Jackson 3.3 Homework Problem Solution

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PROBLEM:

A thin, flat, conducting, circular disc of radius R is located in the x-y plane with its center at the origin, and is maintained at a fixed potential V. With the information that the charge density on a disc at fixed potential is proportional to $(R^2 - \rho^2)^{-1/2}$, where ρ is the distance out from the center of the disc,

(a) show that for $r > R$ the potential is

$$\Phi(r,\theta,\phi) = \frac{2V}{\pi}\frac{R}{r}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{R}{r}\right)^{2l} P_{2l}(\cos\theta)$$

(b) find the potential for $r < R$.

(c) What is the capacitance of the disc?

SOLUTION:

Warning! The solution that Jackson gives is wrong. Let us solve the problem the wrong way (the way Jackson expects), then show why this solution is wrong. Then let us solve the problem the right way and figure out where Jackson went wrong.

The Wrong Way:

(a) The surface charge density was stated to be:

$$\sigma = \frac{S}{\sqrt{R^2 - r^2}} \quad \text{for } r < R \text{ and 0 otherwise}$$

The three-dimensional charge density is then:

$$\rho = S\frac{1}{\sqrt{R^2 - r^2}}\frac{\delta(\theta - \pi/2)}{r} \quad \text{for } r < R \text{ and 0 otherwise}$$

To find out what S is in terms of the potential V, use Coulomb's law which integrates over all the charge density to find the potential at the origin and set it equal to V:

$$\Phi = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}d\mathbf{x}'$$

$$V = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}'|}d\mathbf{x}'$$

$$V=\frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} S\frac{1}{\sqrt{R^2-r'^2}}\frac{\delta(\theta'-\pi/2)}{r'}\frac{1}{r'}r'^2\sin\theta' dr' d\theta' d\phi'$$

$$V=S\frac{1}{2\epsilon_0}\int_0^{R}\frac{1}{\sqrt{R^2-r'^2}}dr'$$

$$V=S\frac{1}{2\epsilon_0}\left[\sin^{-1}\left(\frac{r'}{R}\right)\right]_0^R$$

$$S=\frac{4\epsilon_0 V}{\pi}$$

Now knowing *S*, plugging it back in, we have the final form of the charge density:

$$\boxed{\rho=\frac{4\epsilon_0 V}{\pi}\frac{1}{\sqrt{R^2-r^2}}\frac{\delta(\theta-\pi/2)}{r}}\quad \text{for } r<R \text{ and } 0 \text{ otherwise}$$

Now apply Coulomb's Law to find the potential at any point on the z axis:

$$\Phi=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'$$

Use $\frac{1}{|\mathbf{r}-\mathbf{r}_0|}=\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta')$ to expand this (which is allowed because we are on the z axis).

$$\Phi=\frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\frac{4\epsilon_0 V}{\pi}\frac{1}{\sqrt{R^2-r'^2}}\frac{\delta(\theta'-\pi/2)}{r'}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta')r'^2\sin\theta' dr' d\theta' d\phi'$$

$$\Phi=\frac{2V}{\pi}\sum_{l=0}^{\infty}\int_0^{R}\frac{1}{\sqrt{R^2-r'^2}}\frac{r_<^l}{r_>^{l+1}}P_l(0)r' dr'$$

We have to treat the two regions separately. Let us look at the $r > R'$ region because its integral is easier. For $r > R$ we also have $r > r'$ so that:

$$\Phi=\frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{P_l(0)}{r^{l+1}}\int_0^{R}\frac{1}{\sqrt{R^2-r'^2}}r'^{l+1}dr'$$

Make a change of variables $u=\sqrt{R^2-r'^2}$ and $u\,du=-r'dr'$

$$\Phi=\frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{P_l(0)}{r^{l+1}}\int_0^{R}(R^2-u^2)^{l/2}du$$

Now $P_l(0)$ is zero for l odd, so only even l terms contribute. Let us relabel to take account for this fact:

$$\Phi = \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{1}{r^{2l+1}}\int_0^R P_{2l}(0)(R^2-u^2)^l\,du$$

If we do the integral case by case for l = 0, 1, 2... the integration is trivial and we soon see a pattern:

$$\Phi(r,\theta,\phi) = \frac{2V}{\pi}\frac{R}{r}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{R}{r}\right)^{2l}$$

Now we must remember that this is only valid on the z axis. We can make use of the handy theorem that for problems with azimuthal symmetry, the general solution is just the solution on the z-axis, multiplied by $P(\cos\theta)$:

$$\boxed{\Phi(r,\theta,\phi) = \frac{2V}{\pi}\frac{R}{r}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{R}{r}\right)^{2l}P_{2l}(\cos\theta)\quad \text{for } r > R}$$

This is the solution (the wrong one Jackson expects) to the potential in the exterior region.

