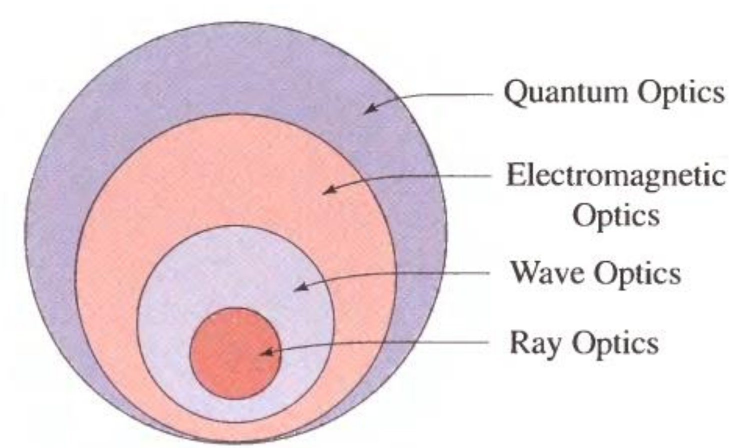
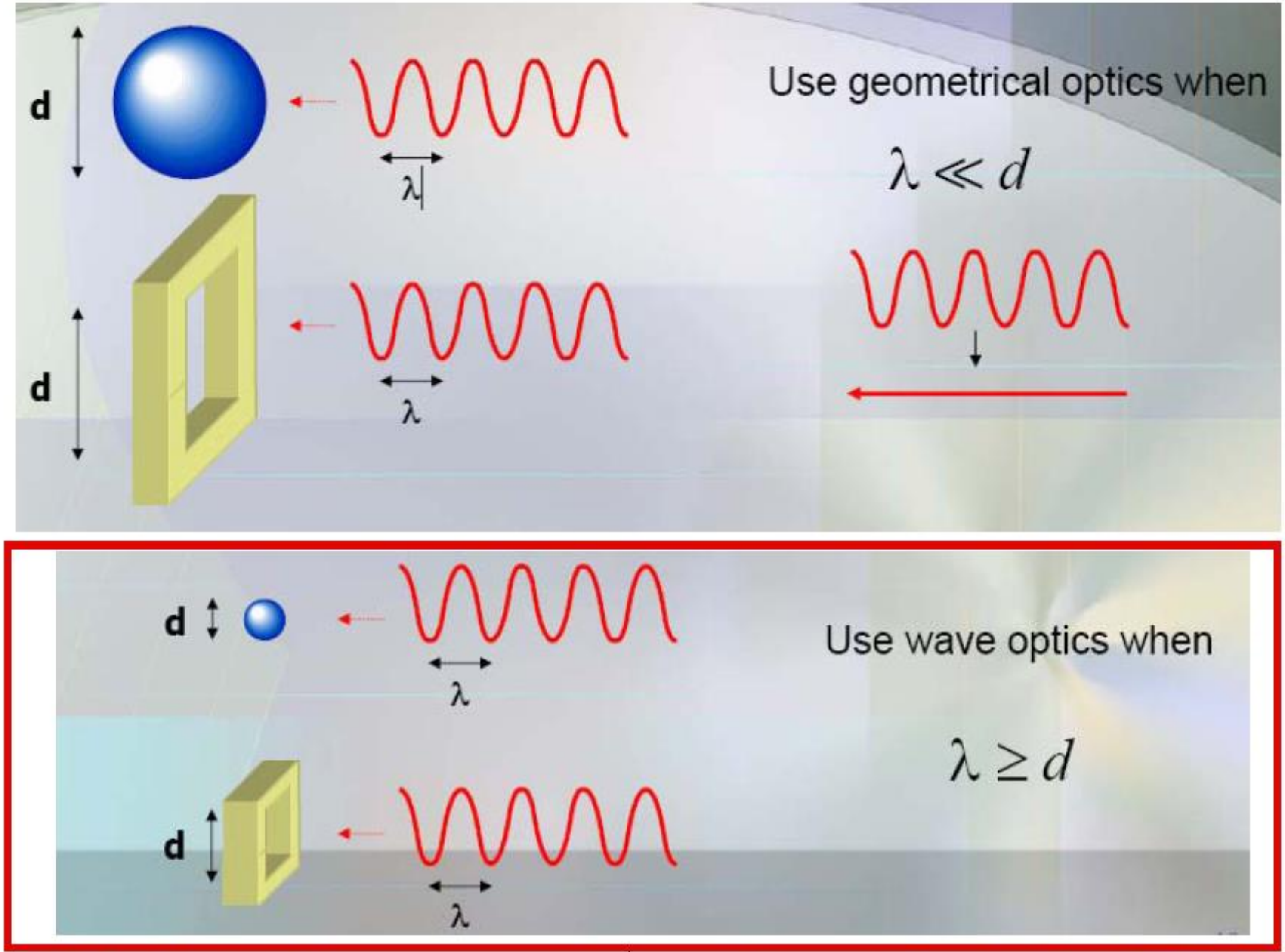
Chapter 2. Wave Optics



When do we use Wave Optics?



Lih Y. Lin, http://www.ee.washington.edu/people/faculty/lin_lih/EE485/

2-1. Postulates of Wave Optics

Wave Equation

$$\boxed{\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0,}$$

$$c = \frac{c_o}{n}$$

$$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$$

Because the wave equation is linear, the **principle of superposition** applies; i.e., if $u_1(\mathbf{r}, t)$ and $u_2(\mathbf{r}, t)$ represent optical waves, then $u(\mathbf{r}, t) = u_1(\mathbf{r}, t) + u_2(\mathbf{r}, t)$ also represents a possible optical wave.

Intensity, Power, and Energy

$$I(\mathbf{r}, t) = 2\langle u^2(\mathbf{r}, t)\rangle$$

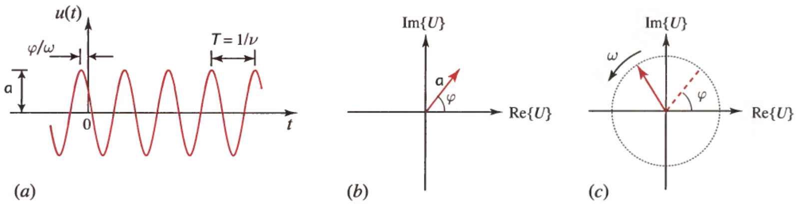
$$P(t) = \int_A I(\mathbf{r}, t)\, dA$$

The optical energy (units of joules) collected in a given time interval is the time integral of the optical power over the time interval.

A monochromatic wave is represented by a wavefunction with harmonic dence,

$$u(\mathbf{r},t) = \mathfrak{a}(\mathbf{r})\cos[2\pi\nu t + \varphi(\mathbf{r})]\,,$$

as illustrated in Fig. 2.2-1(a), where



$$\mathfrak{a}(\mathbf{r}) = \text{amplitude}$$
$$\varphi(\mathbf{r}) = \text{phase}$$
$$\nu = \text{frequency (cycles/s or Hz)}$$
$$\omega = 2\pi\nu = \text{angular frequency (radians/s or s}^{-1})$$
$$T = 1/\nu = 2\pi/\omega = \text{period (s)}.$$

Complex Representaion and the Helmholtz equation

It is convenient to represent the real wavefunction $u(\mathbf{r},t)$ in (2.2-1) in terms of a complex function

$$U(\mathbf{r},t) = \mathfrak{a}(\mathbf{r})\exp[j\varphi(\mathbf{r})]\exp(j2\pi\nu t)\,, \tag{2.2-2}$$

so that

$$u(\mathbf{r},t) = \mathrm{Re}\{U(\mathbf{r},t)\} = \tfrac{1}{2}[U(\mathbf{r},t) + U^*(\mathbf{r},t)]\,, \tag{2.2-3}$$

where the symbol * signifies complex conjugation. The function $U(\mathbf{r},t)$, known as the **complex wavefunction**, describes the wave completely; the wavefunction $u(\mathbf{r},t)$ is simply its real part. Like the wavefunction $u(\mathbf{r},t)$, the complex wavefunction $U(\mathbf{r},t)$ must also satisfy the wave equation

$$\boxed{\nabla^2 U - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = 0\,.} \tag{2.2-4}$$

Wave Equation

The two functions satisfy the same boundary conditions.

Helmholtz equation

Substituting $U(\mathbf{r}, t) = U(\mathbf{r}) \exp(j2\pi\nu t)$ into the wave equation

$$\boxed{(\nabla^2 + k^2)U(\mathbf{r}) = 0,} \qquad k = \frac{2\pi\nu}{c} = \frac{\omega}{c} \quad \text{(wavenumber)}$$

: Helmholtz equation

➡ ***"The wave equation for monochromatic waves"***

The optical intensity

$$I(\mathbf{r}) = |U(\mathbf{r})|^2$$

$$I(\mathbf{r}, t) = 2\langle u^2(\mathbf{r}, t)\rangle$$

$$\begin{aligned} 2u^2(\mathbf{r}, t) &= 2a^2(\mathbf{r}) \cos^2[2\pi\nu t + \varphi(r)] \\ &= |U(\mathbf{r})|^2\{1 + \cos(2[2\pi\nu t + \varphi(\mathbf{r})])\} \end{aligned}$$

The intensity of a monochromatic wave does not vary with time.

Gauss's law	$\nabla \cdot \mathbf{D} = \rho_V$
Faraday's law	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
No magnetic charges (Gauss's law for magnetism)	$\nabla \cdot \mathbf{B} = 0$
Ampère's law	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$

Maxwell equations in vacuum

$$\begin{cases} \nabla \cdot \vec{E} = \cancel{\frac{\rho_{free}}{\epsilon}} = 0 \\ \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \\ \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{H} = \cancel{J_{free}} + \varepsilon \frac{\partial \vec{E}}{\partial t} \end{cases}$$

$$\begin{cases} \nabla \cdot \vec{E} = 0 \\ \nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \\ \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \end{cases}$$

$$\begin{cases} \nabla \times (\nabla \times \vec{E}) = -\mu_0 \frac{\partial (\nabla \times \vec{H})}{\partial t} \\ \nabla \times \vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \end{cases}$$

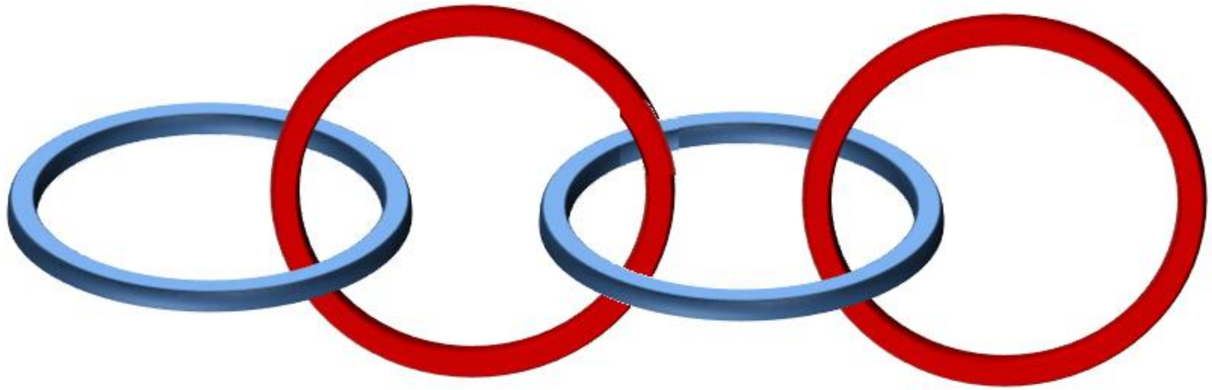
$$\boxed{\nabla \times (\nabla \times \vec{E}) = -\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial^2 t}}$$

$$\boxed{\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial^2 t}}$$

3D wave equation

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E}$$

$$\frac{1}{v^2} = \mu_0 \varepsilon_0$$



(예제 1)

$$\boxed{\begin{aligned} \frac{\partial E(x)}{\partial x} &= -B(x) \\[1em] \frac{\partial B(x)}{\partial x} &= +E(x) \end{aligned}}$$

$$\boxed{\frac{d^2E(x)}{dx^2} = -k^2E(x)}$$

$$E(x) = \cos(kx)$$

예제 2

$$\frac{1}{\partial x} \quad \boxed{\frac{\partial E(x,t)}{\partial x} = \frac{\partial B(x,t)}{\partial \mathrm{t}}}$$

$$\frac{1}{\partial \mathrm{t}} \quad \boxed{\frac{\partial B(x,t)}{\partial x} = \frac{\partial E(x,t)}{\partial \mathrm{t}}}$$

$$\boxed{\frac{\partial^2 E(x,t)}{\partial x^2} = G\,\frac{\partial^2 E(x,t)}{\partial t^2}}$$

$$E(x) = \cos(kx \pm \omega t)$$

$$G = \left(\frac{k}{\omega}\right)^2 = \left(\frac{1}{u}\right)^2$$

$$\begin{cases} u = \pm\dfrac{\omega}{\mathrm{k}} \\ \omega = \dfrac{2\pi}{T} = 2\pi f: \text{각속도} \\ k = \dfrac{2\pi}{\lambda}: \text{파수} \end{cases}$$

$$\frac{1}{u^2}\frac{\partial^2\psi}{\partial t^2}=\frac{\partial^2\psi}{\partial x^2}$$

