Homework #8 solutions

1. This is true.

   a. Proof 1 (directly from the definitions):

$$\begin{aligned}
x \in (B \cup C) - A &\leftrightarrow x \in (B \cup C) \wedge x \notin A. && \text{[Defn. of set difference]}\\
&\leftrightarrow (x \in B \vee x \in C) \wedge x \notin A. && \text{[Defn. of union]}\\
&\leftrightarrow (x \in B \wedge x \notin A) \vee (x \in C \wedge x \notin A) && \text{[Distribution]}\\
&\leftrightarrow (x \in B - A) \vee (x \in C - A) && \text{[Defn. of set difference]}\\
&\leftrightarrow x \in (B - A) \cup (C - A) && \text{[Defn. of union]}
\end{aligned}$$

   b. Proof 2 (Using the rules on the sheet:)

$$\begin{aligned}
(B \cup C) - A &= (B \cup C) \cap A^c && \text{[Alternate Representation of set difference]}\\
&= (B \cap A^c) \cup (C \cap A^c) && \text{[Distributive Law]}\\
&= (B - A) \cup (C - A) && \text{[Alternate Representation of set difference]}
\end{aligned}$$

2. This is true.

   a. Proof 1 (directly from the definitions):
   Assume (by way of contradiction) that $(A \cap C) - (C \cup A)$ is *not* empty.
   Let $x \in (A \cap C) - (C \cup A)$.
   Then $x \in A \cap C$, but $x \notin C \cup A$. [Defn. of set difference.]
   Since $x \in A \cap C$, $x \in A$ and $x \in C$. [Defn. of intersection.]
   Therefore $x \in A \cup C$. [Defn. of union.]
   But this contradicts the above.
   This contradiction shows that $(A \cap C) - (C \cup A)$ is empty.

   b. Proof 2 (Using the rules on the sheet:)

$$\begin{aligned}
(A \cap C) - (C \cup A) &= (A \cap C) \cap (C \cup A)^c && \text{[Alternate Representation of set difference]}\\
&= (A \cap C) \cap (C^c \cap A^c) && \text{[DeMorgan's Law]}\\
&= (A \cap A^c) \cap (C \cap C^c) && \text{[Associative Law.]}\\
&= \emptyset \cap \emptyset && \text{[Intersection with Compliment]}\\
&= \emptyset && \text{[Intersection with Empty Set.]}
\end{aligned}$$

3. This is true.

   a. Proof 1 (directly from the definitions):

$$\begin{aligned}
x \in (A \cap B) \cap C &\leftrightarrow x \in A \wedge x \in B \wedge x \in C. && \text{[Defn. of intersection]}\\
&\leftrightarrow x \in A \wedge \sim(x \notin B \vee x \notin C) && \text{[DeMorgan's Law from propositional logic]}\\
&\leftrightarrow x \in A \wedge \sim(x \in B^c \vee x \in C^c) && \text{[Defn. of compliment]}\\
&\leftrightarrow x \in A \wedge \sim(x \in B^c \cup C^c) && \text{[Defn. of union]}\\
&\leftrightarrow x \in A \wedge x \in (B^c \cup C^c)^c && \text{[Defn. of Compliment]}\\
&\leftrightarrow x \in A - (B^c \cup C^c) && \text{[Defn. of set difference]}
\end{aligned}$$

b. Proof 2 (using the rules from the sheet):

$$
\begin{aligned}
(A \cap B) \cap C &= A \cap (B \cap C) \quad \text{[Associative Property]} \\
&= A \cap (B^c \cup C^c)^c \quad \text{[DeMorgan's Law]} \\
&= A - (B^c \cup C^c) \quad \text{[Alternate Representation of Set Difference]}
\end{aligned}
$$

4. This is false.

   Counterexample: $A = \{1\}$, $B = \{1\}$, $C = \emptyset$.

   Now $((A \cup B) - C) \cup (A \cap B) = \{1\}$, while $((A - B) \cup (B - A)) - C = \emptyset$

   Any example where $A \cap B$ is non-empty will suffice as a counterexample.

5. Counterexample: Let $A = \{1\},\ B = \{2\},\ C = \emptyset$

   Now the ordered pair $(1, 2) \in A \times (B \cup C)$, but $(1, 2) \notin (A \times B) \cap (A \times C)$.

6. Counterexample: Let A be any set of size 2, and let $B = A$.

   Now $|\mathcal{P}(A \times B)| = 16 = |\mathcal{P}(A) \times \mathcal{P}(B)|$

7. Counterexample: Let $A = \emptyset$, and let $B = \{1\}$.

   Now $A - B = \emptyset$, but $A \neq B$.

8. Counterexample: Let $A = \emptyset$, let $B = \emptyset$, and let $C = \emptyset$.

   Now $(A - B) \cup (B - A) = \emptyset$, and $(A \cup B) - (A \cap B \cap C) = \emptyset$.

9. This is true.

   Assume $A \cap B = A$. [We will show $A \cup B = B$.]

   Part I: [Show $A \cup B \subseteq B$.]

   Let $x \in A \cup B$. By the definition of "Union", that means either $x \in B$, as desired, or else $x \in A$. In the case where $x \in A$, we apply our assumption that $A \cap B = A$, to get $x \in A \cap B$. But now (by the definition of intersection) $x \in A \wedge x \in B$, hence $x \in B$ (specialization).

   Part II: [Show $B \subseteq A \cup B$]

   If $x \in B$, then we can apply "generalization" to say $x \in B \vee x \in A$, hence $x \in A \cup B$, by the definition of union.

10. This is true.

    Assume $A \cap B = A$, and $B \cap C = B$. [We will show $A \cap C = A$.]

    Part I: [Show $A \cap C \subseteq A$.]

    Let $x \in A \cap C$. $x \in A \wedge x \in C$, by the definition of intersection. Applying specialization, we get $x \in A$.

Part II: [Show $A \subseteq A \cap C$.]

Let $x \in A$. Since we have assumed $A \cap B = A$, we have $x \in A \cap B$. Since we have assumed $B \cap C = B$, we have $x \in A \cap (B \cap C)$. This means that x is in all three sets, so in particular $x \in A \cap C$ (by the definition of intersection).

11. This is true.

    Let $x \in \mathcal{P}(A) \cup \mathcal{P}(B)$. Then either $x \in \mathcal{P}(A)$ or $x \in \mathcal{P}(B)$, so either $x \subseteq A$ or $x \subseteq B$. If $x \subseteq A$ then $x \subseteq A \cup B$ (since $A \subseteq A \cup B$). Similarly, if $x \subseteq B$ then $x \subseteq A \cup B$. So in either case $x \subseteq A \cup B$, hence $x \in \mathcal{P}(A \cup B)$.