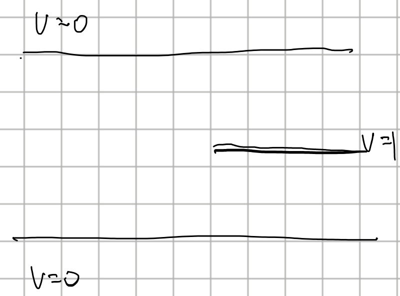
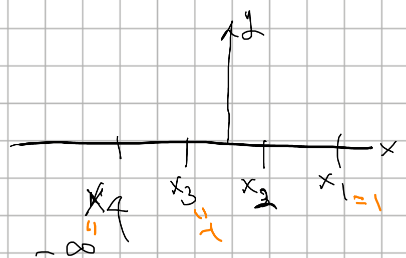
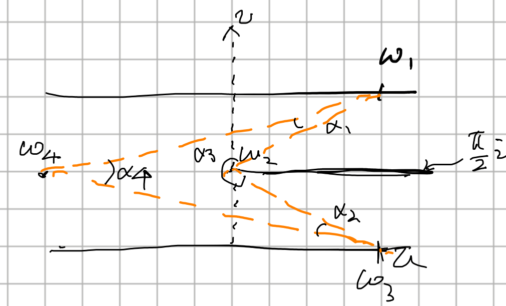
Ex 5)

Potential distribution?





$w=f(z)$ $\rightarrow$



$-\infty - x_3 \longrightarrow w_3 - w_4$ $\qquad$ $w_1, w_3 \longrightarrow +\infty$

$x_3 - x_2 \longrightarrow w_2 - w_3$ $\qquad$ $w_4 \longrightarrow -\infty$

$x_2 - x_1 \longrightarrow w_2 - w_1$ $\qquad$ $\alpha_1 = 0, \alpha_2 = 2\pi, \alpha_3 = 0$

$x_1 - \infty \longrightarrow w_1 - w_4$

$x_1 = 1, x_3 = -1$

$$\frac{dw}{dz} = C_1 \frac{1}{z+1}(z-x_2)\frac{1}{z-1} = \frac{C_1}{2}\left(\frac{1+x_2}{z+1} + \frac{1-x_2}{z-1}\right)$$

$$w = \int \frac{dw}{dz} = \frac{C_1}{2}\left[(1+x_2)\ln(z+1) + (1-x_2)\ln(z-1)\right] + C_2$$

$C_1 = C_1' + C_1''$, $C_2 = C_2' + C_2''$

$z \pm 1 = |z \pm 1| e^{i\theta_\pm}$, $\theta_\pm = \arg(z \pm 1)$

$$= \frac{C_1'}{2}\left[(1+x_2)\ln|z+1| + (1-x_2)\ln|z-1|\right]$$
$$- \frac{C_1''}{2}\left[(1+x_2)\arg(z+1) + (1-x_2)\arg(z-1)\right] + C_2'$$
$$+ i\left\{\frac{C_1''}{2}\left[(1+x_2)\ln|z+1| + (1-x_2)\ln|z-1|\right]\right.$$
$$\left. + \frac{C_1'}{2}\left[(1+x_2)\arg(z+1) + (1-x_2)\arg(z-1)\right] + C_2''\right\}$$

i) $z = x < -1$: $w_4 - w_1$ line (u-axis, $v=0$)

$$\arg(z+1) = \pi = \arg(z-1)$$

$v=0$: $z = x \to -1$, $\ln|z+1| \to -\infty \Rightarrow C_1'' = 0$

$$0 = \frac{C_1'}{2}\left[(1+x_2)\pi + (1-x_2)\pi\right] + C_2'' \quad \to -\pi C_1' = C_2''$$

ii) $-1 < z = x \le x_2$: $w_3 - w_2$ line ($u \ge 0$, $v = \frac{\pi}{2}$)

$$\arg(z+1) = 0 \,, \quad \arg(z-1) = \pi$$

$$\frac{\pi}{2} = \frac{C_1'}{2}(1-x_2)\pi + C_2''$$

iii) $z = x > 1$: $w_1 - w_4$ line ($w = u + i\frac{\pi}{2}$)

