# Statistics with SpaRRows II

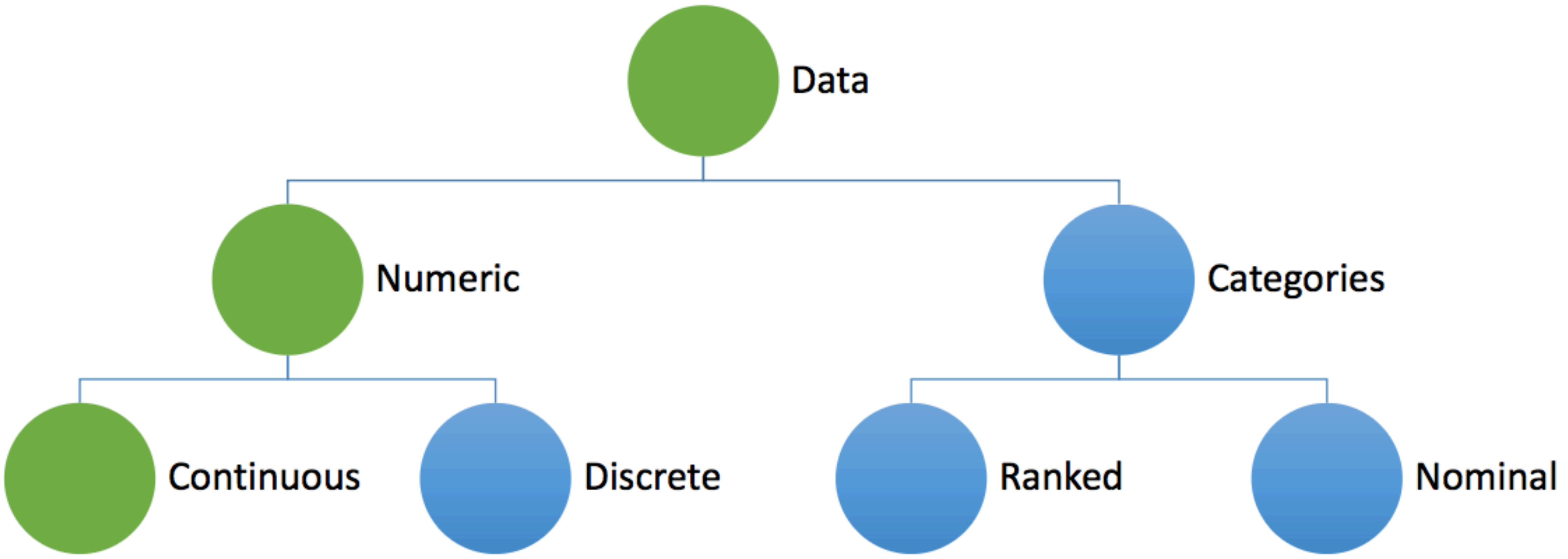
## Many models, matrices, and magic

Julia Schroeder

Julia.schroeder@imperial.ac.uk

# Remember this?

## Data types



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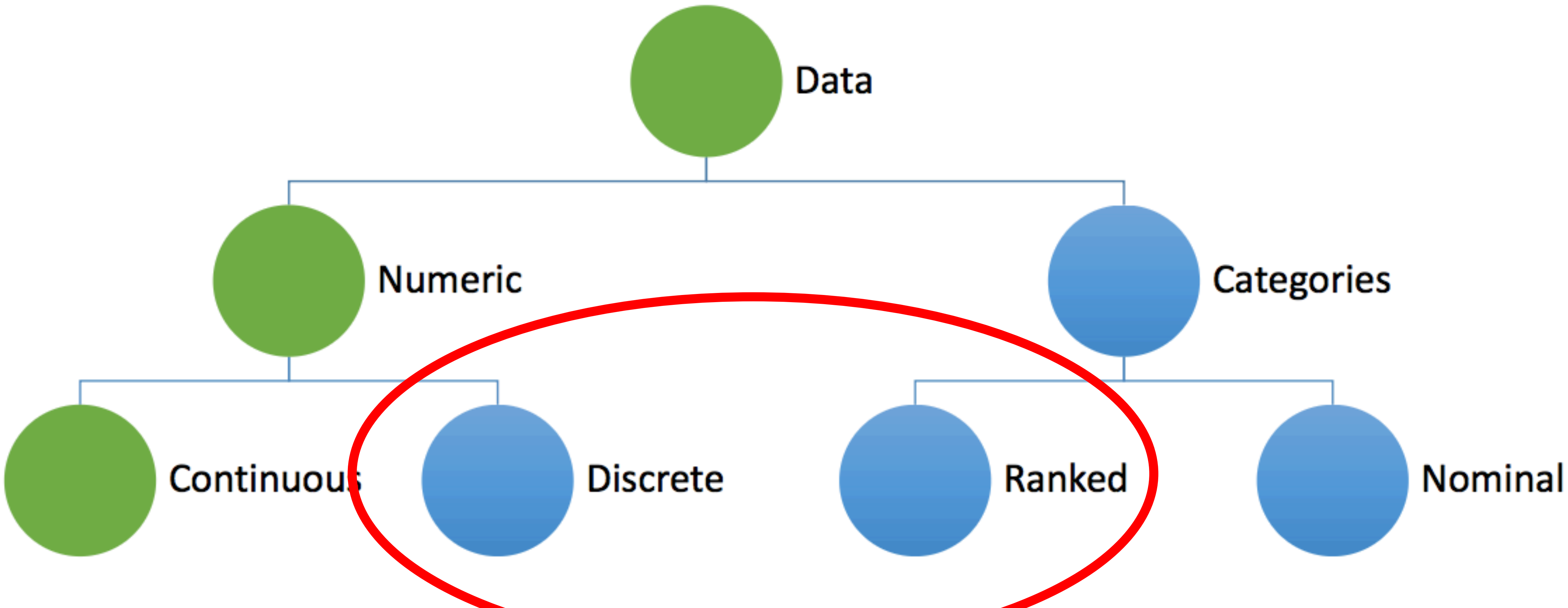

# Generalized Linear Models

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- Don’t panic

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- Extension of Linear Models

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- General philosophy is the same

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- Extension of Linear Models
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# Problematic response variables

- Count data

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- Number of trees in a plot

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- Number of offspring

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- Number of trees in a plot
- Number of offspring
- Number of species observed

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- $\geq 0$

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- Cannot be less than zero
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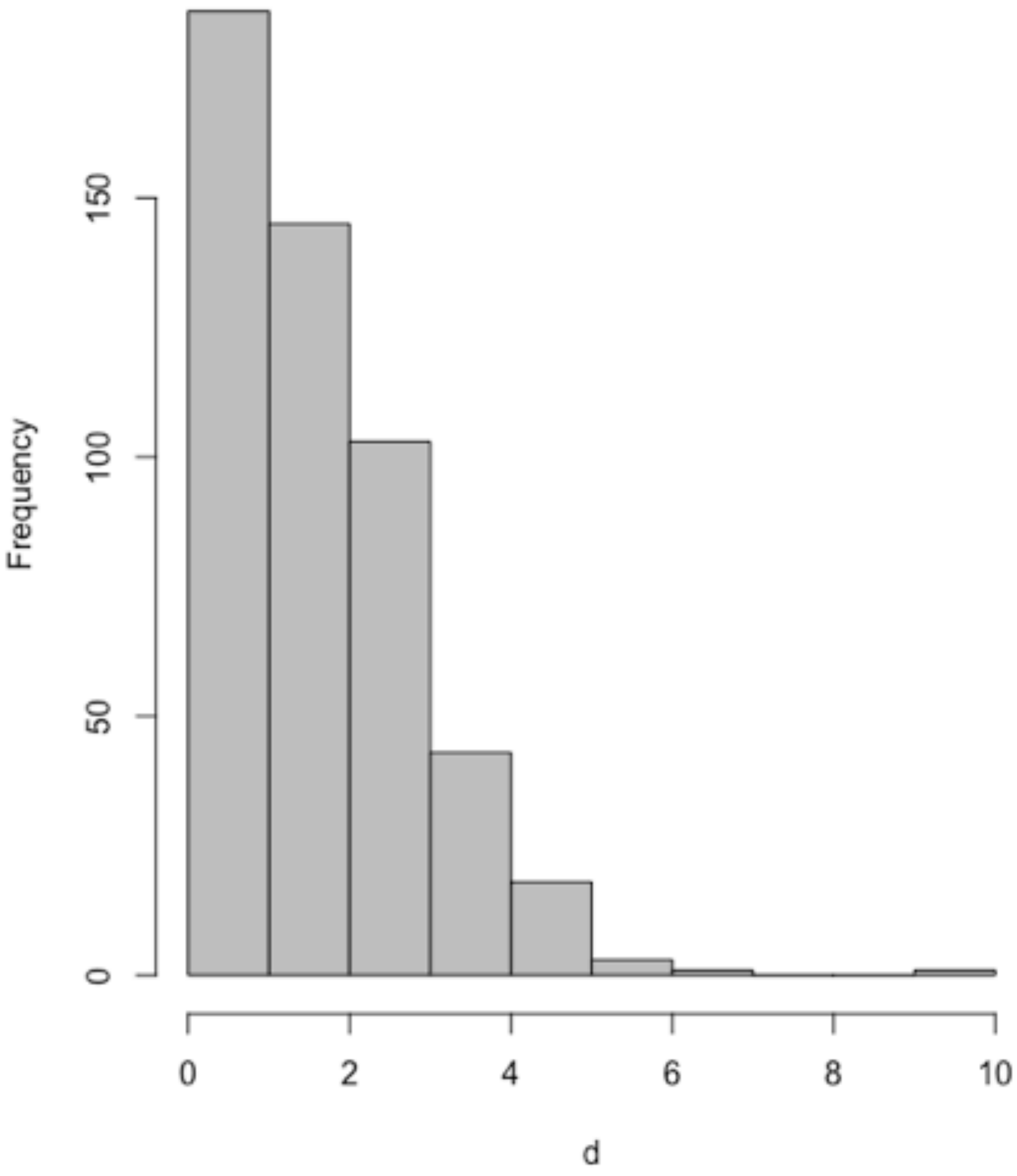
- Right-tailed

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# Problematic response variables

- Count data
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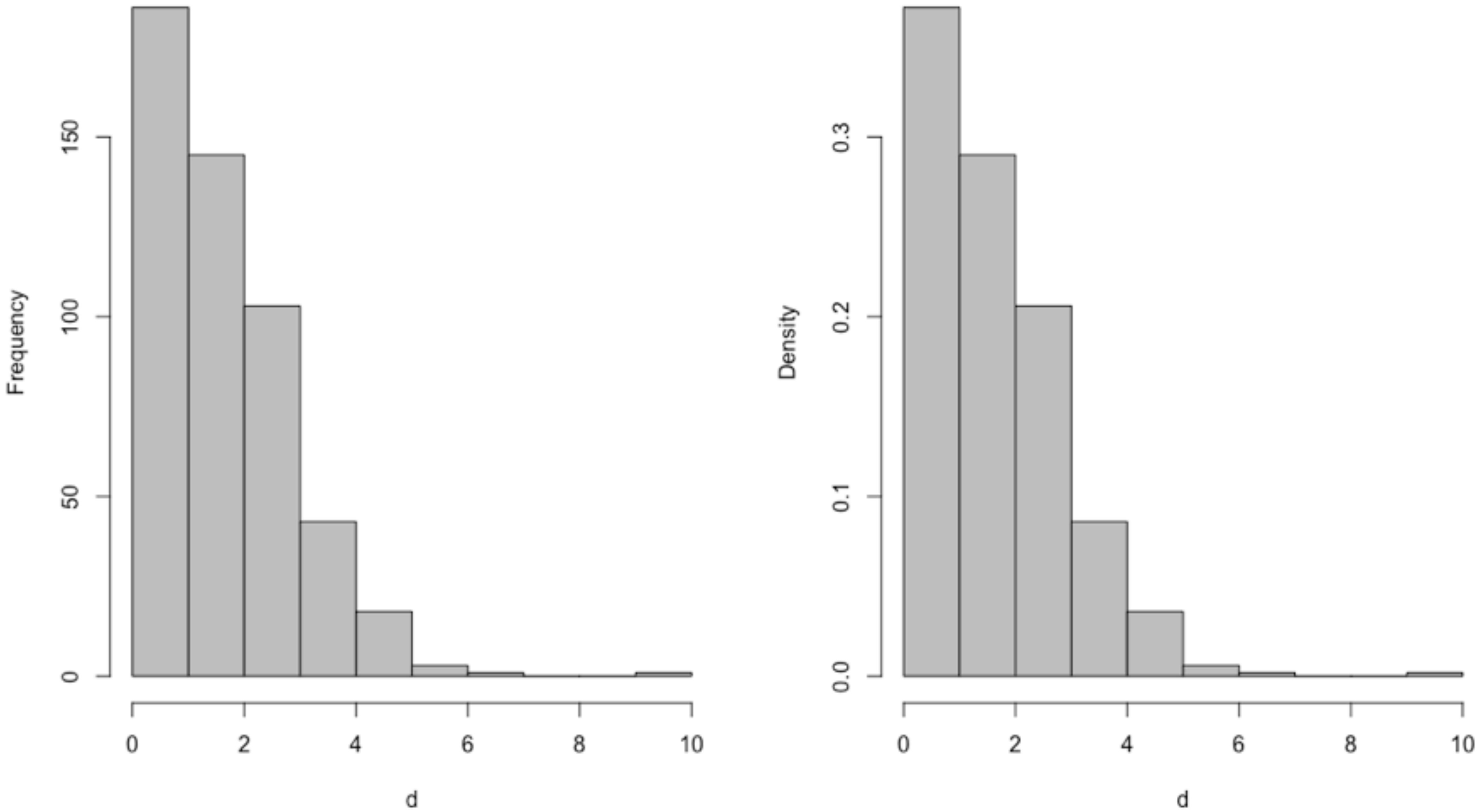

