D) Hertz Potential

A & Φ can be combined into one Potential

$J(x,t) = \int_{-\infty}^{\infty} d\omega\, J(x,\omega) e^{-i\omega t}$ $\rho(x,t) = \int_{-\infty}^{\infty} d\omega\, \rho(x,\omega) e^{-i\omega t}$

$S = \int_{-\infty}^{\infty} d\omega \frac{J(x,\omega)}{(-i\omega)} e^{-i\omega t}$ can be used to

$$\frac{\partial \vec{S}}{\partial t} = \int d\omega \frac{J}{-i\omega}(-i\omega) e^{-i\omega t} = J(x,t)$$

$$\nabla\cdot\vec{S} = \int d\omega \frac{\nabla\cdot J}{(-i\omega)} e^{-i\omega t} = \int d\omega\, \rho(x,\omega) e^{-i\omega t} = -\rho(x,t)$$

$\nabla\cdot J + \frac{\partial}{\partial t}\rho = 0$

$\nabla\cdot J(x,\omega) + (-i\omega)\rho(\vec{x},\omega) = 0$

Then $\nabla^2\vec{\gamma} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{\gamma} = -4\pi\vec{S}$ in Lorentz equation.

$\vec{\gamma}$: Hertz Potential, $\vec{A} = \frac{1}{c}\frac{\partial\vec{\gamma}}{\partial t}$, $\phi = -\nabla\cdot\vec{\gamma}$

$$\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial}{\partial t}\phi = \nabla\cdot\left(\frac{1}{c}\frac{\partial\vec{\gamma}}{\partial t}\right) + \frac{1}{c}\frac{\partial}{\partial t}(-\nabla\cdot\vec{\gamma}) = 0$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = \frac{1}{c}\frac{\partial}{\partial t}\left(\nabla^2\vec{\gamma} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{\gamma}\right) = \frac{1}{c}\frac{\partial}{\partial t}(-4\pi\vec{S}) = -\frac{4\pi}{c}\vec{J}$$

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = -\nabla\cdot\left(\nabla^2\vec{\gamma} - \frac{1}{c^2}\frac{\partial^2\vec{\gamma}}{\partial t^2}\right) = 4\pi\nabla\cdot\vec{S} = -4\pi\rho$$

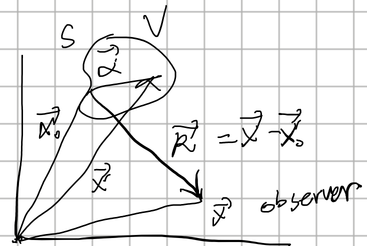
$\Rightarrow$ Solving Maxwell's equations is equivalent to solving $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{\gamma} = -4\pi \vec{S}$

Ex) Find the far field of E & B from a localized source, using Hertz Potential.

$$\vec{\gamma}(x,t) = \int_{-\infty}^{\infty} d\omega\, \gamma_\omega(\vec{x}) e^{-i\omega t}$$

$$\nabla^2 \vec{\gamma} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{\gamma} = -4\pi S \quad \Rightarrow \quad \nabla^2 \vec{\gamma}_\omega + \frac{\omega^2}{c^2}\vec{\gamma}_\omega = -4\pi S_\omega$$

Solution: $\vec{\gamma}_\omega = \int d^3x' \, \frac{\vec{S}_\omega(\vec{x}') \, e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}$ $\qquad r = |\vec{x}-\vec{x}'|$



① Source dimension is small compared to the distance to the observer

$$|\vec{\alpha}(\vec{x}')| \ll |\vec{x} - \vec{x}'|$$

② Source size is small to the wavelength of radiation

$$|\alpha(\vec{x}')| \ll \lambda = \frac{2\pi}{k} \quad \text{or} \quad k|\alpha| \ll 2\pi$$

$$\vec{x} - \vec{x}' = \vec{x} - (\vec{x}_0 + \vec{\alpha}) = \vec{R} - \vec{\alpha} \qquad |\vec{R}| \gg |\vec{\alpha}|$$

$$\frac{e^{ikr}}{r} = ik \sum_{n=0}^{\infty} (2n+1) P_n(\cos\Theta)\, j_n(k\alpha)\, h_n(kR)$$

Lengendre polynomial of order n

spherical Bessel function of order n

Henkel function of order n

Θ : angle between $\vec{\alpha}$ and $\vec{R}$

$k x \ll 1 \qquad j_n(kx) \simeq \frac{2^n n!}{(2n+1)!}(kx)^n$

$kR \gg 1 \qquad h_n(kR) \simeq (-i)^{n+1} \frac{e^{ikR}}{kR}$

$$\boxed{\vec{V}_\omega = \int_V d^3x \; \vec{S}_\omega(\vec{x}) \frac{e^{ikR}}{R} \sum_{n=0}^{\infty} \frac{2^n n!}{(2n)!} (-i)^n (kx)^n P_n(\cos \Theta)}$$

$\sum_{n=0} \rightarrow$ multiple expansion

$n=0$ electric dipole

$n=1 \begin{cases} \text{electric quadrapole} \\ \text{magnetic dipole} \end{cases}$