(b) To find the potential in the near region ($r < R$), first note that the problem has azimuthal symmetry and no charge in the near region, so the general solution to the Laplace equation holds:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-l-1})P_l(\cos\theta)$$

We need a finite solution at the origin, so the solution must have the form:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta)\quad \text{for } r < R$$

This solution for the potential in the outer region must match the solution for the potential in the inner region at the interface where they touch, $r = R$:

$$\frac{2V}{\pi}\sum_{l=0,even}^{\infty}\frac{(-1)^{l/2}}{l+1}P_l(\cos\theta) = \sum_{l=0}^{\infty}A_l R^l P_l(\cos\theta)$$

The Legendre polynomials are orthogonal, so the coefficients must match up separately, leading to:

$$A_l = \frac{2V}{\pi}\frac{(-1)^{l/2}}{l+1}R^{-l}\quad \text{and } A_l = 0 \text{ for } l \text{ odd}$$

The solution for the near region is therefore:

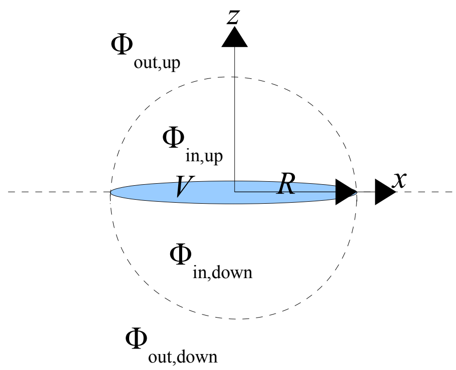
$$\boxed{\Phi(r,\theta,\phi) = \frac{2V}{\pi}\sum_{l=0}^{\infty}\frac{(-1)^l}{2l+1}\left(\frac{r}{R}\right)^{2l}P_{2l}(\cos\theta)}$$

Now, the plate is held at V, so the solution to the potential should reduce down to the constant V for

$\theta=\pi/2$ and $r < R$, independent of r. It should be obvious that the solution above does not reduce to V on the disc. The factor $P_l(0)$ is zero only for l odd, but this solution only has l even. The correct solution will have only l odd values.

The Correct Way:

Note that there are really four regions that we need to treat, separately, as indicted in the diagram. In each region, there is no charge, there is azimuthal symmetry, and the poles are included, so the solution to the potential has the form:



$$\Phi(r,\theta,\phi)=\sum_{l=0}^{\infty}(A_l r^l+B_l r^{-l-1})P_l(\cos\theta)$$

The inner regions include the origin, so they must have all B_l zero to have a finite solution at the origin. Similarly, the outer regions include infinity, which we can assume to have zero potential, leading to all A_l being zero in these regions. Our solutions in all regions therefore become:

$$\Phi_{\text{out,up}}(r,\theta,\phi)=\sum_{l=0}^{\infty} B_l r^{-l-1} P_l(\cos\theta) \quad \text{for } r>R \text{ and } \theta<\pi/2$$

$$\Phi_{\text{out,down}}(r,\theta,\phi)=\sum_{l=0}^{\infty} B_{l,\text{down}} r^{-l-1} P_l(\cos\theta) \quad \text{for } r>R \text{ and } \theta>\pi/2$$

$$\Phi_{\text{in,up}}(r,\theta,\phi)=\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \quad \text{for } r<R \text{ and } \theta<\pi/2$$

$$\Phi_{\text{in,down}}(r,\theta,\phi)=\sum_{l=0}^{\infty} A_{l,\text{down}} r^l P_l(\cos\theta) \quad \text{for } r<R \text{ and } \theta>\pi/2$$

First, due to symmetry, the potential at any point in an upper region must equal the potential at the mirror point across the x-y plane:

$$\Phi(z)=\Phi(-z)$$

$$\Phi_{\text{in,up}}(\cos\theta)=\Phi_{\text{in,down}}(-\cos\theta) \quad \text{and} \quad \Phi_{\text{out,up}}(\cos\theta)=\Phi_{\text{out,down}}(-\cos\theta)$$

$$\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)=\sum_{l=0}^{\infty} A_{l,\text{down}} r^l P_l(-\cos\theta) \quad \text{and} \quad \sum_{l=0}^{\infty} B_l r^{-l-1} P_l(\cos\theta)=\sum_{l=0}^{\infty} B_{l,\text{down}} r^{-l-1} P_l(-\cos\theta)$$

$$A_{l,\text{down}}=A_l(-1)^l \quad \text{and} \quad B_{l,\text{down}}=B_l(-1)^l$$