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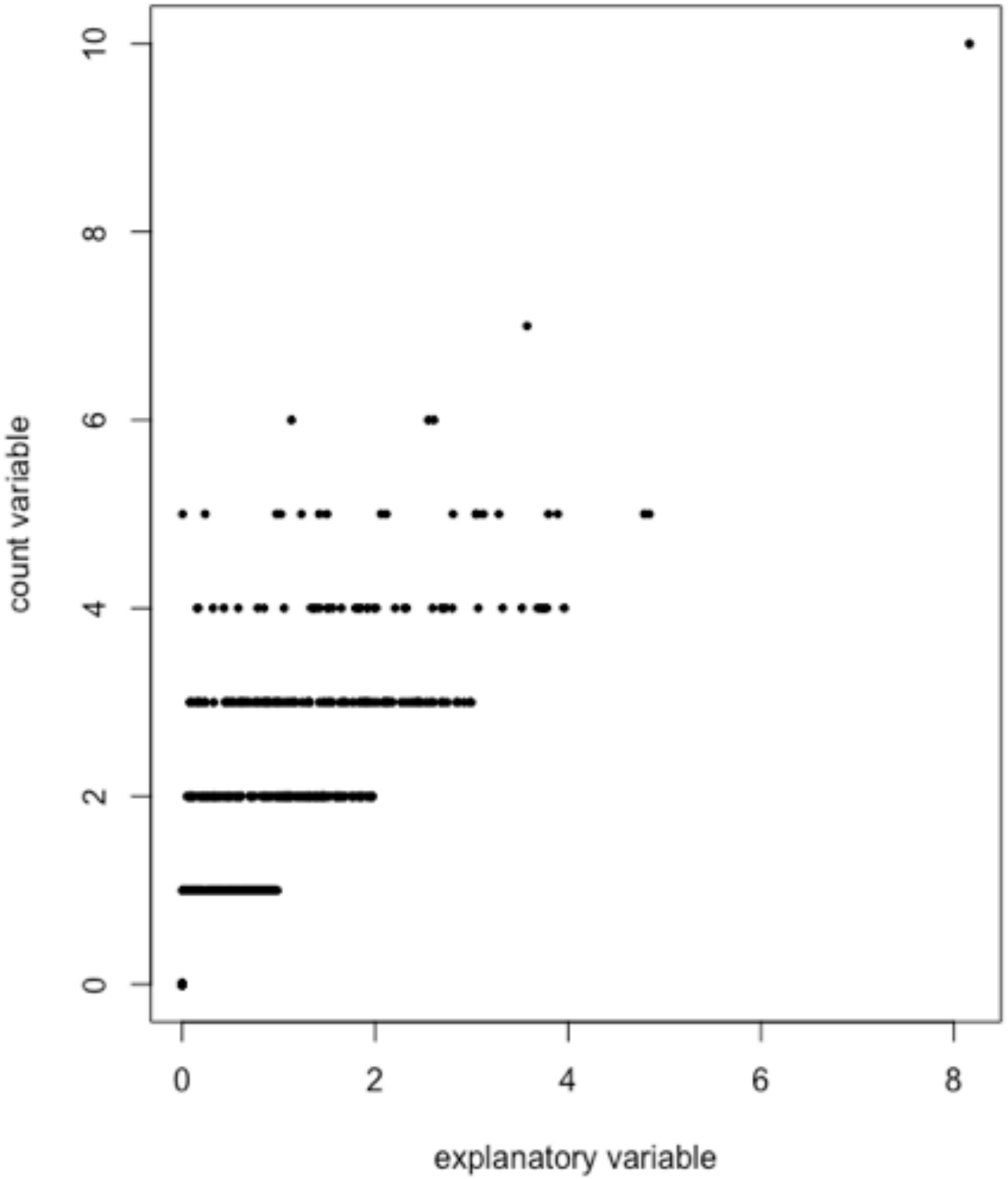

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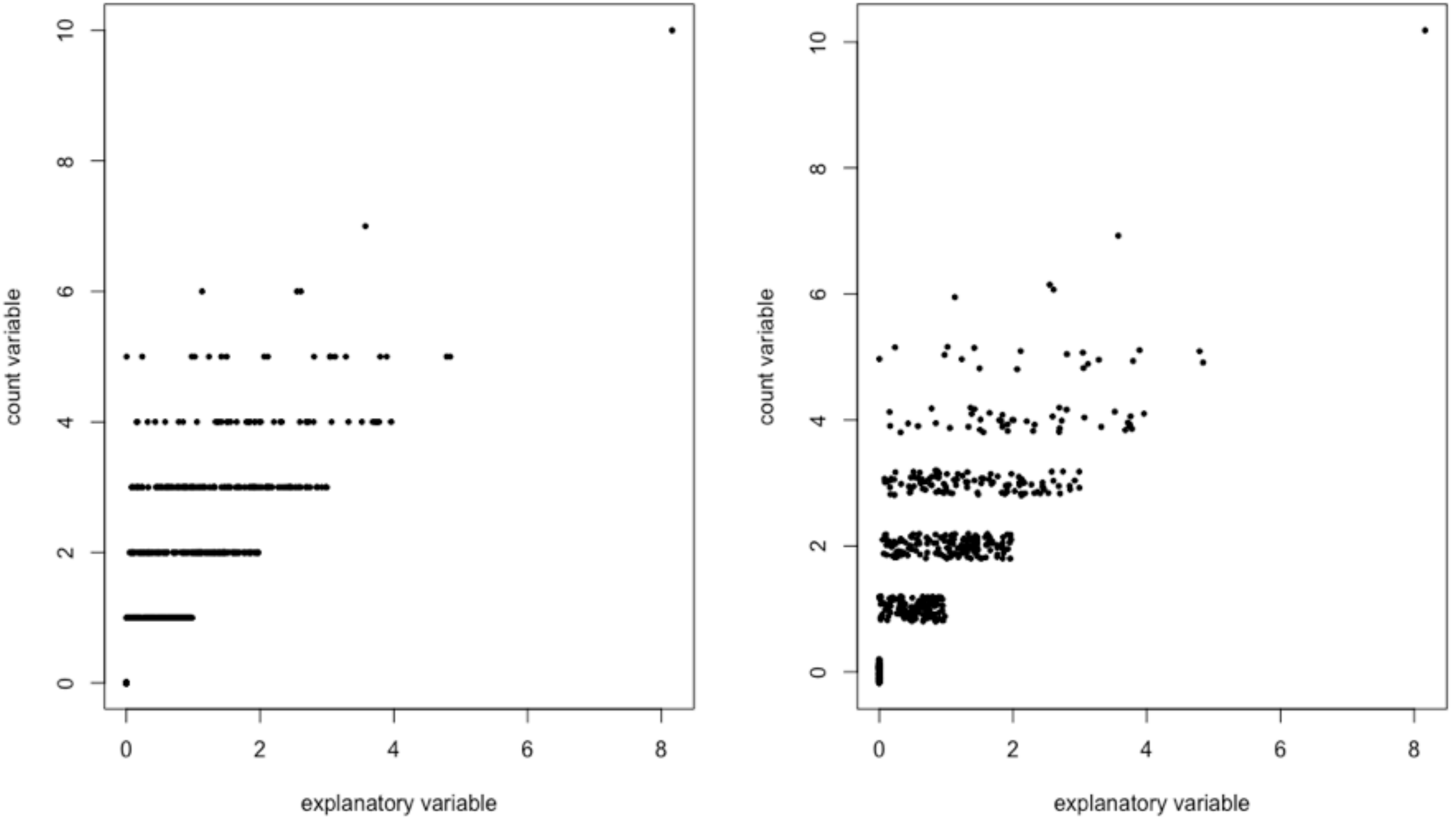

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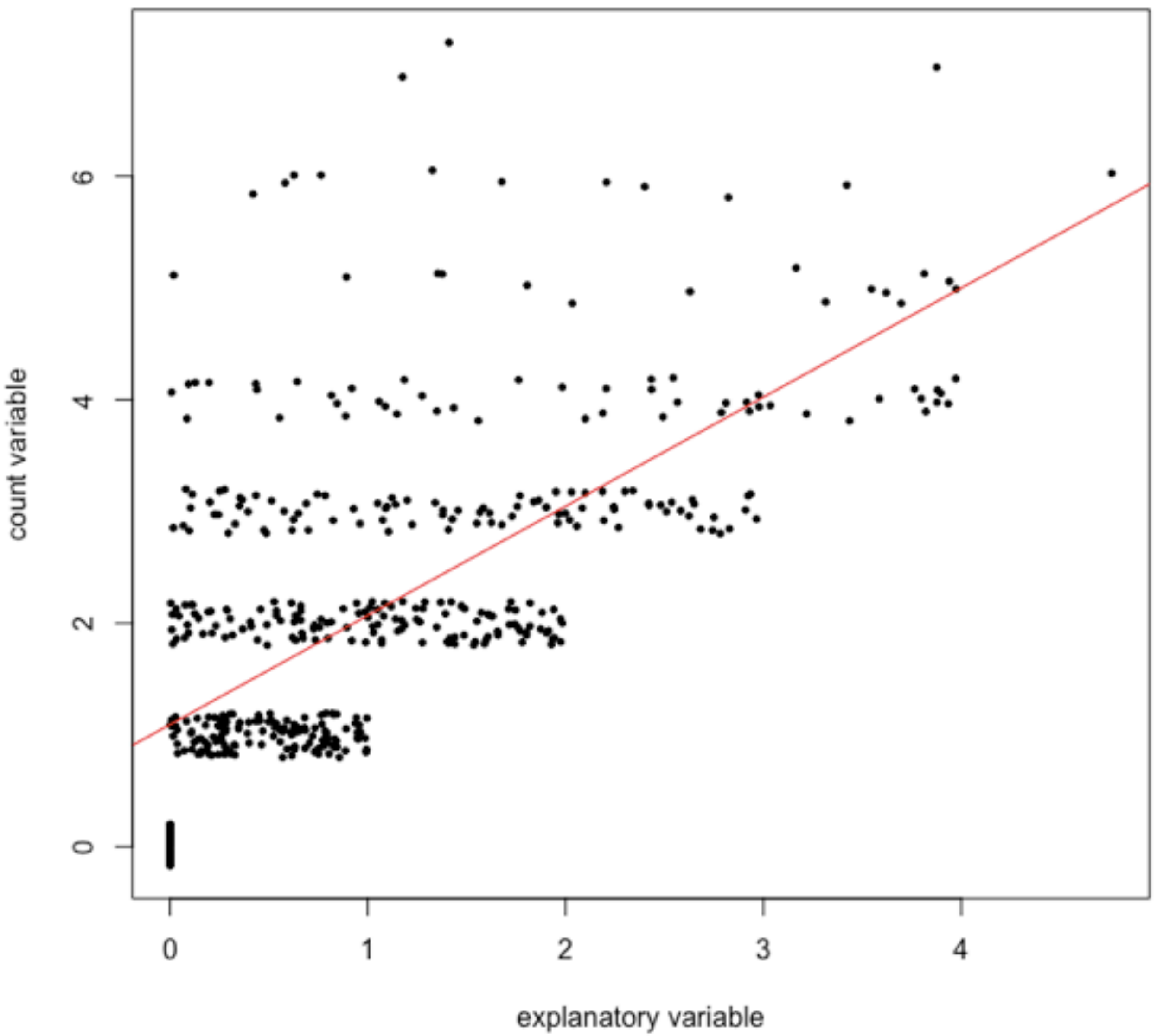

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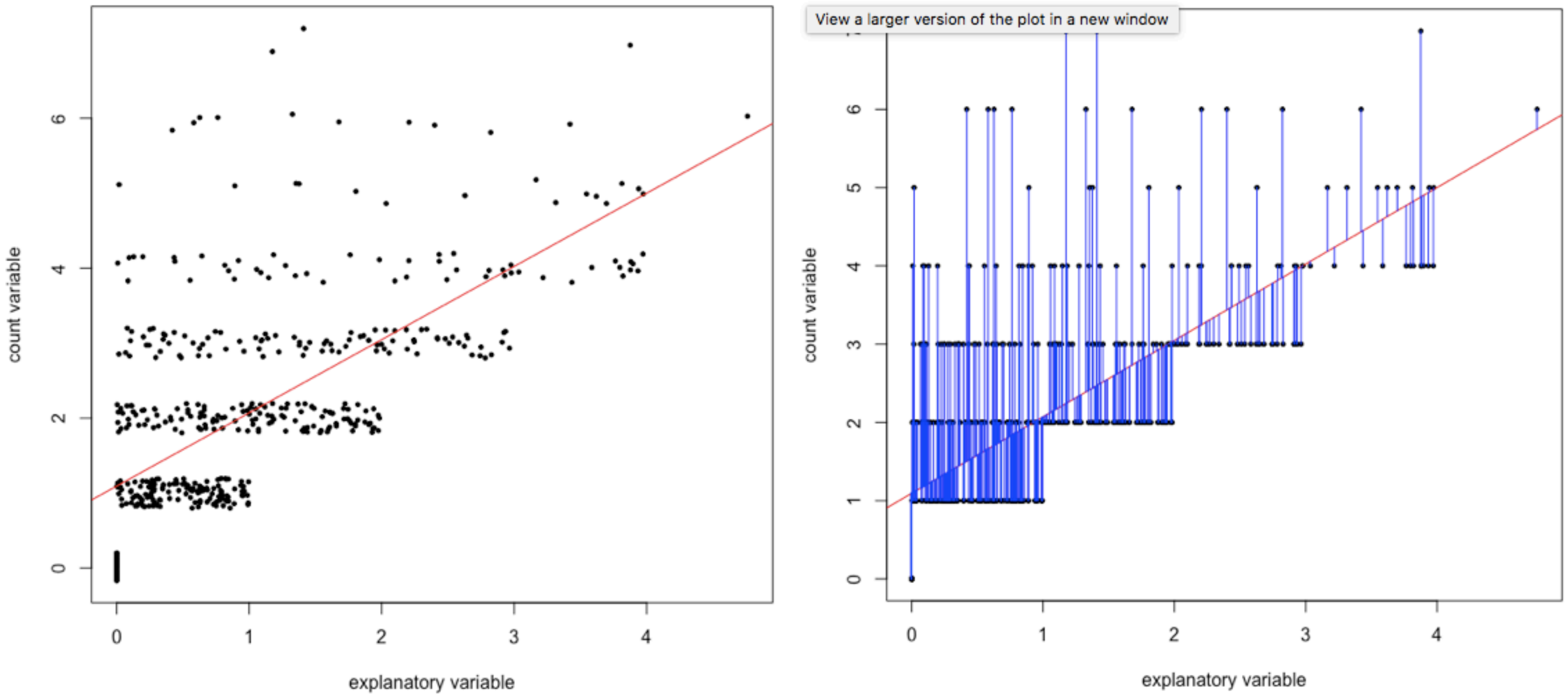
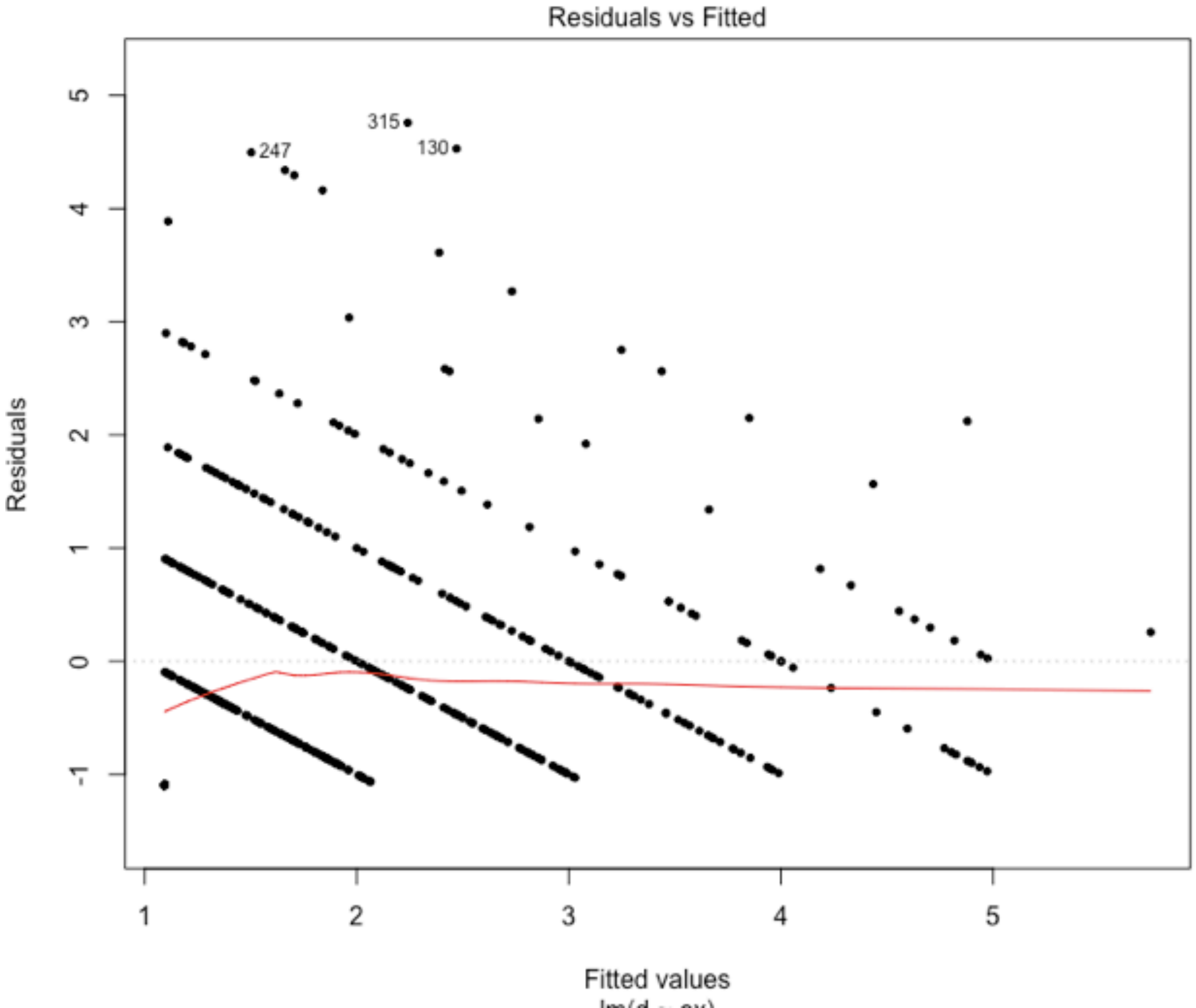


