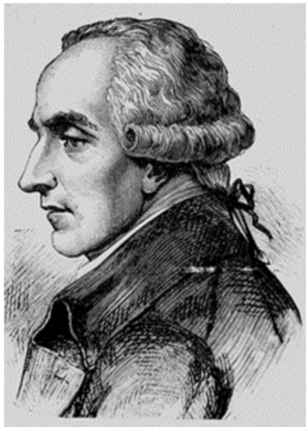
Chapter 4
INTEGRAL TRANSFORMS

Joseph Fourier
(1768-1830)
Math/Physics
Fourier Series/Transform

Lecture 14

4.2 Fourier Transform



Pierre-Simon Laplace
(1749-1827)
Math/Physic
Laplace Transform
Laplace Equation
(Scalar Potential Theory)

Unilateral Fourier Transform (Fourier-Laplace Transform)

In physics problems, for instance, the **Kubo formalism**, the causality require a unilateral FT or a Fourier-Laplace transform of $f(t)$:

convolution for frequency domain.

$$F_L(\omega) = \int_0^\infty dt\, f(t) e^{i\omega t} = \int_{-\infty}^\infty dt\, u(t) f(t) e^{i\omega t} \tag{4.37}$$

then we have

$$\boxed{F_L(\omega) = \frac{1}{2} F(\omega) + \frac{i}{2\pi} P \int_{-\infty}^\infty d\omega' \frac{F(\omega - \omega')}{\omega'}} \tag{4.38}$$

where the ordinary bilateral FT is given by

$$F(\omega) = \int_{-\infty}^\infty dt\, f(t) e^{i\omega t} \tag{4.39}$$

Proof) The inverse FTs are given by

$$\begin{aligned} u(t) &= \frac{1}{2\pi} \int_{-\infty}^\infty d\omega\, U(\omega) e^{-i\omega t} \\ f(t) &= \frac{1}{2\pi} \int_{-\infty}^\infty d\omega\, F(\omega) e^{-i\omega t} \end{aligned} \tag{4.40}$$

Substituting (4.40) into (4.37), we have

$$F_L(\omega) = \int_{-\infty}^{\infty} dt \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega' U(\omega') e^{-i\omega' t} \right) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega'' F(\omega'') e^{-i\omega'' t} \right) e^{i\omega t}$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} d\omega'' U(\omega') F(\omega'') \int_{-\infty}^{\infty} dt\, e^{i(\omega - \omega' - \omega'')t} \leftarrow 2\pi \text{ for normalize } \delta$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} d\omega'' U(\omega') F(\omega'') \delta(\omega - \omega' - \omega'') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega'\, U(\omega') F(\omega - \omega')$$

Using the FT of the unit step function as given in (4.14),

$$U(\omega') = \frac{i}{\omega' + i0^+} = \pi\delta(\omega') + iPV \frac{1}{\omega'}$$

we obtain

$$F_L(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega' \left(\pi\delta(\omega') + iPV \frac{1}{\omega'} \right) F(\omega - \omega')$$

$$= \frac{1}{2} F(\omega) + \frac{i}{2\pi} PV \int_{-\infty}^{\infty} d\omega' \frac{F(\omega - \omega')}{\omega'}$$

$$\int_{-\infty}^{\infty} dt = \underbrace{\int_{-\infty}^{0} dt}_{F_L^-} + \underbrace{\int_{0}^{\infty} dt}_{F_L^+}$$

E4.1 Maxwell's Equations in Spatial-Time and Spectral Domains

Differential Vector Equations

$$\nabla\times\mathbf{E}(\mathbf{r},t)=\partial_t\mathbf{B}(\mathbf{r},t)$$
$$\nabla\times\mathbf{H}(\mathbf{r},t)=\mathbf{J}(\mathbf{r},t)+\partial_t\mathbf{D}(\mathbf{r},t)$$
$$\nabla\cdot\mathbf{D}(\mathbf{r},t)=\rho(\mathbf{r},t)$$
$$\nabla\cdot\mathbf{B}(\mathbf{r},t)=0$$
$$\mathbf{F}(\mathbf{r},t)=q\left[\mathbf{E}(\mathbf{r},t)+\mathbf{v}(t)\times\mathbf{B}(\mathbf{r},t)\right]$$

FT

Algebraic Vector Equations

$$\mathbf{k}\times\mathbf{E}(\mathbf{k},\omega)=-\omega\mathbf{B}(\mathbf{k},\omega)$$
$$\mathbf{k}\times\mathbf{H}(\mathbf{k},\omega)=-i\mathbf{J}(\mathbf{k},\omega)-\omega\mathbf{D}(\mathbf{k},\omega)$$
$$\mathbf{k}\cdot\mathbf{D}(\mathbf{k},\omega)=-i\rho(\mathbf{k},\omega)$$
$$\mathbf{k}\cdot\mathbf{B}(\mathbf{k},\omega)=0$$
$$\mathbf{F}(\mathbf{k},\omega)=q\left[\mathbf{E}(\mathbf{k},\omega)+\mathbf{v}(\omega)\times\mathbf{B}(\mathbf{k},\omega)\right]$$

$$\mathbf{D}(\mathbf{k},\omega)=\boldsymbol{\varepsilon}(\mathbf{k},\omega)\cdot\mathbf{E}(\mathbf{k},\omega)$$
$$\mathbf{B}(\mathbf{k},\omega)=\boldsymbol{\mu}(\mathbf{k},\omega)\cdot\mathbf{H}(\mathbf{k},\omega)$$

$$D = \varepsilon \cdot E = \varepsilon_0 \cdot E + P$$
$$H = \mu^{-1} \cdot B = \mu_0^{-1} \cdot B - M$$

fundamental field.

IFT

double convolution (space, time)

$$\mathbf{D}(\mathbf{r},t)=\boldsymbol{\varepsilon}(\mathbf{r},t)*\mathbf{E}(\mathbf{r},t)$$
$$\mathbf{B}(\mathbf{r},t)=\boldsymbol{\mu}(\mathbf{r},t)*\mathbf{H}(\mathbf{r},t)$$

Review for Exam-2 (2017-05-02)