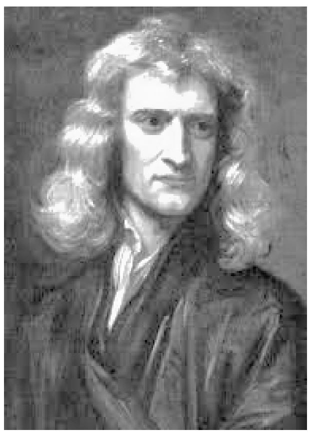
# Chapter 5
# DIFFERENTIAL EQUATIONS



**Isaac Newton**
(1642-1726)
Math/Physics
**Universal Gravity**
**Newtonian Mechanics**
***Differential* Calculus**

## Lecture 18

5.2 Ordinary Differential Equations

**Gottfried Wilhelm Leibniz**
(1646-1716)
Math/Physic
***Integral* Calculus**
**Leibniz Notation**

# 5.2 Ordinary Differential Equations

An ordinary differential equation (ODE) has one or several derivatives of an unknown function $y(x)$ and may also include $y(x)$ itself:

$$G[x, y, y^{(1)}, y^{(2)}, \cdots, y^{(n)}(x)] = 0 \tag{5.2}$$

## 5.2.1 First-Order First-Degree ODEs

A first-order first-degree ODE is defined as

$$\frac{dy}{dx} = F(x, y) = -\frac{A(x, y)}{B(x, y)} \tag{5.3}$$

or equivalently

$$A(x, y)dx + B(x, y)dy = 0 \tag{5.4}$$

Which of the two above forms is the more useful for finding a solution should depend on the specific types of physics problems. Here, we consider several different types of first-degree first-order ODEs.

$$\int_0^\infty dk \frac{J_0(kr)}{k + k_0} e^{-kd}$$

constant

$k_0 = k_0' \pm i|k_0''|$

gain

loss

**Separable-Variable ODEs**

A first-order first-degree ODE is by a general form:

$$\frac{dy}{dx} = F(x,y) = -\frac{A(x)}{B(y)} \tag{5.5}$$

So we have

$$\int dyB(y) + C_y = -\int dxA(x) + C_x \quad \text{or} \quad \int dyB(y) = -\int dxA(x) + C \tag{5.6}$$

Of course, the integral constant $C = C_x - C_y$ needs to be known to get the full solution.

Ex) Solve $y' = x(1+y)$. Do it yourself.

$\frac{dy}{dx} = x(1+y)$ $\frac{dy}{1+y} = x\,dx$ $\ln|1+y| = \frac{1}{2}x^2 + C_1$

**Exact ODEs**

If and only if there exists a continuously differentiable function $U(x,y)$ with a **zero total differential** defined by $A(x,y)$ and $B(x,y)$ in (5.3) and (5.4) such that

$$dU(x,y) = 0 = \frac{\partial U(x,y)}{\partial x}dx + \frac{\partial U(x,y)}{\partial y}dy = A(x,y)dx + B(x,y)dy \tag{5.7}$$

*i.e.*,

$$A(x,y) = \frac{\partial U(x,y)}{\partial x}, \quad B(x,y) = \frac{\partial U(x,y)}{\partial y} \tag{5.8}$$

then we have an **exact** ODE that is subject to the exactness condition:

$$\boxed{\frac{\partial A(x,y)}{\partial y} = \frac{\partial B(x,y)}{\partial x} = \frac{\partial U(x,y)}{\partial x \partial y}} \quad \textbf{Exactness Condition} \tag{5.9}$$

Ex) Solve $xy' + 3x + y = 0$.

Rearranging into the form in (5.7),

$$(3x + y)dx + xdy = 0 \quad \rightarrow \quad A(x, y) = \frac{\partial U}{\partial x} = 3x + y, \quad B(x, y) = \frac{\partial U}{\partial y} = x$$

we can see that it is an exact ODE:

$$\frac{\partial A(x, y)}{\partial y} = \frac{\partial B(x, y)}{\partial x} = 1$$

Integrating $A(x, y)$ along $x$,

$$U(x, y) = \int dx\,(3x + y) + C(y) = \frac{3}{2}x^2 + yx + C(y)$$

Taking $\partial U / \partial y$,

$$\frac{\partial U(x, y)}{\partial y} = x + \frac{\partial C(y)}{\partial y} = B(x, y) = x \quad \rightarrow \quad \frac{\partial C(y)}{\partial y} = 0 \quad \rightarrow \quad C(y) = \text{const}$$

Therefore, we have

$$U(x, y) = \frac{3}{2}x^2 + yx + \text{const}$$

depends on reference.

