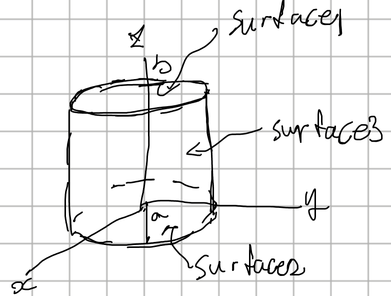
Ex)



$S_1: \psi(\rho, \theta, z=b)$
$S_2: \psi(\rho, \theta, z=0)$
$S_3: (\rho=a, \theta, z)$
} given

$\psi_{inside}(\rho, \theta, z) = ?$

By the superposition principle,

$$\psi = \psi_1 + \psi_2 + \psi_3, \quad \nabla^2 \psi_i = 0, \quad i=1,2,3$$

		S_1	S_2	S_3
with	$\psi_1(\rho,\theta,z=b) =$	$\psi(\rho,\theta,z=b)$	$\psi_1=0$	$\psi_1=0$
	ψ_2	0	$\psi(\rho,\theta,z=0)$	0
	ψ_3	0	0	$\psi(\rho=a,\theta,z)$

General solution: $\psi = \sum_{m,k} \left[A_m J_m(k\rho) + B_m N_m(k\rho)\right]$

$$\cdot \left[C_m \sin(m\theta) + D_m \cos(m\theta)\right]$$
$$\left[E \sinh(kz) + F\cosh(kz)\right]$$

i) ψ_1

① $\psi_1 = 0$ at $z=0 \Rightarrow F=0$

② ψ_1 is finite at $\rho=0 \Rightarrow B_m = 0$

($\because N_m \to \infty$ as $\rho \to 0$)

③ $\psi_1 = 0$ at $\rho=a$

$J_m(ka)=0, \quad k_{mn}a = X_{mn} \quad (X_{mn}: n^{th}$ root of $J_m(x)=0)$

$$k = k_{mn} = X_{mn}/a$$

$$\psi_1 = \sum_{m=0,n=1}^{\infty} \left[C_m \sin(m\theta) + D_m \cos(m\theta)\right] J_m(k_{mn}\rho)\sinh(k_{mn}z)$$

orthogonal

$\rightarrow D_m$ vanish

④ $\psi_1 = \psi(\rho, \theta, z=b)$ at $z=b$

$$\int_0^a \rho d\rho \int_0^{2\pi} d\theta \psi(\rho, \theta, z=b) J_{m'}(k_{m'n'}\rho) \sin(m'\theta)$$

$$= \int \rho d\rho \int d\theta \sum_{m,n} \left[C_m \sin(m\theta) + D_m \cos(m\theta) \right] \sin(m'\theta) J_{m'}(k_{m'n'}\rho) J_m(k_{mn}\rho) \sinh(k_{mn}b)$$

$$= \sum_{m,n} C_{mn} \sinh(k_{mn}b) \underbrace{\int_0^{2\pi} d\theta \sin(m\theta) \sin(m'\theta)}_{=\pi \delta_{mm'}} \underbrace{\int_0^a \rho d\rho J_m(k_{mn}\rho) J_m(k_{mn'}\rho)}_{=\frac{a^2}{2} J_{m+1}^2(k_{mn}a)\delta_{nn'}}$$

$$= C_{m'n'} \sinh(k_{m'n'}b) \pi \cdot \frac{a^2}{2} J_{m+1}^2(k_{m'n'}a)$$

$$\Rightarrow \begin{Bmatrix} C_{mn} \\ D_{mn} \end{Bmatrix} = \frac{\int \rho d\rho \int d\theta \psi(\rho, \theta, b) J_m(k_{mn}\rho) \begin{Bmatrix} \sin(m\theta) \\ \cos(m\theta) \end{Bmatrix}}{\frac{\pi a^2}{2} J_{m+1}^2(k_{mn}a) \sinh(k_{mn}b)}$$

