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XII. *On the Diffraction of an Object-glass with Circular Aperture.* By

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THE investigation of the form and brightness of the rings or rays surrounding the image of a star as seen in a good telescope, when a diaphragm bounded by a rectilinear contour is placed upon the object-glass, though sometimes tedious is never difficult. The expressions which it is necessary to integrate are always sines and cosines of multiples of the independent variable, and the only trouble consists in taking properly the limits of integration. Several cases of this problem have been completely worked out, and the result, in every instance, has been entirely in accordance with observation. These experiments, I need scarcely remark, have seldom been made except by those whose immediate object was to illustrate the undulatory theory of light. There is however a case of a somewhat different kind; which in practice recurs perpetually, and which in theory requires for its complete investigation the value of a more difficult integral; I mean the usual case of an object-glass with a circular aperture. The desire of submitting to mathematical investigation every optical phænomenon of frequent occurrence has induced me to procure the computation of the numerical values of the integral that presents itself in this inquiry: and I now beg leave to lay before the Society the calculated table, with a few remarks upon its application.

Let a be the radius of the aperture of the object-glass, f the focal length, b the lateral distance of a point (in the plane which is normal to the axis of the telescope) from the focus. Then, the lens being supposed aplanatic, and a plane wave of light being supposed incident, the immediate effect of the lens is to give to this wave a spherical shape, its centre being the focus of the lens. Every small portion of the wave, as limited by the form of the object-glass, must now be supposed to be the origin of a little wave, whose intensity is proportional to the surface of that small portion; and the phases of all these little waves, at the time of leaving the spherical surface above alluded to, must be the same. If then $\delta x \times \delta y$ be the area of a very small part of the object-glass, q the distance of that part from the point defined by the distance b, the displacement of the ether at that point, caused by this small wave, will be represented by

$$\delta x \times \delta y \times \sin \frac{2\pi}{\lambda}(vt - q - A);$$

and the whole displacement caused by the small waves coming from every part of the spherical wave will be the integral of

$$\sin \frac{2\pi}{\lambda}(vt - q - A)$$

through the whole surface of the object-glass, q being expressed in terms of the co-ordinates of any point of the spherical surface.

Now let x be measured from the center of the lens in a direction parallel to b; y perpendicular to x and also perpendicular to the axis of the telescope; and z from the focus parallel to the axis of the telescope. Then

$$q = \sqrt{\{(x-b)^2 + y^2 + z\}} = \sqrt{(x^2 + y^2 + z^2 - 2bx)}$$

omitting squares and superior powers of b. But $x^2 + y^2 + z^2 = f^2$, since the wave is part of a sphere whose centre is the focus; therefore,

$$q = \sqrt{(f^2 - 2bx)} = f - \frac{b}{f}x \text{ nearly};$$

and the quantity to be integrated is

$$\sin \frac{2\pi}{\lambda}(vt - f - A + \frac{b}{f}x).$$

The first integration with regard to y is simple, as y does not enter into the expression, which is therefore to be considered as constant. Putting y_1 and y_2 for the smallest and greatest values of y corresponding to x, the first integral is

$$(y_2 - y_1) \times \sin\frac{2\pi}{\lambda}(vt - f - A + \frac{b}{f}x).$$

To this point of the investigation the expressions are general, including every form of contour of the object-glass.

We must now substitute the values of y_1 and y_2 in terms of x, before integrating with regard to x. For a circular aperture

$$y_2 - y_1 = 2\sqrt{a^2 - x^2}$$

where the sign of the radical is essentially positive. Hence the displacement of the ether at the point defined by the distance b is represented by

$$2\int_x \sqrt{a^2 - x^2} \,.\, \sin\frac{2\pi}{\lambda}(vt - f - A + \frac{b}{f}x)$$

$$= 2\sin\frac{2\pi}{\lambda}(vt - f - A)\int_x \sqrt{a^2 - x^2} \,.\, \cos\frac{2\pi}{\lambda} \,.\, \frac{b}{f}x$$

$$+ 2\cos\frac{2\pi}{\lambda}(vt - f - A)\int_x \sqrt{a^2 - x^2} \,.\, \sin\frac{2\pi}{\lambda} \,.\, \frac{b}{f}x,$$

and the limits of integration are from $x = -a$ to $x = +a$. Between these limits it is evident that

$$\int_x \sqrt{a^2 - x^2} \,.\, \sin\frac{2\pi}{\lambda} \,.\, \frac{b}{f}x = 0,$$

