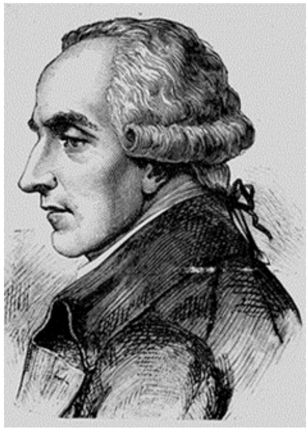
Chapter 4
INTEGRAL TRANSFORMS

Joseph Fourier
(1768-1830)
Math/Physics
Fourier Series/Transform

Lecture 12

4.1 General Form of Integral Transforms
4.2 Fourier Transform



Pierre-Simon Laplace
(1749-1827)
Math/Physic
Laplace Transform
Laplace Equation
(Scalar Potential Theory)

4.1 General Form of Integral Transforms

In physics and engineering problems, we frequently encounter integral transforms. For example, we use integral transforms to convert **differential equations into algebraic equations**, which makes the problems much easier.

There are several kinds of integral transforms, and we can find a general form for these integral transforms:

$$\boxed{F(k) = \int_a^b dx\, K(k,x)\, f(x)} \qquad (4.1)$$

where $K(x,k)$ is called as the **kernel function** of the integral transform.

Integral Transform as an Operator

In analogy to differential operators, the integral transform can be defined as an operator that transforms $f(x)$ from x space to k space:

$$\boxed{\begin{array}{c} F(k) = I(x,k)\, f(x) \\ f(x) \xrightarrow{I(x,k)} F(k) \end{array}} \qquad (4.2)$$

with an integral operator defined as

$$\boxed{I(x,k) = \int_a^b dx\, K(k,x)} \qquad (4.3)$$

There are four widely used kernels for integral transforms:

$$\boxed{F(k) = \begin{cases} \mathscr{F}[f(x)] = \int_{-\infty}^{\infty} dx\, e^{ikx} f(x) \\ \mathscr{H}[f(x)] = \int_{0}^{\infty} dx\, x\, J_n(kx) f(x) \\ \mathscr{L}[f(x)] = \int_{0}^{\infty} dx\, e^{-kx} f(x) \\ \mathfrak{M}[f(x)] = \int_{0}^{\infty} dx\, x^{k-1} f(x) \end{cases}} \qquad (4.4)$$

Fourier Transform
Hankel Transform
Laplace Transform
Mellin Transform

We can also define the inverse integral transforms, which will be found in the next sections.

4.1 Fourier Transform

The Fourier transform (FT) pairs are usually used to transform a function between two domains, between **space-time ($\mathbf{r},t$) and spatial-temporal frequency** ($\mathbf{k},\omega$) domains. Note that it's physical interpretation is based on the **superposition principle**.

For a **traveling wave in *n*-D space**, a main topic in this lecture, the FT pair is defined as

$$\boxed{\begin{aligned} F(\mathbf{k},\omega) &= \int_{-\infty}^{\infty} d\mathbf{r}^n \int_{-\infty}^{\infty} dt\, f(\mathbf{r},t) e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \\ f(\mathbf{r},t) &= \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} d^n\mathbf{k} \int_{-\infty}^{\infty} d\omega\, F(\mathbf{k},\omega) e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \end{aligned}} \tag{4.5}$$

with the convergence (or existence) conditions of FT:

1) $f(\mathbf{r},t)$ is **piece-wise continuous**, and at the discontinuity

$$f(\mathbf{r},t) = \frac{1}{2}\left[f(\mathbf{r}^+,t^+) + f(\mathbf{r}^-,t^-) \right] \tag{4.6}$$

2) $f(\mathbf{r},t)$ is **square-integrable** :

$$\int_{-\infty}^{\infty} d^n\mathbf{r} \int_{-\infty}^{\infty} dt \left| f(\mathbf{r},t) \right|^2 < \infty \tag{4.7}$$

where, n is the spatial dimension, $\mathbf{k}$ is the wave vector (spatial frequency vector), and ω is the angular frequency.