## Integrating Factors: Inexact-to-Exact Conversion

Many ODEs can be cast into a total differential form of $A(x,y)dx + B(x,y)dy = 0$, but **without** the exactness condition:

$$\boxed{\frac{\partial A(x,y)}{\partial y} \neq \frac{\partial B(x,y)}{\partial x}} \quad \textbf{Inexactness Condition} \qquad (5.10)$$

However, for some cases, we can still convert it from an inexact differential into an exact differential by multiplying an **Integrating Factor** $\mathbf{X(x,y)}$ to $A(x,y)$ and $B(x,y)$ such that

$\frac{\partial[XA]}{\partial y} = \frac{\partial[XB]}{\partial x}$

$$\boxed{\frac{\partial[X(x,y)A(x,y)]}{\partial y} = \frac{\partial[X(x,y)B(x,y)]}{\partial x}} \quad \textbf{Modified Exactness Condition} \qquad (5.11)$$

Unfortunately, there is **no general method** of finding integrating factors for the exact forms. Here, we just show some of the common integrating factors for two cases:

1) Function of $x$ alone:

$$\boxed{X(x) \equiv \frac{1}{B(x,y)}\left[\frac{\partial A(x,y)}{\partial y} - \frac{\partial B(x,y)}{\partial x}\right] \quad \Rightarrow \quad I(x,y) = \exp\left(\int dx\, X(x)\right)}$$

2) Function of $y$ alone:

$$\boxed{Y(y) = \frac{1}{A(x,y)}\left[\frac{\partial A(x,y)}{\partial y} - \frac{\partial B(x,y)}{\partial x}\right] \quad \Rightarrow \quad I(x,y) = \exp\left(\int dy\, Y(y)\right)}$$

3) Function of $x$ and $y$ : $A(x,y) = yf(xy)$ and $B(x,y) = yg(xy)$

$$\boxed{\begin{aligned} A(x,y) &= yf(xy) \\ B(x,y) &= xg(xy) \end{aligned} \Rightarrow \quad I(x,y) = \frac{1}{xA(x,y) - yB(x,y)}}$$

Special function.
- Bessel, Legendre, ...

## Some Integrating Factors

| Terms | $I(x,y)$ | Exact Differential |
|---|---|---|
| $ydx - xdy$ | $-\dfrac{1}{x^2}$ | $\dfrac{xdy - ydx}{x^2} = d\left(\dfrac{y}{x}\right)$ |
| $ydx - xdy$ | $\dfrac{1}{y^2}$ | $\dfrac{ydx - xdy}{y^2} = d\left(\dfrac{x}{y}\right)$ |
| $ydx - xdy$ | $-\dfrac{1}{xy}$ | $\dfrac{xdy - ydx}{xy} = d\left(\ln\dfrac{y}{x}\right)$ |
| $ydx - xdy$ | $-\dfrac{1}{x^2+y^2}$ | $\dfrac{xdy - ydx}{x^2+y^2} = d\left(\operatorname{atan}\dfrac{y}{x}\right)$ |
| $ydx + xdy$ | $\dfrac{1}{xy}$ | $\dfrac{ydx + xdy}{xy} = d(\ln xy)$ |
| $ydx + xdy$ | $\dfrac{1}{(xy)^n},\ n>1$ | $\dfrac{ydx + xdy}{(xy)^n} = d\left[\dfrac{-1}{(n-1)(xy)^{n-1}}\right]$ |
| $ydx + xdy$ | $\dfrac{1}{x^2+y^2}$ | $\dfrac{ydx + xdy}{x^2+y^2} = d\left[\dfrac{1}{2}\ln(x^2+y^2)\right]$ |
| $ydx + xdy$ | $\dfrac{1}{(x^2+y^2)^n},\ n>1$ | $\dfrac{ydx + xdy}{x^2+y^2} = d\left[\dfrac{-1}{2(n-1)(x^2+y^2)^{n-1}}\right]$ |
| $aydx + bxdy$ | $x^{a-1}$ | $x^{a-1}y^{b-1}(aydx + bxdy) = d(x^a y^b)$ |