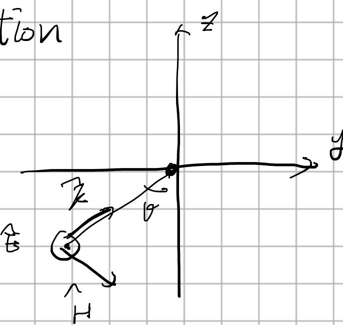
4.3 EM wave propagation in a stratified dielectric medium

$\varepsilon = \varepsilon(z)$, $\mu = \mu(z)$ $\qquad z$: propagation direction

multilayer : a pile of the films

- anti reflection ...
- enhancement of refractivity
- beam splitter.



yz-plane : plane of incidence

harmonic plane wave, TE-mode (s-polarization) $E_x \neq 0,\ E_y = E_z = 0$

Maxwell's equation.

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} + i\varepsilon\frac{\omega}{c}E_x = 0, \qquad \frac{i\omega\mu}{c}H_x = 0$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = 0, \qquad \frac{\partial E_x}{\partial z} - \frac{i\omega\mu}{c}H_y = 0$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0, \qquad \frac{\partial E_x}{\partial y} + \frac{i\omega\mu}{c}H_z = 0$$

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)E_x + n^2 k_0^2 E_x = \frac{d}{dz}(\log(\mu))\frac{\partial E_x}{\partial z}, \quad n^2 = \varepsilon\mu, \quad k_0 = \frac{\omega}{c} = \frac{2\pi}{\lambda_0}$$

(cf. $\nabla^2\vec{E} + \frac{n^2\omega^2}{c^2}\vec{E} + (\nabla\log\mu)\times(\nabla\times\vec{E}) + \nabla(\vec{E}\cdot\nabla\log\varepsilon) = 0 \qquad \vec{E} \perp \hat{z},\ \nabla\log\varepsilon \propto \hat{z}$)

solving this equation by Separate of variables method. $E(y,z) = Y(y)\,U(z)$

$$\underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{= -k_0^2\alpha^2} = -\frac{1}{U}\frac{d^2U}{dz^2} - n^2k_0^2 + \frac{d}{dz}(\log\mu)\frac{1}{U}\frac{dU}{dz} \qquad \left(\text{cf. } k\cos(90^\circ - \theta) = k\sin\theta = \frac{k_0 n\sin\theta}{\alpha}\right)$$

$$Y = \text{const.}\ e^{ik_0\alpha y} \qquad \frac{d^2U}{dz^2} - \frac{d}{dz}(\log\mu)\frac{dU}{dz} + (n^2-\alpha^2)k_0^2 U = 0$$

$$E_x = U(z)e^{i(k_0\alpha y - \omega t)}, \quad H_y = V(z)e^{i(k_0\alpha y - \omega t)}, \quad H_z = W(z)e^{i(k_0\alpha y - \omega t)}$$

From Maxwell's equation,

$$V' = ik_0(\alpha W + \epsilon U), \quad U' = ik_0\mu V, \quad \alpha U + \mu W = 0,$$

$$U' = ik_0\mu(z)V, \quad V' = ik_0\left(\epsilon - \frac{\alpha^2}{\mu}\right)U, \quad \frac{d^2V}{dz^2} - \frac{d}{dz}\left[\log\left(\epsilon - \frac{\alpha^2}{\mu}\right)\right]\frac{dV}{dz} + k_0^2(n^2 - \alpha^2)V = 0$$

A second-order linear equation for both U, V two independentes solutions

U, V can be expressed as a linear combination of two particular solutions U_1, U_2 ; V_1, V_2.

$$\frac{d}{dz}(U_1V_2 - U_2V_1) = 0 \Rightarrow \begin{vmatrix} U_1 & U_2 \\ V_1 & V_2 \end{vmatrix} = \text{Const.}$$

$U_2(z=0) = V_1(z=0) = 1$, $U_1(z=0) = V_2(z=0) = 0$ with $U(z=0) = U_0$, $V(z=0) = V_0$

$$U(z) = U_1V_0 + U_2U_0$$
$$V(z) = V_1V_0 + V_2U_0$$

$$\begin{pmatrix} U(z) \\ V(z) \end{pmatrix} = \begin{pmatrix} U_2 & U_1 \\ V_2 & V_1 \end{pmatrix}\begin{pmatrix} U(z=0) \\ V(z=0) \end{pmatrix}$$

$$\vec{Q} = \overleftrightarrow{N}\cdot\vec{Q_0}, \quad \vec{Q_0} = \overleftrightarrow{M}\cdot Q(z), \quad \overleftrightarrow{M} = \overleftrightarrow{N}^{-1} = \begin{pmatrix} V_1 & -U_1 \\ -V_2 & U_2 \end{pmatrix}$$: characteristic matrix

$$|\overleftrightarrow{M}| = V_1U_2 - U_1V_2 = V_1(0)U_2(0) - U_1(0)V_2(0) = 1$$

Ex) Characteristic Matrix for homogeneous dielectric film TE mode, ϵ, μ, constant.

$$\frac{d^2U}{dz^2} + (k_0^2n^2\cos^2\theta)U = 0, \quad \frac{d^2V}{dz^2} + (k_0^2n^2\cos^2\theta)V = 0$$

U, V : combination of $\sin(k_0 n\cos\theta)$, $\cos(k_0 n\cos\theta)$

subject to $U' = ik_0\mu V$, $V' = ik_0\left(\epsilon - \frac{\alpha^2}{\mu}\right)U = ik_0\epsilon\cos^2\theta\, U$

$$U(z) = A\cos(k_0 n z\cos\theta) + B\sin(k_0 n z\cos\theta)$$

$$V(z) = \frac{1}{i}\sqrt{\frac{\varepsilon}{\mu}}\cos\theta\left[B\cos(k_0 n z\cos\theta) - A\sin(k_0 n z\cos\theta)\right]$$