ii) ψ_2

① $\psi_2 = 0$ at $z=b$

$$E \sinh(kb) + F \cosh(kb) = 0$$

$$F = -E \frac{\tanh(kb)}{\cosh(kb)}$$

the same procedure as in ψ_1

$$\begin{Bmatrix} C_m \\ D_m \end{Bmatrix} = \frac{\int \rho d\rho \int d\theta \psi_1(\rho, \theta, z=0) J_m(k_{mn}\rho) \begin{Bmatrix} \sin m\theta \\ \cos m\theta \end{Bmatrix}}{''}$$

iii) ψ_3

① $\psi_3 = 0$ at $z=0$ & b $\Rightarrow$ $F=0$, $E\sinh(kb)=0$, $kb = i\pi n$

$n = 1, 2, \cdots$

③ ψ_3 is finite at $\rho = 0$ $\Rightarrow$ $B_m = 0$

$$\psi_3 = \sum_{m,n} \left[C_m' \sin(m\theta) + D_m' \cos(m\theta)\right] J_m\left(i \frac{\pi n}{b} \rho\right) i \sin\left(\frac{\pi n}{b} z\right)$$

$$\downarrow$$

$$I_m\left(\frac{\pi n}{b}\rho\right)$$

$$= \sum_{m,n} \left[C_m \sin(m\theta) + D_m \cos(m\theta)\right] I_m\left(\frac{\pi n}{b}\rho\right) \sin\left(\frac{\pi n}{b} z\right)$$

$$\begin{Bmatrix} C_m \\ D_m \end{Bmatrix} = \frac{2}{\pi b\, I_m\left(\frac{\pi n}{b} a\right)} \int_0^b dz \int_0^{2\pi} d\rho\, \psi(\rho, \theta, z) \sin\left(\frac{\pi n}{b} z\right) \begin{Bmatrix} \sin m\theta \\ \cos m\theta \end{Bmatrix}$$

$$\Rightarrow \psi_{general} = \psi_1 + \psi_2 + \psi_3$$

e) Green's function revisited

Green's function: solution for Poisson equation

charge distribution

$$\nabla^2_{\vec{x}} G(\vec{x},\vec{x}') = -4\pi\delta(\vec{x}-\vec{x}') \quad , \quad G(\vec{x},\vec{x}')=0 \text{ for } \vec{x},\vec{x}' \text{ on a boundary surface}$$

(Dirichlet BVP)

$$= -4\pi\left[\frac{1}{r^2}\delta(r-r')\,\delta(\phi-\phi')\,\delta(\cos\theta-\cos\theta')\right]$$

$$= -4\pi\frac{1}{r^2}\delta(r-r')\underbrace{\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\phi')\,Y_{lm}(\theta,\phi)}_{\text{closure selection}}$$

$$G(\vec{x},\vec{x}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm}(r,r',\theta',\phi')\,Y_{lm}(\theta,\phi)$$

$$A_{lm} = g_l(r,r')\,Y^*_{lm}(\theta',\phi')$$

$$\phi(\vec{x}) = \int_V d^3x\, \rho(\vec{x}')\,G(\vec{x},\vec{x}') - \frac{1}{4\pi}\oint_S da'\,\phi(\vec{x}')\boxed{\frac{\partial G(\vec{x},\vec{x}')}{\partial n'}}$$

For $g_l(r,r')$

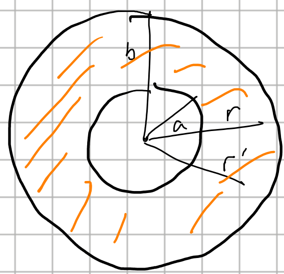
$$\frac{1}{r}\frac{d^2}{dr^2}\left(r\,g_l(r,r')\right) - \frac{l(l+1)}{r^2}g_l(r,r') = -\frac{4\pi}{r^2}\delta(r-r')$$

$$g_l(r,r') = \begin{cases} A_l r^l + \dfrac{B_l}{r^{l+1}} & r<r' \\[2mm] A_l' r^l + \dfrac{B_l'}{r^{l+1}} & r>r' \end{cases}$$