With these findings, our solutions now become:

$$\Phi_{\text{out,up}}(r,\theta,\phi)=\sum_{l=0}^{\infty} B_l r^{-l-1} P_l(\cos\theta) \quad \text{for } r>R \text{ and } \theta<\pi/2$$
$$\Phi_{\text{out,down}}(r,\theta,\phi)=\sum_{l=0}^{\infty} B_l(-1)^l r^{-l-1} P_l(\cos\theta) \quad \text{for } r>R \text{ and } \theta>\pi/2$$
$$\Phi_{\text{in,up}}(r,\theta,\phi)=\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \quad \text{for } r<R \text{ and } \theta<\pi/2$$
$$\Phi_{\text{in,down}}(r,\theta,\phi)=\sum_{l=0}^{\infty} A_l(-1)^l r^l P_l(\cos\theta) \quad \text{for } r<R \text{ and } \theta>\pi/2$$

Note that by forcing the upper and lower region potentials to be mirror images, we automatically made them match up at their interface, and have already taken care of this boundary condition.
Next, the potential in the inner regions must become V on the disc.

$$V=\sum_{l=0}^{\infty} A_l r^l P_l(\cos(\pi/2)) \quad \text{and} \quad V=\sum_{l=0}^{\infty} A_l(-1)^l r^l P_l(\cos(\pi/2))$$

leading to:

$$A_0=V \quad , \quad \sum_{l=1}^{\infty} A_l r^l P_l(0)=0 \quad \text{, and} \quad \sum_{l=1}^{\infty} A_l(-1)^l r^l P_l(0)=0$$

Note that $P_l(0) = 0$ for all l odd, in which case the last two equations are automatically satisfied. For l even, $P_l(0)$ is not zero, so:

$$A_l=0 \text{ for } l \text{ even}, l>0$$

Our solution so far is:

$$\Phi_{\text{out,up}}(r,\theta,\phi)=\sum_{l=0}^{\infty} B_l r^{-l-1} P_l(\cos\theta) \quad \text{for } r>R \text{ and } \theta<\pi/2$$
$$\Phi_{\text{out,down}}(r,\theta,\phi)=\sum_{l=0}^{\infty} B_l(-1)^l r^{-l-1} P_l(\cos\theta) \quad \text{for } r>R \text{ and } \theta>\pi/2$$
$$\Phi_{\text{in,up}}(r,\theta,\phi)=V+\sum_{l=1,3,5..}^{\infty} A_l r^l P_l(\cos\theta) \quad \text{for } r<R \text{ and } \theta<\pi/2$$
$$\Phi_{\text{in,down}}(r,\theta,\phi)=V+\sum_{l=1,3,5...}^{\infty} A_l(-1)^l r^l P_l(\cos\theta) \quad \text{for } r<R \text{ and } \theta>\pi/2$$

Note that now that l is odd, $(-1)^l$ is always just -1. The potential in the inner-down region therefore becomes $\Phi_{\text{in,down}}(r,\theta,\phi)=V-\sum_{l=1,3,5...}^{\infty} A_l r^l P_l(\cos\theta)$. The solutions in the two inner regions can now be combined into $\Phi_{\text{in}}(r,\theta,\phi)=V+sgn(\cos\theta)\sum_{l=1,3,5...}^{\infty} A_l r^l P_l(\cos\theta)$ where sgn(cos θ) is +1 for $\theta<\pi/2$ and -1 for $\theta>\pi/2$. Also note that because the potential in the inner regions and outer regions must match at $r = R$, and due to orthogonality, only the l = odd terms will contribute in the outer regions as well. The outer region solutions can therefore be combined in the same way. Our solution so far is thus:

$$\Phi_{\text{out}}(r,\theta,\phi)=\frac{B_0}{r}+sgn(\cos\theta)\sum_{l=1,3,5...}^{\infty} B_l r^{-l-1} P_l(\cos\theta)$$
$$\Phi_{\text{in}}(r,\theta,\phi)=V+sgn(\cos\theta)\sum_{l=1,3,5...}^{\infty} A_l r^l P_l(\cos\theta)$$