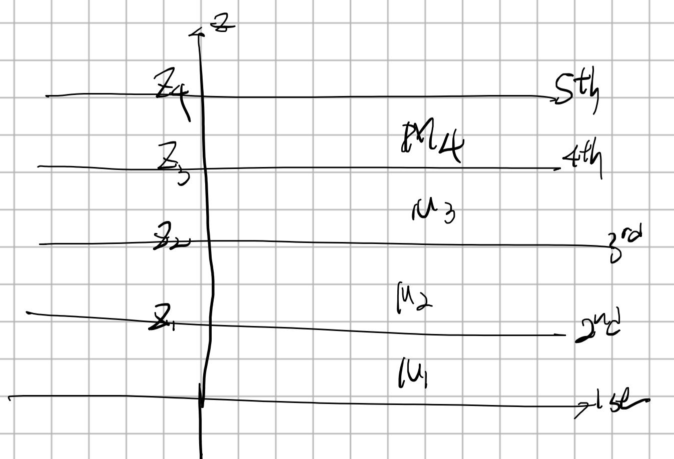
Particular solutions with $V_1(0)=1,\ V_2(0)=0$, $U_1(0)=0,\ U_2(0)=1$ & $\begin{vmatrix} U_1 & U_2 \\ V_1 & V_2 \end{vmatrix} = \text{const.}$

$$V_1 = \cos(k_0 n z\cos\theta),\quad V_2 = i\sqrt{\frac{\varepsilon}{\mu}}\cos\theta\sin(k_0 n z\cos\theta)$$

$$U_1 = \frac{i}{\cos\theta}\sqrt{\frac{\mu}{\varepsilon}}\sin(k_0 n z\cos\theta),\quad U_2 = \cos(k_0 n z\cos\theta)$$

Then
$$\overleftrightarrow{M} = \begin{bmatrix} \cos(k_0 n z\cos\theta) & -\frac{i}{p}\sin(k_0 n z\cos\theta) \\ -ip\sin(k_0 n z\cos\theta) & \cos(k_0 n z\cos\theta) \end{bmatrix} = \begin{bmatrix} \cos\beta & -\frac{i}{p}\sin\beta \\ -ip\sin\beta & \cos\beta \end{bmatrix}$$

$$p = \sqrt{\frac{\varepsilon}{\mu}}\cos\theta,\quad \beta = k_0 n z\cos\theta \qquad = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$



$$Q_0 = M_1(z_1)\,Q(z_1)$$

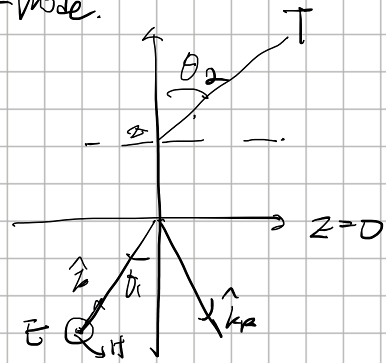
$$Q_1(z) = M_2(z_2 - z_1)\,Q(z_2)$$

$$Q = M_1(z_1)\,M_2(z_2 - z_1)\,Q_1(z_2)$$

$$Q = M_1(z_1)\,M_2(z_2 - z_1)\cdots M_N(z_N - z_{N-1})\,Q_1(z_N)$$

Reflectivity & Transmittance

TE mode.



$E_{in} = A\hat{x}$, $E_R = R\hat{x}$

$\vec{k}_{in} = \cos\theta\hat{z} - \sin\theta\hat{y}$

$\vec{H} = \sqrt{\frac{\epsilon}{\mu}}\,\hat{k}\times\vec{E}$, $\vec{k}_R = -\cos\theta\hat{z} - \sin\theta\hat{y}$

$\mu = 1$ for convenience.

$U_0 = U(z=0) = E_{in}(z=0) + E_R(z=0) = A+R$

$V_0 = V(z=0) = H_{y,in}(z=0) + H_{y,R}(z=0)$

$= P_1(A-R)$ $\quad P_1 = n_1\cos\theta_1$

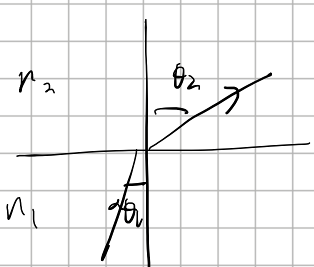
$U(z) = T$, $V(z) = pT$, $p = n\cos\theta_2$, $Q_0 = MQ$, $\begin{pmatrix}A+R\\ P_1(A-R)\end{pmatrix} = M\begin{pmatrix}T\\ pT\end{pmatrix}$

$$r = \frac{R}{A} = \frac{(m_{11}+m_{12}p)p_1 - (m_{21}+m_{22}p)}{(m_{11}+m_{12}p)p_1 + (m_{21}+m_{22}p)}$$

$$t = \frac{2p_1}{(\quad)}$$

$$R = |r|^2, \quad T = \frac{p}{p_1}|t|^2$$

i) One interface



TE mode $p_2 = n_2\cos\theta_2$, $p_1 = n_1\cos\theta_1$, $\beta = k_0 n_2 z\cos\theta_2 = 0$
$z=0$.

$$r_{12} = \frac{p_1 - p_2}{p_1 + p_2} = \frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2}$$

$$t_{12} = \frac{2p_1}{p_1 + p_2} = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + n_2\cos\theta_2}$$

the same as Fresnel's formation.