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# Characterizing conflicts in fair division of indivisible goods using a scale of criteria

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**Abstract** We investigate five different fairness criteria in a simple model of fair resource allocation of indivisible goods based on additive preferences. We show how these criteria are connected to each other, forming an ordered scale that can be used to characterize how conflicting the agents' preferences are: for a given instance of a resource allocation problem, the less conflicting the agents' preferences are, the more demanding criterion this instance is able to satisfy, and the more satisfactory the allocation can be. We analyze the computational properties of the five criteria, give some experimental results about them, and further investigate a slightly richer model with $k$-additive preferences.

**Keywords** Computational social choice · Resource allocation · Fair division · Indivisible goods · Preferences

## 1 Introduction

The problem of fairly allocating some resources to a set of economically motivated agents is an important and common problem. Fair division of indivisible goods in particular, on which we focus in this paper, arises in a wide range of real-world applications, including auctions, divorce settlements, airport traffic management, spatial resource allocation [31], fair scheduling, allocation of tasks to workers, articles to reviewers, courses to students [39].

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In this paper, we focus on fair division of *indivisible* goods. In this setting, a finite set of indivisible objects has to be allocated to a finite set of agents. We assume that the process is *centralized*, that is, a supposedly benevolent arbitrator is in charge of allocating the objects to the agents according to their revealed preferences. Moreover, we rule out the use of any divisible resource (such as money) as a mean of compensation for the agents: in other words, there can be no monetary transfers between the agents and the arbitrator, nor between the agents themselves, during the allocation process or afterwards.

An important point in this context is how agents express their preferences. In a centralized allocation process, the agents have first to communicate and hence explicitly describe their preferences over the objects. Two main approaches are appropriate for that. The first one rests on a purely ordinal expression of preferences, such as a weak (partial or total) order. The second one exploits a numerical expression of preferences taking the form of *utility functions*. This article, for convenience, is based on the second approach, but many of the results presented here could be transposed in the first, purely ordinal one.

Another important point about preferences is whether the agents should be allowed to express some preferential dependencies between goods or not. In this paper, we assume that the agents have *additive* preferences (except in Sect. 8 where we consider a more general model). This model may seem restrictive, but it offers a natural trade-off between simplicity and expressiveness. Moreover, it has been largely investigated in the literature about fair division (see for example [1,3,5,13,15,21,28,30,32,33]).

The last critical point about fair division is how fairness should be defined and how it can be evaluated. Once again two main options are available. The first one consists of defining a collective utility function (CUF) aggregating individual agents' utilities. If the CUF is well chosen, its outcome when applied to individual utilities reflects the fairness (and possibly other desirable properties) of a given allocation. The arbitrator just looks for an allocation maximizing this CUF. The second option consists of defining, by means of a Boolean (logical) criterion, what is considered as fair. This is the approach followed by most works in fair division of divisible goods—cake-cutting—or by Lipton et al. [32] for envy-freeness in the context of indivisible goods. This article explores mainly this logical option.

While most papers in fair division focus on a specific criterion, here we consider five of them and investigate their connection to each other. Four of these criteria are classical or already known, namely: max-min fair-share (MFS), proportional fair-share (PFS), envy-freeness (EF) and competitive equilibrium from equal incomes (CEEI), and we introduce an original one: min-max fair-share (mFS). All these criteria have a natural interpretation and a very appealing quality: they do not need a common scale of agents' utilities.[1]

Summing things up, we focus in this paper on a simple model of fair division, which is based on the following assumptions:

- a set of indivisible objects must be distributed among a set of agents;
- agents have numerical additive preferences over the objects;
- the allocation process is centralized, that is, it is decided by a neutral arbitrator or computation, taking into account only agents' preferences, in a single step;
- no monetary transfer is possible between agents.

*Our contribution* Some instances of fair resource allocation problems are more conflicting than others. When the number of objects is high and the agents prefer somewhat different

[1] Whereas most CUF—except Nash—only make sense if the utilities are expressed on a common scale or normalized.

objects, a well-balanced allocation, satisfying all participants, is likely to be found. On the contrary, when agents have similar preferences (they want more or less the same objects with the same intensity), or when there are only a few objects to allocate, the sharing out will for sure be conflicting.

Our main and original contribution is the following. Starting from the simple model of fair division sketched above, we show that the five aforementioned criteria form a linear scale of increasing requirements, that can be used to characterize formally the level of fairness of a given allocation. The more demanding criterion an allocation is able to satisfy, the fairer, more harmonious and less conflicting this allocation will be.

This scale of properties can be used to characterize not only an allocation, as said before, but also a resource allocation problem instance: the non-conflicting degree of an instance can be measured by the level of the most demanding criterion an allocation from this instance can satisfy.

This article is structured as follows. Section 2 gives an overview of related work. Section 3 describes the model: fair division of indivisible objects under numerical additive preferences. The scale of five properties characterizing the fairness of an allocation, as well as associated computational complexity results are exposed in Sect. 4. We go back to the collective utility function approach in Sect. 5 to connect the important egalitarian CUF to the scale of criteria. Section 6 is devoted to a bunch of interesting restricted cases. Some experimental results on the scale of criteria are presented in Sect. 7. Extending the model to $k$-additive preferences, Sect. 8 presents a quite different perspective. At the end, the short Sect. 9 gathers the problems left open.

## 2 Related work

The problem of finding an allocation of indivisible goods to agents that satisfies a given fairness criterion has already been considered in a lot of papers in the setting sketched above. As mentioned in the introduction, several of these papers focus—like we do—on additive numerical preferences. Unless otherwise mentioned, the works we cite in this related work section concern additive preferences.

Among the five criteria that we consider, the most classical one is probably *envy-freeness* (which originates from Foley [25] in a more general setting). This criterion has been re-investigated more recently, in particular by Lipton et al. [32]. Among other results, they prove the $\mathsf{NP}$-completeness of the problem of existence of a complete (that is in which all goods are allocated) envy-free allocation. They extend this result by relaxing envy-freeness with several quantitative measures of envy, and give some polynomial-time approximation algorithms, refining a previous result that was known from Dall'Aglio and Hill [20] for a more general model mixing divisible and indivisible resources. This work has been slightly extended by de Keijzer et al. [30] who prove the $\Sigma_2^{\mathsf{P}}$-completeness of the problem of determining whether an envy-free and Pareto-efficient allocation exists. A very recent work by Dickerson et al. [22] has further investigated envy-freeness by providing several theoretical and experimental results about the probability of existence of a complete and envy-free allocation. In particular, they show analytically that under several assumptions on the probability distribution of the agents' preferences, an envy-free allocation is unlikely to exist up to a given threshold on the ratio between the number of goods and the number of agents, and very likely to exist beyond. Experimental results show an interesting phenomenon of phase-transition.

Another important fairness (and efficiency) concept, which is classical in microeconomics, is the *Competitive Equilibrium from Equal Incomes* (CEEI). Roughly speaking, a CEEI is obtained when the "supply" (the objects with some public prices) meets the "demand" (the agents' preferences for objects), each agent having a fixed budget to buy objects. Fairness comes from the fact that prices and budgets are the same for all agents. This concept has only been brought recently to computer science, in particular by Budish [16]. He introduces *approximate-CEEI*, which is a parameterized approximation of the CEEI solution concept. He shows that under some assumption on the approximation parameters, an approximate-CEEI always exists (namely, approximately fair, approximately efficient and feasible).[2] However, this result is based on a non-constructive proof. Othman et al. [39] give a heuristic algorithm to compute approximate-CEEI, but this algorithm sometimes fails to find the approximation guaranteed by Budish's result. More recently, Othman et al. [40] have proved that the problem of finding an approximate-CEEI in Budish's sense is actually intracatable (PPAD-complete). The approximate-CEEI solution is particularly interesting in the context of fair course scheduling, in which objects—the seats in courses—are available in many copies (the seats of the same course). When applied to our model (described in Sect. 3), where each object exists in a single copy, the approximate-CEEI solution is much less interesting.

Budish [16] also introduces an interesting solution concept, namely, the *max–min fair-share* criterion (actually called *Maximin Share Guarantee* in Budish's paper), on which part of our work is based, and particularly well adapted to the fair allocation of indivisible goods. Suppose an agent is allowed to divide the whole bundle as she wants, but cannot choose her own share. The max–min fair-share of this agent is the maximum utility she can guarantee for herself in this allocation game, and so can be considered as a lower bound of her utility.[3] This concept has been further studied by Procaccia and Wang [43], who prove, among other results, that there exists some instances where it is not possible to find any allocation giving her max–min fair-share to all agents, which is a rather surprising result (see Sect. 4.1). They also exhibit an approximation ratio of the max–min fair-share it is always possible to satisfy, giving a constructive proof to this result.

The problem of maximizing the egalitarian collective utility function (hence trying to find the maximal amount of absolute utility it is possible to guarantee to each agent) has been investigated for years. Under additive numerical preferences, this problem is now known as the *Santa-Claus problem* after Bansal and Sviridenko [3]—but several other works focus on this problem (mainly from the algorithmical point of view), under various denominations (e.g [1,6]). As quickly mentioned in the introduction, we point out that there is an important difference between these works and the criteria we investigate here: since the egalitarian approach is based upon utility maximization, it only makes sense if the individual utilities are expressed on a common scale (such as money). This is not the case with our Boolean criteria: the agents may each have their own utility scales.[4] Nevertheless, it is worth considering the links between egalitarianism and the approach based on criteria, as [11,14,15] do (although in a bit different model based on ordinal preferences), and as we further investigate in Sect. 5.

Some papers have also paid attention to fair division with ordinal preferences, where here *separable preferences* are the counterpart of additive numerical preferences. Although a bit less related to our work, these papers are worth being mentioned because most of

2 Actually, this result remains valid even with unrestricted combinatorial preferences.

3 Moulin [37], although in a continuous context (divisible goods), introduces a seminal idea of this concept under a "though experiment" in which an agent considers the case of other agents having the same preference as her, deriving from that experiment a lower bound of her utility.

4 Moreover, even if a common utility scale is used, an agent could manipulate the game to her advantage by decreasing her utilities. A way to overcome this problem is to normalize utilities such that the whole set of objects gives the same utility to each agent. This is known as the Kalai-Smorodinsky approach, see for example [35], p 67.

the criteria we consider are purely ordinal, in the sense that they could be translated to ordinal preferences. Brams et al., Herreiner and Puppe, and Brams and King [11,14,15,28] all consider a model where the agents express linear orders on objects, with a dominance relation roughly based on separability and monotonicity. As mentioned before, the authors study the links between envy-freeness and an ordinal version of egalitarianism. Bouveret et al. [9] work on a similar model, but focus on envy-freeness and Pareto-efficiency, extended to deal with incomplete preferences. Aziz et al. [2] extend these results by dealing with a more general model of preferences, whose semantics is based on stochastic dominance. Although introducing several fairness criteria of different strengths, the two latter works only focus on envy-freeness for the fairness aspects, and do not investigate any of the other criteria that are dealt with in our paper.

As said above, Brams and King [11] study the problem of allocating indivisible goods with linear order preferences (strict ranking of objects). Among other contributions, they consider "maximin allocations", in which the lowest-ranked object that any agent receives is as high as possible (an ordinal version of egalitarianism). Notably, they introduce the notion of *sequence of sincere choice* that is reused in Sect. 6 (in a different way), in connection to the efficiency property (Pareto-optimality) which is also considered in our Sect. 4 together with envy-free and CEEI. However, they use an ordinal notion of domination (hence Pareto-optimality) which is defined ordinally only over bundles of equal cardinality.

Some recent works focus on specialized versions of fair division of indivisible goods without money. Xia [49] studies the special case where objects are partitioned in categories, and each agent is required to get exactly or at least one item from each category. Ferraioli et al. [23] focus on approximating solutions to a constrained version of the problem of finding the allocation that maximizes the egalitarian (min) utility (see our Sect. 5), where each agent receives exactly the same number of objects.

Our contribution questions the role of similarities/dissimilarities in agents' preferences. In the fair allocation problem, similar preferences are conflicting and dissimilar ones are more compatible. However it is exactly the opposite situation in the context of voting (similar preferences are compatible and dissimilar ones are conflicting). The question of measuring preference diversity in a group has been addressed, from a voting theoretic point of view, by Hashemi and Endriss [27], who propose several ways of measuring the diversity in a preference profile and show a correlation between this measure and the existence of a Condorcet winner for the given profile.

Finally, as quickly mentioned in the introduction, even if we focus on fair allocation of indivisible goods, it has some indubitable connections with fair division of divisible goods, also known as *cake-cutting*. Most works in this area focus on the development and analysis of cake-cutting protocols satisfying some given criteria such as proportionality or envy-freeness. As we shall see in Sect. 4, two of the criteria that we study in this paper (namely, max–min fair-share and min–max fair-share) are connected to cake-cutting protocols. The interested reader can refer to the seminal book by Brams and Taylor [12] or to the paper by Procaccia [42] for a more recent overview.

## 3 Model

Let $\mathscr{A} = \{1, \ldots, n\}$ be a set of *agents* and $\mathscr{O} = \{1, \ldots, m\}$ be a set of indivisible *objects*. An *allocation* of the objects to the agents is a vector $\overrightarrow{\pi} = \langle \pi_1, \ldots, \pi_n \rangle$, where $\pi_i \subseteq \mathscr{O}$ is the *bundle* of objects allocated to agent $i$, called agent $i$'s *share*. An allocation $\overrightarrow{\pi}$ is said to be *admissible* if and only if it satisfies the two following conditions: (1) $i \neq j \Rightarrow \pi_i \cap \pi_j = \emptyset$

(each object is allocated to at most one agent) and (2) $\cup_{i \in \mathscr{A}} \pi_i = \mathscr{O}$ (all the objects are allocated). We will denote by $\mathscr{F}_{n,m}$ the set of admissible allocations for a given set of $n$ agents and $m$ objects ($n$ and $m$ will be omitted when the context is clear). All the allocations considered in this paper are implicitly admissible.