# Count data as response variable - validation



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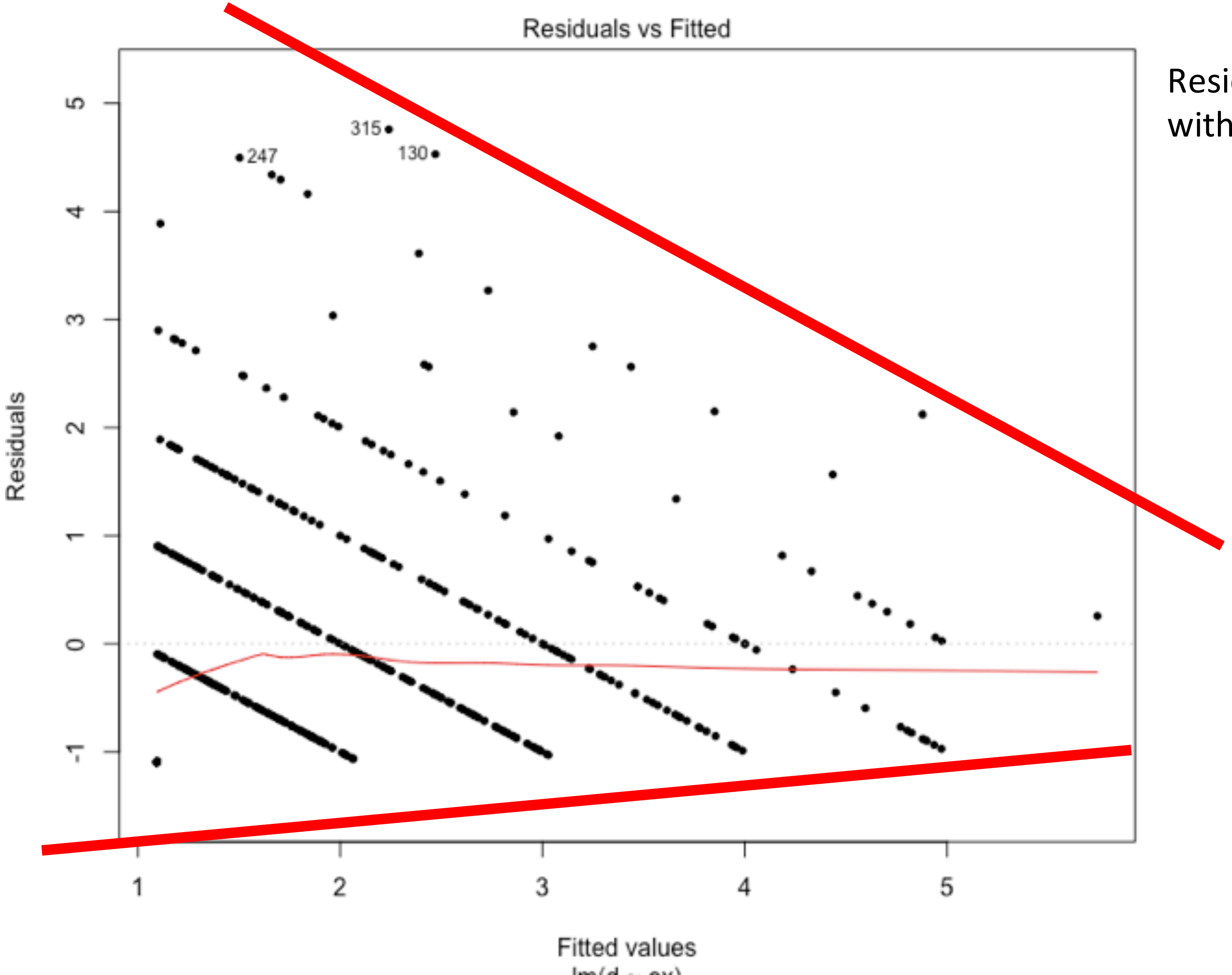


Residual variance decreases with Y

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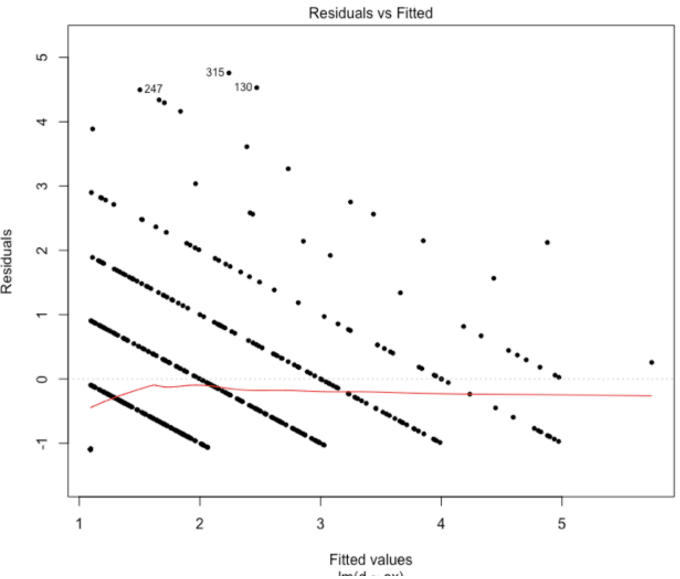

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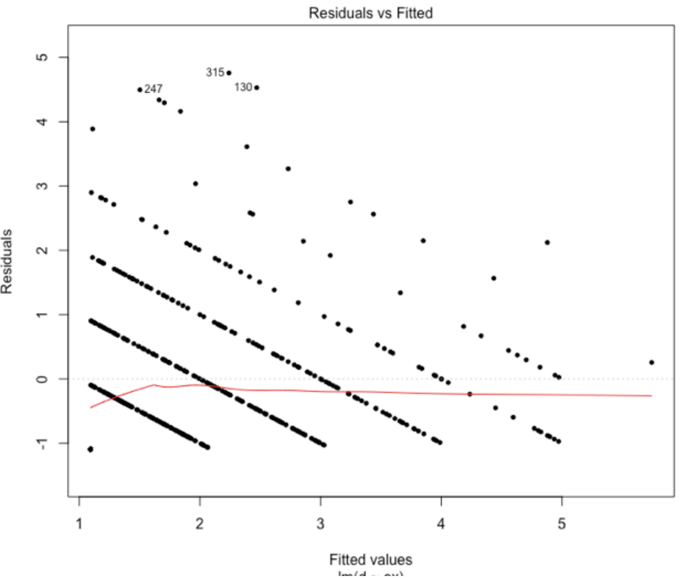


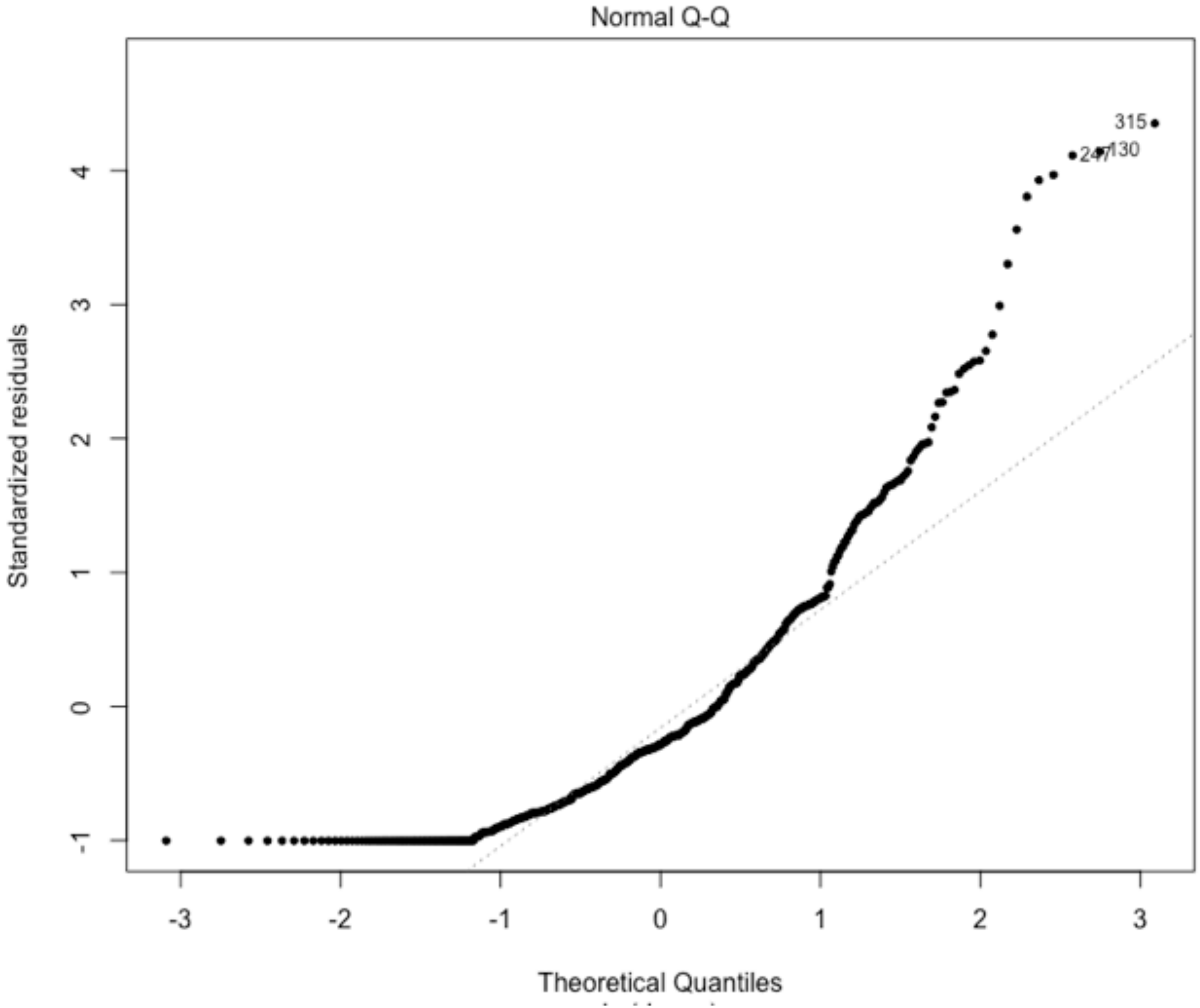

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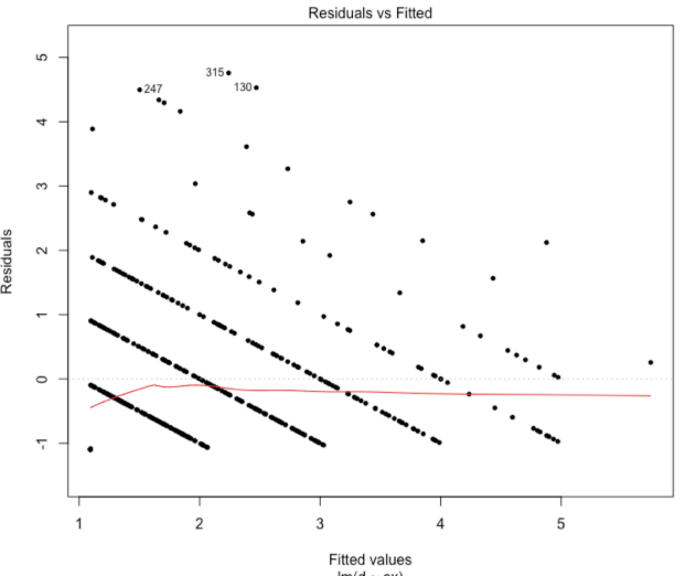


