Phase correction for a Michelson interferometer with misaligned mirrors

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The phase correction for a Michelson interferometer with misaligned mirrors in converging light is shown to give rise to a quadratic phase shift. In general, the calculation of a spectrum from the measured interferogram needs phase correction. Phase corrections have been well worked out for the cases of a linear phase shift and a phase that is slowly varying. The standard procedures for correcting calculated spectra need to be modified, however, to remove any phase errors resulting from misaligned mirrors.

I. Introduction

In Michelson interferometry the calculated spectrum is obtained after a Fourier transform of the measured interferogram. The calculated spectrum is related to the true spectrum by the relationship

$$B_c(\sigma') = \int_{-\infty}^{\infty} B(\sigma)H(\sigma',\sigma)\,d\sigma, \qquad (1)$$

where $B_c(\sigma')$ is the calculated spectrum, and $B(\sigma)$ is the true spectrum. $H(\sigma', \sigma)$ is a function that depends (1) upon the mathematical processes involved in transforming the interferogram and (2) on the physical properties of the instrument. For these reasons $H(\sigma', \sigma)$ is referred to as the instrumental function.

Since the spectral distribution $B(\sigma)$ is a real function, while $H(\sigma', \sigma)$ can be complex, the calculated spectrum $B_c(\sigma')$ may be complex. The complex nature of $H(\sigma', \sigma)$ indicates that both the amplitude and phase of the radiation at wavenumber σ can be changed by the mathematics used or by the instrument.

If the phase of $H(\sigma', \sigma)$ is linear or does not vary too rapidly, the calculated spectrum $B_c(\sigma')$ can be corrected.[1-3] For an ideal instrument the phase of $H(\sigma', \sigma)$ is zero; therefore, a nonzero phase represents a less than ideal instrument. In general, phase corrections are necessary when the zero path difference in the interferogram is not known, when dispersive elements are in the optical path, or when the mathematical transform is not symmetric.

I shall show that a phase correction is also necessary when a converging cone of light is incident on an interferometer with misaligned mirrors. Well known methods for correcting linear phase errors in the calculated spectrum will not correct for phase errors caused by misaligned mirrors, because, as I will show, the phase correction is quadratic.

It is shown below that neither a converging beam of radiation on perfectly aligned mirrors nor collimated radiation on misaligned mirrors results in the need for phase correction of the measured spectrum. The two effects treated separately result in a decrease in modulation but still produce symmetric interferograms. When treated together, however, these effects produce an asymmetric interferogram resulting in the need for phase correction of the calculated spectrum.

II. Instrumental Profiles

For an ideal Michelson interferometer, the measured interferogram $I'(x)$ is related to the spectrum of the source $B(\sigma)$ by the equation[4]

$$I'(x) = \int_{0}^{\infty} B(\sigma)(1 + \cos 2\pi\sigma x)\,d\sigma,$$

where x is the optical path difference, measured from zero retardation, and σ is the wavenumber of radiation. The value of the interferogram at $x = 0$ represents a dc term that can always be subtracted out giving

$$I(x) = I'(x) - \frac{1}{2}I'(0) = \int_{0}^{\infty} B(\sigma)\,\cos 2\pi\sigma x\,d\sigma$$

as the oscillatory part of the interferogram with which we will deal from now on.

Since $I(x)$ is real we can write

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$$I(x) = \frac{1}{2}\int_{-\infty}^{\infty} B(\sigma)\exp(i2\pi\sigma x)\,d\sigma, \tag{2}$$

which assumes $B(\sigma) = B(-\sigma)$.

For a nonideal Michelson interferometer the interferogram $I(x)$ is related to the spectrum $B(\sigma)$ by

$$I(x) = \int_{-\infty}^{\infty} B(\sigma)F(x,\sigma)\exp(i2\pi\sigma x)\,d\sigma, \tag{3}$$

where $F(x, \sigma)$ is a function that depends upon the instrument. The effect of $F(x, \sigma)$ is to decrease the modulation of the fringes and introduce phase delays as x increases, where x is the optical path difference that the radiation of wavenumber σ undergoes in the Michelson interferometer.

