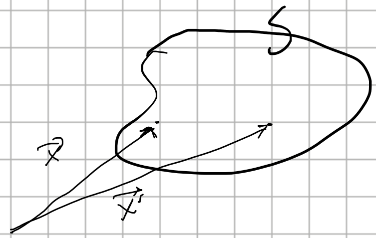
④ $\phi(\vec{x}) = \int d\vec{x}' \, \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}$ with $\phi(\vec{x}=\infty)=0$



$\boxed{\text{Using Green theorem}}$

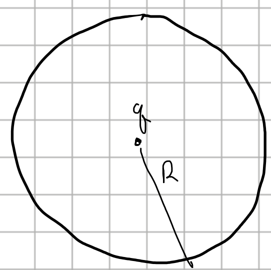
inside V

$$\phi(\vec{x}) = \int dv' \, \frac{\rho(\vec{x}')}{r} + \frac{1}{4\pi}\int d\vec{s}\cdot\left(-\phi\frac{\vec{x}}{r^2}+\frac{\nabla'\phi}{r}\right)$$

outside V

$$0 = \underbrace{\int dv' \, \frac{\rho(\vec{x}')}{r}}_{\text{inside}} + \underbrace{\frac{1}{4\pi}\int d\vec{s}\cdot\left(-\phi\frac{\vec{x}}{r^2}+\frac{\nabla'\phi}{r}\right)}_{\text{surface}}$$

Ex)



$\sigma = \frac{q}{4\pi R^2}$

$\tau = \frac{q}{4\pi R}$

$$\phi_\sigma = \begin{cases} -\frac{q}{R} & r<R \\ -\frac{q}{r} & r>R \end{cases}$$

$$\phi_\tau = \int d\vec{s}\cdot\frac{\vec{r}}{r^3}\,\tau = \frac{q}{4\pi R}\int d\vec{s}\cdot\frac{\vec{r}}{r^3} = \begin{cases} \frac{q}{R} & r<R \\ 0 & r>R \end{cases}$$

$\phi_q = q/r$ for all r

Potential is additive

$r<R$ $\quad \phi = \phi_q + \phi_\sigma + \phi_\tau = \frac{q}{r} - \frac{q}{R} + \frac{q}{R} = \frac{q}{r}$

$r>R$ $\quad \phi = \frac{q}{r} - \frac{q}{r} + 0 = 0$

Surface forms, when ϕ & $\nabla\phi$ are properly evaluated on the surface, are precisely those necessary to cancle the field by charge inside S in the outer region.

⑤ For a charge - free volume.

$$\phi = \frac{1}{4\pi} \int d\vec{S} \cdot \left(-\phi \frac{\vec{x}}{r^3} + \frac{\nabla'\phi}{r} \right)$$

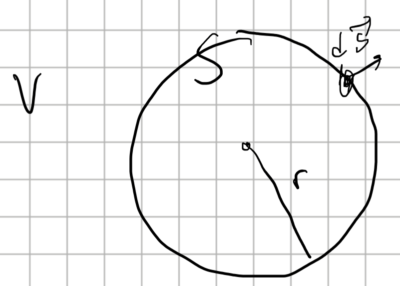
is uniquely determined by the potential & its normal derivate on the surface enclosing the volume

Dirichlet Boundary Condition : $\phi = 0$

Neumann " : $\frac{\partial \phi}{\partial n} = 0$

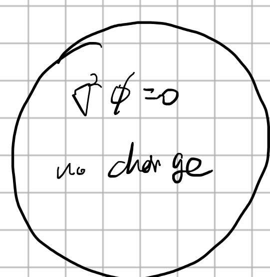
Mean - Value theorem

For a charge - free space, the value of the electrostatic potential at any point $\equiv$ the average of the potential over the surface of any sphere centered on that point.



$$\phi(0) = \frac{1}{4\pi r^2} \oint_S \phi(r)\, ds$$

a) ϕ_{max} is on the surface

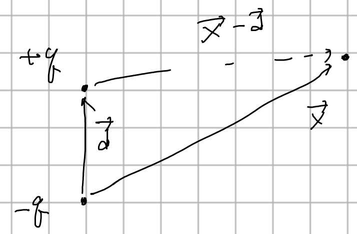


b) $\nabla^2 \phi(\vec{x}) = 0$ at all points with $\phi(\infty) =$ finite

$\phi = const.$

c) For $\phi(\infty) = 0$, $\phi = 0$.