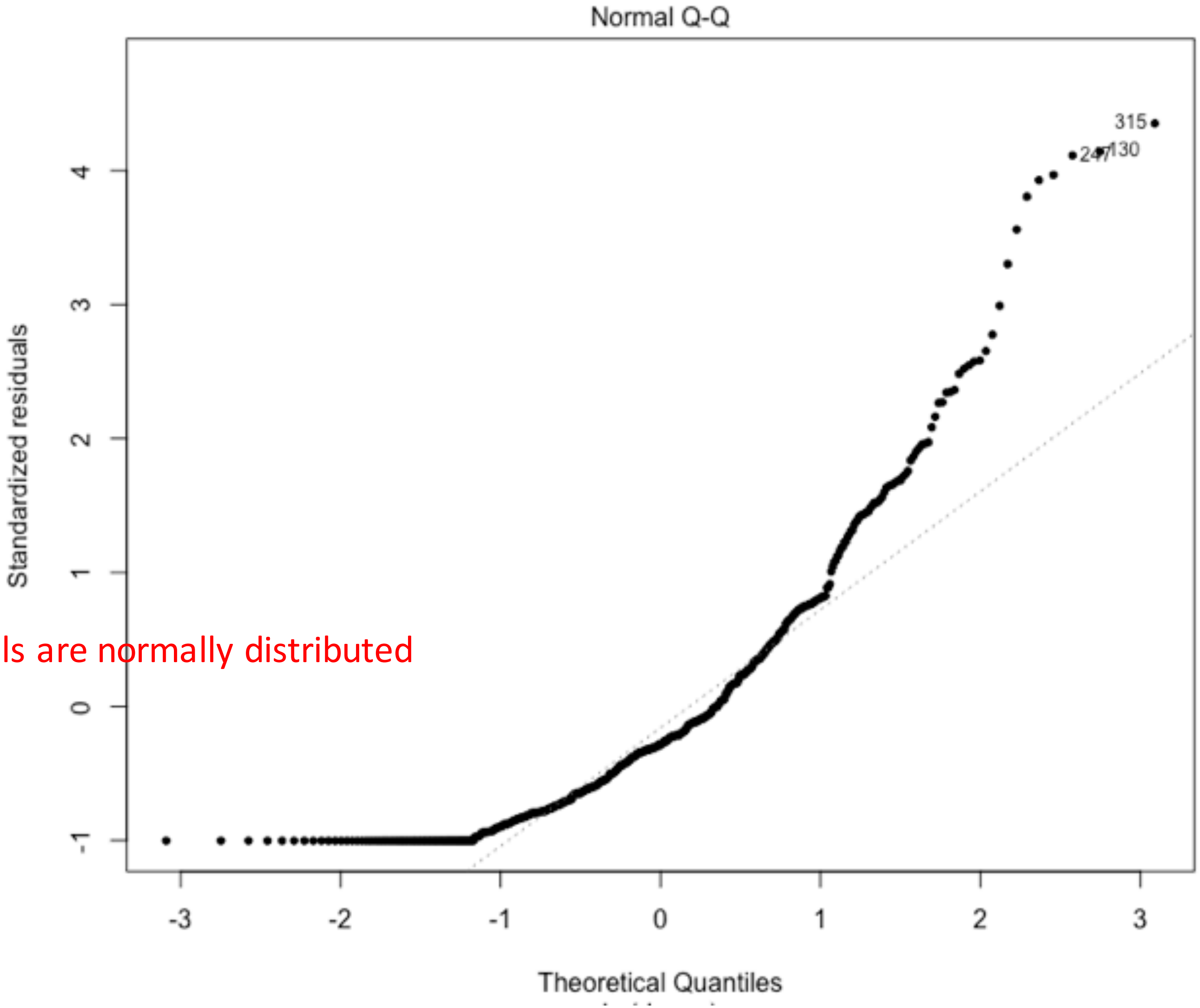


Violation of the assumption that residuals are normally distributed

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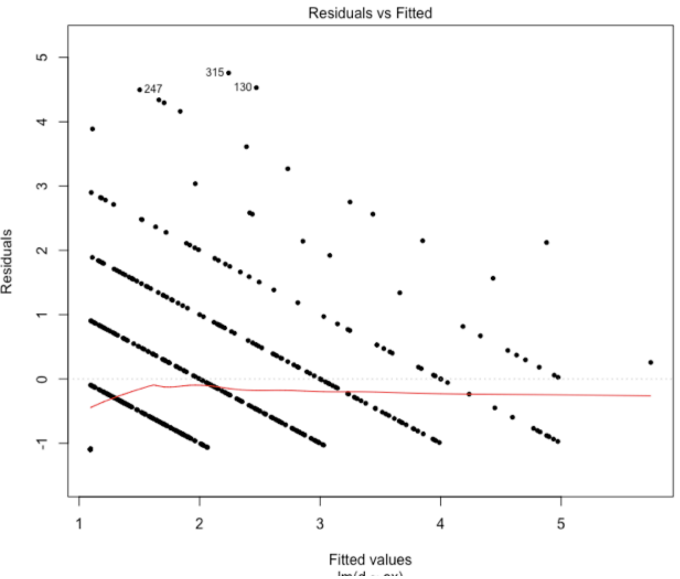


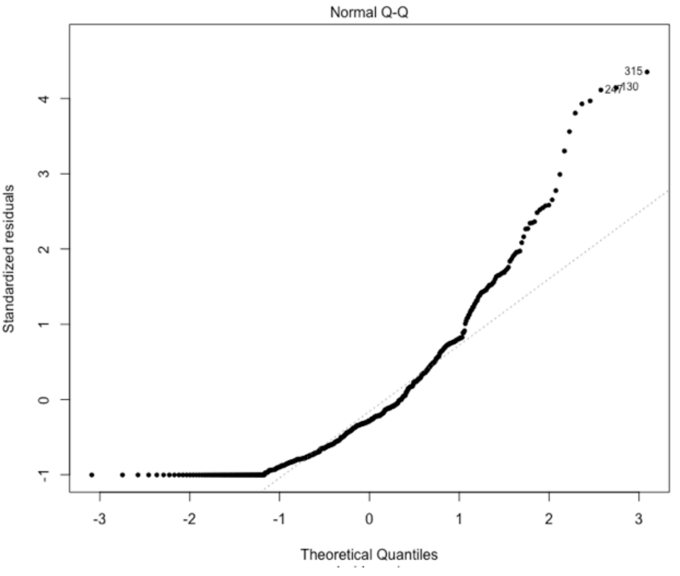


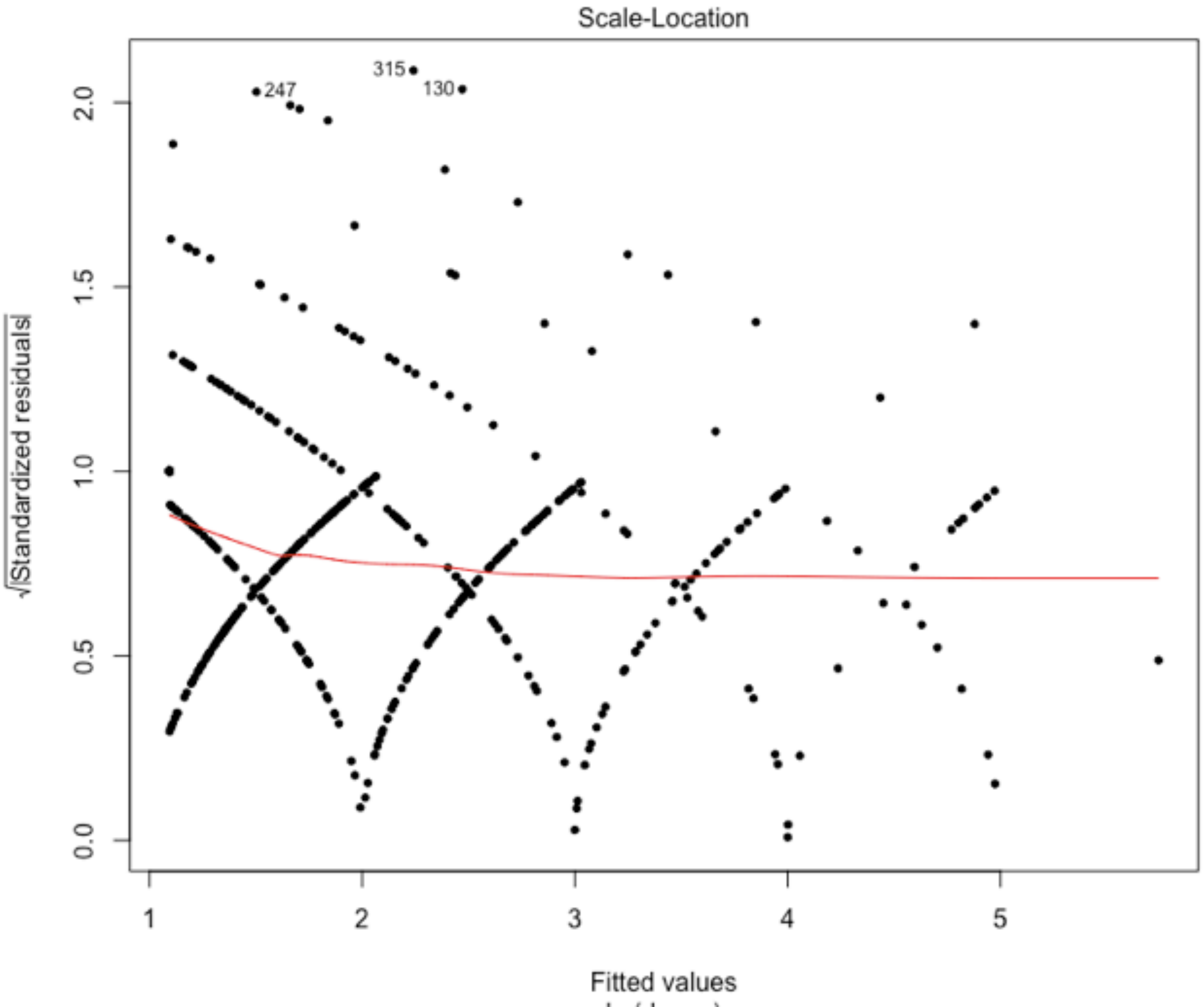

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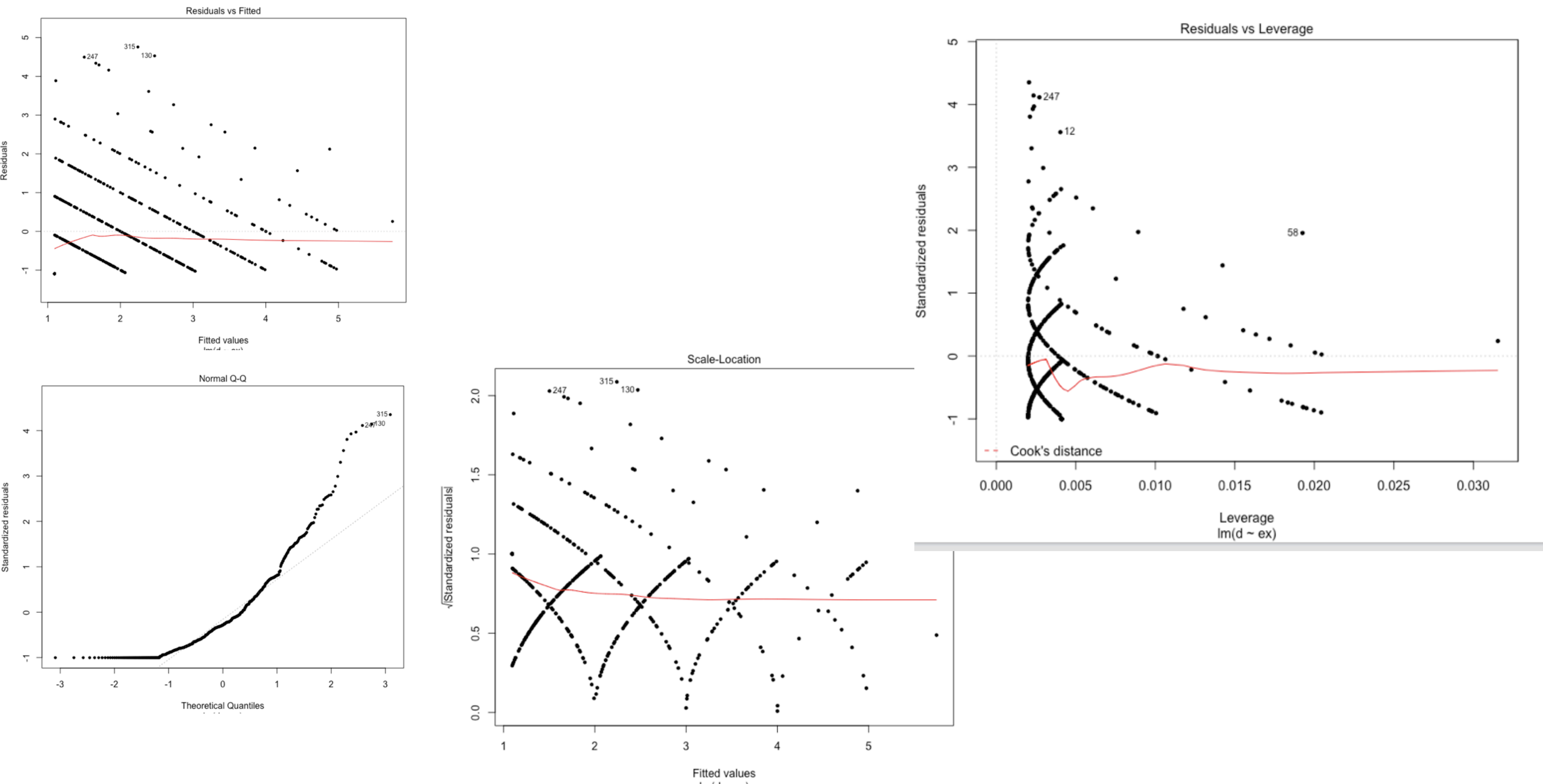