(as every positive value is destroyed by an equal negative value); and the displacement is therefore represented by

$$2\sin\frac{2\pi}{\lambda}(vt - f - A)\int_x \sqrt{a^2 - x^2} \,.\, \cos\frac{2\pi}{\lambda} \,.\, \frac{b}{f}x,$$

the integral being taken between the limits $x = -a$, $x = +a$.

If we make $\frac{x}{a}=w$, $\frac{2\pi}{\lambda}\cdot\frac{ba}{f}=n$, the expression becomes

$$2a^2 \cdot \sin\frac{2\pi}{\lambda}(vt-f-A)\int_w \sqrt{1-w^2}\cdot\cos nw, \text{ from } w=-1 \text{ to } w=+1,$$

$$\text{or } 4a^2 \cdot \sin\frac{2\pi}{\lambda}(vt-f-A)\int_w \sqrt{1-w^2}\cdot\cos nw, \text{ from } w=0 \text{ to } w=1.$$

It does not appear, so far as I am aware, that the value of this integral can be exhibited in a finite form either for general or for particular values of w. The definite integral

$$\int_n \sqrt{1-w^2}\cdot\cos nw \text{ (from } w=0 \text{ to } w=1,)$$

(which will be a function of n only) being expressed by N, it may be shewn that N satisfies the linear differential equation

$$N+\frac{3}{n}\cdot\frac{dN}{dn}+\frac{d^2N}{dn^2}=0,$$

which may be depressed to an equation of the first order that does not appear to yield to any known methods of solution.

If we solve the equation by assuming a series proceeding by powers of n, or if we expand $\cos nw$ and integrate each term separately, we arrive (by either method) at this expression for the integral

$$\frac{\pi}{4}\times\left(1-\frac{n^2}{2.4}+\frac{n^4}{2.4^2.6}-\frac{n^6}{2.4^2.6^2.8}+\&c.\right)$$

The table appended to this paper contains the values of the series in the bracket, for every 0,2 from $n=0$ to $n=12$. Each value has been calculated separately, the logarithms used in the calculation have been systematically checked, and the whole process has been carefully examined. The calculations were carried to one place further than the numbers here exhibited. I believe that they will seldom be found in error more than a unit of the last place; except perhaps in some of the last values, where the rapid divergence of the series for the first five or six terms made it difficult to calculate them accurately by logarithms.

In the use of this table n must be taken $= \frac{2\pi}{\lambda} \cdot \frac{ba}{f}$. If instead of using the linear distance b to define the point of the field at which we wish to ascertain the illumination, we use the number of seconds s, then $b = f.s.\sin 1''$, and n must be taken $= \frac{2\pi}{\lambda}\, as \sin 1''$. If λ be taken for mean rays $= 0{,}000022$ inch, n must be taken $= 1{,}3846 \times as$, a being expressed in inches. From this expression, and from the numbers of the table, we draw the following inferences.

1. The image of a star will not be a point but a bright circle surrounded by a series of bright rings. The angular diameters of these (or the value of s corresponding to a given value of n) will depend on nothing but the aperture of the telescope, and will be inversely as the aperture.

