Chapter 2
VECTOR AND TENSOR ANALYSES

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Lecture 4

2.1 Summation Convention and Special Symbols
2.2 Vectors and Tensors
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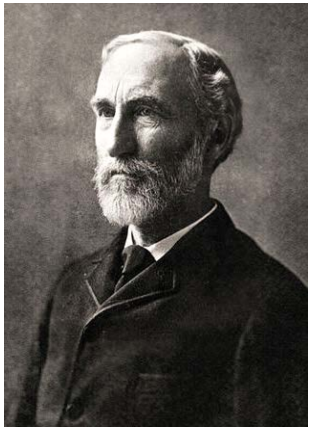

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2.4 Helmholtz Theorem

So far we have defined an n-dimensional vector $\mathbf{F}$ by specifying all of the n components, F_i $(i=1,2,\cdots n)$. However, there is an another way to uniquely define a vector.

Helmholtz Theorem

A vector can uniquely be defined by

$$\mathbf{F}(\mathbf{r}) = -\nabla\left(\int d\mathbf{r}' \frac{\nabla'\cdot\mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) + \nabla\times\left(\int d\mathbf{r}' \frac{\nabla'\times\mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) \qquad (2.x)$$

if $\nabla\cdot\mathbf{F}(\mathbf{r})$ and $\nabla\times\mathbf{F}(\mathbf{r})$ are known and subject to three boundary conditions:

$$\lim_{r\to\infty}\mathbf{F}(\mathbf{r}) = 0$$
$$\lim_{r\to\infty} r^2\nabla\cdot\mathbf{F}(\mathbf{r}) = 0 \qquad (2.x)$$
$$\lim_{r\to\infty} r^2\nabla\times\mathbf{F}(\mathbf{r}) = 0$$

*The proof will be assigned to a homework problem.

2.5 Transvers and Longitudinal Components

In solid-state physics, the **material response functions** to external excitation is usually based on the decomposition of electromagnetic fields into **transverse and longitudinal components**.

This decomposition process by itself introduces the **generalized scalar and vector potentials**, which results from the Helmholtz theorem.

Generalized Scalar and Vector Potentials

From the Helmholtz theorem, we can always define an arbitrary vector field $\mathbf{F}(\mathbf{r})$ as

$$\boxed{\mathbf{F}(\mathbf{r}) = -\nabla\phi(\mathbf{r}) + \nabla\times\mathbf{V}(\mathbf{r})} \qquad (2.x)$$

where the generalized scalar and vector potentials are given by

$$\phi(\mathbf{r}) = \int d\mathbf{r}' \frac{\nabla'\cdot\mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} \qquad \text{Scalar Potential} \qquad (2.x)$$

$$\mathbf{V}(\mathbf{r}) = \int d\mathbf{r}' \frac{\nabla'\times\mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} \qquad \text{Vector Potential} \qquad (2.x)$$

Transverse and Longitudinal Decomposition: Interpretation of Helmholtz Theorem

$\nabla \times = 0$

From (2.xx), it is always possible to decompose a vector field into **longitudinal** **(curl-free or irrotational)** component $\mathbf{F}_L(\mathbf{r})$ and **transverse** **(divergence-free, rotational or solenoidal)** component $\mathbf{F}_T(\mathbf{r})$:

$\nabla \cdot = 0$

$$\mathbf{F}(\mathbf{r}) = \mathbf{F}_L(\mathbf{r}) + \mathbf{F}_T(\mathbf{r}) \tag{2.x}$$

where the generalized scalar and vector potentials are given by

$F_L(r) = -\nabla\phi(r) = \nabla\left(\int dr' \frac{\nabla\cdot F(r')}{4\pi|r-r'|}\right)$

$$\boxed{\mathbf{F}_L(\mathbf{r}) = \nabla\phi(\mathbf{r}) = \nabla\left(\int d\mathbf{r}' \frac{\nabla'\cdot\mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}\right)} \quad \text{Longitudinal Component} \tag{2.x}$$

$$\boxed{\mathbf{F}_T(\mathbf{r}) = \nabla\times\mathbf{V}(\mathbf{r}) = \nabla\times\left(\int d\mathbf{r}' \frac{\nabla'\times\mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|}\right)} \quad \text{Transverse Component} \tag{2.x}$$

Note that the longitudinal and transverse are curl-free and divergence-free, respectively

$\phi(\mathbf{r}) \xrightarrow{F.T.} \phi(\mathbf{k})$

r-space k-space.

$\nabla\phi(\mathbf{r}) \xrightarrow{F.T.} i\mathbf{k}\,\phi(\mathbf{k})$

$F_L(\mathbf{r}) = \nabla\phi(\mathbf{r}) \xrightarrow{F.T.} i\mathbf{k}\,\phi(\mathbf{k})$

parallel

$F_T(\mathbf{r}) = \nabla\times\mathbf{V}(\mathbf{r}) \xrightarrow{F.T.} i\mathbf{k}\times\mathbf{V}(\mathbf{k})$

perpendicular.

2.5 Coordinate Systems

Why study coordinate systems?

Most of physics laws are expressed by partial differential equations such as classical and quantum wave equations. Depending on the geometric symmetry of the problem, the coordinate system should be chosen **to make it easier to solve the equations**.

The Cartesian coordinates is the simplest among the orthogonal coordinates, and has unique properties that all the unit vectors are constant, and perpendicular to each other. However, the Cartesian system is not the most convenient one for problems with spherical or cylindrical symmetries. In some cases, **not just orthogonal coodinates**, even **non-orthogonal (oblique or skew)** coordinates fit much better the problems.

There are **eleven separable coordinate systems** in which the wave equations are separable such that the partial differential equation can be split into ordinary differential equations.

direct lattice $\Rightarrow$ reciprocal lattice.

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