Let's examine $n=0$ term

$$\vec{\gamma}_\omega^{\,n=0} = \frac{e^{ikR}}{R}\int d^3x\, \vec{\zeta}_\omega(\vec{x}) \quad , \quad \vec{\zeta}_\omega(\vec{x}) = \frac{\vec{J}(x,\omega)}{-i\omega}$$

cf. $\int d^3x' \, J_x(\vec{x}') = \int d^3x' \, J_x(x') \frac{dx'}{dx'}$

$$= \int d^3x' \left[\frac{d}{dx'}\left(x' J_x(\vec{x}')\right) - x' \frac{d}{dx'} J_x(\vec{x}')\right]$$

$$= \cancel{\int dy'dz' \, x' J_x(\vec{x}') \Big|_{x'=\text{boundary}}} - \int d^3x' \, x' \frac{d}{dx'} J_x(\vec{x}')$$

localized

$$\int d^3x' \, \vec{J}(x') = -\int d^3x' \, \vec{x}' \, \nabla\cdot\vec{J}(\vec{x}')$$

$$= -\int d^3x' \, \vec{x}'(+i\omega\rho) \qquad \left(\text{cf. } \frac{\partial\rho}{\partial t} + \nabla\cdot J = 0 \,,\ \nabla\cdot J = i\omega\rho\right)$$

$$= -i\omega \int d^3x' \, \vec{x}'\rho$$

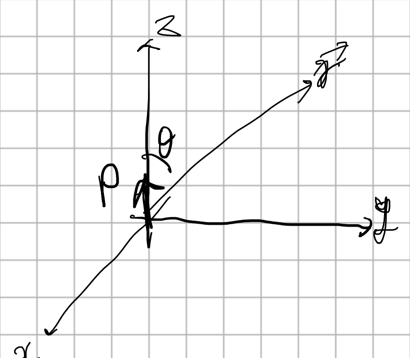
$$\int d^3x' \frac{\vec{J}(\vec{x}')}{-i\omega} = \int d^3x' \, \vec{x}\rho = \mathrm{P}$$

$$= \int d^3x' \, \vec{\zeta}_\omega$$

↳ polarization.

$$\vec{\gamma}_\omega^{\,n=0} = \frac{e^{ikR}}{R}\vec{P} \quad , \quad \vec{P} = p\,\hat{z}$$

$$= \frac{e^{ikR}}{R} \mathrm{P}\hat{z} \,, \quad \hat{z} = \hat{R}\cos\theta - \hat{\theta}\sin\theta$$



$$A = \frac{1}{c}\frac{\partial \vec{P}}{\partial t}$$

$$A_\omega = \frac{-i\omega}{c}\frac{e^{ikR}}{R} \mathrm{P}\hat{z}$$

$$\phi = -\nabla\cdot\vec{\gamma} = -\mathrm{P}\left[\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{e^{ikR}}{R}\cos\theta\right) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(- \quad \cdot\frac{e^{ikR}}{R}\right)\right]\right]$$

$$= -p\cos\theta \frac{e^{ikR}}{R}\left(ik - \frac{1}{R}\right)$$

$$\vec{B}_\omega = \nabla\times\vec{A}_\omega = \hat{R}[\quad] + \hat{\theta}[\quad] + \hat{\phi}[\quad]$$

$$= -\hat{\phi} k^2 \frac{e^{ikR}}{R}\sin\theta\, p + O\left(\frac{1}{R^2}\right)$$

$$= -k^2\frac{e^{ikR}}{R}\vec{p}\times\hat{R} + O\left(\frac{1}{R^2}\right)$$

$$\vec{E}_\omega = -\nabla\phi - \frac{(-i\omega)}{c}\vec{A}_\omega$$

$$= -\hat{R}\frac{\partial}{\partial R}\left[p\cos\theta\frac{e^{ikR}}{R}\left(ik - \frac{1}{R}\right)\right] + \hat{\theta}\frac{1}{R}\frac{\partial}{\partial\theta}\left[p\cos\theta\frac{e^{ikR}}{R}\left(ik-\frac{1}{R}\right)\right] + \hat{\phi}\cdot 0 - \frac{(-i\omega)}{c}\vec{A}_\omega$$

$$= -\hat{\theta}k^2\frac{e^{ikR}}{R}p\sin\theta$$

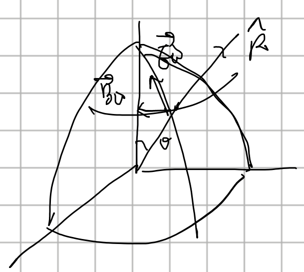
$$= -k^2\frac{e^{ikR}}{R}\underbrace{\hat{R}\times(\hat{R}\times\vec{p})}_{\sin\theta\,\hat{\theta}} + O\left(\frac{1}{R^2}\right)$$

① $\vec{E}_\omega = -\hat{R}\times\vec{B}_\omega$

② $|E_\omega| = |B_\omega|$

③ $\vec{E}_\omega\cdot\hat{R} = 0,\quad \vec{B}_\omega\cdot\hat{R} = 0$

④

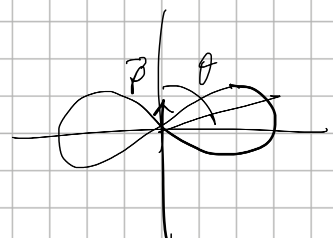


⑥ $E, B \propto \frac{1}{R}$

⑤ Energy flux

poynting vector

$$S = \frac{c}{4\pi}\vec{E}\times\vec{B} \propto \sin^2\theta$$



cf. electrostatic field due to dipole

$$E = \frac{p}{R^3}\sqrt{3\cos^2\theta+1}$$