2. The intensity of the light being expressed (on the principles of the undulatory theory) by the square of the coefficient of

$$\sin \frac{2\pi}{\lambda}(vt - f - A),$$

and the intensity at the center of the circle being taken as the standard, it appears that the central spot has lost half its light when $n = 1{,}616$, or $s = \frac{1{,}17}{a}$; that there is total privation of light, or a black ring, when $n = 3{,}832$, or $s = \frac{2{,}76}{a}$; that the brightest part of the first bright ring corresponds to $n = 5{,}12$, or $s = \frac{3{,}70}{a}$, and that its intensity is about $\frac{1}{57}$ of that at the center; that there is a black ring when $n = 7{,}14$, or $s = \frac{5{,}16}{a}$; that the brightest part of the second bright ring corresponds to $n = 8{,}43$, or $s = \frac{6{,}09}{a}$, and that its intensity is about $\frac{1}{240}$ of that of the center; that there is a black ring when $n = 10{,}17$, or $s = \frac{7{,}32}{a}$; that the brightest part of the third bright ring corresponds to $n = 11{,}63$, or $s = \frac{8{,}40}{a}$, and that its intensity is about $\frac{1}{620}$ of that of the center.

The rapid decrease of light in the successive rings will sufficiently explain the visibility of two or three rings with a very bright star and the non-visibility of rings with a faint star. The difference of the diameters of the central spots (or spurious disks) of different stars (which has presented a difficulty to writers on Optics) is also fully explained. Thus the radius of the spurious disk of a faint star, where light of less than half the intensity of the central light makes no impression on the eye, is determined by making $n = 1{,}616$, or $s = \frac{1{,}17}{a}$: whereas the radius of the spurious disk of a bright star, where light of $\frac{1}{10}$ the intensity of the central light is sensible, is determined by making $n = 2{,}73$, or $s = \frac{1{,}97}{a}$.

The general agreement of these results with observation is very satisfactory. It is not easy to obtain measures of the rings; since when a is made small enough to render them very distinct as to form and separation, the intensity of their light (which varies as a^4) is so feeble that they will not bear sufficient illumination for the use of a micrometer. Fraunhofer however obtained measures agreeing pretty well (as to proportion of diameters, &c.) with the results above.

For verification of the numbers it would probably be best to use an elliptic aperture. By an investigation of exactly the same kind as that above it will be found that the rings will then be ellipses exactly similar to the ellipse of the aperture, but in a transverse position; that the major axes of the rings for the elliptic aperture will be the same as the diameters of the rings for a circular aperture whose diameter = minor axis of ellipse of aperture, but that the intensity will be greater in the proportion of the squares of the axes. I have not yet had an opportunity of examining this in practice.

I shall now apply the numbers of the table to the solution of the following problem. To find the diameters, &c. of the rings when a circular patch, whose diameter is half the diameter of the object-glass, is applied to its center, so as to leave an annular aperture.

The radius of the patch being $\frac{a}{2}$, it is easily seen that the displacement (using the same notation) is

$$2\sin\frac{2\pi}{\lambda}(vt-f-A)\int_x \sqrt{a^2-x^2}\,.\cos\frac{2\pi}{\lambda}\,.\,\frac{b}{f}x \text{ (from } x=-a \text{ to } x=+a)$$

$$-2\sin\frac{2\pi}{\lambda}(vt-f-A)\int_x \sqrt{\frac{a^2}{4}-x^2}\,.\cos\frac{2\pi}{\lambda}\,.\,\frac{b}{f}x \left(\text{from } x=-\frac{a}{2} \text{ to } x=+\frac{a}{2}\right).$$

Putting $\frac{x}{a}=w$, $\frac{2x}{a}=u$, this becomes

$$4a^2\,.\sin\frac{2\pi}{\lambda}(vt-f-A)\int_w \sqrt{1-w^2}\,.\cos\frac{2\pi}{\lambda}\,.\,\frac{ba}{f}w$$

$$-4\,.\,\frac{a^2}{4}\,.\sin\frac{2\pi}{\lambda}(vt-f-A)\int_u \sqrt{1-u^2}\,.\cos\frac{2\pi}{\lambda}\,.\,\frac{ba}{2f}u,$$

the limits of integration both for w and for u being 0 and 1. Omitting the factor $a^4\pi$, the intensity will be expressed by

$$\left\{\phi(n)-\tfrac{1}{4}\phi\left(\frac{n}{2}\right)\right\}^2,$$

where $\phi(n)$ is the number given in the table.