Trial solution

$$\psi(x,t)=\mathrm{X}(x)\mathrm{T}(t)$$

$$\begin{cases} \mathrm{X}\dfrac{1}{u^2}\dfrac{\partial^2\mathrm{T}}{\partial t^2}=\mathrm{T}\dfrac{\partial^2\mathrm{X}}{\partial x^2} \ldots(1) \\ \dfrac{1}{\mathrm{T}}\dfrac{1}{u^2}\dfrac{\partial^2\mathrm{T}}{\partial t^2}=\dfrac{1}{\mathrm{X}}\dfrac{\partial^2\mathrm{X}}{\partial x^2} \ldots(2)=-k^2 \end{cases}$$

$$\begin{cases} \dfrac{1}{\mathrm{T}}\dfrac{1}{u^2}\dfrac{\partial^2\mathrm{T}}{\partial t^2}=-\omega^2 \quad \ldots(1) \\ \dfrac{1}{\mathrm{X}}\dfrac{\partial^2\mathrm{X}}{\partial x^2}=-k^2 \quad \ldots(2) \end{cases}$$

$$k^2u^2=\omega^2$$

$$\begin{cases} \dfrac{\partial^2\mathrm{T}}{\partial t^2}=-\omega^2\mathrm{T} \ldots(1) \\ \dfrac{\partial^2\mathrm{X}}{\partial x^2}=-k^2\mathrm{X} \ldots(2) \end{cases}$$

$$\psi(x,t)=\mathrm{X}(x)\mathrm{T}(t)=\cos(kx)\cos(\omega\mathrm{t})$$

cos(A+B) = cosAcosB - sinAsinB
sin(A+B) = sinAcosB + cosAsinB

$$\begin{aligned}\mathrm{T}(t)&=\mathrm{A}\cos(\omega\mathrm{t})+\mathrm{B}\sin(\omega\mathrm{t})\\ &=\left(\sqrt{A^2+\mathrm{B}^2}\right)\cos(\omega\mathrm{t}+\phi_0),\ \tan\phi_0=-\frac{B}{A}\\ &=A'\cos(\omega\mathrm{t}+\phi_0)\end{aligned}$$

$$\begin{aligned}\mathrm{X}(x)&=\mathrm{C}\cos(kx)+\mathrm{D}\sin(kx)\\ &=\left(\sqrt{C^2+\mathrm{D}^2}\right)\cos(kx+\phi_1),\ \tan\phi_0=-\frac{D}{C}\\ &=B'\cos(kx+\phi_1)\end{aligned}$$

복소해?

$$\mathrm{T}(t)=A'\exp(j(\omega\mathrm{t}+\phi_0))$$

$$\mathrm{X}(x)=B'\exp(j(kx+\phi_1))$$

$$\mathrm{T}(t)\mathrm{X}(x)=A'\exp(j(kx+\omega\mathrm{t}+\phi_2))$$

$$\frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \nabla^2\psi$$

$$\psi(x,y,t) = X(x)Y(y)T(t)$$

$$T'' + \omega^2 T = 0 \qquad \omega^2 = (k_x^2 + k_y^2)v^2$$

$$X'' + k_x^2 X = 0$$

$$Y'' + k_y^2 Y = 0$$

$$\mathbf{E}(x,y,z;t) = Re[\tilde{\mathbf{E}}(x,y,z)e^{j\omega t}]$$

$$\mathbf{H}(x,y,z;t) = Re[\widetilde{\mathbf{H}}(x,y,z)e^{j\omega t}]$$

$$\left\{ \begin{array}{c} \mathbf{E} \\ \mathbf{H} \end{array} \right\}$$

$$\nabla \cdot \tilde{\mathbf{E}} = \frac{\tilde{\boldsymbol{\rho}}_v}{\varepsilon}$$

$$\nabla \times \tilde{\mathbf{E}} = -\mathrm{j}\omega\mu\widetilde{\boldsymbol{H}}$$

$$\nabla \cdot \widetilde{\mathbf{H}} = 0$$

$$\nabla \times \widetilde{\mathbf{H}} = \tilde{\mathbf{J}} + \mathrm{j}\omega\varepsilon\tilde{\mathbf{E}}$$

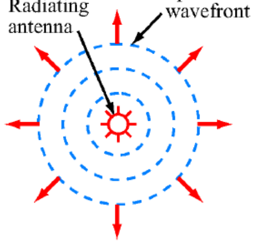
In materials,

$$\left\{\begin{aligned}
&\nabla \cdot \vec{\mathrm{E}} = \frac{\rho_{free}}{\epsilon} = 0 \\
&\nabla \times \vec{\mathrm{E}} = -\mu \frac{\partial \vec{\mathrm{H}}}{\partial \mathrm{t}} \\
&\nabla \cdot \vec{\mathrm{B}} = 0 \\
&\nabla \times \vec{\mathrm{H}} = \mathrm{J}_{\mathrm{free}} + \varepsilon \frac{\partial \vec{\mathrm{E}}}{\partial \mathrm{t}}
\end{aligned}\right.$$

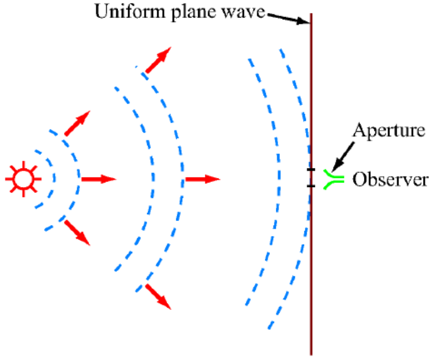
7.1 Time-Harmonic Fields

$$\mathbf{E}(x,y,z;t) = Re[\tilde{\mathbf{E}}(x,y,z)e^{j\omega t}]$$

$$\mathbf{H}(x,y,z;t) = Re[\tilde{\mathbf{H}}(x,y,z)e^{j\omega t}]$$



(a) Spherical wave



(b) Plane-wave approximation

$$\left\{ \begin{matrix} \mathbf{E} \\ \mathbf{H} \end{matrix} \right\}$$

$$\nabla \cdot \tilde{\mathbf{E}} = \frac{\tilde{\rho}_v}{\varepsilon}$$

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\boldsymbol{H}}$$

$$\nabla \cdot \tilde{\mathbf{H}} = 0$$

$$\nabla \times \tilde{\mathbf{H}} = \tilde{\mathbf{J}} + j\omega\varepsilon\tilde{\mathbf{E}}$$

7.1 Time-Harmonic Fields

$$\nabla \times \tilde{\mathbf{H}} = \tilde{\mathbf{J}} + j\omega\varepsilon\tilde{\mathbf{E}}$$

$$= (\boxed{} + j\omega\varepsilon)\tilde{\mathbf{E}} = j\omega(\boxed{\phantom{\varepsilon - j\frac{\sigma}{\omega}}})\tilde{\mathbf{E}}$$

complex permittivity $\varepsilon_c =$

$$\nabla \times \tilde{\mathbf{H}} = j\omega\varepsilon_c\tilde{\mathbf{E}}$$

$$\boxed{\phantom{\nabla \times \tilde{\mathbf{H}} = j\omega\varepsilon_c\tilde{\mathbf{E}}}} \quad \cdots \quad (1)$$

$$\nabla \cdot \nabla \times \tilde{\mathbf{H}} = 0 \quad \rightarrow \nabla \cdot \left(j\omega\varepsilon_c\tilde{\mathbf{E}}\right) = 0 \text{ or } \nabla \cdot \tilde{\mathbf{E}} \rightarrow \widetilde{\rho_v} = 0$$

$$\nabla \cdot \tilde{\mathbf{E}} = \frac{\tilde{\boldsymbol{\rho}}_v}{\varepsilon}$$
$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\widetilde{\boldsymbol{H}}$$
$$\nabla \cdot \tilde{\mathbf{H}} = 0$$
$$\nabla \times \tilde{\mathbf{H}} = \tilde{\mathbf{J}} + j\omega\varepsilon\tilde{\mathbf{E}}$$

$\Rightarrow$

$$\nabla \cdot \tilde{\mathbf{E}} =$$
$$\nabla \times \tilde{\mathbf{E}} =$$
$$\nabla \cdot \tilde{\mathbf{H}} =$$
$$\nabla \times \tilde{\mathbf{H}} =$$

7.1 Time-Harmonic Fields

$$\nabla \cdot \tilde{\mathbf{E}} = 0$$
$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\boldsymbol{H}}$$
$$\nabla \cdot \tilde{\mathbf{H}} = 0$$
$$\nabla \times \tilde{\mathbf{H}} = j\omega\varepsilon_c\tilde{\mathbf{E}}$$

$$\nabla \times \left(\nabla \times \tilde{\mathbf{E}}\right) =$$
$$\nabla \times \left(\nabla \times \tilde{\mathbf{E}}\right) =$$
$$\nabla \times \left(\nabla \times \tilde{\mathbf{E}}\right) =$$

H.W

$$\nabla^2\tilde{\mathbf{E}} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}\right)\tilde{\mathbf{E}}$$

∇^2: Laplacian operator

$$\nabla^2\tilde{\mathbf{E}} + \omega^2\mu\varepsilon_c\tilde{\mathbf{E}} = 0$$

$$\gamma_c^2 = -\omega^2\mu\varepsilon_c$$

Propagation constant

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$\nabla \times (\nabla \times \mathbf{A})$

$$(\nabla \times \mathbf{A}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ \partial_y A_z - \partial_z A_y & \partial_z A_x - \partial_x A_z & \partial_x A_y - \partial_y A_x \end{vmatrix}$$

$$\big(\nabla \times (\nabla \times \mathbf{A})\big)_x = \partial_y \partial_x A_y - \partial_y^2 A_x - \partial_z^2 A_x + \partial_z \partial_x A_z$$

$$\nabla(\nabla \cdot \mathbf{A}) \quad = (\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z) \cdot (\partial_x A_x + \partial_y A_y + \partial_z A_z)$$

$$-\nabla^2 \mathbf{A} \; = -(\partial_x^2 + \partial_y^2 + \partial_z^2) \cdot (\hat{x} A_x + \hat{y} A_y + \hat{z} A_z)$$

$$(\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A})_x = \cancel{\partial_x^2 A_x} + \partial_x \partial_y A_y + \partial_x \partial_z A_z$$

$$\cancel{-\partial_x^2 A_x} - \partial_y^2 A_x - \partial_z^2 A_x$$

7.2 Plane-wave propagation in lossless media

Homogenous wave equation

$$\nabla^2\tilde{\mathbf{E}} + \omega^2\mu\varepsilon_c\tilde{\mathbf{E}} = 0$$

$$\gamma_c^2 = -\omega^2\mu\varepsilon_c$$

Propagation constant

(Non conductor medium)

$$\sigma = 0, \varepsilon_c = \epsilon$$

$$\gamma^2 = -\omega^2\mu\varepsilon_c = -k^2 \ \ (real)$$

k: wave number

$$\nabla^2\tilde{\mathbf{E}} - \gamma_c^2\tilde{\mathbf{E}} = 0$$

$$\tilde{\mathbf{E}} = \hat{\boldsymbol{x}}\tilde{E}_x + \hat{\boldsymbol{y}}\tilde{E}_y + \hat{\mathbf{z}}\tilde{E}_z$$

$$\tilde{\mathbf{E}} = \hat{\boldsymbol{x}}\tilde{E}_x(x,y,z) + \hat{\boldsymbol{y}}\tilde{E}_y\,(x,y,z) + \hat{\mathbf{z}}\tilde{E}_z\,(x,y,z)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(\hat{\boldsymbol{x}}\tilde{E}_x + \hat{\boldsymbol{y}}\tilde{E}_y + \hat{\mathbf{z}}\tilde{E}_z) + k^2(\hat{\boldsymbol{x}}\tilde{E}_x + \hat{\boldsymbol{y}}\tilde{E}_y + \hat{\mathbf{z}}\tilde{E}_z) = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\tilde{E}_x = 0, \quad can\ apply\ to\ \tilde{E}_y, \tilde{E}_z$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)\tilde{H}_x = 0, \quad \tilde{H}_y, \tilde{H}_z$$

6개의 **Helmholtz equations**

헬름홀츠 방정식

$$\nabla^2 f(..) + k^2 f(..) = 0$$

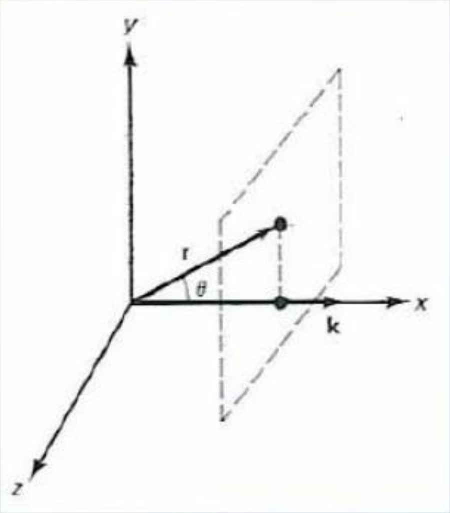
$$\nabla^2 \tilde{\mathbf{E}} - \gamma_c^2 \tilde{\mathbf{E}} = 0$$
$$\nabla^2 \widetilde{\mathbf{H}} - \gamma_c^2 \widetilde{\mathbf{H}} = 0$$

6개의 **Helmholtz equations**

헬름홀츠 방정식

Elementary waves of Helmholtz eq.

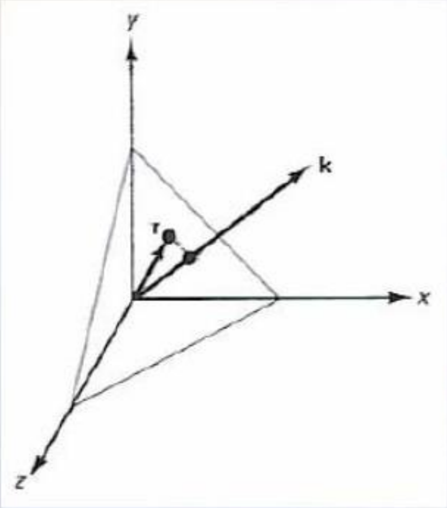
Plane wave in +x-direction



$$\psi = Ae^{i(kr\cos\theta - \omega t)}$$

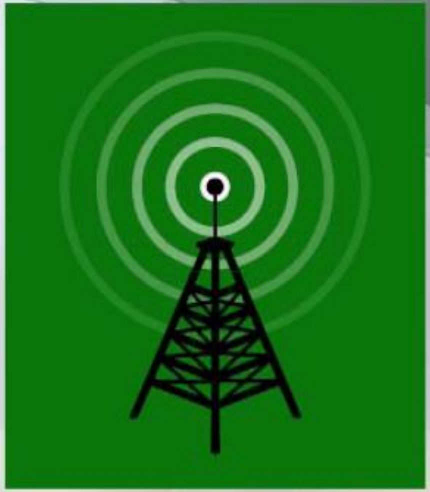
Define **k** to represent the propagation constant and direction.

