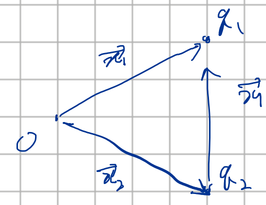
1.1 Review

A Coulomb's law

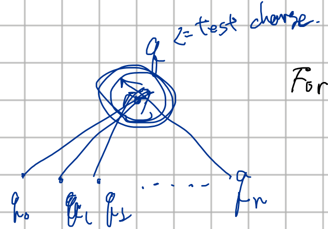
basic experimental facts



$$\vec{F}_{12} = q_1 q_2 \frac{(\vec{x}_1 - \vec{x}_2)}{|\vec{x}_1 - \vec{x}_2|^3}$$

force on q_1 to q_2

$$\propto \begin{cases} q_1 q_2 \\ 1/(\text{distance})^2 \\ \text{along the line between the two charge: central} \end{cases}$$



For a system of charges

$$\vec{F}(\vec{x}) = \sum_i q q_i \frac{(\vec{x} - \vec{x}_i)}{|\vec{x} - \vec{x}_i|^3}, \quad \text{discrete charges}$$

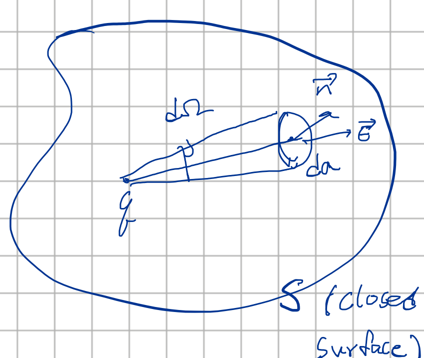
$$= q \int d\vec{x}' \rho(\vec{x}') \frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}, \quad \text{continuous distribution of charges}$$

① field : force per unit charge

$$E = \lim_{q \to 0} \frac{\vec{F}}{q}$$ ⇒ description becomes independent of a test charge

cf. cgs gauss = 1 dyne/1 esu

② Coulomb's law is not the most suitable form for the evaluation of field

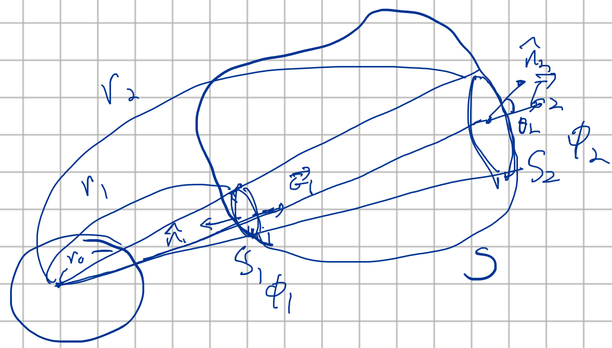


Flux through the area element, da

$= \vec{E} \cdot \hat{n} \; da$

$= \frac{q \cos\theta}{r^2} da = q \, d\Omega$

$$\oint \vec{E} \cdot \hat{n} da = \int q \, d\Omega = \begin{cases} 4\pi q & \text{if the charge inside } S \\ 0 & \text{if the charge outside } S \end{cases}$$



$$\phi_2 = \vec{E}_2 \cdot \hat{n}_2 = E_2 S_2 \cos\theta_2$$
$$= \left[E_0 \left(\frac{r_0}{r_2}\right)^2\right]\left[S_0\left(\frac{r_2}{r_0}\right)^2 \frac{1}{\cos\theta_2}\right]\cos\theta_2 = E_0 S_0$$

$$\phi_1 = \vec{E}_1 \cdot \hat{n}_1 = E_1 S_1 \cos\theta_1$$
$$= \left[E_0 \left(\frac{r_0}{r_1}\right)^2\right]\left[S_0\left(\frac{r_1}{r_0}\right)^2 \frac{1}{\cos\theta_1}\right]\cos\theta_1 = -E_0 S_0$$

$\Rightarrow \phi_1 + \phi_2 = 0$

$\oint \vec{E} \cdot \hat{n}\, da = 0$ for q outside S $\Leftarrow$ ① $\frac{1}{r^2}$ ② central force

$\oint \vec{E} \cdot \hat{n}\, da = 4\pi \int_V \rho(\vec{x})\, d^3x$ due to linear superposition.

Gauss' theoram

$$\int_V d^3x\, \vec{\nabla} = \oint_S da\, \hat{n} \qquad \text{ex)} \int_V d^3x\, \vec{\nabla}\cdot\vec{b} = \oint_S da\, \hat{n}\cdot\vec{b}$$

$$\int_V d^3x\, \nabla\cdot\vec{E} = \int_V d^3x\, (4\pi\rho) \quad \text{for any volume}$$
$$\nabla\cdot\vec{E} = 4\pi\rho(\vec{x})$$
$$\int d^3x\, \nabla\times\vec{E} = \int da\, \hat{n}\times\left(q\frac{\vec{x}}{r^3}\right), \quad \hat{r} = \hat{x}$$
$$= 0$$
$$\nabla\times\vec{E} = 0$$

③ Mathematical Theorem

$\boxed{\text{All vector fields in 3D are uniquely determined if curl \& divergence are given functions of the coordinates at all points in space \& if the source vanishes at infinity}}$

$\vec{V}(x,y,z)$, 3D vector field

$$\nabla\cdot\vec{V} = S, \quad \nabla\times\vec{V} = \vec{C} \qquad \text{w/} \quad V(\infty)=0$$

i) $\vec{V} = -\nabla\phi + \nabla\times\vec{A}$

w/ $\phi(\vec{x}) = \frac{1}{4\pi}\int \frac{S(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'$, $\vec{r} = |\vec{x}-\vec{x}'|$

$$\vec{A}(\vec{x}) = \frac{1}{4\pi}\int \frac{\vec{C}(\vec{x}')}{r} d^3x'$$

then $\nabla\cdot\vec{V} = S$ & $\nabla\times\vec{V} = \vec{C}$

$$\begin{aligned}
\nabla\cdot\vec{V} &= \nabla\cdot(-\nabla\phi + \nabla\times\vec{A}) \\
&= -\nabla^2\phi = -\nabla^2\frac{1}{4\pi}\int d^3x' \frac{S(\vec{x}')}{r} \\
&= -\frac{1}{4\pi}\int d^3x' S(\vec{x}')\nabla^2\left(\frac{1}{r}\right) \\
&= \int d^3x\, S(\vec{x}')\delta(\vec{x}-\vec{x}') = S(\vec{x})
\end{aligned}$$

cf. $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(\vec{x}-\vec{x}')$

$$\begin{aligned}
\nabla\times\vec{V} &= -\nabla\times\nabla\phi + \nabla\times(\nabla\times\vec{A}) \\
&= 0 + \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} \\
&= \nabla\left(\nabla\cdot\frac{1}{4\pi}\int d^3x' \frac{\vec{C}}{r}\right) - \nabla^2\left(\frac{1}{4\pi}\int d^3x' \frac{\vec{C}}{r}\right) \\
&= \frac{1}{4\pi}\int d^3x' (\vec{C}\cdot\nabla)\left(\nabla\frac{1}{r}\right) - \frac{1}{4\pi}\int d^3x'\, \vec{C}\,\nabla^2\left(\frac{1}{r}\right) \\
&= 0 + C(\vec{x})
\end{aligned}$$