To find a "good" allocation, it is necessary to know the agents' preferences over the sets of objects they may receive. We make two common assumptions concerning the way agents express their preferences. First, we consider that they are expressed numerically by a *utility function* $u_i : 2^{\mathscr{O}} \rightarrow \mathbb{R}^+$ specifying, for each agent $i$, the satisfaction $u_i(\pi)$ she enjoys if she receives bundle $\pi$: this is the utilitarian model [34]. Second, we consider (except in Sect. 8) that the agents' preferences are *additive*, which means that the utility function of an agent is defined as follows:

$$u_i(\pi) \stackrel{def}{=} \sum_{\ell \in \pi} w(i, \ell),$$

where $w(i, \ell)$ is the *weight* given by agent $i$ to object $\ell$. This assumption, as restrictive it may seem to be, is made in many studies (e.g [3,32]) and offers a good compromise between preference expressive power and conciseness.

Adapting the terminology from the survey by Chevaleyre et al. [18], we define an *additive MultiAgent Resource Allocation* instance (add-MARA instance for short) as a triple $\langle \mathscr{A}, \mathscr{O}, w \rangle$, where $\mathscr{A}$ is a set of agents, $\mathscr{O}$ is a set of objects, and $w : \mathscr{A} \times \mathscr{O} \rightarrow \mathbb{R}^+$ is a function specifying the weight $w(i, \ell)$ given by agent $i$ to object $\ell$.

In the following, indices $i$ and $j$ will always refer to agents, and $\ell$ to objects. To ease notation, we will adopt a matrix representation $W$ for the weight function $w$, where the element at row $i$ and column $\ell$ represents the weight $w(i, \ell)$. Finally, we will denote by $\mathscr{I}$ the set of all add-MARA instances.

The basic notions of computational complexity (see e.g. [41]) are supposed to be well-known by the reader: $\mathsf{P}$ and $\mathsf{NP}$ refer to the two standard complexity classes; $\Sigma_2^{\mathsf{P}}$ is the class of problems that can be solved in non-deterministic polynomial time by a Turing machine augmented by an $\mathsf{NP}$ oracle.

## 4 Five fairness criteria

Even before any fairness consideration, the most basic desirable criterion for a resource allocation is Pareto-efficiency, of which the definition is recalled here:

**Definition 1** Let $\langle \mathscr{A}, \mathscr{O}, w \rangle$ be an add-MARA instance. We say that allocation $\overrightarrow{\pi}$ *dominates* allocation $\overrightarrow{\pi}'$ if and only if $u_i(\pi_i) \geq u_i(\pi_i')$ for all $i$, with at least one strict inequality. A *Pareto-efficient allocation* is an admissible allocation which is not dominated by another admissible allocation.

Pareto-efficiency conveys the idea that the resource to be allocated should not be wasted or under-exploited, but tells nothing about fairness. Two approaches are possible to deal with the fairness requirement.

1. If the preferences are numerical, we can use a *collective utility function* (CUF) to aggregate the individual preferences into a collective preference, and look for an allocation that maximizes this function. If this function is carefully chosen, it can convey some idea of fairness (like the egalitarian criterion min for example, discussed in Sect. 5).

2. We can choose a Boolean (logical) fairness criterion and look for an allocation that obeys it, if some exists. The two prominent fairness criteria are envy-freeness [25] and proportional fair-share [46].

In this paper, we adopt the second point of view. We will consider five fairness criteria (including envy-freeness and proportional fair-share) and show how they form together a scale of criteria of increasing strength. This scale provides an evaluation of the degree of fairness of a given allocation on the one hand, and can give an idea of the degree of "conflictuality" of a given add-MARA instance. For each one of these criteria, we denote by $\overrightarrow{\pi} \models \mathscr{C}$ the fact that the allocation $\overrightarrow{\pi}$ satisfies criterion $\mathscr{C}$; $\mathscr{I}_{|\mathscr{C}}$ denotes the set of add-MARA instances admitting at least one allocation satisfying criterion $\mathscr{C}$.

### 4.1 Max–min fair-share

One of the most prominent fairness criteria in resource allocation problems is proportional fair-share, that will be discussed in details in Sect. 4.2. This criterion, coined by Steinhaus [46] in the context of continuous fair division (cake-cutting) problems, states that each agent should get from the allocation at least the $n$th of the total utility she would have received if she were alone.[5] However, when one deals with indivisible objects, it is often too demanding: consider for example a problem with one object and two agents, where obviously no allocation can give her proportional fair share to each agent. That is why it has been recently adapted to this context by Budish [16], which defines the *max–min fair-share*,[6] whose original definition is purely ordinal, but which can be defined in our (utilitarian) setting as follows:

**Definition 2** Let $(\mathscr{A}, \mathscr{O}, w)$ be an add-MARA instance. The *max–min fair-share* of agent $i$ for this instance is

$$u_i^{\mathrm{MFS}} \stackrel{def}{=} \max_{\overrightarrow{\pi} \in \mathscr{F}} \min_{j \in \mathscr{A}} u_i(\pi_j)$$

We say that the allocation $\overrightarrow{\pi}$ satisfies the criterion of *max–min fair-share*, denoted by $\overrightarrow{\pi} \models$ MFS, if $u_i^{\mathrm{MFS}} \leq u_i(\pi_i)$ for all $i$ (each agent obtains at least her max–min fair-share in $\overrightarrow{\pi}$).

*Example 1* Let us consider the 2 agents / 4 objects instance defined by the following weight matrix:

$$W = \begin{pmatrix} *7 & 2 & 6 & *10 \\ 4 & *7 & *7 & 7 \end{pmatrix}$$

We have $u_1^{\mathrm{MFS}} = 12$ (with share $\{2, 4\}$) and $u_2^{\mathrm{MFS}} = 11$ (with share $\{1, 2\}$). The allocation $\overrightarrow{\pi} = \langle\{1, 4\}, \{2, 3\}\rangle$ marked with stars satisfies the max–min fair-share criterion, with $u_1(\pi_1) = 17 > 12$ and $u_2(\pi_2) = 14 > 11$.

The max–min fair-share of an agent is the maximal utility that she can hope to get from an allocation if all the other agents have the same preferences as her, when she always receives the worst share. The max–min fair-share can be considered as the minimal amount of utility that an agent could feel to be entitled to, based on the following argument: if all the other agents have the same preferences as me, there is at least one allocation that gives me this utility, and makes every other agent (weakly) better off; hence there is no reason to give me less. It is also the maximum utility that an agent can get for sure in the allocation game

5 Actually, Steinhaus [46] does not give any name for this criterion, later named *Equal Split Guarantee* by Moulin [36].

6 This notion is actually called *maximin share* by Budish [16].

"I cut, I choose last": the agent proposes her best allocation and leaves all the other ones choose one share before taking the remaining one.

The max-min fair-share level is loosely connected to a result from Hill [29], recently refined by Markakis and Psomas [33], which establishes a worst case guarantee on the utility an agent can have. However, this guarantee only depends on the maximum weight of an agent, and so is not very informative, often being just 0.

Beyond its appealing formulation, max–min fair-share has a computational drawback: the computation of the max–min fair-share $u_i^{\text{MFS}}$ itself for a given agent is computationally hard. More precisely, the following decision problem is NP-complete:

| Problem 1 [MFS- Comp] | |
|---|---|
| **Input:** | An add-MARA instance $\langle A, \mathscr{O}, w\rangle$, an agent $i$, an integer $K$. |
| **Question:** | Do we have $u_i^{\text{MFS}} \geq K$? |

**Proposition 1** [MFS- Comp] *is NP-complete, for all $n \geq 2$.*

*Proof* Membership to NP is obvious. NP-hardness can be proved by reduction from the partition problem:

| Problem 2 [Partition] | |
|---|---|
| **Input:** | A set $\mathscr{X} = \{x_1, \ldots, x_n\}$ and a mapping $s : \mathscr{X} \rightarrow \mathbb{N}$ such that $\sum_{x_i \in \mathscr{X}} s(x_i) = 2L$ for some integer $L$. |
| **Question:** | Is there a partition $(\mathscr{X}_1, \mathscr{X}_2)$ of $\mathscr{X}$ such that $\sum_{x_i \in \mathscr{X}_1} s(x_i) = \sum_{x_i \in \mathscr{X}_2} s(x_i) = L$ ? |

From a given instance of [Partition], we create an instance of [MFS- Comp] with two agents and $n$ objects $\{1, \ldots, n\}$, such that $w(1, \ell) = s(x_\ell)$ for all $\ell \in \{1, \ldots, n\}$. We consider agent 1 and integer $K$ is set to $L$, which completes the reduction.[7] It is now easy to see that $u_1^{\text{MFS}} \geq L$ if and only if there exists a solution to the partition problem. □

Let us now focus on the problem [MFS- Exist] of determining, for a given add-MARA instance, if there is an allocation satisfying the max–min fair-share criterion. Strong evidence led us to think that every add-MARA instance had at least one such allocation: it is true in many restricted cases (see Sect. 6), and no counterexample was found in thousands of randomly generated instances (see Sect. 7). However, surprisingly, Procaccia and Wang [43] have recently proved (by a very tricky construction) that there actually exists add-MARA instances for which there is no allocation satisfying max–min fair-share. Put in other words, we thus have $\mathscr{I}_{|\text{MFS}} \subsetneq \mathscr{I}$. Nevertheless, the precise complexity of the decision problem of determining, for a given instance, whether there is an allocation satisfying max–min fair share remains unknown. All that we can say for sure is that this problem belongs to $\Sigma_2^{\mathsf{P}}$, because it can be solved by the following non-deterministic algorithm:

1. Guess an allocation $\overrightarrow{\pi}$.
2. Compute $u_i(\pi_i)$ for each agent $i$.
3. Check that $u_i(\pi_i) \geq u_i^{\text{MFS}}$ for each agent $i$ (requires an NP-oracle).

[7] We use here a very similar idea to the one used by Lipton et al. [32], page 4.

### 4.2 Proportional fair-share

The aforementioned concept of proportional fair-share was originally defined not on the utilities but on the resources themselves [46]. A lot of authors have since given a natural utilitarian interpretation of this notion, like the one that follows:

**Definition 3** Let $\langle \mathscr{A}, \mathscr{O}, w \rangle$ be an add-MARA instance. The *proportional fair-share* of agent $i$ for this instance is

$$u_i^{\text{PFS}} \stackrel{def}{=} \frac{1}{n} u_i(\mathscr{O}) = \frac{1}{n} \sum_{\ell \in \mathscr{O}} w(i, \ell).$$

We say that the allocation $\overrightarrow{\pi}$ satisfies the criterion of *proportional fair-share*, denoted by $\overrightarrow{\pi} \models \text{PFS}$, if $u_i^{\text{PFS}} \leq u_i(\pi_i)$ for all $i$ (that is, each agent obtains at least her proportional fair-share in $\overrightarrow{\pi}$).

The justification for this criterion is the following: in the virtual and perfectly fair allocation obtained by dividing each object into $n$ parts, each one allocated to a different agent, each single agent would enjoy precisely her proportional fair-share.

This criterion is more demanding than max–min fair-share:

**Proposition 2** *Let $\langle \mathscr{A}, \mathscr{O}, w \rangle$ be an add-MARA instance. We have $u_i^{\text{MFS}} \leq u_i^{\text{PFS}}$, for all $i \in \mathscr{A}$. Hence, for all $\overrightarrow{\pi}$, we have $\overrightarrow{\pi} \models \text{PFS} \implies \overrightarrow{\pi} \models \text{MFS}$, and $\mathscr{I}_{|\text{PFS}} \subset \mathscr{I}_{|\text{MFS}}$.*

*Proof* Let $\overrightarrow{\pi}$ be an allocation and $i$ an agent. We have $\sum_{j \in \mathscr{A}} u_i(\pi_j) = u_i(\mathscr{O})$. The minimum of a set of numbers being weakly lower than their mean, we have

$$\min_{j \in \mathscr{A}} u_i(\pi_j) \leq \frac{1}{n} \sum_{j \in \mathscr{A}} u_i(\pi_j) = \frac{1}{n} u_i(\mathscr{O}) = u_i^{\text{PFS}}$$

Hence

$$u_i^{\text{MFS}} \stackrel{def}{=} \max_{\overrightarrow{\pi} \in \mathscr{F}} \min_{j \in \mathscr{A}} u_i(\pi_j) \leq u_i^{\text{PFS}}.$$

The inclusion $\mathscr{I}_{|\text{PFS}} \subset \mathscr{I}_{|\text{MFS}}$ is strict: consider an instance with two agents and one object, for which every allocation satisfies max–min fair-share, but none satisfies proportional fair-share. □

Contrary to max–min fair-share, computing the proportional fair-share for a given agent is easy. However, determining whether a given add-MARA instance has an allocation satisfying proportional fair-share (problem that we shall call [PFS- Exist]) is computationally hard:

**Proposition 3** [PFS- Exist] *is NP-complete, for all $n \geq 2$.*

This proposition can be proved using a similar reduction as the one used in the proof of Proposition 1.

### 4.3 Min–max fair-share

The min–max fair-share criterion that we now present is, to the best of our knowledge, original. It can be seen as the symmetric version of the max–min fair-share criterion defined earlier.