Plane wave in any direction



$$\psi = Ae^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

Spherical wave



$$\psi = \frac{A}{r}e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

Intensity (W/m^2) $\propto \left(\frac{A}{r}\right)^2$

Energy conservation obeyed

A plane wave

Elements of Photonics, Volume I: In Free Space and Special Media. Keigo Iizuka

ISBNs: 0-471-83938-8 (Hardback); 0-471-22107-4 (Electronic)

$E = E_0 e^{j(-\omega t+\beta z)}$ or $E = E_0 e^{j(\omega t-\beta z)}$, the waves are forward waves;

$E = E_0 e^{j(\omega t+\beta z)}$ or $E = E_0 e^{j(-\omega t-\beta z)}$, the waves are backward waves.

$$E = \mathrm{Re}\left[E_0 e^{j(-\omega t+\beta z)}\right] = E_0 \cos(-\omega t + \beta z)$$

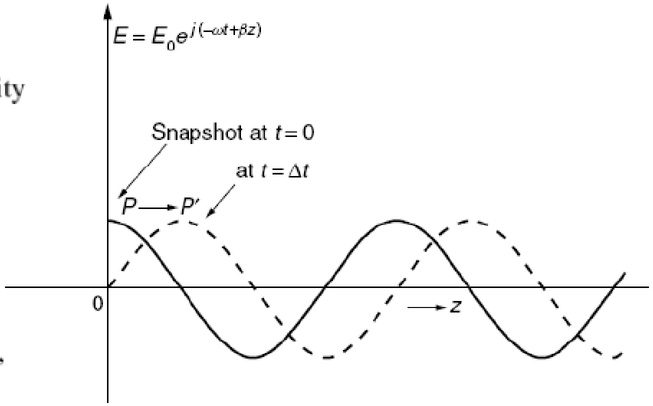
The peak has moved to a positive new location at $\Delta z = (\omega/\beta)\Delta t$.
Thus, this equation represents a forward wave with the phase velocity

$$v_p = \omega/\beta$$

On the other hand, as time increases with

$$E = E_0 \cos(\omega t + \beta z)$$

the peak moves toward the negative z direction, $\Delta z = -(\omega/\beta)\Delta t$, and this represents the backward wave.



$E = E_0\, e^{j(-\omega t+\beta z)}$ is a forward wave.

In this book the convention of $e^{-j\omega t}$ is used, unless otherwise stated, because the forward wave $E = E_0 e^{j(-\omega t+kz)}$ has a positive sign on the z.

		Rectangular Coordinates	Cylindrical Coordinates	Spherical Coordinates
Metric coefficients	h_1	1	1	1
	h_2	1	r	R
	h_3	1	1	$R\sin\theta$

$$\nabla = \hat{\mathbf{u}}_1 \frac{\partial}{h_1 \partial u_1} + \hat{\mathbf{u}}_2 \frac{\partial}{h_2 \partial u_2} + \hat{\mathbf{u}}_3 \frac{\partial}{h_3 \partial u_3}$$

$$\left\{ \begin{aligned} \nabla T &= \hat{\mathbf{r}} \frac{\partial T}{\partial r} + \hat{\boldsymbol{\phi}} \frac{1}{r} \frac{\partial T}{\partial \phi} + \hat{\mathbf{z}} \frac{\partial T}{\partial z} \\ \nabla T &= \hat{\mathbf{R}} \frac{\partial T}{\partial R} + \hat{\boldsymbol{\theta}} \frac{1}{R} \frac{\partial T}{\partial \theta} + \hat{\boldsymbol{\phi}} \frac{1}{R\sin\theta} \frac{\partial T}{\partial \phi} \end{aligned} \right.$$

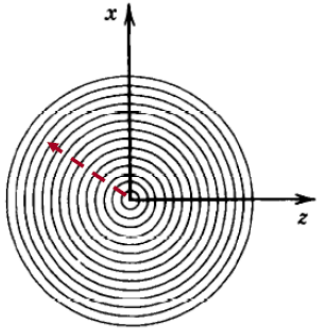
$$\nabla \bullet \mathbf{E} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1}(h_2 h_3 E_1) + \frac{\partial}{\partial u_2}(h_1 h_3 E_2) + \frac{\partial}{\partial u_3}(h_1 h_2 E_3) \right]$$

$$\nabla \times \mathbf{B} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{\mathbf{u}}_1 h_1 & \hat{\mathbf{u}}_2 h_2 & \hat{\mathbf{u}}_3 h_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 B_1 & h_2 B_2 & h_3 B_3 \end{vmatrix}$$

Spherical Wave : $\boxed{U(\mathbf{r}) = \frac{A}{r}\exp(-jkr)}$

$I(\mathbf{r}) = |A|^2/r^2$ is inversely proportional to the square of the distance

the wavefronts are the surfaces $kr = 2\pi q$ or $r = q\lambda$, where q is an integer.



$U(\mathbf{r}) = (A/r)\exp(+jkr)$ is a spherical wave traveling inwardly

$$\nabla^2 = \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y} + \frac{\partial^2}{\partial^2 z}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial^2\phi}$$

Fresnel Approximation of the Spherical Wave; Paraboloidal Wave

$$r = (x^2 + y^2 + z^2)^{1/2} = z(1 + \theta^2)^{1/2} = z\left(1 + \frac{\theta^2}{2} - \frac{\theta^4}{8} + \cdots\right)$$

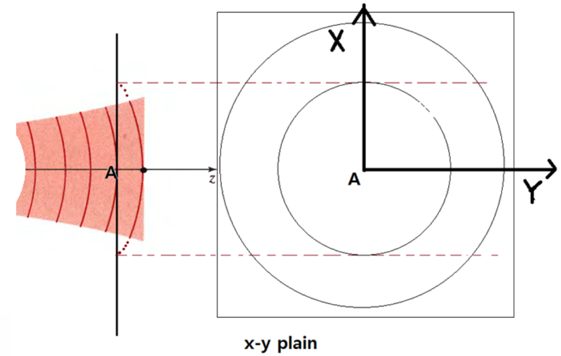
$$\theta^2 = (x^2 + y^2)/z^2 \ll 1$$

$$U(\mathbf{r}) = \frac{A}{r}\exp(-jkr)$$

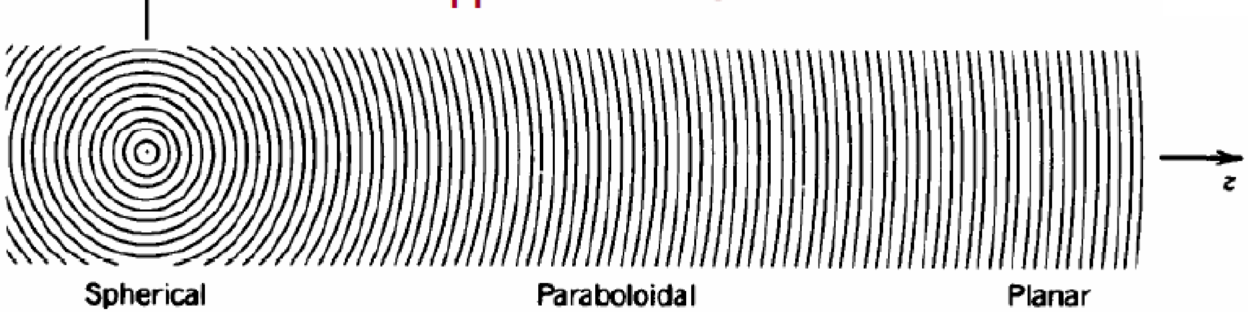
Substituting $r = z + (x^2 + y^2)/2z$ **into the phase,**

and $r = z$ **into the magnitude of**

$$\boxed{U(\mathbf{r}) \approx \frac{A}{z}\exp(-jkz)\exp\left[-jk\frac{x^2 + y^2}{2z}\right].}$$



Fresnel Approximation → Paraboloidal Wave



Fresnel Approximation is valid when

$$kz\theta^4/8 \ll \pi, \text{ or } (x^2+y^2)^2 \ll 4z^3\lambda. \longleftarrow z\left(1+\frac{\theta^2}{2}-\frac{\theta^4}{8}+\cdots\right)$$

$$\theta^2 = (x^2+y^2)/z^2$$

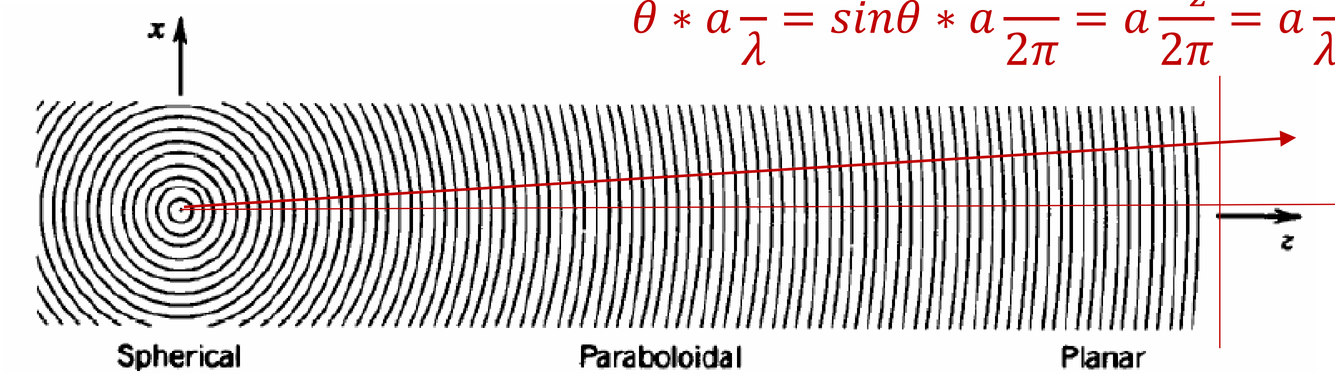
$$(x^2+y^2) = a^2 \Rightarrow a^4 \ll 4z^3\lambda$$

$$\boxed{\frac{N_F\theta_m^2}{4} \ll 1,}$$

where $\theta_m = a/z$ is the maximum angle

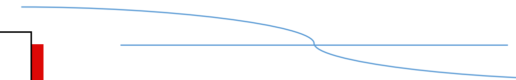
$N_F = \frac{a^2}{\lambda z}$ is known as the **Fresnel number**.

$$\theta * a\frac{1}{\lambda} = sin\theta * a\frac{k}{2\pi} = a\frac{k_z}{2\pi} = a\frac{1}{\lambda_z}$$



Slow varying?

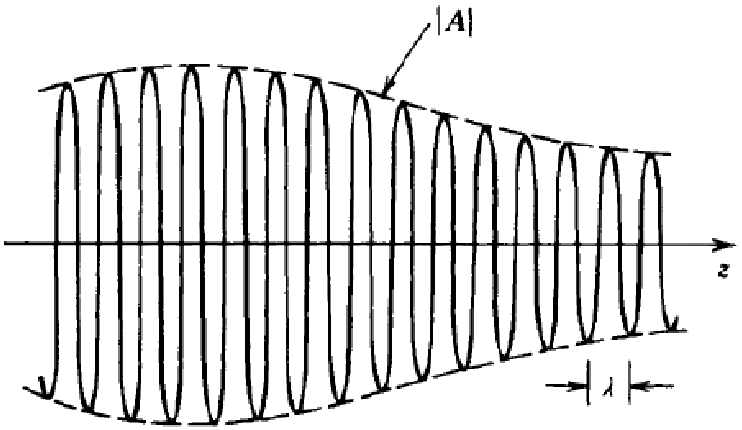
Gradient?

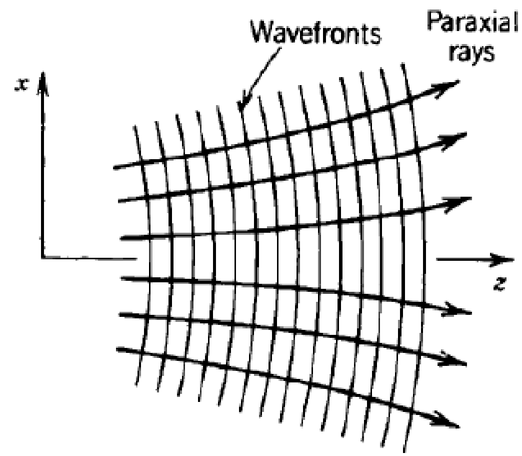


Paraxial waves

Original signal
Small itself?

A wave is said to be paraxial if its wavefront normals are paraxial rays.





$$\boxed{U(\mathbf{r}) = A(\mathbf{r}) \exp(-jkz)}$$

$A(\mathbf{r})$ varies slowly with respect to z

the change ΔA is much smaller than A itself; $\Delta A \ll A$

Paraxial Helmholtz equation

$$U(\mathbf{r}) = A(\mathbf{r}) \exp(-jkz)$$

$$\boxed{\Delta A \ll A}$$

$$\Delta A = (\partial A/\partial z)\,\Delta z = (\partial A/\partial z)\lambda,$$

$$\partial A/\partial z \ll A/\lambda = Ak/2\pi$$

$$\frac{\partial A}{\partial z} \ll kA$$

Similarly, the derivative $\partial A/\partial z$ varies slowly within the distance λ,

$$\partial^2 A/\partial^2 z \ll k\,\partial A/\partial z, \longrightarrow \frac{\partial^2 A}{\partial z^2} \ll k^2 A$$

$$\boxed{\nabla_T^2 A - j2k\frac{\partial A}{\partial z} = 0,} \longleftarrow (\nabla^2 + k^2)U(\mathbf{r}) = 0$$

➔Slowly varying envelope approximation of the Helmholtz equation

➔ Paraxial Helmholtz equation.