Upon forming the numerical values we find that the black rings correspond to values of $n=3{,}15,\ 7{,}18,\ 10{,}97$: and that the intensities of the bright rings (in terms of the intensity of the center) are $\frac{1}{10}$, $\frac{1}{80}$. Thus the magnitude of the central spot is diminished, and the brightness of the rings increased, by covering the central part of the object-glass.

In like manner, if the diameter of the circular patch $=a(1-p)$, the intensity of light would be proportional to $\{\phi(n)-(1-p)^2\,.\,\phi(n-pn)\}^2$.

The quantity under the bracket, if p is very small, is equal to

$$2\,p\,.\,\phi\,(n) + pn\phi'\,(n) = \frac{p}{n}\,.\,\frac{d}{dn}\,\{n^2\phi\,(n)\}.$$

In the case of a very narrow annulus therefore the diameters of the black rings will be determined by making $n^2\phi\,(n)$ maximum or minimum. It appears then that there ought to be only one black ring corresponding to each black ring with the full aperture, and that its diameter ought to be somewhat smaller. This conclusion does not agree with the experiments recorded by Sir J. Herschel, in the Encyc. Metrop. Article Light, page 488: but it is acknowledged there that the results are discordant with Fraunhofer's: and I am inclined therefore to attribute the phænomena observed by Sir J. Herschel to some other cause.

The investigation of cases of diffraction similar to that discussed here appears to me a matter of great interest to those who are occupied with the examination of theories of light. The assumption of transversal vibrations is not necessary here as for the explanation of the phænomena of polarization: and they therefore offer no arguments for the support of that principle. But they require absolutely the supposition of almost unlimited divergence of the waves coming not merely from a small aperture, but also from every point of a large wave: and the results to which they lead us, shew strikingly how small foundation there was for the original objection to the undulatory theory of light, viz. that if waves spread equally in all directions, there could be no such thing as darkness.

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TABLE of the values of $\phi(n) = \frac{4}{\pi} \int_w \sqrt{1-w^2} . \cos nw$ from $w=0$ to $w=1$.

n	$\phi(n)$	n	$\phi(n)$
0,0	+ 1,0000	6,0	− 0,0922
0,2	+ ,9950	6,2	− ,0751
0,4	+ ,9801	6,4	− ,0568
0,6	+ ,9557	6,6	− ,0379
0,8	+ ,9221	6,8	− ,0192
1,0	+ ,8801	7,0	− ,0013
1,2	+ ,8305	7,2	+ ,0151
1,4	+ ,7742	7,4	+ ,0296
1,6	+ ,7124	7,6	+ ,0419
1,8	+ ,6461	7,8	+ ,0516
2,0	+ ,5767	8,0	+ ,0587
2,2	+ ,5054	8,2	+ ,0629
2,4	+ ,4335	8,4	+ ,0645
2,6	+ ,3622	8,6	+ ,0634
2,8	+ ,2927	8,8	+ ,0600
3,0	+ ,2261	9,0	+ ,0545
3,2	+ ,1633	9,2	+ ,0473
3,4	+ ,1054	9,4	+ ,0387
3,6	+ ,0530	9,6	+ ,0291
3,8	+ ,0067	9,8	+ ,0190
4,0	− ,0330	10,0	+ ,0087
4,2	− ,0660	10,2	− ,0013
4,4	− ,0922	10,4	− ,0107
4,6	− ,1116	10,6	− ,0191
4,8	− ,1244	10,8	− ,0263
5,0	− ,1310	11,0	− ,0321
5,2	− ,1320	11,2	− ,0364
5,4	− ,1279	11,4	− ,0390
5,6	− ,1194	11,6	− ,0400
5,8	− ,1073	11,8	− ,0394
6,0	− ,0922	12,0	− ,0372