The calculated spectrum is obtained by taking the Fourier transform of the interferogram

$$\begin{aligned} B_c(\sigma') &= \int_{-\infty}^{\infty} I(x)\exp(-i2\pi\sigma' x)dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B(\sigma)F(x,\sigma)\exp[i2\pi(\sigma-\sigma')x]dxd\sigma \\ &= \int_{-\infty}^{\infty} B(\sigma)H(\sigma',\sigma)d\sigma, \end{aligned} \tag{4}$$

where

$$H(\sigma',\sigma) = \int_{-\infty}^{\infty} F(x,\sigma)\exp[i2\pi(\sigma-\sigma')x]dx. \tag{5}$$

For a real interferometer, the optical path does not extend to infinity but is truncated. Then the instrumental function $H(\sigma', \sigma)$ calculated in Eq. (5) is convoluted with the Fourier transform of the truncation function.

The functional form for $F(x, \sigma)$ and $H(\sigma', \sigma)$ for several different effects is as follows:

A. Ideal Interferometer

For an ideal interferometer, we have from Eq. (2), $F(x, \sigma) = 1$, and Eq. (5) gives $H(\sigma', \sigma) = \delta(\sigma' - \sigma)$, where $\delta(\sigma' - \sigma)$ is the Kroneker delta. The calculated spectrum is then equal to the source spectrum of $B_c(\sigma) = B(\sigma)$.

B. Ideal Interferometer in Converging Radiation

For an interferometer with a cone of radiation of wavenumber σ and solid angle Ω incident upon perfectly aligned mirrors, we have after expressing Eq. (5) of Stroke[5] in the form of Eq. (3)

$$F(x,\sigma) = \frac{\sin\left(\frac{\Omega\sigma x}{2}\right)}{\left(\frac{\Omega\sigma x}{2}\right)}\exp\left(-\frac{i\Omega\sigma x}{2}\right).$$

The instrumental function is then calculated using Eq. (5), and

$$H(\sigma',\sigma) = \frac{2\pi}{\Omega\sigma'}\ \text{for}\ \sigma'\left(1-\frac{\Omega}{2\pi}\right) < \sigma < \sigma',$$

otherwise

$$H(\sigma',\sigma) = 0.$$

The calculated spectrum is related to the source spectrum by Eq. (4) as

$$B_c(\sigma') = \frac{2\pi}{\Omega\sigma'}\int_{\sigma'\left(1-\frac{\Omega}{2\pi}\right)}^{\sigma'} B(\sigma)\,d\sigma.$$

The resolution limit is $\delta\sigma = \Omega\sigma/2\pi$ leading to the well-known relationship between the resolution and solid angle $R\Omega = 2\pi$. The nonsymmetric limits on the integral reflect the fact that the wavenumber σ is replaced by $\sigma(1 + \Omega/4\pi)$.

C. Interferometer with Misaligned Mirrors

For the case of collimated radiation of wavenumber σ incident upon mirrors of radius R, which are misaligned by an angle α, we have, after expressing the equation derived by Williams[6] in the form of Eq. (3),

$$F(x,\sigma) = \frac{2J_1(4\pi\sigma R\alpha)}{4\pi\sigma R\alpha},$$

where J_1 is the common first order Bessel function. The instrumental function is calculated using Eq. (5), and

$$H(\sigma',\sigma) = \frac{2J_1(4\pi\sigma R\alpha)}{4\pi\sigma R\alpha}\delta(\sigma'-\sigma).$$

The calculated spectrum is related to the source spectrum by Eq. (4):

$$B_c(\sigma) = \frac{2J_1(4\pi\sigma R\alpha)}{4\pi\sigma R\alpha}B(\sigma).$$

D. Interferometer with Misaligned Mirrors in Converging Radiation

For the case of a cone of incident radiation of wavenumber σ and solid angle Ω incident upon mirrors misaligned by an amount α, we have, using Eq. (4) from Kunz and Goorvitch[7] in Eq. (3) above,

$$F(x,\sigma) = \frac{\sin\left(\frac{\Omega\sigma x}{2}\right)}{\left(\frac{\Omega\sigma x}{2}\right)}\exp\left(\frac{-i\Omega\sigma x}{2}\right) + \frac{\pi(2D-x)^2\alpha^2\sigma}{x} \times\left[\exp(-i\Omega\sigma x) - \frac{\sin\left(\frac{\Omega\sigma x}{2}\right)}{\left(\frac{\Omega\sigma x}{2}\right)}\exp\left(\frac{-i\Omega\sigma x}{2}\right)\right],$$

where D is the distance from the fixed mirror to the focus and x is the distance of the movable mirror from the position of zero path difference.

