Chapter 3
COMPLEX VARIABLES

Augustin-Louis Cauchy
(1789-1857)
Math/Physics
Complex Analysis
Stress Tensor

Lecture 9

3.5 Cauchy Principal Value and Hilbert Transform

David Hilbert
(1862-1943)
Math
Hilbert Space
(Vector Space for QM)

3.5 Cauchy Principal Value and Hilbert Transform

In physics and engineering problems, we see many integrals. For example, we need to evaluate integrals to find electric and magnetic fields for given charge and current distributions.

However, for some problems it is almost impossible to find the integrals. For these **"improper" integrals**, we need advanced mathematical techniques.

The Cauchy principal value (PV) and the Hilbert transform are frequently encountered in the **linear response theory of solid state physics** and also in **radiation and scattering problems of classical electrodynamics**.

[Definition] Improper Integrals

The improper integral is a definite integral of a real function $f(x)$ on the real axis having:

1) either or both limits are $\pm\infty$ infinite, or/and
2) an integrand becomes $\pm\infty$ in the range of integration.

Cauchy Principal Values of Improper Integrals

Consider an improper integral of a function $f(x)$ with a **simple pole** at $x = x_0$ on the real axis x ($a < x_0 < b$),

$$\int_a^b dx\, f(x)$$

The Cauchy principal value of this improper integral is then defined as a limit:

$$\boxed{PV\int_a^b dx\, f(x) \equiv \lim_{\varepsilon\to 0^+}\left[\int_a^{x_0-\varepsilon} dx\, f(x) + \int_{x_0+\varepsilon}^{b} dx\, f(x)\right]} \qquad (3.26)$$

Now we note that **the first and the only principal term** is most dominant in the Laurent expansion around the simple pole:

$f(x) = \sum_{n=-\infty} A_n(x-x_0)^n = \frac{A_{-1}}{x-x_0} + \sum_{n=0} A_n(x-x_0)^n$

$$f(x) = \sum_{n=-\infty}^{\infty} A_n (x-x_0)^n = \frac{A_{-1}}{x-x_0} + \sum_{n=0}^{\infty} A_n (x-x_0)^n \simeq \frac{A_{-1}}{x-x_0}, \quad x \simeq x_0 \qquad (3.27)$$

$\simeq \frac{A_{-1}}{x-x_0}$

Here, $f(x)$ becomes an odd function at $x \simeq x_0$, and note that the limiting process is just for kind of **balancing or canceling process** since the single principal term for the simple pole is the most dominant one in its Laurent expansion.

Ex) $f(x) = 1/x$ with $a = b = R$

$$PV\int_{-R}^{R} dx\frac{1}{x} = \lim_{\varepsilon\to 0}\left[\int_{-R}^{-\varepsilon} dx\frac{1}{x} + \int_{\varepsilon}^{R} dx\frac{1}{x}\right] = \lim_{\varepsilon\to 0}\left[-\int_{\varepsilon}^{R} dx'\frac{1}{x'} + \int_{\varepsilon}^{R} dx\frac{1}{x}\right] = 0 \quad (x' = -x)$$

Coming back to add

Relation between Analyticity and Causality

when I get a chance!

Hilbert Transform

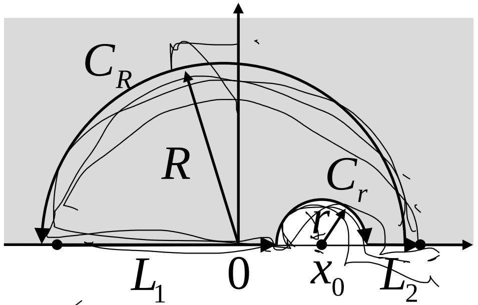

Fig. 2-5 Contour for Hilbert transform

Consider an integral of a complex function $f(z)$ on a contour $C = C_R + L_1 + C_r + L_2$ (Fig. 3-6) under the following conditions:

1) $f(z)$ is analytic in the upper half plane, including the real axis ($y \geq 0$).
2) $\lim_{R\to\infty} |f(z)| = 0$

Here, note that C excludes the simple pole so that the integrand is analytic inside/on C:

$$\oint_C dz \frac{f(z)}{z-x_0} = \left[\int_{C_R} dz + \int_{L_1} dz + \int_{C_r} dz + \int_{L_1} dz\right]\frac{f(z)}{z-x_0} = 0 \quad (3.28)$$

In the limit of $r \to 0$ and $R \to \infty$, using the Cauchy principal value

$$\lim_{R\to\infty}\int_{C_R} dz \frac{f(z)}{z-x_0} = \lim_{R\to\infty}\int_0^\pi iRd\theta_R \frac{f(Re^{i\theta_R})}{Re^{i\theta_R}-x_0} = i\lim_{R\to\infty}\int_0^\pi d\theta_R\, f(Re^{i\theta_R})e^{-i\theta_R} = 0$$

$$\lim_{r\to0}\left[\int_{L_1} dz + \int_{L_2} dz\right]\frac{f(z)}{z-x_0} = \left[\int_{-\infty}^{x_0-r} dx + \int_{x_0+r}^{\infty} dx\right]\frac{f(x)}{x-x_0} = PV\int_{-\infty}^{\infty} dx \frac{f(x)}{x-x_0} \quad (3.29)$$

$$\lim_{r\to\infty}\int_{C_r} dz \frac{f(z)}{z-x_0} = \lim_{r\to0}\int_0^\pi ird\theta_r \frac{f(x_0+re^{i\theta_r})}{re^{i\theta_r}} = i\lim_{r\to0}\int_0^\pi d\theta_r\, f(x_0)e^{-i\theta_r} = -i\pi f(x_0)$$

Substituting (3.29) into (3.28), we find

$$\boxed{PV\int_{-\infty}^{\infty} dx \frac{f(x)}{x-x_0} = i\pi f(x_0)} \tag{3.29}$$

$PV \int_{-\infty}^{\infty} dx \frac{f(x)}{x-x_0} = i\pi f(x_0)$

Split into real and imaginary parts, finally we have

$$\boxed{\begin{aligned} f_{\mathrm{Re}}(x_0) &= \frac{1}{\pi} PV\int_{-\infty}^{\infty} dx \frac{f_{\mathrm{Im}}(x)}{x-x_0} \\ f_{\mathrm{Im}}(x_0) &= -\frac{1}{\pi} PV\int_{-\infty}^{\infty} dx \frac{f_{\mathrm{Re}}(x)}{x-x_0} \end{aligned}} \quad \text{Hilbert Transform Pair} \tag{3.29}$$

Note that if $f_{\mathrm{Re}}(x)=0$ for all x, then $f_{\mathrm{Im}}(x)=0$, and vice versa:

$$\boxed{f_{\mathrm{Re}}(x)=0 \quad \rightleftharpoons \quad f_{\mathrm{Re}}(x)=0} \equiv \boxed{f(x)=0}$$

Dirac Identity (Sokhotski–Plemelj Formula)

If a real-valued function $f(x)$ is continuous on the real axis x and $-a<0<b$, then

$$\lim_{\varepsilon\to 0+}\int_{-\infty}^{\infty}dx\frac{f(x)}{x\pm i\varepsilon}=PV\int_{-\infty}^{\infty}dx\frac{f(x)}{x}\mp i\pi f(0) \tag{3.29}$$

Recast into a more compact form, we have

$$\boxed{\frac{1}{x\pm i0^{+}}=PV\frac{1}{x}\mp i\pi\delta(x)} \quad \text{Dirac Identity} \tag{3.29}$$

The Dirac identity is an extremely powerful for the linear response theory in **solid state (or advanced semiconductor) physics** in general, especially for **theory of optical processes**., and also for many problems in classical electrodynamic.

* The proof of the Dirac identity will be assigned to an homework problem. It will be easy difficult if you can slightly modify the pole and contour used for Hilbert transform.