$g_l(r,r') = g_l(r',r)$

A_l, A_l', B_l, B_l' to be determined by B.C. and symmetry of $g_l(r,r')$

Consider two concentric spherical shells



$*\ g_\ell(r, r') = 0$ for $r = a \,\&\, b$

$$A_\ell a^\ell + \frac{B_\ell}{a^{\ell+1}} = 0 \Rightarrow B_\ell = -A_\ell a^{2\ell+1}$$

$$g_\ell = \begin{cases} A(r')\left(r^\ell - \dfrac{a^{2\ell+1}}{r^{\ell+1}}\right) & r < r' \\[2ex] B(r')\left(\dfrac{1}{r^{\ell+1}} - \dfrac{r^\ell}{b^{2\ell+1}}\right) & r > r' \end{cases}$$

$*\ g_\ell$ is symmetric in $r \,\&\, r'$

$$g_\ell(r', r) = \begin{cases} C(r)\left(r'^\ell - \dfrac{a^{2\ell+1}}{(r')^{\ell+1}}\right) & r' < r \\[2ex] D(r)\left(\dfrac{1}{r^{\ell+1}} - \dfrac{r'^\ell}{b^{2\ell+1}}\right) & r' > r \end{cases}$$

$$\Rightarrow g_\ell = E\left(r_<^\ell - \frac{a^{2\ell+1}}{r_<^{\ell+1}}\right)\left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}}\right)$$

To evaluate E

$$\int_{r'-\epsilon}^{r'+\epsilon} dr\ r\left[\frac{1}{r}\frac{d^2}{dr^2}(r g_\ell) - \frac{\ell(\ell+1)}{r^2} g_\ell\right] = \int_{r'-\epsilon}^{r'+\epsilon} dr\left(-\frac{4\pi}{r}\right)\delta(r-r')\, r$$

$\epsilon \to 0,$

$$\underbrace{\frac{d}{dr} r g_\ell \Big|_{r'-\epsilon}^{r'+\epsilon}} - \cancelto{0}{\int_{r'-\epsilon}^{r'+\epsilon} dr\, \frac{g_\ell}{r}} = -\frac{4\pi}{r'}$$

$$\frac{d}{dr}(r g_\ell)\Big|_{r'+\epsilon} - \frac{d}{dr}(r g_\ell)\Big|_{r'-\epsilon} \qquad \begin{matrix} r' = r_< \\ r = r_> \end{matrix} \qquad \begin{matrix} r' = r_> \\ r = r_< \end{matrix}$$

$$\Rightarrow E = \frac{1}{2\ell+1}\ \frac{4\pi}{1 - \left(\frac{a}{b}\right)^{2\ell+1}}$$

$$\Rightarrow G(\vec{x},\vec{x}') = 4\pi \sum_{l}\sum_{m} \frac{Y_{lm} Y_{lm}^{*}}{(2l+1)\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]} \left(r_<^{l} - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^{l}}{b^{2l+1}}\right)$$

i) $a \to 0$, $b \to \infty$ no boundary

$$G(\vec{x},\vec{x}') = 4\pi \sum\sum \frac{1}{2l+1} \frac{r_<^{l}}{r_>^{l+1}} Y_{lm} Y_{lm}^{*} = \frac{1}{|\vec{x}-\vec{x}'|}$$

ii) $b \to \infty$ outside the spherical surface.

$$G(\vec{x},\vec{x}') = 4\pi \sum\sum \frac{1}{2l+1} \frac{1}{r_>^{l+1}} \left(r_<^{l} - \frac{a^{2l+1}}{r_<^{l+1}}\right) Y_{lm} Y_{lm}^{*}$$

$$= \frac{1}{|\vec{x}-\vec{x}'|} - \frac{a}{r'\left|\vec{x} - \frac{a^2}{r'^2}\vec{x}'\right|}$$

iii) $a \to 0$ inside the spherical surface

$$G(\vec{x},\vec{x}') = 4\pi \sum\sum \frac{1}{2l+1} Y_{lm} Y_{lm}^{*} \, r_<^{l} \left(\frac{1}{r_>^{l+1}} - \frac{r_>^{l}}{b^{2l+1}}\right)$$