Now the factor $(2D - x)^2$ in the second term can be expanded and terms collected to give

$$\begin{aligned} F(x,\sigma) = {} & (1 + 4\pi D^2\alpha^2\sigma)\frac{\sin\left(\frac{\Omega\sigma x}{2}\right)}{\left(\frac{\Omega\sigma x}{2}\right)}\exp\left(\frac{-i\Omega\sigma x}{2}\right) \\ & + \frac{4\pi D^2\alpha^2\sigma}{x}\left[\exp(-i\Omega\sigma x) - \frac{\sin\left(\frac{\Omega\sigma x}{2}\right)}{\left(\frac{\Omega\sigma x}{2}\right)}\exp\left(\frac{-i\Omega\sigma x}{2}\right)\right] \\ & - 4\pi D^2\alpha^2\sigma\exp(-i\Omega\sigma x) + \pi\alpha^2\sigma x\exp(-i\Omega\sigma x) \\ & - \frac{2\pi\alpha^2}{\Omega}\sin\left(\frac{\Omega\sigma x}{2}\right)\exp\left(\frac{-i\Omega\sigma x}{2}\right). \end{aligned} \tag{6}$$

As shown in the Appendix, the instrumental function is

$$H(\sigma',\sigma) = \frac{2\pi}{\Omega\sigma'}(1 + 4\pi D^2\alpha^2\sigma') - 4\pi D^2\alpha^2\sigma'\delta\left(\sigma - \sigma' + \frac{\Omega\sigma'}{2\pi}\right) + i\left\{\frac{16\pi^3D^2\alpha^2(\sigma' - \sigma)}{\Omega} + \frac{\pi\alpha^2}{\Omega}\left[\delta(\sigma - \sigma') - \delta\left(\sigma - \sigma' + \frac{\Omega\sigma'}{2\pi}\right)\right] + \frac{\alpha^2\sigma'}{2}\frac{d}{d\sigma}\delta\left(\sigma - \sigma' + \frac{\Omega\sigma'}{2\pi}\right)\right\} \quad \text{for } \sigma'\left(1 - \frac{\Omega}{2\pi}\right) < \sigma < \sigma', \tag{7}$$

otherwise

$$H(\sigma',\sigma) = 0.$$

The instrumental profile is no longer real but complex. The complex nature of $H(\sigma', \sigma)$ results from the asymmetry in the interferogram. When integrating the last three terms in Eq. (6) over an infinite pathlength, the only contribution to the integral occurs at zero phase. Otherwise, the oscillatory nature of the exponentials leads to a net contribution of zero to the integral. The delta functions in Eq. (7) mathematically represent this result of the integration.

The calculated spectrum $B_c(\sigma')$ is related to the source spectrum $B(\sigma)$ by Eq. (4):

$$B_c(\sigma') = \frac{2\pi}{\Omega\sigma'}(1 + 4\pi D^2\alpha^2\sigma')\int_{\sigma'\left(1-\frac{\Omega}{2\pi}\right)}^{\sigma'} B(\sigma)d\sigma - 4\pi D^2\alpha^2\sigma' B\left(\sigma' + \frac{\Omega\sigma'}{2\pi}\right) + \frac{\alpha^2\sigma' i}{2}\frac{dB(\sigma)}{d\sigma}\Big|_{\sigma=\sigma'+\frac{\Omega\sigma'}{2\pi}} + \frac{16\pi^3D^2\alpha^2 i}{\Omega}\int_{\sigma'\left(1-\frac{\Omega}{2\pi}\right)}^{\sigma'}(\sigma' - \sigma)B(\sigma)d\sigma + \frac{\pi\alpha^2 i}{\Omega}\left[B(\sigma') - B\left(\sigma' + \frac{\Omega\sigma'}{2\pi}\right)\right].$$