**Definition 4** Let $(\mathscr{A}, \mathscr{O}, w)$ be an add-MARA instance. The *min–max fair-share* of agent $i$ for this instance is

$$u_i^{\text{mFS}} \stackrel{def}{=} \min_{\overrightarrow{\pi} \in \mathscr{F}} \max_{j \in \mathscr{A}} u_i(\pi_j)$$

We say that the allocation $\overrightarrow{\pi}$ satisfies the criterion of *min–max fair-share*, denoted by $\overrightarrow{\pi} \models$ mFS, if $u_i^{\mathrm{mFS}} \leq u_i(\pi_i)$ for all $i$ (each agent obtains at least her min–max fair-share in $\overrightarrow{\pi}$).

The min–max fair-share of an agent is the minimal utility that she can hope to get from an allocation if all the other agents have the same preferences as her, when she always receives the best share. It is also the minimal utility that an agent can get for sure in the allocation game "Someone cuts, I choose first". The following result is the equivalent of Proposition 2 and can be proved in a similar way:

**Proposition 4** *Let $\langle \mathscr{A}, \mathscr{O}, w\rangle$ be an add-MARA instance. We have $u_i^{\mathrm{PFS}} \leq u_i^{\mathrm{mFS}}$, for all $i \in \mathscr{A}$. Hence, for all $\overrightarrow{\pi}$, we have $\overrightarrow{\pi} \models \mathrm{mFS} \implies \overrightarrow{\pi} \models \mathrm{PFS}$ and $\mathscr{I}_{|\mathrm{mFS}} \subset \mathscr{I}_{|\mathrm{PFS}}$.*

This inclusion is strict, as the following example shows.

*Example 2* Let us consider the 3 agents / 3 objects instance defined by the following weight matrix:

$$W = \begin{pmatrix} 2 & 2 & *2 \\ 3 & *2 & 1 \\ *3 & 2 & 1 \end{pmatrix}$$

Obviously $u_i^{\mathrm{PFS}} = 2$ for each agent. Hence the allocation marked with stars gives to each agent her proportional fair-share. However, no allocation gives to each agent her min–max fair-share (which is 2 for agent 1 and 3 for the other ones).

Exactly like the max–min fair-share, the computation of the min–max fair-share for a given agent is hard. More precisely, if [MFS- COMP] is the equivalent for min–max fair-share of decision Problem 1, the following proposition holds.

**Proposition 5** [MFS- COMP] *is* coNP*-complete, for all $n \geq 2$.*

The proof is very similar to the one of Proposition 1, and is thus omitted. The decision problem becomes coNP-complete (instead of NP-complete) because it is just the opposite as the regular decision version of an optimization problem: the min–max fair-share is defined as a minimization problem, and we want to know, as for the max–min fair-share, whether the min–max fair-share of a given agent is greater than a given threshold.

Of course, an add-MARA instance may not always have an allocation satisfying min–max fair-share. As for max–min fair share, the decision problem of determining whether there exists one is very likely to be hard, but its precise complexity remains unknown. Once again, all that we can say for sure is that this problem belongs to $\Sigma_2^{\mathsf{P}}$, because it can be solved by a similar algorithm as the one used at the end of Sect. 4.1.

### 4.4 Envy-freeness

The envy-freeness criterion [25] is probably the most prominent one.

**Definition 5** Let $\langle \mathscr{A}, \mathscr{O}, w\rangle$ be an add-MARA instance. The allocation $\overrightarrow{\pi}$ satisfies the criterion of *envy-freeness* (or simply is envy-free), denoted by $\overrightarrow{\pi} \models$ EF, when for all $i, j$ : $u_i(\pi_i) \geq u_i(\pi_j)$ (no agent strictly prefers the share of another agent to her own share).

A known fact is that envy-freeness implies proportionality for additive preferences. The following proposition is actually a bit stronger.

**Proposition 6** *Any envy-free allocation gives to each agent at least her min–max fair-share. In other words, for all $\overrightarrow{\pi}$ : $\overrightarrow{\pi} \models \mathrm{EF} \implies \overrightarrow{\pi} \models \mathrm{mFS}$, and $\mathscr{I}_{|\mathrm{EF}} \subset \mathscr{I}_{|\mathrm{mFS}}$.*

*Proof* In every envy-free allocation, each agent obtains a share which is of maximal utility for her in this allocation. Hence, such a share has a (weakly) greater utility than her min–max fair-share. Formally : let $\overrightarrow{\pi}$ be an envy-free allocation. Then for all $i, j : u_i(\pi_i) \geq \max_{j \in \mathscr{A}} u_i(\pi_j)$ by definition. Since $\overrightarrow{\pi} \in \mathscr{F}$, $u_i(\pi_i) \geq \min_{\overrightarrow{\pi} \in \mathscr{F}} \max_{j \in \mathscr{A}} u_i(\pi_j) = u_i^{\mathrm{mFS}}$. Once again, the inclusion in this proposition is strict, as the following example shows. □

*Example 3* Let us consider the 3 agents / 4 objects instance defined by the following weight matrix:

$$W = \begin{pmatrix} *10 & 6 & 6 & 1 \\ 10 & *6 & *6 & 1 \\ 1 & 6 & 6 & *10 \end{pmatrix}$$

We have $u_i^{\mathrm{mFS}} = 10$ for each agent, thus the marked allocation gives the min–max fair-share to every agent. Now suppose that there exists an envy-free allocation $\overrightarrow{\pi}$. This allocation $\overrightarrow{\pi}$ should give the same utility to agents 1 and 2 since they have the same preferences (otherwise one of them would be envious): either $\overrightarrow{\pi}$ gives nothing to them, or it gives 6 to each of them. In both cases they envy agent 3. Hence there is no envy-free allocation for this instance.

At last, we recall two complexity results related to envy-freeness already stated in the related work section. First, the problem of existence of a complete (that is in which all goods are allocated) envy-free allocation is $\mathsf{NP}$-complete [32]. Second, the problem of deciding whether an envy-free and Pareto-efficient allocation exists is $\Sigma_2^{\mathsf{P}}$-complete [30].

### 4.5 Competitive equilibrium from equal incomes

The last criterion we present is a classical notion in microeconomics (see for example [38], page 177). It has, to the best of our knowledge, almost never been considered in computer science, with the notable exception of the work by Othman et al. [39] about course allocation (building on a preliminary version of the work by Budish [16]). This criterion is based on the following argument: the allocation process should be considered as a search for an equilibrium between the supply (the set of objects, each one having a public price) and the demand (the agents' desires, each agent having the same budget for buying the objects). A competitive equilibrium is reached when the supply matches the demand. The fairness argument is straightforward: prices and budgets are the same for everyone. A lot of variants of this notion exist; the following definition is adapted from Budish [16]. A discussion about this criterion is postponed at the end of this subsection.

**Definition 6** Let $\langle \mathscr{A}, \mathscr{O}, w \rangle$ be an add-MARA instance, $\overrightarrow{\pi}$ an allocation, and $\overrightarrow{p} \in [0, 1]^m$ a price vector. A pair $(\overrightarrow{\pi}, \overrightarrow{p})$ is said to form a *competitive equilibrium from equal incomes* (CEEI), if for each agent $i$,

$$\pi_i \in \mathrm{argmax}_{\pi \subseteq \mathscr{O}} \left\{ u_i(\pi) : \sum_{\ell \in \pi} p_\ell \leq 1 \right\}.$$

In other words, $\pi_i$ is one of the maximal shares that $i$ can buy with a budget of 1, given that the price of object $\ell$ is $p_\ell$.

We say that the allocation $\overrightarrow{\pi}$ satisfies the CEEI criterion (or is a CEEI allocation for short), denoted by $\overrightarrow{\pi} \models$ CEEI, if there exists a price vector $\overrightarrow{p}$ such that $(\overrightarrow{\pi}, \overrightarrow{p})$ forms a CEEI.

*Example 4* Let us consider the 2 agents / 4 objects instance defined by the following weight matrix:

$$W = \begin{pmatrix} *7 & 2 & 6 & *10 \\ 7 & *6 & *8 & 4 \end{pmatrix}$$

The marked allocation, associated to price vector $\langle 0.8, 0.2, 0.8, 0.2\rangle$ forms a CEEI.

The following proposition holds for a lot of continuous resource allocation instances (divisible goods, existence of monetary compensations, etc.). It also holds in our discrete model:

**Proposition 7** *Every CEEI allocation is envy-free. That is, for every allocation $\overrightarrow{\pi}$ : $\overrightarrow{\pi} \models$ CEEI $\Longrightarrow$ $\overrightarrow{\pi} \models$ EF, and $\mathscr{I}_{|\text{CEEI}} \subset \mathscr{I}_{|\text{EF}}$.*

*Proof* Let $\overrightarrow{\pi}$ be a CEEI allocation, and suppose that $u_i(\pi_j) > u_i(\pi_i)$ (agent $i$ envies $j$). Since budgets and prices are the same for everyone, $\pi_i$ is not the maximal utility share which can be bought by agent $i$, which contradicts the definition of the CEEI. Thus $\overrightarrow{\pi}$ is envy-free. Hence $\mathscr{I}_{|\text{CEEI}} \subseteq \mathscr{I}_{|\text{EF}}$. The strictness of this inclusion is proved by Example 6 below. □

The CEEI criterion also has the following interesting property:

**Proposition 8** *When the agents' preferences are strict (i.e. for every agent i and bundles $\pi$ and $\pi'$, $\pi \neq \pi'$ implies $u_i(\pi) \neq u_i(\pi')$), any CEEI allocation is Pareto-efficient.*

*Proof* Let $(\overrightarrow{\pi}, \overrightarrow{p})$ be a CEEI allocation. For a share $\pi$, we use the notation $p(\pi) \overset{def}{=} \sum_{\ell \in \pi} p_\ell$. Suppose that $\overrightarrow{\pi}$ is not Pareto-efficient. Then there is a $\overrightarrow{\pi}'$ such that $u_i(\pi_i) \leq u_i(\pi'_i)$ for all $i$, with at least one strict inequality. Since $\overrightarrow{\pi}$ is optimal under each agent's budget given prices $\overrightarrow{p}$, we have $u_i(\pi_i) < u_i(\pi'_i) \Rightarrow p(\pi_i) < p(\pi'_i)$. But $u_i(\pi_i) = u_i(\pi'_i) \Rightarrow \pi_i = \pi'_i \Rightarrow p(\pi_i) = p(\pi'_i)$ because preferences are strict. Therefore $\sum_{i \in \mathscr{A}} \overrightarrow{p}(\pi_i) < \sum_{i \in \mathscr{A}} \overrightarrow{p}(\pi'_i)$, which is impossible. Thus $\overrightarrow{\pi}$ is Pareto-efficient. □

The following example shows that the strict preference hypothesis, in the previous proposition, is necessary.

*Example 5*

$$W = \begin{pmatrix} *2 & 3 & 3 & *2 \\ 2 & 3 & *4 & 1 \\ 0 & *4 & 2 & 4 \end{pmatrix}$$

In this instance, preferences are not strict. The marked allocation, associated to price vector $\langle 0.5, 1, 1, 0.5\rangle$ forms a CEEI. However, it is dominated by the allocation $\langle (1, 2), (3), (4)\rangle$ which gives utilities $(5, 4, 4)$. The marked allocation is CEEI but not Pareto-efficient.

As a consequence of Propositions 7 and 8, when preferences are strict, a necessary condition for the existence of a CEEI is the existence of an envy-free Pareto-efficient allocation. With this necessary condition, we can prove that the inclusion in Proposition 7 is strict, as the following example shows:

*Example 6* Let us consider the 3 agents / 5 objects instance in which preferences are strict, defined by the following weight matrix:

$$W = \begin{pmatrix} 2 & 12 & *7 & \dagger 15 & *\dagger 11 \\ *\dagger 12 & 15 & \dagger 11 & *7 & 2 \\ 15 & *\dagger 20 & 9 & 2 & 1 \end{pmatrix}$$

It can be proved that the allocation marked with $*$ is the only envy-free allocation. However, this allocation is not Pareto-efficient, as it is dominated by the one marked with †. Hence there is no Pareto-efficient envy-free allocation. The preferences being strict, Proposition 8 implies that there is no CEEI allocation.

We give now some additional insights about the CEEI criterion used in this article and its context. The notion of equilibrium is one of the most important concepts in economic theory, and has long been a subject of investigations, since the end of the 19th century. The CEEI criterion used in the present article is related to the *Fisher model* of equilibrium. Irving Fisher was one of the first economists, with Walras [48], to give, in his Ph. D. dissertation [24], a mathematical model of the equilibrium concept, now quoted after his name.[8] The Fisher model formalizes an ideal micro-market in which a set of buyers are faced with a set of *divisible* goods (goods may be obtained in fractional quantities). Each good exists in limited quantity. The utility function of each buyer, specifying the amount of utility a buyer enjoys for each bundle of goods, is given. Each buyer owns a fixed amount of money. The problem is to find prices for the goods that "clear the market", that is, if every buyer receives a bundle of maximal utility among the bundles that she can afford, then an equilibrium is obtained: no good is over nor under demanded.

The CEEI criterion we use here is related to a subcase of the Fisher model. First—and this is the main difference—goods are indivisible (and each exist in one unit). In other words, the general Fisher model has continuous variables, whereas the CEEI criterion we use has discrete variables. Second, utility functions of buyers (agents) are linear (i.e. additive). Third, each agent is endowed with the same amount of money.

An interesting result is that, when agents' utilities are linear in their consumptions, an equilibrium of the general Fisher model (*i.e.* with divisible goods) always exists, in which all buyers spend all their money. Moreover, in such a competitive equilibrium, prices and quantities allocated can be computed in polynomial time, by the Eisenberg–Gale convex program, which consists of maximizing the Nash CUF[9] under linear constraints. See [47] for details.