Next, the potentials of the inner and outer regions should match at $r = R$:

$$\Phi_{\text{in}}(R,\theta,\phi)=\Phi_{\text{out}}(R,\theta,\phi)$$

$$V+sgn(\cos\theta)\sum_{l=1,3,5...}^{\infty} A_l R^l P_l(\cos\theta)=\frac{B_0}{R}+sgn(\cos\theta)\sum_{l=1,3,5..} B_l R^{-l-1} P_l(\cos\theta)$$

The Legendre polynomials are orthogonal, so we match up coefficients. Matching up all coefficients, we find:

$$B_0=V R \quad \text{and} \quad B_l=A_l R^{2l+1}$$

The solution so far becomes:

$$\Phi_{\text{out}}(r,\theta,\phi)=V\frac{R}{r}+sgn(\cos\theta)\sum_{l=1,3,5...}^{\infty} A_l R^{2l+1} r^{-l-1} P_l(\cos\theta) \quad \text{for } r>R$$
$$\Phi_{\text{in}}(r,\theta,\phi)=V+sgn(\cos\theta)\sum_{l=1,3,5...}^{\infty} A_l r^l P_l(\cos\theta) \quad \text{for } r<R$$

All the A_l at this point are arbitrary, so let us redefine A_l as A_l/R^l to make these equations symmetric, leading to:

$$\Phi_{\text{out}}(r,\theta,\phi)=V\frac{R}{r}+sgn(\cos\theta)\sum_{l=1,3,5...}^{\infty} A_l\left(\frac{R}{r}\right)^{l+1} P_l(\cos\theta) \quad \text{for } r>R$$
$$\Phi_{\text{in}}(r,\theta,\phi)=V+sgn(\cos\theta)\sum_{l=1,3,5...}^{\infty} A_l\left(\frac{r}{R}\right)^{l} P_l(\cos\theta) \quad \text{for } r<R$$

The last set of coefficients can be found by relating the electric field across the plate:

$$(\mathbf{E}_2-\mathbf{E}_1)\cdot\mathbf{n}_{12}=\frac{\sigma}{\epsilon_0}$$

$$(\mathbf{E}_{\text{in,down}}-\mathbf{E}_{\text{in,up}})\cdot\hat{\boldsymbol{\theta}}=\frac{\sigma}{\epsilon_0} \quad \text{where} \quad \sigma=\frac{S}{\sqrt{R^2-r^2}} \quad \text{for a conducting plate}$$

To find out what S is in terms of the potential V, use Coulomb's law to integrate over all the charge density and find the potential at the origin and set it equal to V:

$$V=\frac{1}{4\pi\epsilon_0}\int\frac{\sigma(\mathbf{x}')}{|\mathbf{x}'|}da$$

$$V=\frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{R} S\frac{1}{\sqrt{R^2-r'^2}}\frac{1}{r'}r'\sin\theta' dr' d\phi'$$

$$V=S\frac{1}{2\epsilon_0}\int_0^{R}\frac{1}{\sqrt{R^2-r'^2}}dr'$$

$$V=S\frac{1}{2\epsilon_0}\left[\sin^{-1}\left(\frac{r'}{R}\right)\right]_0^R$$

$$S=\frac{4\epsilon_0 V}{\pi}$$

Using this value, the boundary condition on the electric field across the plate now becomes:

$$(-\nabla\Phi_{\text{in,down}}+\nabla\Phi_{\text{in,up}})\cdot\hat{\boldsymbol{\theta}}=\frac{4V}{\pi}\frac{1}{\sqrt{R^2-r^2}}$$

$$-\frac{1}{r}\frac{\partial\Phi_{\text{in,down}}}{\partial\theta}+\frac{1}{r}\frac{\partial\Phi_{\text{in,up}}}{\partial\theta}=\frac{4V}{\pi}\frac{1}{\sqrt{R^2-r^2}}$$

$$-\frac{\partial\Phi_{\text{in,down}}}{\partial\theta}+\frac{\partial\Phi_{\text{in,up}}}{\partial\theta}=\frac{4V}{\pi}\frac{r}{R}\frac{1}{\sqrt{1-(r/R)^2}}$$