In the case where $B(\sigma)$ does not vary over the resolution element $\delta\sigma = \Omega\sigma/2\pi$, we have $B(\sigma') = B(\sigma' + \Omega\sigma'/2\pi)$, $dB(\sigma)/d\sigma = 0$, and

$$B_c(\sigma') = B(\sigma')(1 - 2\pi\Omega D^2\sigma'^2\alpha^2 i).$$

The resolution limit can be expressed as $\delta\sigma = \sigma/2(f/)^2$. For a $f/10$ converging beam $\delta\sigma = \sigma/200$. Thus the spectrum should not vary over $\sigma/200$ wavenumbers.

The phase is then defined by

$$\phi(\sigma) \equiv \arctan\left[\frac{\mathrm{Im}B_c(\sigma)}{\mathrm{Re}B_c(\sigma)}\right] = -\arctan(2\pi\Omega D^2\alpha^2\sigma^2),$$

where σ' is replaced by σ. Since for an instrument with a small field of view the solid angle Ω is related to the f number by $\Omega = \pi/4(f/)^2$, we have for the phase

$$\phi(\sigma) = -\arctan\left[\frac{\pi^2D^2\alpha^2\sigma^2}{2(f/)^2}\right] = -\arctan(2\pi^2\alpha^2R^2\sigma^2),$$

where αR is just the distance between the first mirror and the image of the second mirror at the radius R when $\tan\alpha \approx \alpha$.[7] The phase plotted against the wavenumber σ is shown in Fig. 1.

When $[\pi D\alpha\sigma/\sqrt{2}(f/)]^2 < 1$, we have for the phase $\Phi(\sigma) = -[\pi D\alpha\sigma/\sqrt{2}(f/)]^2$. In this case the phase is shown to vary quadratically with wavenumber. If σ_{max} is the maximum wavenumber of the radiation incident upon the interferometer, we have, for the maximum allowable mirror misalignment,

$$\alpha < \frac{\sqrt{2}(f/)}{\pi D\sigma_{max}} = \frac{1}{\pi\sqrt{2}R\sigma_{max}}.$$

As an example, consider an interferometer for which $D = 10$ cm, $\sigma_{max} = 200$ cm^{-1}, and the converging beam is $f/10$. Then $\alpha < 2.25 \times 10^{-3}$ rad = 7.74 min of arc.

The requirement on α for phase correction is less stringent than the requirement on α when power loss is considered. Previously, we found that for a power loss of A, due to mirror misalignment,[7]

$$\alpha \leq \frac{(2A)^{1/2}(f/)}{\pi D\sigma_{max}} = \frac{\sqrt{A}}{\pi\sqrt{2}R\sigma_{max}}.$$

Therefore, minimum power loss will insure that the expansion of the arctangent is valid and that the phase correction for any residual mirror misalignment is truly quadratic. A linear phase error may also be present, usually due to missampling at the zero retardation position.

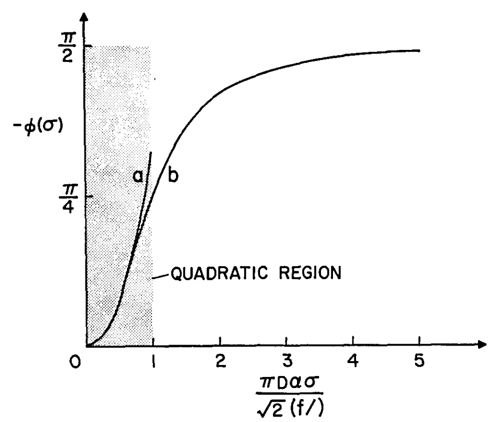

Fig. 1. The phase plotted against wavenumber: (a) $\Phi(\sigma) = -(\pi D\alpha\sigma/\sqrt{2}f/)^2$ and (b) $\Phi(\sigma) = -\arctan(\pi D\alpha\sigma/\sqrt{2}f/)^2$.