EXERCISE 2.2-2 The Paraboloidal Wave and the Gaussian Beam.

기울기 정의식

$$\Delta A = \left(\frac{\partial A}{\partial z}\right)\Delta z = \left(\frac{\partial A}{\partial z}\right)\lambda$$

$$\frac{\Delta A}{\lambda} = \left(\frac{\partial A}{\partial z}\right)$$

Slow varying assumption

$$\Delta z \sim \lambda$$

$$\Delta A \ll A$$

$$\Delta A \ll A$$

$$\frac{\Delta A}{\lambda} \ll \frac{A}{\lambda}$$

$$\left(\frac{\partial A}{\partial z}\right) \ll \frac{A}{\lambda}$$

$$\left(\frac{\partial A}{\partial z}\right) \ll \frac{Ak}{2\pi} \sim Ak$$

$$\left(\frac{\partial^2 A}{\partial^2 z}\right) \ll k\left(\frac{\partial A}{\partial z}\right)$$

$$\boxed{k = \frac{2\pi}{\lambda}}$$

Relation between wave optics and ray optics

Phase

$$U(\mathbf{r}) = a(\mathbf{r}) \exp[-jk_o S(\mathbf{r})] \longrightarrow (\nabla^2 + k^2)U(\mathbf{r}) = 0$$

$$\Rightarrow k_o^2\left[n^2 - |\nabla S|^2\right]a + \nabla^2 a - jk_o[2\nabla S \cdot \nabla a + a\nabla^2 S] = 0,$$

Equating the real part to zero $\longrightarrow |\nabla S|^2 = n^2 + \left(\frac{\lambda_o}{2\pi}\right)^2 \nabla^2 a / a$

a varies slowly over the distance λ_o means that $\lambda_o^2 \nabla^2 a / a \ll 1$, in the limit $\lambda_o \to 0$

$$\Rightarrow \boxed{|\nabla S|^2 \approx n^2.}$$

Eikonal Equation

- The scalar function $S(\mathbf{r})$ is the eikonal of ray optics.
- The eikonal equation is the limit of the Helmholtz equation when $\lambda_o \to 0$.
- Fermat's principle can be derived from the eikonal equation.

$$\longrightarrow \frac{d}{ds}\left(n \frac{d\mathbf{r}}{ds}\right) = \nabla n$$

: Ray equation can be also derived

2-4. Simple optical components

Reflection from a Planar Mirror

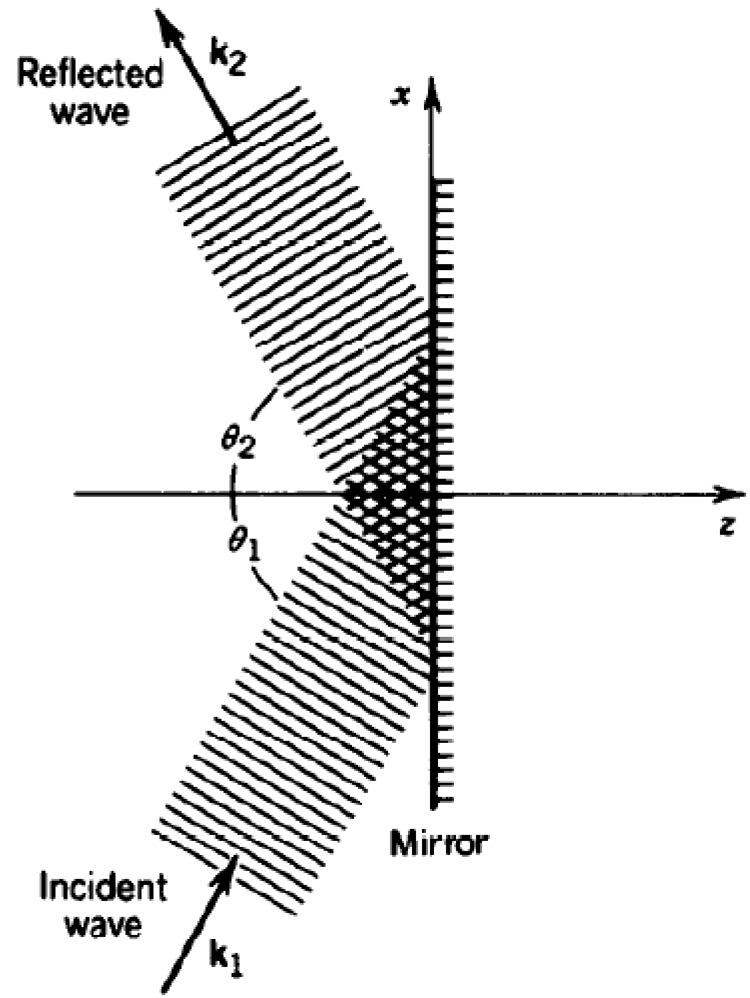
At the boundary,
the wavefronts of the two waves match,
i.e., the phase must be equal,

$\Rightarrow$ $\mathbf{k}_1 \cdot \mathbf{r} = \mathbf{k}_2 \cdot \mathbf{r}$ for all $\mathbf{r} = (x, y, 0)$

$$\mathbf{r} = (x, y, 0),$$
$$\mathbf{k}_1 = (k_o \sin\theta_1, 0, k_o \cos\theta_1),$$
$$\mathbf{k}_2 = (k_o \sin\theta_2, 0, -k_o \cos\theta_2)$$

$\Rightarrow$ $k_o \sin(\theta_1) x = k_o \sin(\theta_2) x$

$\Rightarrow$ $\theta_1 = \theta_2$

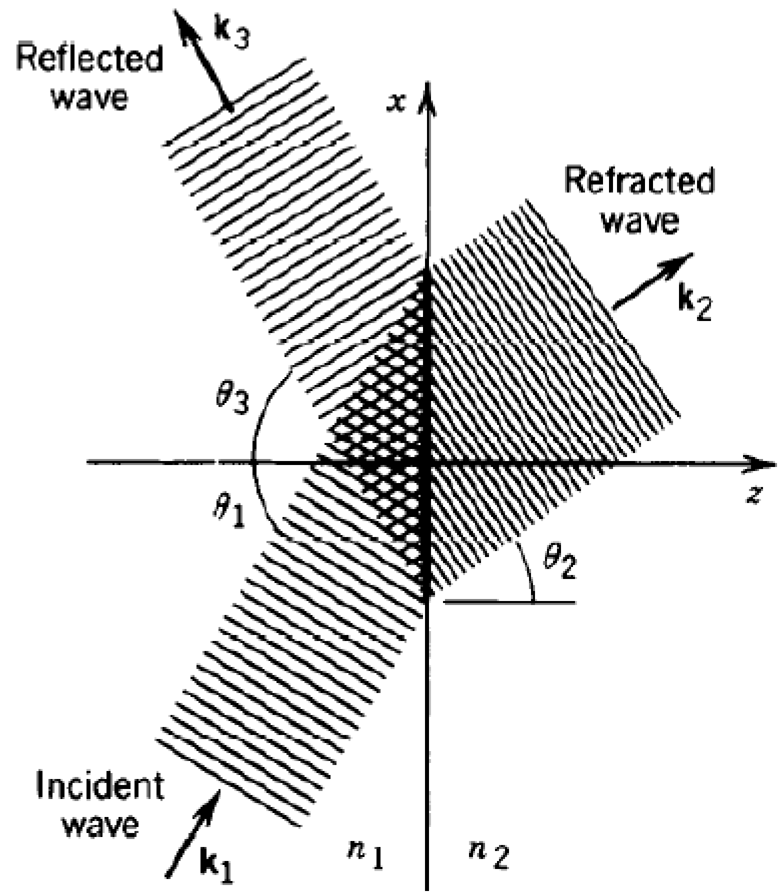


Reflection and refraction at a planar dielectric boundary

$$\mathbf{k}_1 \cdot \mathbf{r} = \mathbf{k}_2 \cdot \mathbf{r} = \mathbf{k}_3 \cdot \mathbf{r} \qquad \text{for all } \mathbf{r} = (x, y, 0)$$

➡ $\theta_1 = \theta_3$

➡ $n_1 \sin\theta_1 = n_2 \sin\theta_2$



BOUNDARY CONDITIONS

Wavelength (phase) matching at the boundary = Snell's law

Suppose that at a particular instance and at a particular location of the boundary,
the oscillation of the incident wave is at its maximum;
then both reflected and transmitted waves have to be at their maxima.

In other words,
the wavelengths along the interface surface
must have the same temporal and spatial variation.

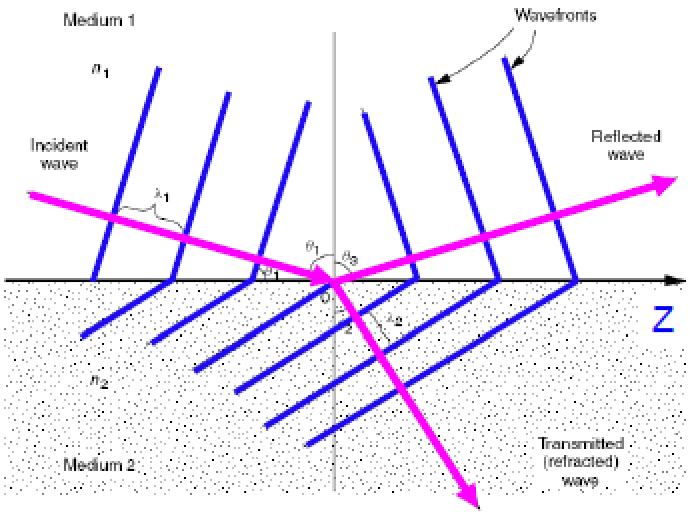


Figure 2.1 Wavefront and ray directions near the boundary.

$$\lambda_{z1} = \lambda_{z2} = \lambda_{z3}$$

→ Propagation constant : $\beta_i = \dfrac{2\pi}{\lambda_{zi}} = \text{constant}$

→ Called also as β, k, phase, or momentum matching

→ But all mean the same thing: **wavelength matching at the boundary!**

$$\frac{\lambda_1}{\sin\theta_1} = \frac{\lambda_1}{\sin\theta_3} = \frac{\lambda_2}{\sin\theta_2}$$ → **Snell's law :** $n_1 \sin\theta_1 = n_2 \sin\theta_2$

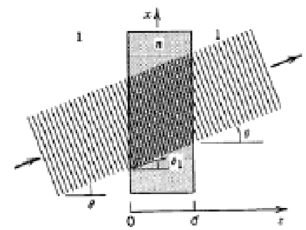
$$t(x,y) = U(x,y,d)/U(x,y,0)$$

B. Transmission Through Optical Components

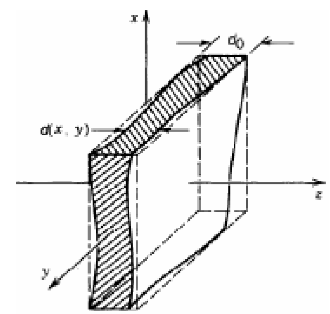
Transmission Through a Transparent Plate

$$t(x, y) = \exp[-jnk_o(d\cos\theta_1 + x\sin\theta_1)]$$

Complex Amplitude Transmittance of a Transparent Plate



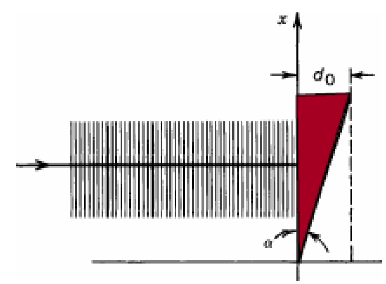
Thin Transparent Plate of Varying Thickness



$$t(x, y) \approx h_0 \exp[-j(n-1)k_o d(x, y)]$$

Transmission Through a Prism

$$t(x, y) = h_0 \exp[-j(n-1)k_o \alpha x], \text{ where } h_0 = \exp(-jk_o d_0)$$



$$d(x,y) = d_0 - \left[R - \sqrt{R^2 - (x^2 + y^2)}\right].$$

$$\sqrt{R^2 - (x^2 + y^2)} = R\sqrt{1 - \frac{x^2 + y^2}{R^2}} \approx R\left(1 - \frac{x^2 + y^2}{2R^2}\right)$$

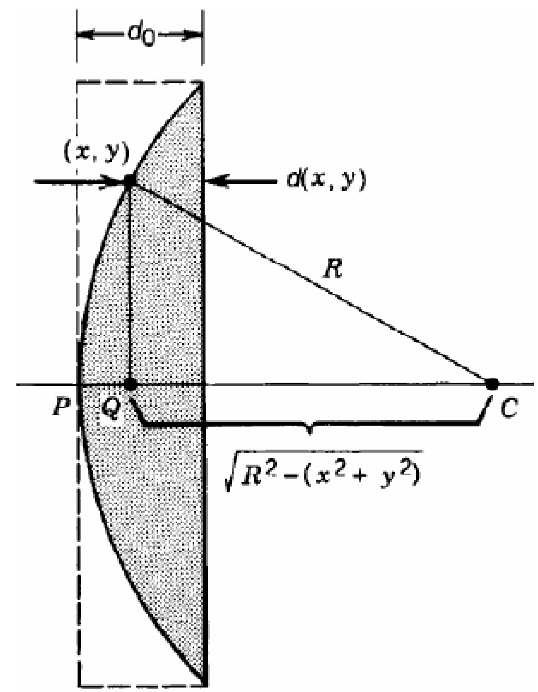
$$d(x,y) \approx d_0 - \frac{x^2 + y^2}{2R}.$$

$$t(x,y) \approx h_0 \exp[-j(n-1)k_o d(x,y)]$$

$$\boxed{t(x,y) \approx h_0 \exp\left[jk_o \frac{x^2 + y^2}{2f}\right],}$$

$$f = \frac{R}{n-1}$$

$$h_0 = \exp(-jnk_o d_0)$$



EXERCISE 2.4-3

Focusing of a Plane Wave by a Thin Lens. Show that when a plane wave is transmitted through a thin lens of focal length f in a direction parallel to the axis of the lens, it is converted into a paraboloidal wave (the Fresnel approximation of a spherical wave) centered about a point at a distance f from the lens, as illustrated in Fig. 2.4-9. What is the effect of the lens on a plane wave incident at a small angle θ?

