infinitesimal change of H ,

$$dH = \sum_i \frac{\partial H}{\partial \mathbf{p}_i} \bullet d\mathbf{p}_i + \sum_i \frac{\partial H}{\partial \mathbf{q}_i} \bullet d\mathbf{q}_i + \frac{\partial H}{\partial t} dt$$

$$= \sum_i \dot{\mathbf{q}}_i . d\mathbf{p}_i + \cancel{\sum_i \mathbf{p}_i \bullet d\dot{\mathbf{q}}_i} - \left(\sum_i \frac{\partial L}{\partial \mathbf{q}_i} \bullet d\mathbf{q}_i + \underbrace{\cancel{\sum_i \frac{\partial L}{\partial \dot{\mathbf{q}}_i} \bullet d\dot{\mathbf{q}}_i}}_{\sum_i \mathbf{p}_i \bullet d\dot{\mathbf{q}}_i \quad \text{from} \quad \mathbf{p}_i = \frac{\partial L}{\partial \dot{\mathbf{q}}_i}} + \frac{\partial L}{\partial t} dt \right)$$

$$\Rightarrow \begin{cases} \dot{\mathbf{q}}_I = \dfrac{\partial H}{\partial \mathbf{p}_i} \\ \dfrac{\partial L}{\partial \mathbf{q}_i} = -\dfrac{\partial H}{\partial \mathbf{q}_i} \quad \text{and} \quad \mathbf{F}_i = \dfrac{\partial L}{\partial \mathbf{q}_i} = \mathbf{p}_i \\ \dfrac{\partial L}{\partial t} = \dfrac{\partial H}{\partial t} \end{cases}$$

$$\Rightarrow \boxed{\begin{array}{l} \dot{\mathbf{q}}_i = \dfrac{\partial H}{\partial \mathbf{p}_i} \\ \dot{\mathbf{p}}_i = -\dfrac{\partial H}{\partial \mathbf{q}_i} \end{array}} \quad : \quad \text{Hamilton's Equation} \quad \text{with} \quad \frac{\partial H}{\partial t} = \frac{\partial L}{\partial t}$$

A Point Charge in EM fields

start from the Newton's equation of motion (EOM)

$$\frac{d\mathbf{p}}{dt}=Q\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)$$

$\Rightarrow$ Express $\mathbf{F}$ by EM potentials V and $\mathbf{A}$ with $\left[\begin{array}{l}\mathbf{E}=-\nabla V-\dfrac{\partial\mathbf{A}}{\partial t}\\ \mathbf{B}=\nabla\times\mathbf{A}\end{array}\right.$

$$\Rightarrow\ \frac{d\mathbf{p}}{dt}=Q\left(-\nabla V-\frac{\partial\mathbf{A}}{\partial t}+\mathbf{v}\times\nabla\times\mathbf{A}\right)$$

Use $\left(\begin{array}{l}\nabla\left(\mathbf{v}\bullet\mathbf{A}\right)=\mathbf{v}\bullet\nabla A+\mathbf{v}\times\nabla\times\mathbf{A}\\ \dfrac{d\mathbf{A}}{dt}=\dfrac{\partial\mathbf{A}}{\partial t}+\dfrac{\partial\mathbf{r}}{\partial t}\bullet\nabla\mathbf{A}=\dfrac{\partial\mathbf{A}}{\partial t}+\mathbf{v}\bullet\nabla\mathbf{A}\end{array}\right.$

$$\Rightarrow\ \frac{dp}{dt}=Q\left[-\nabla V-\frac{d\mathbf{A}}{dt}+\cancel{\mathbf{v}\bullet\nabla\mathbf{A}}+\nabla\left(\mathbf{v}\bullet\mathbf{A}\right)-\cancel{\mathbf{v}\bullet\nabla\mathbf{A}}\right]$$

$$\Rightarrow\ \frac{d}{dt}\left(\mathbf{p}+Q\mathbf{A}\right)=-Q\nabla\left(V-\mathbf{v}\bullet\mathbf{A}\right)$$

$$\Rightarrow\ \boxed{\frac{d}{dt}\boldsymbol{\pi}=-\nabla U}$$

With $\boxed{\boldsymbol{\pi}=\mathbf{p}+Q\mathbf{A}}$: $\text{Total}\left(\text{Canonical}\right)\text{Momentum}$

Kinetic Momentum Field Momentum

$\boxed{U=V-Q\mathbf{A}\bullet\mathbf{v}}$: Total Potential

$$\Rightarrow\ \boxed{\begin{array}{l}L=K-U\\ H=K+U\end{array}}$$