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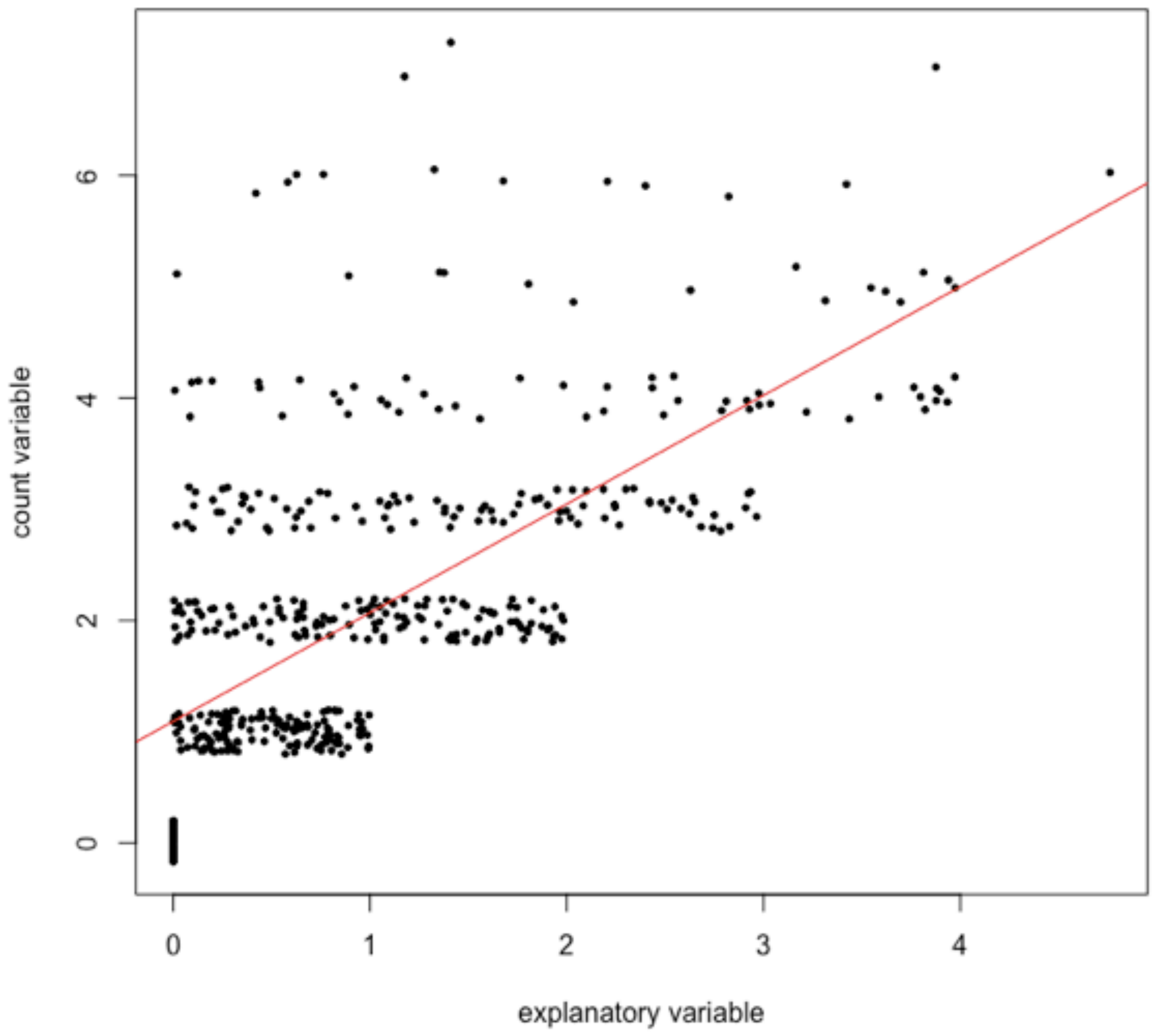


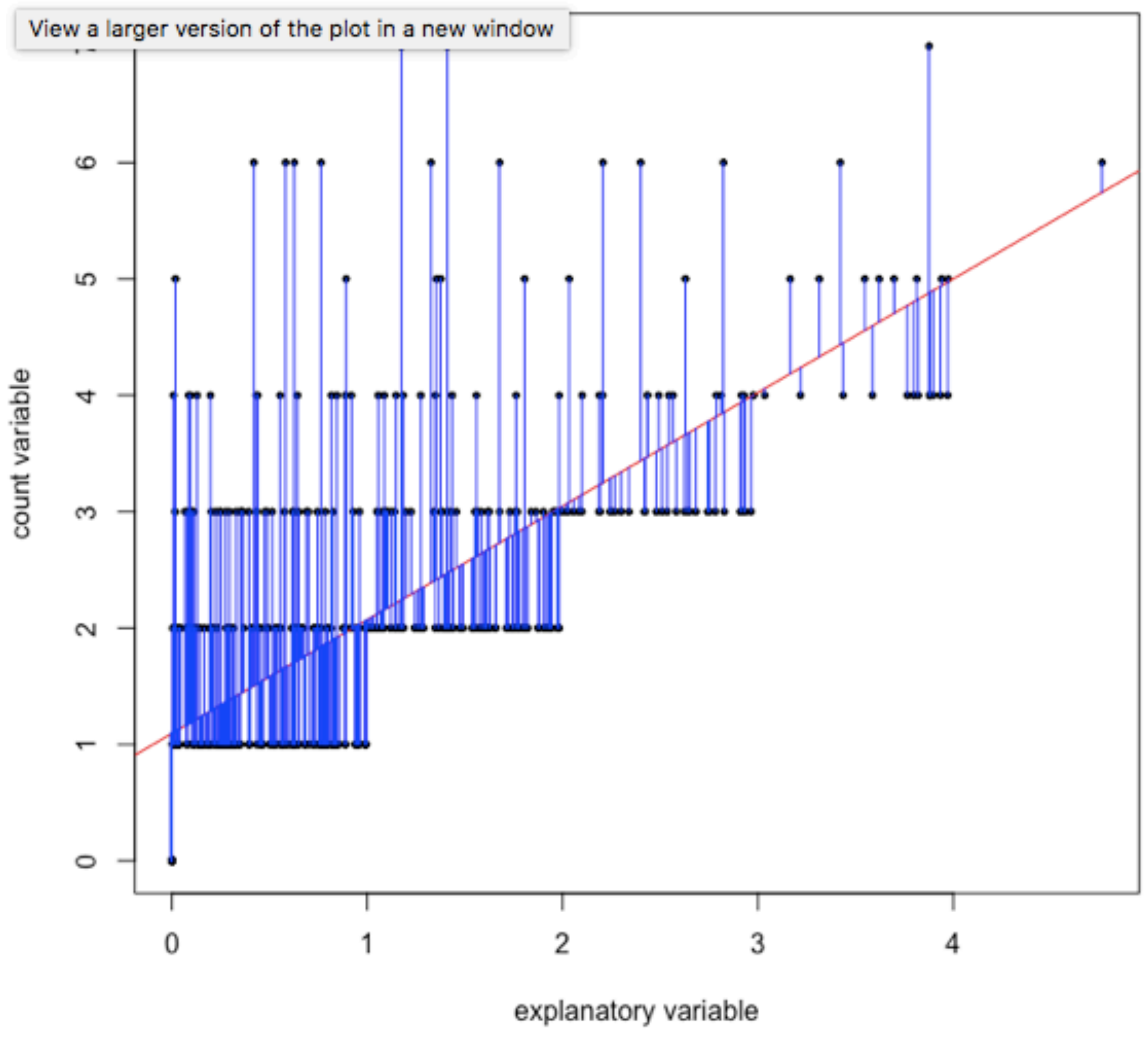

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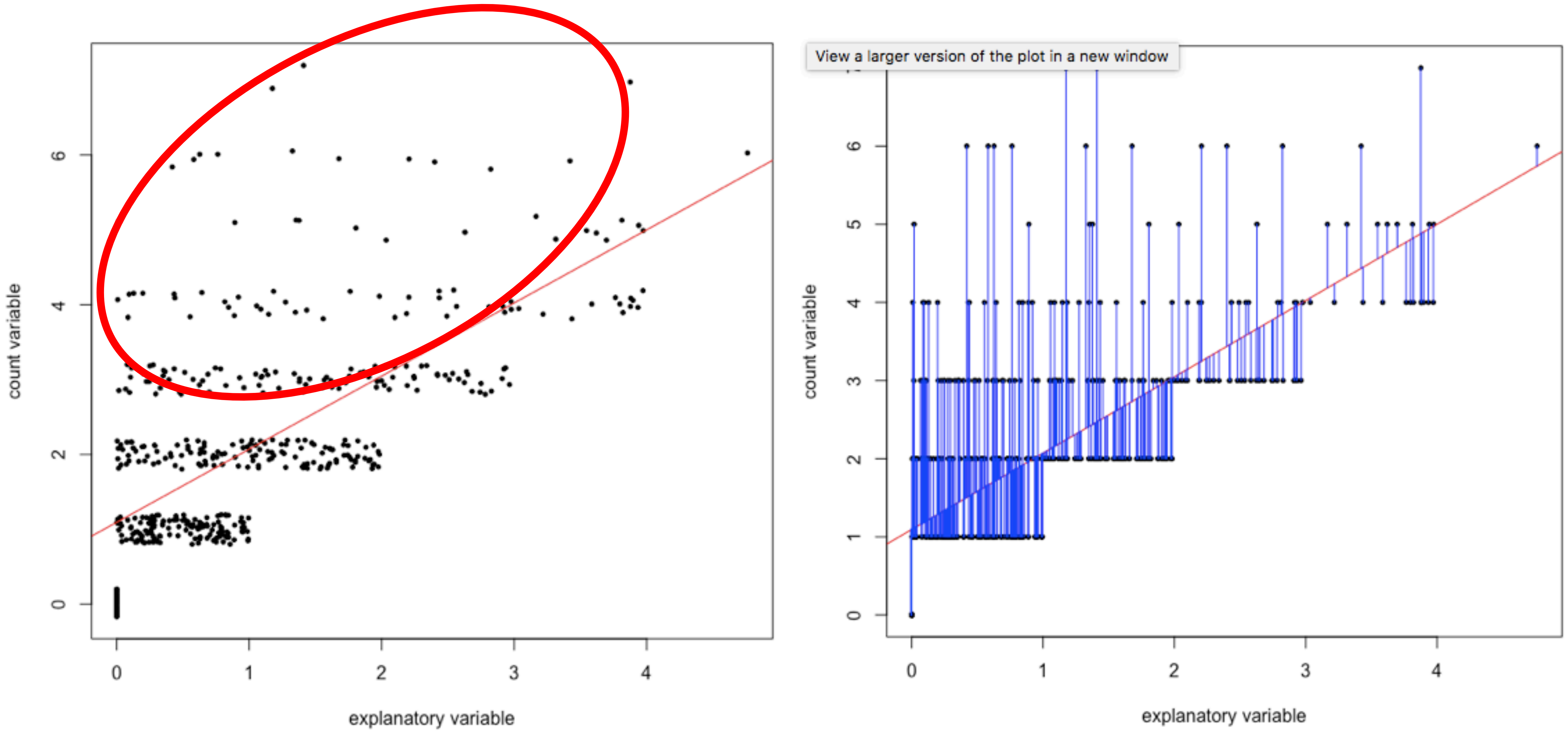

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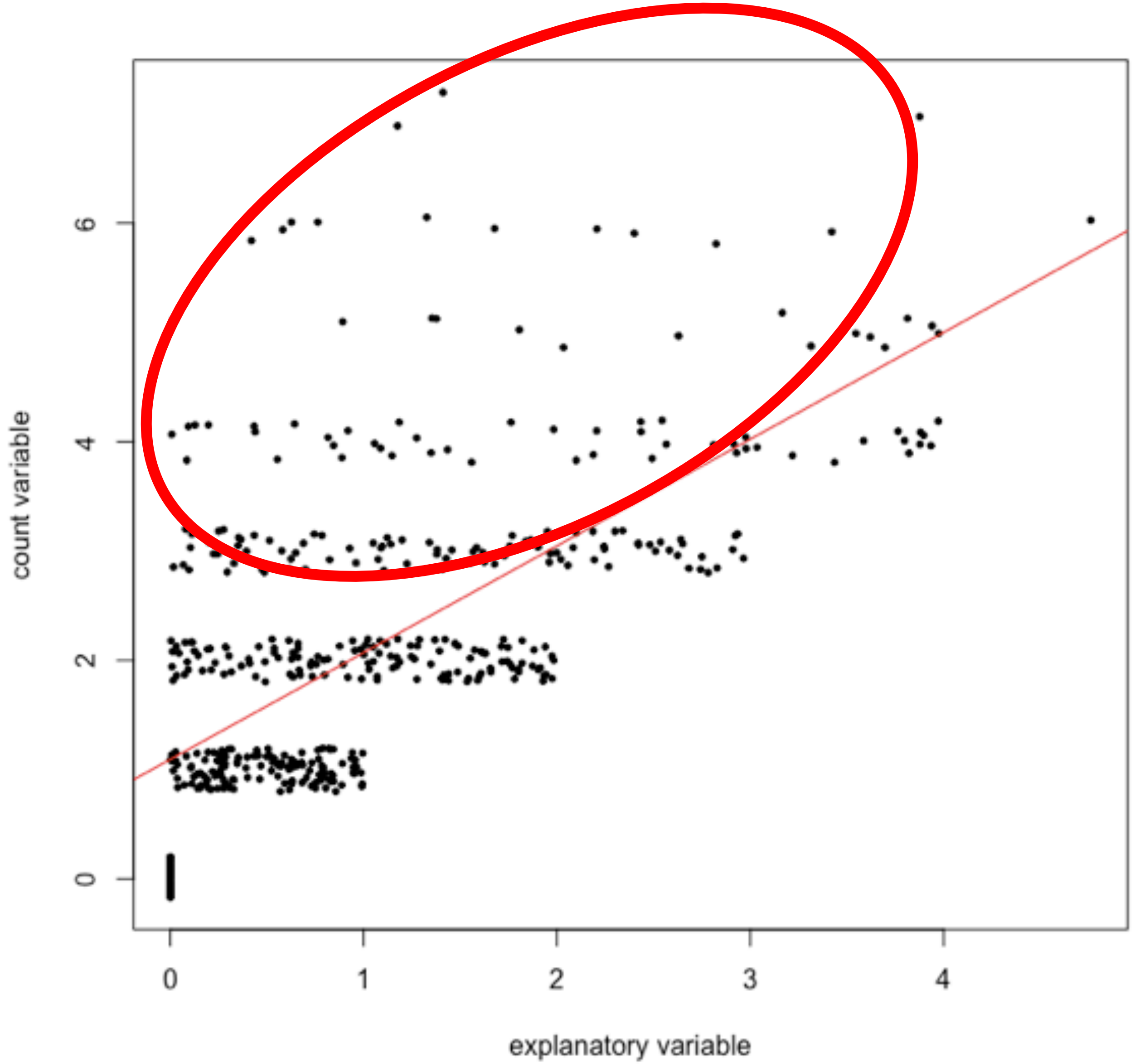