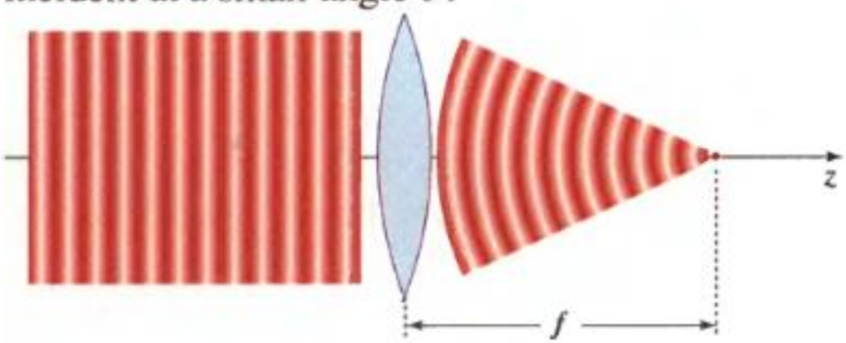


Figure 2.4-9 A thin lens transforms a plane wave into a paraboloidal wave.

2.5 INTERFERENCE

$$U(\mathbf{r}) = U_1(\mathbf{r}) + U_2(\mathbf{r})$$

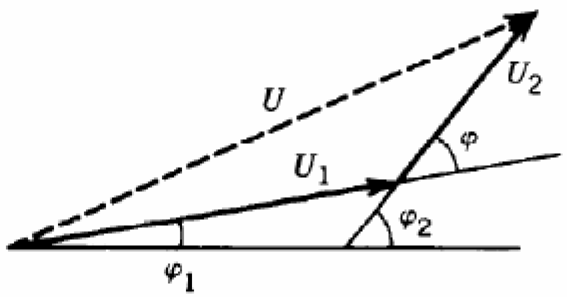
$$I = |U|^2 = |U_1 + U_2|^2 = |U_1|^2 + |U_2|^2 + U_1^* U_2 + U_1 U_2^*$$

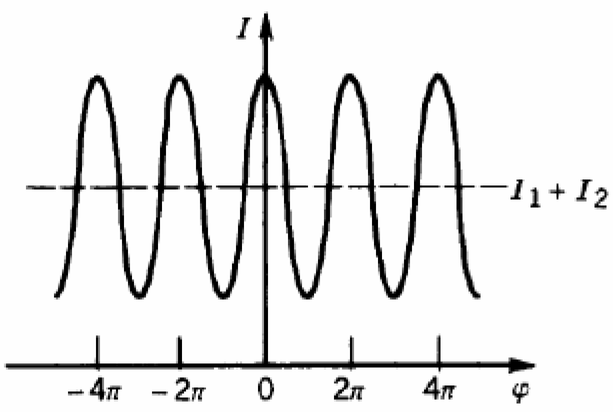
$$\boxed{I = I_1 + I_2 + 2(I_1 I_2)^{1/2} \cos\varphi,}$$

$$U_1 = I_1^{1/2} \exp(j\varphi_1)$$

$$U_2 = I_2^{1/2} \exp(j\varphi_2)$$

$$\varphi = \varphi_2 - \varphi_1$$



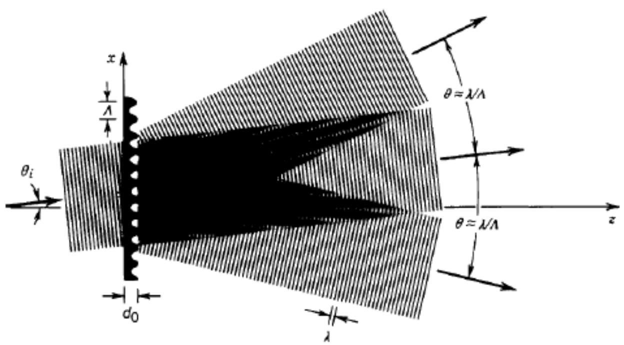


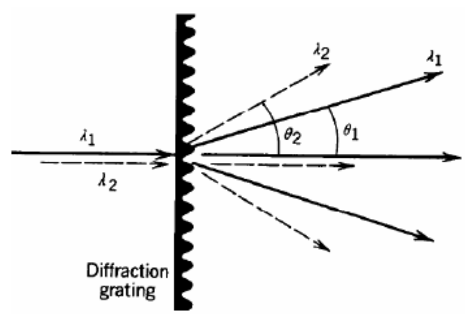
Diffraction gratings

$$\boxed{\sin\theta_q = \sin\theta_i + q\frac{\lambda}{\Lambda}}$$: **Grating Equation**

$$q = 0, \pm 1, \pm 2, \ldots,$$

q is called the diffraction order





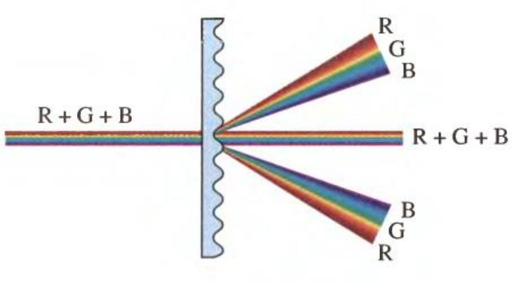
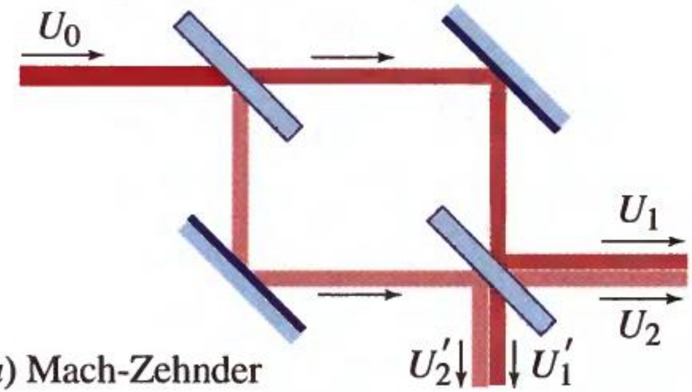
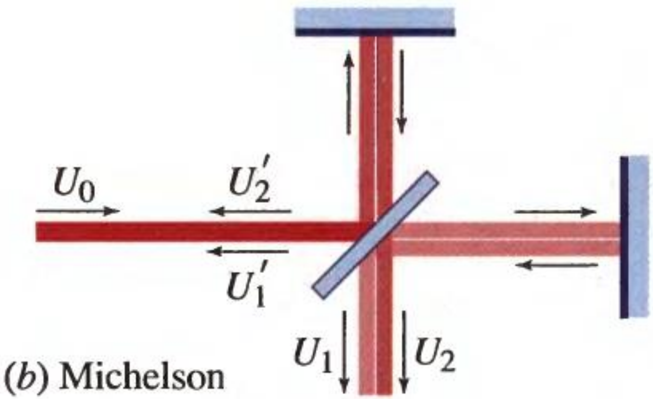


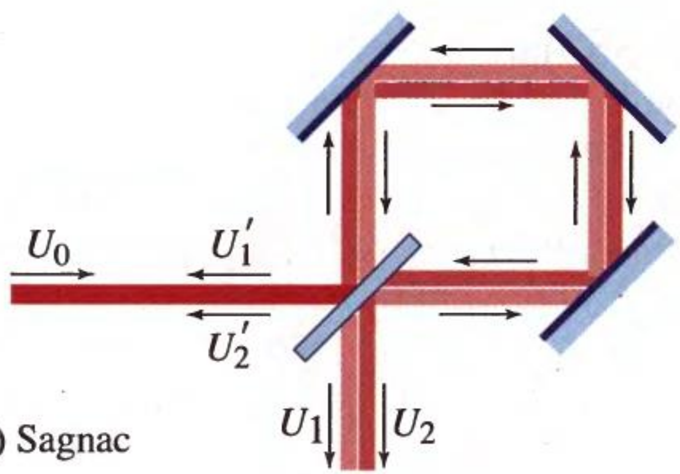
Figure 2.4-12 A diffraction grating directs two waves of different wavelengths, λ_1 and λ_2, into two different directions, θ_1 and θ_2. It therefore serves as a spectrum analyzer or a spectrometer.



(*a*) Mach-Zehnder



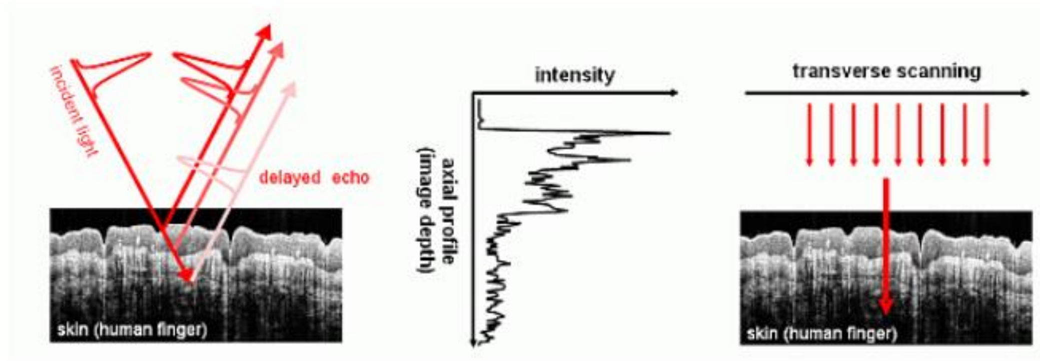
(*b*) Michelson

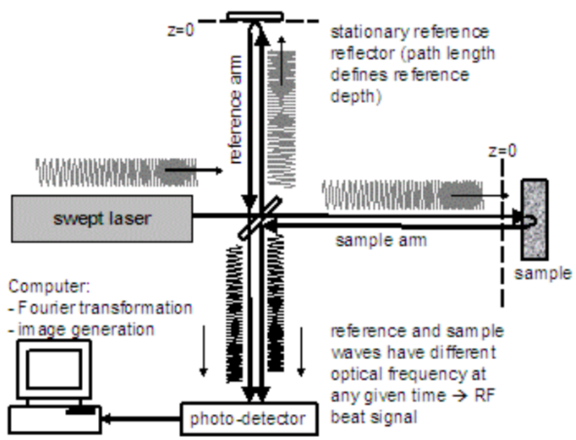


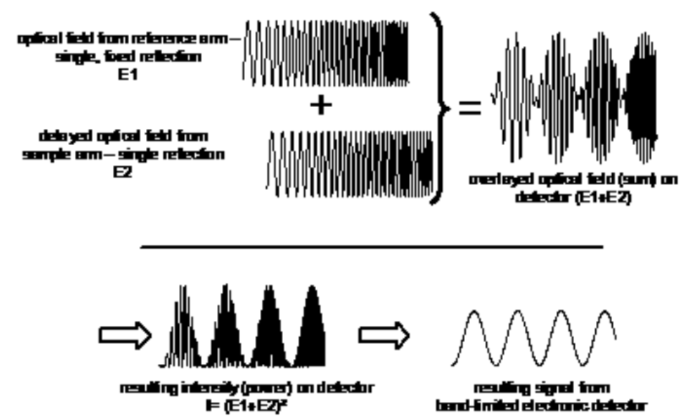
(*c*) Sagnac

Figure 2.5-3 Interferometers: A wave U_0 is split into two waves U_1 and U_2 (they are shown as shaded light and dark for ease of visualization but are actually congruent). After traveling through different paths, the waves are recombined into a superposition wave $U = U_1 + U_2$ whose intensity is recorded. The waves are split and recombined using beamsplitters. In the Sagnac interferometer the two waves travel through the same path, but in opposite directions.

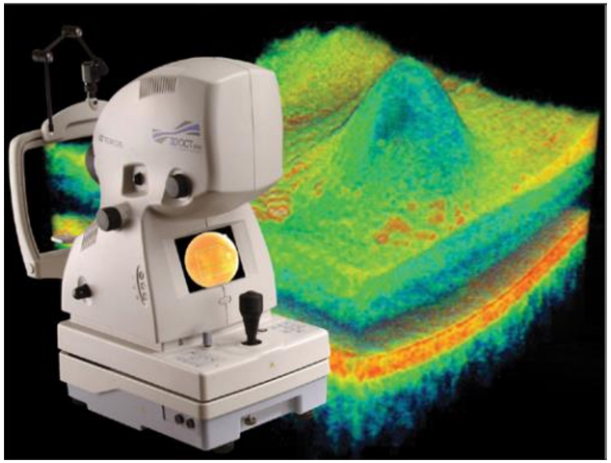
Concept of swept source OCT:

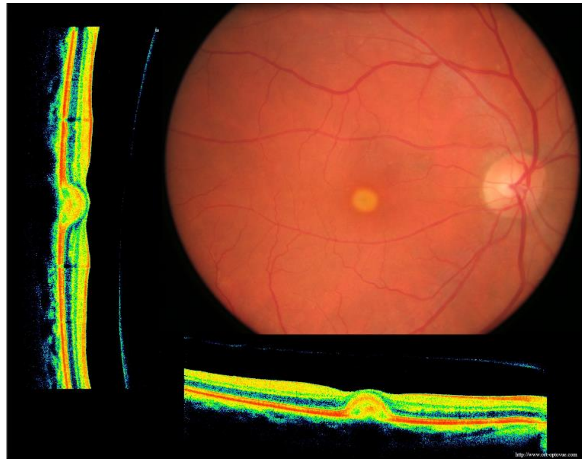


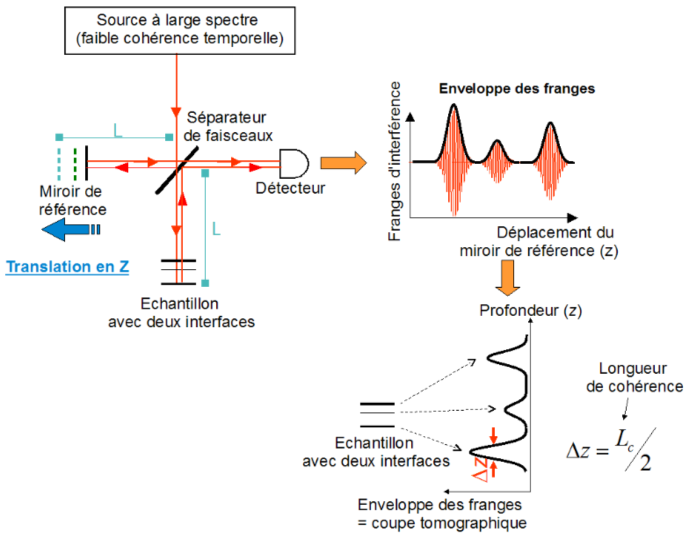




OCT (Optical Coherence Tomography)