6 Dipole field



$$\vec{E} = -q\frac{\vec{x}}{r^3} + q\,\frac{\vec{x}-\vec{d}}{|\vec{x}-\vec{d}|^3}$$
$$\simeq 3\vec{x}\cdot\frac{\vec{x}\cdot\vec{p}}{r^3} - \frac{\vec{p}}{r^3},$$
$$= -\vec{p}\cdot\nabla\left(\frac{\vec{x}}{r^3}\right)$$

$r = |\vec{x}|$
$\vec{p} = q\vec{d}$
dipole moment

defined by $\vec{x}$ & $\vec{p}$

axially symmetric about $\vec{p}$

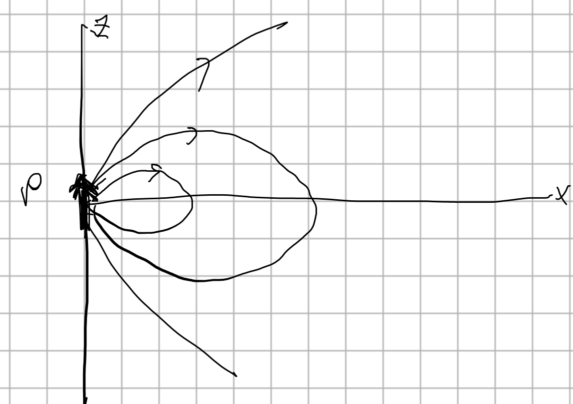
a) field line

For convenience, we consider the xz plane

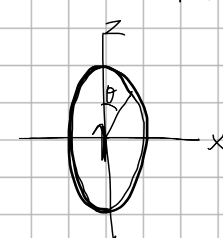
$$\vec{p} = (0,0,p), \quad \vec{x} = (x,0,z)$$

$$E_x = 3\,\frac{xpz}{r^5}, \quad E_z = \frac{3z^2p - r^2p}{r^5}$$

field line $\quad \frac{dz}{dx} = \frac{E_z}{E_x} = \frac{3z^2 - r^2}{3xz} \Rightarrow x^2 + z^2 = C\,x^{2/3}$



Field strength: $\frac{|E|}{|p|/r^3} = \sqrt{3\cos^2\theta + 1}$



for a given r

b) $\nabla\cdot\vec{E} = \nabla\cdot\left(-\vec{p}\cdot\nabla\left(\frac{\vec{x}}{r^3}\right)\right)$

$$= -\vec{p}\cdot\vec{\nabla}\underbrace{\left(\nabla\cdot\left(\frac{\vec{x}}{r^3}\right)\right)}_{4\pi\delta(\vec{x})}, \quad p \text{ is constant}$$

$$= 4\pi\left(-\vec{p}\cdot\nabla\delta\right)$$

E_{dipole} is considered to be produced by a charge density: $-\vec{p}\cdot\nabla\delta(\vec{x})$

point charge	dipole
$q\,\delta(\vec{x})$	$(-\vec{p}\cdot\nabla)\,\delta(x)$
$q\,\frac{\vec{x}}{r^3}$	$(-\vec{p}\cdot\nabla)\frac{\vec{x}}{r^3}$
$q\,\frac{1}{r}$	$(-\vec{p}\cdot\nabla)\frac{1}{r}$

c) force on the dipole p at $\vec{x}_1$ by an external field

$$\vec{F}(\vec{x}_1) = \int d^3x\ \rho(\vec{x})\vec{E}(\vec{x})$$
$$= \int d^3x\ \left(-\vec{p}\cdot\nabla\delta(\vec{x}-\vec{x}_1)\right)\vec{E}(\vec{x})$$
$$= \int d^3x\ \delta(\vec{x}-\vec{x}_1)\,\vec{p}\cdot\nabla E(\vec{x})$$

cf. $\int dx\, f(x)\frac{d}{dx}\delta(x) = \int dx\,\delta(x)\left(-\frac{df}{dx}\right)$ with $f(x=\infty)=0$

$$= \vec{p}\cdot\nabla E(x)\big|_{x=x_1}$$

cf. $\vec{p}\times(\nabla\times\vec{E}) = 0 = \nabla(\vec{p}\cdot\vec{E}) - \vec{p}\cdot\nabla\vec{E}$

$$= \nabla(\vec{p}\cdot\vec{E})$$

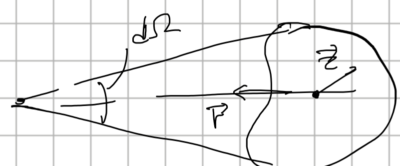
$$W = \int(-\vec{F})\cdot dl = -\vec{p}\cdot\vec{E}$$

7 Dipole layer surface

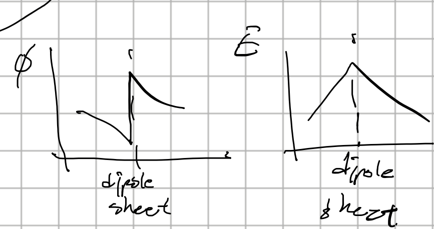
dipole moment/unit area : τ

$$\vec{p} = \vec{\tau}\,ds$$

$$\phi = \int -\vec{\tau}\cdot\nabla\left(\frac{1}{r}\right)ds = \int\frac{\tau\cdot\vec{r}}{r^3}ds = \tau\int\frac{\vec{r}}{r^3}ds = -|\tau|\Omega$$



$\phi_1 \to -2\pi\tau$, $\phi_2 \to +2\pi\tau$



cf. charge density