In our discrete setting of the Fisher model, the situation is quite different: a CEEI allocation may not exist, even with additive utilities.[10] Moreover, in a CEEI allocation, a buyer may not spend all of her money.[11]

The introduction of the CEEI concept raises the natural and non-trivial question of how to compute a CEEI allocation for a given add-MARA instance, if such an allocation exists. The problem turns out to be computationally difficult, as discussed by ([39] section 3.1) in a similar context. The reason is of course the discrete setting of the problem. To the best of our knowledge, no algorithm has been proposed to solve this problem exactly.[12] In the following, we sketch such an algorithm, however rather naive and probably inefficient. Due to the discrete nature of the model, the set of admissible allocations of a given instance can

8 See also [10] for a summary of the original Fisher's equations and for a brilliant account of his work (actually, I. Fisher had also developed a hydraulic device (!) for calculating equilibrium prices). Vazirani [47] is a recent key reference on this subject, building on the linear case of the Fisher model (hence eliminating the notion of marginal utility present in the original one).

9 In our settings, the Nash CUF is the function $g_N : \overrightarrow{\pi} \mapsto \prod_{i \in \mathscr{A}} u_i(\pi_i)$.

10 For instance, consider an instance for which no envy-free allocation exists.

11 Note that in this discrete model, money is not "real", in the sense that it is just a modelling artifact used to define the "choice set" of buyers. This is an important difference with the combinatorial auction problem [19], in which a buyer is supposed to keep the money not spent if any. This important point is discussed in details by Othman et al. [39].

12 The work by Othman et al. [39] is devoted to the computation of *approximate* competitive equilibria.

be enumerated (there are $m^n$ such allocations). So the problem comes down to determining whether a given allocation $\overrightarrow{\pi}$ is CEEI. Thanks to Proposition 7, if $\overrightarrow{\pi}$ is not envy-free, the question is solved negatively. Otherwise, the problem comes down to finding values for variables $p_1, \ldots, p_m$ (object prices) satisfying the following set of contraints:

$$0 \leq p_\ell \leq 1 \quad \text{for all} \quad \ell \in [\![1, m]\!] \tag{1}$$

$$\sum_{\ell \in \pi_i} p_\ell \leq 1 \quad \text{for all} \quad i \in [\![1, n]\!] \tag{2}$$

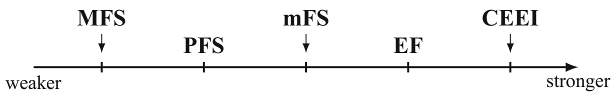
$$\sum_{\ell \in \pi'} p_\ell > 1 \quad \text{for all} \quad i \in [\![1, n]\!] \quad \text{and} \quad \pi' \in 2^{\mathscr{O}} \quad \text{such that} \quad u_i(\pi') > u_i(\pi_i). \tag{3}$$

Constraints (2) model the fact that each agent can afford her share. Constraints (3) express the optimality of $\overrightarrow{\pi}$ given the prices: each better share $\pi'$ for $i$ is unaffordable. Unfortunately, as we can observe, this problem is very likely to be hard to solve in practice because: (1) the number of constraints of kind (3) is not polynomially bounded, and (2) these constraints are not even linear (because of the strict inequality).

### 4.6 A scale of criteria

Putting Propositions 2, 4, 6 and 7 together leads to the following implication sequence, for any allocation $\overrightarrow{\pi}$: $(\overrightarrow{\pi} \vDash \text{CEEI}) \Rightarrow (\overrightarrow{\pi} \vDash \text{EF}) \Rightarrow (\overrightarrow{\pi} \vDash \text{mFS}) \Rightarrow (\overrightarrow{\pi} \vDash \text{PFS}) \Rightarrow (\overrightarrow{\pi} \vDash \text{MFS})$.

In other words, these criteria can be ranked from the least to the more demanding as follows:




As the propositions also show, these results can also be interpreted the other way around, in terms of add-MARA instances: $\mathscr{I}_{|\text{CEEI}} \subset \mathscr{I}_{|\text{EF}} \subset \mathscr{I}_{|\text{mFS}} \subset \mathscr{I}_{|\text{PFS}} \subset \mathscr{I}_{|\text{MFS}} \subset \mathscr{I}$, all these inclusions being strict (see Fig. 1).

These five criteria can thus be used to characterize the level of conflict inherent to a given add-MARA instance. In an instance for which it is proved that there exists an allocation satisfying the CEEI criterion, the level of conflict is very low, and thus it is possible to find an allocation which is quite satisfactory for everyone. On the other hand, an instance for which

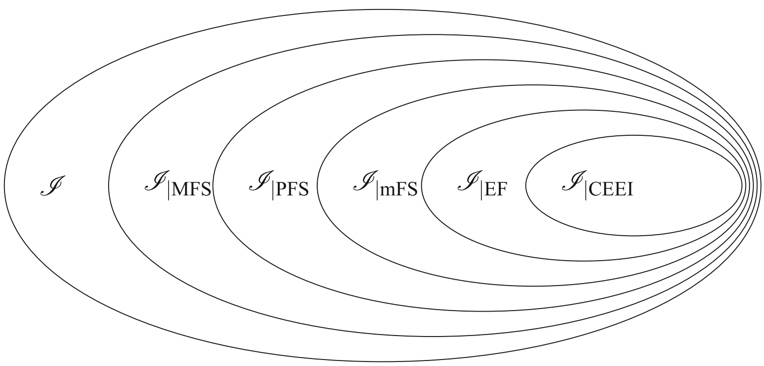



**Fig. 1** Inclusion properties of add-MARA instances satisfying different fairness criteria

we cannot even find an allocation satisfying the max-min fair-share criterion is very prone to conflicts, and in that case, the benevolent arbitrator will have no choice but to leave some agents unsatisfied.

It can be noticed that all these criteria have a kind of "distributed" (or decentralized) flavor. The max-min fair-share, proportional fair-share and min-max fair-share criteria are of similar nature: every agent, only considering her own share, is able to judge whether she is satisfied or not. Envy-freeness requires the additional knowledge of the other shares, but each agent is still able to assert on her own whether she is envious or not. As for the CEEI criterion, once the prices are fixed by the arbitrator, each agent is able to compute her own share (up to some equivalent shares).

Beyond their differences, these five criteria all have a common very appealing feature: they are not based on any interpersonal comparison of utilities: the definition of any criterion does not involve any comparison nor arithmetical operation between utilities of two distinct agents, so each agent may use her own utility scale.[13] This leads to the following (easy) proposition:

**Proposition 9** *The max–min fair-share, proportional fair-share, min–max fair-share, envy-freeness and CEEI criteria are preserved by any linear dilatation of individual utility scales. Formally, if $\langle \mathscr{A}, \mathscr{O}, w \rangle$ is an add-MARA instance and $\overrightarrow{\pi}$ an allocation satisfying criterion $\mathscr{C}$, then $\overrightarrow{\pi}$ also satisfies $\mathscr{C}$ for any instance $\langle \mathscr{A}, \mathscr{O}, w_K \rangle$, where $K : \mathscr{A} \to \mathbb{R}^+$ and $w_K$ is defined as follows: $w_K(i, \ell) = K(i) \times w(i, \ell)$.*

Finally, the max-min fair-share, proportional fair-share and min-max fair-share criteria have an interesting feature, which comes from the fact that they are all defined as minimum thresholds to satisfy for the agents utilities: if for a given add-MARA instance there is an allocation satisfying one of these three criteria, then either this allocation is Pareto-efficient, or there exists another allocation which both satisfies Pareto-efficiency and this criterion.[14] This is not the case for the envy-free criterion : as Example 6 shows, one can find instances having envy-free allocations, none of them being Pareto-efficient.

## 5 The egalitarian approach

As pointed out at the beginning of Sect. 4, an orthogonal approach for ensuring fairness in resource allocation problems is to choose a CUF and find an allocation that maximizes it. The most prominent one is probably the egalitarian CUF, which can be defined as follows in our context:

**Definition 7** Let $\langle \mathscr{A}, \mathscr{O}, w \rangle$ be an add-MARA instance. The egalitarian CUF is the function $g_e : \overrightarrow{\pi} \mapsto \min_{i \in \mathscr{A}} u_i(\pi_i)$. Any allocation maximizing the egalitarian CUF will be called *min-optimal*.

This CUF is the formal translation of Rawlsian egalitarianism [44], which recommends to maximize the utility of the least well-off agent.

At this point a natural question arises: what are the links between on one hand the egalitarian approach of fairness (the min CUF) and on the other hand the approach based on criteria

[13] Even if in our examples we use normalized weights. Actually, four of the criteria are even purely ordinal—proportional fair-share is not.

[14] If an allocation is not Pareto-efficient, then it is dominated by at least one Pareto-efficient allocation, which satisfies the criterion as well.

(Sect. 4)? Actually, some criteria are not fully compatible with egalitarianism. For example, an envy-free allocation can be far away from being min-optimal. This question is raised by Brams and King [11]. However, as we will see just below, egalitarianism is more compatible with proportional fair-share as well as with max-min fair-share.

Contrary to the five criteria considered in Sect. 4, the egalitarian approach only makes sense if the utilities of the agents are commensurable (since to compute a min-optimal allocation, one must be able to compare the utilities different agents obtain). To cope with this difficulty, we assume in this section that the agents have *normalized* weights, namely: there is a constant $K$ such that for all $i$, $\sum_{\ell \in \mathscr{O}} w(i, \ell) = K$.

**Proposition 10** *If there is an allocation satisfying the proportional fair-share criterion (with normalized weights), then any min-optimal allocation satisfies this criterion.*

*Proof* For all $i$, $u_i^{\text{PFS}} = K/n$. If there is an allocation $\overrightarrow{\pi}$ such that $\overrightarrow{\pi} \models \text{PFS}$, then $K/n \leq \min_{i \in \mathscr{A}} u_i(\pi_i)$. Let $\overrightarrow{\pi}^\star$ be a min-optimal allocation. By definition $\min_i u_i(\pi_i) \leq \min_i u_i(\pi_i^\star)$, hence $K/n \leq \min_{i \in \mathscr{A}} u_i(\pi_i^\star)$ and $K/n \leq u_i(\pi_i^\star)$, for all $i$. □

The following example shows that the normalization hypothesis, in the previous proposition, is necessary.

*Example 7* Consider the following instance, given by its (non normalized) weight matrix.

$$\begin{pmatrix} 5 & *4 & \dagger 3 \\ 0 & \dagger 2 & *1 \\ *\dagger 4 & 7 & 1 \end{pmatrix} \begin{array}{l} \rightarrow u_1^{\text{PFS}} = 4 \\ \rightarrow u_2^{\text{PFS}} = 1 \\ \rightarrow u_3^{\text{PFS}} = 4 \end{array}$$

The proportional fair-share of each agent is given on the right of the matrix. The only allocation satisfying the proportional fair-share criterion is marked with '$*$'. The only min-optimal allocation, marked with '$\dagger$', does not satisfy the proportional fair-share criterion.

Things are less clear for max-min fair-share. On the one hand, the latter result does not hold for max-min fair-share,[15] as the following counter-example shows (with $K = 100$).

*Example 8* Consider the following instance, given by its weight matrix.

$$\begin{pmatrix} 58 & \dagger 15 & *\dagger 19 & 8 \\ \dagger 63 & *5 & 25 & *7 \\ 37 & 10 & *27 & \dagger 26 \end{pmatrix} \begin{array}{l} \rightarrow u_1^{\text{MFS}} = 19 \ / \ u(\pi_1^\dagger) = 34 \\ \rightarrow u_2^{\text{MFS}} = 12 \ / \ u(\pi_2^\dagger) = 63 \\ \rightarrow u_3^{\text{MFS}} = 27 \ / \ u(\pi_3^\dagger) = 26 \end{array}$$

The max-min fair-share of each agent is given at the right of the matrix, and the corresponding shares are marked with '$*$' in the matrix. A min-optimal allocation[16] and the corresponding utilities are marked with '$\dagger$'. In this min-optimal allocation, the third agent does not get her max-min fair-share (expecting at least 27 but getting only 26).

For this instance, there are allocations satisfying the max-min fair-share criterion, for example $\langle\{2, 4\}, \{1\}, \{3\}\rangle$, but none of them are min-optimal. Moreover, the min-optimal

[15] Actually a similar result holds if weights are normalized such that $u_i^{\text{MFS}}$ is equal for all agents.

[16] This min-optimal allocation is also leximin-optimal. The leximin ordering [45] is a refinement of the min ordering for which a lexicographic comparison of sorted vectors of weigths is used, instead of comparing their min values.

allocation does not provide her proportional fair-share to agent 3 ($26 < 100/3$). Hence from Proposition 10, we know that this instance admits no allocation satisfying the proportional fair-share criterion, and from Propositions 6 and 7, it admits no allocation satisfying the min-max fair-share, envy-freeness or CEEI criteria.

On the other hand however, such an instance is rare: for example, using a uniform generation process similar to the impartial culture in voting theory (see Sect. 7), for 3 agents and 4 objects, approximately only one instance over 3500 is a counter-example similar to Example 8, in which a min-optimal allocation does not satisfy the max-min fair-share criterion. This shows that the max-min fair-share criterion has a good correlation with the egalitarian approach in practice.

## 6 Restricted cases

In this section we examine the behaviour of our criteria—and especially the max-min fair-share one—in some restricted cases, giving to these criteria an additional insight. These restrictions concern the agents' preferences and the number of agents and objects. The main result here is that for all these restrictions (even if some of them are very general), it is always possible to find an allocation satisfying the max-min fair-share criterion.