Perform a binomial expansion on the right side, using $\frac{1}{\sqrt{1-x^2}}=1+\frac{1}{2}x^2+\frac{1\cdot 3}{2\cdot 4}x^4+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6+...$

$$-\frac{\partial\Phi_{\text{in,down}}}{\partial\theta}+\frac{\partial\Phi_{\text{in,up}}}{\partial\theta}=\frac{4V}{\pi}\left[\frac{r}{R}+\frac{1}{2}\left(\frac{r}{R}\right)^3+\frac{1\cdot 3}{2\cdot 4}\left(\frac{r}{R}\right)^5+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{r}{R}\right)^7+...\right]$$

$$-\frac{\partial\Phi_{\text{in,down}}}{\partial\theta}+\frac{\partial\Phi_{\text{in,up}}}{\partial\theta}=\frac{4V}{\pi}\left[\sum_{l=1,3,5...}\frac{(l-2)!!}{(l-1)!!}\left(\frac{r}{R}\right)^l\right]$$

$$-\frac{\partial\Phi_{\text{in,down}}}{\partial\theta}+\frac{\partial\Phi_{\text{in,up}}}{\partial\theta}=\frac{4V}{\pi}\left[\sum_{l=1,3,5...}(-1)^{\frac{l-1}{2}}P_{l-1}(0)\left(\frac{r}{R}\right)^l\right]$$

A Legendre polynomial identity was used in the last step to get the right side in a form that we anticipate will be on the left side. Now evaluate the derivatives:

$$2\sum_{l=1,3,5...}^{\infty}A_l\left(\frac{r}{R}\right)^l\left[\frac{\partial}{\partial\theta}P_l(\cos\theta)\right]_{\theta=\pi/2}=\frac{4V}{\pi}\left[\sum_{l=1,3,5...}(-1)^{\frac{l-1}{2}}P_{l-1}(0)\left(\frac{r}{R}\right)^l\right]$$

$$-2\sum_{l=1,3,5...}^{\infty}A_l\left(\frac{r}{R}\right)^l\left[\frac{\partial}{\partial x}P_l(x)\right]_{x=0}=\frac{4V}{\pi}\left[\sum_{l=1,3,5...}(-1)^{\frac{l-1}{2}}P_{l-1}(0)\left(\frac{r}{R}\right)^l\right]$$

$$-2\sum_{l=1,3,5...}^{\infty} A_l\left(\frac{r}{R}\right)^l\left[\frac{l\,x\,P_l(x)-l\,P_{l-1}(x)}{x^2-1}\right]_{x=0}=\frac{4V}{\pi}\left[\sum_{l=1,3,5...}(-1)^{\frac{l-1}{2}}P_{l-1}(0)\left(\frac{r}{R}\right)^l\right]$$

$$-2\sum_{l=1,3,5...}^{\infty} A_l\left(\frac{r}{R}\right)^l\left[l\,P_{l-1}(0)\right]=\frac{4V}{\pi}\left[\sum_{l=1,3,5...}(-1)^{\frac{l-1}{2}}P_{l-1}(0)\left(\frac{r}{R}\right)^l\right]$$

$$\boxed{A_l=\frac{2V}{\pi l}(-1)^{\frac{l+1}{2}}}$$

Our final solution is therefore:

$$\boxed{\begin{array}{ll}\Phi_{\text{out}}(r,\theta,\phi)=V\frac{R}{r}+sgn(\cos\theta)\frac{2V}{\pi}\sum_{l=1,3,5...}^{\infty}\frac{(-1)^{\frac{l+1}{2}}}{l}\left(\frac{R}{r}\right)^{l+1}P_l(\cos\theta) & \text{for } r>R\\ \Phi_{\text{in}}(r,\theta,\phi)=V+sgn(\cos\theta)\frac{2V}{\pi}\sum_{l=1,3,5...}^{\infty}\frac{(-1)^{\frac{l+1}{2}}}{l}\left(\frac{r}{R}\right)^{l}P_l(\cos\theta) & \text{for } r<R\end{array}}$$

Note that this solution obeys all the boundary conditions that it should. On the plate, $P_l(\cos\theta)$ becomes $P_l(0)$, which is zero for l odd, leaving just the constant V as it should.

So where did Jackson go wrong? The symmetry of the problem requires Legendre polynomials with l odd, but Jackson's solution had l even, indicating that he got the symmetry wrong.

(c) The total charge is:

$$Q=\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\rho(r',\theta',\phi')r'^2\sin\theta' dr'd\theta'd\phi'$$

$$Q=8\epsilon_0 V\int_0^R\frac{1}{\sqrt{R^2-r'^2}}r'dr'$$

$$Q=8\epsilon_0 V\left[-\sqrt{R^2-r'^2}\right]_0^R$$

$$Q=8\epsilon_0 V R$$

$$\frac{Q}{V}=8\epsilon_0 R$$

This is the capacitance:

$$\boxed{C=8\epsilon_0 R}$$