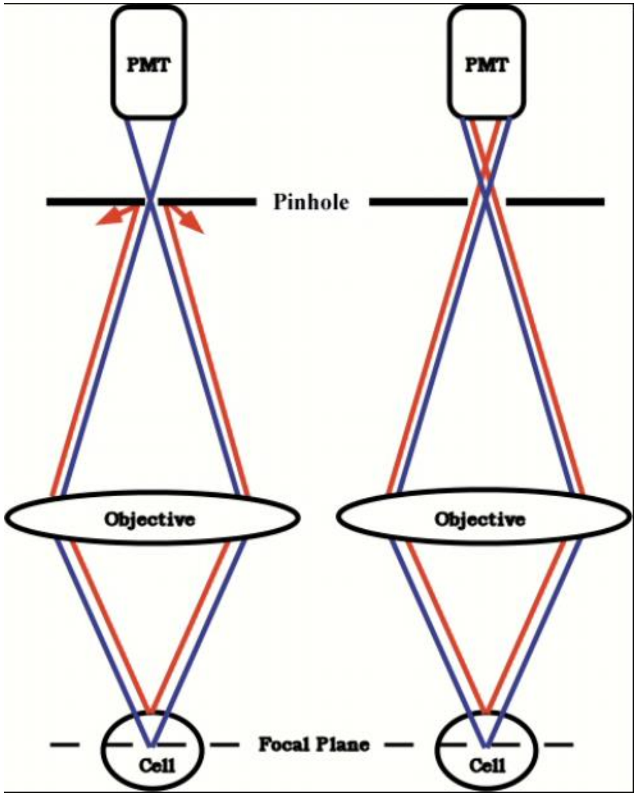


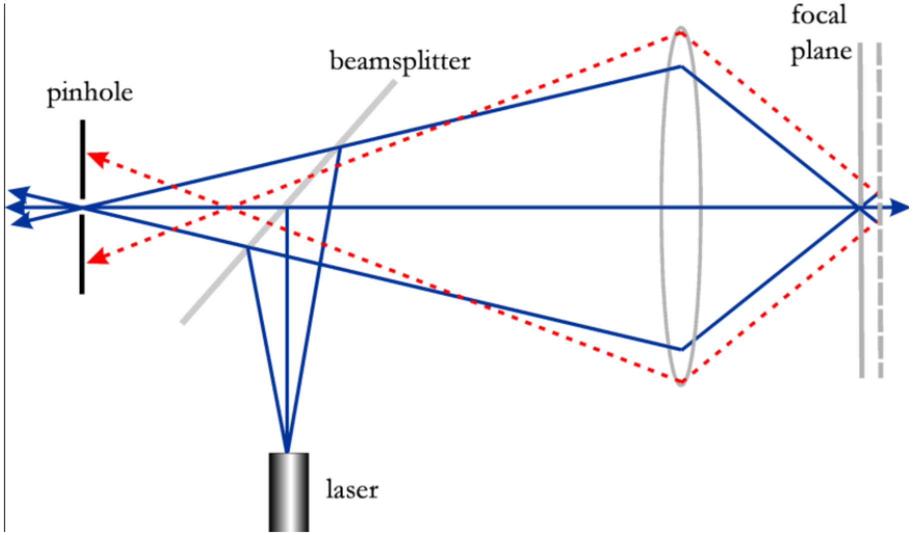
Time domain

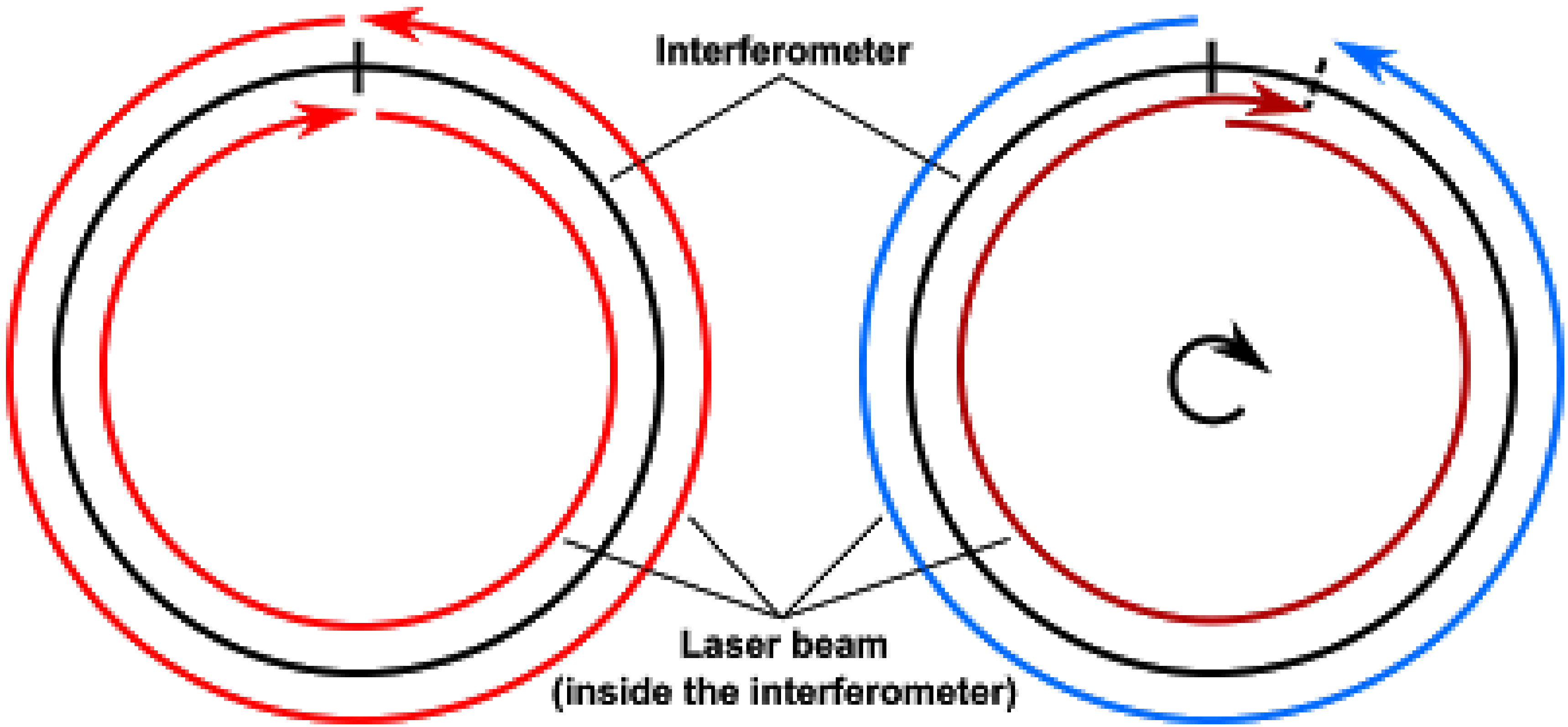
Frequency domain

Coherence length $L = \dfrac{c}{n\,\Delta f},$

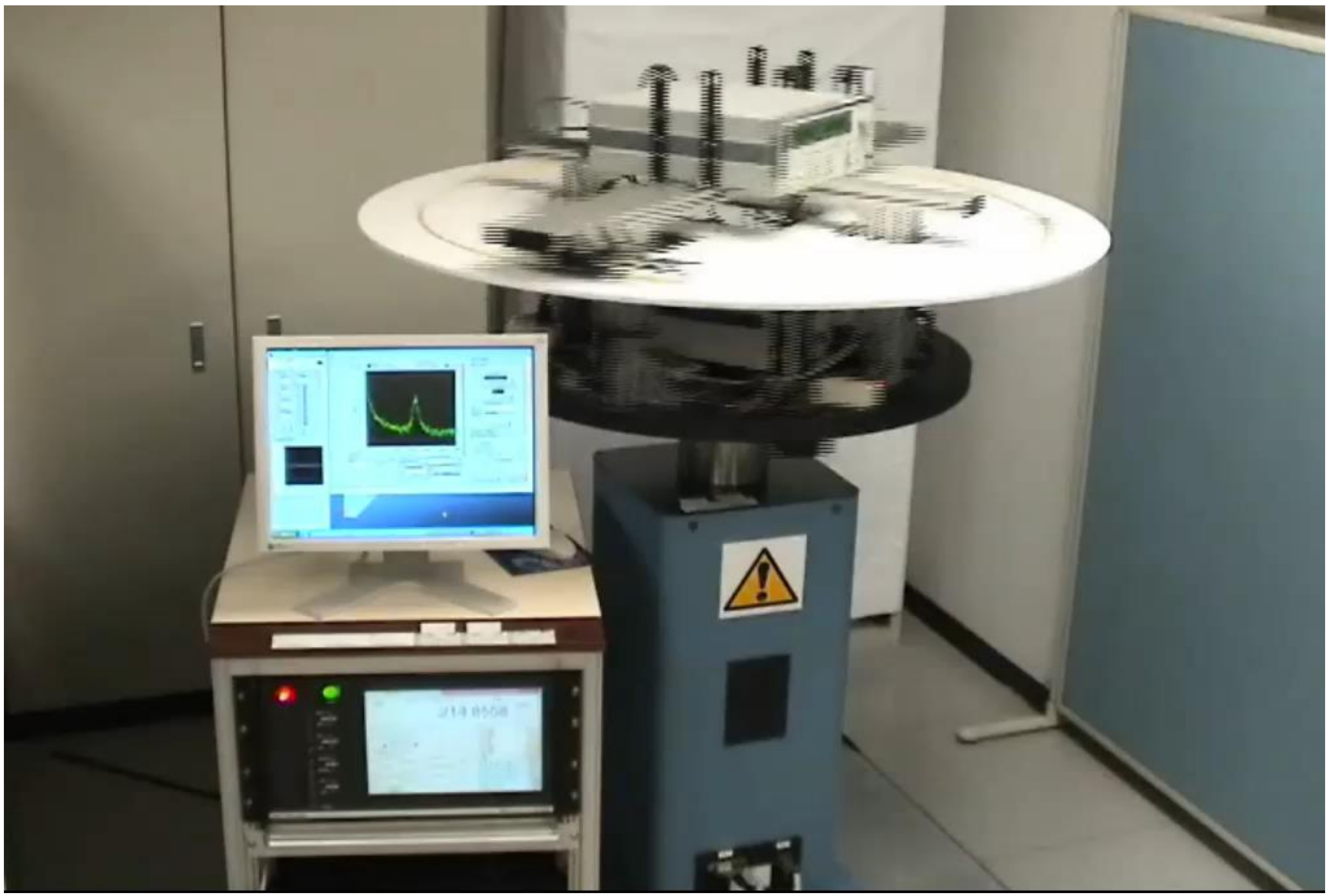
Confocal microscopy






Interferometer
Laser beam
(inside the interferometer)

Optical Gyroscope



A. Interference of Two Waves

When two monochromatic waves with complex amplitudes $U_1(\mathbf{r})$ and $U_2(\mathbf{r})$ are superposed, the result is a monochromatic wave of the same frequency that has a complex amplitude

$$U(\mathbf{r}) = U_1(\mathbf{r}) + U_2(\mathbf{r})\,. \tag{2.5-1}$$

In accordance with (2.2-10), the intensities of the constituent waves are $I_1 = |U_1|^2$ and $I_2 = |U_2|^2$, while the intensity of the total wave is

$$I = |U|^2 = |U_1 + U_2|^2 = |U_1|^2 + |U_2|^2 + U_1^* U_2 + U_1 U_2^*. \tag{2.5-2}$$

The explicit dependence on $\mathbf{r}$ has been omitted for convenience. Substituting

$$U_1 = \sqrt{I_1}\,\exp(j\varphi_1) \qquad \text{and} \qquad U_2 = \sqrt{I_2}\,\exp(j\varphi_2) \tag{2.5-3}$$

into (2.5-2), where φ_1 and φ_2 are the phases of the two waves, we obtain

$$\boxed{I = I_1 + I_2 + 2\sqrt{I_1 I_2}\,\cos\varphi\,,} \tag{2.5-4}$$

Interference Equation

with

$$\varphi = \varphi_2 - \varphi_1\,. \tag{2.5-5}$$

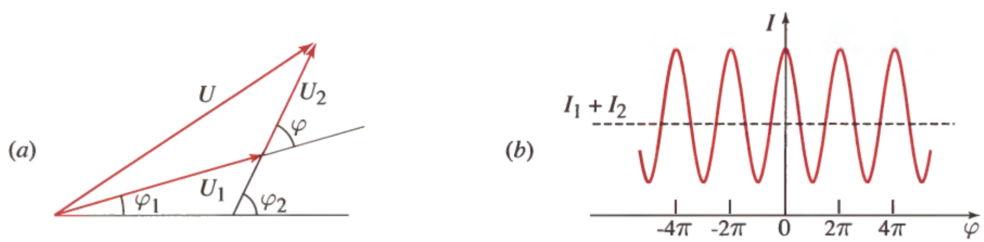


Figure 2.5-1 (*a*) Phasor diagram for the superposition of two waves of intensities I_1 and I_2 and phase difference $\varphi = \varphi_2 - \varphi_1$. (*b*) Dependence of the total intensity I on the phase difference φ.