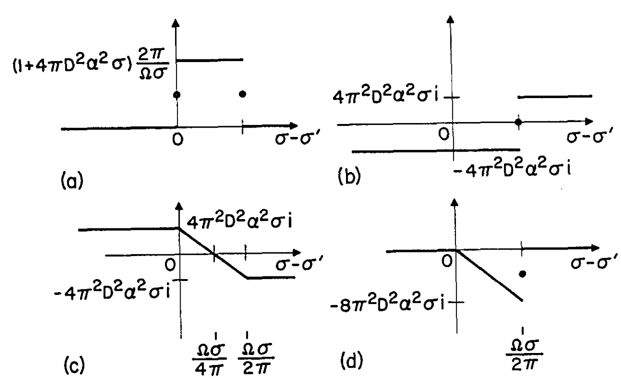

Fig. 2. Instrumental profile: (a) Real part; (b) and (c) steps intermediate in calculating the imaginary part; and (d) imaginary part. A fuller explanation is given in the Appendix.

III. Conclusion

The phase correction due to misaligned mirrors with an incident converging beam of radiation has been calculated. The instrumental profile is complex, reflecting the asymmetry in the interferogram. When the spectrum does not vary over the resolution limit, the source spectrum $B(\sigma)$ is related to the calculated spectrum $B_c(\sigma)$ by

$$B(\sigma) = B_c(\sigma)\exp[-i\Phi(\sigma)],$$

where $\Phi(\sigma) = -[\pi D\alpha\sigma/\sqrt{2}(f/)]^2$ represents the phase correction.

The phase of a spectrum is determined by plotting the arctangent of the imaginary part of the calculated spectrum divided by the real part of the calculated spectrum vs wavenumber. Any quadratic dependence will appear as a deviation from a linear curve.

A quadratically varying phase error, which persists after the calculated spectrum has been corrected for the well-known linear phase error, may signify the existence of mirror misalignment.

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Appendix: Calculation of the Instrumental Profile

The calculation of the instrumental profile for a Michelson interferometer with misaligned mirrors in converging light makes use of the theory of complex integration. The instrumental profile

$$H(\sigma',\sigma) = \int_{-\infty}^{\infty} F(x,\sigma)\exp[i2\pi(\sigma-\sigma')x]dx,$$

where $F(x, \sigma)$ is given by Eq. (6).

Integration of the first term in the complex plane gives[8]

$$(1 + 4\pi D^2\alpha^2\sigma)\int_{-\infty}^{\infty}\frac{\sin\left(\frac{\Omega\sigma x}{2}\right)}{\left(\frac{\Omega\sigma x}{2}\right)}\cdot \exp\left\{i\left[2\pi(\sigma-\sigma') - \frac{\Omega\sigma}{2}\right]x\right\}dx$$

$$= (1 + 4\pi D^2\alpha^2\sigma)\begin{cases}\frac{2\pi}{\Omega\sigma} & 0 < \sigma - \sigma' < \frac{\Omega\sigma}{2\pi}\\ \frac{\pi}{\Omega\sigma} & \sigma - \sigma' = 0 \text{ and } \sigma - \sigma' = \frac{\Omega\sigma}{2\pi}\\ 0 & \text{otherwise.}\end{cases}$$

This term shown in Fig. 2(a) gives a shifted rectangle because of the converging radiation. The resolution limit is $\delta\sigma = \Omega\sigma/2\pi$, and the wavenumber scale is scaled to $\sigma(1 + \Omega/4\pi)$.

The second term integrates to give[9]

$$4\pi D^2\alpha^2\sigma\int_{-\infty}^{\infty}\frac{\exp\{i[2\pi(\sigma-\sigma') - \Omega\sigma]x\}}{x}dx$$

$$= \begin{cases}4\pi^2D^2\alpha^2\sigma i & \sigma - \sigma' > \frac{\Omega\sigma}{2\pi}\\ 0 & \sigma - \sigma' = \frac{\Omega\sigma}{2\pi}\\ -4\pi^2D^2\alpha^2\sigma i & \sigma - \sigma' < \frac{\Omega\sigma}{2\pi},\end{cases}$$

which is complex and is shown in Fig. 2(b).