This means that the convergence condition should satisfy a weak one :

$$\lim_{\mathbf{r},t\to\pm\infty} f(\mathbf{r},t) = 0 \tag{4.8}$$

where, n is the spatial dimension, $\mathbf{k}$ is the wave vector (spatial frequency vector), and ω is the angular frequency.

Symmetry and Antisymmetry of Real-Valued Function

If $f(t)$ is a real-valued function, representing a physical quantity in time domain, then

$$\boxed{F^*(\omega) = F(-\omega)} \tag{4.9}$$

from which we find that the real and imaginary parts of $F(\omega)$ are symmetric (or even) and antisymmetric (or odd) functions, respectively,

$$\boxed{\begin{aligned} \operatorname{Re} F(\omega) &= \frac{1}{2}\left[F(\omega) + F(-\omega)\right] \\ \operatorname{Im} F(\omega) &= \frac{1}{2}\left[F(\omega) - F(-\omega)\right] \end{aligned}} \tag{4.10}$$

The proof is easy and will be assigned as a homework or quiz problem.

Dirac Delta Function

The Dirac delta function is frequently encountered in many physics problems, and can be defined as a Fourier transform of $f(t)=1$:

$$\boxed{\begin{aligned}\delta(t)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\, e^{\pm i\omega t}\\ \delta(\omega)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}dt\, e^{\pm i\omega t}\end{aligned}} \tag{4.10}$$

Proof) Although $f(t)=1$ is not square-integrable, we still define the Dirac delta function, which is not an ordinary function, but a generalized function (or distribution), as a Fourier integral. Using a limiting process, we have

$$\mathfrak{F}[1]=\int_{-\infty}^{\infty}dt\, e^{i\omega t}=\lim_{R\to\infty}\int_{-R}^{R}dt\, e^{i\omega t}=\lim_{R\to\infty}\frac{e^{i\omega R}-e^{-i\omega R}}{i\omega}$$

$$=\lim_{R\to\infty}\frac{2\sin\omega R}{\omega}=2\pi\lim_{R\to\infty}\frac{1}{\pi}\frac{\sin\omega R}{\omega}=2\pi\delta(\omega)$$

Actually, we see that the FT pairs includes a definition of the Dirac delta function by substituting the first equation into the second one in (4.5), and vice versa as follows:

$$f(\mathbf{r},t)=\frac{1}{(2\pi)^{n+1}}\int_{-\infty}^{\infty}d^n\mathbf{k}\int_{-\infty}^{\infty}d\omega\left(\int_{-\infty}^{\infty}d^n\mathbf{r}'\int_{-\infty}^{\infty}dt'\ f(\mathbf{r}',t')e^{-i(\mathbf{k}\cdot\mathbf{r}'-\omega t')}\right)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

$$=\frac{1}{(2\pi)^{n+1}}\int_{-\infty}^{\infty}d^n\mathbf{r}'\int_{-\infty}^{\infty}dt'\left(\int_{-\infty}^{\infty}d^n\mathbf{k}\int_{-\infty}^{\infty}d\omega\ f(\mathbf{r}',t')e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}e^{-i\omega(t-t')}\right)$$

$$=\int_{-\infty}^{\infty}d^n\mathbf{r}'\int_{-\infty}^{\infty}dt'f(\mathbf{r}',t')\left(\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}d^n\mathbf{k}e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega e^{-i\omega(t-t')}\right)$$

Now we remember the sifting property of the Dirac delta function

$$f(\mathbf{r},t)=\int_{-\infty}^{\infty}d^n\mathbf{r}'\int_{-\infty}^{\infty}dt'f(\mathbf{r}',t')\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

and therefore we obtain

$$\boxed{\begin{aligned}\delta(\mathbf{r}-\mathbf{r}')&=\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}d^n\mathbf{r}'e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\\ \delta(t-t')&=\frac{1}{2\pi}\int_{-\infty}^{\infty}dt'e^{-i\omega(t-t')}\end{aligned}}$$