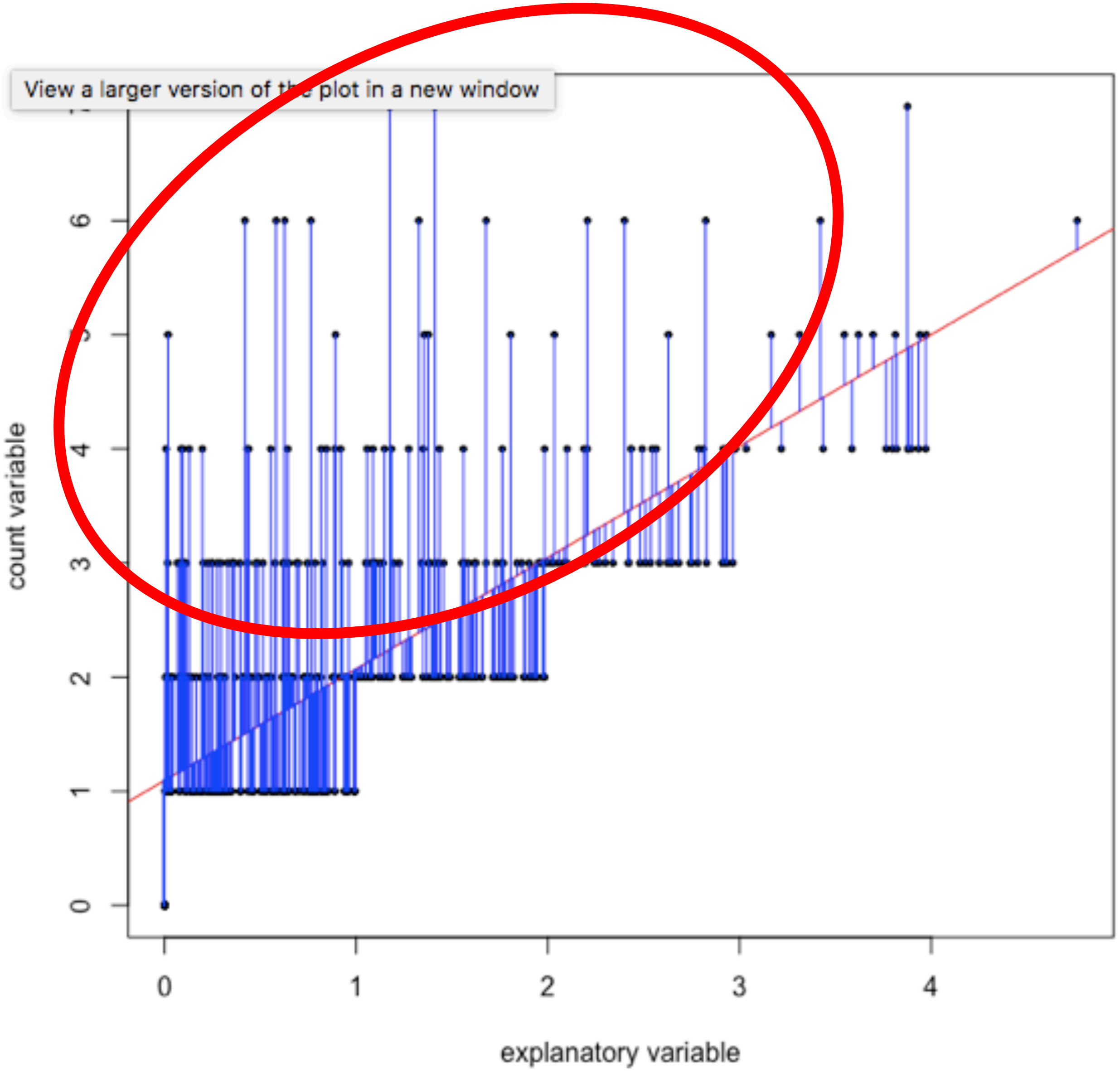

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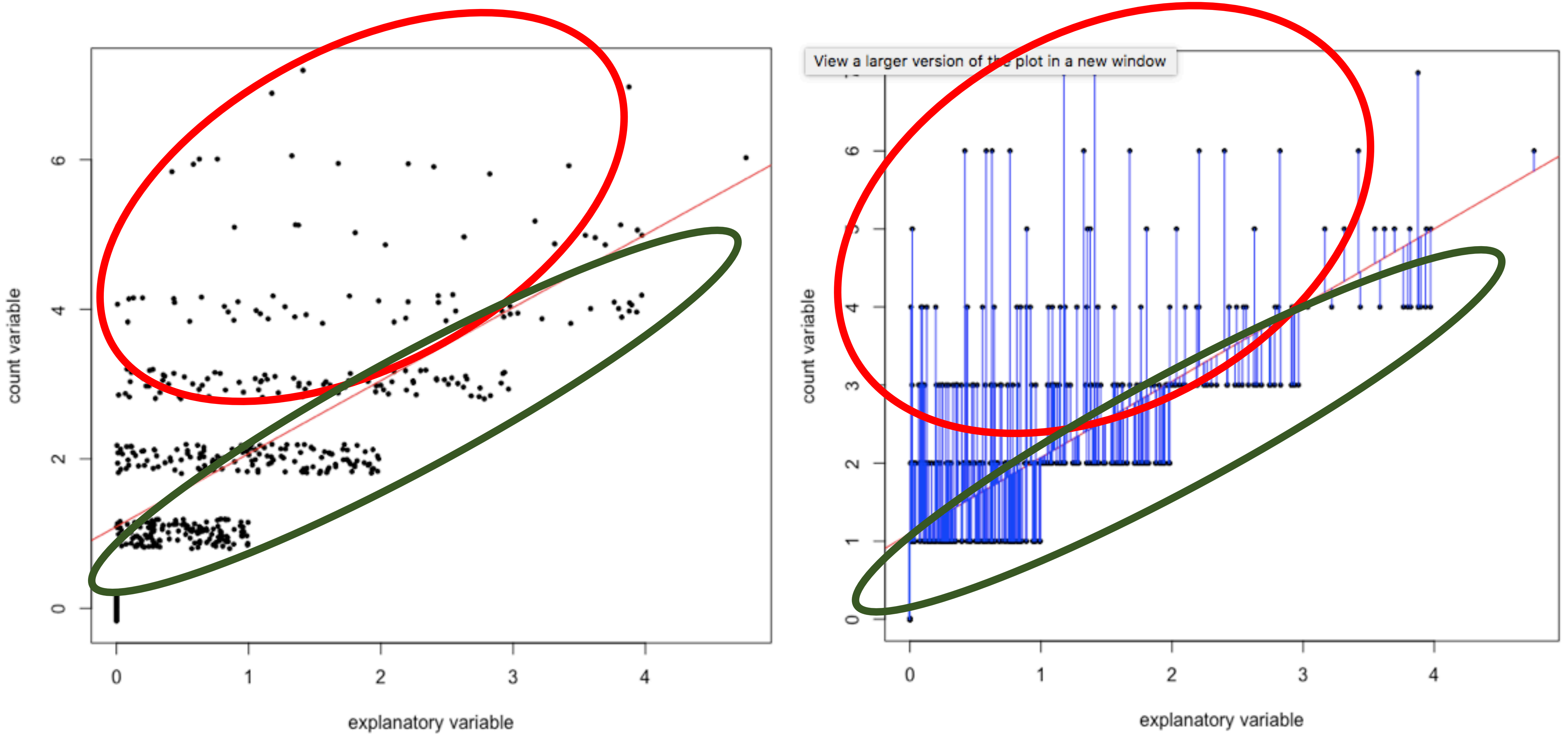

# What is the problem here?

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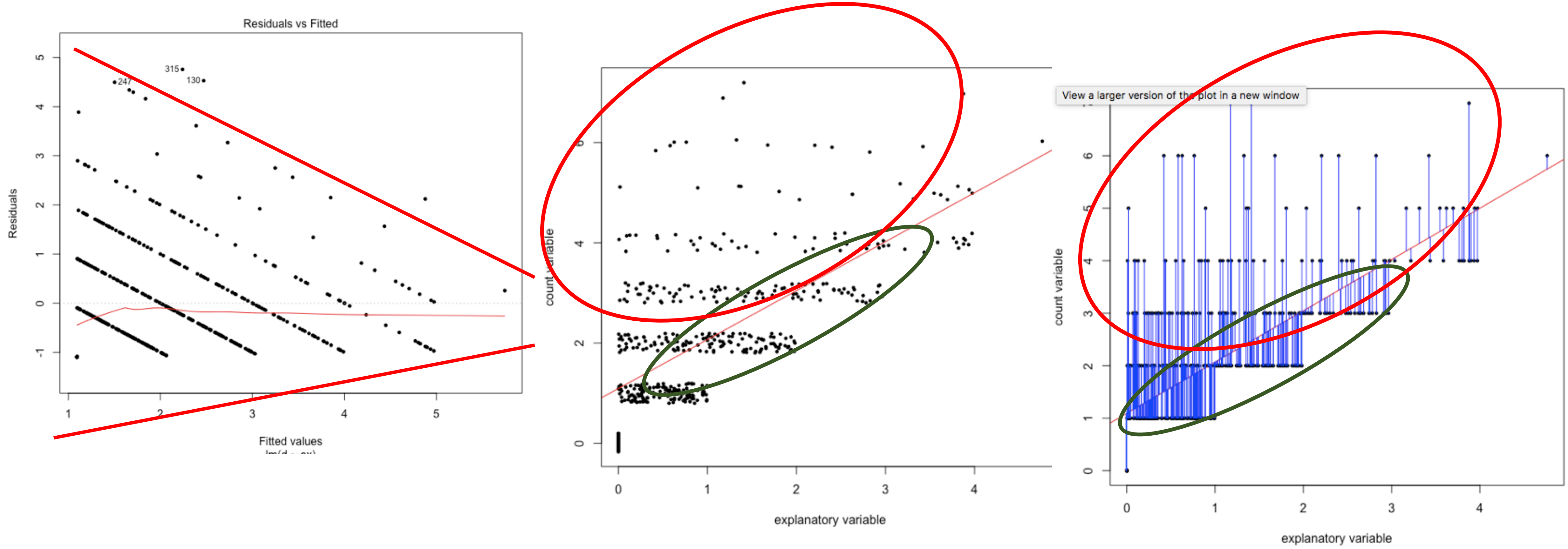
- Residual variance decreases with increasing Y



# What is the problem here?

- Residual variance decreases with increasing Y
- Still a problem when transformed
  - And then, zeroes are hard to account for

# Assumptions of a linear model:

$$\hat{Y} = \mathrm{XB} + E, \qquad y_i \sim N(\bar{Y}_i, \sigma^2)$$

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Predicted values

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$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \end{bmatrix}$$

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Parameter estimates of linear model

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$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \\ .. \\ \beta_i \end{bmatrix}$$

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Error

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Predicted values

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Error

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And a constant variance

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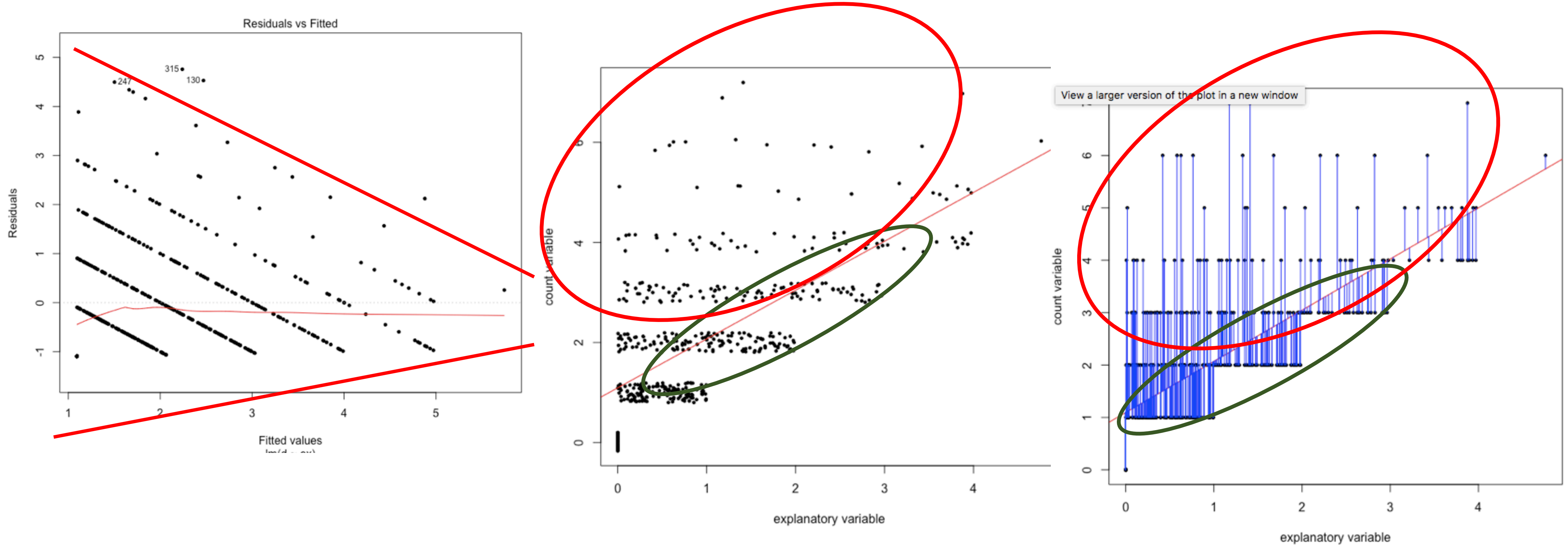
# Variance CHANGES with y



# This clearly does not work for count data

- Variance increases with predicted mean

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- Variance increases with predicted mean
- Errors follow *Poisson* distribution

# What is a *Poisson* distribution?

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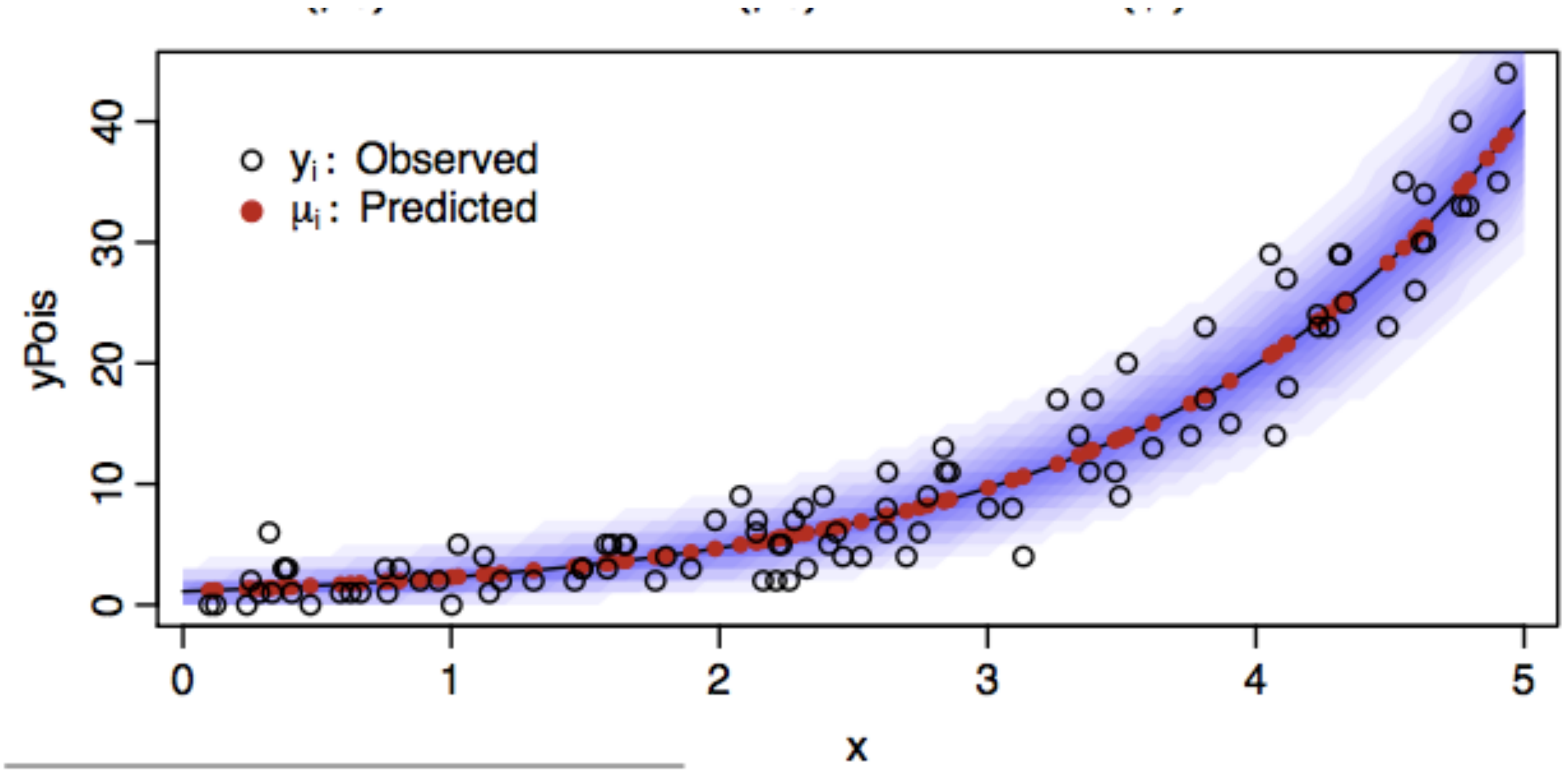