### 6.1 Restricted preferences

#### *6.1.1 0–1 preferences*

We first consider the case where the weights are binary, which corresponds to the MARA version of approval voting. Interestingly, we can prove that an allocation satisfying max-min fair-share can always be found, using a decentralized protocol where each agent takes in turn, according to a predefined sequence, one of her preferred (approved, here) objects among the remaining ones. The outcome of such a *picking protocol* is an allocation called *product of sincere choices* by Brams and King [11]. Using this protocol—also called *elicitation-free sequential protocol* [8]—with an alternating sequence of agents always results in an allocation satisfying max–min fair-share (if every agent acts sincerely):

**Proposition 11** *(Approval resource allocation) Any add-MARA instance with weights restricted to* 0, 1 *belongs to* $\mathscr{I}_{|\mathrm{MFS}}$.

*Proof* In an instance with $n$ agents and $m$ objects, the max–min fair-share of agent $i$ is $\lfloor \frac{s_i}{n} \rfloor$, with $s_i = \sum_{\ell=1}^{m} w(i, \ell)$. The following very simple algorithm (a picking protocol) gives any agent her max–min fair-share:

```
while ( true )
    for i = 1 to n
        Allocate to agent i an object ℓ not allocated yet such that w(i, ℓ) = 1 if any,
        Otherwise allocate to i any remaining object of weight 0.
        If all objects are allocated, exit.
```

The algorithm passes $\lceil \frac{m}{n} \rceil$ times in the for loop, $\lfloor \frac{m}{n} \rfloor$ of these passes being complete. During each complete pass, $n$ objects are allocated, one to each agent. We have $s_i \leq m$, so during each of the first $\lfloor \frac{s_i}{n} \rfloor$ complete passes at least, agent $i$ receives an object of weight 1. $\square$

#### 6.1.2 Identical preferences

When agents have identical utility functions, our scale of criteria collapses into two levels: the max-min fair-share criterion on the one hand, and all others on the other hand.

**Proposition 12** *If agents have identical preferences (for all $i, j, \ell : w(i, \ell) = w(j, \ell)$), then:*

1. *there always exists an allocation satisfying the max-min fair-share criterion, and in particular any min-optimal allocation satisfies it;*
2. *if preferences are strict (i.e. for every agent $i$ and bundles $\pi$ and $\pi'$, $\pi \neq \pi'$ implies $u_i(\pi) \neq u_i(\pi')$), no allocation satisfies the proportional fair-share criterion, and thus none satisfies the three more demanding criteria;*
3. *for any allocation $\overrightarrow{\pi}$, the following five propositions are equivalent: (i) each agent in $\overrightarrow{\pi}$ gets the same utility; (ii) $\overrightarrow{\pi} \models$ CEEI; (iii) $\overrightarrow{\pi} \models$ EF; (iv) $\overrightarrow{\pi} \models$ mFS; (v) $\overrightarrow{\pi} \models$ PFS.*

*Proof* 1. Consider a min-optimal allocation $\overrightarrow{\pi}^\star$. Then for each agent $i$:

$$\begin{aligned} u_i^{\text{MFS}} &\stackrel{def}{=} \max_{\overrightarrow{\pi} \in \mathscr{F}} \min_{j \in \mathscr{A}} u_i(\pi_j) \\ &= \max_{\overrightarrow{\pi} \in \mathscr{F}} \min_{j \in \mathscr{A}} u_j(\pi_j) \quad \text{(because of identical preferences)} \\ &= \min_{j \in \mathscr{A}} u_j(\pi_j^\star) \leq u_i(\pi_i^\star) \end{aligned}$$

2. Because preferences are strict, for any allocation $\overrightarrow{\pi}$, the $n$ numbers $u_i(\pi_i)$ are different. Then, at least one of them is strictly smaller than their mean.
3. Let $\overrightarrow{\pi}$ be an allocation in which agents get the same utility, which is $u_i(\mathscr{O})/n$ for any agent $i$. Consider the following price vector: $p_\ell = nw(i, \ell)/u_i(\mathscr{O})$. The total price is $n$, and the price of every share of $\overrightarrow{\pi}$ is exactly 1. So each agent can buy any share of $\overrightarrow{\pi}$, and any share that would provide higher utility costs necessarily more. Hence $\overrightarrow{\pi} \models$ CEEI. The other three implications follow from the scale of criteria (Sect. 4). The implication closing the cycle can be easily proved: if $\overrightarrow{\pi} \models$ PFS then

$$u_i(\pi_i) \geq u_i(\mathscr{O})/n \quad \text{for all} \quad i.$$

Suppose for contradiction that agent $j$ gets strictly more than her proportional fair-share:

$$u_j(\pi_j) > u_j(\mathscr{O})/n$$

Summing inequalities over all agents we get

$$\sum_i u_i(\pi_i) > u_i(\mathscr{O})$$

which is impossible. Hence $u_i(\pi_i) = u_i(\mathscr{O})/n$, the same for all $i$. □

The first point of this proposition says that as soon as the agents have identical preferences, the max-min fair-share criterion is always satisfiable. It turns out that it is even true for a slightly weaker condition than fully identical preferences:

**Proposition 13** *If all agents but one have identical preferences, then there always exists an allocation satisfying the max-min fair-share criterion.*

*Proof* The $n - 1$ first agents having identical preferences can agree on the same allocation $\overrightarrow{\pi}$ of which any share gives their max-min fair-share to anyone of them. Let the last agent choose first in $\overrightarrow{\pi}$: she gets her min-max fair-share so her max-min fair-share too. □

### 6.1.3 Same-order preferences (SOP)

Intuitively, the more similar the agents preferences are, the more likely they are in conflict, and the harder it will be to satisfy the aforementioned fairness criteria. This notion of similarity is well captured by the concept of same-order preferences (SOP for short). Formally, an add-MARA instance satisfies SOP (we will say that the instance is SOP) if for all $i, \ell, \ell'$ : $\ell < \ell' \Rightarrow w(i, \ell) \geq w(i, \ell')$. In other words, the agents agree on the same ranking of objects (object 1 is the best one—or one of the best if there are ties, object $m$ is the worst one—or one of the worst), but can give them different weights.[17] For any weight function $w$, we will denote by $w^{\uparrow}$ the function $i, \ell \mapsto w(i, \sigma_i(\ell))$, where $\sigma_i$ is a permutation of $[\![1, m]\!]$ such that $\ell < \ell' \Rightarrow w(i, \sigma_i(\ell)) \geq w(i, \sigma_i(\ell'))$. Obviously, $w^{\uparrow}$ is a "SOP" version of $w$. It turns out that if we can find an allocation satisfying max-min fair-share for a given SOP add-MARA instance, then we can find one for every permutation derived from it:

**Proposition 14** *Let $\langle \mathscr{A}, \mathscr{O}, w\rangle$ be an add-MARA instance. We have $\langle \mathscr{A}, \mathscr{O}, w^{\uparrow}\rangle \in \mathscr{I}_{|\text{MFS}} \implies \langle \mathscr{A}, \mathscr{O}, w\rangle \in \mathscr{I}_{|\text{MFS}}$.*

*Proof* Let $\langle \mathscr{A}, \mathscr{O}, w\rangle$ be an add-MARA instance, and let $\overrightarrow{\pi}^{\uparrow}$ be an allocation satisfying the max-min fair-share criterion for the SOP instance $\langle \mathscr{A}, \mathscr{O}, w^{\uparrow}\rangle$. Let $S = S_1, S_2, \ldots S_m$ be the sequence of agents defined as follows: $S_\ell$ is the agent who receives object $\ell$ in $\overrightarrow{\pi}^{\uparrow}$. Such a sequence which depends on $\overrightarrow{\pi}^{\uparrow}$ always exists because the instance is SOP, and because each object is given to exactly one agent.

The key is to notice that the "product of sincere choices" $\overrightarrow{\pi}$ obtained by the picking protocol[18] using sequence $S$, applied to the original instance $\langle \mathscr{A}, \mathscr{O}, w\rangle$, will make every agent at least as well-off as in $\overrightarrow{\pi}^{\uparrow}$. To see it, notice that before step $p$ in the building of $\overrightarrow{\pi}$, exactly $p-1$ objects have been chosen, so the worst object that agent $S_p$ could have at step $p$ is the object $p$ obtained in $\overrightarrow{\pi}^{\uparrow}$. Consequently, for each agent $i$ and each object of $\pi_i^{\uparrow}$, there is an object in $\pi_i$ which is weakly better for $i$: the utility of $i$ weakly increases from $\overrightarrow{\pi}^{\uparrow}$ (in $\langle \mathscr{A}, \mathscr{O}, w^{\uparrow}\rangle$) to $\overrightarrow{\pi}$ (in $\langle \mathscr{A}, \mathscr{O}, w\rangle$).

Since the max-min fair-share of an agent only depends on the set of weights (not on their ordering), it is the same for the SOP instance and the original one. Since $\overrightarrow{\pi}^{\uparrow} \models$ MFS, and $\overrightarrow{\pi}$ makes everyone at least as well-off, we conclude $\overrightarrow{\pi} \models$ MFS. □

Here is an example giving the intuition supporting this proof.

*Example 9* Consider the following instance, given by its weight matrix.

$$w = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 5 & 2 & 10 & 3 \end{pmatrix} \begin{matrix} \rightarrow u_1^{\text{MFS}} = 5 \\ \rightarrow u_2^{\text{MFS}} = 10 \end{matrix}$$

A SOP version of this instance, with a marked allocation satisfying the max-min fair-share criterion is

$$w^{\uparrow} = \begin{pmatrix} 4 & *3 & *2 & 1 \\ *10 & 5 & 3 & *2 \end{pmatrix}$$

This last allocation corresponds to the sequence of sincere choices $\langle 2, 1, 1, 2\rangle$. Applying this sequence to the original instance yields the following marked allocation

$$w = \begin{pmatrix} *4 & *3 & 2 & 1 \\ 5 & 2 & *10 & *3 \end{pmatrix}$$

[17] This property is sometimes known as full-correlation [8].

[18] For the meaning of the term "picking protocol", see the beginning of Sect. 6.1.

in which each agent is better-off than in $w^{\uparrow}$ above. The reason is clearly because $w$ is less conflicting than $w^{\uparrow}$. So for example when agent 2, obeying the sequence, chooses first in $w$, she takes the object 3 (of weight 10), but she frees the object 1, which is now available for agent 1.

Because any instance can be considered as a derivation (by permutations of weights) of a SOP one, this proposition shows that SOP instances are the most difficult ones as far as the max-min fair-share criterion is concerned.[19] So, to prove that all instances of a given subset satisfy this criterion, we only need to prove that any SOP instances of that subset satisfies it. In the following, we will often consider only the SOP instances of the subsets of interest, and hence the results obtained for them will be valid also for all instances of the subset.

#### *6.1.4 Weights defined by a scoring function*

We consider here the case where agents express their preferences using exactly the same multiset of weights (formally, for all $(i, j) \in \mathscr{A}^2$, $\{\!\{w_{i,\ell} \mid \ell \in \mathscr{O}\}\!\} = \{\!\{w_{j,\ell} \mid \ell \in \mathscr{O}\}\!\}$, where denotes a multiset). Equivalently we could say that agents use the same *scoring function*. A scoring function is simply a weakly decreasing function $g : [\![1, m]\!] \to \mathbb{R}^+$. It can be used to convert a purely ordinal expression of preferences into to a numerical one, in the following way. Consider that each agent ranks strictly the objects from 1 (the most preferred) to $m$ (the least preferred). If $r(i, \ell)$ is the rank given to object $\ell$ by agent $i$, then the weight $w(i, \ell)$ is defined as $g(r(i, \ell))$. This framework, which is standard in social choice, is the basis of well-known procedures in voting theory (plurality, veto, Borda scores for examples). It has been already considered in fair division of indivisible goods [4,8] and social choice [7].

**Proposition 15** *Any add-MARA instance in which preferences are defined by the same scoring function is in $\mathscr{I}_{|\text{MFS}}$, and any min-optimal allocation satisfies max-min fair-share in this case. Conclusions* 2 *and* 3 *of Proposition 12 are also valid.*

*Proof* By Proposition 14 it is enough to consider SOP instances, which are in this case instances with identical preferences. Then use Proposition 12. □

### 6.2 Restrictions upon the number of agents and objects

#### *6.2.1 Two agents*

Despite its simplicity, the two agents case is interesting because an allocation satisfying the max–min fair-share criterion can always been obtained by the famous cut-and-choose game.

**Proposition 16** *Any 2-agents add-MARA instance belongs to $\mathscr{I}_{|\text{MFS}}$.*

*Proof* Agent 1 cuts (meaning that she makes the two shares), then guaranteeing her max-min fair-share. Agent 2 chooses first, so she gets her min-max fair-share, therefore her max-min fair-share too (by the property of the ordered scale, Sect. 4.6). □

#### *6.2.2 No more objects than agents*

If there are strictly less objects than agents, the scale of criteria is reduced to only one level, and hence is of no help.[20] The case with as many objects as agents highlights the min-max fair-share criterion.

[19] This also seems to be true for more demanding criteria as our experiments show, see Sect. 7.

[20] The best resort in this case would be a normalized leximin-optimal allocation.

**Proposition 17** *If there are strictly less objects than agents, any allocation satisfies the max-min fair-share criterion, but none satisfies the other criteria.*

*If there are as many objects as agents, then*

1. *any allocation which is a matching (giving to each agent one object) satisfies the max-min fair-share criterion.*
2. *any allocation satisfying the min-max fair-share criterion is a matching, envy-free, Pareto-efficient and CEEI.*

*Proof* Case $m < n$. In any allocation, one agent at least receives no object, hence $u_i^{\text{MFS}} = 0$ for all $i$. As $0 \leq u_i(\pi_i)$ for all $i$, each agent gets her max-min fair-share. Of course no allocation satisfies the proportional fair-share criterion.