B. Multiple-beam interference

Interference of M Waves of Equal Amplitudes and Equal Phase Differences

$$U_m = I_0^{1/2} \exp[j(m-1)\varphi], \qquad m = 1, 2, \ldots, M.$$

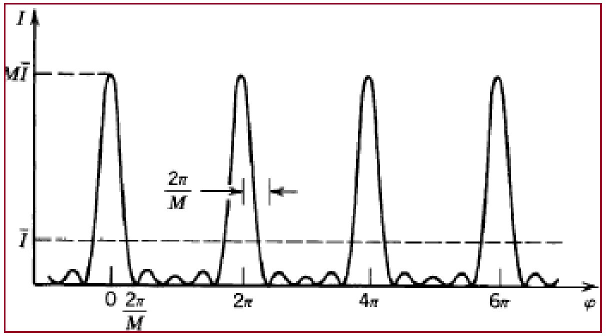
$$\longrightarrow \quad h = \exp(j\varphi), \quad \longrightarrow \quad U_m = I_0^{1/2} h^{m-1}.$$

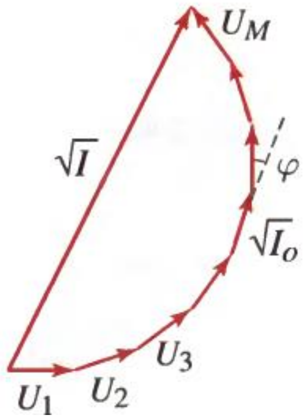
$$U = U_1 + U_2 + \cdots + U_M$$

$$= I_0^{1/2}(1 + h + h^2 + \cdots + h^{M-1}) = I_0^{1/2} \frac{1 - h^M}{1 - h} = I_0^{1/2} \frac{1 - \exp(jM\varphi)}{1 - \exp(j\varphi)}$$

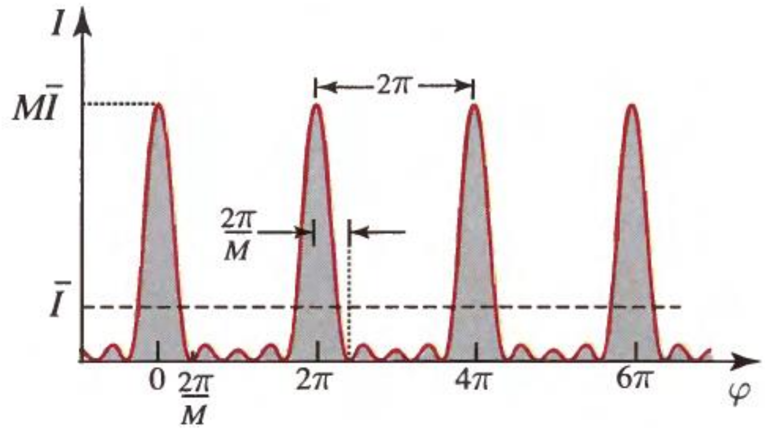
$$I = |U|^2 = I_0 \left| \frac{\exp(-jM\varphi/2) - \exp(jM\varphi/2)}{\exp(-j\varphi/2) - \exp(j\varphi/2)} \right|^2.$$

$$\Rightarrow \quad \boxed{I = I_0 \frac{\sin^2(M\varphi/2)}{\sin^2(\varphi/2)}.} \quad \longrightarrow$$





U_M
$\sqrt{I}$
φ
$\sqrt{I_o}$
U_3
U_1
U_2



I
$M\bar{I}$
2π
$\frac{2\pi}{M}$
$\bar{I}$
0
$\frac{2\pi}{M}$
2π
4π
6π
φ

Interference of an Infinite Number of Waves

of Progressively Smaller Amplitudes and Equal Phase Differences

$$U_1 = I_0^{1/2}, \quad U_2 = hU_1, \quad U_3 = hU_2 = h^2 U_1, \quad \ldots,$$

where $h = r e^{j\varphi}$, $|h| = r < 1$

$$U = U_1 + U_2 + U_3 + \cdots = I_0^{1/2}(1 + h + h^2 + \cdots) = \frac{I_0^{1/2}}{1 - h} = \frac{I_0^{1/2}}{1 - r e^{j\varphi}}$$

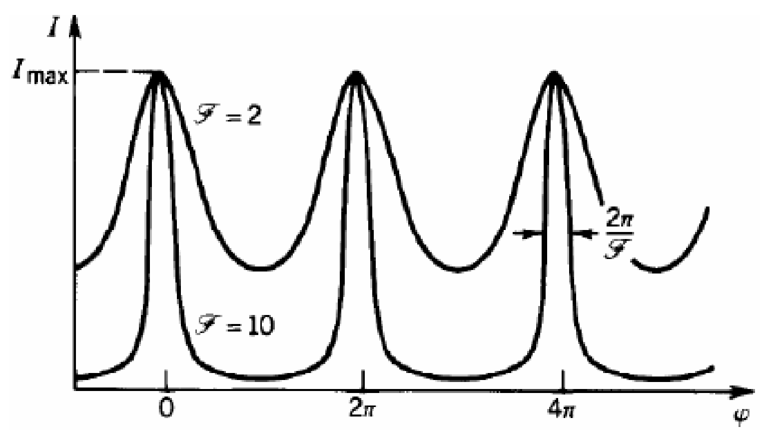
$$I = |U|^2 = I_0/|1 - r e^{j\varphi}|^2 = I_0/[(1 - r\cos\varphi)^2 + r^2 \sin^2\varphi],$$

$$I = \frac{I_0}{(1-r)^2 + 4r\sin^2(\varphi/2)}$$

$$\boxed{I = \frac{I_{max}}{1 + (2\mathscr{F}/\pi)^2 \sin^2(\varphi/2)}},$$

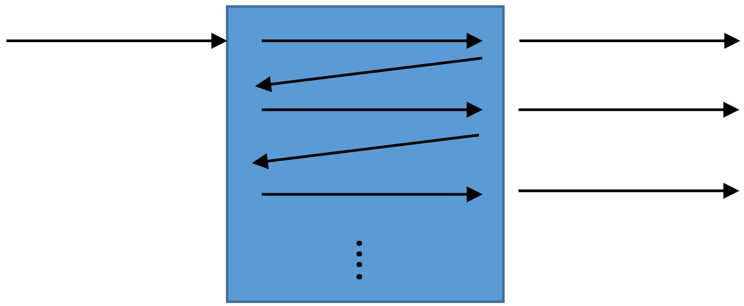
$$I_{max} = \frac{I_0}{(1-r)^2}$$

$$\mathscr{F} = \frac{\pi r^{1/2}}{1 - r}$$: **Finesse**



$$\phi = nkd$$

$z_0 = 1$ $\quad$ z



$$= tt^* e^{i\phi}$$

$$= tt^* r^2 e^{i3\phi}$$

$$= tt^* r^4 e^{i5\phi}$$

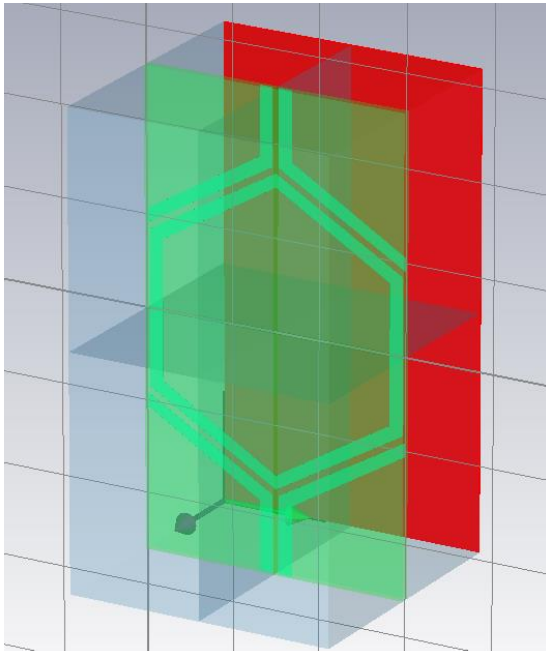
$t\ , r$ $\quad$ t^*, r

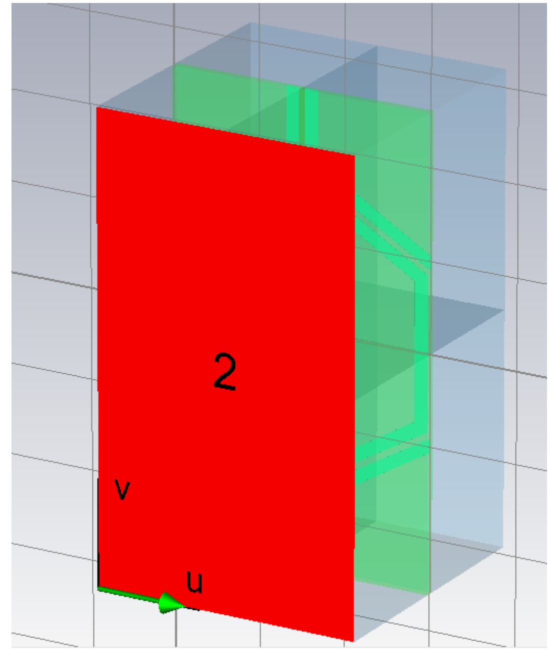
$$r = \frac{z-1}{z+1}, t = \frac{2z}{z+1}\ , t^* = \frac{2}{z+1}$$

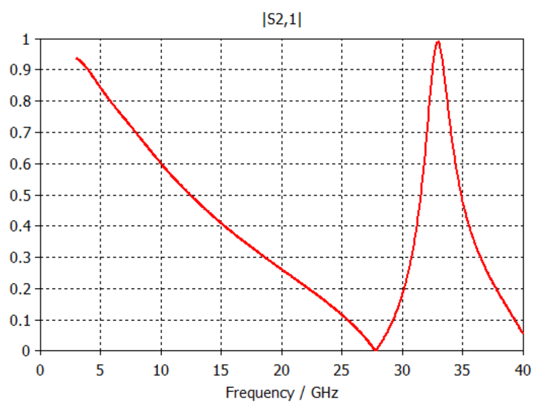
h=$r^2 e^{i\phi}$=$r^2 e^{inkd}$

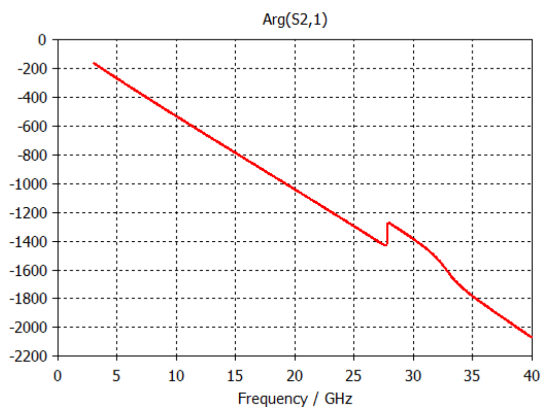
$$\mathrm{T} = \frac{tt^* e^{i\phi}}{1 - r^2 e^{i2\phi}}$$

$$T = \frac{\frac{4 * Ns(j)}{(Ns(j)+1)^2} * \exp\left(-2\,\pi i * \left(Ns(j)\right) * f(j) * \frac{d}{c}\right)}{1 - \left(\left(\frac{Ns(j)-1}{Ns(j)+1}\right)^2\right) * \exp\left(-4\pi i * Ns(j) * f(j) * \frac{d}{c}\right)};$$

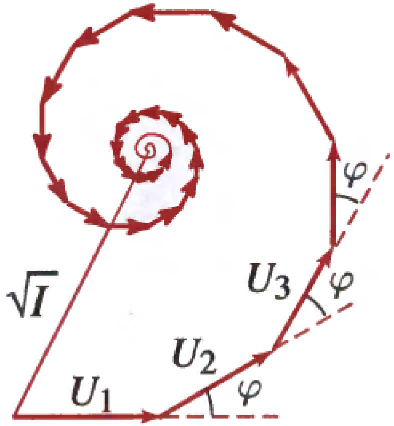


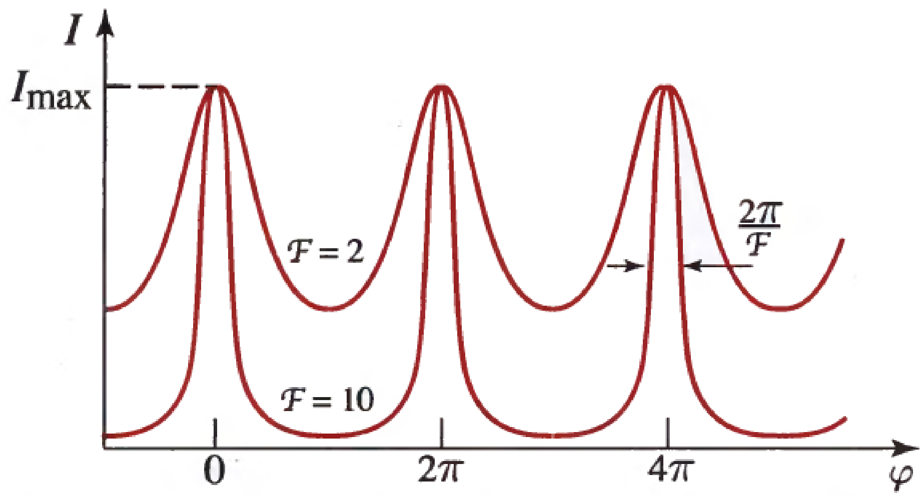






$$Transmission = S_{21} = |S_{21}|\exp(j\phi)$$





$$h = T_1 T_2 e^{2jkd} \text{ or } \quad R_1 R_2 e^{2jkd}$$

$$|h| = T_1 T_2 \qquad \varphi = 2jkd = \frac{4j\pi d}{\lambda} = \frac{4\pi j f d}{c}$$

$$\sin\theta_{+,-} = \sin\left[\frac{(\omega_{+,-})nd}{c}\right] = \pm\frac{1-(R_1R_2)^{1/2}}{(R_1R_2)^{1/4}} \tag{6.3.6a}$$