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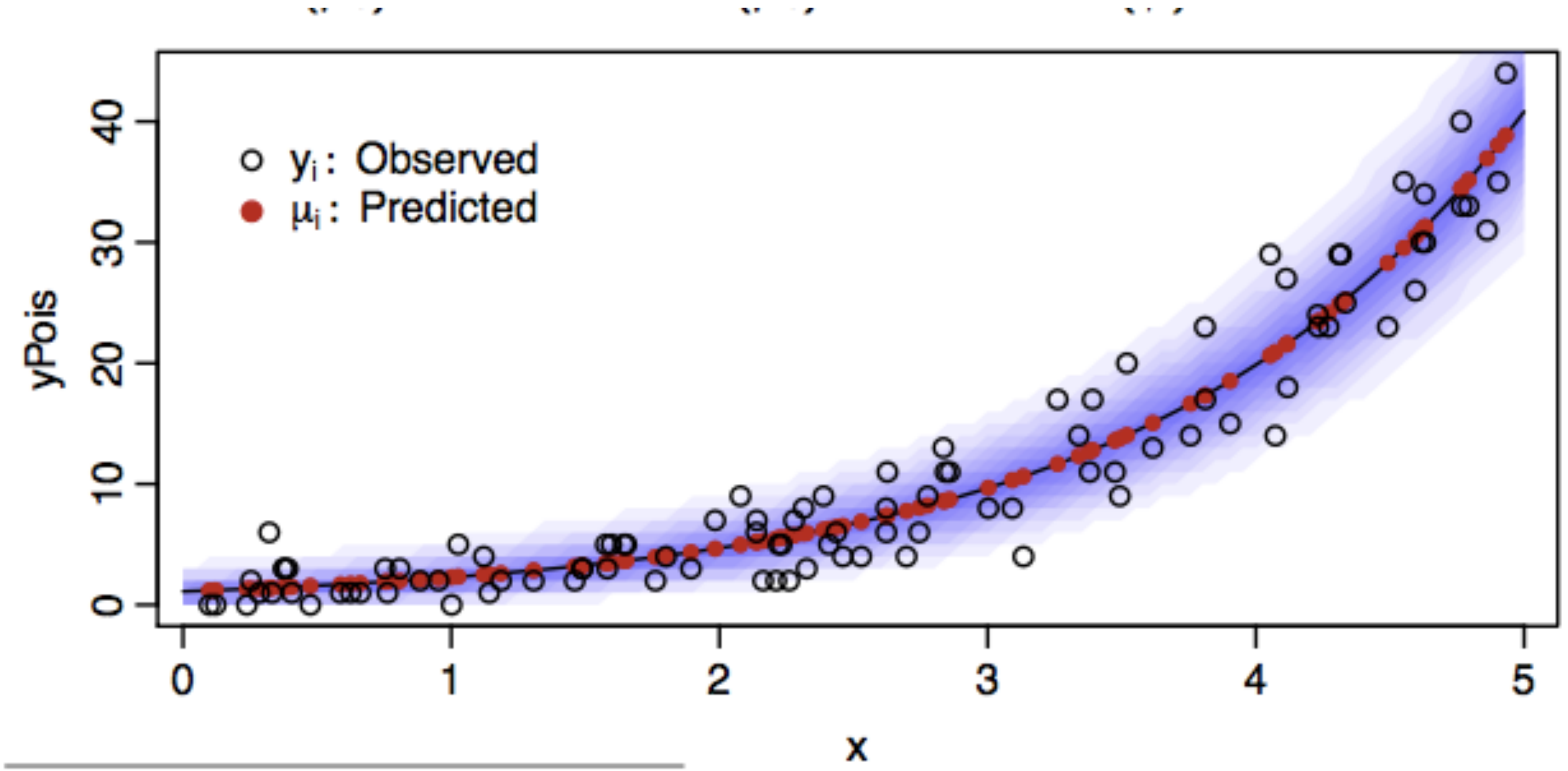


Variance = mean

# GLMs to the rescue!

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- *Log link* and *Logit link*

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- *Log link* and *Logit link* for
- COUNT and BINARY data

# Count data – log link– *Poisson* model

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Scale parameter

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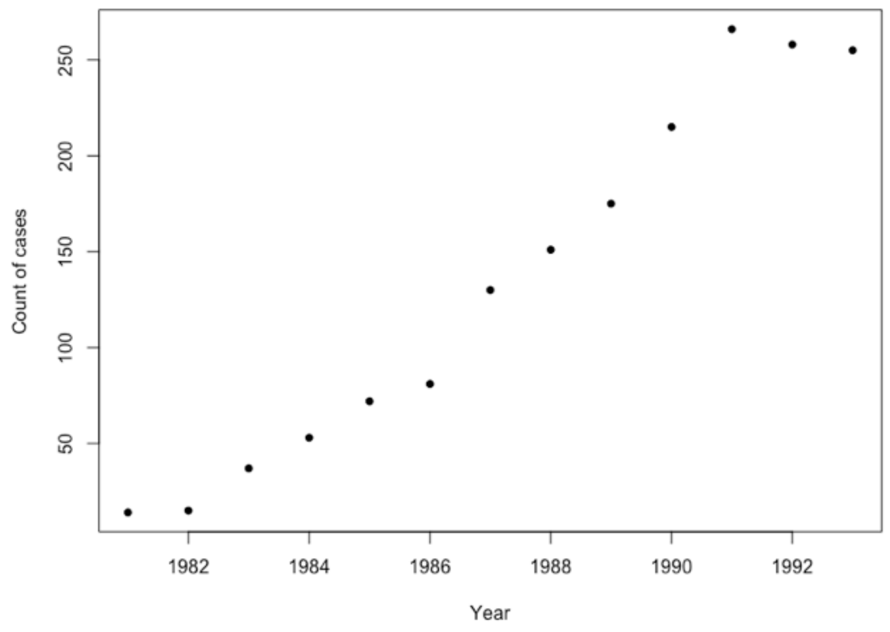

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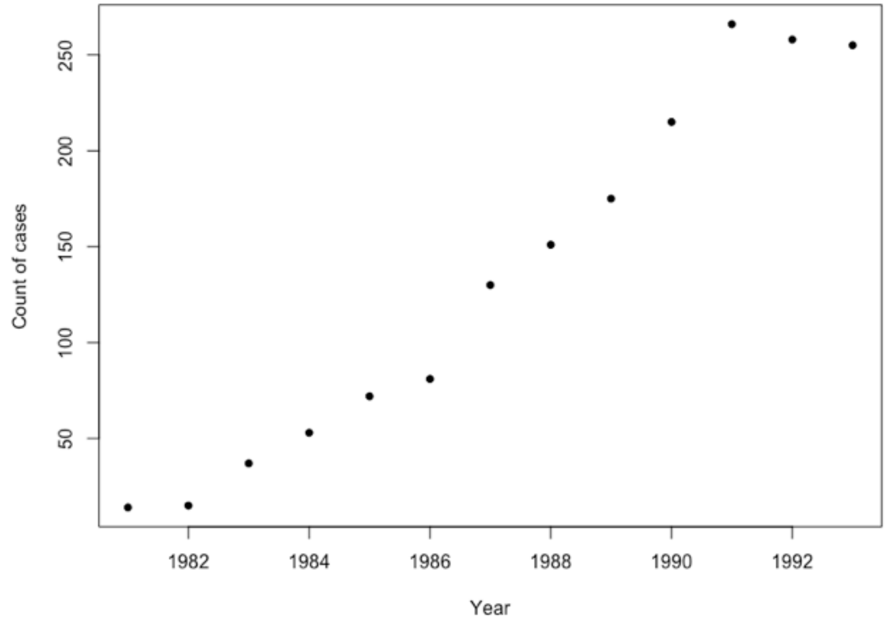


```
> summary(m1)

Call:
glm(formula = cases ~ yr, family = "poisson")

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-4.6398  -1.3494  -0.2574   2.1500   2.6866

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept) -3.915e+02  1.495e+01  -26.19   <2e-16 ***
yr           1.994e-01  7.513e-03   26.54   <2e-16 ***
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Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for poisson family taken to be 1)
```
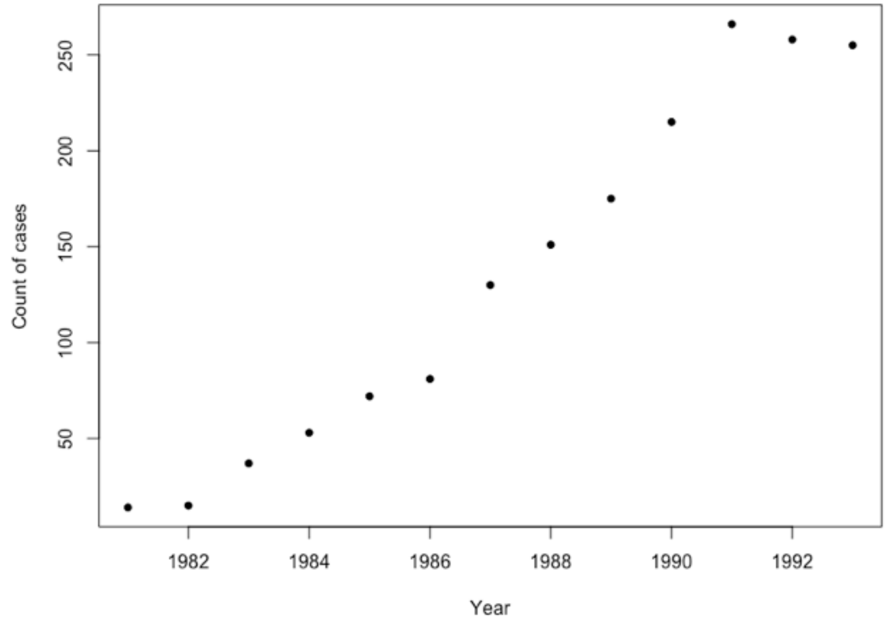

# How to interpret?

$$y_i = Poisson(\exp(-39.15 + 0.20x_{i1}))$$

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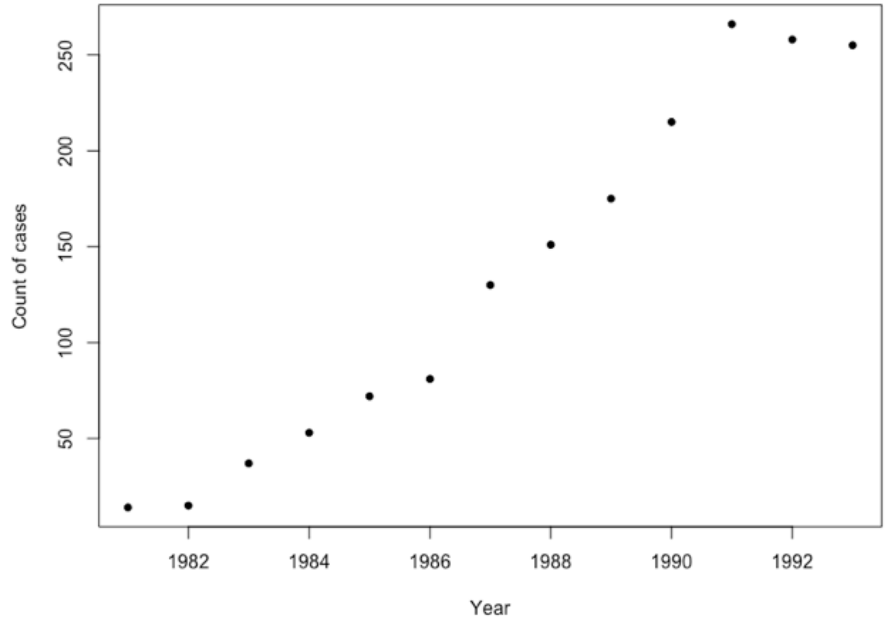


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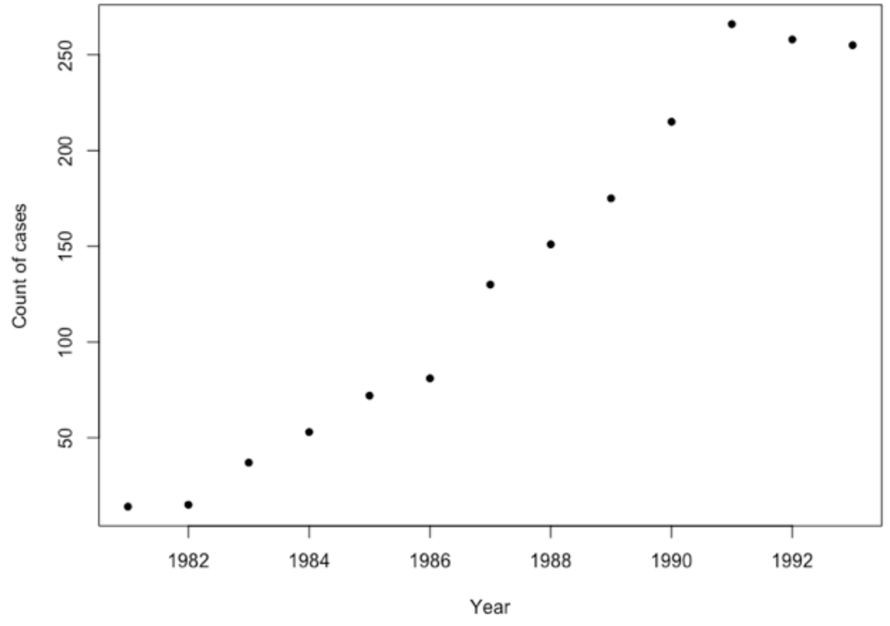


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Intercept. Cases when year = 0. Irrelevant

Slope. Increase in cases over time. On LINK scale.