Case $m = n$.

1. We have easily $u_i^{\text{MFS}} = \min_{\ell \in \mathscr{O}} w(i, \ell)$, hence each agent receives her max-min fair-share in a matching.
2. We have also easily $u_i^{\text{mFS}} = \max_{\ell \in \mathscr{O}} w(i, \ell)$. In an allocation satisfying the min-max fair-share criterion each agent receives a preferred object. The allocation is hence an envy-free matching. It is Pareto-efficient because for an agent to get strictly more utility, she necessarily has to take another object from another agent, strictly reducing this agent's utility. A price of 1 for each object provides a CEEI allocation, because any increase of utility must be paid more. □

### *6.2.3 Up to three more objects than agents*

As mentioned in Sect. 4.1, Procaccia and Wang [43] have proved that for every $n > 2$ it is possible to find an instance with $m = n^n$ objects for which there is no allocation satisfying the max-min fair-share criterion (*i.e.* not in $\mathscr{I}_{|\text{MFS}}$). This raises an interesting question: is it possible to find an instance not in $\mathscr{I}_{|\text{MFS}}$, with $m < n^n$ ? Actually, the answer is positive for $n = 3$, as shown also by the same authors who have found such an instance with 3 agents and 12 ($< 3^3$) objects. The general question can hence be formulated as follows: what is, for a given number of agents $n > 2$, the smallest number of objects $m_n$ for which a $(n, m_n)$ instance is not in $\mathscr{I}_{|\text{MFS}}$ ? (equivalently, for any given $n > 2$, what is the maximum number of objects $m_n$ such that any add-MARA instance with $n$ agents and less than $m_n$ objects is guaranteed to have an instance satisfying the max-min fair-share criterion ?) Obviously, as shown by Proposition 17, $n < m_n$. In this section, we increase this lower bound by showing that if $m$ is no more than $n + 3$, then it is always possible to find an allocation satisfying the max-min fair-share criterion (that is $n + 3 < m_n$ and in particular $7 \leq m_3 \leq 12$).

We begin by the case $m = n + 1$.

**Proposition 18** *For* $n \geq 2$*, any add-MARA instance with* $n$ *agents and* $n+1$ *objects belongs to* $\mathscr{I}_{|\text{MFS}}$.

*Proof* Thanks to Proposition 14, we can focus on SOP instances. Since objects $n$ and $n + 1$ are the worst ones, it is not difficult to see that all the shares from allocation $\langle \{1\}\{2\} \ldots \{n - 1\}\{n, n + 1\}\rangle$ give to each agent her max-min fair-share. □

To continue with $m = n + 2$ and $m = n + 3$, we need first a convenient definition of the extension of an instance obtained by adding $p$ agents and $q$ objects to a given instance, and an additional notation.

**Definition 8** Let $I = \langle \mathscr{A}, \mathscr{O}, w \rangle$ be an add-MARA instance. A $(p, q)$-extension of $I$ is an add-MARA instance $I_{+p,+q} = \langle \mathscr{A}', \mathscr{O}', w' \rangle$ such that $\mathscr{A}' = \mathscr{A} \cup \{n+1, \ldots, n+p\}$, $\mathscr{O}' = \mathscr{O} \cup \{m+1, \ldots, m+q\}$, and $w'(i, \ell) = w(i, \ell)$ for all $(i, \ell) \in \mathscr{A} \times \mathscr{O}$.

We denote by $u_i^{\mathrm{MFS}}(I)$ the max-min fair-share of agent $i$ in instance $I$.

Then we give some preliminary lemmas. The first one shows the behavior of $u_i^{\mathrm{MFS}}(I)$ when $k$ additional agents and objects are added to $I$.

**Lemma 1** *Let $I = \langle \mathscr{A}, \mathscr{O}, w \rangle$ be an add-MARA instance. Then for all $i \in \mathscr{A}$, $u_i^{\mathrm{MFS}}$ does not strictly increase from $I$ to any $(k, k)$-extension of $I$. More formally, for all integer $k > 0$, and all $(k, k)$-extension $I_{+k,+k}$ of $I$, $u_i^{\mathrm{MFS}}(I_{+k,+k}) \leq u_i^{\mathrm{MFS}}(I)$.*

*Proof* Let $I$ be an instance with $n$ agents and $m$ objects, and $I'$ a $(1, 1)$-extension of it. Start from an allocation $\overrightarrow{\pi}'$ of $I'$ that gives her max-min fair-share to agent $i$ in it, that is:

$$\min_{j=1}^{n+1}(u_i(\pi'_j)) = u_i^{\mathrm{MFS}}(I') \tag{4}$$

Removing from $\overrightarrow{\pi}'$ the share containing object $m+1$ yields a valid (possibly incomplete) allocation $\overrightarrow{\pi}$ for $I$. Hence,

$$\min_{j=1}^{n}(u_i(\pi_j)) \leq u_i^{\mathrm{MFS}}(I) \tag{5}$$

Removing a number from a set cannot strictly decrease its minimum, so we have

$$\min_{j=1}^{n+1}(u_i(\pi'_j)) \leq \min_{j=1}^{n}(u_i(\pi_j)) \tag{6}$$

By Eqs. 4–6 we conclude

$$u_i^{\mathrm{MFS}}(I') \leq u_i^{\mathrm{MFS}}(I) \tag{7}$$

Iterating the previous result from 1 to $k$ completes the proof. □

The next (easy) lemma gives a very simple expression of max-min fair-share, proportional fair-share and min-max fair-share when all weights are equal.

**Lemma 2** *Let $I$ be an instance with $n$ agents and $m$ objects, all weights being equal to 1. Then $u_i^{\mathrm{MFS}}(I) = \lfloor \frac{m}{n} \rfloor$, $u_i^{\mathrm{PFS}}(I) = \frac{m}{n}$, $u_i^{\mathrm{mFS}}(I) = \lceil \frac{m}{n} \rceil$, for all $i \in \mathscr{A}$.*

*Proof* First equality: $m = n\lfloor \frac{m}{n} \rfloor + r = (n-r)\lfloor \frac{m}{n} \rfloor + r(\lfloor \frac{m}{n} \rfloor + 1)$ with $0 \leq r < n$. So in an allocation satisfying the max-min fair-share criterion, $n - r$ agents receive $\lfloor \frac{m}{n} \rfloor$ and $r$ agents receive $\lfloor \frac{m}{n} \rfloor + 1$. The second equality is just the definition of the proportional fair-share. For the third equality, the proof is similar to the first. □

The next lemma gives an upper bound of the max-min fair-share of an agent.

**Lemma 3** *Let $I$ be an add-MARA instance. For any agent $i$, the following inequality holds: $u_i^{\mathrm{MFS}}(I) \leq \lfloor \frac{m}{n} \rfloor \max_{\ell=1}^{m} w(i, \ell)$. In particular when $m < 2n$ then $u_i^{\mathrm{MFS}}(I) \leq \max_{\ell=1}^{m} w(i, \ell)$.*

*Proof* Let $I'$ be the instance $I$ in which all weights of agent $i$ are replaced by $\max_{\ell=1}^{m} w(i, \ell)$. Since $u_i^{\mathrm{MFS}}(I)$ is obviously a weakly increasing function of each $w(i, \ell)$, we have $u_i^{\mathrm{MFS}}(I) \leq u_i^{\mathrm{MFS}}(I')$. It is clear also that if all the weights given by agent $i$ are scaled by a constant $k$, then the max–min fair-share of this agent is also scaled by the same constant $k$. Now, take $k = \max_{\ell=1}^{m} w(i, \ell)$. From this last fact it follows that $u_i^{\mathrm{MFS}}(I') = k \cdot u_i^{\mathrm{MFS}}(I'')$, where $I''$ is $I$ with all weights of agent $i$ replaced by 1. The conclusion then follows from the first equality of Lemma 2. □

**Proposition 19** *For $n \geq 2$, any add-MARA instance with $n$ agents and $n+2$ objects belongs to $\mathscr{I}_{|\text{MFS}}$.*

*Proof* Thanks to Proposition 14, we can restrict the proof to SOP instances. We will prove the result by induction, using the following induction hypothesis: $(H_n)$ any SOP instance $I$ with $m = n + 2$ belongs to $\mathscr{I}_{|\text{MFS}}$. The base case $n = 2$ and $m = 4$ is directly given by Proposition 16.

Let us now suppose that $(H_n)$ is true. Then we take any SOP instance $I'$ with $n+1$ agents and $m+1$ objects, with $(m+1) = (n+1)+2$. We have to prove that there is an allocation $\overrightarrow{\pi}'$ for $I'$ such that $\overrightarrow{\pi}' \vDash$ MFS. $I'$ is SOP, so $w(i,1) \geq w(i,2) .... \geq w(i,m) \geq w(i,m+1)$ for all $i$, $1 \leq i \leq n+1$.

We restrict $I'$ by removing agent $n+1$ and object 1. We obtain a SOP $(n,m)$-instance $I$, with $m = n+2$ which, by the induction hypothesis, has an allocation $\overrightarrow{\pi}$ such that $\overrightarrow{\pi} \vDash$ MFS. We extend this allocation to :

$$\overrightarrow{\pi}' = \langle \pi_1, \cdots, \pi_n, \{1\} \rangle \tag{8}$$

($\overrightarrow{\pi}'$ is $\overrightarrow{\pi}$ augmented with a new share built with the object 1 alone). $\overrightarrow{\pi}'$ is a valid allocation for $I'$. We will now show that $\overrightarrow{\pi}' \vDash$ MFS.

From $m = n+2$ and $n \geq 2$ it follows that $m+1 < 2(n+1)$. Hence, by Lemma 3:

$$u_i^{\text{MFS}}(I') \leq \max_{\ell=1}^{m+1} w(i,\ell) = w(i,1) \tag{9}$$

proving that agent $n+1$ obtains her max–min fair-share in $\overrightarrow{\pi}'$. As for other $n$ first agents, Lemma 1 says that this is also the case for them in $\overrightarrow{\pi}'$ (they get the same share in $I$ and $I'$, hence same value, and their max–min fair-share cannot strictly increase from $I$ to $I'$). This proves that $(H_n) \Rightarrow (H_{n+1})$, which completes the proof. □

The case with $n$ agents and $m = n+3$ objects can also be proved the same way, but for that we need to prove the base case with 3 agents and 6 objects.

**Lemma 4** *Any add-MARA instance with 3 agents and 6 objects belongs to $\mathscr{I}_{|\text{MFS}}$.*

Before proving this lemma, we will introduce a small (but tedious) technical lemma:

**Lemma 5** *Let $I$ be a SOP $(3,6)$-instance with $w(i,1) \geq w(i,2) \geq w(i,3) \geq w(i,4) \geq w(i,5) \geq w(i,6)$ for all $i$, and $w(i,1) < u_i^{\text{MFS}}$. Then the allocation $\overrightarrow{\pi}^* = \langle \{1,6\}\{2,5\}\{3,4\} \rangle$ satisfies the max-min fair-share criterion.*

*Proof* Since $w(i,1) < u_i^{\text{MFS}}$, we have $w(i,\ell) < u_i^{\text{MFS}}$ for all $i$ and $\ell$. So, any allocation satisfying the max-min fair-share criterion cannot include a share with a single object, and must have 2 objects in each share.

For convenience, we adopt the following convention: instead of $\langle \{1,6\}\{2,5\}\{3,4\} \rangle$ we just use the notation $\langle 16\ 25\ 34 \rangle$. There are 15 allocations with 2 objects in each share (up to permutations of the shares):

$$\begin{aligned}
\overrightarrow{\pi}^1 &= \langle 12\ 34\ 56 \rangle; \overrightarrow{\pi}^2 = \langle 12\ 35\ 46 \rangle; \overrightarrow{\pi}^3 = \langle 12\ 36\ 45 \rangle; \\
\overrightarrow{\pi}^4 &= \langle 13\ 24\ 56 \rangle; \overrightarrow{\pi}^5 = \langle 13\ 25\ 46 \rangle; \overrightarrow{\pi}^6 = \langle 13\ 26\ 45 \rangle; \\
\overrightarrow{\pi}^7 &= \langle 14\ 23\ 56 \rangle; \overrightarrow{\pi}^8 = \langle 14\ 25\ 36 \rangle; \overrightarrow{\pi}^9 = \langle 14\ 26\ 35 \rangle; \\
\overrightarrow{\pi}^{10} &= \langle 15\ 23\ 46 \rangle; \overrightarrow{\pi}^{11} = \langle 15\ 24\ 36 \rangle; \overrightarrow{\pi}^{12} = \langle 15\ 26\ 34 \rangle; \\
\overrightarrow{\pi}^{13} &= \langle 16\ 23\ 45 \rangle; \overrightarrow{\pi}^{14} = \langle 16\ 24\ 35 \rangle; \overrightarrow{\pi}^{15} = \overrightarrow{\pi}^* = \langle 16\ 25\ 34 \rangle.
\end{aligned}$$

Now we have to check that for every allocation $\overrightarrow{\pi}^k$ ($k \in [\![1, 14]\!]$), for every share $\pi_i^* \in \overrightarrow{\pi}^*$ there is a share $\pi_j^k \in \overrightarrow{\pi}^k$ such that $\pi_j^k$ has less or equal utility than $\pi_i^*$, proving that $\min_{i=1}^3 u(\pi_i^*) \geq \min_{i=1}^3 u(\pi_i^k)$. This is the case for $\overrightarrow{\pi}^1$, because we have $w(i, 1) + w(i, 6) \geq w(i, 5) + w(i, 6)$, $w(i, 2) + w(i, 5) \geq w(i, 5) + w(i, 6)$ and $w(i, 3) + w(i, 4) \geq w(i, 5) + w(i, 6)$. Another example, for $\overrightarrow{\pi}^{14}$ : check that $w(i, 1) + w(i, 6) = w(i, 1) + w(i, 6)$, $w(i, 2) + w(i, 5) \geq w(i, 3) + w(i, 5)$ and $w(i, 3) + w(i, 4) \geq w(i, 3) + w(i, 5)$. The other comparisons can be verified the same way. □

With this technical result, we are know ready to prove Lemma 4.