The third term gives[10]

$$-4\pi D^2\alpha^2\sigma\int_{-\infty}^{\infty}\frac{\sin\left(\frac{\Omega\sigma x}{2}\right)}{\frac{\Omega\sigma x}{2}}\frac{\exp\left\{i\left[2\pi(\sigma-\sigma') - \frac{\Omega\sigma}{2}\right]x\right\}}{x}dx$$

$$= 4\pi^2D^2\alpha^2\sigma i\begin{cases}1 & \sigma - \sigma' \leq 0\\ 1 - \frac{4\pi(\sigma-\sigma')}{\Omega\sigma} & 0 \leq \sigma - \sigma' \leq \frac{\Omega\sigma}{2\pi}\\ -1 & \sigma - \sigma' \geq \frac{\Omega\sigma}{2\pi},\end{cases}$$

which is also complex and is shown in Fig. 2(c).

When these three terms are added together we obtain

$$G_1(\sigma',\sigma) = (1 + 4\pi D^2\alpha^2\sigma)\frac{2\pi}{\Omega\sigma} - \frac{16\pi^3D^2\alpha^2(\sigma-\sigma')}{\Omega}i$$

for

$$0 \leq \sigma - \sigma' \leq \frac{\Omega\sigma}{2\pi}$$

and

$$G_1(\sigma',\sigma) = 0 \text{ otherwise.}$$

The values at the discontinuities have been neglected, since these points do not contribute to the integral. The real part of $G_1(\sigma', \sigma)$ is shown in Fig. 2(a) while the imaginary part of $G_1(\sigma', \sigma)$ is shown in Fig. 2(d).

Using the relationships[11]

$$\int_{-\infty}^{\infty}\exp(-i2\pi\sigma x)\,dx = \delta(\sigma)$$

$$\text{and } \int_{-\infty}^{\infty}x\exp(-i2\pi\sigma x)dx = \frac{i}{2\pi}\delta'(\sigma),$$

where $\delta(\sigma)$ is the Dirac delta function and $\delta'(\sigma)$ is its first derivative, we can evaluate the last three terms in Eq. (6) to obtain

$$G_2(\sigma',\sigma) = -4\pi D^2\alpha^2\sigma\delta\left(\sigma' - \sigma + \frac{\Omega\sigma}{2\pi}\right) + \frac{\alpha^2\sigma i}{2}\delta'\left(\sigma' - \sigma + \frac{\Omega\sigma}{2\pi}\right) + \frac{i\pi\alpha^2}{\Omega}\left[\delta(\sigma' - \sigma) - \delta\left(\sigma' - \sigma + \frac{\Omega\sigma}{2\pi}\right)\right].$$

The total instrumental function is the sum of $G_1(\sigma', \sigma)$ and $G_2(\sigma', \sigma)$ and is given in Eq. (7).

References

1. M. L. Forman, W. H. Steel, and G. A. Vanasse, J. Opt. Soc. Am. 56, 59 (1966).
2. G. A. Vanasse and H. Sakai, in *Progress in Optics*, E. Wolf, Ed. (North-Holland, Amsterdam, 1967), Vol. 6.
3. H. Saki, G. A. Vanasse, and M. L. Forman, J. Opt. Soc. Am. 58, 84 (1968).
4. A. A. Michelson, *Studies in Optics* (University of Chicago Press, Chicago, 1927), p. 36.
5. G. W. Stroke, J. Opt. Soc. Am. 47, 1097 (1957).
6. C. S. Williams, Appl. Opt. 5, 1084 (1966).
7. L. W. Kunz and D. Goorvitch, Appl. Opt. 13, 1077 (1974).
8. I. S. Gradshteyn and I. M. Ryzik, *Table of Integrals, Series, and Products* (Academic, New York, 1965), p. 414, 3.741.2.
9. Ref. 8, p. 406, 3.722.2 and 3.722.6.
10. Ref. 8, p. 414, 3.741.3.
11. R. Bracewell, *The Fourier Transform and Its Applications* (McGraw-Hill, New York, 1965).