Mathematics
2013 Doctoral Qualifying Exam

Caution!!!
Use separate answer books for Problems 1-5 (Math.-A) and 6-7 (Math.-B).

Math.-A

[**1**](10pt) Express $f(x, y) = x^2 y \cos y$ in the Taylor series expansion up to second order around $(x, y) = (1, 0)$. Should be written in the following form

$$f(x,y) = f(1,0) + [\ \cdot\ \ \cdot\ ]\begin{bmatrix} \cdot \\ \cdot \end{bmatrix} + \frac{1}{2}[\ \cdot\ \ \cdot\ ]\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}.$$

[**2**](10pt) Show that the eigenvalues are invariant under the similarity transformation, i.e., for matrix $A \in \mathcal{R}^{n\times n}$ that has distinct n eigenvalues, show that

$$det(sI - A) = det(sI - T^{-1}AT) = 0$$

where $T \in \mathcal{R}^{n\times n}$, $I \in \mathcal{R}^{n\times n}$ is the identity matrix, s is a scalar variable, and $det(A)$ implies the determinant of a matrix A.

[**3**](10pt) Evaluate

$$\int_C 3x^2 y ds$$

clockwise along the circle $C\ :\ x^2 + y^2 = 1$ from (0,1) to (1,0).

[**4**](10pt) Find the solution using the Laplace transformation:

$$y'' + y = 2\cos t, \qquad y(0) = 2, \qquad y'(0) = 0.$$

[**5**](10pt) Find the limit of

$$\lim_{n\to\infty}\left(\frac{2n}{2n-1}\right)^n.$$

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Math.-B

Problem 6. (25 points) When $\underline{x}$ is a length-m column vector of complex entries, we define the p-norm of $\underline{x}$ as

$$\|\underline{x}\|_p \triangleq \left(\sum_{i=1}^{m} |x_i|^p\right)^{\frac{1}{p}}$$

where m is a natural number, x_i satisfying $|x_i| < \infty$ is the i-th entry of $\underline{x}$, and $p \geq 1$. Let j denote $\sqrt{-1}$ and answer the following questions.

(a) (5 points) When $\underline{x} = [-3,\ 4\mathrm{j}\ ]^T$, find the p-norms of $\underline{x}$ for $p = 1$ and 2, respectively.

(b) (5 points) When $m = 2$ and x_i's are all real-valued, sketch on two-dimensional plane the sets of $\underline{x}$ such that $\|\underline{x}\|_p = 1$ for $p = 1$ and 2, respectively.

(c) (10 points) When $m = 2$, find

$$\lim_{p\to\infty} \|\underline{x}\|_p,$$

in terms of $|x_1|$ and $|x_2|$. (Hint. Consider all three different cases: i) $|x_1| > |x_2|$, ii) $|x_1| = |x_2|$, and iii) $|x_1| < |x_2|$.)

(d) (5 points) Based on the result in (c), introduce a definition of the ∞-norm $\|\underline{x}\|_\infty$ of $\underline{x}$ that holds for all $m \geq 1$.

Problem 7. (25 points) Suppose that $s(t)$ is a finite-energy strictly-bandlimited signal with bandwidth B, i.e., its Fourier transform $S(f)$ has no energy outside the interval $|f| < B$. Answer the following questions.

(a) (5 points) State the analysis and the synthesis equations that show the relationship between $s(t)$ and $S(f)$.

(b) (10 points) When $T > 0$, define $p(t)$ as

$$p(t) = \sum_{n=-\infty}^{\infty} s(t - nT).$$

Find the Fourier transform $P(f)$ of $p(t)$ in terms of $S(f)$.

(c) (10 points) Find the necessary and sufficient condition on T for $p(t)$ in (b) to be a constant function of t for any finite-energy strictly-bandlimited signal $s(t)$ with bandwidth $B > 0$.