*Proof* (Lemma 4) As usual, we consider a SOP (3, 6)-instance $I'$, and without loss of generality, we suppose that $w(i, 1) \geq w(i, 2) \geq w(i, 3) \geq w(i, 4) \geq w(i, 5) \geq w(i, 6)$ for all $i$. We will show that $I'$ belongs to $\mathscr{I}_{|\text{MFS}}$ by building an allocation $\overrightarrow{\pi}'$ such that $\overrightarrow{\pi}' \models \text{MFS}$. We consider two subcases.

(1) If for an agent, say agent 3, we have $u_3^{\text{MFS}} \leq w(3, 1)$, then give the share $\{1\}$ to this agent, so she gets her max–min fair-share. Remains a (2, 5)-instance $I$ that belongs to $\mathscr{I}_{|\text{MFS}}$ by Proposition 16. Hence there exists an allocation $\overrightarrow{\pi}$ which gives her max–min fair-share to both agents (1) and (2) in $I$. Extend $\overrightarrow{\pi}$ to $\overrightarrow{\pi}' = \langle \pi_1, \pi_2, \{1\} \rangle$ ($\overrightarrow{\pi}'$ is $\overrightarrow{\pi}$ augmented with a new share built with the object 1 alone). $\overrightarrow{\pi}'$ is a valid allocation for $I'$. Agent 3 gets her max–min fair-share in $I'$, as said before. Agents 1 and 2 get the same share in $I$ and $I'$, hence the same value. By Lemma 1, their max–min fair-share cannot strictly increase from $I$ to $I'$, so they also get their max–min fair-share in $I'$.
(2) Otherwise, we have $w(i, 1) < u_i^{\text{MFS}}$ and then by Lemma 5, allocation $\overrightarrow{\pi}^* = \langle \{1, 6\}\{2, 5\}\{3, 4\} \rangle$ satisfies the max-min fair-share criterion. □

**Proposition 20** *For $n \geq 2$, any add-MARA instance with $n$ agents and $n+3$ objects belongs to $\mathscr{I}_{|\text{MFS}}$.*

*Proof* The case $n = 2$ and $m = 5$ directly follows from Proposition 16. For $n \geq 3$, the proof is by induction, similar to the proof of Proposition 19. The base case is given by Lemma 4. From $n \geq 3$ and $m = n+3$ it follows again that $m+1 < 2(n+1)$, so Lemma 3 still applies. □

## 7 Experiments

Tables 1 and 2 give some experimental results concerning our scale of criteria. We have generated random instances for $n$, the number of agents, ranging from 3 to 5, and for $m$, the number of objects, ranging from 1 to 11. For each combination of $n$, $m$, we have generated 1000 pairs of instances, the first one being non SOP, and the second one being the SOP version of the first. In Table 1, weights are uniformly drawn from [0, 1]. In Table 2, weights are drawn from a Gaussian distribution, mean 0.5 and standard deviation 0.1.

The number on line $n$, $m$ and column $\mathscr{C}$ gives the number of instances, out of 1000, which satisfy the criterion $\mathscr{C}$. The last column is not devoted, as could be expected, to the CEEI criterion, but to the EFP criterion which means envy-free and Pareto-efficient. In fact, it is computationally difficult to characterize exactly an instance having a CEEI allocation in general—see the paper by Othman et al. ([39], Section 3) and the discussion end of Sect. 4.5—so in experiments we have replaced CEEI by EFP.[21]

[21] We have seen in Sect. 4.5 that EFP is a necessary condition for having a CEEI when preferences are strict. We believe that the CEEI and EFP criteria are not equivalent in the context of this discrete model.

**Table 1** Experimental results with a uniform distribution of weights

| Uniform | | Non SOP instances | | | | | SOP instances | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *n* | *m* | MFS | PFS | mFS | EF | EFP | MFS | PFS | mFS | EF | EFP |
| 3 | 1 | 1000 | 0 | 0 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 3 | 2 | 1000 | 0 | 0 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 3 | 3 | 1000 | 618 | 231 | 231 | 231 | 1000 | 0 | 0 | 0 | 0 |
| 3 | 4 | 1000 | 821 | 563 | 318 | 318 | 1000 | 340 | 2 | 2 | 2 |
| 3 | 5 | 1000 | 829 | 730 | 530 | 477 | 1000 | 652 | 237 | 218 | 218 |
| 3 | 6 | 1000 | 991 | 967 | 933 | 890 | 1000 | 775 | 500 | 374 | 374 |
| 3 | 7 | 1000 | 1000 | 999 | 997 | 989 | 1000 | 942 | 780 | 615 | 611 |
| 3 | 8 | 1000 | 1000 | 999 | 997 | 995 | 1000 | 990 | 958 | 869 | 831 |
| 3 | 9 | 1000 | 1000 | 1000 | 1000 | 1000 | 1000 | 1000 | 995 | 983 | 965 |
| 3 | 10 | 1000 | 1000 | 1000 | 1000 | 1000 | 1000 | 1000 | 1000 | 1000 | 990 |
| 3 | 11 | 1000 | 1000 | 1000 | 1000 | 1000 | 1000 | 1000 | 1000 | 1000 | 999 |
| 4 | 1 | 1000 | 0 | 0 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 4 | 2 | 1000 | 0 | 0 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 4 | 3 | 1000 | 0 | 0 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 4 | 4 | 1000 | 746 | 86 | 86 | 86 | 1000 | 0 | 0 | 0 | 0 |
| 4 | 5 | 1000 | 945 | 511 | 130 | 130 | 1000 | 159 | 0 | 0 | 0 |
| 4 | 6 | 1000 | 927 | 744 | 217 | 192 | 1000 | 563 | 2 | 1 | 1 |
| 4 | 7 | 1000 | 920 | 843 | 530 | 434 | 1000 | 868 | 500 | 241 | 240 |
| 4 | 8 | 1000 | 998 | 998 | 978 | 923 | 1000 | 972 | 751 | 442 | 433 |
| 4 | 9 | 1000 | 1000 | 1000 | 998 | 984 | 1000 | 1000 | 952 | 752 | 706 |
| 4 | 10 | 1000 | 1000 | 1000 | 1000 | 999 | 1000 | 1000 | 999 | 962 | 912 |
| 4 | 11 | 1000 | 1000 | 1000 | 1000 | 1000 | 1000 | 0 | 0 | 0 | 0 |
| 5 | 1 | 1000 | 0 | 0 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 5 | 2 | 1000 | 0 | 0 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 5 | 3 | 1000 | 0 | 0 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 5 | 4 | 1000 | 0 | 0 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 5 | 5 | 1000 | 839 | 43 | 43 | 43 | 1000 | 0 | 0 | 0 | 0 |
| 5 | 6 | 1000 | 991 | 376 | 38 | 38 | 1000 | 62 | 0 | 0 | 0 |
| 5 | 7 | 1000 | 989 | 726 | 73 | 61 | 1000 | 430 | 0 | 0 | 0 |
| 5 | 8 | 1000 | 970 | 835 | 178 | 130 | 1000 | 764 | 0 | 0 | 0 |
| 5 | 9 | 1000 | 964 | 903 | 561 | 387 | 1000 | 896 | 70 | 29 | 29 |
| 5 | 10 | 1000 | 1000 | 997 | 985 | 953 | 1000 | 941 | 449 | 142 | 138 |
| 5 | 11 | 1000 | 1000 | 1000 | 1000 | 998 | 1000 | 987 | 732 | 302 | 286 |

These experiments aim only at getting an insight into the behavior of concrete instances over the scale of criteria. They are not motivated by performance concerns, as this article is not about algorithm design. All the algorithms used in these experiments to characterize instances are mainly based on a simple enumeration of all possible allocations, with some straightforward heuristics.[22] This, together with the high theoretical complexity of

[22] For example, when computing the most demanding criterion for a given instance, we first consider proportionality and envy-freeness, as these criteria can be tested in polynomial time.

**Table 2** Experimental results with a Gaussian distribution of weights

| Gauss | | Non SOP instances | | | | | SOP instances | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *n* | *m* | MFS | PFS | mFS | EF | EFP | MFS | PFS | mFS | EF | EFP |
| 3 | 1 | 1000 | 0 | 0 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 3 | 2 | 1000 | 0 | 0 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 3 | 3 | 1000 | 610 | 221 | 221 | 221 | 1000 | 2 | 0 | 0 | 0 |
| 3 | 4 | 1000 | 26 | 1 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 3 | 5 | 1000 | 3 | 2 | 1 | 1 | 1000 | 3 | 2 | 2 | 2 |
| 3 | 6 | 1000 | 994 | 960 | 915 | 886 | 1000 | 647 | 218 | 218 | 218 |
| 3 | 7 | 1000 | 737 | 256 | 41 | 40 | 1000 | 185 | 55 | 45 | 44 |
| 3 | 8 | 1000 | 223 | 181 | 153 | 122 | 1000 | 223 | 151 | 123 | 123 |
| 3 | 9 | 1000 | 1000 | 1000 | 1000 | 1000 | 1000 | 999 | 967 | 870 | 839 |
| 3 | 10 | 1000 | 998 | 935 | 663 | 624 | 1000 | 908 | 738 | 653 | 645 |
| 3 | 11 | 1000 | 873 | 852 | 847 | 782 | 1000 | 873 | 847 | 829 | 807 |
| 4 | 1 | 1000 | 0 | 0 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 4 | 2 | 1000 | 0 | 0 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 4 | 3 | 1000 | 0 | 0 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 4 | 4 | 1000 | 740 | 92 | 92 | 92 | 1000 | 0 | 0 | 0 | 0 |
| 4 | 5 | 1000 | 82 | 0 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 4 | 6 | 1000 | 2 | 1 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 4 | 7 | 1000 | 0 | 0 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 4 | 8 | 1000 | 999 | 996 | 961 | 918 | 1000 | 767 | 86 | 86 | 85 |
| 4 | 9 | 1000 | 993 | 393 | 20 | 16 | 1000 | 267 | 21 | 14 | 13 |
| 4 | 10 | 1000 | 622 | 219 | 19 | 13 | 1000 | 114 | 24 | 14 | 13 |
| 4 | 11 | 1000 | 268 | 224 | 191 | 120 | 1000 | 268 | 216 | 157 | 140 |
| 5 | 1 | 1000 | 0 | 0 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 5 | 2 | 1000 | 0 | 0 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 5 | 3 | 1000 | 0 | 0 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 5 | 4 | 1000 | 0 | 0 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 5 | 5 | 1000 | 843 | 57 | 57 | 57 | 1000 | 0 | 0 | 0 | 0 |
| 5 | 6 | 1000 | 254 | 0 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 5 | 7 | 1000 | 8 | 0 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 5 | 8 | 1000 | 1 | 0 | 0 | 0 | 1000 | 0 | 0 | 0 | 0 |
| 5 | 9 | 1000 | 2 | 1 | 0 | 0 | 1000 | 2 | 0 | 0 | 0 |
| 5 | 10 | 1000 | 1000 | 1000 | 994 | 969 | 1000 | 854 | 57 | 57 | 56 |
| 5 | 11 | 1000 | 1000 | 608 | 6 | 6 | 1000 | 400 | 13 | 6 | 6 |

Mean = 0.5, standard deviation = 0.1

some of the problems addressed,[23] explains the relatively small size of instances generated.

From these experiments, several facts can be noticed, which confirm our theoretical results.

[23] As a reminder, testing the existence of a complete and envy-free allocation (fourth column) is NP-complete [32], and testing the existence of a Pareto-efficient and envy-free allocation (last column) is $\Sigma_2^{\mathsf{P}}$-complete [30].

– Main result : the scale of properties is really significant, of course when $n \leq m$. The numbers weakly decrease from left to right, and often strictly decrease, showing that the scale is not trivial.
– SOP instances are more conflicting than non SOP ones, in accordance with Proposition 14.
– In Table 1 (uniform distribution of weights) for a fixed number of agents, instances are less conflict-prone as the number of objects increases: intuitively, we get closer to the continuous (divisible) case. Note that this remark also concurs with the theoretical and experimental results by Dickerson et al. [22] about envy-freeness, which show that when the number of objects exceeds a given threshold, an envy-free allocation is very likely to exist.
– In Table 2 (Gaussian distribution of weights, mean 0.5, standard deviation 0.1): instances where $m$ is close to a multiple of $n$ are less conflict-prone than others, which is not very surprising.
– All generated instances belong to $\mathscr{I}_{|\text{MFS}}$. This shows that it is actually very unlikely to find an instance not in $\mathscr{I}_{|\text{MFS}}$ (at least with uniform or Gaussian generation of weights) even if such instances exist [43].

## 8 Beyond additive preferences

As noticed in the introduction of this paper, the main interest of additive preferences is their simplicity. However, a major drawback of this kind of preferences is their limited expressivity, which is a main motivation for several work to investigate other preference representation contexts (see *e.g.* [17], for an example). In this section, we give some insights about the extension of our approach to non-additive preferences.