$$\arg(z+1) = \arg(z-1) = 0$$

$$\pi = C_2''$$

iv) $z = 0$ ($x=0$, $y=0$), $w = 0 + \frac{\pi}{2}i$

$$\arg(z+1) = 0, \quad \arg(z-1) = \pi$$

real part : $0 = 0 + C_2'$

$\Rightarrow C_1' = -1$, $C_1'' = 0$, $C_2' = 0$, $C_2'' = \pi$, $x_2 = 0$

$C_1 = -1$, $C_2 = i\pi$, $x_2 = 0$

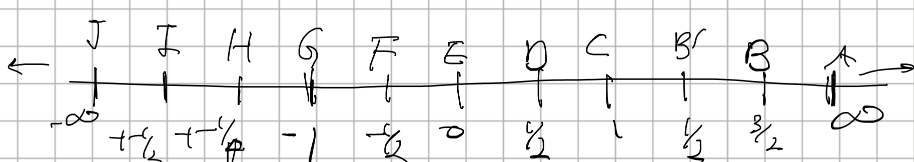
$$w = -\frac{1}{2}\left[\ln(z+1) + \ln(z-1)\right] + i\pi$$

or $z^2 = 1 + e^{-2w}$

Check: $z-1 = r_1 e^{i\theta_1}$, $z+1 = r_2 e^{i\theta_2}$

$$w = -\frac{1}{2}(\ln r_1 r_2) + i\left(\pi - \frac{\theta_1+\theta_2}{2}\right)$$

	z	r_1	r_2	θ_1	θ_2	w
A	∞	∞	∞	0	0	$-\infty + i\pi$
B	$3/2$	$5/2$	$1/2$	0	0	$-\frac{1}{2}\ln\frac{5}{4} + i\pi$
B'	$1+1/4$	$9/4$	$1/4$	0	0	$\frac{1}{2}\ln\frac{16}{9} + i\pi$
C	1	$3/2$	0	0	0	$+\infty + i\pi$
D	$1/2$	$3/2$	$1/2$	π	0	$\frac{1}{2}\ln 4/3 + i\frac{\pi}{2}$
E	0	1	1	π	0	$i\pi/2$
F	$-1/2$	$1/2$	$3/2$	π	0	$1/2 \ln 4/3 + i\pi/2$
G	-1	0	2	π	0	$+\infty + i\frac{\pi}{2}$
H	$-1-\frac{1}{4}$	$1/4$	$9/4$	π	π	$\frac{1}{2}\ln\frac{16}{9} + i0$
I	$-1-\frac{1}{2}$	$1/2$	$5/2$	π	π	$-\frac{1}{2}\ln\frac{5}{4} + i0$
J	$-\infty$	∞	∞	π	π	$-\infty + i0$



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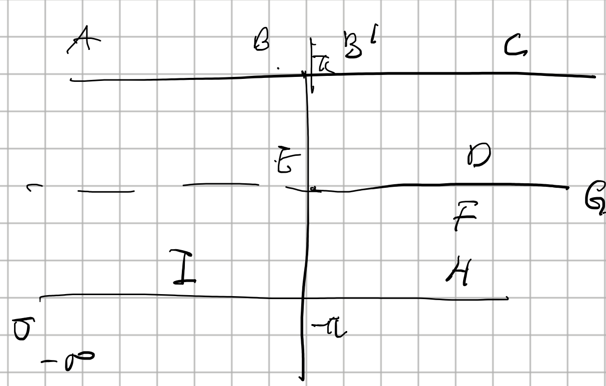

