Linear Algebra - HW12

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1 8.1.1

For the following matrix, find the eigenvalues and their corresponding eigenvectors:

$$M = \begin{bmatrix} 3.0 & 0 \\ 0 & -0.6 \end{bmatrix}$$

We can start with the fundamental eigentheory expression:

$$M\vec{x} = \lambda\vec{x} \tag{1}$$

$$M\vec{x} - \lambda\vec{x} = M\vec{x} - \lambda I\vec{x} = \vec{0} \tag{2}$$

$$(M - \lambda I)\vec{x} = \vec{0} \tag{3}$$

Given our matrix, we can begin solving for our eigenvalues:

$$M - \lambda I = \begin{bmatrix} 3.0 - \lambda & 0 \\ 0 & -0.6 - \lambda \end{bmatrix} \tag{4}$$

Translating this into our previous equation:

$$\begin{bmatrix} 3.0 - \lambda & 0 \\ 0 & -0.6 - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5}$$

Because we want to find non-trivial solutions, we must find values for λ to make the determinant of this vector 0 (note that this will result in an infinite amount of solutions for x_1 and x_2).

$$\left|M - \lambda I\right| = \begin{vmatrix} 3.0 - \lambda & 0 \\ 0 & -0.6 - \lambda \end{vmatrix} = (3.0 - \lambda)(-0.6 - \lambda) = 0 \tag{6}$$

$$\lambda_1 = 3,\ \lambda_2 = -0.6 \tag{7}$$

Now we can find the eigenvectors corresponding to each eigenvalue. So for $\lambda_1 = 3$:

$$\begin{bmatrix} 0 & 0 \\ 0 & -3.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} \tag{8}$$

Note that a can be any arbitrary value— it will still make the equation true. Moving to $\lambda_2 = -0.6$:

$$\begin{bmatrix} 3.6 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix} \tag{9}$$

Again, b can be any arbitrary value for this eigenvector.

2 8.1.2

For the following matrix, find the eigenvalues and their corresponding eigenvectors:

$$M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Just seeing that all of this matrix's elements are 0, the expression $M\vec{x}$ will result in a vector whose elements are all 0. Considering (1), we know that expression $\lambda\vec{x}$ should also be a vector of 0's. Thus, the λ scalar must be 0 (because we're looking for non-trivial solutions), and the eigenvectors are arbitrary:

$$\lambda = 0, \; \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \tag{10}$$

3 8.1.3

For the following matrix, find the eigenvalues and their corresponding eigenvectors:

$$M = \begin{bmatrix} 5 & -2 \\ 9 & -6 \end{bmatrix}$$

Following the same process as in 8.1.1, we can find our system in terms of λ to find our eigenvalues:

$$\begin{bmatrix} 5-\lambda & -2 \\ 9 & -6-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{11}$$

Solving for our eigenvalues:

$$\begin{vmatrix} 5-\lambda & -2 \\ 9 & -6-\lambda \end{vmatrix} = (5-\lambda)(-6-\lambda) - (-2)(9) = 0 \tag{12}$$

$$\lambda_1 = -4, \; \lambda_2 = 3 \tag{13}$$

We can now begin finding eigenvectors corresponding to each eigenvalue. For λ_1:

$$\begin{bmatrix} 9 & -2 \\ 9 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a \begin{bmatrix} 2 \\ 9 \end{bmatrix} \tag{14}$$

a is yet another arbitrary constant; x_1 simply needs to be $\frac{2}{9}$ of x_2. Now solving for the eigenvector corresponding to λ_2:

$$\begin{bmatrix} 2 & -2 \\ 9 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{15}$$

4 8.1.14

For the following matrix, find the eigenvalues and their corresponding eigenvectors:

$$M = \begin{bmatrix} 2 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

Making the expression to find our eigenvalues:

$$\begin{bmatrix} 2-\lambda & 0 & -1 \\ 0 & \frac{1}{2}-\lambda & 0 \\ 1 & 0 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{16}$$

Setting the matrix's determinant to 0:

$$(2-\lambda)(\frac{1}{2}-\lambda)(4-\lambda) + (\frac{1}{2}-\lambda) = 0 \implies \lambda_1 = \frac{1}{2}, \; \lambda_2 = 3 \tag{17}$$

For λ_1:

$$\begin{bmatrix} \frac{3}{2} & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & \frac{7}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} \tag{18}$$

Since there is only one redundancy in this system, x_1 and x_2 must be 0. Now solving for the eigenvector for λ_2:

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & -\frac{5}{2} & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b \\ 0 \\ -b \end{bmatrix} \tag{19}$$

5 8.2.7

Find the limit state of the Markov process modeled by the given matrix.

$$T = \begin{bmatrix} 0.2 & 0.5 \\ 0.8 & 0.5 \end{bmatrix}$$

The Markov steady state representation is simply the Eigen expression with the λ scalar as 1.

$$T\vec{s_s} = \lambda\vec{s_s} = \vec{s_s} \tag{20}$$

$$T - \lambda I = \begin{bmatrix} 0.2 - \lambda & 0.5 \\ 0.8 & 0.5 - \lambda \end{bmatrix} = \begin{bmatrix} -0.8 & 0.5 \\ 0.8 & -0.5 \end{bmatrix} \tag{21}$$

Now we can solve for our steady state vector, knowing that $\lambda = 1$:

$$\begin{bmatrix} -0.8 & 0.5 \\ 0.8 & -0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a \begin{bmatrix} 5 \\ 8 \end{bmatrix} \tag{22}$$

6 8.2.8

For the following matrix, find the eigenvalues and their corresponding eigenvectors:

$$T = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.3 & 0.6 & 0.1 \\ 0.3 & 0.1 & 0.6 \end{bmatrix}$$

Following the same process as in 8.2.7, we can find the steady state vector knowing that $\lambda = 1$:

$$T\vec{s_s} = \lambda\vec{s_s} = \vec{s_s} \tag{23}$$

$$T - \lambda I = \begin{bmatrix} 0.4 - \lambda & 0.3 & 0.3 \\ 0.3 & 0.6 - \lambda & 0.1 \\ 0.3 & 0.1 & 0.6 - \lambda \end{bmatrix} = \begin{bmatrix} -0.6 & 0.3 & 0.3 \\ 0.3 & -0.4 & 0.1 \\ 0.3 & 0.1 & -0.4 \end{bmatrix} \tag{24}$$

Now solving for the elements in our steady-state vector:

$$\begin{bmatrix} -0.6 & 0.3 & 0.3 \\ 0.3 & -0.4 & 0.1 \\ 0.3 & 0.1 & -0.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{25}$$