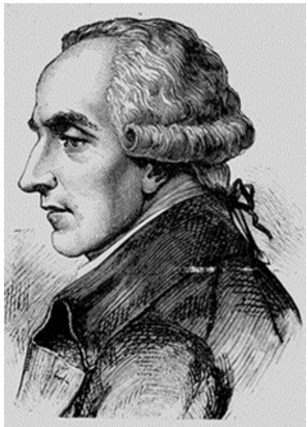
FEM (Finite Element Method)

Chapter 4
INTEGRAL TRANSFORMS

Joseph Fourier
(1768-1830)
Math/Physics
Fourier Series/Transform

Lecture 15

4.3 Laplace Transform



Pierre-Simon Laplace
(1749-1827)
Math/Physic
Laplace Transform
Laplace Equation
(Scalar Potential Theory)

4.3 Laplace Transform

Laplace transform (LT) is also a powerful tool to find solutions of differential equations like Fourier transform (FT). The LT of a function $f(t)$ is a complex function $F(s)$ of a complex variable while the FT of a function is a complex function of a real variable. Compared with the FT, the LT tends to be a well-behaved function.

The LT pair is defined as

ik − ωt

$$\boxed{\begin{aligned} F(s) &= \int_0^{\infty} ds\, e^{-st} f(t) \\ f(t) &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} ds\, e^{st} F(s) \end{aligned}} \quad (4.41)$$

Fourier

where the complex variable or **complex frequency** is given by $s = \sigma + i\omega$.

The existence conditions of LT are

1) $f(t)$ is **piece-wise continuous**, in every finite interval $(0,T)$.
2) $f(t)$ has an **exponential order** of $e^{\sigma t}$:

superposition principle

$$\boxed{\lim_{t\to\infty} f(t) = O(e^{at}) \propto Ke^{at}} \quad (4.7)$$

Proof) $\left|F(s)\right| = \left|\int_{-\infty}^{\infty} dt\, e^{-st} f(t)\right| \le \int_{-\infty}^{\infty} dt \left|e^{-st} f(t)\right| \le K\int_{-\infty}^{\infty} dt\, e^{-(\sigma-a)t} = \dfrac{K}{s-a}$ for $\mathrm{Re}[s] = \sigma > a$

Heaviside D-Calulus: The Beginning of Laplace Transform

기술자

Between 1880 and 1887, Heaviside, an electrician, developed the operational calculus (D-calculus) for the differential operator, to solve differential equations by an algebraic method. This caused a great deal of controversy, owing to its lack of rigor. He famously said, **"Mathematics is an experimental science, and definitions do not come first, but later on."**

For example, Heaviside tried to solve a differential equation,

$$\frac{d^2 f(t)}{dt^2} + 3\frac{df(t)}{dt} + 2f(t) = e^{it}$$

Defining an D operator, $D \equiv d/dt$, he obtained

$$(D^2 + 3D + 2)f(t) = (D+2)(D+1)f(t) = e^{it} \rightarrow f(t) = \left(\frac{1}{D+1} - \frac{1}{D+2}\right)e^{it}$$

Using the "Geometric Series Expansions of the two fractions,

$$f(t) = \left(\sum_{j=0}^{\infty}(-1)^j D^j\right)e^{it} + \left(\sum_{j=0}^{\infty}(-1)^j \frac{1}{2^{j+1}} D^j\right)e^{it} = \left(\frac{1}{1+i} - \frac{1}{2+i}\right)e^{it} = \frac{1-3i}{10}e^{it}$$

he successfully got the particular solutions.

Inverse Laplace Transform: Heuristic Approach

The Laplace transform is useful for many physics and engineering problems. However, there are **main difficulties for finding the inverse Laplace transform (ILT)** unlike the FTs. Although in practice we can find the ILTs using LT tables, we may have to use the ILT formula which may require **numerical analyses** in some cases.

Using the analogy of IFT, we can first try a heuristic approach, $s = i\omega \; (\mathrm{Re}[s] = \sigma = 0)$,

$$f(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds\, F(s)e^{st} = \frac{1}{2\pi}\int_{-\infty}^{\infty} ds\, F(i\omega)e^{i\omega t}$$

However, this formula does not ensure the convergence, and we multiply $e^{-\sigma t}$,

$$f(t)e^{-\sigma t} = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds\, \mathscr{L}[f(t)e^{-\sigma t}]e^{st}$$

Using the shift property, we have

$$f(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds\, F(s+a)e^{(s+a)t} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} ds'\, F(s')e^{s't}$$

Some Laplace Transforms

$f(t)$	$F(s)$	convergence
e^{at}	$\frac{1}{s-a}$	$\mathrm{Re}(s) > \mathrm{Re}(a)$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$	$\omega \in \mathbb{R}$ and $\mathrm{Re}(s) > 0$
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$	$\omega \in \mathbb{R}$ and $\mathrm{Re}(s) > 0$
$\cosh\beta t$	$\frac{s}{s^2-\beta^2}$	$\beta \in \mathbb{R}$ and $\mathrm{Re}(s) > \lvert\beta\rvert$
$\sinh\beta t$	$\frac{\beta}{s^2-\beta^2}$	$\beta \in \mathbb{R}$ and $\mathrm{Re}(s) > \lvert\beta\rvert$
t^n	$\frac{n!}{s^{n+1}}$	$n = 0, 1, \ldots$ and $\mathrm{Re}(s) > 0$
$e^{at} f(t)$	$F(s-a)$	convergence
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	same as for $F(s)$
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\, d\sigma$	same as for $F(s)$
$f_\tau(t)$	$e^{-s\tau} F(s)$	$\tau > 0$ and same as for $F(s)$
$\delta(t-\tau)$	$\theta(t-\tau)\, e^{-s\tau}$	none
$f^{(n)}(t)$	$s^n F(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0)$	$\lim_{t\to\infty} f^{(k)}(t) e^{-st} = 0$
$\int_0^t f(\tau)\, d\tau$	$\frac{F(s)}{s}$	same as for $F(s)$

steady state → Fourier.

transient state → Laplace

Fourier versus Laplace Transform

[Q] What are their main application?