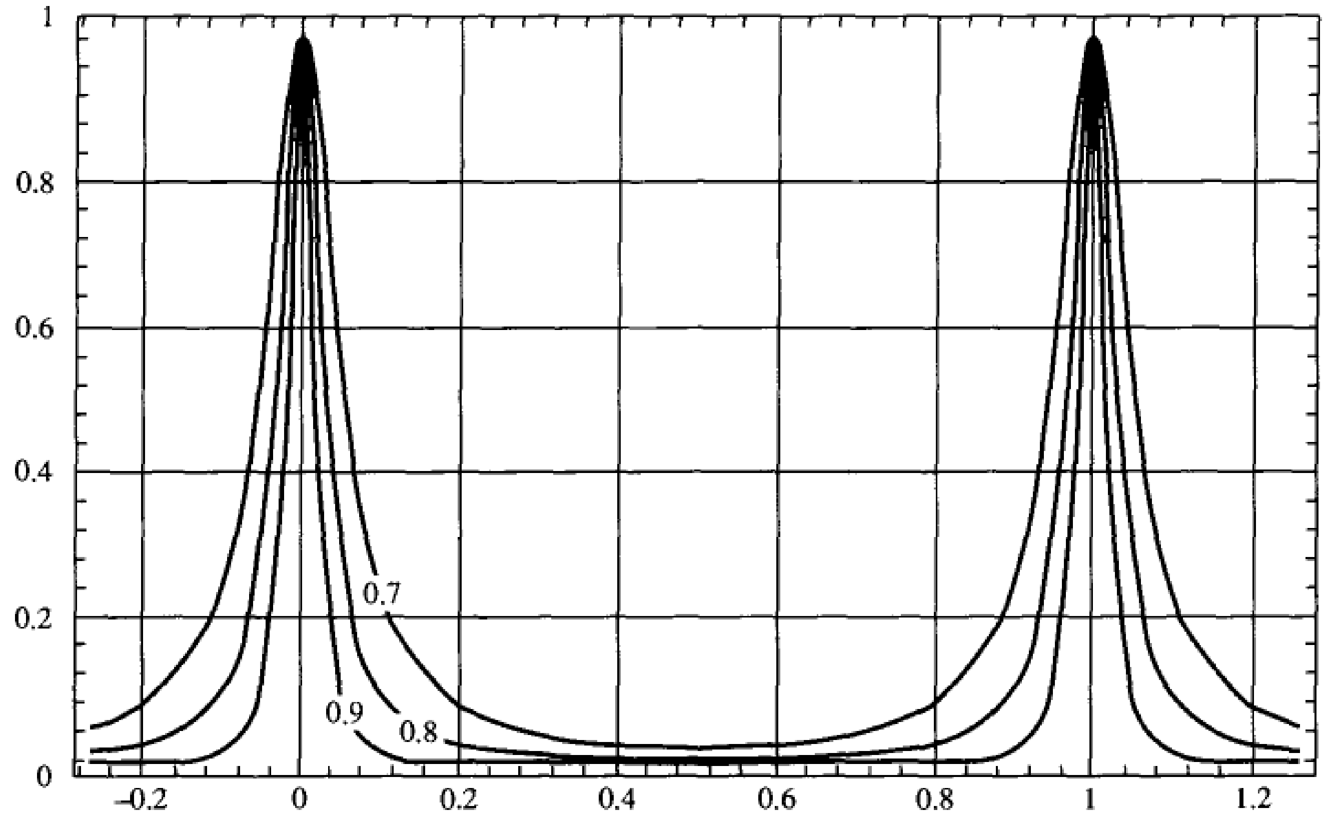


FIGURE 6.3. The transmission through a Fabry-Perot cavity as a function of the electrical length measured in units of $\theta/\pi - q$. The three curves were plotted for $R_1 = R_2 = 0.9$, 0.8, and 0.7.

Furier series

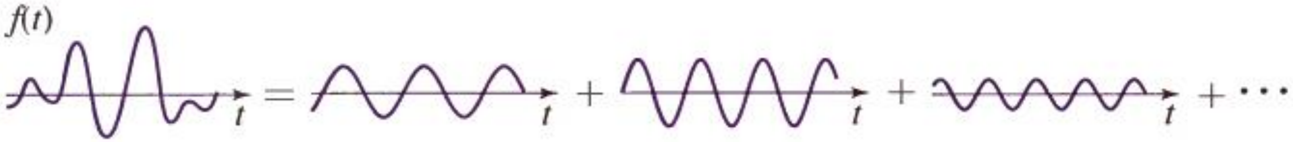


Figure 4.0-1 An arbitrary function $f(t)$ may be analyzed as a sum of harmonic functions of different frequencies and complex amplitudes.

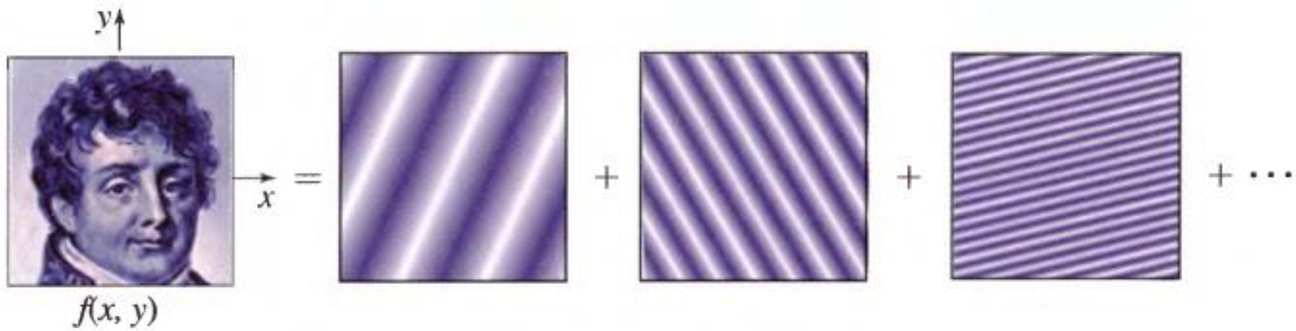
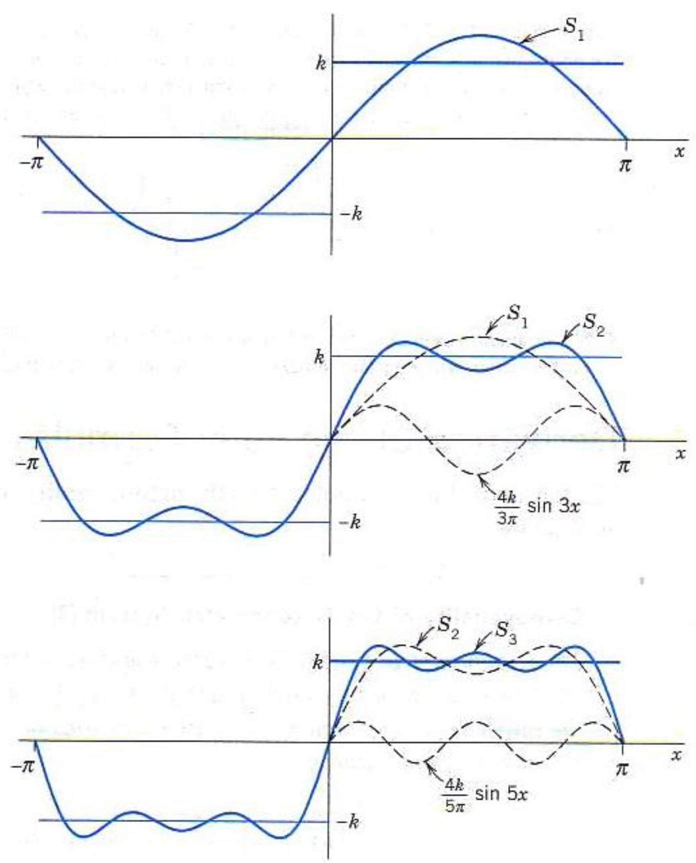


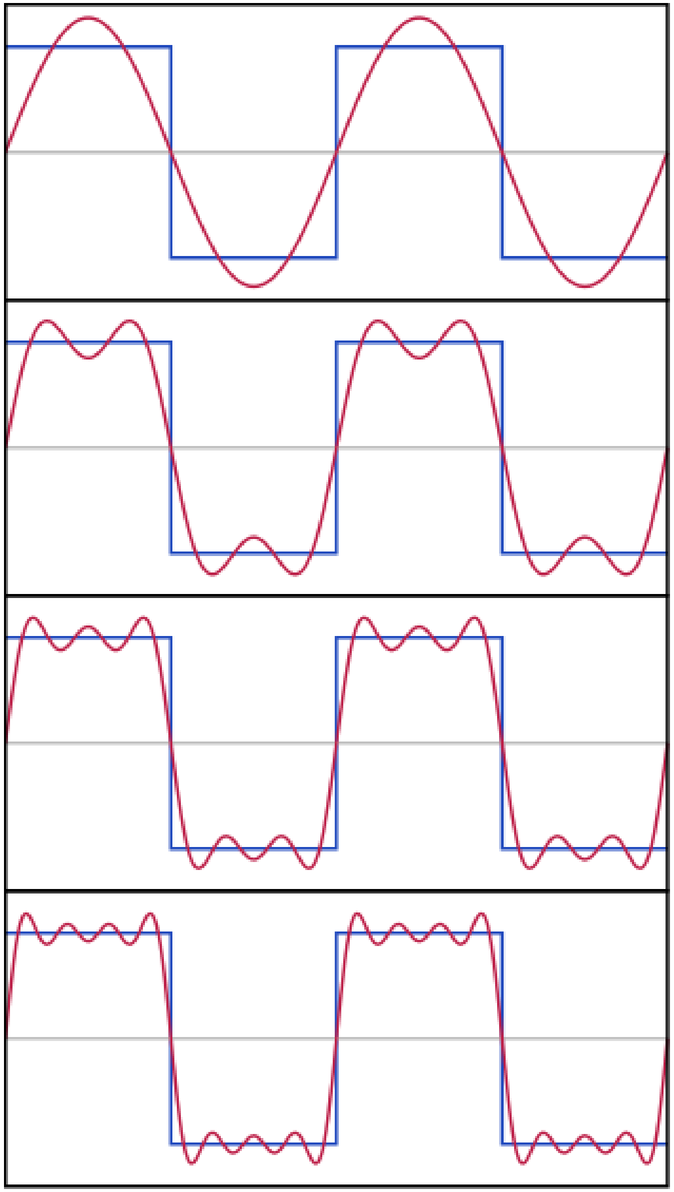
Figure 4.0-2 An arbitrary function $f(x, y)$ may be analyzed as a sum of harmonic functions of different spatial frequencies and complex amplitudes, drawn here schematically as graded blue lines.

$$f(t) = \underbrace{\frac{f(t)+f(-t)}{2}}_{f_{\text{even}}(t)} + \underbrace{\frac{f(t)-f(-t)}{2}}_{f_{\text{odd}}(t)}$$
$$= f_{\text{even}}(t) + f_{\text{odd}}(t)$$

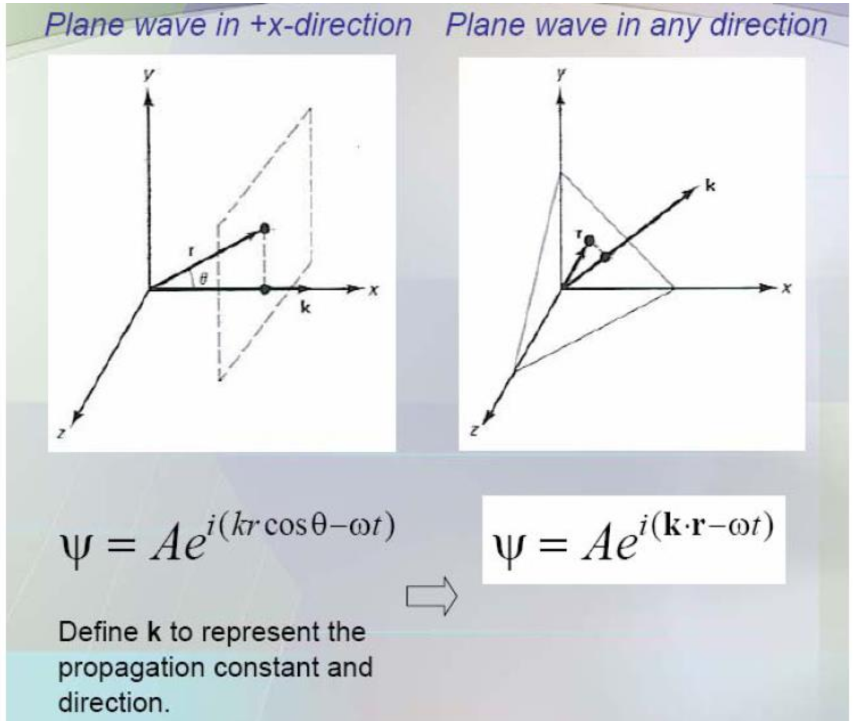
$$\begin{aligned} f(x) & \\ &= \sum_{n=1}^{\infty} \frac{2k}{n\pi}[1-\cos(n\pi)] \\ &= \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right) \end{aligned}$$

그래프의 모양을 살펴보자.





General solution of Plane wave



$$\nabla^2\Psi + k^2\Psi = 0 \qquad \Psi = Ae^{j(kx-\omega t)} \qquad Ae^{j\left(k_x x + k_y y + k_z z - \omega t\right)} = Ae^{j\left(\vec{k}\cdot\vec{r} - \omega t\right)}$$

$$k_x x + k_y y + k_z z = (k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}} + k_z\hat{\mathbf{z}})\cdot(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) = \vec{k}\cdot\vec{r}$$

$$\left|\vec{k}\right| = k^2, \qquad k_x^2 + k_y^2 + k_z^2 = k^2$$