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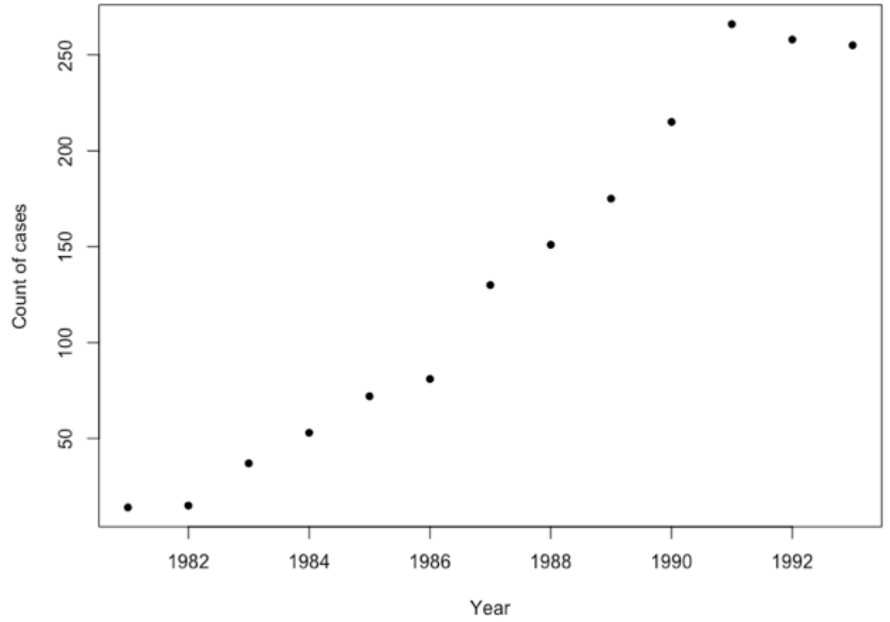
$$y_i = Poisson(\exp(-39.15 + 0.20x_{i1}))$$

$$e^{b_1} = e^{0.2}$$




```
> summary(m1)

Call:
glm(formula = cases ~ yr, family = "poisson")

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-4.6398  -1.3494  -0.2574   2.1500   2.6866

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept) -3.915e+02  1.495e+01  -26.19   <2e-16 ***
yr           1.994e-01  7.513e-03   26.54   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for poisson family taken to be 1)
```

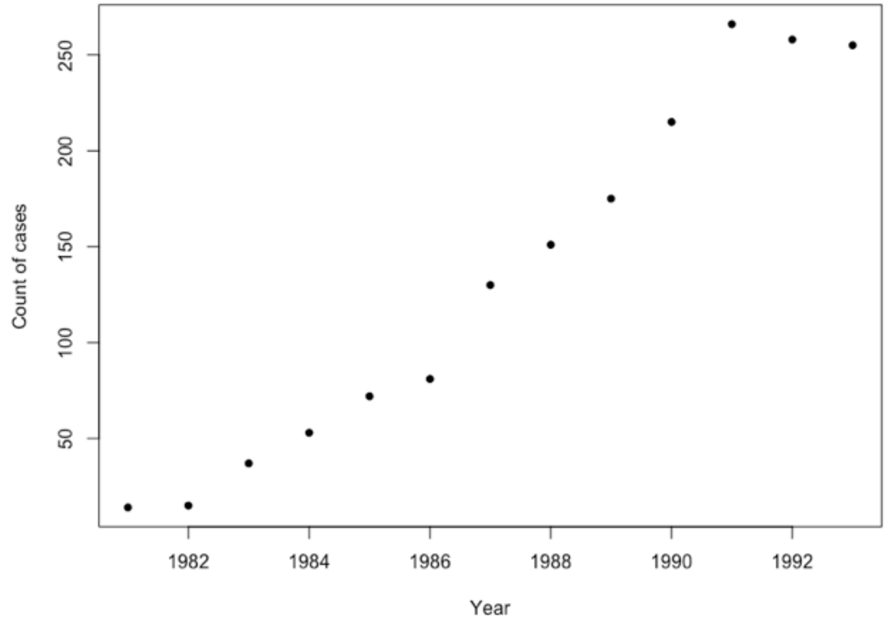
Intercept. Cases when year = 0. Irrelevant

Slope. Increase in cases over time. On LINK scale.

# How to interpret?

$$y_i = Poisson(\exp(-39.15 + 0.20x_{i1}))$$

$$e^{b_1} = e^{0.2} = 1.22$$

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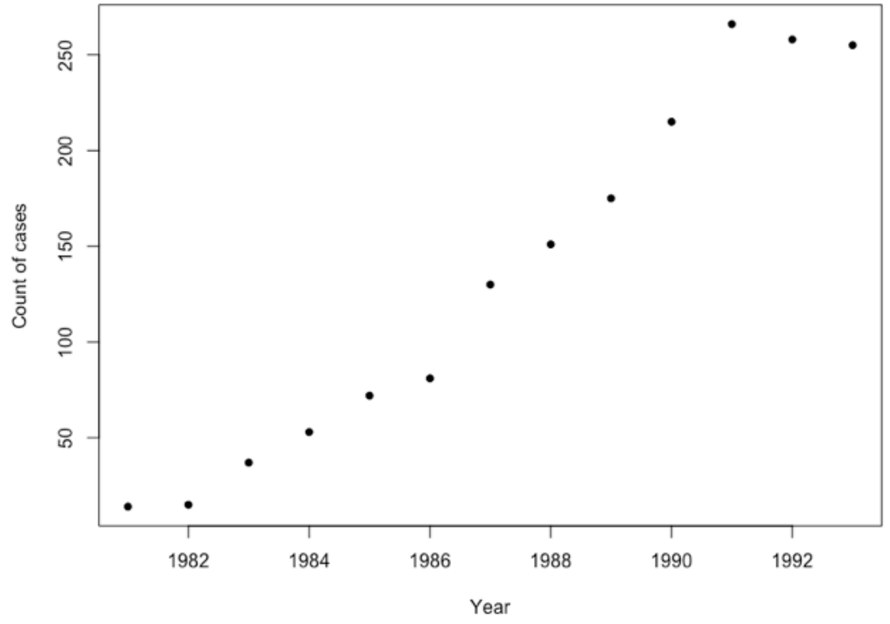
Every consecutive year, there are 1.22% more cases than the year before

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- Deviance is a similar idea, but slightly different estimation
- Not based on sums of squares
- You used deviance in likelihood ratio test, or AIC to select best models

# Detour - AIC

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- Rule of thumb: $\Delta AIC < 2$ not statistically significant

# AIC in R

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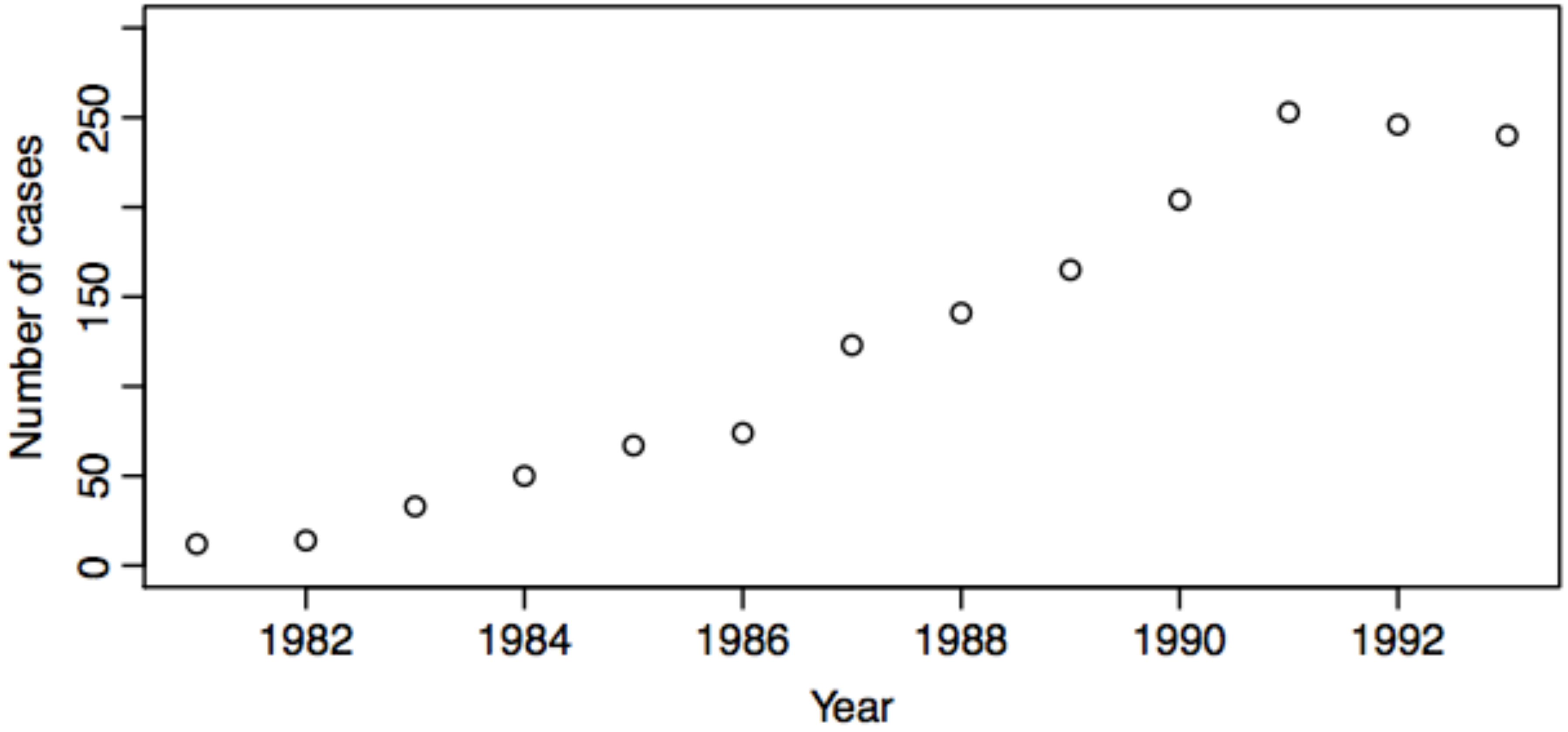
> AIC(m0)-AIC(m1)
[1] 21.60864
> |
```

# Example

- Number of AIDS cases in Belgium from 1981 to 2013.
- Models with linear, quadratic and cubic linear predictors.



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    family = poisson(link = log))
am3 <- glm(cases ~ year + I(year^2) + I(year^3),
    data = belg.aids, family = poisson(link = log))
```

# Example

```
AIC(null, am1, am2, am3)
##       df   AIC
## null   1 955.9
## am1    2 166.4
## am2    3  96.9
## am3    4  98.7
```

# Example

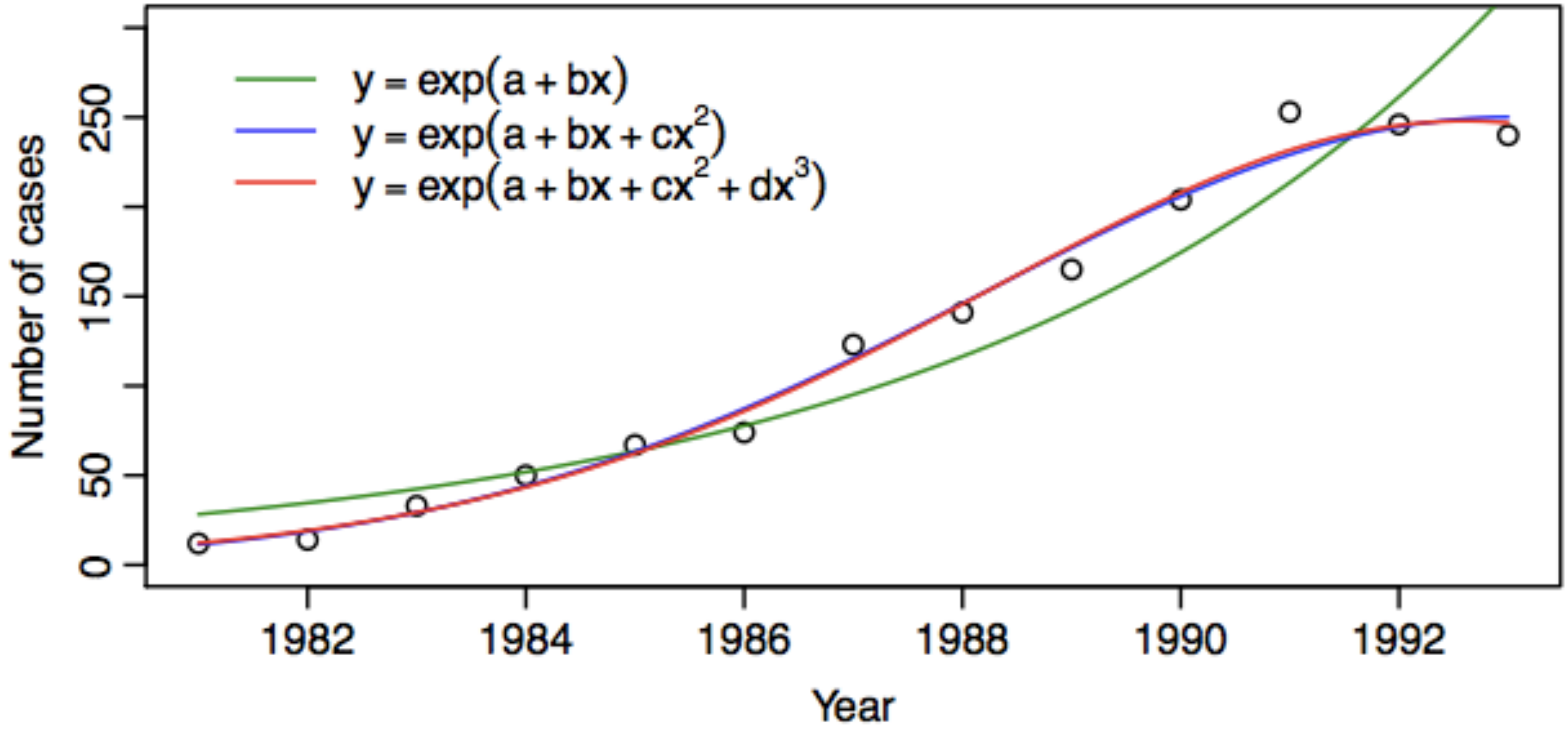
- And we can use linear models summaries to look at the significance of coefficients
- Importantly, model coefficients are estimated and reported *on the scale of the linear predictor*

```
summary(am2)

...
##
## Coefficients:
##              Estimate Std. Error z value Pr(>|z|)
## (Intercept) -8.48e+04   1.05e+04   -8.07  7.3e-16
## year         8.51e+01   1.06e+01    8.05  8.4e-16
## I(year^2)   -2.14e-02   2.66e-03   -8.03  9.8e-16
##
...
```

# Example

- The model is described on the scale of the linear predictor
- Showing those models on the original data:



# HO – Poisson models