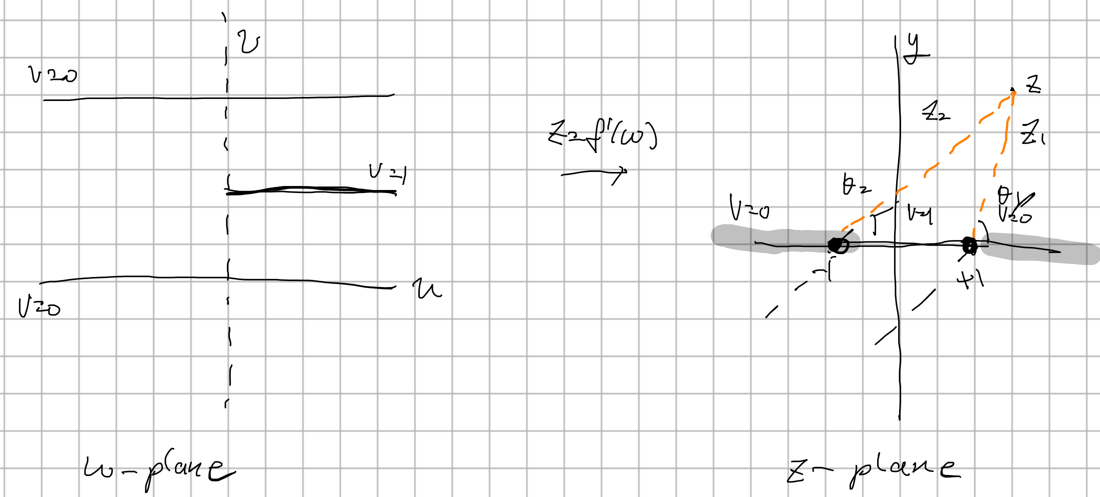

cf. potential by a line charge: $\phi = -2\lambda \ln r$

Consider two line charge at $x = \pm 1$

$$\phi(z) = -2\lambda_1 \ln(z-1) - 2\lambda_2 \ln(z+1)$$

$$= -2\lambda_1 \ln r_1 - 2\lambda_2 \ln r_2 - 2i(\lambda_1\theta_1 + \lambda_2\theta_2)$$

B.C. $x < -1, \ \phi = 0$ $\qquad \theta_1 = \theta_2 = \pi, \ (\lambda_1 + \lambda_2)\pi = 0$

$-1 < x < 1, \ \phi = 1$ $\quad \Rightarrow \quad \theta_1 = \pi, \ \theta_2 = 0, \ -2\lambda_1\pi = 1$

$x > 1, \ \phi = 0$ $\qquad \lambda_1 = -\lambda_2, \ -2\lambda_1 = 1/\pi$

$$\phi(z) = \frac{1}{\pi} \ln \frac{|z-1|}{|z+1|} + i\frac{1}{\pi}(\theta_1 - \theta_2)$$

A solution satisfying the boundary conditions

$\phi = \frac{1}{\pi}(\theta_1 - \theta_2)$ or

$$\tan(\pi\phi) = \tan(\theta_1 - \theta_2) = \frac{2y}{x^2+y^2-1}$$

$$= \frac{1}{s}\sqrt{e^{-4u} - s^2}$$

$z = 1 + e^{-2w}$

$$s = -1 + \sqrt{1 + 2e^{-2u}\cos 2v + e^{-4u}}$$

Ⅴ. Method by separation of variables

- the most general method
- a solution is given by an expansion in orthogonal functions.

$\rho \neq 0$ $\quad \nabla^2 \phi = -4\pi\rho$ (Poisson' equation)

Green's function method

$\nabla^2 F = 0$ needs to be solved

$\rho = 0$ $\quad \nabla^2 \phi = 0$ (Laplace equation)

a) Basics

① A set of orthogonal functions $\{f_n\}$, $a \le x \le b$, $n = 1, \cdots, \infty$

$$\int_a^b dx\, f_n(x)\, f_m(x) = \begin{cases} 0 & m \neq n \\ C & m = n \end{cases}$$

$$\int_a^b dx\, f_n(x)\, f_m(x) \equiv \delta_{mn}$$

orthonormal, f is normalized

Then any function $F(x)$ can be represented in terms of $\{f_n\}$

$$F(x) = \sum_{n=1}^{\infty} a_n f_n(x), \quad a_n = \int_a^b dx\, F(x)\, f_n(x)$$

② the most famous example: Fourier expansion or series

$$\sqrt{\frac{2}{a}}\sin\left(m\frac{2\pi}{a}x\right),\quad \sqrt{\frac{2}{a}}\cos\left(m\frac{2\pi}{a}x\right)$$

$$-\frac{a}{2}\le x\le\frac{a}{2}$$

$$F(x)=\frac{1}{2}A_0+\sum_{m=1}^{\infty}\left[A_m\cos\left(m\frac{2\pi}{a}x\right)+B_m\sin\left(m\frac{2\pi}{a}x\right)\right]$$

$$A_m=\frac{2}{a}\int_{-\frac{a}{2}}^{a/2}dx\;F(x)\cos\left(m\frac{2\pi}{a}x\right)$$

$$B_m=\frac{2}{a}\int_{-\frac{a}{2}}^{a/2}dx\;F(x)\sin\left(m\frac{2\pi}{a}x\right)$$