Even if, as we have seen earlier, it is almost always possible, for a given add-MARA instance, to find an allocation satisfying the max-min fair-share criterion, things are surprisingly different for more general non-additive preferences. The most natural way of relaxing preference additivity while keeping some conciseness is to allow limited synergies (complementarities or substitutabilities) between objects. This is the exact idea behind $k$-additive functions originally introduced in the context of fuzzy measures [26], and also used in the context of resource allocation [17].

Formally, we consider in this section $k$-additive multiagent resource allocation instances ($k$-add-MARA instances for short), defined as triples $\langle \mathscr{A}, \mathscr{O}, w\rangle$, where $w$ is now a mapping from $\mathscr{A} \times 2^{\mathscr{O}}$ to $\mathbb{R}$ such that $w(i, \pi) = 0$ for every agent $i$ and every subset $\pi$ such that $|\pi| > k$. As before, the utility function is defined additively: $u_i(\pi) = \sum_{\pi' \subseteq \pi} w(i, \pi')$.

Obviously, 1-additive functions are the additive functions (so the 1-add-MARA instances are exactly the add-MARA instances considered earlier, corresponding to the model presented in Sect. 3), and thus forbids any preferential interdependence between objects. A 2-additive function allows such interdependence. For example, the weight $w(\{1, 2\})$ stands for the proper interest of the pair of objects $\{1, 2\}$ beyond these two individual objects: if $w(\{1, 2\}) > 0$, the value of this pair is more important than the intrinsic value of the two separated objects (which shows that they are complementary); if $w(\{1, 2\}) < 0$, they are substitutable.

To be consistent with the model presented in Sect. 3, we will require in the following that $u_i(\pi) \geq 0$ for all agent $i$ and share $\pi$: a share has always a positive effect on the agent concerned. This implies that $w(i, \{\ell\}) \geq 0$ for every agent and every singleton, but, contrary to previous sections, it does not imply that *all* the weights should be positive: as we have seen earlier, this is not the case when two given objects are substitutable.

As soon as we switch from 1-additive to 2-additive functions, finding an instance not belonging to $\mathscr{I}_{|\text{MFS}}$ (that is for which no allocation satisfying the max-min fair-share criterion exists) is not challenging anymore:

*Example 10* Let us consider the 2 agents / 4 objects instance defined by the following weight functions:

- $w(1, \{1, 2\}) = w(1, \{3, 4\}) = 1$
- $w(2, \{1, 3\}) = w(2, \{2, 4\}) = 1$
- $w(i, \pi) = 0$ for every other share $\pi$.

It is not hard to see that $u_i^{\text{MFS}} = 1$ for both agents, and no allocation giving at least 1 to both agents exists.

Actually, the problem of determining whether there exists an allocation satisfying the max–min fair-share criterion (further referred to as [$k$- ADD- MFS- EXIST]) is even hard:

**Proposition 21** *[$k$-ADD- MFS- EXIST] is NP-hard, for $k \geq 2$ and $n \geq 3$.*

*Proof* NP-hardness can be proved by reduction from the partition problem (Problem 2). Let $\langle\{x_1, \ldots, x_n\}, s\rangle$ be an instance of this problem. We suppose w.l.o.g. that all the weights are even (if it is not the case, one can multiply all the weights by two and obtain an equivalent problem). From this instance, we create a 3-agents / $n + 4$ objects $k$-add-MARA instance, where the agents' preferences are defined as follows:

- for all $i$, $w(i, \{\ell\}) = s(x_\ell)$ and $w(i, \{\ell, n + \ell'\}) = -s(x_\ell)/2$ for all $\ell \in [\![1, n]\!]$ and $\ell' \in [\![1, 4]\!]$
- $w(1, \{n + 1, n + 2\}) = w(1, \{n + 3, n + 4\}) = L$
- $w(2, \{n + 1, n + 3\}) = w(2, \{n + 2, n + 4\}) = L$
- $w(3, \{n + 1, n + 4\}) = w(3, \{n + 2, n + 3\}) = L$
- $w(i, \pi) = 0$ for every other share $\pi$.

We can first observe that $u_i(\pi) \geq 0$ for every share $\pi$ and every agent $i$. Indeed, consider for example agent 1. If $|\pi \cap [\![n+1, n+4]\!]| \leq 2$, then $u_1(\pi) \geq \sum_{\ell\in\pi} s(x_\ell) - 2 \times \sum_{\ell\in\pi} s(x_\ell)/2 \geq 0$. If $|\pi \cap [\![n + 1, n + 4]\!]| > 2$, then $u_1(\pi) \geq \sum_{\ell\in\pi} s(x_\ell) + L - 4 \times \sum_{\ell\in\pi} s(x_\ell)/2 \geq \sum_{\ell\in\pi} s(x_\ell) + L - L - 2 \times \sum_{\ell\in\pi} s(x_\ell)/2 \geq 0$.

Let us compute the max–min fair-share for each agent. Let us consider the allocation $(\{1, \ldots, n\}, \{n + 1, n + 2\}, \{n + 3, n + 4\})$. The evaluation of these three shares by agent 1 gives respectively $2L$, $L$, and $L$. Hence $u_1^{\text{MFS}} \geq L$.

Let now $\overrightarrow{\pi}$ be a custom allocation. We distinguish four cases, according to how the objects from $[\![n+1, n+4]\!]$ are split up among the different shares: either one share contains the four objects (case (i)), or one share contains three objects (case (ii)), or at least one share contains two objects (cases (iii) and (iv)).

(i) The objects $\ell' > n$ only appear in one share (say w.l.o.g. $\pi_1$), possibly containing objects in $[\![1, n]\!]$. In that case, the objects $\ell \leq n$ which are not in $\pi_1$ are split between the two shares $\pi_2$ and $\pi_3$. Since the utility function of agent 1 is additive on the objects of $\{1, \ldots, n\}$ and that $\sum_{\ell=1}^{n} w(1, \{\ell\}) = 2L$, we have $u_1(\pi_2) \leq L$ or $u_1(\pi_3) \leq L$.

(ii) One share (say w.l.o.g. $\pi_1$) contains exactly three objects $\ell' > n$ (and possibly some in $[\![1, n]\!]$). Suppose that $\pi_1$ does not contain any object in $[\![1, n]\!]$. In that case, obviously, $u_1(\pi_1) = L$. Suppose now that $\pi_1$ contains some objects from $[\![1, n]\!]$. Then it can be observed that for each $\ell \leq n$, $u_1(\pi_1) = u_1(\pi_1 \setminus \{\ell\}) + s(x_\ell) - 3 \times s(x_\ell)/2 < u_1(\pi_1 \setminus \{\ell\})$. If we apply this reasoning by iteratively removing all the objects $\ell < n$, we get that $u_1(\pi_1) < L$.

(iii) One share (say w.l.o.g. $\pi_1$) is either $\{n+1, n+2\} \cup \mathscr{O}'$ or $\{n+3, n+4\} \cup \mathscr{O}'$, where $\mathscr{O}' \subseteq [\![1, n]\!]$. Suppose that $\mathscr{O}' = \emptyset$. Then obviously $u_1(\pi_1) = L$. Suppose now that $\pi_1$ contains some objects from $[\![1, n]\!]$. Then it can be observed that for each $\ell \leq n$, $u_1(\pi_1) = u_1(\pi_1 \setminus \{\ell\}) + s(x_\ell) - 2 \times s(x_\ell)/2 = u_1(\pi_1 \setminus \{\ell\})$. If once again we apply this reasoning by iteratively removing all the objects $\ell < n$, we get that $u_1(\pi_1) = L$.

(iv) No share contains neither $\{n+1, n+2\}$ nor $\{n+3, n+4\}$. Suppose in that case that one share (say $\pi_1$) has a utility greater than or equal to $L$. It means that $\sum_{\ell \in \pi_1} s(x_\ell) \geq L$. It thus means that we have $\sum_{\ell \in \pi_i} s(x_\ell) \leq L$ for both $i \in \{2, 3\}$. Since none of the shares contain $\{n+1, n+2\}$ nor $\{n+3, n+4\}$, we have that $u_1(\pi_i) \leq L$ for both remaining shares.

In cases (i), (ii), (iii) and (iv), $\min_{i \in \mathscr{A}} u_1(\pi_i) \leq L$. Hence $u_1^{\text{MFS}} = L$. The other agents' case can be treated similarly.

We will now prove that there is an allocation satisfying the max–min fair-share criterion if and only if the initial instance is a yes-instance from the partition problem. First we can prove that every allocation $\overrightarrow{\pi}$ such that no share contains all the objects from $[\![n+1, n+4]\!]$ is weakly Pareto-dominated. To see that, we can observe that for every allocation $\overrightarrow{\pi}$, there are at least two agents that are not satisfied by objects $\ell' > n$ (meaning that they do not receive at least $L$ from these objects). Let us assume that 2 and 3 are these two unsatisfied agents. Then, obviously, transferring objects $\ell' > n$ from $\pi_2$ and $\pi_3$ to $\pi_1$ is a weak Pareto-improvement: it does neither hurt agents 2 and 3 (because objects $\ell' > n$ have a negative influence unless they form a satisfactory combination), nor agent 1, who will receive an extra utility $L$ which will compensate the potential negative synergies with objects from $[\![1, n]\!]$. Hence we can restrict our proof to allocations such that one share contains all the objects from $[\![n+1, n+4]\!]$.

Let $\overrightarrow{\pi}$ be such an allocation, and assume w.l.o.g. that $\pi_1$ contains $[\![n+1, n+4]\!]$. Obviously, $u_1(\pi_1) \geq L$. Since the utility function of the agents restricted to objects from $[\![1, n]\!]$ is additive and that the total utility of these objects is $2L$, the only way of giving their max–min fair-share ($L$) to both agents 2 and 3 is to distribute all the objects from $[\![1, n]\!]$ to them, such that $\sum_{\ell \in \pi_2} s(x_\ell) = \sum_{\ell \in \pi_3} s(x_\ell) = L$: this is equivalent to finding a partition in the initial instance of [Partition]. □

It can be noticed that Proposition 21 only gives a NP-hardness result, as it is not known yet whether [$k$-Add- MFS- Exist] belongs to NP. We can only say that this problem belongs to $\Sigma_2^{\mathsf{P}}$, because it can be solved by the same non-deterministic algorithm as in the additive case (see end of Sect. 4.1). Moreover, the utility function used in the reduction is neither sub- nor super-modular. We do not know whether the hardness result still holds if we restrict to these particular kinds of $k$-additive utility functions.

## 9 Open problems

This work raises several open questions that are listed below. Notice that questions 1 to 3, concerning the CEEI criterion, are somewhat connected.

1. The precise complexity classes of the following problems are unknown: MFS- Exist, mFS- Exist, CEEI- Test, CEEI- Exist, and $k$ - Add- MFS- Exist. $\mathscr{C}$ - Exist is the decision problem of determining, given an add-MARA instance, whether there exists an allocation satisfying the criterion $\mathscr{C}$. CEEI- Test is the problem of determining whether a given allocation is CEEI.
2. It is not known whether the following statement is true or false: *When the agents' preferences are strict (i.e. for every agent $i$ and bundles $\pi$ and $\pi'$, $\pi \neq \pi'$ implies*

$u_i(\pi) \neq u_i(\pi')$), *then any envy-free and Pareto-efficient allocation is also CEEI*. With Propositions 7 and 8, this statement, if true, would give (under the strict preference hypothesis) the equivalence between the CEEI criterion and the conjunction of envy-freeness and Pareto-efficiency. A counter-example of this result would be an instance with strict preferences, having an envy-free and Pareto-efficient allocation but not CEEI.
3. We have sketched at the end of Sect. 4.5 an algorithm to compute a CEEI allocation if there is one (and answer "no" when no such allocation exists), but this approach is very naive and very likely to be inefficient in practice. Hence the question of efficiently computing such a CEEI allocation remains open.
4. For a given number of agents $n > 2$, what is the maximum number of objects $m_n$ such that every add-MARA instance with less than $m_n$ is guaranteed to be in $\mathscr{I}_{|\text{MFS}}$? (see the beginning of Sect. 6.2.3).

## 10 Conclusion and future work

In this paper we have considered five fairness criteria for resource allocation, two of which being classical, two of which being less well-known, and one being original. We have shown how these criteria form, in the context of the fair allocation of indivisible goods with additive preferences, an ordered scale that can be used as a basis not only for finding satisfactory (fair) allocations, but also for measuring to which extent it is possible to find some. We have also run some experiments that give some insights on how instances divide up on this scale of properties, and finally we have shown that the extension of these criteria to more general preferences is likely to have quite different properties.

Beyond the open problems presented in the previous section, this work raises several interesting and more general questions. Firstly, it would be interesting to investigate to which extent the similarity and dissimilarity of preference profiles influences the probability of existence of allocations satisfying each criterion (the analysis of SOP instances is a start but deserves to be refined). For example, it could be interesting to experimentally test the assumption that the more dissimilar the preferences are, the more likely fair allocations exist; which seems to be the exact opposite in voting theory (see e.g [27]).

Then, from a more theoretical point of view, the question of extending the results to non-additive problems is worth being further investigated.

Lastly, since four of the five criteria considered are purely ordinal (the proportional fair-share criterion is not), it would be interesting to analyze to which extent our results carry over to an ordinal setting with separable[24] preferences: unlike numerical additivity, ordinal separability leaves many pairs of allocations incomparable. Hence, even if the criteria themselves can be directly expressed ordinally, the way they must be adapted to deal with incomparable pairs is not so clear and deserves further investigation.

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